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CONTRACTION-FREE SEQUENT CALCULI FOR INTUITIONISTIC LOGIC: A CORRECTION

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Abstract. We present a much-shortened proof of a major result (originally due to Vorob'ev) about intuitionistic propositional logic: in essence, a correction of our 1992 article, avoiding several unnecessary definitions.

§1. Introduction. One of the standard sequent calculi for **Int** (intuitionistic propositional logic) (with a language $\mathcal{L}$, based on atoms P (i.e., atoms p, q, r etc) using absurdity $\bot$, conjunction, disjunction and implication $\to$) is **G3ip** [4]. Sequents are of the form $\Gamma \Rightarrow A$ where Γ is a multiset of formulae and A is a formula. Combination of two multisets uses multiset sum. Its rules are

$$\frac{}{\Gamma, P \Rightarrow P}\ At \qquad \frac{}{\Gamma, \bot \Rightarrow E}\ L\bot$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ R\to \qquad \frac{\Gamma, A \to B \Rightarrow A \quad \Gamma, B \Rightarrow E}{\Gamma, A \to B \Rightarrow E}\ L\to$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ R\wedge \qquad \frac{\Gamma, A, B \Rightarrow E}{\Gamma, A \wedge B \Rightarrow E}\ L\wedge$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ R\vee \qquad \frac{\Gamma, A \Rightarrow E \quad \Gamma, B \Rightarrow E}{\Gamma, A \vee B \Rightarrow E}\ L\vee.$$

Note that (unless a loop-checker is added) root-first proof search in **G3ip** is nonterminating, because of the $L\to$ rule; so various authors proposed the trick of replacement of the $L\to$ rule by

$$\frac{\Gamma, P, B \Rightarrow E}{\Gamma, P, P \to B \Rightarrow E}\ L0\to \qquad \frac{\Gamma, D \to B \Rightarrow C \to D \quad \Gamma, B \Rightarrow E}{\Gamma, (C \to D) \to B \Rightarrow E}\ L\to\to$$

$$\frac{\Gamma, C \to (D \to B) \Rightarrow E}{\Gamma, (C \wedge D) \to B \Rightarrow E}\ L\wedge\to \qquad \frac{\Gamma, C \to B, D \to B \Rightarrow E}{\Gamma, (C \vee D) \to B \Rightarrow E}\ L\vee\to$$

and thus we have the calculus **G4ip** for **Int**; it can be shown to be equivalent to **G3ip**. (Half of this is straightforward, using admissibility of *Cut* in **G3ip**.) Its crucial feature is that if formulae and then sequents are appropriately weighted, each rule's premisses are of weight less than the conclusion; so root-first proof search

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terminates. This both allows applications to uniform interpolation, as in (e.g.) [3] and provides an easily implemented calculus for a simple logic where backtracking is required (this may have some pedagogical value).

This article corrects the argument for equivalence given in [1], which is much too complicated—its author believes he understood it in about 1991, but struggles to convince himself. See [2] for references to other arguments.

§2. Routine results.

Weakening (on the left) is easily shown to be admissible in the calculus **G3ip**. Routinely then, for any Γ and A, the sequent $\Gamma, A \Rightarrow A$ is derivable. *Contraction* and *Cut* are then shown to be admissible. This calculus then formalises the intuitionistic propositional logic **Int**—as axiomatised by the Hilbert system based on *Modus Ponens* and all axiom schemata (e.g., from Heyting's book) of **Int**. Checking this to be the case is routine.

PROPOSITION 2.1. *The rules $R\to$, $L\wedge$, $R\wedge$, $L\vee$ (and the second premiss of $L\to$) are invertible in* **G3ip**.

PROOF. Routine. ⊣

PROPOSITION 2.2. *The rules $L0\!\to$, $L\wedge\!\to$, $L\vee\!\to$ (and the second premiss of $L\!\to\!\to$) of* **G4ip** *are invertible in* **G3ip**.

PROOF. By admissibility of *Cut* in **G3ip**. ⊣

We return to the question of equivalence between the two calculi. We recall from the Introduction that there is a definition of weight of sequents ensuring that root-first proof search terminates; crucially, this also allows use of induction on sequent weight, as illustrated below.

§3. Main old result, with new proof.

THEOREM 3.1. *Any sequent derivable in* **G3ip** *is derivable in* **G4ip**.

PROOF. By induction on the weight of the sequent. Without loss of generality, we may assume it is not of the form $\Gamma, P \Rightarrow P$ and its antecedent does not contain $\bot$. Using the invertibility results and the induction hypothesis, we may also assume that the succedent is not an implication or a conjunction, and that the antecedent contains no conjunction, disjunction, implication of the form $(C \wedge D) \to B$ or of the form $(C \vee D) \to B$, or pair of the form $P, P \to B$; in other words, it is *irreducible*.

Consider, among all derivations in **G3ip** of this sequent, one $\mathcal{D}$ with a shortest leftmost branch. (When computing this length for a $R\wedge$ or $L\vee$ step, the maximum of the lengths of the two premisses is used.) By the irreducibility, the last step must be by one of $R\vee$ or, with principal formula of the form $(C \to D) \to B$ or $P \to B$, the rule $L\to$; in the ultimate case, P is not in the antecedent. We consider the possible cases in turn:

1. $R\vee$ is a rule common to the two calculi: the induction hypothesis deals with it.
2. $L\to$ with principal formula $(C \to D) \to B$: so the final step of $\mathcal{D}$ has premisses $\Gamma, (C \to D) \to B \Rightarrow C \to D$ and $\Gamma, B \Rightarrow E$. We deal with the first premiss by using the invertibility (in **G3ip**) of $R\to$ (twice), the equivalence in **G3ip** (in the presence of C) of $D \to B$ and $(C \to D) \to B$ and the induction hypothesis

on the sequent $\Gamma, D \to B \Rightarrow C \to D$; then we deal with the second premiss by the induction hypothesis on $\Gamma, B \Rightarrow E$; then we use a $L\to\to$ step.

3. $L\to$ with principal formula $P \to B$: so $\mathcal{D}$ is as follows:

$$\dfrac{\dfrac{\mathcal{D}'}{P\to B, \Gamma' \Rightarrow P} \qquad \dfrac{\mathcal{D}''}{B, \Gamma' \Rightarrow E}}{P\to B, \Gamma' \Rightarrow E}\, L\to.$$

By irreducibility, $P \notin \Gamma'$. So $\mathcal{D}'$ must end in a step by a left rule (and by irreducibility that means the $L\to$ rule). If $P \to B$ is the principal formula then we can (by removing a step) shorten the leftmost branch of $\mathcal{D}$ and thus contradict the property of $\mathcal{D}$ of being a derivation of its end-sequent with shortest leftmost branch. So the principal formula must be a different implication; let it be $D \to F$, with $\Gamma' = D \to F, \Gamma''$. Then $\mathcal{D}$ is

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{P\to B, D\to F, \Gamma'' \Rightarrow D} \qquad \dfrac{\mathcal{D}_1}{F, P\to B, \Gamma'' \Rightarrow P}}{P\to B, D\to F, \Gamma'' \Rightarrow P}\, L\to \qquad \dfrac{\mathcal{D}''}{B, D\to F, \Gamma'' \Rightarrow E}}{P\to B, D\to F, \Gamma'' \Rightarrow E}\, L\to.$$

As in [1], using an inversion to obtain $\mathcal{D}_2$ from $\mathcal{D}''$, we can permute the proof $\mathcal{D}$ to obtain

$$\dfrac{\dfrac{\mathcal{D}_0}{P\to B, D\to F, \Gamma'' \Rightarrow D} \qquad \dfrac{\dfrac{\mathcal{D}_1}{P\to B, F, \Gamma'' \Rightarrow P} \qquad \dfrac{\mathcal{D}_2}{B, F, \Gamma'' \Rightarrow E}}{F, P\to B, \Gamma'' \Rightarrow E}\, L\to}{P\to B, D\to F, \Gamma'' \Rightarrow E}\, L\to,$$

which has a shorter leftmost branch than $\mathcal{D}$ and yet the same end-sequent. So this case cannot arise.

$\dashv$

It follows that **G4ip** and **G3ip** are equivalent. This proof replaces that in Section 3 of [1]. It avoids complicated definitions of otherwise useless concepts like "awkward", "clumsy" and "sensible", and the uses of these definitions; it cuts the hard part of the proof from two pages to one; the proof no longer feels 'hard'; it gives a correct definition of "length" of proof. This may encourage wider extension to logics extending **Int**.

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