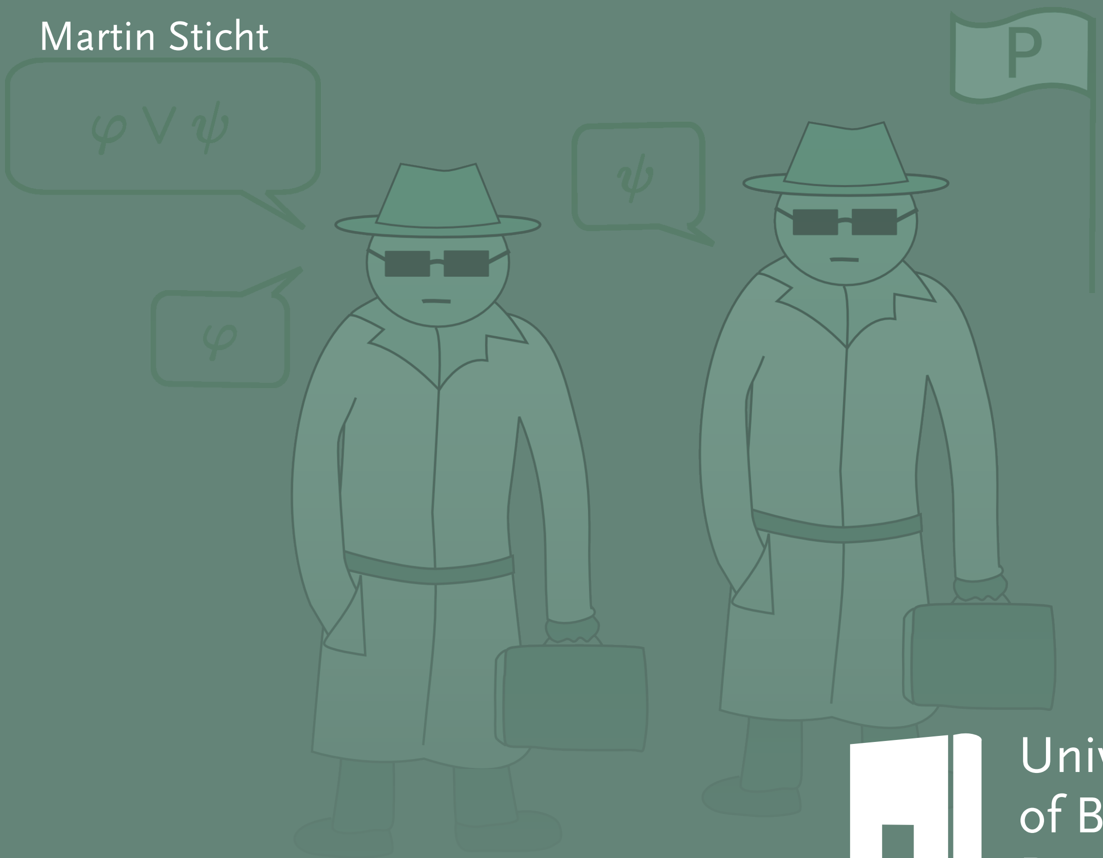
32
Schriften aus der Fakultät Wirtschaftsinformatik und Angewandte Informatik der Otto-Friedrich-Universität Bamberg

Proof Search in Multi-Agent Dialogues for Modal Logic

Martin Sticht



University of Bamberg Press

32 Schriften aus der Fakultät Wirtschaftsinformatik und Angewandte Informatik der Otto-Friedrich-Universität Bamberg

Contributions of the Faculty Information Systems and Applied Computer Sciences of the Otto-Friedrich-University Bamberg

Schriften aus der Fakultät Wirtschaftsinformatik und Angewandte Informatik der Otto-Friedrich-Universität Bamberg

Contributions of the Faculty Information Systems and Applied Computer Sciences of the Otto-Friedrich-University Bamberg

Band 32

University of Bamberg Press
2018

Proof Search in Multi-Agent Dialogues for Modal Logic

Martin Sticht

University
of Bamberg
Press
2018

Bibliographische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliographie; detaillierte bibliographische Informationen sind im Internet über http://dnb.d-nb.de/ abrufbar.

Diese Arbeit hat der Fakultät Wirtschaftsinformatik und Angewandte Informatik der Otto-Friedrich-Universität Bamberg als Dissertation vorgelegen.
1. Gutachter: Prof. Michael Mendler, PhD
2. Gutachter: Ao. Prof. DI. Dr. Christian Fermüller
Tag der mündlichen Prüfung: 09.07.2018

Herstellung und Druck: docupoint, Magdeburg
Umschlaggestaltung: University of Bamberg Press, Larissa Günther
Umschlagbilder: © Martin Sticht

ISSN: 1867-7401
ISBN: 978-3-86309-599-4 (Druckausgabe)
eISBN: 978-3-86309-600-7 (Online-Ausgabe)
URN: urn:nbn:de:bvb:473-opus4-525276
DOI: http://dx.doi.org/10.20378/irbo-52527

Acknowledgements

First of all, I want to thank my supervisor Michael Mendler for his support, visions and ideas as source of inspiration. He offered me opportunities to visit different conferences all over the world which was important to establish contacts to other scientists. Finally, he taught me logic, which elicited a great fascination that eventually resulted in this work. His claim for mathematical accuracy supported me to work properly and prevent mistakes.

I am grateful to Chris Fermüller for his support and advises. He made it possible that I could spend some time in Vienna where I was able to discuss my scientific stuff with colleagues who worked in the same area, an opportunity I have not had in Bamberg in this extent. I like to thank Björn Lellmann, Roman Kuznets, Jesse Alama, Hannah Karpenko, Sebastian Krebs, Dominik Klein, Erik Krabbe, Helge Rückert, and Shahid Rahman for fruitful discussions and inspirations; and Manuel Bärenz, Celia Reuter, and Stephan Scheele for reading parts of the thesis and revealing linguistic errors. Thanks go to the Austrian Science Fund (FWF) for supporting my reasearch in this work (project P25417-G15 LOGFRADIG).

I like to thank my partner Peter, my parents Werner and Elfi, and my siblings Maria and Lars for their steady support, especially when things did not go so well. Also, I am grateful to my former colleague and friend Stephan Scheele for his many advises (of how to survive) and encouragements. Many thanks to all my friends and/or former colleagues for their friendly words and support, in particular to Joaquín Aguado, Manuel Bärenz, Aboubakr Benabbas, Linus Dietz, Philipp Eittenberger, Benjamin Greim, Kathrin and Marcel Großmann, Simon Harrer, Benedikt Holzinger, Nasr Kasrin, Stefan Kufer, Michael Mahr, Bastian and Celia Reuter, Ute Schmid, Michael Siebers, Klaus Stein, Katharina Weitz, Alexander Werner, Diedrich Wolter, Christina Zeller, the members of the MfI1-team and others not mentioned here explicitly, but not forgotten.

Finally, I thank all my friends of the theater groups Pipperlapupp and Laienspielgruppe spectaculum e.V. They gave me the opportunity to do that kind of crazy stuff I was not supposed to do at university. Without them I would have probably lost my sanity ;-)

Abstract

In computer science, and also in philosophy, modal logics play an important role in various areas. They can be used to model knowledge structures among software-agents, behaviour of computer systems, or ontologies. They also provide mathematical tools to perform reasoning in these models, e.g., to extract common knowledge of agents, check whether security-relevant problems might occur when running a program, or to detect contradictions in a set of terminological definitions. Intuitionistic or constructive propositional logic can be considered as a special kind of modal logic. Constructive modal logics, as a combination of intuitionistic propositional logic and classical modal logics, describe a family of modal systems which are, compared to the classical variant, more restrictive concerning the validity of formulas.

To prove validity of a statement formalized in such a logic, various reasoning procedures (also called calculi) have been investigated. There are especially many variants of sequent and tableau systems which can be used easily to find proofs by applying given syntactical rules one after another. Sometimes there are different possibilities to find a proof for the same formula within the same calculus. It also happens that a bad choice of non-invertible rule applications at the wrong time makes it impossible to finish the proof successfully, although the formula is provable. For this reason, a normalization of deductions in a calculus is desired. This restricts the possibilities to apply rules arbitrarily and emphasizes the situations in which significant, non-invertible rule applications are necessary. Such a normalization is enforced in so-called focused sequent systems.

Another attempt to find a normalized calculus leads to dialogical logic, a game-theoretic reasoning technique. Usually, two players, one proponent and one opponent, argue about an assertion, expressed as a formula and stated by the proponent at the beginning of the play. The kinds of arguments, namely attacks and defences, are bound to special game rules. These are designed in such a way that the proponent has a winning strategy in the game if and only if his initial statement is a valid formula. The dialogical approach is very flexible as the game rules can be adjusted easily. Sets of rules exist to perform reasoning in many different kinds of logic, however proving soundness and completeness of dialogical calculi is complex and, if at all, often only considered very roughly in the literature. The standard two-player dialogues do not have much potential to enforce normalization like focus sequent systems. However, it turns out that introducing further proponent-players who fight against one opponent in a round-based setting leads to a normalization as described above. The flexibility of two-player games is largely preserved in multi-proponent dialogues. Other ordinary sequent systems can easily be transferred into the dialectic setting to achieve a normalization. Further, the round-based scheduling induces a method to parallelize the reasoning process. Modifying the game rules makes it possible to construct new intermediate or even more restrictive logics.

In this work, dialogical systems with multiple proponents are presented for intuitionistic propositional logic and modal logics S4 and CS4. Starting with the former one, it is shown that the normalization can be transferred easily to both the latter systems. Informal game rules are introduced and, to make them concrete and unambiguous, translated into the dialogical sequent-style calculi DiaSeqI, DiaSeqS4, and DiaSeqCS4. An extra system for intuitionistic logic, which guarantees termination in proof searches, even if the target formula is not valid, is also provided. Soundness and completeness of all these presented dialogical sequent calculi is proven formally, by showing that it is always possible to translate derivations in the game-oriented approach into another sound and complete sequent system and vice versa. Thereby, a new (ordinary) multi-conclusion sequent calculus for CS4 is introduced for which adequateness is shown, too.

The multi-proponent dialogical systems of this work are compared to different sequent calculi and other dialogical attempts found in literature. A comprehensive survey of such approaches is also part of this thesis.

Zusammenfassung

Modallogiken spielen in verschiedenen Gebieten der Informatik und auch in der Philosophie eine wichtige Rolle. Sie ermöglichen es, Wissensstrukturen von Softwareagenten, Verhalten von Computersystemen oder Begriffsysteme (Ontologien) zu modellieren. Mithilfe mathematischer Werkzeuge lassen sich Schlussfolgerungen in diesen Modellen ziehen, z.B. um gemeinschaftliches Wissen der Agenten zu extrahieren, um zu überprüfen, ob sicherheitsrelevante Probleme beim Ausführen von Programmen auftreten können, oder um Widersprüche in Begriffsdefinitionen aufzudecken. Intuitionistische oder konstruktive Aussagenlogik kann als eine spezielle Art von Modallogik betrachtet werden. Eine mögliche Kombination aus intuitionistischer Aussagenlogik und klassischer Modallogik stellt die Familie der konstruktiven Modallogiken dar, die im Vergleich zur klassischen Variante restriktiver bezüglich der Gültigkeit von Formeln sind.

Um die Gültigkeit einer in einer solchen Logik formalisierten Aussage zu beweisen, wurden verschiedene Verfahren (auch Kalküle genannt) entwickelt, darunter etliche Varianten an Sequenzen- und Tableausystemen, mit deren Hilfe die Beweissuche auf einfache Art und Weise durch die sukzessive Anwendung von syntaktischen Regeln ermöglicht wird. Manchmal gibt es dabei mehrere Möglichkeiten, einen Beweis für die gleiche Formel in ein und demselben Kalkül zu konstruieren. Es kann auch vorkommen, dass eine falsche Wahl bezüglich der Anwendung einer nicht invertierbaren Regel zu einem falschen Zeitpunkt zum Scheitern des Beweises führt, obgleich die entsprechende Formel beweisbar ist. Aus diesem Grund ist eine Normalisierung von Deduktionen in einem Kalkül wünschenswert. Diese schränkt die Möglichkeiten ein, Regeln in beliebiger Reihenfolge anzuwenden und hebt Situationen heraus, in denen nicht-invertierbare Anwendungen notwendig sind. Eine solche Normalisierung wird in den sogenannten Fokussierten Sequenzenkalkülen forciert.

Der Gedanke der Normalisierung führt auch zur Dialogischen Logik, einem spieltheoretischen Verfahren: Im Normalfall führen zwei Spieler, nämlich ein Proponent und ein Opponent, eine Diskussion über eine als Formel dargestellte Aussage, die der Proponent zu Beginn des Spiels äußert. Die möglichen Argumentationsweisen (Angriffe und Verteidigungen) richten sich dabei nach speziellen Spielregeln. Diese sind so gestaltet, dass der Proponent genau dann eine Gewinnstrategie hat, wenn seine initiale Äußerung eine valide Formel ist. Der dialogische Ansatz ist sehr flexibel, da die Spielregeln leicht angepasst werden können, um, wie in der Vergangenheit geschehen, viele verschiedene Arten von Logiken zu bedienen. Korrektheit und Vollständigkeit sind für diese dialogischen Kalküle jedoch schwer nachzuweisen und dieser Punkt wird in der Literatur oftmals nur sehr oberflächlich behandelt oder komplett außer Acht gelassen. Die gewöhnlichen Zweispieler-Dialoge sind für Normalisierungen, wie sie in Fokussierten Sequenzenkalkülen realisiert werden, wenig geeignet. Werden jedoch weitere Proponenten eingeführt, welche sich dem Opponenten gemeinsam in einer rundenbasierten Umgebung stellen, so führt dies zu einer Normalisierung wie sie oben beschrieben ist. Die Flexibilität der Zweispieler-Spiele bleibt in den Multi-Proponenten-Dialogen weitestgehend erhalten. Andere gewöhnliche Sequenzensysteme können auf einfache Art und Weise in die dialektische Umgebung überführt werden, wodurch eine Normalisierung erreicht wird. Des Weiteren weist der rundenbasierte Ablauf auf Möglichkeiten zur Parallelisierung des Beweisverfahrens hin. Das Abändern der Spielregeln ermöglicht es, neue intermediäre oder noch restriktivere Logiken zu entwickeln.

In dieser Arbeit werden dialogische Systeme mit mehreren Proponenten für intuitionistische Aussagenlogik und die Modallogiken S4 und CS4 vorgestellt. Beginnend mit der ersteren wird gezeigt, dass die Normalisierung einfach auf die beiden letzteren Systeme übertragen werden kann. Informelle Spielregeln werden eingeführt und in die dialogischen Kalküle DiaSeqI, DiaSeqS4 und DiaSeqCS4, welche eine ähnliche Struktur wie Sequenzenkalküle aufweisen, überführt. Damit werden die Regeln konkretisiert und von Mehrdeutigkeiten befreit. Ein weiteres System für intuitionistische Logik, welches die Terminierung der Beweissuche auch dann garantiert, wenn die Zielformel nicht allgemeingültig ist, wird ebenfalls bereitgestellt. Korrektheit und Vollständigkeit werden für all diese vorgestellten dialogischen Sequenzenkalküle formell bewiesen, indem gezeigt wird, dass sich jede Herleitung im spielorientierten Ansatz in die eines anderen korrekten und vollständigen Sequenzensystems überführen lässt, und umgekehrt. Nebenbei wird ein neues (gewöhnliches) Multi-Conclusion-Sequenzenkalkül für CS4 vorgestellt, für das ebenfalls Korrektheit und Vollständigkeit nachgewiesen wird.

Die Multi-Proponenten-Dialogsysteme dieser Arbeit werden mit verschiedenen Sequenzenkalkülen und anderen dialogischen Ansätzen aus der Literatur verglichen. Eine umfassende Übersicht dieser Konzepte ist ebenfalls Teil der vorliegenden Dissertation.

Contents

List of Figures

1 Introduction

1.1 Motivation

1.1.1 Modal Logic

Propositional modal logics (in the following simply *modal logics*) provide a flexible formalism to express complex statements in a way which is mostly more compact and easier to interpret than an equivalent expression in first-order predicate logic. The semantics are usually defined by way of *Kripke models* consisting of *possible worlds* connected via binary relations, and a valuation function. In computer science, modal logics have a great impact especially in knowledge representation, software verification, and type theory. For example, formal knowledge representation in terms of ontologies is easily possible with *description logics*.[1] There are also forms of modal knowledge representation languages for modelling the knowledge distribution of independent software agents, e.g., to derive conclusions about common knowledge. *Temporal Logics* [125, 121] (which are also modal logics) provide possibilities to describe situations that are admissible or forbidden while running an algorithm, and can therefore be used in software verification[2]. Intuitionistic Logic, which is applied in theorem provers like *Coq*[3], can be interpreted as a special kind of modal logic as well. *Security logics*, e.g. the

[1]Description logics are closely related to modal logic. Several description logical systems have counterparts in modal logic, e.g., the basic system $\mathcal{ALC}$ corresponds exactly to the multi-modal logic K_n [133, 8].

[2]e.g., as it is the case in the SPIN model checker, c.f. [71].

[3]`https://coq.inria.fr/`

so-called *BAN-logic* [20], can be used to verify properties of authentication protocols.

There are many extensions of modal logic which increase its expressiveness. With *hybrid logic*, one can also make assertions about certain Kripke worlds within formulas, which is not possible in standard modal logic. *Propositional Dynamic logic* (PDL) is used to model control-flow structures of computer programs. A special *dynamic* logic is *public announcement logic*[4] (PAL) where one can make assertions about situations in which certain states (worlds) are not relevant anymore, i.e., where the underlying model is changed dynamically due to new information that is *announced*. This is especially useful when one wants to analyse the distribution of knowledge after certain information is made globally available.

Other variants are non-standard modal logics which are restricted concerning the validity of formulas, e.g., the intuitionistic modal logics IK [55, 135] and CK [110]. These are especially interesting with respect to type theory (as for the variant CS4, e.g., see [36]) or when modelling incomplete or evolving knowledge [111].

As deduction techniques in modal logic, several systems have been presented by various authors. A natural deduction calculus is proposed by Fitch [56]. Since the publication of the Kripke semantics, a tableau calculus introduced by Kripke [87] (based on the tableau system by Beth [11]) made it possible to perform intuitive reasoning in the standard systems K, KT, KB, S4, and S5. With modified Kripke semantics for intuitionistic or constructive modal logics, other tableaux calculi were proposed such as a system for the description logic c$\mathcal{ALC}$ (which corresponds to multi-modal CK) by Scheele [132].

The *sequent system* that goes back to Gentzen [60] has also been extended in many ways for modal logic, some making explicit use of the underlying

[4]It goes back to Plaza [119].

Kripke structure by labelling formulas (e.g. [145, 116]), others rejecting this direction, often for philosophical reasons.

1.1.2 Dialogical Logic and Game Semantics

In many calculi different proofs can be found for the same formula. If the reasoning procedure provides rules which are not invertible, it can also happen that applying adverse rules at the wrong time makes it impossible to complete the proof successfully although the target formula is derivable. Therefore, the normalization of a reasoning process is a reasonable aim in order to reduce the number of possible proof attempts and to highlight the situations in which delicate applications are executed. In this work, to enforce such a normalization, we make use of a different approach for reasoning procedures which are called game-theoretic semantics. One such attempt, the *dialogical logic*, was introduced by Lorenzen and Lorenz [100, 98, 103] for which modal extensions exist as well.

Game-theoretic reasoning procedures provide a different way of proving a logical statement than the usual calculi. In general, two players *argue* about the validity of a given formula: one player tries to *verify* a statement, the other one wants to *falsify* it. In the game-theoretical system proposed by Hintikka [70], a logical model is given and the players perform actions according to the outer-most operators of the formula for which the truth value shall be obtained. Both players have their own set of logical connectors for which they are responsible, i.e., for which they may perform an action. The truth values of atomic formulas are usually determined due to the underlying model.

By contrast, in the dialogical games proposed by Lorenzen and Lorenz, the players take turns one after another and the proof structure is rather argumentative. Further, truth is usually not bound to a fixed model, i.e., derivations are usually model-independent.[5] However, the so-called *material dialogues*[6] can also be played with an underlying model. Therefore, the dialogical approach can be seen as the *more general* one and so we put the focus on this and do not consider Hintikka games in detail.

In dialogical logic, two players (called *proponent* P and *opponent* O) argue about the validity of some formula/statement which is *stated* by P at the beginning of the dialogue. The opponent *attacks* that statement, while P must either *defend* himself against this attack or counter-attack it. The possible moves depend on the underlying rules. One part of these rules (the *particle rules*) refer to the syntax of the formula that is to be attacked/defended, while others (the *structural rules*) describe possible game moves and winning situations. Enforcing stronger restrictions concerning possible moves allows us to change the semantics, e.g., from classical to intuitionistic logic. Then, with a well-defined set of such rules, a winning-strategy for P corresponds to a proof for the theorem stated by P at the beginning of the dialogue.

There have also been several attempts to use dialogues for classical modal logics. The only such systems, for which there are adequateness proofs, are—as far as I currently know—the systems introduced by Krabbe [83, 86] and Clerbout [29, 30]. Other attempts have been suggested by various authors, but usually without proofs (more in Chapter 2.4).

In general, dialogical logic provides an area of research which is not as well explored as other conventional proof systems. However, it is a very interesting topic of its own. Nowadays, distributed independent computer systems or software agents work together to solve a shared problem and need to communicate and exchange information making use of predefined protocols. Dialogical logic could support this process by providing a framework to *convince* the other system with facts in a persuasive way. This would in fact be a contribution in the area of artificial intelligence.

[5] Hintikka calls Lorenzen-games *indoor games* with *"verbal 'challenges' and 'responses' "*, while his own system is rather *"related to the uses of logical symbols in finding out something about the world"* [70].

[6] introduced by Lorenz [99] as *relative Dialogspiele*.

In this work, we consider game semantics as deductive systems. Game-theoretic accounts to proof theory provide a different view on reasoning procedures and let us learn more about the underlying logic. For example, Lorenzen came up with the idea of dialogues also to provide a different, more intelligible justification for intuitionistic logic than Brouwer's original attempts [100, 101]. Also, the game-based/dialogical approaches have another perspectives on the rules which do not only work on the syntactical structure of the formulas like in most calculi. Instead, a set of game rules is provided that puts restrictions on the players' moves. Modifying these rules even allows us to create new logics or improve proof search.

For example, the semantics of *independence-friendly logic* (IF) is defined in terms of Hintikka games [106], and Abramsky [1] proposes semantics which are the result of introducing more players than just one verifier and one falsifier. Modifying underlying game rules (of Hintikka games or Lorenzen dialogues) makes it possible to obtain reasoning procedures for different kinds of logical systems, e.g., for relevance logic [128], linear logic [16] or many-valued/fuzzy logic [52, 50]. Furthermore, there have been attempts to adjust rules to obtain new sub-intuitionistic systems [143]. A theory defining what *reasonable* modifications could be is provided in [2].

Concerning proof search, the proponent is the player who tries to show validity of a certain formula, his hypothesis. This is done by finding a so-called *winning strategy*, i.e., he has to find a sequence of moves that make him win, no matter how the opponent behaves. It is possible to introduce further game rules in order to lead the proponent in this searching process, i.e., to prevent him in advance from performing wrong moves. For example, Galmiche et al. [59] point out that a rule by Rahman and Keiff [127] limits the number of repetitive moves P may do.[7] Alama [3] tests some strategy preferences for P and concludes that it is important not to make the restrictions

[7]Similar rule restrictions were already presented much earlier, most notable in the work by Barth and Krabbe [10]. More on this follows in Chapter 2.3.4.

too stringent, as this can result in a loss of completeness of the dialogical calculus.

In this work we mainly investigate how dialogues can be used to do reasoning in modal logic and how they can be used to normalize sequent proofs. In particular, we show that the normalization that results from the multi-proponent dialogical interpretation of intuitionistic logic can be transferred to the modal logics S4 and CS4.

1.2 Contribution

The main contributions of this thesis are:

- A new multi-proponent dialogical approach, which is used to do reasoning in intuitionistic (Chapter 3) and modal logic (Chapter 4), is described. The presented systems comprise scheduling mechanisms that lead to a normalization of proofs.
- A sequent-style system, which serves as a framework to formalise informal dialogical rules, making them concrete, is introduced (Chapters 3.2, 4.1.2, and 4.2.1).
- Soundness and completeness of these sequent-style systems are shown in terms of detailed constructive proofs. These induce an algorithm to translate normalized derivations in dialogical sequent systems into proofs of ordinary sequent calculi and vice versa (Chapters 3.3 and 4.3).
- As by-product, a new sound and complete ordinary multi-conclusion sequent system for the constructive modal logic CS4 is introduced (Chapter 2.2.2).
- A detailed survey on sequent systems and dialogical approaches found in the literature is presented (Chapter 2) in order to contrast them with the new multi-proponent approach.

Dialogical systems which are used to do reasoning in modal logics are discussed and a new approach is presented in which the usual setting of one proponent and one opponent is given up. Instead we let a syndicate of proponents fight one opponent.

Such a system can be used to enforce a *round-based scheduling* in the proving process. This leads to a new level of strategies which have not yet been discussed in the literature and which results in a normalization of sequent proofs: instead of analysing the behaviour of a single proponent player, collective decisions among proponent agents are put into focus. These (and only these) are significant for finding a proof in this new dialogical system.

The normalization of sequent proofs puts more restrictions on possibilities for rule applications compared to what is possible in usual sequent systems. Those applications which are significant for the success of the proof are treated separately, while the others can be performed arbitrarily without having a difference in the result. This behaviour shows similarities to sequent systems with *focus* (Chapter 2.1.5), which also aim for a normalization of proofs. However, the normalization process is different as the automated scheduling differs. Due to the phase structure in the games it is possible to parallelize the moves of proponent agents at certain points of the dialogue, while the collective decisions are isolated in other phases. The proposed scheduling process can therefore be seen as an attempt to implement the reasoning procedure in a concurrent way, where each proponent player acts independently of the others until some point in which they are all synchronised to make a common decision. This paradigm shall be useful for further investigations in agent-orient programming and concurrent systems.

Formal proofs for soundness and completeness (we use the term *adequateness* to comprise both properties) of dialogical systems are quite rare in literature. Sequent-style systems that interpret the dialogical rules in an unambiguous way are presented in this work, which makes it possible to show adequateness for all the described multi-proponent systems, i.e, it is formally shown that these systems are both sound and complete and can be used as deduction systems for intuitionistic propositional logic (IPL), S4, and CS4 accordingly. A *completely formalised* proof for a dialogical S4-system is provided for the first time.[8] Additionally, a new adequate multi-conclusion sequent system for CS4 is introduced. A dialogical approach for a constructive modal logic, for which soundness and completeness is proven, is presented for the first time. For the case of intuitionistic propositional logic we show formally that it is possible to guarantee termination of the proof searching process due to some additional game rules and without making use of *ranks*.[9]

The modal dialectics we consider here mainly deal with S4 and CS4, however it is possible to modify the systems to make them suitable for K and KT. Also other modal systems are thinkable. We concentrate here on S4 for three simple reasons:

1. It provides reasonable semantics for dialectics.
2. In computer science it has a big scope of application such as model-checking.
3. It can be used to simulate intuitionistic behaviour.

Regarding number 1, we refer to Krabbe's idea of *non-cumulative dialectics* [83, 86, 84]: the opponent is given the chance to *withdraw commitments* under certain circumstances. This is a real-world application in terms of argumentation theory.[10] Concerning number 2, we refer to *temporal logic* [125, 121]. The relation between propositional intuitionistic logic and S4 is briefly discussed in Section 1.5.3.

As it is strongly related to S4 and also of interest on its own, we start with a multi-proponent system we use for reasoning in IPL in Chapter 3. Then we construct the S4-approach from this (Chapter 4). As a strongly restricted

[8]Beside that of my own work in [139]. Note that there are other proofs for dialogical modal systems in the literature, but these are rather informal or contain gaps.

[9]Ranks are one possibility to guarantee termination in dialogical logic. They are discussed in Chapter 2.3.4.

[10]Details follow in Chapter 2.4.1.

version of S4 we also consider a dialogue system for CS4 and provide an idea of how the informal underlying game rules look like. These can be modified to obtain reasoning procedures for intermediate modal logics.

Dialogues are discussed especially with respect to *proof theory*, i.e., the aim is to find a reasoning procedure that makes use of features of Lorenzen dialogues. Informal game rules are also presented to follow the tradition of Lorenzen and Lorenz and to indicate possibilities to construct new logical systems. However, we do not try to give a philosophical justification for the reasonableness of these rules. Instead, they turn out to be quite complex when used for constructive modal logic.

It is not clear whether the time complexity of the proposed dialogical sequent systems is better than that of ordinary sequent systems. It is rather the aim of the work to provide a foundation in terms of the scheduling mechanism which can be the basis for more efficient systems.

Outline

The rest of this chapter provides conventions and foundations, in particular concerning intuitionistic and modal logic and logical deduction.

In Chapter 2 we discuss various sequent systems and dialogical approaches for intuitionistic propositional logic and several propositional modal logics. General similarities and differences between sequent systems and dialogues are described.

A multi-proponent dialogical approach for intuitionistic propositional logic is presented in Chapter 3. Informal game rules are proposed and translated into a sequent-style system. One variant which guarantees termination is also discussed. Adequateness of these systems is shown formally. In Chapter 4 the rules are extended to establish sound and complete dialogical systems for the modal logics S4 and CS4. All these systems are compared to different sequent systems and dialectics of Chapter 2.

The thesis concludes in Chapter 5 with a short summary and an outlook on possible future work.

1.3 Conventions

In the following, the usual abbreviation "iff" is used for "*if and only if*". The set of *natural numbers* $\mathbb{N}$ contains all non-negative integers, i.e., all positive integers and 0.

1.3.1 Sequences, Sets, and Multi-Sets

We distinguish *sets*, *sequences*, and *multi-sets* as described in the following:

- *Sets* are defined as usual with the known operators *union* ($\cup$), *intersection* ($\cap$), *difference* ($\setminus$), *element* ($\in$), *subset* ($\subseteq$), ... The *power set* of set Γ is written $\mathcal{P}(\Gamma)$.
- In *sequences* or *lists*, the order of the elements is relevant and multiple occurrences are possible which is not the case for sets.
- *Multi-sets* are an intermediate form in which an element may occur several times, but the order of all elements is not relevant.

It is nowadays usual to use the comma ',' in deduction systems as a symbol for concatenation (sequences) or union (sets/multi-sets). For example, let us assume that Γ and Δ are multi-sets defined as $\Gamma =_{df} \{P, Q\}$ and $\Delta =_{df} \{P, R\}$. Then Γ, Δ is the multi-set $\{P, Q, P, R\}$. We also use the comma to add single elements to sequences, sets, and multi-sets, e.g., Γ, R is an abbreviation of $\Gamma, \{R\}$ which indicates the (multi-)set $\{P, Q, R\}$. Table 1.1 shows the outcome of using the comma for an exemplary assignment of Γ and Δ for sequences, sets, and multi-sets.

	sequence	set	multi-set
Γ (example)	P, Q	$\{P, Q\}$	$\{P, Q\}$
Δ (example)	P, R	$\{P, R\}$	$\{P, R\}$
Γ, Δ	P, Q, P, R	$\{P, Q, R\}$	$\{P, Q, P, R\} = \{P, P, Q, R\}$
Γ, R, Δ	P, Q, R, P, R	$\{P, Q, R\}$	$\{P, Q, R, P, R\} = \{P, P, Q, R, R\}$

Table 1.1: Comma-Convention for Sequences, Sets, and Multi-Sets

1.3.2 Variables

In general, we use different letters/symbols as variables for different mathematical structures.

- As variables representing arbitrary logical formulas (propositional, modal, or first-order formulas), we use both Roman capital letters and Greek lowercase letters (A, B, C, ..., ϕ, φ, ψ, ...).
- *Propositional variables* are usually represented by uppercase roman letters P, Q, and R.
- We use Greek uppercase letters (Γ, Δ, Φ, Ψ, Θ, Λ) for *sequences* (*lists*), *sets* and *multi-sets* of formulas.
- The Roman lowercase letters h, i, k, m, n, o are used for *natural numbers*, and u, v, w for *Kripke worlds* (see Section 1.5).
- Concerning *first-order logic*, x, y and z refer to *individual variables* while c and d are used for *individual constants*, and t, s, r for *terms*.

The arrow '$\rightarrow$' is used for *binary relations* of different kinds. If $(a, b) \in \rightarrow$, we usually write $a \rightarrow b$.

1.3.3 Syntax and Language

Propositional Language

The language of *propositional logic* is defined as follows:

$$
\begin{array}{lllll}
A, B & \longrightarrow & \bot & \mid & \text{(bottom/false)} \\
 & & P & \mid & \text{(propositional variable)} \\
 & & A \wedge B & \mid & \text{(conjunction, "and")} \\
 & & A \vee B & \mid & \text{(disjunction, "or")} \\
 & & A \supset B & \mid & \text{(implication, "implies")} \\
 & & \neg A & & \text{(negation, "not")}
\end{array}
$$

The symbols $\wedge$, $\vee$, $\supset$, and $\neg$ can be read as indicated in the parentheses. Propositional variables as formulas are called *prime formulas* (sometimes abbreviated as *primes*). Prime formulas and the constant $\bot$ are *atomic formulas*.

Formula equivalence $A \equiv B$ is not stated here explicitly, as we will barely use it. Note that in general, $A \equiv B$ equals to $(A \supset B) \wedge (B \supset A)$ and is therefore dispensable. Also, negation $\neg A$ is redundant as it is equal to $A \supset \bot$. However, we keep it in our definition as it will occur in the different systems we discuss in Chapters 2, 3, and 4.

The negation binds stronger than conjunction, disjunction, and implication, which have an equal binding strength. Therefore, $\neg A \wedge B$ is equal to $(\neg A) \wedge B$, but not equal to $\neg(A \wedge B)$.

In a conjunction $A \wedge B$, A and B are called the *left* and the *right conjunct* respectively. Accordingly, in $A \vee B$, they are the *left* and the *right disjunct*. In an implication $A \supset B$, A is referred to as *antecedent* and B as *consequent*.

First-Order Language

The language of *first-order logic* (FOL) copes with *relations* instead of propositional atoms and introduces terms and quantifiers:

$$
\begin{array}{rcll}
A, B & \longrightarrow & \bot & | \quad \text{(bottom/false)} \\
 & & R(t_1, \ldots, t_n) & | \quad \text{(relation)} \\
 & & A \wedge B & | \quad \text{(conjunction, “and”)} \\
 & & A \vee B & | \quad \text{(disjunction, “or”)} \\
 & & A \supset B & | \quad \text{(implication, “implies”)} \\
 & & \neg A & | \quad \text{(negation, “not”)} \\
 & & \forall x\, A & | \quad \text{(universal quantifier, “for all”)} \\
 & & \exists x\, A & \quad\; \text{(existential quantifier, “there is”)}
\end{array}
$$

The quantifiers bind stronger than conjunction, disjunction, and implication, but equally strong as negation. Usually, *equality* of terms is also considered as an extra relation. This is not covered here.

For a formula or sub-formula that starts with a quantified variable ($\forall x\ A$ or $\exists x\ A$), we say that all occurrences of x in A are called *bound variables*. Variables in a formula which are not in a quantifier's context (i.e., not written directly after some quantifier) are called *free variables* or *Eigenvariablen*. For any formula A of the first-order language, $A[x/t]$ expresses the *substitution* of variable x in A by term t, i.e., all free occurrences of x in A are replaced by t.

Many calculi we consider in Chapter 2 refer to first-order logic and therefore the syntax is given here. However, it is not of big importance in this work and therefore we do not discuss its properties here in detail.

Modal Language

The language of *propositional mono-modal logic* is an extension of that of propositional logic, where two further unary operators are introduced. The additional construction rules are defined as follows:

$$
\begin{array}{rcll}
A & \longrightarrow & \Box A & | \quad \text{(box modality, “necessarily”, “obligatory”, “always”,\ldots)} \\
 & & \Diamond A & \quad\; \text{(diamond modality, “possibly”, “permissibly”, “eventually”,\ldots)}
\end{array}
$$

The modal operators bind equally strong as negation and stronger than conjunction, disjunction, and implication.

In the following, we mainly refer to *mono-modal* languages. However, *multi-modal* variants are also usual, where $\Box$ and $\Diamond$ are replaced by $[\mathfrak{l}]$ and $\langle \mathfrak{l} \rangle$ with the label $\mathfrak{l}$ being an element of a predefined index set I, also called *modal signature*. The mono-modal language can be considered as a special case of multi-modal logic where I is a singleton set.

Formulas and Subformulas

We define Form to be the set of all possible formulas of the propositional or modal language defined above.[11] Let Sub be a mapping

$$\mathit{Form} \Rightarrow \mathcal{P}(\mathit{Form}) \ .$$

For any formula ϕ of Form the set of *subformulas* of ϕ, written $\mathit{Sub}(\phi)$, is defined as follows:

$$\mathit{Sub}(P) =_{df} \{P\} \quad \text{if } P \text{ is an atomic formula.} \tag{1.1}$$

$$\mathit{Sub}(A \otimes B) =_{df} \{A \otimes B\} \cup \mathit{Sub}(A) \cup \mathit{Sub}(B) \quad \text{for } \otimes \in \{\wedge, \vee, \supset\} \tag{1.2}$$

$$\mathit{Sub}(\nabla A) =_{df} \{\nabla A\} \cup \mathit{Sub}(A) \quad \text{for } \nabla \in \{\Box, \Diamond, \neg\} \tag{1.3}$$

A formula is in *negation normal form* iff the negation operator $\neg$ occurs only directly in front of prime formulas.

1.3.4 Rules and Deduction

In this work we discuss several deductive systems, especially sequent calculi and dialogical proof systems. Such calculi operate on the syntax of given

[11] We do not need the set with respect to the first-order language, so we ignore it.

formulas. They usually provide rules which define valid transformations from given formulas (*premises*) to other formulas (*conclusions*). We make use of the following notation which goes back to Gentzen [60]:

$$\frac{\textit{premises}}{\textit{conclusion}}$$

Simple examples for such rules are:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A \supset B \quad A}{B}$$

The first two examples show rules that transform $A \wedge B$ to A and B respectively. The third one can be used to compose A and B to $A \wedge B$, i.e., one can read it as "*from* A *and* B *follows* $A \wedge B$". The last one is the famous rule called *modus ponens* which states that from $A \supset B$ and A follows B.

Premises and conclusions do not need to consist only of single formulas, but can also have more complex structures, e.g., sequences or sets of formulas. If the rules are *correct*, i.e., they interpret a given semantics in an adequate way, sequences of such rule applications can be used to construct proofs for the validity of formulas.

In the following, we use the terms *(proof) system*[12], *calculus*, and *decision procedure* for the same thing: some data structure and a set of deduction rules which can be applied in the structure to derive formulas in a syntactic way. We usually require that the rules are sound and complete with respect to the semantics of some predefined logic, although this is not always the case. Informally, a system is *sound* iff it is not possible to derive false conclusions, i.e., to derive formulas which are not valid according to the underlying semantics. The system is *complete* iff every formula which is valid according to the semantics can be derived in the system.

[12]The term *system* is also used in another context. We call *modal logics* like S4 sometimes *modal systems*.

1.3.5 Trees

We use deduction rules to build *derivation trees*. For any such tree, the terms *root* and *leaf* are defined as usual. A tree t is either a leaf or it is a *node* n having at least one *child*. All the children can be seen as roots of sub-trees again.

$$
\begin{array}{lllll}
t & \longrightarrow & n & \mid & \text{(single node, leaf)} \\
 & & (n, t_1, \ldots, t_m) & & \text{(node } n \text{ (root) with attached trees } t_1 \text{ to } t_m\text{)}
\end{array}
$$

The *height* of a tree t is defined recursively as follows:

$$height(n) =_{df} 1 \quad \text{if } n \text{ is a node / leaf of tree.} \tag{1.4}$$

$$height((n, t_1, \ldots, t_m)) =_{df} 1 + max(height(t_1), \ldots, height(t_m)) \tag{1.5}$$

The variables t_1 to t_m represent the trees attached to the corresponding n. The function max simply returns the highest number of a given list of numbers.

A *path* of tree t is a list of nodes from the root to one leaf. We define a relation

$$Path \subseteq Tree \times Node^+$$

where $Tree$ is a set containing all possible trees and $Node^+$ indicates all possible lists of nodes of such trees.

$$
\begin{array}{rl}
Path(n, n) & \text{if } n \text{ is a leaf.} \\
Path((n, t_1, \ldots, t_m), (k, \Sigma)) & \text{if } k = n \text{ and } Path(t_l, \Sigma) \text{ for some } t_l \in \{t_1, \ldots, t_m\}.
\end{array}
$$

IPL1	$A \supset (B \supset A)$
IPL2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
IPL3	$A \supset (B \supset (A \wedge B))$
IPL4	$(A \wedge B) \supset A$ and $(A \wedge B) \supset B$
IPL5	$A \supset (A \vee B)$ and $B \supset (A \vee B)$
IPL6	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
IPL7	$\bot \supset A$

Figure 1.1: Axioms of IPL

1.4 Intuitionistic Logic

As some variants of modal logic we discuss later follow intuitionism, we have a short look on *intuitionistic propositional logic* (IPL). The syntax is the same as that of classical propositional logic but semantics are more restrictive, i.e., there are less theorems in intuitionistic than in classical logic, though every intuitionistic theorem is also a classical one.

The idea of intuitionism comes from L. E. J. Brouwer [18]. The logic was formalized by Arend Heyting [69]. The key aspect of intuitionistic logic is the rejection of the *law of the excluded middle* or *tertium non datur* $A \vee \neg A$.

IPL can be axiomatized with the axioms shown in Figure 1.1[13] together with the deduction rule of *modus ponens* (MP):

$$\frac{A \supset B \qquad A}{B}\,\text{MP}$$

The axioms establish with the rule a deduction system which is called *Hilbert calculus*. From these axioms it is possible to derive further tautologies.

[13] There are slightly different versions of axiomatizations for IPL. We refer to that of [39].

A Hilbert deduction system for classical propositional logic (CPL) is obtained by adding the axiom $\neg\neg A \supset A$ to those of Figure 1.1.

1.5 Classical and Constructive Modal Logic

1.5.1 Axioms of Classical Modal Logic

An early account to modern modal logic was given by C. I. Lewis who proposed axioms for different systems called S1 to S5 (Appendix II of [95]).[14] Systems S1 to S3 are now of minor interest and are said to be *not normal* as they do not obey axiom **K**:

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$

With this axiom, together with those of CPL, the MP-rule and the additional deduction rule of *necessitation* (Nec) given as

$$\frac{A}{\Box A}\,\text{Nec}$$

a deduction system for the basic normal modal logic K is obtained which can be used to derive its tautologies. To prove assertions that include the $\Diamond$-operator, another axiom needs to be added, the axiom of *duality*:

(Dual) $\Box A \supset \neg\Diamond\neg A$ and $\neg\Diamond\neg A \supset \Box A$

There are many other axioms that can be added to the set we have seen so far. The most famous are:

T $\Box A \supset A$
4 $\Box A \supset \Box\Box A$
D $\Box A \supset \Diamond A$
B $A \supset \Box\Diamond A$
5 $\Diamond A \supset \Box\Diamond A$

[14] For a detailed account to the history of modal logic since Lewis see Ballarin [9].

Combining these with the rules MP and Nec leads to deduction systems for different modal systems.

- Taking the axioms of CPL together with axioms **(Dual)**, **K**, **T**, and **4**, leads to a system which is equal to Lewis' S4.
- Taking the axioms of CPL together with axioms **(Dual)**, **K**, **T**, **4**, and **5**; or **(Dual)**, **K**, **T**, **4**, and **B**, leads to a system which is equal to Lewis' S5.

1.5.2 Kripke Semantics for Classical Modal Logic

Semantics of modal logics are usually defined as *Kripke structures* named after Saul Kripke who proposed them in 1963 [87]. These are also called *possible world semantics*.

The set Var is defined to consist of *propositional variables* / prime formulas. Let W be a set of *possible worlds* and $\rightarrow \subseteq (W \times W)$ be an *accessibility relation* over these worlds. Together they form a *(Kripke) frame* $\mathcal{F} = (W, \rightarrow)$. For all worlds w and u such that $w \rightarrow u$ (also written $(w, u) \in \rightarrow$), we say that u is a *successor* of w.

The *valuation* $\mathcal{V}$ is a function accepting a world and returning a set of propositional variables which are assigned to hold in the world, i.e., which are valid in the world.

$$\mathcal{V} \;:\; W \Rightarrow \mathcal{P}(\mathit{Var})$$

The frame and valuation form together a *(Kripke) model* $\mathcal{M} = (\mathcal{F}, \mathcal{V})$. Given a model $\mathcal{M} = (W, \rightarrow, \mathcal{V})$ and some world $w \in W$ of that model, the *satisfaction relation* $\models$ decides whether an arbitrary formula of the modal language is valid in w:

T	reflexivity	$\forall w \in W$: $w \rightarrow w$
D	seriality	$\forall w \in W$: $\exists u \in W$: $w \rightarrow u$
B	symmetry	$\forall w, u \in W$: if $w \rightarrow u$ then $u \rightarrow w$
4	transitivity	$\forall w, u, v \in W$: if $w \rightarrow u$ and $u \rightarrow v$ then $w \rightarrow v$
5	euclidicity	$\forall w, u, v \in W$: if $w \rightarrow u$ and $w \rightarrow v$ then $u \rightarrow v$

Table 1.2: Frame axioms and accessibility relations

$$\begin{array}{lll}
\mathcal{M}, w \models P & \text{iff} & P \in \mathcal{V}(w) \\
\mathcal{M}, w \models \neg A & \text{iff} & \mathcal{M}, w \not\models A \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B \\
\mathcal{M}, w \models A \supset B & \text{iff} & \mathcal{M}, w \not\models A \text{ or } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \Box A & \text{iff} & \forall u \in W\text{: if } w \rightarrow u \text{ then } \mathcal{M}, u \models A \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \exists u \in W\text{: } w \rightarrow u \text{ and } \mathcal{M}, u \models A
\end{array}$$

Worlds w with $\mathcal{M}, w \models \bot$ do *not exist* by definition.

If a formula A is valid in all worlds of a model $\mathcal{M}$, then A is said to be *valid* in $\mathcal{M}$, written $\mathcal{M} \models A$. A formula A is valid in a frame $\mathcal{F} = (W, \rightarrow)$, written $\mathcal{F} \models A$, iff for all valuations $\mathcal{V}$ it holds $(W, \rightarrow, \mathcal{V}) \models A$.

This defines the semantics of normal modal logic K. Accordingly, axiom **K** holds in all models and all frames which are defined as described above. Axiom **T** holds in a frame $\mathcal{F} = (W, \rightarrow)$ iff $\rightarrow$ is a *reflexive relation*, while axiom **4** expresses the property of *transitivity*, axiom **D** of *seriality*, axiom **B** *symmetry*, and **5** *euclidicity*. These properties, their meanings and the correspondence to the axioms are summarised in Table 1.2.

As the modal system S4 can be constructed as a combination of axioms **K**, **T**, and **4**, the corresponding frame structure is *reflexive* and *transitive* which therefore establishes a *preorder* over $\rightarrow$. S5-frames are additionally symmetric (axiom **B** is added) and therefore the accessibility relation corresponds to *equivalence classes*.

1.5.3 Kripke Semantics for Intuitionistic Propositional Logic

Gödel [63] observed that IPL formulas can be translated into formulas of modal logic when obeying the S4-axioms.[15] From this, Kripke [87] derived semantics for IPL based on the possible worlds metaphor:

$$\begin{array}{lll}
\mathcal{M}, w \models P & \text{iff} & P \in \mathcal{V}(w) \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \neg A & \text{iff} & \forall u \in W\text{: if } w \to u \text{ then } \mathcal{M}, u \not\models A \\
\mathcal{M}, w \models A \supset B & \text{iff} & \forall u \in W\text{: if } w \to u \text{ then } \mathcal{M}, u \not\models A \text{ or } \mathcal{M}, u \models B
\end{array}$$

If $\mathcal{M}, w \models A$ then $\forall u \in W$ such that $w \to u$: $\mathcal{M}, u \models A$. The accessibility relation is *reflexive* and *transitive*. Note that the implication $\supset$ and negation $\neg$ have a special status in the semantics as these have a certain impact on other worlds.

1.5.4 Axioms of Intuitionistic and Constructive Modal Logic

There have been many attempts for intuitionistic modal logics. Differences between several ideas with respect to specific features such as validity of the duality axiom or certain other axioms are discussed exhaustively in Simpson's PhD thesis [135]. Here, we are only interested in proof-theoretic features of intuitionistic modal logics and consider only two of these variants.

The first one is the IK-family which was introduced by Fischer-Servi [55] and Plotkin and Stirling [120] and examined in detail by Simpson [135]. The members of the other, even more restrictive CK-family are usually called

[15] Gödel uses the letter '*B*' as modal operator for the German word *beweisbar*, which means *provable*. It corresponds to the $\Box$. For details see the note by Troelstra in [46], p.296–299.

IK1	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
IK2	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
IK3	$\neg\Diamond\bot$
IK4	$\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
IK5	$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

Figure 1.2: Modal axioms of IK [120]

constructive modal logics to distinguish them from the *intuitionistic* IK-family [149, 110, 13, 4].

Both *families* reject the duality of $\Box$ and $\Diamond$, i.e., the **(Dual)**-axiom is not valid anymore. The propositional fragment[16] of IK and CK corresponds to IPL in each case. Because the $\Diamond$-operator cannot be defined in terms of the $\Box$ anymore, further axioms are necessary. Those for IK include all of IPL plus these shown in Figure 1.2 proposed by Plotkin and Stirling. As Hilbert deduction rules the known *modus ponens* and *necessitation* are used. Note that Fischer-Servi proposed a different set of axioms with one more (redundant) axiom before Plotkin and Stirling published their paper. These alternative axioms correspond to the same modal system IK [120, 135].

The IK-axioms can be extended, inter alia by **T**, **4**, **D**, **B**, and **5**. However, as most of these are only defined with respect to the $\Box$-operator and the $\Diamond$ is not taken into account, variants need to be considered as well. For example, in classical modal logic, the axiom **4** $\Box A \supset \Box\Box A$ and the alternative axiom **4'** $\Diamond\Diamond A \supset \Diamond A$ are equivalent and as frame axioms they can both be used to enforce transitivity of $\rightarrow$. However, in the classical setting **4'** is redundant due to the duality of $\Box$ and $\Diamond$ which is now given up. The intuitionistic frame axioms corresponding to **T**, **4**, **D**, **B**, and **5**, are shown in Figure 1.3.

We obtain a modal system named IS4 when combining the axioms **IK**, **T**, **T'**, **4**, and **4'**. Adding **5** and **5'** leads to IS5 accordingly [135].

[16] i.e., where the syntax is restricted to formulas that do not contain any modal operators.

D	$\Diamond\top$		
T	$\Box A \supset A$	**T'**	$A \supset \Diamond A$
B	$\Diamond\Box A \supset A$	**B'**	$A \supset \Box\Diamond A$
4	$\Box A \supset \Box\Box A$	**4'**	$\Diamond\Diamond A \supset \Diamond A$
5	$\Diamond\Box A \supset \Box A$	**5'**	$\Diamond A \supset \Box\Diamond A$

Figure 1.3: Frame axioms of Intuitionistic Modal Logic [135]

The CK-family is even more restrictive. From the IK-axioms of Figure 1.2, **IK3**, **IK4**, and **IK5** are dropped [110]. So the only remaining axiom that concerns the $\Diamond$ is **IK2**. The intuitionistic frame axioms of Figure 1.3 can be used to extend the CK-system accordingly, e.g., to obtain the modal systems CS4 and CS5 (see Arisaka et al. [6]).

1.5.5 Kripke Semantics for Intuitionistic and Constructive Modal Logic

Several semantics have been proposed for intuitionistic and constructive modal logics, also different variants of Kripke semantics. We consider combinations of Kripke semantics for classical modal logic and for IPL, that use two different kinds of relations.

Kripke Semantics for IK

The semantics for IK we present here was suggested by Plotkin and Stirling [120]. The frame $\mathcal{F}$ is redefined as *intuitionistic (Kripke) frame* or *Fischer Servi Frame* (as called by Grefe [64]) $\mathcal{F} = (W, \to, \preceq)$. The arrow $\to$ corresponds to the accessibility relation of classical modal logic, while the *refinement relation* $\preceq$ is the reflexive and transitive relation of the IPL Kripke semantics.

The satisfaction relation $\models$ for IK is defined as follows (c.f. [120]):

$$
\begin{array}{lll}
\mathcal{M}, w \models P & \text{iff} & P \in \mathcal{V}(w) \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \neg A & \text{iff} & \forall w' \in W\text{: if } w \preceq w' \text{ then } \mathcal{M}, w' \not\models A \\
\mathcal{M}, w \models A \supset B & \text{iff} & \forall w' \in W\text{: if } w \preceq w' \text{ then } \mathcal{M}, w' \not\models A \text{ or } \mathcal{M}, w' \models B \\
\mathcal{M}, w \models \Box A & \text{iff} & \forall w', u \in W\text{: if } w \preceq w' \text{ and } w' \to u \text{ then } \mathcal{M}, u \models A \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \exists u \in W\text{: } w \to u \text{ and } \mathcal{M}, u \models A
\end{array}
$$

Again, worlds w with $\mathcal{M}, w \models \perp$ do *not exist* [120]. As in the semantics of IPL, if $\mathcal{M}, w \models A$ then $\mathcal{M}, w' \models A$ for all w' such that $w \preceq w'$. Further, the following two additional *frame properties* hold in IK [120, 135]:

$$\forall w, w', u \in W: \text{ if } w \preceq w' \text{ and } w \to u \text{ then } \exists u' \in W \text{, such that } w' \to u' \text{ and } u \preceq u' \quad (1.6)$$

$$\forall w, u, u' \in W: \text{ if } w \to u \text{ and } u \preceq u' \text{ then } \exists w' \in W \text{, such that } w \preceq w' \text{ and } w' \to u' \quad (1.7)$$

Note that the operators $\neg$, $\supset$, and $\Box$ are stronger than the others due to their influence on all worlds which are *refinements* ($\preceq$-successors) of the current one, i.e., all worlds w' such that $w \preceq w'$.

Kripke Semantics for CK

The semantics we consider here makes also use of the two different relation types. It was first presented by Mendler and de Paiva [110].

$$
\begin{array}{lll}
\mathcal{M}, w \models P & \text{iff} & P \in \mathcal{V}(w) \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\
\mathcal{M}, w \models A \vee B & \text{iff} & \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \neg A & \text{iff} & \forall w' \in W\text{: if } w \preceq w' \text{ then } \mathcal{M}, w' \not\models A \\
\mathcal{M}, w \models A \supset B & \text{iff} & \forall w' \in W\text{: if } w \preceq w' \text{ then } \mathcal{M}, w' \not\models A \text{ or } \mathcal{M}, w' \models B \\
\mathcal{M}, w \models \Box A & \text{iff} & \forall w', u \in W\text{: if } w \preceq w' \text{ and } w' \to u \text{ then } \mathcal{M}, u \models A \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \forall w' \in W\text{: if } w \preceq w' \text{ then } \exists u \in W\text{: } w' \to u \text{ and } \mathcal{M}, u \models A
\end{array}
$$

CK has the finite model property and is decidable [110]. By contrast to IK, worlds w such that $\mathcal{M}, w \models \bot$ are possible and called *fallible worlds*. For these, there are extra properties which are defined as follows [110]:[17]

$\forall \mathcal{M}, w$: if $\mathcal{M}, w \models \bot$ then

$$\forall w' \in W\text{: if } w \to w' \text{ or } w \preceq w' \text{ then } \mathcal{M}, w' \models \bot \quad \text{and} \tag{1.8}$$

$$\forall P \in \mathit{Var}\text{:} \quad \mathcal{M}, w \models P \tag{1.9}$$

Remember that axiom **IK3** $\neg\Diamond\bot$ is not a theorem of CK. In IK it is automatically valid as $\bot$ holds in no world, but CK has these fallible worlds.[18] The $\Diamond$-operator is now also *strong*, as its truth influences all worlds w' which are *refinements* of the current one, i.e., such that $w \preceq w'$. The first frame property of IK (1.6) follows directly from this, the second property (1.7) is dropped. These changes cause that axioms **IK4** and **IK5** are not tautologies of CK [132].

1.5.6 The S5-Cube

Different combinations of modal axioms lead to different systems. If we take the modal system K and its axioms as foundation, we can simply add any combination of **T**, **4**, **D**, **B**, and **5** to obtain different modal systems, of which KT4 corresponds to S4, and KT45 or KTB4 correspond to S5. The relation between the different systems is usually illustrated as a *cube* as shown in Figure 1.4. Many combinations do not appear there, as these can be defined as other combinations which are shown, e.g., KB4 and KB5 describe the same logic, as well as KT45, KTB4, and KT5.[19]

[17] Wijesekera [149] proposed Kripke semantics for a variant of CK before. Concerning the axioms, **IK3** is kept and only **IK4** and **IK5** are dropped. Alternative special Kripke semantics for CS4 are proposed by Alechina et al. [4].

[18] see also [4, 110, 112, 132].

[19] The shown cube is based on that for the CK-family of [6]. A different one for the K-family can for example be found in [26]. Note that in the literature (also in [6, 24]) the letter 'K' is often omitted as **K** is an axiom of all the shown systems.

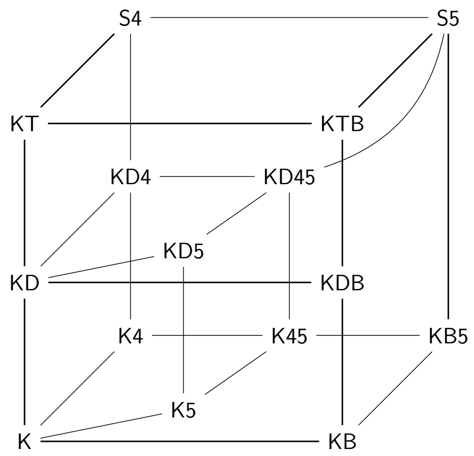


Figure 1.4: The S5-Cube

The same cube exists also for the IK-family. We simply prefix every modal system of the S5-cube with an 'I' [24]. This we call the IS5-cube. Accordingly, for the CK-family we prefix the systems with a 'C' and obtain the CS5-cube [6].

In order to check the equality of two logical systems one can show that the combination of frame properties in one system implies the properties of the other one and vice versa.

2 Sequent Systems and Dialogues

In the first part of this chapter, we consider different sequent systems, especially for classical and intuitionistic propositional logic (Section 2.1), as well as for several propositional modal logics (Section 2.2). There are different approaches which all have their own flavours, advantages or disadvantages. Dyckhoff [41] provides a historical overview of systems for intuitionistic propositional logic. For our purpose, we concentrate on proof searching issues and on systems which are relevant for the further progress of this work. There are, inter alia, calculi that embed λ-terms, making use of the Curry-Howard isomorphism, which we do not discuss here. We also introduce a multi-conclusion calculus for the modal system CS4 (Section 2.2.2).

In the second part, we turn to dialogical games and have a look at historical approaches of differend kinds. Most of these systems are related to intuitionistic and classical first-order logic. In Section 2.3 we focus on the propositional fragment (although we also have a short look at the first-order variants) and afterwards on several modal-logical approaches (Section 2.4). Finally, in Section 2.5, we point out similarities of sequent systems and dialogical calculi concerning proof-searching strategies.

2.1 Sequent Systems for Classical and Intuitionistic Logic

2.1.1 Gentzen's Systems LK and LJ

Sequent systems go back to Gerhard Gentzen. In [60] he presents the systems NJ and NK for *natural deduction*[1], as well as the systems LJ and LK which are usually called *sequent systems*. All four calculi are used to perform deduction in first-order logic. NJ and LJ are intuitionistic variants of NK and LK which are used for classical logic.

As Gentzen states in [60], his natural deduction techniques provide a formal way of how a mathematical proof is usually done. LJ and LK are rather introduced to make it possible to formulate and prove his *Hauptsatz*, namely that it is possible to transform any formal proof to some *normal form* [60], which is the result of an admissible technique for sequent calculi that is nowadays widely known as *cut elimination*. Unless stated otherwise, for all sequent systems we consider in the following, it has been shown that the cut rules are dispensable with respect to completeness.

In Gentzen's systems LJ and LK, every *state* of a proof can be expressed as a *sequent* (that is where the name comes from) of the form

$$A_1, A_2, \ldots A_n \Rightarrow B_1, B_2, \ldots B_m \ ,$$

so in each sequent we have two *sequences* (or *lists*) of formulas separated by the $\Rightarrow$ symbol. The left sequence (the A's) is called the *antecedent* of the sequent, the right part (the B's) is called *succedent* [81]. In fact, the general sequent from above can be read as the implication: *from* $\bigwedge_{i=1}^{n} A_i$ *follows* $\bigvee_{j=1}^{m} B_j$, or to express it in words: the *conjunction* of all antecedent formulas of a sequent implies the *disjunction* of all succedent formulas (of the sequent).

[1] He gave the natural deduction system the German name *Kalkül des natürlichen Schließens*.

We say that a sequent is *derivable* or *valid*, iff it is an *axiom* or it can be derived due to *rules* which a sequent system provides. For any arbitrary formula A, the sequent

$$A \Rightarrow A$$

is derivable/valid. This sequent is called an *axiom* (*Anfangssequenz* [60]).

When we consider a sequent that is not an axiom, we have to apply *rules*[2] to justify its validity. Applying such rules, one after another, leads to a *derivation* of a sequent by constructing a *sequent tree*.

Gentzen distinguishes *logical* and *structural rules*. The former depend on the outer-most operator of some formula. The formulas of antecedent and succedent also refer to different sets of rules. We call rules for formulas of the antecedent *left-hand rules* or *antecedent rules* and those for formulas of the succedent *right-hand rules* or *succedent rules*. The differentiation between these sets is necessary as they have other semantic meanings, namely the antecedent sequents are interpreted as conjunctions and the succedent sequents as disjunctions. *Structural rules* are mainly used to add or remove formulas to/from sequences or to exchange the order of formulas within a sequence.

As in Chapter 1.3.4, a rule consists of zero, one or two (or sometimes even more) *premises* which are in our case sequents written above a horizontal line, and exactly one *conclusion* below the line. For example, there are two rules for conjunction in the antecedent which look thus:

$$\frac{A, \Gamma \ \Rightarrow \ \Theta}{A \wedge B, \Gamma \ \Rightarrow \ \Theta} \qquad \frac{B, \Gamma \ \Rightarrow \ \Theta}{A \wedge B, \Gamma \ \Rightarrow \ \Theta}$$

Both rules have one premise. The Greek letters Γ and Θ stand for arbitrary (possibly empty) sequences of formulas. The sequence A, Γ is a sequence

[2]Gentzen [60]: *Schlußfigurenschemata*

that starts with a formula A, which again stands for an arbitrary formula, and is continued by sequence Γ (see also Chapter 1.3).

Rules can be interpreted as a deduction which says: *if all premises are derivable/valid, then the conclusion is derivable/valid as well.* In the first case from above, one could interpret the rule as: *if* A *and the conjunction of all formulas of* Γ *imply the disjunction of formulas of* Θ*, then* $A \wedge B$ *and the conjunction of all formulas of* Γ *imply the disjunction of formulas of* Θ. To find a proof/deduction for a sequent, one usually applies the rules *backwards*, i.e., from bottom to top until one reaches axioms in all branches of the sequent tree. We say that a sequent tree is *closed* iff axioms occur at all its leaves.

For the conjunction in the succedent, Gentzen provides only one sequent rule with two premises:

$$\frac{\Gamma \Rightarrow \Theta, A \qquad \Gamma \Rightarrow \Theta, B}{\Gamma \Rightarrow \Theta, A \wedge B}$$

Note that in general, antecedent rules operate on the first formula of the antecedent sequence and rules for the right-hand side on the last formula, respectively. In the displayed rule, the last formulas of the succedents are replaced (A and B are replaced by $A \wedge B$ while Θ is left untouched). The formulas A and B are usually called *active formulas*.[3] The result $A \wedge B$ in the conclusion is called *principal formula*, and all other formulas of Γ and Θ are *side formulas*.

In the following, when we say that a rule is *applied*, we refer to the principal formula and read the application from bottom to top. The application then results in the premise sequent(s) if the rule is not applied in an axiom sequent.

As structural rules Gentzen provides three pairs plus the *cut-rule*. These are shown in Figure 2.1.[4] He introduces two different sequent systems (LK

[3] We use the terms proposed in [142]. Kleene [81] called the active formulas *side formulas*, while Troelstra and Schwichtenberg name the *untouched* formulas *side formulas*.

[4] We use other symbols here to keep consistency with respect to the systems we consider later.

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

Contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Exchange

$$\frac{\Gamma, A, B, \Theta \Rightarrow \Delta}{\Gamma, B, A, \Theta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}$$

Cut

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

Figure 2.1: Structural Rules of Gentzen's original sequent system

and LJ) that make it possible to derive statements, which are valid in the corresponding logic (classical or intuitionistic first-order logic), syntactically. This is simply done by applying the provided rules to obtain a sequent tree, where in each of its leaves an axiom sequent occurs. He uses the same logical and structural rules in both systems but puts a restriction to LJ on the meta level to comply with the intuitionistic semantics (more on this follows in the next section).

2.1.2 Getting Rid of Structural Rules

Based on Gentzen's LK and LJ, Kleene [81] introduced system G3 considering sequences to be equal iff they are *cognate* to each other, i.e., iff they simply contain the same formulas, no matter in which order. This makes the structural rules (with the exception of *Cut*) dispensable [81] and actually leads us to the use of *multi-sets* or *sets* instead of sequences.

$$\frac{}{P, \Gamma \Rightarrow \Delta, P}\, ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\, \bot l$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, \wedge l \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, \wedge r$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, \vee l \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, \vee r$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\, \supset l \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\, \supset r$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\, \neg l \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\, \neg r$$

$$\frac{\forall x\, A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x\, A, \Gamma \Rightarrow \Delta}\, \forall l \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x\, A}\, \forall r$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x\, A, \Gamma \Rightarrow \Delta}\, \exists l \qquad \frac{\Gamma \Rightarrow \Delta, A[x/t], \exists x\, A}{\Gamma \Rightarrow \Delta, \exists x\, A}\, \exists r$$

Figure 2.2: Rules of GCPC/G3c (c.f. [39, 142])

In the meantime, Ketonen [79] simplified some of Gentzen's logical rules. Combining both alternations leads to systems like GCPC and GHPC by Dragalin [39] for classical and intuitionistic first-order logic respectively. Based on these, Troelstra and Schwichtenberg [142] present G3c and G3i. For each, soundness and completeness and the admissibility of *cut-elimination*[5], which says that the cut-rule (Figure 2.1) is dispensable, are shown. From now on, unless stated otherwise, we consider the Greek capital letters Γ and Δ as multi-sets of formulas instead of sequences.

The rules of GCPC/G3c[6] are shown in Figure 2.2. For the rules $\forall r$ and $\exists l$ we have to make sure that the variable y does not occur as free variable (*Eigenvariable*) in the conclusion (c.f. [60, 39, 142]). The letter t stands for an arbitrary term. In the ax-rule P is an *atomic* formula.

[5] This corresponds to Gentzen's *Hauptsatz*.

[6] The systems GCPC and G3c are equal. However, GHPC and G3i have different sets of rules and are here considered separately.

$$\frac{}{P, \Gamma \Rightarrow P}\ ax \qquad \frac{}{\bot, \Gamma \Rightarrow A}\ \bot l$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \land B, \Gamma \Rightarrow C}\ \land l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \land B}\ \land r$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \lor B, \Gamma \Rightarrow C}\ \lor l \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \lor B}\ \lor r \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \lor B}\ \lor r$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ \supset l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ \supset r$$

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow C}\ \neg l \qquad \frac{A, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg r$$

$$\frac{\forall x\, A, A[x/t], \Gamma \Rightarrow C}{\forall x\, A, \Gamma \Rightarrow C}\ \forall l \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x\, A}\ \forall r$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x\, A, \Gamma \Rightarrow C}\ \exists l \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x\, A}\ \exists r$$

Figure 2.3: Rules of G3i (c.f. [142])

So far, we have not made any distinction between LK and the intuitionistic version LJ. The reason is that the difference cannot be found in the rules themselves. Gentzen puts a restriction on a meta level: in every LJ sequent, the succedent may only contain one or no formula [60]. Considering the sequences as sets again, this leads us to system G3i, where this meta rule is implemented directly.

The rules of G3i (in the variant we consider in the following) are displayed in Figure 2.3. The restrictions for the free variables are the same as in G3c. The differences are as follows:

- The succedent of all sequents contains exactly one formula. This means that we need two different rules for $\lor r$, one for the left and one for the right disjunct.

- Reading the rules from bottom to top, in rule $\supset l$, $A \supset B$ is kept in the left premise. Accordingly, $\neg A$ is kept in the premise of rule $\neg l$.

Sequent systems in which the succedent contains *at most one* formula, like G3ɪ, are called *single-conclusion sequent systems* or *single-conclusion calculi*. By contrast, if we permit more than one formula in the succedent, like in G3ᴄ, we say that the sequent calculus is a *multi-conclusion sequent system* or *multi-conclusion calculus*.

The requirement that the succedent of each sequent consists of only one formula makes it impossible to derive non-intuitionistic tautologies like the law of the *excluded middle* $P \vee \neg P$:

$$\frac{\Rightarrow P}{\Rightarrow P \vee \neg P}\vee r \qquad \frac{\dfrac{P \Rightarrow \bot}{\Rightarrow \neg P}\neg r}{\Rightarrow P \vee \neg P}\vee r$$

Both of these derivations cannot be completed with a *closing rule* (ax or $\bot l$). The reason is that when constructing the tree (reading it from bottom to top), resolving the $\vee$ forces us to drop one of the disjuncts which is needed to obtain the axiom sequent finally. By contrast, in GCPC/G3ᴄ, we can keep both disjuncts and the derivation closes:

$$\frac{\dfrac{\overline{P \Rightarrow P}\ ax}{\Rightarrow P, \neg P}\neg r}{\Rightarrow P \vee \neg P}\vee r$$

The second difference between G3ɪ and G3ᴄ, namely keeping the principal formula in the rules $\supset l$ and $\neg l$, is necessary to ensure completeness of the proof system.[7] If we kept the left contraction rule (Figure 2.1) as non-dispensable, this measure would not be necessary, as it would be possible to duplicate formulas arbitrarily (reading the rules from bottom to top). Therefore, we call the rules $\supset l$, $\neg l$ and $\forall l$ of G3ɪ *duplication rules*. The duplication is only necessary because information is lost when some of the G3ɪ-rules are applied, e.g., one disjunct is lost when we use $\vee r$. As such loss of information is restricted in G3ᴄ, duplication is only necessary in $\forall l$ and $\exists r$ there.

[7] Note that negation is a special case of implication as $\neg A$ can be written as $A \supset \bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P \vee \neg P), P \Rightarrow P}\;ax}{\neg(P \vee \neg P), P \Rightarrow P \vee \neg P}\;\vee r}{\neg(P \vee \neg P), P \Rightarrow \bot}\;\neg l}{\neg(P \vee \neg P) \Rightarrow \neg P}\;\neg r}{\neg(P \vee \neg P) \Rightarrow P \vee \neg P}\;\vee r}{\neg(P \vee \neg P) \Rightarrow \bot}\;\neg l$$

Figure 2.4: Necessity of duplication in G3I

An example (mentioned in [40]) which shows the necessity of this duplication is shown in Figure 2.4. As it can be seen, the derivation is not possible if the principal formula $\neg(P \vee \neg P)$ is dropped from the premise of $\neg l$.

2.1.3 Multi-Conclusion Sequents for Intuitionistic Logic

In 1954, Maehara [104] introduced a multi-conclusion version L'J of LJ that is actually based on LK, with the difference that the rules for $\supset r$, $\neg r$ and $\forall r$ are replaced by variants in which the succedents of the premises are empty besides the active formulas. In other words, reading these rules from bottom to top again, formulas are *dropped* when they are applied. We call these rules *critical* [44] or *non-invertible* rules, while the others are called *non-critical* or *invertible rules*. More details follow in Chapter 3.3.1.

Again, we consider a variant of L'J that gets along without the structural rules. This we call G3I^{M} which is based on Dragalin's GHPC [39].[8] The rules of G3I^{M} are the same as those of G3C with the exception that we replace rules $\supset l$, $\neg l$, $\supset r$, $\neg r$ and $\forall r$ by those shown in Figure 2.5. G3I^{M} is a multi-conclusion version of G3I. In the following chapters, we make use of G3I^{M} at several points, so it has an important role in this work.

Duplication is needed again in $\supset l$ and $\neg l$ (and $\forall l$ as before). To apply the rule $\forall r$, it must be ensured that y does not occur free in the conclusion [142].

[8] It is called m-G3i in [142] and LJ_{mc} in [44]. The difference to GHPC is explained below.

$$\frac{A \supset B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \supset l \qquad \frac{\neg A, \Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \neg l$$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B} \supset r \qquad \frac{A, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \Delta, \neg A} \neg r \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \Delta, \forall x\, A} \forall r$$

Figure 2.5: Rules of G3I^{M} (c.f. [142])

The only difference to GHPC is the rule for $\supset l$ which differs from that of G3I^{M} in the conclusion of the left premise. By contrast to $\supset l$ of G3I^{M}, this is critical as well [39]:

$$\frac{A \supset B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \supset l$$

From now on we concentrate on propositional logic, because the main topic of this work is propositional modal logic, in which the quantifiers do not occur. Therefore, for the following sequent systems we do not consider the rules $\forall l$, $\forall r$, $\exists l$ or $\exists r$ anymore. Also, because $\neg A$ is equal to $A \supset \bot$[9] some of the following calculi do not provide rules for the negation $\neg$.

2.1.4 Termination in Intuitionism

We already observed that the duplications in $\supset l$ and $\neg l$ are necessary in systems G3I and G3I^{M} for completeness. On the other hand, they are a problem: every non-duplicating rule (read from bottom to top) that is related to a logical connective, decomposes a formula which means that after a certain amount of rule applications, we finally obtain atomic formulas in the leaves of a proof tree. However, the duplication rules keep the principal formula in (at least) one premise which prevents the system to terminate when the conclusion sequent is not valid.

[9] To verify this, we can use one of the systems from above to show that $\neg A \Rightarrow A \supset \bot$ and $A \supset \bot \Rightarrow \neg A$ are both valid sequents.

For example, let us consider *Peirce's Law* $((P \supset Q) \supset P) \supset P$. This formula is a tautology in classical propositional logic, but not in intuitionistic. Nevertheless, if we try to prove it in G3I$^{\mathrm{M}}$, we obtain the following tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots}{(P \supset Q) \supset P, P, P \Rightarrow Q}}{(P \supset Q) \supset P, P \Rightarrow Q, P \supset Q}\,{\supset}r \qquad P, P \Rightarrow Q}{(P \supset Q) \supset P, P \Rightarrow Q}\,{\supset}l}{(P \supset Q) \supset P \Rightarrow P, P \supset Q}\,{\supset}r \qquad \dfrac{}{P \Rightarrow P}\,ax}{(P \supset Q) \supset P \Rightarrow P}\,{\supset}l}{\Rightarrow ((P \supset Q) \supset P) \supset P}\,{\supset}r
$$

At the position indicated by "…" further rule applications of $\supset l$ are possible which would not help us to reach an axiom sequent anyway. We say that the proof does *not terminate* and therefore the systems G3I and G3I$^{\mathrm{M}}$ do not guarantee termination.

If we have a closer look at the top left sequent of the tree, we see that actually no new information is added compared to two levels below. We only have one extra P in the antecedent. Although we could apply rules again and again, we would not gain any new facts. We actually have some kind of *loop* here which can be detected as recurrences within a tree [40]. However, if one wants to implement an automated theorem prover based on these sequent systems it will be helpful to have one that prevents such loops from the start using special rules. But of course, completeness and soundness must be preserved.

Implications in Detail

Dyckhoff [40] presents a variant of G3I he traces back to Vorob'ev [146] and which has been *rediscovered* by Hudelmaier [74]. The rule $\supset l$ is replaced by the four rules of Figure 2.6. Rule $\supset l_1$ may only be applied if P is atomic. Note that in this calculus negations $\neg A$ are interpreted as implications $A \supset \bot$. This system is often called G4IP[10]; in the following, we simply call it G4I.

$$\frac{B, P, \Gamma \Rightarrow D}{P \supset B, P, \Gamma \Rightarrow D} \supset l_1 \text{ [P is atomic]} \qquad \frac{A \supset (B \supset C), \Gamma \Rightarrow D}{(A \wedge B) \supset C, \Gamma \Rightarrow D} \supset l_2$$

$$\frac{A \supset C, B \supset C, \Gamma \Rightarrow D}{(A \vee B) \supset C, \Gamma \Rightarrow D} \supset l_3 \qquad \frac{B \supset C, \Gamma \Rightarrow A \supset B \qquad C, \Gamma \Rightarrow D}{(A \supset B) \supset C, \Gamma \Rightarrow D} \supset l_4$$

Figure 2.6: Rules of G4ɪ (c.f. [40])

If we try to prove Peirce's Law in G4ɪ, we obtain the following finite sequent tree where no further rule applications are possible:

$$\dfrac{\dfrac{\dfrac{Q \supset P, P \Rightarrow Q}{Q \supset P \Rightarrow P \supset Q} \supset r \qquad \dfrac{}{P \Rightarrow P}\, ax}{(P \supset Q) \supset P \Rightarrow P} \supset l_4}{\Rightarrow ((P \supset Q) \supset P) \supset P} \supset r$$

This is the only possible saturated[11] G4ɪ-tree for Peirce's Law, if P and Q are prime formulas. In G4ɪ, termination is guaranteed [40]. From the proof-searcher's point of view, the rules do not only take the *outer-most operator* of the principal formula into account, but consider also the structure of the principal's antecedent. So we have four possibilities for this structure: it can be an atom ($\supset l_1$), a conjunction ($\supset l_2$), a disjunction ($\supset l_3$), or an implication ($\supset l_4$). When it is not an atom, the active formulas are in general new implications, which are *not* subformulas of the principal. However all rules of the other sequent systems considered so far have this *subformula property* which is given up in G4ɪ [33, 32].

Rule Restrictions

Another approach which guarantees termination in intuitionistic propositional logic, is given by Corsi and Tassi. Their system IG [32, 33] is based on $G3\text{ɪ}^{M}$, but has an additional rule named *a fortiori*:

[10] The ᴘ indicates that only propositional formulas are considered in the system.
[11] i.e., no further rule applications are possible.

$$\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ a.f$$

This does not replace $\supset$r. Instead, *meta-rules* (or *external restrictions*[12]) make clear in which case one may use $\supset$r, *a fortiori* ($a.f$ for short), or $\supset$l to guarantee termination (adapted from [33]):

i) Rule $\supset$r may only be used *once* with the same principal formula within one path of the sequent tree.

ii) If a formula $A \supset B$ is principal in the application of $a.f$ in a sequent tree t, then below this application in t, $A \supset B$ is principal in an application of $\supset$r on the same path.

iii) If $\supset$l is applied twice with the same principal $A \supset B$ within one path of a sequent tree, then there is an application of $\supset$r on the same path between these two $\supset$l-applications.

If these rules are obeyed, the IG-derivation is called to be *regular*, or IG^r-derivation for short. Considering any implication $A \supset B$ on any path of a sequent tree from bottom to top, point i) ensures that $\supset$r is only applied once with $A \supset B$ as principal. Together with point iii), we may have $\supset$l a limited number of times with the same principal. Trying to find a derivation for Peirce's Law in IG^r leads to the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{(P \supset Q) \supset P, P \Rightarrow Q, Q}{(P \supset Q) \supset P, P \Rightarrow P \supset Q, Q}\ a.f \qquad P, P \Rightarrow Q}{\dfrac{(P \supset Q) \supset P, P \Rightarrow Q}{(P \supset Q) \supset P \Rightarrow P, P \supset Q}\ \supset r}\ \supset l \qquad \dfrac{}{P \Rightarrow P}\ ax}{(P \supset Q) \supset P \Rightarrow P}\ \supset l}{\Rightarrow ((P \supset Q) \supset P) \supset P}\ \supset r$$

The leaves of this tree are *blocked*, i.e., no further rules can be used above. The reason is restriction iii): as there is no application of $\supset$r above the second (read from bottom to top) application of $\supset$l on $(P \supset Q) \supset P$, we are not

[12]We say *external* because they are not implemented directly in the sequent rules.

allowed to put another $\supset$l with the same principal on top. Using $\supset$r instead of $a.f$ is also not allowed because of restriction i). This is the only possible saturated IG^{r}-tree for Peirce's law with P and Q being prime formulas.

IG^{r} is sound and complete, and termination is guaranteed [32, 33]. The sub-formula property is preserved. The idea of enforcing termination with meta-rules suits game semantics well. In Chapter 3, we refer to IG^{r} when establishing our intuitionistic multi-proponent dialectics with termination guarantee.

An earlier calculus which guarantees termination in a similar way as IG^{r}, is $\mathrm{IPC}^{RP}_{(\wedge,\supset)}$ by Heuerding et al. [68] which makes use of a *history set* that is extended in the left branch of $\supset$l-applications. It is defined for the fragment of propositional intuitionistic logic having only $\wedge$ and $\supset$ as logical connectives. It was extended by Howe [73] to handle disjunctions (and negations) as well.[13] It features also two versions of the $\supset$r-rule (where one can be seen as a variant of *a fortiori*). Further restrictions are built into the rules themselves. We consider a variant for the modal logic S4 in Section 2.2.3. Howe himself also proposes an alternative approach making also use of history sets [72]. See [72, 73] for a detailed comparison.

Another such system making use of some history-like set is proposed by Ferrari et al. [53]. In their work, there is only one left-hand rule and one right-hand rule for implication but the former has three premises instead of the usual two. The latter has two premises instead of one.

2.1.5 Focusing

It is often the case that one can find different derivations for the same sequent in one and the same system, simply by reordering some rule applications or having applications which are not expedient for the derivation. For example, as we saw before, $\supset$l can be applied infinitely often in G3i

[13] Howe calls the method the *Swiss history* [73].

and $G3I^M$. To prevent such futile applications in proof searches, Andreoli [5] introduced a technique for *linear logic* called *focusing*. There are several adaptions for intuitionistic logic.

Focusing enforces a normalization of sequent proofs. In Chapters 3 and 4, our aim will be to find normalized proofs with techniques of dialogical logic. Focused proofs are therefore relevant in this work although their strategy is different. Similarities and differences to multi-proponent dialectics are discussed in Section 3.4.2.

Uniform Proofs

Uniform proofs [114] were created to define behaviour of inference machines in logical programming languages and can be seen as early approaches of focused sequent systems (e.g., c.f. [96]). An intuitionistic single-conclusion sequent system is used as basis. Uniform proofs have the following restriction: if the succedent formula of a sequent is not atomic then it is principal of a rule application [114]. The result is that antecedent formulas can only be principal if a right-hand rule cannot be applied.

However, this restriction is too strong and there is not a uniform proof for every valid sequent, for example $\Rightarrow (A \vee B) \supset (B \vee A)$ cannot be derived with a uniform proof. Because of this incompleteness we do not consider uniform proofs further.

Isolating Critical Rule Applications

Searching for an isomorphism between special lambda-terms and proofs in sequent systems, Herbelin [66] introduced a simple focused system LJT for the fragment of intuitionistic logic having only the implication as possible connective. In this single-conclusion system, the sequents have the form

$$\frac{}{\Gamma; A \Rightarrow A}\,ax \qquad \frac{\Gamma, A; A \Rightarrow B}{\Gamma, A; \Rightarrow B}\,Cont$$

$$\frac{\Gamma; \Rightarrow A \qquad \Gamma; B \Rightarrow C}{\Gamma; A \supset B \Rightarrow C}\supset l \qquad \frac{\Gamma, A; \Rightarrow B}{\Gamma; \Rightarrow A \supset B}\supset r$$

Figure 2.7: Rules of LJT (c.f. [66])

$$\Gamma; \Rightarrow C \qquad \text{or} \qquad \Gamma; A \Rightarrow C$$

i.e., it is possible to have a focused formula A in the antecedent which is explicitly separated from Γ due to a semicolon. This formula is called *stoup* [66].[14] Before a rule can be applied on an implication of the antecedent, this principal needs to be stored to the stoup with a special kind of contraction rule (*Cont*). The rules of LJT are shown in Figure 2.7.

Later, Herbelin introduced a sequent system LJQ which is based on G3I but has a restriction that says that above the left premise of $\supset l$, only the rules ax, $\wedge r$, $\vee r$ and $\supset r$ are allowed [67]. This *external restriction* is internalized in the variants LJQ′ and LJQ* presented by Dyckhoff and Lengrand [42].

LJQ′ is a single-conclusion system in which two kinds of sequents are distinguished. The *ordinary sequents* have the usual form $\Gamma \Rightarrow A$ with Γ being a multi-set. *Focus sequents* simply have another implication arrow ($\rightarrow$ instead of $\Rightarrow$). The new condition is that in a derivation tree, the conclusion of left-hand rule applications must always be an ordinary sequent, while the conclusion of right-hand rules and rule ax are always focus sequents. There is also a special rule Der (for *dereliction* [42]) to transform an ordinary sequent into a focused one (read from bottom to top). The rules of LJQ′ are displayed in Figure 2.8 (top). The sequent $P \vee Q \rightarrow P \vee Q$ is not derivable in LJQ′ [42], because $\vee l$ can only be applied on *ordinary* sequents, but not in focus sequents. However, $P \vee Q \Rightarrow P \vee Q$ is derivable, of course.

[14] The term *stoup* in such a setting goes back to Girard [61] (c.f. [66]).

LJQ′ (single-conclusion)

$$\frac{}{\bot, \Gamma \Rightarrow A}\ \bot l \qquad \frac{\Gamma \rightarrow A}{\Gamma \Rightarrow A}\ Der \qquad \frac{}{P, \Gamma \rightarrow P}\ ax$$

$$\frac{A \supset B, \Gamma \rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ \supset l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \rightarrow A \supset B}\ \supset r$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \wedge l \qquad \frac{\Gamma \rightarrow A \qquad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}\ \wedge r$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ \vee l \qquad \frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}\ \vee r \qquad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}\ \vee r$$

LJQ* (multi-conclusion)

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot l \qquad \frac{\Gamma \rightarrow A; \Delta}{\Gamma \Rightarrow A, \Delta}\ Der \qquad \frac{}{P, \Gamma \rightarrow P; \Delta}\ ax$$

$$\frac{A \supset B, \Gamma \rightarrow A; \emptyset \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ \supset l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \rightarrow A \supset B; \Delta}\ \supset r$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge l \qquad \frac{\Gamma \rightarrow A; \Delta \qquad \Gamma \rightarrow B; \Delta}{\Gamma \rightarrow A \wedge B; \Delta}\ \wedge r$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \vee l \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \rightarrow A \vee B; \Delta}\ \vee r$$

Figure 2.8: Rules of LJQ′ and LJQ* (c.f. [42])

An important feature is that the left premise of $\supset$l is a focus sequent which makes it impossible to have left-hand rule applications directly above. By contrast, the premise of $\supset$r is an ordinary sequent again. The combination of both ensures that we can have $\supset$l twice with the same principal formula (instance) on the same path of a proof tree only if there is an application of $\supset$r in between.

Dyckhoff and Lengrand also present system LJQ* [42] as a multi-conclusion variant of LJQ′ based on a system of the same name by Herbelin [67]. The focus sequents have now the form $\Gamma \to A; \Delta$ where the formula A is separated by the semicolon from Δ. Rules above such sequents must be right-hand rules with A as principal. These focused formulas are also called *stoup* (this time in the succedent). So changing from an ordinary sequent to a focus sequent (from bottom to top) requires to choose the right formula to put into focus. Also note that the left premise of $\supset$l has an empty succedent (besides the stoup) which is similar to Dragalin's $\supset$l of GHPC [42]. For both systems LJQ′ and LJQ*, termination is *not* guaranteed, e.g., searching for a proof of Peirce's Law leads to the same problems as in G3ɪ and G3ɪ^{M}.

Focus with Polarities

Liang and Miller introduced a general version of focusing sequent systems for intuitionistic and classical first-order logic [96, 97] which is based more on Andreoli's attempt for linear logic and which makes use of *polarities* as in Girard's work [61, 62]. Formulas can always be categorised according to their outermost connective into different polarities. These then define in which phase of the proof a rule can be applied to the formula.

In the following, we have a look at their system LJF for intuitionistic logic but we consider only the propositional fragment to keep things simpler. In LJF all formulas either have a positive or a negative polarity, which is assigned according to the following rules [96]:

- Prime formulas have an arbitrary polarity (either positive or negative), but the assignment must be consistent, i.e., once chosen, the atoms with the same name also have the same polarity.
- *Positive* formulas are positive primes (P^+), $\top$, $\bot$, disjunctions ($A \vee B$), and *positive* conjunctions ($A \wedge^+ B$).
- *Negative* formulas are negative primes (P^-), implications ($A \supset B$), and *negative* conjunctions ($A \wedge^- B$).

Note that a negative polarity has nothing to do with negation. It is simply a categorisation which is related to invertibility of rule applications.

In LJF, there are both *positive* and *negative* conjunctions. The different polarities increase flexibility so it is possible to *embed* other systems (like LJQ') into LJF. Concerning provability, it does not matter which polarities are assigned to the different conjunctions, but the choice has an impact on the proof strategy, i.e., the structure of the proof tree. Therefore, for now we can assign polarities to conjunctions in an arbitrary fashion [96].

Liang and Miller define four types of sequents with two types of focus [96]. We call formulas of $[\Gamma]$ or $[C]$ *focus candidates* as a focus can be put on these formulas in a sequent of type (ii):

(i) $[\Gamma], \Theta \longrightarrow \mathcal{C}$
This sequent has no focus. $\mathcal{C}$ is either a formula C or a bracketed formula $[C]$. In such a sequent, principals can be formulas of Θ or C if it is not bracketed.

(ii) $[\Gamma] \longrightarrow [C]$
This is a special variant of (i), where Θ is empty and C is bracketed. In this sequent, a focus can be selected, i.e., put on a formula of Γ or on C.

(iii) $[\Gamma] \xrightarrow{A} [C]$
The focus is set on formula A which originates from the antecedent of some sequent below. A is the principal of the rule application above.

(iv) $[\Gamma] -_{A} \rightarrow$

The focus is set on formula A which originates from the succedent of some sequent below. A is the principal of the rule application above.

The rules of LJF are shown in Figure 2.9. Formulas A^{+} and A^{-} have positive or negative polarities respectively. P^{+} and P^{-} are prime formulas. $D^{=}$ must be a formula of negative polarity or a prime formula of positive polarity, whereas $E^{\#}$ is a positive formula or a negative prime (c.f. [96]).

Rules $I\mathrm{r}$ and $I\mathrm{l}$ correspond to ax in G3I and are only applicable on atoms which are in focus. Foci can be put on negative focus candidates in the antecedent ($F\mathrm{l}$), or on the positive focus candidate in the succedent ($F\mathrm{r}$). After the polarity of the focused formula has changed, it can be taken out of focus again ($R\mathrm{l}$, $R\mathrm{r}$). Unfocused formulas of the antecedent which are negative, and positive prime formulas of the antecedent can become focus candidates ($[\,]\mathrm{l}$). The same holds for positive formulas and negative primes of the succedent ($[\,]\mathrm{r}$). As *introduction rules*, we have *unfocused rules* where the principal is not in focus ($\bot\mathrm{l}$, $\top\mathrm{l}$, $\wedge^{+}\mathrm{l}$, $\wedge^{-}\mathrm{r}$, $\vee\mathrm{l}$, $\supset\mathrm{r}$), *left-focus rules* with a focused principal that originates from some antecedent (sequent type (iii) from above; $\wedge^{-}\mathrm{l}$, $\supset\mathrm{l}$), and *right-focus rules* with a focused principal originating from the succedent (sequent type (iv) from above; $\top\mathrm{r}$, $\wedge^{+}\mathrm{r}$, $\vee\mathrm{r}$).

The following simple tree shows a derivation for $Q \supset (P \supset (Q \wedge P))$ in LJF. We give Q a positive polarity and P a negative one. The conjunction $Q \wedge P$ is considered to be positive. Other assignments of polarities are possible as well but lead to other structures of the proof tree.

Decision/Reaction Rules

$$\frac{[A^-,\Gamma] \xrightarrow{A^-} [C]}{[A^-,\Gamma] \longrightarrow [C]}\, F\mathfrak{l} \qquad \frac{[\Gamma] \;{-}_{A^+}{\to}}{[\Gamma] \longrightarrow [A^+]}\, F\mathfrak{r}$$

$$\frac{[\Gamma], A^+ \longrightarrow [C]}{[\Gamma] \xrightarrow{A^+} [C]}\, R\mathfrak{l} \qquad \frac{[\Gamma] \longrightarrow A^-}{[\Gamma] \;{-}_{A^-}{\to}}\, R\mathfrak{r}$$

$$\frac{[D^=,\Gamma],\Theta \longrightarrow \mathcal{C}}{[\Gamma],\Theta,D^= \longrightarrow \mathcal{C}}\, [\,]\mathfrak{l} \qquad \frac{[\Gamma],\Theta \longrightarrow [E^\#]}{[\Gamma],\Theta \longrightarrow E^\#}\, [\,]\mathfrak{r}$$

Initial Rules

$$\frac{}{[P^+,\Gamma] \;{-}_{P^+}{\to}}\, I\mathfrak{r} \qquad \frac{}{[\Gamma] \xrightarrow{P^-} [P^-]}\, I\mathfrak{l}$$

Introduction Rules

$$\frac{}{[\Gamma],\Theta,\bot \longrightarrow \mathcal{C}}\, \bot\mathfrak{l} \qquad \frac{[\Gamma],\Theta \longrightarrow \mathcal{C}}{[\Gamma],\Theta,\top \longrightarrow \mathcal{C}}\, \top\mathfrak{l} \qquad \frac{}{[\Gamma] \;{-}_{\top}{\to}}\, \top\mathfrak{r}$$

$$\frac{[\Gamma] \xrightarrow{A} [C]}{[\Gamma] \xrightarrow{A\wedge^- B} [C]}\, \wedge^-\mathfrak{l} \qquad \frac{[\Gamma] \xrightarrow{B} [C]}{[\Gamma] \xrightarrow{A\wedge^- B} [C]}\, \wedge^-\mathfrak{l} \qquad \frac{[\Gamma],\Theta \longrightarrow A \qquad [\Gamma],\Theta \longrightarrow B}{[\Gamma],\Theta \longrightarrow A \wedge^- B}\, \wedge^-\mathfrak{r}$$

$$\frac{[\Gamma],\Theta,A,B \longrightarrow \mathcal{C}}{[\Gamma],\Theta,A \wedge^+ B \longrightarrow \mathcal{C}}\, \wedge^+\mathfrak{l} \qquad \frac{[\Gamma] \;{-}_{A}{\to} \qquad [\Gamma] \;{-}_{B}{\to}}{[\Gamma] \;{-}_{A\wedge^+ B}{\to}}\, \wedge^+\mathfrak{r}$$

$$\frac{[\Gamma],\Theta,A \longrightarrow \mathcal{C} \qquad [\Gamma],\Theta,B \longrightarrow \mathcal{C}}{[\Gamma],\Theta,A \vee B \longrightarrow \mathcal{C}}\, \vee\mathfrak{l} \qquad \frac{[\Gamma] \;{-}_{A}{\to}}{[\Gamma] \;{-}_{A\vee B}{\to}}\, \vee\mathfrak{r} \qquad \frac{[\Gamma] \;{-}_{B}{\to}}{[\Gamma] \;{-}_{A\vee B}{\to}}\, \vee\mathfrak{r}$$

$$\frac{[\Gamma] \;{-}_{A}{\to} \qquad [\Gamma] \xrightarrow{B} [C]}{[\Gamma] \xrightarrow{A\supset B} [C]}\, \supset\mathfrak{l} \qquad \frac{[\Gamma],\Theta,A \longrightarrow B}{[\Gamma],\Theta \longrightarrow A \supset B}\, \supset\mathfrak{r}$$

Figure 2.9: Rules of LJF (c.f. [97]).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow_{\mathsf{Q}^+}}\ I\mathrm{r} \quad \dfrac{\dfrac{\dfrac{}{[\mathsf{Q}^+, \mathsf{P}^-] \xrightarrow{\mathsf{P}^-} \mathsf{P}^-}\ I\mathrm{l}}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow \mathsf{P}^-}\ F\mathrm{l}}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow_{\mathsf{P}^-}}\ R\mathrm{r}}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow_{\mathsf{Q}^+ \wedge^+ \mathsf{P}^-}}\ \wedge^+\mathrm{r}}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow [\mathsf{Q}^+ \wedge^+ \mathsf{P}^-]}\ F\mathrm{r}}{[\mathsf{Q}^+, \mathsf{P}^-] \longrightarrow \mathsf{Q}^+ \wedge^+ \mathsf{P}^-}\ []\mathrm{r}}{[\mathsf{Q}^+], \mathsf{P}^- \longrightarrow \mathsf{Q}^+ \wedge^+ \mathsf{P}^-}\ []\mathrm{l}}{[\,], \mathsf{Q}^+, \mathsf{P}^- \longrightarrow \mathsf{Q}^+ \wedge^+ \mathsf{P}^-}\ []\mathrm{l}}{[\,], \mathsf{Q}^+ \longrightarrow \mathsf{P}^- \supset (\mathsf{Q}^+ \wedge^+ \mathsf{P}^-)}\ \supset\mathrm{r}}{[\,] \longrightarrow \mathsf{Q}^+ \supset (\mathsf{P}^- \supset (\mathsf{Q}^+ \wedge^+ \mathsf{P}^-))}\ \supset\mathrm{r}
$$

Reading the proof from bottom to top, we first have to apply all rules until every formula is bracketed, so we can set a focus. We start with rule $\supset\mathrm{r}$ being applied twice.[15] Afterwards, every formula can be made a focus candidate ($[]\mathrm{l}$, $[]\mathrm{l}$, $[]\mathrm{r}$) and we can set a focus. We set it on $\mathsf{Q}^+ \wedge^+ \mathsf{P}^-$ ($F\mathrm{r}$). Note that it would also be possible to choose P^-, but this would make the proof terminate without success as no rule could be applied above. Afterwards, we apply $\wedge^+\mathrm{r}$ on the focused conjunction. The left branch is finished immediately ($I\mathrm{r}$), the right one is not, because the polarity of P^- does not fit to the initial rule. So the focus needs to be changed: it is taken from P^-, which is moved into the succedent ($R\mathrm{r}$), and put on P^- of the antecedent ($F\mathrm{l}$) which finishes the proof.

This simple example demonstrates how LJF is used. Further examples can be found in [96, 97]. As we see, the rules put strong restrictions on possible applications which leads to a normalization of proofs that depends only on the selection of polarities for atoms and conjunctions. However, an important observation is that proof-relevant decisions are not only made when the focus is selected, but also when rules $\wedge^-\mathrm{l}$ or $\vee\mathrm{r}$ are applied. Termination is not guaranteed, because focus candidates of the antecedent stay focus candidates, also after being put into focus. This also means that it is possible

[15]In the succedent, only positive formulas and negative atoms can be bracketed, but implications are negative.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A^+ \wedge^- B^-, A^+] \xrightarrow{B^-} [B^-]}\ I\mathfrak{l}}{[A^+ \wedge^- B^-, A^+] \xrightarrow{A^+\wedge^- B^-} [B^-]}\ \wedge^-\mathfrak{l}}{[A^+ \wedge^- B^-, A^+] \longrightarrow [B^-]}\ F\mathfrak{l}}{[A^+ \wedge^- B^-], A^+ \longrightarrow [B^-]}\ []\mathfrak{l}}{[A^+ \wedge^- B^-] \xrightarrow{A^+} [B^-]}\ R\mathfrak{l}}{[A^+ \wedge^- B^-] \xrightarrow{A^+\wedge^- B^-} [B^-]}\ \wedge^-\mathfrak{l}}{[A^+ \wedge^- B^-] \longrightarrow [B^-]}\ F\mathfrak{l}}{[\,], A^+ \wedge^- B^- \longrightarrow B^-}\ []\mathfrak{l}, []\mathfrak{r}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{[A^+ \wedge^- B^-] \xrightarrow{B^-} [B^-]}\ I\mathfrak{l}}{[A^+ \wedge^- B^-] \xrightarrow{A^+\wedge^- B^-} [B^-]}\ \wedge^-\mathfrak{l}}{[A^+ \wedge^- B^-] \longrightarrow [B^-]}\ F\mathfrak{l}}{[\,], A^+ \wedge^- B^- \longrightarrow B^-}\ []\mathfrak{l}, []\mathfrak{r}
$$

Figure 2.10: LJF-Proofs with unnecessary rule applications (left) and without (right)

to have rule applications in a proof tree which do not contribute to the proof progress in a reasonable way as the two derivation trees in Figure 2.10 show.

LJF can be used to embed systems LJT, LJQ′ and uniform proofs by fixing the polarities of the formulas in a specific way. Details are given in [96, 97]. The authors also present a variant of LJF for classical logic, called LKF.

More on Foci

Simmons [134] introduces an extension of LJF where the order of rule applications in parts with unfocused sequents is fixed, and with a different treatment of atoms. Chaudhuri et al. [22] present a *multi-focus* sequent system for the linear logic MALL. Here, it is possible to put more than one formula into focus. A multi-focus system for classical first-order logic is proposed in [23]. We do not discuss these here.

2.1.6 Hypersequents in Intuitionistic Logic

Avron [7] was the first to use hypersequents as reasoning systems for *intermediate logics*. These are logics lying *between* classical and intuitionistic logics, i.e., in which the intuitionistic theorems hold but not necessarily all classical ones. In this context, hypersequents calculi for intuitionistic propositional logic have been proposed by Ciabattoni and Ferrari [27], and Fermüller [49]. In each node of the proof tree, instead of considering single sequents, we operate on sequences of sequents. For intermediate logics we can then have an interaction between these sequents of a hypersequent. Regarding intuitionistic logic, it is possible to duplicate a whole sequent before applying a non-invertible rule and still keep book of the result. Calculi using hypersequents are a useful tool for proof searches. In Section 2.3.5, we consider a multi-proponent dialogical system which is based on them.

In general, hypersequents have the following form [7]:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

so they are simply sequences of ordinary (sub-)sequents separated by vertical bars, although we rather consider them here as (possibly empty) multisets. As meta-variables representing arbitrary hypersequents, we use the letters $\mathcal{G}$ and $\mathcal{H}$.

If $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ and $\mathcal{H} = \Gamma_{n+1} \Rightarrow \Delta_{n+1} \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$, then $\mathcal{G} \mid \mathcal{H}$ is defined as $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \mid \Gamma_{n+1} \Rightarrow \Delta_{n+1} \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$.

To operate on hypersequents, we need *structural external rules* which allow us to duplicate sequents or drop them from a hypersequent. The structural external rules are displayed in Figure 2.11 together with the logical rules of the intuitionistic system HLI' proposed by Fermüller [49].

We assume that the logical rules can be applied on *any* of the ordinary sequents, not only on the one written at the leftmost position of the hypersequent. This calculus has the advantage that we can duplicate an ordinary sequent and take out different decisions. For example, consider again the

External Structural Rules

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ EW \qquad \frac{\Gamma \Rightarrow C \mid \Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma \Rightarrow C \mid \mathcal{H}}\ EC$$

Axioms

$$\frac{}{A, \Gamma \Rightarrow A \mid \mathcal{H}}\ ax \qquad \frac{}{\bot, \Gamma \Rightarrow C \mid \mathcal{H}}\ \bot l$$

Logical Rules

$$\frac{A, A \vee B, \Gamma \Rightarrow C \mid \mathcal{H} \qquad B, A \vee B, \Gamma \Rightarrow C \mid \mathcal{H}}{A \vee B, \Gamma \Rightarrow C \mid \mathcal{H}}\ \vee l$$

$$\frac{\Gamma \Rightarrow A \mid \mathcal{H}}{\Gamma \Rightarrow A \vee B \mid \mathcal{H}}\ \vee r \qquad \frac{\Gamma \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \vee B \mid \mathcal{H}}\ \vee r$$

$$\frac{A, A \wedge B, \Gamma \Rightarrow C \mid \mathcal{H}}{A \wedge B, \Gamma \Rightarrow C \mid \mathcal{H}}\ \wedge l \qquad \frac{B, A \wedge B, \Gamma \Rightarrow C \mid \mathcal{H}}{A \wedge B, \Gamma \Rightarrow C \mid \mathcal{H}}\ \wedge l$$

$$\frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \wedge B \mid \mathcal{H}}\ \wedge r$$

$$\frac{A \supset B, \Gamma \Rightarrow A \mid \mathcal{H} \qquad B, A \supset B, \Gamma \Rightarrow C \mid \mathcal{H}}{A \supset B, \Gamma \Rightarrow C \mid \mathcal{H}}\ \supset l \qquad \frac{A, \Gamma \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \supset B \mid \mathcal{H}}\ \supset r$$

Figure 2.11: Rules of HLI′ (c.f. [49])

law of the excluded middle of classical logic $A \vee \neg A$ which is not a theorem of intuitionistic logic:[16]

$$\dfrac{\dfrac{\dfrac{\Rightarrow A \mid A \Rightarrow \bot}{\Rightarrow A \mid \Rightarrow \neg A}\ \neg r}{\Rightarrow A \vee \neg A \mid \Rightarrow A \vee \neg A}\ \vee r\ \text{(twice)}}{\Rightarrow A \vee \neg A}\ EC$$

Reading the tree from bottom to top, we first duplicate the sequent with the external contraction rule (EC), because, as this is a single-conclusion calculus, we have to decide whether to keep the left or the right disjunct when using $\vee$r. Now we can look at both possibilities in *parallel*. As we are soon not able to do anything in the left sequent, we can apply $\neg$r in the right one which leads to a dead-end as well.

There are several ways to put these sub-sequents together again in order to implement systems for different intermediate logics (e.g. c.f. [7, 28, 27]). To do reasoning in *classical* logic, we simply need to add the following rule:[17]

$$\dfrac{\Gamma, \Gamma' \Rightarrow A \mid \mathcal{H}}{\Gamma \Rightarrow \bot \mid \Gamma' \Rightarrow A \mid \mathcal{H}}\ CL$$

So if one sub-sequent has $\bot$ as succedent, it can be combined with another sub-sequent. This allows us to finish the derivation from above putting the following on the top:

$$\dfrac{\dfrac{}{A \Rightarrow A}\ ax}{\Rightarrow A \mid A \Rightarrow \bot}\ CL$$

We consider such *merging* of sequents in the setting of dialogues briefly in Section 2.3.5. Hypersequents play also an important role in modal logic.

[16]The example is adapted from [49].
[17]The rule is taken from [28]. In [27] and [49], more general versions are used.

A sequent system which uses *sequent lists* instead of hypersequents is SIC by Corsi and Tassi [33]. Instead of considering all individual sequents at the same time, only the last one of the sequent list is taken into account, which differs to hypersequents [33]. It makes use of the *a fortiori* rule and guarantees termination. Restrictions of IG^r are directly implemented by using special labels attached to formulas.

Nested sequents are an alternative to hypersequents where the nodes of the derivation tree do not contain hypersequents but the sequents have a nested structure. Fitting [58] provides such systems for both intuitionistic propositional and first-order logic. Lellmann [92] gives an alternative version, in which sequents are nested *linearly*.

Both hypersequent and nested sequent calculi are also used to do reasoning in modal logics. More on this follows in Section 2.2.5. Next we consider various modal extensions of the propositional systems we examined so far, starting with the simple, ordinary sequent structures.

2.2 Sequent Systems for Modal Logic

There are many sequent systems for modal logic as well. Early attempts like that by Curry [35] and Ohnishi and Matsumoto [117, 118] were presented before Kripke came up with his famous possible world semantics.

There are too many of these calculi to discuss all of them here. We consider only a selection of systems in detail, which are relevant for our purposes. In particular, we concentrate on sequent systems for the logics K, KT and S4, and their intuitionistic/constructive variants. A survey of (unlabelled) sequent calculi for normal modal logic can be found in [148] by Wansing. A more recent, general survey of sequent systems for normal modal logics, including Display calculus, is provided by Poggiolesi [123], which however does not cover standard hypersequents or focused sequents, but also takes logics S5, GL and different combinations of axioms B, T and 4 into account. The reader is referred to her book for a detailed overview on calculi for classical modal logic.

2.2.1 Ordinary Modal Sequent Systems

First, we have a look at modal extensions of the systems discussed in Sections 2.1.1 to 2.1.3. Like Wansing [148] and Poggiolesi [123], we call these *ordinary sequent systems* (not to mix up with *ordinary sequents* of focus calculi): all sequents have the simple structure $\Gamma \Rightarrow \Delta$ where Γ and Δ are either sequences, sets, or multi-sets of formulas.

An early extension of Gentzen's LK for modal logic K was proposed by Leivant [91] in 1981. He treats the sequences as sets and suggests to add a single rule:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \Box\mathrm{I}$$

There is no left- or right-hand rule for $\Box$, but only this *introduction*. The set $\Box\Gamma$ is defined as $\{\Box\phi \mid \phi \in \Gamma\}$. Accordingly, we use this notation for other sequent systems hereinafter, also we let $\Diamond\Delta$ be $\{\Diamond\phi \mid \phi \in \Delta\}$. In the succedent of Leivant's rule, there is only a single formula both in the premise and in the conclusion. Therefore, when we use this rule in proof search, we have to make extensive use of the weakening rules, so that we have only boxed formulas in the antecedent, before we apply $\Box$I. There is no rule for $\Diamond$.

Several years before, in 1957, Ohnishi and Matsumoto [117] presented rules for logics S4 (of which the $\Box$-rules were already proposed earlier by Curry [34, 35]), T, and S5.[18] Both left- and right-hand rules for $\Box$ and $\Diamond$ were introduced to extend LK. The additional rules for KT are

[18]They use the name M for T and also propose rules for another modal system Q2 we do not consider here at all. For the S5 version, cut elimination does not work [87].

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,\Box l \qquad \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\Box r \qquad \frac{A \Rightarrow \Delta}{\Diamond A \Rightarrow \Diamond\Delta}\,\Diamond l \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}\,\Diamond r$$

and for $\mathsf{S4}$ they are:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,\Box l \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\Box r \qquad \frac{A \Rightarrow \Diamond\Delta}{\Diamond A \Rightarrow \Diamond\Delta}\,\Diamond l \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}\,\Diamond r$$

Differences between KT and $\mathsf{S4}$ occur in rules $\Box r$ and $\Diamond l$. Their application can be considered as changing the current world of the Kripke model. Note that these rules are based on LK and not on G3c. The contraction and weakening rules are required for certain derivations. Even worse, the systems are not complete, as for example, $\neg\Box\neg A \supset \Diamond A$ cannot be shown to be valid [87].

Fitting [57] proposes a variant based on his tableau system. When applying his sequent rules, formulas of the conclusion are somehow kept in the premise, although he does not mention this explicitly. Contraction-free $\Box$-rules for K, KT and $\mathsf{S4}$ can be found in [75], referring to [57]. These are as follows:

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box A}\,\Box\mathsf{K} \qquad \frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,\Box\mathsf{T} \qquad \frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box A}\,\Box 4$$

Note that Γ and Γ' are distinguished multi-sets. For modal logic K we use only rule $\Box\mathsf{K}$, for KT we use both $\Box\mathsf{K}$ and $\Box\mathsf{T}$, and for $\mathsf{S4}$ we forget about $\Box\mathsf{K}$ and use both $\Box\mathsf{T}$ and $\Box 4$ instead [75]. The rules $\Box\mathsf{K}$ and $\Box 4$ can be seen as right-hand rules for $\Box$, while $\Box\mathsf{T}$ is then the left-hand rule. The logic K therefore does not have a left-hand rule for $\Box$. Analogous rules were already proposed by Kanger [78] in 1957 for KT and $\mathsf{S4}$ (without adequateness proofs for the resulting systems).

Due to the duality of $\Box$ and $\Diamond$, i.e., $\Box\varphi \equiv \neg\Diamond\neg\varphi$ and $\Diamond\varphi \equiv \neg\Box\neg\varphi$, we can extend G3c by the rules shown in Figure 2.12 to obtain G3K for K, G3KT for KT, and G3S4 for $\mathsf{S4}$.

The rules of G3S4 are exactly the same as those of G3s presented by Troelstra and Schwichtenberg [142]. The admissibility of the $\Diamond$-rules in G3K and

G3K

$$\frac{\Gamma \Rightarrow \Delta, A}{\Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta', \Box A}\ \Box r \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta'}\ \Diamond l$$

G3KT

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box l \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta', \Box A}\ \Box r$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta'}\ \Diamond l \qquad \frac{\Gamma \Rightarrow \Delta, \Diamond A, A}{\Gamma \Rightarrow \Delta, \Diamond A}\ \Diamond r$$

G3S4

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box l \qquad \frac{\Box\Gamma \Rightarrow \Diamond\Delta, A}{\Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta', \Box A}\ \Box r$$

$$\frac{A, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond\Delta, \Delta'}\ \Diamond l \qquad \frac{\Gamma \Rightarrow \Delta, \Diamond A, A}{\Gamma \Rightarrow \Delta, \Diamond A}\ \Diamond r$$

Figure 2.12: Rules of G3K, G3KT and G3S4

G3KT can be shown easily due to the duality of the modal operators. Actually, these systems correspond to Fitting's [57] analytic tableaux.

Note that in all three cases, the rules $\Box r$ and $\Diamond l$ are *non-invertible*, as some formulas of the conclusions do not occur in the premises. Also, $\Box l$ and $\Diamond r$ are *duplication rules*, because the principal formulas stay in the premises.

There are not many *ordinary* sequent systems for the intuitionistic modal logical counterparts IK/CK, IKT/CKT or IS4/CS4. For CS4, a *single-conclusion* system was created by Bierman and de Paiva [13]. We consider a slight variant of it proposed by Kuznets et al. [88] who also prove soundness and completeness of systems for CK, CKD (not considered here) and CKT.[19] The rules consist of those of system G3ɪ plus the corresponding modal rules displayed in Figure 2.13.

[19]They refer to the cut-elimination techniques presented by Lellmann and Pattinson [93].

G3CK

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Box A}\,\Box r \qquad \frac{A, \Gamma \Rightarrow C}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond C}\,\Diamond l$$

G3CKT

$$\frac{\Box A, A, \Gamma \Rightarrow C}{\Box A, \Gamma \Rightarrow C}\,\Box l \qquad \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Box A}\,\Box r \qquad \frac{A, \Gamma \Rightarrow C}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond C}\,\Diamond l \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}\,\Diamond r$$

G3CS4

$$\frac{\Box A, A, \Gamma \Rightarrow C}{\Box A, \Gamma \Rightarrow C}\,\Box l \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Box A}\,\Box r \qquad \frac{A, \Box\Gamma \Rightarrow \Diamond C}{\Diamond A, \Box\Gamma, \Gamma' \Rightarrow \Diamond C}\,\Diamond l \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}\,\Diamond r$$

Figure 2.13: Rules of G3CK, G3CKT and G3CS4 (c.f. [88])

A striking feature is that in *all* $\Diamond$-rules, the succedent of the conclusion must be a $\Diamond$-formula. The other rules are what we probably expect from an intuitionistic single-conclusion calculus.

For the modal systems IK, IKT and IS4, I was not able to find an ordinary sequent calculus in the literature. It seems that it is not easy or even impossible to construct such systems, especially because of the **IK5** axiom. Assume we have a single-conclusion sequent system for IK and we try do derive the corresponding formula:

$$\dfrac{\dfrac{\dfrac{???}{\Diamond A \supset \Box B \Rightarrow \Diamond A} \qquad \dfrac{\dfrac{\dfrac{}{B, A \Rightarrow B}\,ax}{B \Rightarrow A \supset B}\,\supset r}{\Box B \Rightarrow \Box(A \supset B)}\,\Box r}{\Diamond A \supset \Box B \Rightarrow \Box(A \supset B)}\,\supset l}{\Rightarrow (\Diamond A \supset \Box B) \supset \Box(A \supset B)}\,\supset r$$

Of course, the left branch cannot be brought to an axiom. It is also no option to apply $\Box l$ instead of $\supset l$ as this would cause that $\Diamond A \supset \Box B$ is dropped from the antecedent (change of Kripke world). Another possibility would be to set up an ordinary multi-conclusion calculus which results in thus:

$$\dfrac{\dfrac{???}{\Diamond A \supset \Box B \Rightarrow \Box(A \supset B), \Diamond A} \quad \dfrac{\dfrac{\dfrac{}{B, A \Rightarrow B}\, ax}{B \Rightarrow A \supset B}\supset r}{\Box B \Rightarrow \Box(A \supset B)}\Box r}{\dfrac{\Diamond A \supset \Box B \Rightarrow \Box(A \supset B)}{\Rightarrow (\Diamond A \supset \Box B) \supset (\Box(A \supset B))}\supset r}\supset l$$

Now, of course it would be possible to apply $\Box r$ on top of the left branch and keep the A of $\Diamond A$. But then there is actually no difference to the modal rules of G3K which would cause that $\Box$ and $\Diamond$ are dual (the sequent $\Rightarrow \neg\Box\neg A \supset \Diamond A$ could be derived). The Kripke semantics of IK (p. 24) could make us believe that $\Box r$ is a critical rule.[20] But if we then apply it on top of the left branch, the new premise would be $\Rightarrow A \supset B$ for which there is also no derivation. Simpson [135] solves the problem with labels (see Section 2.2.4) but thereby leaves the area of ordinary sequent systems.

2.2.2 A Multi-Conclusion Sequent System for CS4

In this section, we introduce a cut-free multi-conclusion variant of G3iCS4. The reason is that the multi-proponent dialogues we consider in Chapter 4 are related to multi-conclusion sequent systems and the adequateness of the dialogical sequent system DiaSeqCS4, which we introduce in 4.2, is based on this.

We call the following system G3iCS4^{M}. The rules are displayed in Figure 2.14. Those of propositional logic are exactly the same as of G3i^{M} (Section 2.1.3). The new rules are $\Box l$, $\Box r$, $\Diamond l$, and $\Diamond r$, of which only $\Box l$ is invertible. Note that $\Diamond l$ can only be used when $\Diamond\Delta$ is not empty.[21] We show the adequateness of G3iCS4^{M} by proving that every G3iCS4^{M}-derivation can be

[20] $\mathcal{M}, w \models \Box A$ holds iff for *all refinements* w' of w, in all successors u of w', it holds $\mathcal{M}, u \models A$. The reference to the refinements w' of w also occur for implication $\mathcal{M}, w \models A \supset B$ and negation $\mathcal{M}, w, \models \neg A$. For both (and only for these) the right-hand rule is critical in G3i^{M}.

[21] This restriction is added here next to the premise in the rule. We do not consider it as a *premise* in terms of derivation rules, but rather as an additional constraint.

$$\frac{}{P, \Gamma \Rightarrow \Delta, P}\,ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot l$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,\wedge l \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\,\wedge r$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,\vee l \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\,\vee r$$

$$\frac{A \supset B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\,\supset l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\,\supset r$$

$$\frac{\neg A, \Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\,\neg l \qquad \frac{A, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \Delta, \neg A}\,\neg r$$

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\,\Box l \qquad \frac{\Box \Gamma \Rightarrow A}{\Box \Gamma, \Gamma' \Rightarrow \Box A, \Delta}\,\Box r$$

$$\frac{\Box \Gamma, A \Rightarrow \Diamond \Delta \qquad |\Diamond \Delta| \geqslant 1}{\Box \Gamma, \Gamma', \Diamond A \Rightarrow \Diamond \Delta, \Delta'}\,\Diamond l \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A, \Delta}\,\Diamond r$$

Figure 2.14: Rules of G3ICS4$^{\mathrm{M}}$

transformed into a G3CS4-derivation and vice-versa. Soundness and completeness of G3CS4 then imply soundness and completeness of G3ICS4$^{\mathrm{M}}$.

Definition 2.1 (Deducibility (G3CS4, G3ICS4$^{\mathrm{M}}$))**.** *For an arbitrary* G3CS4/ G3ICS4$^{\mathrm{M}}$ *sequent* $\Gamma \Rightarrow \Delta$ *we write for some* $n \in \mathbb{N}$*:* $\Vdash^{S}_{n} \Gamma \Rightarrow \Delta$ *iff there is a closed* G3CS4*-tree with* $\Gamma \Rightarrow \Delta$ *as* root sequent *and which has a height* $h \leqslant n$ *with* $n \in \mathbb{N}$*. We write* $\Vdash^{M}_{n} \Gamma \Rightarrow \Delta$ *iff there is a closed* G3ICS4$^{\mathrm{M}}$*-tree with* $\Gamma \Rightarrow \Delta$ *as* root sequent *and which has a height* $h \leqslant n$*. We also say that* $\Gamma \Rightarrow \Delta$ *is* deducible *in* n deductive steps*, or that there is a* derivation *of height* n *for* $\Gamma \Rightarrow \Delta$*.*

The S in $\Vdash^{S}_{n}$ refers to *single-conclusion* and the M to *multi-conclusion*.

Lemma 2.1 (Admissibility of Weakening (G3CS4, G3ɪCS4$^{\mathrm{M}}$)[22])**.**
For all φ*,* Γ*,* Δ*:*

1. *If* $\Vdash^{S}_{n} \Gamma \Rightarrow \Delta$ *then* $\Vdash^{S}_{n} \Gamma, \varphi \Rightarrow \Delta$.
2. *If* $\Vdash^{M}_{n} \Gamma \Rightarrow \Delta$ *then* $\Vdash^{M}_{n} \Gamma, \varphi \Rightarrow \Delta$.
3. *If* $\Vdash^{M}_{n} \Gamma \Rightarrow \Delta$ *then* $\Vdash^{M}_{n} \Gamma \Rightarrow \varphi, \Delta$.

The proof of this lemma is omitted here, as we discuss weakening in more detail in Chapters 3.3.1, 4.3.2, and 4.3.3.

Theorem 2.1 (G3ɪCS4$^{\mathrm{M}}$ Completeness)**.** *Every* G3CS4 *proof tree can be transformed into a* G3ɪCS4$^{\mathrm{M}}$ *proof tree.*

Proof. We consider the G3CS4-derivation $\Vdash^{S}_{n} \Gamma \Rightarrow C$ with Γ being an arbitrary multi-set of formulas and C a single formula. We want to show that for some m and for all Δ it holds: $\Vdash^{M}_{m} \Gamma \Rightarrow C, \Delta$.

We perform an induction on n.

Base Case: $n = 1$ — The only rule which is used in the derivation is either ax or $\bot l$.

- If the rule is ax, the root sequent has the form $\Gamma, P \Rightarrow P$. The rule ax of G3ɪCS4$^{\mathrm{M}}$ can also be used in the sequent $\Gamma, P \Rightarrow P, \Delta$, so this is no problem.
- The case of $\bot l$ works accordingly.

Inductive Step: Assume we have a G3CS4-derivation $\Vdash^{S}_{n+1} \Gamma \Rightarrow C$. We have to perform a case analysis on the rule which is applied in the root sequent, i.e., the lowest rule application in the tree. We consider three cases here. The others are handled similarly.

[22] The lemma is adapted from [39] and [142].

1. Assume that C is a disjunction $A \vee B$ with arbitrary formulas A and B. Also assume that $A \vee B$ is the principal formula of the lowest rule application. Then we have either $\Vdash^{S}_{n} \Gamma \Rightarrow A$ or $\Vdash^{S}_{n} \Gamma \Rightarrow B$. By hypothesis, we obtain $\Vdash^{M}_{m} \Gamma \Rightarrow A, \Delta$ (or with B instead of A respectively). Then by *weakening* we achieve $\Vdash^{M}_{m} \Gamma \Rightarrow A, B, \Delta$. Now we append the application of $\wedge r$ below and get $\Vdash^{M}_{m+1} \Gamma \Rightarrow A \wedge B, \Delta$ as root.

2. Assume that $\Vdash^{S}_{n+1} \Box\Gamma, \Gamma' \Rightarrow \Box A$ and $\Box A$ is the principal. Then $\Vdash^{S}_{n} \Box\Gamma \Rightarrow A$. By hypothesis, we then get $\Vdash^{M}_{m} \Box\Gamma \Rightarrow A$ for $\Delta = \emptyset$. Appending $\Box r$ of G3ICS4^{M} results in $\Vdash^{M}_{m+1} \Box\Gamma, \Gamma' \Rightarrow \Box A, \Delta$ for an arbitrary Δ.

3. Assume that $\Vdash^{S}_{n+1} \Box\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond C$ and $\Diamond A$ is the principal of the lowest rule application. We want to have $\Vdash^{M}_{m+1} \Box\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond C, \Delta$. We define $\Diamond\Delta'$ as the part of Δ which contains $\Diamond$-formulas, and let Δ'' be the rest. Then from $\Vdash^{S}_{n} \Box\Gamma, A \Rightarrow \Diamond C$ we get by hypothesis:
$\Vdash^{M}_{m} \Box\Gamma, A \Rightarrow \Diamond C, \Diamond\Delta'$ and adding $\Diamond l$ of G3ICS4^{M} leads to
$\Vdash^{M}_{m+1} \Box\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond C, \Diamond\Delta', \Delta''$.

□

For the transformation from multi-conclusion to single-conclusion sequents we adapt the technique by Maehara [104]. Assume that Δ is the multi-set of formulas $\delta_1, \delta_2, \ldots, \delta_n$. Then the *disjunction over* Δ, written $\bigvee\Delta$, is defined as $\delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$.

We also need an additional lemma.

Lemma 2.2 (Alternative Succedents in G3CS4). *For any* Γ, n, *and formulas* A *and* B*: if* $\Vdash^{S}_{n} \Gamma \Rightarrow A \vee B$, *then* $\Vdash^{S}_{n} \Gamma \Rightarrow A$ or $\Vdash^{S}_{n} \Gamma \Rightarrow B$.

Proof. By induction on n:

Base Case: $n = 1$ — In this case, the only rule application in the derivation must be $\bot l$, which can also be applied in $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$.

Inductive Step: We consider the lowest rule application of the derivation tree for $\Vdash^S_{n+1} \Gamma \Rightarrow A \vee B$.

- If it is an application of $\wedge l$, $\vee l$, or $\Box l$, we can simply apply the hypothesis on the premise of the application and append it afterwards again.
- If it is an application of $\supset l$ on some formula $C \supset D$, then we have $\Vdash^S_n \Gamma, C \supset D \Rightarrow C$ and $\Vdash^S_n \Gamma, D \Rightarrow A \vee B$.
  By hypothesis then $\Vdash^S_n \Gamma, D \Rightarrow A$ or $\Vdash^S_n \Gamma, D \Rightarrow B$. Therefore, appending $\supset l$ again results in $\Vdash^S_{n+1} \Gamma, C \supset D \Rightarrow A$ or $\Vdash^S_{n+1} \Gamma, C \supset D \Rightarrow B$.
- The case of a $\neg l$-application works accordingly.
- The only other possibility is an application of $\vee r$, but both thinkable premises fulfil our target properties.

□

Theorem 2.2 (G3ɪCS4^M Soundness)**.** *Every* G3ɪCS4^M *proof tree can be transformed into a* G3CS4 *proof tree.*

Proof. What we have to show is that for any n, Γ, and Δ: $\Vdash^M_n \Gamma \Rightarrow \Delta$ implies for some m: $\Vdash^S_m \Gamma \Rightarrow \bigvee\Delta$. We do this by induction on n.

Base Case: $n = 1$ — We have two possibilities:

- The only rule application is ax in a sequent such that $\Vdash^M_1 \Gamma, P \Rightarrow P, \Delta$. This corresponds to $\Vdash^S_m \Gamma, P \Rightarrow P \vee \bigvee\Delta$. There is obviously such a derivation: we simply make use of $\vee r$ and ax of G3CS4 and obtain the following tree:

$$\dfrac{\overline{\Gamma, P \Rightarrow P}\;ax}{\Gamma, P \Rightarrow P \vee \bigvee\Delta}\;\vee r$$

  If P is not the left-most disjunct we simply make use of $\vee r$ several times until it is dissolved out of $\bigvee\Delta$.

- The only rule application is $\bot l$. Then $\bot l$ can also be applied directly in the sequent of G3CS4.

Inductive Step: We have a look at three cases:

1. Assume $\Vdash^{M}_{n+1} \Gamma \Rightarrow A \vee B, \Delta$ with the lowest rule application being $\vee r$ applied on $A \vee B$. Then $\Vdash^{M}_{n} \Gamma \Rightarrow A, B, \Delta$. By hypothesis, this corresponds to $\Vdash^{S}_{m} \Gamma \Rightarrow A \vee B \vee \bigvee\Delta$. This is already what we were looking for.[23]

2. Assume $\Vdash^{M}_{n+1} \Gamma, A \supset B \Rightarrow \Delta$ with the lowest rule application being $\supset l$ applied on $A \supset B$. Then $\Vdash^{M}_{n} \Gamma, A \supset B \Rightarrow A, \Delta$ and $\Vdash^{M}_{n} \Gamma, B \Rightarrow \Delta$. By hypothesis then for some l and m: $\Vdash^{S}_{l} \Gamma, A \supset B \Rightarrow A \vee \bigvee\Delta$ and $\Vdash^{S}_{m} \Gamma, B \Rightarrow \bigvee\Delta$. According to Lemma 2.2, the former leads us to two cases: either $\Vdash^{S}_{l} \Gamma, A \supset B \Rightarrow A$ or $\Vdash^{S}_{l} \Gamma, A \supset B \Rightarrow \bigvee\Delta$.

   - If $\Vdash^{S}_{l} \Gamma, A \supset B \Rightarrow A$, then together with $\Vdash^{S}_{m} \Gamma, B \Rightarrow \bigvee\Delta$, we can use $\supset l$ to obtain $\Gamma, A \supset B \Rightarrow \bigvee\Delta$.

   - If $\Vdash^{S}_{l} \Gamma, A \supset B \Rightarrow \bigvee\Delta$, we are already finished.

3. Assume $\Vdash^{M}_{n+1} \Box\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond\Delta, \Delta'$ and the lowest applied rule is $\Diamond l$ (on $\Diamond A$). Then due to the premise we get $\Vdash^{M}_{n} \Box\Gamma, A \Rightarrow \Diamond\Delta$ with $|\Diamond\Delta| \geqslant 1$, and by hypothesis $\Vdash^{S}_{m} \Box\Gamma, A \Rightarrow \bigvee\Diamond\Delta$ for some m.

   - If $\bigvee\Diamond\Delta$ contains only a single disjunct, i.e., is some $\Diamond C$, then we can simply put $\Diamond l$ below and append $\bigvee\Delta'$ with rule $\vee r$.

   - If it contains more than one disjunct, we apply Lemma 2.2 (if necessary multiple times) and locate the one which is relevant for the derivation. We call it $\Diamond E$. Then it is again easy to build the root of the G3CS4-derivation:

[23] Note that this requires that *associativity* of '$\vee$' is admissible in the succedent of any G3CS4-sequent. However, using Lemma 2.2 makes it possible to show easily that $\Vdash^{S}_{n} \Gamma \Rightarrow A \vee (B \vee C)$ iff $\Vdash^{S}_{m} \Gamma \Rightarrow (A \vee B) \vee C$.

$$\dfrac{}{\Rightarrow \Sigma \,;\, P, \neg P, \Delta}\, ax \qquad \dfrac{\Rightarrow \Sigma \,;\, A, B, \Delta}{\Rightarrow \Sigma \,;\, A \vee B, \Delta}\, \vee \qquad \dfrac{\Rightarrow \Sigma \,;\, A, \Delta \qquad \Rightarrow \Sigma \,;\, B, \Delta}{\Rightarrow \Sigma \,;\, A \wedge B, \Delta}\, \wedge$$

$$\dfrac{\Rightarrow \Diamond A, \Sigma \,;\, A, \Delta}{\Rightarrow \Sigma \,;\, \Diamond A, \Delta}\, \Diamond \qquad \dfrac{\Rightarrow \emptyset \,;\, A, \Sigma}{\Rightarrow \Diamond \Sigma \,;\, \Box A, \Delta}\, \Box$$

Figure 2.15: Rules of KT^{su} (c.f. [68])

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\Box\Gamma, A \Rightarrow \Diamond E}}{\Box\Gamma, \Gamma', \Diamond A \Rightarrow \Diamond E}\,\Diamond l}{\Box\Gamma, \Gamma', \Diamond A \Rightarrow \bigvee \Diamond \Delta}\,\vee r \text{ (multiple times)}}{\Box\Gamma, \Gamma', \Diamond A \Rightarrow \bigvee \Diamond \Delta \vee \bigvee \Delta'}\,\vee r \text{ (multiple times)}$$

□

2.2.3 Termination in Modal Proof Search

Like rule $\supset l$ in G3I, rules $\Box l$ and $\Diamond r$ of G3KT and G3S4 are *duplication rules* which might cause non-termination when searching for a proof, because these rules can be applied infinitely often on the same principal.

Heuerding et al. [68] provide *single-sided* sequent systems, i.e., the sequents' antecedents are always empty. Formulas are only handled when they are in negation normal form, which is no problem, as in classical modal logic it is possible to translate every formula into negation normal form. The axiom rule is then applied if there is a contradiction with respect to some atom in the sequent. The authors use separate multi-sets Σ and H for keeping book of certain formulas on which rules have already been applied [68]. The following systems by Heuerding et al. are based on algorithmic approaches by Ladner [89].

In system KT^{su} for the modal logic KT, H is not yet used. The rules (in our altered notation) are displayed in Figure 2.15. In rule *ax*, P is an atom. The only duplicating rule ($\Diamond$) adds the principal formula to Σ and prevents us to apply this rule on the same formula again, until we use the $\Box$-rule. This

$$\frac{}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{P}, \neg \mathsf{P}, \Delta} \; ax$$

$$\frac{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A}, \mathsf{B}, \Delta}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A} \vee \mathsf{B}, \Delta} \; \vee \qquad \frac{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A}, \Delta \quad \Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{B}, \Delta}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A} \wedge \mathsf{B}, \Delta} \; \wedge$$

$$\frac{\Rightarrow \emptyset \; ; \; \Diamond \mathsf{A}, \Sigma \; ; \; \mathsf{A}, \Delta}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \Diamond \mathsf{A}, \Delta} \; \Diamond_0 \qquad \frac{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A}, \Delta}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \Diamond \mathsf{A}, \Delta} \; \Diamond_1 \qquad \frac{\Rightarrow \mathsf{A}, \Theta, \mathsf{H} \; ; \; \Sigma \; ; \; \mathsf{A}, \Sigma}{\Rightarrow \mathsf{H} \; ; \; \Sigma \; ; \; \Box \mathsf{A}, \Box \Theta, \Delta} \; \Box$$

Figure 2.16: Rules of $KS4^{su}$ (c.f. [68])

corresponds to a world change in the Kripke model and moves the formulas out of the separated multi-set (the prefixed $\Diamond$'s removed). So a repetition is also possible if it is done above the application of the non-invertible $\Box$-rule, which is however enough for KT to terminate the proof search [68].

For S4, things are not that easy, as an additional history (multi-)set is needed. The rules for $KS4^{su}$ are shown in Figure 2.16. The calculus has two rules for $\Diamond$ and restrictions for rule applications are as follows (c.f. [68]):

- Rule $\Diamond_0$ may only be applied if $\Diamond A \notin \Sigma$.
- Rule $\Diamond_1$ may only be applied if $\Diamond A \in \Sigma$.
- Rule $\Box$ may only be applied if $A \notin H$.

As before, rule $\Box$ causes a world change in the Kripke model. It is the only rule that extends the history H. The use of $\Diamond_0$ empties the history, further applications on the same principal (with $\Diamond_1$) do not touch H. System $KS4^{su}$ is sound and complete and termination is guaranteed [68].

Howe [73] uses two separate history sets to set up a sequent system for the $\Box$-fragment[24] $IS4_{\Box}$ [12] of IS4/CS4.[25] Another, more recent terminating sequent system for S4 is proposed by Fiorentini [54]. It deals with an extended

[24] $\Box$-fragment means that the $\Diamond$ is not considered to be part of the language, whereas the $\Box$ and the propositional connectives are.

[25] Howe states that he considers IS4, but when the $\Diamond$ is not taken into account as an admissible operator, there is no difference between IS4 and CS4, as axioms **IK2** to **IK5** contain $\Diamond$-formulas, and are therefore dropped anyway. The relevant axioms of $IS4_{\Box}$ are given in [12].

modal language (two extra modalities) and so-called *cluster-sequents* which go beyond the scope of this work.

2.2.4 Labelled Modal Sequent Systems

Instead of dropping formulas from sequents when *non-invertible* rules are applied, formulas can be equipped by *labels*, which refer to Kripke worlds, in which the corresponding formulas shall hold. When the relations are also added explicitly to the sequents, a single sequent can represent a whole Kripke model.

One of the first labelled sequent systems for modal logics is presented by Mints [115] for modal logics K, KT, K4, KB, KTB, S4, and S5. Instead of labelling single formulas, he considers several sequents at once, i.e., sequences of labelled sequents. Whenever rule $\Box r$ is applied, a new sequent with a new label, representing a new Kripke world, is added.

Viganò [145] made the relational structure explicit. He introduced *relational formulas* of the form xRy, indicating the accessibility between two worlds x and y via a relation R. A labelled sequent then has either the form $\Gamma, \rho \Rightarrow \Delta$ or $\rho \Rightarrow xRy$, where Γ and Δ are multi-sets of labelled formulas and ρ is a multi-set of relational formulas. The second kind of sequent (*relational sequents*) is only used to reason about the relational structure of the model which is internally constructed. In Viganò's sequent systems, the $\Box r$ rule adds relational formulas to the sequents, while above $\Box l$ we need to give reason that some relational formula can be derived. For these we then have different possibilities according to the corresponding modal system's frame properties (which depend on the modal system) with additional rules. A cut rule is not provided.

In [116] Negri presents similar rules without using relational sequents. In this simplification, sequents have the form $\Gamma \Rightarrow \Delta$ with Γ now containing both labelled and relational formulas. In the following, we refer to Negri's sequent calculi as $G3K^l$, $G3KT^l$, and $G3S4^l$. Several years before, Simpson [135] introduced a labelled single-conclusion sequent calculus for IK with an additional general rule that can be used to include different frame properties, including reflexivity and transitivity.

Adding such labels to the syntax of formulas leads us to *hybrid logics*, for which there are several sequent systems, e.g., proposed in [14, 17]. However, hybrid logic is a topic of its own and we do not discuss it hereinafter.

Systems with labelled sequents are more flexible than the ordinary ones, as one can cope with different frame properties using rules that are applied on relational formulas. These rules can simply be added or removed to the basic set, instead of replacing $\Box$- and $\Diamond$-rules as in ordinary sequents. Further, it is possible to cope with modal logics like S5, for which there are only quite complex or no adequate ordinary systems at all, in an easy way. As Negri [116] states, all her rules are invertible, which is not the case for ordinary systems.

2.2.5 Avoiding Explicit Relations

Hypersequents

Labelled systems as those we just considered, strongly rely on Kripke semantics, which is seen as a problem by some authors.[26] This is one of the reasons why Avron [7] proposed a *hypersequent* system for S5. An earlier approach was proposed in an abstract by Pottinger [124] for logics KT and S4, where sequents consist of sequences instead of multi-sets.

Lahav [90] provides a method of how to transform frame properties to hypersequent rules, and thereby obtains calculi for a bunch of classical modal logics, inter alia for K, KT and S4.

[26] A discussion on this topic can for example be found in [7], and Chapter 1.9 of [123].

Nested Sequents

Another similar possibility to *hide* the relational Kripke structure is the usage of *nested sequents*. These are often used as *single-sided* sequents, i.e., multisets, that contain formulas and again other nested sequents. Due to this structure, it is possible to map the frame of an underlying model without the usage of labels or relational formulas.

Brünnler [19] provides a flexible system for the most common classical modal logics, inter alia K, KT, S4, S5..., where every frame property has its own rule. He states that for some combinations of frame properties his system is not cut-free. Poggiolesi introduces independently the similar method of *tree-hypersequents* for K, KD, K4, and KD4 [122], which she later extends for all modal systems of the S5-cube (not cut-free for K5 and not complete for KT5) [123].

Nested sequent systems for the intuitionistic modal logic IK with combinations of the extensions D, T, B, 4, and 5 are presented by Straßburger [141] and Marin and Straßburger [107]. All formulas of a single-sided nested sequent either have an *input polarity* corresponding to left-hand formulas of two-sided sequents, or an *output polarity* which can be seen as the right-hand formula, of which there is only one in each sequent. So the system can be considered as a single-conclusion calculus. It was later modified by Arisaka et al. [6] for the logics of the CS5-cube.

An earlier approach for multi-modal CK, which can be interpreted as a special case of nested sequents, is the multi-conclusion sequent calculus without cut rule by Mendler and Scheele [112], based on the so-called *2-sequent calculus* by Masini [108]. Here, the antecedent and the succedent of a sequent are departed into several sections, each representing another Kripke world.

Linear nested sequents for the classical modal logics of the S4-cube (S5-cube without combinations of B or 5) plus combinations with 45 are provided by Lellmann [92].

Looking just one step ahead

It is worth to note Cerrato's calculus [21] for all logics of the S5-cube, in which formulas can be bracketed with [] and ⟨⟩ to mark that all of them ([]) or one of them (⟨⟩) is valid in a succeeding world. When all formulas of a sequent are bracketed, further rules can then be used to move to there, dropping all these brackets. Cut-elimination is only shown for the K-case. A similar idea let Mendler and Scheele [111] introduce a calculus (without cut-rule) for the constructive description logic c$\mathcal{ALC}$, a syntactic variant of multi-modal CK.[27]

2.2.6 Focusing and Modal Logic

As for propositional intuitionistic logic and linear logic, there are also some focusing attempts for modal logics.

Miller and Volpe [113] transform LKF (Section 2.1.5) to a single-sided focused sequent system LMF for logic K. They exploit the fact that modal formulas can be translated to first-order formulas. The usage of this *standard translation* has the side effect that negated relational atoms may occur in the sequents. Other (classical) modal systems can be obtained by embedding rules which correspond to the frame properties [113].

Chaudhuri et al. present a focusing nested sequent calculus for the classical modal logics of the S5-cube [25], as well as for the logics of the IS5-cube [24] based on the calculus of [141]. Lellmann and Pimentel [94] provide a focusing version of the linear nested sequents for classical modal logics [92] mentioned above. However, I was not able to find a focusing calculus which is based on *ordinary* modal sequent systems, the simplest kind of modal sequent systems. Instead, those with focus found in literature make use

[27] The authors state that c$\mathcal{ALC}$ is "related to" CK as the standard logic of description logic, $\mathcal{ALC}$, "is related to" K [111].

of labels or nested sequents. Also, there is currently no focusing calculus dealing with logics of the CK-family.

2.3 Lorenzen Dialogues as Proof Systems for Classical and Intuitionistic Logic

It is now time to leave sequent systems and turn to dialogical games in more detail. In this section we discuss dialogues which can be used as proof systems for classical and intuitionistic logic. We consider several approaches with different advantages and disadvantages. Some of them deal with first-order logic, while others are restricted to propositional logic. The former is of minor interest for this work, but these dialogues are the original ones and also have some features which will be useful when we turn towards modal logic in Section 2.4.

2.3.1 Preliminaries

Here, we only give informal definitions for dialogues to illustrate the idea of the methodology and because in the literature the game rules are also often defined informally. We do not want to prove validity or other properties of the dialogical systems presented in this chapter. Later, when the multi-proponent reasoning procedure is constructed in Chapters 3 and 4, we get more concrete and formal. Different mathematical definitions for dialogues and their rules can for example be found in [80, 47, 48, 143, 140].

Lorenzen first presented his ideas of dialogues in 1958 [100]. He argues that using the logical connectives in a way that corresponds to some —as he claims— natural meanings in a dialogue, leads to a proof system for intuitionistic logic [101]. At the beginning of a dialogue one of the two players, the *proponent* P, states a *hypothesis* in terms of a formula. The other player who is called *opponent* O *attacks* this formula according to some predefined rules. P may then defend against this attack with a new formula and the game continues. Instead of defending, it is often also possible to *counter-attack* a previous assertion of the attacking player.

A dialogical *game* or *run* is a sequence of *moves*, each performed by one player. A move can be an *attack*, a *defence*, or the claim of the *hypothesis*. Further, a move is either an *assertion* which is a formula of some language, or a *demand* which we indicate with a question mark (**?**) and some additional symbol specifying the type of attack.

An assertion stated by O is also called *commitment* or *concession*. Sometimes we allow O to state one or more concessions before P claims the hypothesis. These are called *initial concessions* and P can make use of them during the game as if they were stated then.

A *closed round* as we consider it in *this* chapter for two-player dialogues, consists of two moves, namely one attack and one defence against this attack. A round in a game is *open* iff it consists of an attack but there is no corresponding defence.

A p-*strategy* of a player $p \in \{O, P\}$ is a function that defines the next move of p in a given non-empty game. The move needs to comply with the game rules. The strategy can therefore be seen as the player's mind that leads him through the game.

In general, a player wins a game, if the other one is not able to move anymore. A p-strategy is a p-*winning-strategy* iff, when consecutively applied, it makes p win the game, no matter what the other player's moves are or will be. So, if a player has a winning strategy, he is always capable to win the game.

Assert	$A \wedge B$		$A \vee B$		$A \supset B$	$\neg A$	$\forall x\, A$	$\exists x\, A$
Attack	$?_l$	$?_r$	$?_\vee$		A	A	$?_t$	$?_\exists$
Defence	A	B	A	B	B	—	$A[x/t]$	$A[x/t]$

Figure 2.17: Particle Rules for the First-Order Language based on [98]

2.3.2 D-Dialogues, E-Dialogues, and Ipse Dixisti for Validity

Lorenz [98] defined the game rules for systems we call (as Felscher [47]) *D-dialogues* in the following. He considers two separate sets: the *particle rules* (*allgemeine Spielregel*[28]) define how to handle compound formulas *locally*, while the *structural rules* (*spezielle Spielregel*[29]) concern the players' assertions *globally* [98], i.e., they define which player may move in which situation and also when a game finishes and who the winner is.

Lorenz presents the *particle rules* for the first-order language in the form of a table. Figure 2.17 shows the rules in a slightly modified notation to serve our purpose. The letters A and B represent arbitrary first-order-formulas, x is a variable and t is an *object* [98] or *term* [47, 48]. As before, we write $A[x/t]$ to express a *substitution*, where all occurrences of x in A are replaced by t. The first line of the table shows possible assertions that can be stated by one of the players. The second line defines how the other player can *attack* an assertion of the form given in the first line, while the last line expresses how to *defend* against the corresponding attack. The symbols $?_\vee$, $?_l$, $?_r$, $?_t$, and $?_\exists$ are the *demands*.

For example, if one player states a conjunction $A \wedge B$, then the other one can attack it with one of the demands $?_l$ or $?_r$, asking the first player to defend either with the left or the right conjunct. The attacked player can then react

[28]German for *general game rule*. In [47, 48] the rules are named *argumentation forms*, in [10] *strip rules*. We follow Lorenz's later terminology and call them *particle rules* (German: *Partikelregeln*) [99].

[29]German for *special game rule*. Also called *Rahmenregel* [99] (German: frame rule). We follow Rahman and Keiff [127] and call them *structural rules*.

to this with the defence by stating the corresponding part of the previous assertion, i.e., with A or B respectively.

A disjunction $A \vee B$ can only be attacked with $?_\vee$, while the defending player may choose whether to defend with the left or the right disjunct. An implication $A \supset B$ is attacked by stating the antecedent A and defended with the consequent B. A negation $\neg A$ is attacked directly due to the contrary statement A. Such an attack cannot be defended. When a universal statement $\forall x\, A$ is given, then the attacking player may choose an object/term t, which must be used by the defender to give a concrete instance, i.e., by means of a substitution of x with t. For the existential statement $\exists x\, A$ the defender may choose the object t for which he/she claims the assertion A.

In a variant of the particle rules, Felscher [47, 48] adds a player variable to each statement indicating that the attacking and the defending player are different and that the attacking player always attacks statements by the other one.

D-Dialogues

When P is supposed to show general *validity* of his hypothesis, then the following rule has to be obeyed (called *characteristic rule*, "*charakteristische Spielregel*" [98]):

(D0) P may state a prime formula only if it was stated by O before.

Further, Lorenz proposes the following structural rules for intuitionistic dialogues ("*effektive Spielregel*") [98]:[30]

(D1O) An assertion by P may be attacked once in any following move.

(D2P) An assertion by O may be attacked arbitrarily often in any following move.

[30]As we do not rely on his terminology or definitions here, the rules are paraphrased to simply give the main idea.

(D3) A defence may take place once and only if all rounds after the corresponding attack are closed.

Rule **(D3)** can also be seen as restriction, that players may only defend against the *last open attack*, i.e., the last attack by the other player for which there has been no defence yet [127]. As Felscher we call these rules *D-rules* and dialogues obeying them *D-dialogues* (c.f. Felscher [47, 48]).

Also *classical rules* are presented, for which rule **(D3)** is replaced by the following [98]:

(D3O) O may defend an assertion only if all rounds after the corresponding attack are closed.

(D3P) An assertion by P may be defended arbitrarily often in a following move, once in an attack round if all rounds after the corresponding attack are closed, or in any following defence round.

Here, *attack rounds* are the usual rounds consisting of an attack and a defence. By contrast, a *defence round* consists of a single defence against an attack which has already been defended before [98].

Later, Lorenz modifies the structural rules, making things more concrete. He defines who is the winner of the game (the last one who can perform a move), who begins, and so forth [99]. The particle rules are then included as part of the structural rules. Lorenz also introduces *ranks* (*Schranken*) which we discuss later in Section 2.3.4.

We have a look at a simple example which is often used in the literature to show how dialogues work (first published in a similar way in [101]). It shows that P does not have a winning strategy for the formula $A \vee \neg A$ in intuitionistic dialogues. We assume that A is a propositional variable and therefore a prime formula. Figure 2.18 shows a *dialogue tableau* for this hypothesis. Different styles of such tableaux are proposed in the literature, e.g., [101, 98, 10, 127]. Our variant is a table consisting of three columns. The

	O		P	
1				$A \vee \neg A$
2	$[\mathbf{?}, 1]$	$\mathbf{?}_\vee$	$[\mathbf{!}, 2]$	$\neg A$
3	$[\mathbf{?}, 2]$	A	—	—

Figure 2.18: The Excluded Middle in an intuitionistic dialogue

first contains the row numbers, the second moves performed by O, and the last P's moves. Each presented move (with the exception of the hypothesis) consists of two parts. The right one is the player's statement (assertion or demand), the left one, written in square brackets [ ], indicates whether the move is an attack (**?**) or a defence (**!**) and the row number to which the move refers. For example $[\mathbf{?}, 1]$ in O's move in row 2 means that it is an *attack* against P's move of row 1. P's answer $\neg A$ with $[\mathbf{!}, 2]$ means that it is a *defence* against O's attack which was performed in row 2.

The dialogue starts with P stating the *hypothesis* $A \vee \neg A$ in row 1. This is attacked by O with the demand $\mathbf{?}_\vee$ which is in fact her only possibility to react. P had the choice to defend with the left or the right disjunct, namely A or $\neg A$. However, as he may only state prime formulas which were asserted by O before **(D0)**, A is not an option and therefore P must defend with $\neg A$. O attacks this in row 3 stating the contradiction. P cannot defend against the previous attack again **(D3)** and attacking prime formulas is also not possible. Therefore, he has no possibility to move and loses. P has no winning strategy which matches the fact that the law of the excluded middle is *not* valid in intuitionistic logic.

If we replace the intuitionistic **(D3)** by the classical rules **(D3P)** and **(D3O)**, P may defend in row 3 against O's attack of row 2 again. Now, as O stated A in the meantime, P may defend with the left disjunct of the hypothesis A:

3	$[\mathbf{?}, 2]$	A	$[\mathbf{!}, 2]$	A

O cannot react to this as for her it is also not possible to attack prime formulas. We conclude that P has a winning strategy in the *classical* setting. Note that from the argumentative perspective the intuitionistic structural rules are more reasonable (c.f. [101]) as P should not have the right to withdraw a previous decision and make use of O's commitments which she made because of different assumptions.

It is quite clear that P and O have different rights and obligations and therefore the structural rules are *asymmetric* [98], whereas the particle rules are the same for both players and therefore symmetric. Lorenz provides a proof attempt that ought to show the equivalence of P-*winning-strategies* in games obeying classical rules, and proofs in a classical sequent system[31], i.e., he tried to show for classical logic that if P has a winning strategy for some *hypothesis* A, then there is a sequent derivation for $\Rightarrow A$, and vice versa. Such a proof, provided that it is correct, would ensure soundness and completeness of dialogues and make them usable as adequate calculus. However as Kindt [80] points out, Lorenz's proofs are only *hints/indications*, and some of his claims are even wrong. There are other, different proof attempts concerning D-dialogues with underlying first-order Logic, as well as variants considering altered rules, e.g., [136, 80, 65, 109, 126, 127, 31]. However, these are not important for our purposes as we go into another, more convenient direction.

From D to E

In 1969 Lorenzen proposed a strong restriction for O in terms of the following rule:

> "The opponent may either attack the sentence asserted by the proponent in the preceding move or he may defend himself against the attack of the proponent in the preceding move."
>
> (Lorenzen [102], p. 29)

[31]He defines an intermediate system between LK and G3

This means that O always has to react immediately to P's previous move. Felscher calls this the *E-rule*. Adding it to the *D-rules* from above leads us to the so-called *E-dialogues* [47, 48]. Felscher [47] shows that P-winning-strategies in E-dialogues can be transformed to LJ-proofs and vice versa, which corresponds to proving soundness and completeness of E-dialogues as intuitionistic calculus. He also gives evidence that strategies in *D-dialogues* can be transformed to strategies of *E-dialogues* which means that the E-rule can be omitted without losing neither soundness nor completeness [47]. In general, the E-rule is often used to restrict the number of possible moves in a dialogue and to simplify the adequateness proof for dialogues as reasoning procedure.

O's Contradictions

Kamlah and Lorenzen [77] also discuss a variant of dialogues where the characteristic rule **(D0)** is not implemented directly. P is simply not allowed to attack prime formulas stated by O, however O may do it [77]. The new *winning rule* (*"Gewinnregel"*) says that P wins if he has to defend a prime formula O already stated before [77].

Barth and Krabbe [10] introduce the *ipse dixisti!* remark (*"you said so yourself!"*) that allows P to win the game when O attacks a prime which is given by herself.[32] The authors also provide a rule that deals with uttered $\perp$'s. The *absurdum dixisti* remark (*"you said something absurd!"*) can be used to win the game if the other player states a $\perp$ [10].

Barth and Krabbe present in [10] a framework in terms of different sets of structural rules. They also interpret P-winning-strategies as proofs in classical, intuitionistic and minimal logic, with variants of taking $\perp$ into account

[32]First, the authors allow the *ipse dixisti* remark (hypothetically) for both players and also for all kinds of assertions, i.e., also for compound formulas ([10], Def. 17, p. 68). But for a dialogical game with P stating the only hypothesis, it makes sense to give only P the right to use it ([10], p. 70). Finally, P may only use it when it refers to an atom ([10], p. 142).

	O		P	
1				$A \vee \neg A$
2	$[?, 1]$	$?_{\vee}$	$[!, 2]$	A
3	$[?, 2]$	$?_{A}$	$[!, 2]$	$\neg A$
4	$[?, 3]$	A	$[!, 3]$	$!!$

Figure 2.19: The law of the Excluded Middle with *ipse dixisti*

and not.[33] Using their own rule set they state that the corresponding systems are *equivalent* to those by Lorenzen and Lorenz ([10], p.55), obviously including the *E-rule*[34].

Figure 2.19 shows a winning strategy for $A \vee \neg A$ in classical logic making use of *ipse dixisti*. In this setting P is allowed to state A with his defence in row 2. O may attack prime formulas now (only O may do so), asking P to give reason for A. P then defends again against the previous attack (row 2) defending with the right disjunct. As before, this is only possible in classical dialectics, not in intuitionistic. O attacks with A. Answering to the attack of row 3, where O asked P to explain A, P can now answer with *you said so yourself* (*ipse dixisti*). We indicate this as **!!** and treat the move as a defence.[35] O cannot react to this and loses the game.

The statement of *ipse dixisti* corresponds to the closure of a standard tableau or the usage of the rule ax in a sequent system. Barth and Krabbe [10] also introduced a sequent-style formalism for dialogical states which we consider in the following section.

[33]They actually use the symbol Λ instead of $\bot$ but with the same intended meaning.
[34]Indicated as rule "FD D7" in [10, Chapter 3]; see also remark in [48].
[35]Barth and Krabbe call the move a *remark* instead of a defence.

2.3.3 Dialogue Sequents

According to Barth and Krabbe [10], a *dialogue sequent* is a tuple consisting of six components[36] $(\Pi, \Phi, \mathsf{T}, \mathsf{N}, \varphi, \Psi)$, usually written in a more *compact* notion as follows:

$$\Pi; [\Phi]/\mathsf{T}/_{\mathsf{N}}\ \varphi; [\Psi]\ .$$

- N indicates the player whose turn it is currently, i.e., $\mathsf{N} \in \{\mathsf{P}, \mathsf{O}\}$.
- T is the *current local thesis* in intuitionistic dialogues, i.e., the last statement by P which has been attacked but not yet been defended. For classical dialogues, T is a set consisting of the current local thesis plus all the previous local theses. The current local thesis can be empty, indicated by ϵ.
- Π contains all of O's concessions (assertions) in the current game/run.
- φ is either ϵ or the last assertion by P.
- Φ is a set consisting of the assertions O may state as defences.
- Ψ is a set consisting of the assertions P may state as defences.

We consider this dialogue sequent system with its rules the same way as the systems of Section 2.1, i.e., we interpret Π, Φ, and Ψ as sets of formulas and use the comma accordingly. Dialogues are therefore read from bottom to top. The rules in Figure 2.20 are derived from the definitions and descriptions for intuitionistic dialectics in [10, 85]. The resulting sequent-style system is called CND, following the designation *constructive-NOT dialectics* in [10].

Every move in a dialogue corresponds to a transformation step from one dialogue sequent to another one. A branching in a rule indicates two (or more) possibilities of how to perform the move. Note that if we use this calculus to show validity of a formula, we consider only one P-winning

[36] Some of the symbols Barth and Krabbe use are different but have the same meaning.

O-moves: attacks

$$\frac{\Pi; [\,]/A \wedge B/_{\mathsf{P}}\ \epsilon; [A] \qquad \Pi; [\,]/A \wedge B/_{\mathsf{P}}\ \epsilon; [B]}{\Pi; [\Phi]/T/_{\mathsf{O}}\ A \wedge B; [\Psi]}\ \mathsf{O}\mathbf{?}\wedge$$

$$\frac{\Pi; [\,]/A \vee B/_{\mathsf{P}}\ \epsilon; [A, B]}{\Pi; [\Phi]/T/_{\mathsf{O}}\ A \vee B; [\Psi]}\ \mathsf{O}\mathbf{?}\vee \qquad \frac{A, \Pi; [\,]/A \supset B/_{\mathsf{P}}\ \epsilon; [B]}{\Pi; [\Phi]/T/_{\mathsf{O}}\ A \supset B; [\Psi]}\ \mathsf{O}\mathbf{?}\supset$$

$$\frac{A, \Pi; [\,]/\neg A/_{\mathsf{P}}\ \epsilon; [\,]}{\Pi; [\Phi]/T/_{\mathsf{O}}\ \neg A; [\Psi]}\ \mathsf{O}\mathbf{?}\neg \qquad \frac{\Pi; [\,]/P/_{\mathsf{P}}\ \epsilon; [\,]}{\Pi; [\Phi]/T/_{\mathsf{O}}\ P; [\Psi]}\ \mathsf{O}a\mathbf{?}$$

P-moves: attacks

$$\frac{A \wedge B, \Pi; [A]/T/_{\mathsf{O}}\ \varphi; [\Psi] \qquad A \wedge B, \Pi; [B]/T/_{\mathsf{O}}\ \varphi; [\Psi]}{A \wedge B, \Pi; [\Phi]/T/_{\mathsf{P}}\ \varphi; [\Psi]}\ \mathsf{P}\mathbf{?}\wedge$$

$$\frac{A \vee B, \Pi; [A, B]/T/_{\mathsf{O}}\ \varphi; [\Psi]}{A \vee B, \Pi; [\Phi]/T/_{\mathsf{P}}\ \varphi; [\Psi]}\ \mathsf{P}\mathbf{?}\vee$$

$$\frac{\neg A, \Pi; [\,]/T/_{\mathsf{O}}\ A; [\Psi]}{\neg A, \Pi; [\Phi]/T/_{\mathsf{P}}\ \varphi; [\Psi]}\ \mathsf{P}\mathbf{?}\neg \qquad \frac{A \supset B, \Pi; [B]/T/_{\mathsf{O}}\ A; [\Psi]}{A \supset B, \Pi; [\Phi]/T/_{\mathsf{P}}\ \varphi; [\Psi]}\ \mathsf{P}\mathbf{?}\supset$$

P/O-moves: defences

$$\frac{A_1, \Gamma; [\,]/T/_{\mathsf{P}}\ \epsilon; [\Psi] \qquad \ldots \qquad A_n, \Gamma; [\,]/T/_{\mathsf{P}}\ \epsilon; [\Psi]}{\Pi; [A_1, \ldots A_n]/T/_{\mathsf{O}}\ \varphi; [\Psi]}\ \mathsf{O}\mathbf{!}$$

$$\frac{\Pi; [\,]/\epsilon/_{\mathsf{O}}\ A_1; [\,] \qquad \ldots \qquad \Pi; [\,]/\epsilon/_{\mathsf{O}}\ A_n; [\,]}{\Pi; [\Phi]/T/_{\mathsf{P}}\ \varphi; [A_1, \ldots A_n]}\ \mathsf{P}\mathbf{!} \qquad \frac{\textit{Ipse dixisti!}}{P, \Pi; [\Phi]/P/_{\mathsf{P}}\ \varphi; [\Psi]}$$

Figure 2.20: Rules of CND (adapted from [10])

$$\dfrac{\dfrac{\emptyset;[\,]/A/_{\mathsf{P}}\ \epsilon;[\,]}{\emptyset;[\,]/\epsilon/_{\mathsf{O}}\ A;[\,]}\ \mathsf{O?}a \qquad \dfrac{A;[\,]/\neg A/_{\mathsf{P}}\ \epsilon;[\,]}{\emptyset;[\,]/\epsilon/_{\mathsf{O}}\ \neg A;[\,]}\ \mathsf{O?}\neg}{\dfrac{\emptyset;[\,]/A\vee\neg A/_{\mathsf{P}}\ \epsilon;[A,\neg A]}{\emptyset;[\,]/\epsilon/_{\mathsf{O}}\ A\vee\neg A;[\,]}\ \mathsf{O?}\vee}\ \mathsf{P!}$$

Figure 2.21: Excluded Middle in CND

strategy and therefore only one of the branches starting with $\mathsf{P?}\wedge$ or $\mathsf{P!}$ needs to be *closed*, i.e., end with the *ipse dixisti* remark.

The E-rule is implemented due to emptying $[\Phi]$ after O performs an attack or a defence. Thereby φ always becomes ϵ when O defends. Both features together force O to react to P's last move every time.

Figure 2.21 shows a complete CND-tree with all possible strategies for the hypothesis $A \vee \neg A$. It makes clear that P has no possibility to win and therefore no winning strategy. The dialogue starts with $A \vee \neg A$ as P's *last assertion* (φ). It is O's turn who attacks it ($\mathsf{O?}\vee$). The formula becomes the local thesis. P's possibilities to defend are put into the square brackets on the right side. When he defends, he may choose one of these and assert it (φ). We consider both strategies: in the left branch he defends with A and in the right one with $\neg A$.

- In the left branch O attacks the atom A ($\mathsf{O?}a$) making it the current local thesis. P has no possibility to react to this as there is no corresponding rule and there is no concession by O which can be attacked or otherwise made use of. P loses this branch/game.
- In the right branch O attacks the negation $\neg A$ ($\mathsf{O?}\neg$) which also makes it become the current local thesis. A is added to O's concessions on the left side of the sequent. However, there is nothing to defend for P and as he may not attack prime formulas he loses this branch/game as well.

Barth and Krabbe also provide *mechanical procedures* of how to transform dialogues to other calculi, which serves as a proof for a soundness theorem.

	O		P	
1				$\neg\neg(A \vee \neg A)$
2	$[?, 1]$	$\neg(A \vee \neg A)$	$[?, 2]$	$A \vee \neg A$
3	$[?, 2]$	$?_{\vee}$	$[!, 3]$	$\neg A$
4	$[?, 3]$	A	$[?, 2]$	$A \vee \neg A$
5	$[?, 4]$	$?_{\vee}$	$[!, 5]$	A

Figure 2.22: Repetition in an intuitionistic dialogue

Completeness is shown in a similar way [10]. The system is extended for first-order logic in Krabbe's doctoral thesis [83]. Proofs are only sketched there.

2.3.4 Termination

A problem regarding dialogical proof search occurs with respect to termination. The rules we have considered so far might force O to react only to P's last move (E-rule) which does not allow her to attack earlier statements again and again. However, for P we do not have this restriction, and if we introduced it, it would not lead to a complete deductive dialogical proof system.

For example, if O states a conjunction $A \wedge B$, then P must have the right to ask for both conjuncts with two separate moves. In intuitionistic dialogues it is even necessary that P is allowed to attack the same implication or negation more than once. The problem is the same as for intuitionistic sequent systems where we need the *duplication* (see Chapter 2.1.2). It gets even worse when $\forall$ and $\exists$ are allowed as connectives, because in certain cases, P needs to be able to instantiate the quantified formulas with different terms.

Figure 2.22 shows an intuitionistic E-dialogue.[37] Here, P attacks O's statement of row 2 twice: once in row 2 and once in row 4. Only because of this *repetition* it is possible for him to win. Note that the ability of attacking negations (and implication) more often than once corresponds to the duplication in rules $\neg l$ and $\supset l$ of G3I.[38] In the literature, there are two different approaches to cope with this issue: *ranks* and *structural preventing rules*.

Ranks

Lorenz introduces the idea of *attack ranks* (*Angriffsschranken*) in [99]. Instead of using the E-rule, he proposes that before the actual dialogue starts, each player announces one rank in terms of natural numbers n and m. During the dialogue, they are not allowed to attack the same statement more often than n or m times respectively. This new rule replaces **(D1O)** and **(D2P)** of the D-dialogues explained above in Section 2.3.2. If the players may only defend against the last open attack, this measure should guarantee termination.

Clerbout, considering only classical first-order logic, does not distinguish attacking or defending ranks, but instead speaks of *repetition ranks* [31]. Of course, this cannot be done in intuitionistic dialogues, where the repetition rank for defences must be fixed to one (no repetition). Clerbout also provides procedures to translate his ranked dialogues for classical first-order logic to tableaux and vice versa. These serve as proofs for soundness and completeness.

In general, ranks can be seen as an elegant game-theoretic way to tackle non-termination. However, for proof-searching purposes, they are probably not the best choice, as many different repetition ranks might have to be taken into account, and it is probably hard to say beforehand which are the right

[37] Allowing P to state primes before O did does not change the problem. The repetition is still necessary.

[38] More on the relation between repetitions in dialogues and duplication in sequent systems follows in Chapter 3.

	O		P	
1				$((P \supset Q) \supset P) \supset P$
2	$[?, 1]$	$(P \supset Q) \supset P$	$[!, 2]$	P
3	$[?, 2]$	$?_P$	$[?, 2]$	$P \supset Q$
4	$[?, 3]$	P	$[!, 4]$	Q
5	$[?, 4]$	$?_Q$	$[?, 2]$	$P \supset Q$
6	$[?, 5]$	P	—	—

Figure 2.23: Termination with F-Rules

ones to ensure that validity of a given formula cannot be proven in terms of a P-winning-strategy.

Structural Preventing Rules

Krabbe [83] also suggests ranks in his outlook on first-order logic. However, for propositional logic, he and Barth propose some informal structural rules which are *not* implemented in their sequent system [10]. We call them *F-rules* which are defined as follows:[39]

(F1) If a player can perform a counter-attack, he/she may not attack the same statement by the other player again in the same manner, unless the current local thesis changed in the meantime.

(F2) After O attacks a statement by P, P may not repeat a previous assertion which was already attacked by O, unless in the meantime O made a new concession she has not made before.

Attacking in "the same manner" means that the very same demand symbols are used. For example $?_l$ and $?_r$ are different manners of attacking one conjunction. Figure 2.23 shows an E-dialogue that obeys the F-rules and with P being allowed to state atoms anytime. As hypothesis we use *Peirce's*

[39]Paraphrased from [10, pp. 81, 85] as we use a different terminology. The names **(F1)** and **(F2)** are chosen here to distinguish the rules from the D-rules and the E-rule.

Law again. In row 5, P repeats the attack against O's commitment of line 2. He may not repeat it a second time in line 6 because it would be a counter-attack while the local thesis has not changed (it is still $P \supset Q$) **(F1)**. He may also not defend with Q, because he already stated it in row 4 and O has not made a new concession since then **(F2)**. Therefore P is not able to move and loses.

Barth and Krabbe show that in their dialogical system (i.e., with *ipse dixsiti*), these rules together with the E-rule guarantee termination. Another attempt to enforce finite games is a rule proposed by Rahman and Keiff. For intuitionistic dialogues it says that P may only repeat a move (in the same manner) after O introduces a new prime formula P can then make use of [127].[40] However, neither soundness nor completeness proofs are provided, nor a proof which shows that termination is actually guaranteed.

2.3.5 Multi-Proponent Dialogues (Forking and Merging)

Fermüller and Ciabattoni [51] propose multi-proponent dialogues to cope with intuitionistic and the intermediate *Gödel-Dummett Logic*. The approach was later extended by Fermüller [49] for further intermediate logics. In the following, we refer to these kinds of dialogues as *multi-proponent dialectics with forking and merging* (MPD^{FM}). They correspond to hyper-sequent systems. For intuitionistic logic, the appropriate sequent system is HLI' [49] (Section 2.1.6). The structural rules contain the E-rule. The proponent may state prime formulas and wins either with *ipse dixisti* or *absurdum dixisti*.

The idea is that instead of one P, we can now have an army of P's that fights against O in a parallel way. Before performing an attack or a defence, which are called *internal moves*, a single P may do an *external move*: with a *fork* move, any proponent can duplicate himself at any time before an own attack or defence. Each of these P's then performs his own move towards

[40] Note that their dialogues do not allow ipse dixisti remarks and therefore P may only state prime formulas when O committed herself to them before.

O. When it is O's turn, she has to react to all of the proponents' previous attacks and defences separately. O's concessions towards the various P's are individual, i.e., if O states a formula A towards some agent P_i, then this is invisible for all the other proponents P_k, so they cannot make use of it [51, 49].

Another external move available to the proponents is *cancel* which removes a P-player from the game [51]. Depending on the logic, the proponents have also the ability to *merge* again. This means that they can communicate and share the information O has provided towards them in terms of commitments. Several proponents are thereby merged to one, collecting all of O's commitments towards the individual proponents [51, 49].

In *intuitionistic* MPD^{FM}-dialogues merging is not allowed. In classical dialogues, two proponents may merge iff one of them stated $\bot$. For other intermediate logics there are other merging conditions [49].

To represent a dialogue state, Fermüller and Ciabattoni use sets of single-conclusion sequents, called *components*, each representing a parallel game with a different proponent.

$$\{\Pi_1 \vdash_1 C_1, \ldots, \Pi_n \vdash_n C_n\}$$

Here, Π_i contains all of O's concessions towards P_i, and C_i is the last assertion made by P_i. However, this is not enough to represent parallel game states, so a state machine is introduced that indicates whose turn it is and what the possibilities for the players are to move [51, 49]. A state of such a machine must be assigned to each component of the sequent.

Note that the *fork* move corresponds to rule EC, *cancel* to EW, and classical *merge* to rule CL of HLI' [49]. Fermüller and Ciabattoni translate P-winning-strategies to proofs in HLI' and vice versa, to show soundness and completeness of the multi-proponent dialogical systems with respect to the different intermediate logics, including intuitionistic and classical propositional logic [51, 49].

To summarise: in the dialogical setting proposed by Fermüller and Ciabattoni, *parallel* games are considered. A proponent may *fork* arbitrarily before performing an internal move. O's concessions are not global, but instead made only towards individual P's. Instead of the dialogical sequent system by Barth and Krabbe [10], the system presented here is closer to a usual sequent system for which soundness and completeness are shown (HLI'), but the MPD^{FM}-sequents are not capable to represent complete dialogical states.

2.3.6 Material Dialogues and Hintikka Games

In material dialogues the proponent does not want to show general validity for a given formula. Instead, a model is given in which a truth value is assigned to each propositional variable. Material dialogues were first presented by Lorenz [99] as *relative Dialogspiele*,[41] and later also discussed extensively by Krabbe [82, 83] and Barth and Krabbe [10]. Dialogue sequents (see Section 2.3.3) for material dialogues are also provided in [10].

Note that material dialogues correspond to Hintikka games [70] with the difference that in the dialogues, the players still take turn one after another, while in Hintikka games the parties may move in a more jumbled way. A detailed, rather philosophically oriented comparison between dialogues and Hintikka games can be found in [131].

Although both material dialogues and Hintikka games are interesting topics of their own, we do not consider them further, as we restrict ourselves to dialogues we can use to prove general validity of formulas.

[41] literal translation: *relative dialogue games*

Assert	$\Box A$
Attack	$?_{\Box}$
Defence	A

Figure 2.24: Particle Rule for modal logic (c.f. [102])

2.4 Lorenzen Dialogues as Proof Systems for Modal Logic

2.4.1 Ordinary Modal Dialogues

Also for modal logics, several dialogical attempts have been proposed. We first look at a type of dialogues we call *ordinary*. They are named this way as they come along without designated contexts (like ordinary sequent systems) and with only the two usual players.

Early Attempts

The earliest dialogical approach dealing with modalities comes from Lorenzen himself. He proposes a very simple particle rule [102] (Figure 2.24). As structural rule he introduces the following *defence rule*:

> "If the proponent defends a Δ-formula he may attack only the Δ-formulae (the beginning Δ deleted) put by the opponent beforehand."
>
> (Lorenzen [102], p. 64)

Note that Lorenzen uses the symbol Δ instead of □ but with the same intended meaning.

According to Krabbe [83], the structural rule can be interpreted in a fallible way. As example he states that P cannot defend $\Box A$ when O's initial concession is $B \wedge \Box A$. Actually it is not completely clear what Lorenzen meant with his structural rule. It could also mean that after P defends against a $\Box$-attack, he may only counter-attack O's $\Box$-formulas *after his defence*. Then P would have a winning strategy for Krabbe's example. However, if P's defence requires that *only* $\Box$-formulas may be attacked by P, then this would not allow him to attack the conjunction at all (see also [83], pp. 207, 208).

Lorenzen does not provide any proofs that his idea for modal dialogues leads to sound or complete reasoning procedures. More on his work on modal dialectics is summarised in Krabbe's dissertation [83]. Here, we only discuss the part which is relevant when we consider dialogues as reasoning procedures.

Strict Commitments and Withdrawals

Krabbe [83, 86] suggests a different interpretation of the $\Box$ than the usual *necessarily* or *obligatory*. Instead, he introduces *strictness levels*. Each statement can be assigned to a strictness class. These are related due to a strict partial ordering indicating which assertions are stricter than others. An index $i \in I$ is used to extend the modal operator and indicate the strictness class. For simplification we assume that I is the set of natural numbers $\mathbb{N}$ and that the class of i is stricter than the class of j iff $i > j$. The statements then have the form $\boxed{i}\varphi$, where $i \in \mathbb{N}$ and φ is an arbitrary formula which may again contain other boxes with indexes. If $i = 0$, $\boxed{i}$ can be omitted [83], i.e., $\boxed{0}\varphi \equiv \varphi$.

Krabbe's idea is that O can *withdraw* earlier, weaker commitments whenever P defends against a $\boxed{i}$-attack. The particle rule for $\boxed{i}$ is the same as Lorenzen's (Figure 2.24), if we replace $\Box$ by $\boxed{i}$. The structural rule is then as follows:[42]

[42]As Krabbe uses a different terminology, the formulation is slightly adjusted.

> If P defends against an attack on $\boxed{i}V$ with V, then "O shall have a right to assume the neutral position to any or all concessions that are not of the form $\boxed{j}W$", where $i \leqslant j$, "immediately before O attacks V." (Krabbe [83], p. 224)

This means that if P's statement starting with $\boxed{i}$ was attacked and he defends, then O can *withdraw* earlier concessions which are less strict than i. These statements cannot be used by P for attacks or defences anymore, e.g., atoms which were available to perform an *ipse-dixisti*-move before can now be withdrawn if they have not at least the same strictness level as i. Note that the rules are placed on top of the system by Barth and Krabbe [10] and that P may therefore state atoms which have not been stated by O before.

Here is a simple example:

	O		P	
1				$A \supset \boxed{1}A$
2	$[?, 1]$	A	$[!, 2]$	$\boxed{1}A$
3	$[?, 2]$	$?_\Box$	$[!, 3]$	A

So far, everything seems to be fine for P. Nevertheless, he has not yet won, as in these rules O may attack prime formulas which P has to defend with *ipse dixisti*. Further, as he has just defended against a $\boxed{i}$-attack, O may *withdraw* all commitments with strictness level 0, namely the A of row 2. We indicate the withdrawal by crossing out the concerned concession.

	O		P	
1				$A \supset \boxed{1}A$
2	$[?, 1]$	~~A~~	$[!, 2]$	$\boxed{1}A$
3	$[?, 2]$	$?_\Box$	$[!, 3]$	A
4	$[?, 3]$	$?_A$	—	—

Now P cannot make use of this commitment and loses.

Krabbe also provides an axiomatic system for the underlying logic [83]:

1.	$\boxed{i}(\phi \supset \psi) \supset (\boxed{i}\phi \supset \boxed{i}\psi)$	(for each $i \in \mathbb{N}$)
2.	$\boxed{i}\phi \supset \boxed{j}\phi$	(for each $i, j \in \mathbb{N}$ such that $j < i$)
3.	$\boxed{i}\phi \supset \phi$	(for each $i \in \mathbb{N}$)
4.	$\boxed{i}\phi \supset \boxed{i}\boxed{i}\phi$	(for each $i \in \mathbb{N}$)

Besides the deduction rule of modus ponens, the rule of *necessitation* is also valid:

$$\frac{\phi}{\boxed{i}\phi}\,\text{Nec}$$

It is quite clear that we have here a multi-modal variant of S4. If we set $I = \{0, 1\}$, we actually obtain the axioms of S4 [83]. A proof that shows adequacy of the dialogical system with respect to the axiomatic system is sketched in [86, 83]. Concerning *termination*, Krabbe also proposes an alternative variant of rule **(F2)** (see p. 84):[43]

(F2^{S4})

a) After O attacks a statement by P, P may not repeat a previous assertion which was already attacked by O, unless in the meantime O made a new concession (not yet withdrawn) she has not made before.

b) After P defended against an attack on $\boxed{i}\phi$ stating ϕ, he may not defend with ϕ with the same particle rule again, unless O stated a new, not yet withdrawn commitment $\boxed{j}\psi$ such that $i \leqslant j$.

Krabbe also states that his system is suitable for intuitionistic dialectics [83]. However, note that there is no particle rule for the $\Diamond$-operator (or an adequate replacement). Therefore, as $\Diamond$ can only be defined in terms of $\Box$, the present dialogical system in the current form cannot be used for IS4 or CS4.

[43]Paraphrased from [86, pp. 198, 199] as we use a different terminology.

2.4.2 Multi-Opponent Dialogues

Multi-opponent dialogues were presented by Van Dun already in 1972 [144]. He suggests a *coalition* of opponents fighting one proponent. At the start of a dialogue, P states his thesis as usual and a single opponent (his communication partner) can attack. P has to defend towards her. However, he could have *strengthened* his hypothesis with a $\Box$.

> "[The] proponent may wish to strengthen his claim by inviting his opponent to call in the help of a coalition-partner or co-player. If he wishes to defend the formula a in this way, he can make his intentions clear by prefixing the symbol L to the formula."
>
> (Van Dun [144], p. 126)

Note that Van Dun uses the letter L for $\Box$. So when P states an assertion $\Box\phi$, the opponent can delegate her problem to a *coalition partner*, i.e., another opponent towards whom P has to defend with ϕ, changing P's communication partner. Van Dun gives a rather philosophical justification for his approach.

> "[C]ommitment is usually the result of moral, contractual or even legal obligations. Indeed, most of the time one is not committing oneself — one is simply committed." (Van Dun [144], p. 124)

The special thing is that every O-player is bound to her own commitments, i.e., if one opponent O_1 commits herself to an atom A, but another opponent O_2 does not, P cannot make use of the A when he is currently talking to O_2. The reason is that O_2 could argue that he never claimed A and that she is not responsible for O_1's concessions.

Roughly spoken, P's communication partner is changed whenever any player defends a $\Box$ or a $\Diamond$. However, a modal assertion by P has another semantic meaning than a modal assertion by an O-agent.

- If P states a modal assertion, the meaning can be interpreted as follows:

  $\Box\phi$ *I can give reason for* ϕ *towards* all *of your coalition partners.* You choose!

  $\Diamond\phi$ *I can give reason for* ϕ *towards at least* one *of your coalition partners.* I choose!

- If O states a modal assertion, the meaning can be interpreted as follows:

  $\Box\phi$ All *of my coalition partners are ready to commit themselves to* ϕ*.* You choose!

  $\Diamond\phi$ *At least* one *of my coalition partners is ready to commit herself to* ϕ*.* I choose!

So the player who attacks a $\Box$ may choose P's communication partner. For the $\Diamond$, the defender may choose P's communication partner. Van Dun introduces two particle rules for the $\Box$ and two for the $\Diamond$, i.e., the opponents use other particle rules than the proponent. This *asymmetry* in particle rules is an exception among dialogical systems.

Opponents are related due to a *coalition relation* which is usually *reflexive*. New opponents are introduced by any of the present O-players who attacks a $\Box$ or defends a $\Diamond$. Then the freshly introduced opponent player is a coalition partner of the one who called her in, so their connection is recorded in the coalition relation. Note that the relation is not necessarily *symmetric* which means that if O_1 is a coalition partner of O_2 it does not need to be the case that O_2 is also a coalition partner of O_1. When P wants to change his communication partner, he may only choose one who was introduced before and who is a coalition partner of his current communication partner [144].

Actually, the opponents correspond to Kripke worlds and the coalition relation to a frame structure which is reflexive.[44] It is also possible to enforce

[44] Philosophically, it does not make sense that a player is not a coalition partner of herself [144].

that the coalition relation is transitive and symmetric which leads to modal systems that seem to correspond to S4 and S5 [144]. However, Van Dun does not prove that these dialogues implement a sound and complete reasoning procedure for a formally defined modal logic.

2.4.3 Dialogue Contexts

The last type of modal dialectics we consider here are the two-player games dealing with *dialogue contexts*. The idea can be traced back to an article by Inhetveen [76] where the author presents early-stage work by Marijan Marčinko who introduced *dialogue levels*[45]. Each assertion of a dialogue is assigned to such a level.

Rückert and Rahman [130] later proposed the idea of *dialogue contexts*[46] which are, from the technical point of view, more or less the same. The rules are based on the D-dialogue setting by Lorenz. The contexts of the dialogues are *nested*, i.e., any context has an arbitrary number of *context successors*.

At the beginning of a dialogue, P states his hypothesis in an initial context. Attacks and defences happen in this context as long as only propositional connectives are involved. If O attacks a $\Box$ she may ask explicitly for another context in which P has to defend. Thereby, O can introduce a new context which is then a *successor* of the current one. If she attacks a $\Diamond$, P may defend in any existing context which is a successor. However, P may only defend in contexts which were introduced by O before. When P attacks a $\Box$ he may decide accordingly in which O must defend. When P attacks a $\Diamond$, O may defend in any successor context and thereby introduce a new one [130].

By contrast to Van Dun's approach, the particle rules are symmetric, i.e., they are the same for both players. Only O may introduce new contexts which is defined due to a structural rule. The authors also propose different

[45] translated from German "Dialogebenen"
[46] translated from German "Dialogkontexte"

levels of rights for P defining which contexts he may select when defending a $\Diamond$ or attacking a $\Box$, e.g., a rule is proposed that allows P to stay in the current context instead of changing it, or to select a context which is not a direct successor, but instead the successor of a successor. The authors claim that reasoning procedures for various classical modal systems, such as K, T, B, S4, and S5, are obtained in this way.

From the proof-theoretic view, contexts are nothing else than Kripke worlds, again connected in a frame structure which is constructed depending on the underlying structural rules. So it sounds quite natural that the corresponding modal systems are implemented in this way. However, like Lorenzen and Van Dun, the authors do not provide any adequateness proofs.

In [29] Clerbout combines ranks[47] with the approach by Rückert and Rahman to guarantee termination. In his PhD thesis [30], he provides an algorithm for transforming P-winning-strategies to tableau proofs and vice versa, which serves as an adequateness proof regarding the modal systems K and S5.

Rebuschi [129] claims that using Lorenz's rules for intuitionistic logic instead of classical logic within the setting by Rückert and Rahman might lead to a dialogical reasoning system for IK. He also presents some examples but no actual proof.

Another approach based on the work by Rückert and Rahman is a dialogical system for the constructive description logic $c\mathcal{ALC}$, where the nesting of the contexts is adjusted so that it corresponds to the Kripke semantics of constructive modal logic. It is presented in my diploma thesis [137], but does not contain adequateness proofs either. I improved the work later: instead of considering all refinements in the Kripke structure as individual contexts, the constructiveness can also be obtained by restricting P's possible moves [140].

[47] See Section 2.3.4 for details on ranks.

2.5 Sequents and Dialogues – Common Features and Differences

Lorenz [99] says that the usual calculi can be seen as *monologues* where one player/agent tries to show validity of some formula applying different rules step by step.[48] However, it is quite obvious that sequent systems and dialogical games share many similarities concerning proof theory.

Considering a sequent $\Gamma \Rightarrow \Delta$, Γ corresponds to concessions by O and Δ to the hypothesis/assertions by P. The relation becomes more obvious when we look at the single-conclusion sequent calculus for intuitionistic logic, G3I (rules on page 33). Here, Δ contains at most one formula. This again can be associated with the structural rule **(D3)** that allows P to defend only against the last open attack, i.e., the attack on the statement that thereby becomes the *current local thesis* as it is called by Barth and Krabbe [10], whereas in classical dialogues he may defend against an earlier attack more often. P's decisions actually have an influence on the local thesis. For example, if $A \vee B$ was stated by P and O attacks this, then P can defend with A or with B. The other formula is lost as P may not repeat this defence. Therefore, the decision might be highly significant for winning or losing the dialogue. Rule $\vee r$ of G3I is interpreted similarly when read from bottom to top:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\vee r \qquad\qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\vee r$$

In both cases, one disjunct is dropped and the information that it was present is lost as well. Accordingly, once P decided to defend with B, he cannot change his mind later and state A.

Barth and Krabbe also connect dialogues to sequents by introducing their dialogical sequent systems that implement the rules directly. However, their sequent structure is different to usual Gentzen sequents. Fermüller and Ciabattoni make the relation between $\mathrm{MPD}^{\mathrm{FM}}$-dialectics and hypersequents

[48] He calls such calculi *entartete Dialoge*, which literally means *degenerated dialogues*.

explicit and use it to prove soundness and completeness of their dialogical calculus. A cut rule for dialogues was proposed by Lorenz [99], the application of this however cannot be eliminated [80]. The composition of dialogues, which correspond to the application of a cut rule in a sequent system, is investigated by Alama [2].

A big difference between (ordinary) sequent systems and dialogues is that the latter have some *scheduling mechanism* which occurs due to the subsequent moves performed by the parties. In particular, the restrictions concerning repetitions, which are fixed in the structural rules, put a limit on the possible moves. In ordinary sequent systems, rules can be applied on the different formulas in any order in a more arbitrary way. However, in this chapter we saw some exceptions, e.g., sequent systems that guarantee termination of the proof search or the sequent systems with focus which enforce a normalization of the proofs. In fact, Herbelin shows in his PhD-thesis [67] that proofs of his sequent system LJQ (Section 2.1.5) correspond to P-winning-strategies in E-dialogues, for the fragment of intuitionistic logic not containing disjunction. To include the $\vee$-operator, he also introduced another variant of LJQ he calls LJQ* which is however different to the LJQ* we considered in Section 2.1.5.

Anyway, the scheduling in D- and E-dialogues is limited as the players are very free in their choices. Although they take turn one after another, P has many possibilities (in D-dialogues O as well), as he can perform attacks on all non-atomic commitments stated by O. Intuitionistic dialectics are usually related to *single-conclusion* sequent systems, while multi-conclusion variants are rather relevant for classical logics. In these classical dialectics, P has even more possibilities to perform moves as the defence restriction is lifted. On the other hand, for intuitionistic logic, multi-conclusion sequent calculi are seen as the better alternative when searching for a proof (e.g., c.f. [44]) as crucial decisions are postponed.

In the following, we investigate a method to restrict P further in his possibilities to perform moves, but take advantage of multi-conclusion calculi for intuitionistic logic. The result is a multi-proponent approach of intuitionistic and modal dialogical logic. In these systems scheduling becomes more essential than in the dialectics presented so far.

3 Multi-Proponent Dialogues for Intuitionistic Logic

In this chapter we introduce[1] an approach of dialogical games for intuitionistic propositional logic (IPL), which we call *multi-proponent intuitionistic dialectics* (MPID for short). These dialogues shall be used for proof searches in IPL. Like the dialectics presented in Chapter 2.3.5, we have not only one opponent and one proponent as players, but instead a *syndicate* of *proponent-agents* fighting one opponent. However, this system has many differences to the one by Fermüller and Ciabattoni, that are discussed in detail in Section 3.4.

In the following, we first consider the game rules for MPID and a variant which enforces termination. Afterwards, in 3.2, we have a look at the dialogical sequent-style systems DiaSeqI and DiaSeqI$^+$ which implement these game rules in a concrete way and which are used to prove soundness and completeness of the dialogues when used as reasoning procedures for IPL. These proofs are presented and explained in Section 3.3. At the end, we compare DiaSeqI and DiaSeqI$^+$ with sequent calculi and dialogical approaches presented in Chapter 2.

[1] I already introduced parts (MPID and DiaSeqI) of this chapter in [138, 139]. The proofs presented here go into more detail and adequateness of the terminating variant (DiaSeqI$^+$) is shown for the first time.

Assert	$A \wedge B$		$A \vee B$		$A \supset B$	$\neg A$	$\bot$	P
Attack	$?_L$	$?_R$	$?_\vee$		A	A	$?_\bot$	$?_P$
Defence	A	B	A	B	B	—	—	**!!**

Figure 3.1: Particle Rules for the propositional language based on [98, 10] (c.f. [138, 139])

3.1 Game Rules

3.1.1 Without Termination Guarantee

As it is usually the case for dialogues, we distinguish two sets of game rules, namely the *particle rules* and the *structural rules* (see Chapter 2.3.2 for details).

Our *particle rules* are actually the same of Figure 2.17 (p. 72), but without those for the quantifiers $\forall$ and $\exists$, because we consider only propositional logic here. Additionally, as we let the proponents state atoms and O attack these (just like Barth and Krabbe [10]), we need a particle rule for prime formulas as well. We also consider an extra rule for $\bot$. Note that the $\neg$-rule is then actually redundant, because $\neg A \equiv A \supset \bot$, but nevertheless we offer both options for historical reasons. The particle rules are displayed in Figure 3.1. We write P for an arbitrary prime formula and A and B for propositional formulas. Double exclamation marks (**!!**) represent the *ipse dixisti remark*. We do not have an *absurdum dixisti* but instead there is simply no possibility to defend against a $\bot$-attack.

We consider two *parties*. On the one hand, we have the single opponent O, on the other hand the P-agents $\mathit{Propos} =_{df} \{\mathsf{Pi} \mid i \in \mathbb{N}\}$.

The *structural rules* for MPID are as follows [139]:

I1 At the beginning of a dialogue, O states *initial concessions* and a single P-agent states the *hypothesis*.

I2 A *round* consists of a sequence of moves by O, followed by moves of all active P-agents. A *dialogue run* is a sequence of such rounds. The first round starts after the assertion of the hypothesis.

I3 If possible, all players are obliged to perform moves. A P-agent may postpone a move until succeeding rounds if he is forced to react to a *critical attack* (see rule I6), but commitments by O made with the critical attack are only conceded as soon as the critically attacked agent reacts. Whenever a P-agent has several possibilities of how to react to an O-move, new P-agents are introduced to take out these remaining possibilities.

I4 A dialogue is *won* by the proponents iff the opponent cannot react to all of the proponents' moves of the previous round. The opponent wins iff no P-agent can react to any of O's statement of the same round (either with an attack or a defence).

I5 Only O is allowed to attack *prime formulas*. P-agents may defend against these attacks only if O has stated the prime formula herself towards a P-agent who is not *deactivated* in the same round.

I6 Attacks on negations and implications are considered to be *critical attacks*. Other attacks are *non-critical*.

I7 Whenever a P-agent reacts to a *critical attack*, all other active proponent agents are immediately *deactivated*, i.e., they may not perform defences or counter-attacks.

I8 A P-agent may repeat critical attacks on the same assertion only after any P-agent reacted to a critical attack performed by O. Other repetitions are not allowed.

Rules I6 to I8 are significant for intuitionism. I6 and I7 (in combination with I3) put the necessary restrictions on the proponents which correspond to the non-invertible rules ($\supset$r and $\neg$r) of multi-conclusion sequent calculi like

	O		P0		P1		P2	
1			$(A \wedge B) \supset (A \supset B)$					
2	$[\textbf{?}, 1]^0$	$A \wedge B$	$[\textbf{!}, 2]$	$A \supset B$	$[\textbf{?}, 2]$	$\textbf{?}_l$	$[\textbf{?}, 2]$	$\textbf{?}_r$
3	$[\textbf{?}, 2]^0$	A	$[\textbf{!}, 3]$	B				
	$[\textbf{!}, 2]^1$	A			—	—		
	$[\textbf{!}, 2]^2$	B					—	—
4	$[\textbf{?}, 3]^0$	$\textbf{?}_B$	$[\textbf{!}, 4]$	$\textbf{!!}$	—	—	—	—

Figure 3.2: A simple MPID-example

$G3i^M$. Rule I8 allows the P's to repeat attacks on implications and negations which correspond to the duplication rules ($\supset l$ and $\neg l$) that are however restricted here as well, as these repetitions may only be performed after a P-agent reacted to a critical attack. Note that rule I6 refers to specific connectives, namely $\supset$ and $\neg$. This is quite unusual, as in the literature, structural rules are normally independent of the particle rules and are therefore not to be related to specific logical operators. However, we give up this convention to ensure higher flexibility of our dialogical system. Later, we will change the set of critical moves and thereby generate other dialectics for other logics.

The dialogical tableau of the first example is shown in Figure 3.2. In the first row the initial proponent P0 states the hypothesis $(A \wedge B) \supset (A \supset B)$. In row 2, O attacks this assertion (the 0 attached to the brackets is the proponent number O refers to). With her attack she states the antecedent of the hypothesis. P0 can now either defend with the consequent or counter-attack O's commitment. He defends with the antecedent but calls in the new agents P1 and P2 to perform the counter-attacks. There are two of these. One agent asks for the left conjunct of O's concession, the other one asks for the right one. In the next round, O has to react to all of the proponents' moves. She attacks P0's implication and defends against the attacks of P1 and P2 stating the conjuncts towards them. P0 reacts with a defence stating the consequent B. He is not allowed to counter-attack A, as only O may attack primes. That

	O		P0		P1	
1			$A \vee \neg A$			
2	$[\mathbf{?}, 1]^0$	$\mathbf{?}_\vee$	$[\mathbf{!}, 2]$	A	$[\mathbf{!}, 2]$	$\neg A$
3	$[\mathbf{?}, 2]^0$	$\mathbf{?}_A$	—	—		
	$[\mathbf{?}, 2]^1$	A			—	—

Figure 3.3: The Excluded Middle in MPID

is also the reason why P1 and P2 cannot move now and miss that turn.[2] In the last round, O attacks P0's B who answers that O stated it herself (towards P2) which is enough to win the game, as O cannot react to this.

The second example (Figure 3.3) concerns again the law of the excluded middle which is not valid in intuitionistic logic. The first attack by O is defended by P0 with the left and by P1 with the right disjunct. Then O attacks both in row 3. The attack against $\neg A$ is critical. This means that if P1 reacts to this, P0 is deactivated. On the other hand, O's concession of A towards P1 is not conceded until the agent reacts to this which means that P0 cannot make use of it to perform an *ipse dixisti* defence. Also P1 cannot react, as it is not possible for him to attack primes and the particle rules also do not allow defences against $\neg$-attacks. No proponent is able to move and therefore the opponent wins.

3.1.2 With Termination Guarantee

Rule I8 is not sufficient to guarantee termination in the intuitionistic case. With *Peirce's Law* we have again a counter-example.

Now we have a look at the sequent calculus IG^r again (Chapter 2.1.4) which fulfils our requirements. As the restrictions of a *regular derivation* are exactly

[2]Note that they are actually *deactivated*, but as they are not to defend or counter-attack, this has actually no influence on them. They simply cannot move, because they are not allowed to attack prime formulas.

	O		P0		P1		P2	
1				$((P \supset Q) \supset P) \supset P$				
2	$[?, 1]^0$	$(P \supset Q) \supset P$	$[!, 2]$	P	$[?, 2]$	$P \supset Q$		
3	$[?, 2]^0$	$?_P$	—	%				
	$[?, 2]^1$	P			$[!, 3]$	Q	$[?, 2]$	$P \supset Q$
4	$[?, 3]^1$	$?_Q$			—	%		
	$[?, 3]^2$	P					$[!, 4]$	Q
5	$[?, 4]^2$	$?_Q$					—	—

Figure 3.4: Peirce's Law in MPID with termination

what we need, we simply have to implement these in terms of a structural rule. It would be rather hard to transfer G4ɪ instead of IG^r into the dialogical setting because of the lack of the subformula property. Further, G4ɪ is a single-conclusion calculus, whereas IG has the multi-conclusion property which fits better to the multi-proponent idea (as we will see later). The restrictions of IG^r-derivations are also already provided as meta-rules which makes it easy to translate these to a structural game rule. The other approaches mentioned in Chapter 2.1.4 are either more complicated or the blocking mechanism is hidden within the rules and harder to extract.

This new rule is simply added to those on pages 100/101:

$I9^+$ If a P-agent reacted to a critical attack against some assertion, then, if O attacks an assertion she already attacked before, the P-agents may only defend non-critically against this attack. A counter-attack is then not possible. (c.f. [139])

This corresponds to the *a fortiori* rule of IG^r. Rule I8 implements the other restriction. The combination of both enforces termination (the proof is given later in Section 3.3). Peirce's Law is then no problem anymore as we see in Figure 3.4. After the defence and the counter-attack by the proponent agents in row 2, O attacks P0's prime formula P and P1's implication. The latter is *critical*. As P0 cannot defend, the proponents decide to *deactivate* him (indicated with %). P1 continues with his defence. As this is a reaction to a critical attack, an earlier critical attack against O may be repeated (I8). P2 is introduced to do this, i.e., he counter-attacks O's commitment of row 2 again. Now in row 4, O attacks P1's prime formula Q and P2's implication, stating P. Again, the latter one is critical and P1 is deactivated due to the lack of other options. P2 defends with Q. Another attack against O's commitment of row 2 is not possible due to our new rule I9$^+$. Finally, O attacks P2's prime formula and the last remaining P-agent has no possibility to answer this. The opponent wins. If we allowed the proponents to repeat attacks against O's commitment of row 2, new proponents would be introduced again and again, and the dialogue would never end.

3.2 The Systems DiaSeqI and DiaSeqI$^+$

In this section we introduce the sequent-style calculus DiaSeqI that represents multi-agent dialogues for IPL. This calculus provides an adequate way to define the players' moving possibilities and displays various strategies. Then we discuss a variant DiaSeqI$^+$ for the first time, which enforces termination of the games.

3.2.1 Interpreting Dialogues in DiaSeqI

A proof in DiaSeqI has a similar structure as a proof in G3i^M (see Chapter 2.1.3) but divides the proof flow in different sections. It is a tree consisting of *sequents*. We have one single *root* at the bottom, which we call the *initial sequent* or *root sequent*. At the top of each branch we have the *leaves*.

A *sequent* in DiaSeqI has the form $\Phi \vdash_X \Psi$ where Φ and Ψ are sets of *signed formulas* and $X \in \{O, PN, PD\}$. The *turnstile* $\vdash_X$ indicates the party whose turn it is to move next, i.e. $\vdash_O$ means that it is O's turn whereas $\vdash_{PN}$ and $\vdash_{PD}$ are used for the proponent agents.

A *signed formula* includes (beside the formula) an *announcer* with an optional *mark* above (a vertical bar, sometimes with a label L or R, or a tilde) and sometimes an *addressee*:

$$\Phi \subseteq \{o_p : \varphi,\ \overline{o_p} : \varphi,\ \overline{o_p}^L : \varphi,\ \overline{o_p}^R : \varphi,\ \widetilde{o_p} : \varphi \mid p \in \mathit{Propos}, \varphi \in \mathit{Form}\} \quad (3.1)$$

$$\Psi \subseteq \{p : \varphi,\ \overline{p} : \varphi,\ \overline{p}^L : \varphi,\ \overline{p}^R : \varphi \mid p \in \mathit{Propos}, \varphi \in \mathit{Form}\} \quad (3.2)$$

So every formula is assigned by the *announcer label* o (O) or p (a P-agent), where the signed formulas announced by O are elements of Φ and the others of Ψ. Addressees are added to that label only in the case that the announcer is O, because when the announcer is a proponent, then the addressee must be O anyway. The addressee labels are written as subscript next to the announcer label. As mark, a line over the announcer label indicates that the announced formula has been attacked but not yet defended. The tilde in $\widetilde{o_p}$ indicates that the formula is *isolated* to implement rule I8. These isolated (or *blocked*) formulas form some kind of history set, similarly as in the work by Heuerding et al. [68] (see Sections 2.1.4 and 2.2.3). More on that follows later.

The rules of the DiaSeqI-calculus are given in Figure 3.5. We use the terms *rule application*, *premise*, *conclusion*, *active formula*, *principal formula*, and *side formula* as described in Chapter 2.1.1. Rule applications are read from bottom to top. In a DiaSeqI-sequent $s = \Phi \vdash_X \Psi$, we say that Φ is the *antecedent* of s and Ψ the *succedent* of s. As the usual sequent trees, a DiaSeqI-proof is called *closed* iff axiom rules are applied on all leaves. These are P!! and P?$\bot$.

Rules starting with O?/O! are attacks/defences performed by O, accordingly, P?/P! are moves performed by the P-agents. We also have extra *trigger rules* for negations and implications instead of defences.[3] These rules are O$*\neg$, O$*\supset$ and P$*\neg$. Concerning the implication stated by O in O$*\supset$, the rule includes the defence.

[3]Of course, there should not be a defence for negations anyway, but we need a process that gives the other party the opportunity to decide to react to this formula and therefore we need these triggering rules.

o-rules

$$\frac{\Phi \vdash_O \Psi,\ \overline{p} : A \supset B}{\Phi \vdash_O \Psi,\ p : A \supset B}\ \mathsf{O?}{\supset} \qquad \frac{\widetilde{\overline{o_p}} : A \supset B,\ \Phi \vdash_O \Psi,\ p : A \qquad o_p : B,\ \Phi \vdash_O \Psi}{\overline{o_p} : A \supset B,\ \Phi \vdash_O \Psi}\ \mathsf{O}{*}{\supset}$$

$$\frac{\Phi \vdash_O \Psi,\ \overline{p}^L : A \wedge B \qquad \Phi \vdash_O \Psi,\ \overline{p}^R : A \wedge B}{\Phi \vdash_O \Psi,\ p : A \wedge B}\ \mathsf{O?}{\wedge} \qquad \frac{\Phi \vdash_O \Psi,\ \overline{p} : A \vee B}{\Phi \vdash_O \Psi,\ p : A \vee B}\ \mathsf{O?}{\vee}$$

$$\frac{o_p : A,\ \Phi \vdash_O \Psi}{\overline{o_p}^L : A \wedge B,\ \Phi \vdash_O \Psi}\ \mathsf{O!L} \qquad \frac{o_p : A,\ \Phi \vdash_O \Psi \qquad o_p : B,\ \Phi \vdash_O \Psi}{\overline{o_p} : A \vee B,\ \Phi \vdash_O \Psi}\ \mathsf{O!}{\vee}$$

$$\frac{o_p : B,\ \Phi \vdash_O \Psi}{\overline{o_p}^R : A \wedge B,\ \Phi \vdash_O \Psi}\ \mathsf{O!R} \qquad \frac{\Phi \vdash_O \Psi,\ \overline{p} : \neg A}{\Phi \vdash_O \Psi,\ p : \neg A}\ \mathsf{O?}{\neg} \qquad \frac{\widetilde{\overline{o_p}} : \neg A,\ \Phi \vdash_O \Psi,\ p : A}{\overline{o_p} : \neg A,\ \Phi \vdash_O \Psi}\ \mathsf{O}{*}{\neg}$$

$$\frac{\Phi \vdash_O \Psi,\ \overline{p} : A}{\Phi \vdash_O \Psi,\ p : A}\ \mathsf{O?}a \qquad \frac{\Phi \vdash_O \Psi}{\Phi \vdash_O \Psi,\ p : \bot}\ \mathsf{O?}{\bot} \qquad \frac{\Phi \vdash_{PD} \Psi}{\Phi \vdash_O \Psi}\ c_O$$

only applicable if no other rule application is possible

p-rules – decide phase

$$\frac{o_q : A,\ \Phi^\delta \vdash_{PN} p : B}{\Phi \vdash_{PD} \Psi,\ \overline{p} : A \supset B}\ \mathsf{P!}{\supset} \qquad \frac{o_p : A,\ \Phi^\delta \vdash_{PN} \emptyset}{\Phi \vdash_{PD} \Psi,\ \overline{p} : \neg A}\ \mathsf{P}{*}{\neg} \qquad \frac{\Phi \vdash_{PN} \Psi}{\Phi \vdash_{PD} \Psi}\ \mathsf{PN}$$

$\Phi^\delta =_{df} (\Phi \setminus \{\widetilde{\overline{o_p}} : \varphi \mid p \in \mathit{Propos}, \varphi \in \mathit{Form}\}) \cup \{o_s : \varphi \mid \widetilde{\overline{o_p}} : \varphi \in \Phi, p \in \mathit{Propos}\}$,

each s is a new P-agent.

p-rules – normal phase

$$\frac{\overline{o_p} : A \supset B,\ \Phi \vdash_{PN} \Psi}{o_p : A \supset B,\ \Phi \vdash_{PN} \Psi}\ \mathsf{P?}{\supset} \qquad \frac{\overline{o_p} : \neg A,\ \Phi \vdash_{PN} \Psi}{o_p : \neg A,\ \Phi \vdash_{PN} \Psi}\ \mathsf{P?}{\neg} \qquad \frac{\Phi \vdash_{PN} \Psi,\ p : A}{\Phi \vdash_{PN} \Psi,\ \overline{p}^L : A \wedge B}\ \mathsf{P!L}$$

$$\frac{\overline{o_p}^L : A \wedge B,\ \overline{o_q}^R : A \wedge B,\ \Phi \vdash_{PN} \Psi}{o_p : A \wedge B,\ \Phi \vdash_{PN} \Psi}\ \mathsf{P?}{\wedge} \qquad \frac{\Phi \vdash_{PN} \Psi,\ p : B}{\Phi \vdash_{PN} \Psi,\ \overline{p}^R : A \wedge B}\ \mathsf{P!R}$$

$$\frac{\overline{o_p} : A \vee B,\ \Phi \vdash_{PN} \Psi}{o_p : A \vee B,\ \Phi \vdash_{PN} \Psi}\ \mathsf{P?}{\vee} \qquad \frac{\Phi \vdash_{PN} \Psi,\ p : A,\ q : B}{\Phi \vdash_{PN} \Psi,\ \overline{p} : A \vee B}\ \mathsf{P!}{\vee}$$

$$\frac{}{o_p : A,\ \Phi \vdash_{PN} \Psi,\ \overline{r} : A,\ \Psi}\ \mathsf{P!!} \qquad \frac{}{o_p : \bot,\ \Phi \vdash_{PN} \Psi,\ r : A}\ \mathsf{P?}{\bot}$$

$$\frac{\Phi \vdash_O \Psi}{\Phi \vdash_{PN} \Psi}\ c_P$$

only applicable if no other rule application is possible

q is a new P-agent in rules P!⊃, P?∧, and P!∨.
r is an arbitrary active P-agent in rules P!! and P?⊥.

Figure 3.5: Rules of DiaSeqI (c.f. [138, 139])

Now assume that the proponents want to prove that an assertion φ is valid in IPL. We start with an initial sequent $\vdash_O p0 : \varphi$, so O starts the play. Whenever she has a choice for several moves, a branching of the proof tree is created, e.g., if O attacks a conjunction $p : A \wedge B$ then she can choose if she wants to demand the left or the right conjunct, i.e., the next sequent contains $\overline{p}^L : A \wedge B$ or $\overline{p}^R : A \wedge B$. Note that with respect to usual sequent calculi it might make more sense to establish the branching when $A \wedge B$ is decomposed, namely when the P-agent performs his move and states either the left or the right disjunct. However, from the dialogical point of view, a branching occurs when O has several possibilities to perform one move and therefore we establish the branching with her attack.

Actually, the calculus could be simplified by removing all announcer labels and keeping only the marks. It also works without the addressees at the left-hand formulas. However, we keep all this extra information to highlight the connection to the structural rules and the interaction between the opponent and the unique proponent agents.

The turnstile $\vdash_X$ of a sequent indicated the current *phase* of the round. This can be changed with the rules c_O, c_P, PN and some critical rule applications (see also Figure 3.6). The rules c_O and c_P can only be used if no other application (from bottom to top) is possible. This is needed to implement rule I3 which is significant for the scheduling mechanism.

As long as the turnstile is '$\vdash_O$', O can perform attacks (mark agent-signed formulas with bars) or defend against attacked o-signed formulas (remove bars). As soon as there are no formulas with an attack mark left on the left side of the turnstile, and every formula on the right side is marked as attacked, no more rules can be applied with the exception of c_O which changes $\vdash_O$ to $\vdash_{PD}$.

Then we are in the *proponent decide phase* where the P's have to agree on how to continue. They can either react to a critically attacked formula ($P!\supset$ or $P{*}\neg$) and deactivate the other P-agents, or continue by switching directly

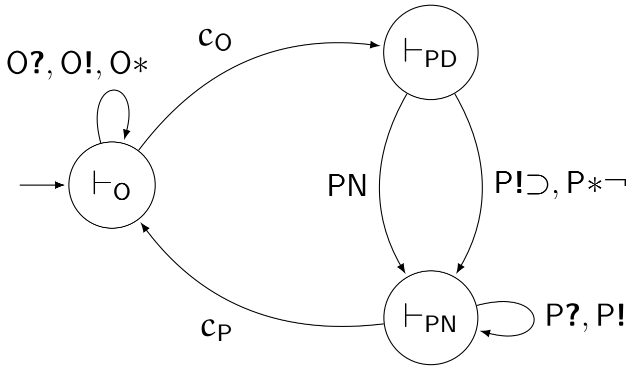


Figure 3.6: Proof cycle in DiaSeqI (c.f. [138, 139])

to the *normal phase* with rule PN which means that the defence of critically attacked formulas is delayed or abandoned. No matter what the decision is, the turnstile changes to $\vdash_{PN}$ and the normal phase is entered where the remaining attacks can be defended and statements announced by O are attacked. Finally, c_P is applied and the whole cycle starts again as illustrated in a state transition graph in Figure 3.6.

Theorem 3.1 (DiaSeqI Closure Property (c.f. [139]))**.** *A* DiaSeqI*-tree* t *is closed iff* P *has a winning strategy, i.e., in all dialogue runs in which* O *takes a decision, she finally loses according to rule I4.*

Proof. We consider only an informal proof here, as the result should be quite obvious.

(⇒) Each branch of t is closed either with P**!!** or P**?**⊥. Both are moves O cannot react to. So considering the paths as dialogical runs, O is not able to react to these moves and therefore loses.

(⇐) Due to rule I5, no P-agent may attack prime formulas. However, when a P-agent attacks a formula, O is always able to defend or counter-attack. As well, when the P-agent defends a formula without P**!!**, he thereby states a formula which can be attacked by O, even if it is a prime formula. Note that the trigger of a negation is the consequence of an attack and therefore does not count as a move. So, as long as the P-agents perform moves which are

$$
\begin{array}{cl}
\hline
o_{p0} : A,\ o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{PN}} \overline{p0} : B & \mathsf{P!!} \\
\hline
o_{p0} : A,\ o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{O}} \overline{p0} : B & c_{\mathsf{O}},\ \mathsf{PN} \\
\hline
o_{p0} : A,\ o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{O}} p0 : B & \mathsf{O?}a \\
\hline
o_{p0} : A,\ o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{PN}} p0 : B & c_{\mathsf{P}} \\
\hline
o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{PD}} \overline{p0} : A \supset B & \mathsf{P!}\supset \\
\hline
o_{p1} : A,\ o_{p2} : B \vdash_{\mathsf{O}} \overline{p0} : A \supset B & c_{\mathsf{O}} \\
\hline
\overline{o_{p1}}^{L} : A \wedge B,\ \overline{o_{p2}}^{R} : A \wedge B \vdash_{\mathsf{O}} \overline{p0} : A \supset B & \mathsf{O!L},\ \mathsf{O!R} \\
\hline
\overline{o_{p1}}^{L} : A \wedge B,\ \overline{o_{p2}}^{R} : A \wedge B \vdash_{\mathsf{O}} p0 : A \supset B & \mathsf{O?}\supset \\
\hline
\overline{o_{p1}}^{L} : A \wedge B,\ \overline{o_{p2}}^{R} : A \wedge B \vdash_{\mathsf{PN}} p0 : A \supset B & c_{\mathsf{P}} \\
\hline
o_{p1} : A \wedge B \vdash_{\mathsf{PN}} p0 : A \supset B & \mathsf{P?}\wedge \\
\hline
\vdash_{\mathsf{PD}} \overline{p0} : (A \wedge B) \supset (A \supset B) & \mathsf{P!}\supset \\
\hline
\vdash_{\mathsf{O}} p0 : (A \wedge B) \supset (A \supset B) & \mathsf{O?}\supset
\end{array}
$$

Figure 3.7: A simple example of a DiaSeqI-proof

no defences of prime formulas or attacks against ⊥'s, O can still react to all these moves of the previous round [139]. The only option for the proponents to win the game is to close the tree. □

Figure 3.7 shows the DiaSeqI-variant of the dialogue we have already looked at in Figure 3.2 (p. 102). The procedure is the very same as that of the referred example. The other examples of Section 3.1 can be constructed accordingly.

3.2.2 Enforcing Termination with DiaSeqI$^+$

As discussed in Section 3.1.2, we can make the dialogues terminate in all cases if we put some restrictions on P which we defined as rule I9$^+$ (p. 104). For this, we need to introduce an *a fortiori*-equivalent: a new, non-critical defence for $\supset$.

For simplification reasons, we omit rules for negation in DiaSeqI$^+$. Remember that $\neg A \equiv A \supset \bot$, so concerning completeness this is no problem. The rules O?¬, P?¬, O∗¬, and P∗¬ are omitted. Instead, we add the non-critical

$$\frac{\Phi \vdash_{\mathsf{PN}} \Psi,\ p : B}{\Phi \vdash_{\mathsf{PN}} \Psi,\ \overline{p} : A \supset B}\ \mathsf{P!}{\supset}^{+}$$

Figure 3.8: Non-critical defence in DiaSeqI$^+$

defence of an implication (Figure 3.8). This is of course not enough, we still need to implement the restriction of rule I9$^+$. It would of course be possible to extend the sequents of DiaSeqI$^+$ by adding a *history set* or something similar, e.g., in a way as done by Heuerding et al. [68] or Howe [73] (see again Chapter 2.1.4).

However, we define *meta-rules* just as Corsi and Tassi [33]. In fact we can simply adjust the rules of *regular* IG-derivations (see p. 39) as follows:

1. Rule $\mathsf{P!}{\supset}$ may only be used *once* with the same principal formula within one path of the sequent tree.

2. If a formula $A \supset B$ is principal in the application of $\mathsf{P!}{\supset}^{+}$ in a sequent tree t, then below this application in the same path of t, $A \supset B$ is principal of an application of $\mathsf{P!}{\supset}$.

The third rule of regular IG-derivations is already implemented due to the blocking mechanism of DiaSeqI (I8). An alternative possibility is to use the $\tilde{}$-mark on implications which were defended with $\mathsf{P!}{\supset}$ and add the restriction explicitly to $\mathsf{P!}{\supset}^{+}$, so that it is only allowed to apply it if $A \supset B$ is present and marked with a tilde.

3.3 Adequateness of DiaSeqI and DiaSeqI$^+$

In the following, we prove *adequateness*, i.e., soundness and completeness, of DiaSeqI and DiaSeqI$^+$. This is done relying on the adequateness of $G3i^M$ (Chapter 2.1.3) and IG^r (Chapter 2.1.4). Soundness and completeness of $G3i^M$ were first shown by Maehara [104] relying on Gentzen's original sequent system[4], and later by Dragalin [39][5] who makes use of the axiomatic system of IPL (such procedures are also described in detail in [142]). Adequateness of IG^{r} is shown by Corsi and Tassi [32, 33].

Instead of making use of axioms or semantics, we show that it is always possible to convert DiaSeqI-proofs to $G3i^{M}$-proofs (soundness) and vice versa (completeness). We do the same with $DiaSeqI^{+}$ and IG^{r}. Short versions of the adequateness proofs for DiaSeqI were published in [138, 139].

3.3.1 Completeness of DiaSeqI

The transformation from $G3i^{M}$ to DiaSeqI (and also from IG^{r} to $DiaSeqI^{+}$) is not trivial. We first need to introduce some terminology. The $G3i^{M}$-derivation needs be be transferred into some normal form by resorting rule applications, i.e, *permuting* them. The approach has similarities to the normalization proposed by Egly and Schmitt who translated $G3i^{M}$-derivations to G3i-derivations [43, 44], i.e., the derivation tree is divided into several parts which are tackled one after another. However, our transformation has another aim and the parts/blocks are defined differently.

The transformation process is illustrated in Figure 3.9 and starts with an arbitrary $G3i^{M}$-proof-tree which is made free of redundant rule applications. All of its paths are divided into *macro blocks* (Definition 3.3). Ends of macro blocks mark the application of a non-invertible rule or a closure. All macro blocks are then *saturated* (Definition 3.6), i.e., non-invertible rule applications are added as far as possible. The saturated blocks consist of *micro blocks* (Definition 3.4) which are then saturated as well (Definition 3.7). Micro blocks shall correspond to rounds in the dialogue, so after the saturation the rule applications within these blocks can be rearranged to obtain the de-

[4] This uses sequences instead of multi-sets and has some other rules, so it is actually a variant of $G3i^{M}$ for which Maehara shows adequateness.

[5] Dragalin's GHPC has another rule for $\supset l$ but the rest is the same as in $G3i^{M}$.

Figure 3.9: The transformation process from G3I$^{\text{M}}$ to DiaSeqI

sired order. The DiaSeqI-tree can then be derived more or less easily from this normalized G3I$^{\text{M}}$-proof.

Terminology

The following definitions refer to derivations in G3I$^{\text{M}}$ and IG. They can therefore be seen as an extension of the definitions in Chapter 2.1.

A closed sequent tree is a proof for some implication which is represented by the root sequent $\Gamma \Rightarrow \Delta$. If there is such a tree, we say that $\Gamma \Rightarrow \Delta$ is *deducible*.

Definition 3.1 (Deducibility (G3I$^{\text{M}}$, IG)). *In general, for an arbitrary* G3I$^{\text{M}}$/IG-*sequent* $\Gamma \Rightarrow \Delta$ *we write for some* $n \in \mathbb{N}$*:* $\Vdash_n \Gamma \Rightarrow \Delta$ *iff there is a closed* G3I$^{\text{M}}$/IG-*tree with* $\Gamma \Rightarrow \Delta$ *being the* root sequent*, which has a height* $h \leqslant n$*. We also say that* $\Gamma \Rightarrow \Delta$ *is* deducible *in* n deductive steps*, or there is a* derivation *of height* n *for* $\Gamma \Rightarrow \Delta$*.*

To make a distinction between deducibility in G3I$^{\text{M}}$ and IG, we write $\Vdash^{\mathsf{I}}_n$ for deducibility in G3I$^{\text{M}}$ and $\Vdash^{\mathsf{IG}}_n$ for that in IG. If it is not relevant whether G3I$^{\text{M}}$ or IG is used, we simply write $\Vdash_n$.

As mentioned in Chapter 2.1.3, when we talk about *critical rules* in G3I$^{\text{M}}$, we refer to the rules $\supset$r and $\neg$r which reduce the number of formulas on the right-hand side of the sequent arrow $\Rightarrow$. The other G3I$^{\text{M}}$-rules are called *non-critical rules*. Accordingly, *critical formulas* are implications or negations in intuitionistic logic. Rules ax and $\bot ax$ *close* the branches of a sequent tree and are therefore called *closing rules*. All other rules are *non-closing rules*.

Further, any sequent can be considered as a set of formula *occurrences*, i.e., formulas occur in sequents of a tree. Particularly, a formula φ occurs in a sequent $\Phi \Rightarrow \Psi$ in the left part (if $\varphi \in \Phi$) or in the right part ($\varphi \in \Psi$). We call the occurrence *left-occurrence / left-hand formula* or *right-occurrence / right-hand formula*, respectively.

For any sequent tree t, read from bottom to top, on every branch we can find a sequent s in which a formula φ occurs for the first time, e.g., as an *active formula*, but it can also be the root sequent of t. We say that φ is *introduced* in s.

Immediately after the application of a *critical rule*, *implications* and *negations* in the antecedent of the sequent are said to be *reinitialized* with the application of the critical rule.

If some formula φ is principal in some sequent of tree t then we can find the sequent in which φ is introduced or reinitialized. If φ is reinitialized in t below the rule application, then we call the occurrence of this reinitialization the *formula instance* $\hat{\varphi}$ of φ. Otherwise, the formula instance $\hat{\varphi}$ is the occurrence of the *introduced* φ. The formula instance always refers to a certain sequent in t, namely the sequent in which it is introduced or reinitialized. If a formula instance $\hat{\varphi}$ appears in a sequent $\Phi \Rightarrow \Psi$ such that $\hat{\varphi} \in \Phi$ then it is called a *left instance*. If $\hat{\varphi} \in \Psi$, it is a *right instance*.

Here is an example. Consider the following $G3\text{I}^{M}$-derivation-tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, C, C \supset D, E, D \Rightarrow D}\,ax \qquad \dfrac{}{A, B, C, C \supset D, E \Rightarrow D, C}\,ax}{A, B, C, C \supset D, E \Rightarrow D}\,{\supset}l}{A, B, C, C \supset D \Rightarrow E \supset D}\,{\supset}r}{A \wedge B, C, C \supset D \Rightarrow E \supset D}\,{\wedge}l$$

The first (lowest) rule application is $\wedge l$ with principal formula $A \wedge B$. This $A \wedge B$ is the *instance* to which the application refers, as $A \wedge B$ is introduced with the root sequent. The same holds for the application of $\supset r$: it refers to the instance $E \supset D$ of the *root sequent*, as it is introduced there. Now, as this is a critical rule, all implications of the antecedent are *reinitialized*. Therefore, the instance the application of $\supset$l refers to is not the implication $C \supset D$ of the root sequent, but instead the occurrence directly below the application, as $C \supset D$ was just reinitialized with $\supset$r. The application of *ax* in the right branch refers to two instances of C (one in the antecedent and one in the succedent). The left one is the instance of C that occurs in the root sequent. The right one is the instance which occurs directly below the application, as it is introduced there with $\supset$l.

In the following, when we talk about a *rule application*, we refer to a certain rule (name) of some sequent calculus and an *instance* of a formula. The sequent, in which the application occurs, is not necessarily the sequent in which the formula is introduced or reinitialized but it is possible to trace back the instance of the formula somewhere in the sequent tree. We consider rule applications as pairs $(r, \hat{\varphi})$ where r is a rule name and $\hat{\varphi}$ is called the *principal formula instance*. A rule application in some sequent tree either results in one or two sequents written above the sequent in which the rule is applied, or in an empty sequent. We usually refer to a rule application with $a_l = (r, \hat{\varphi})$ where l is some level of some branch in some tree.

The Normalization of G3i^m-Proofs

To prove *completeness* of DiaSeqI, we show that every proof in G3i^M can be transformed into an equivalent proof in DiaSeqI. As G3i^M is complete, this implies that DiaSeqI is complete as well. To achieve this translation, we first have to transform the proof in G3i^M into a *normal form* that represents a *scheduled* version of the proof, by moving rule applications from top to bottom.

Definition 3.2 (Rule Application Dependence [138]). *Given two applications* $a_l = (r_l, \hat{\varphi}_l)$ *and* $a_m = (r_m, \hat{\varphi_m})$ *of a derivation tree, we say that* a_l depends directly on a_m*, written* $a_l \ltimes a_m$*, iff both* r_l *and* r_m *are applied in the same path of the tree and* $\hat{\varphi}_l$ *is* introduced *with* a_m *or* reinitialized *with* a_m.

$$
\begin{array}{cl}
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A, C & ax \\
\hdashline
\neg A, B, \neg(A \supset C), A \Rightarrow C & \neg l \\
\hline
\neg A \wedge B, \neg(A \supset C), A \Rightarrow C & \wedge l \\
\hline\hline
\neg A \wedge B, \neg(A \supset C) \Rightarrow A \supset C & \supset r \\
\hline
\neg A \wedge B, \neg(A \supset C) \Rightarrow & \neg l \\
\hline\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \neg r \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \vee r \\
\hline\hline
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) & \supset r
\end{array}
$$

Figure 3.10: Macro and micro blocks in an $G3i^M$-proof (c.f. [139])

For applications a_l, a_m, *and* a_n, *we say that* a_l depends on a_n, *written* $a_l \propto a_n$, *iff* $a_l \ltimes a_n$, *or* $a_l \ltimes a_m$ *and* $a_m \propto a_n$, *so* $\propto$ *is defined as the* transitive closure *of* $\ltimes$. *If* a_l *does not depend on* a_m, *we say that* a_l *is* independent *of* a_m, *written* $a_l \not\propto a_m$.

Definition 3.3 (Macro Block)**.**

- *A macro block is a path of a sequent tree which contains only non-critical rule applications.*
- *A macro-block is* closed *iff a closing rule is applied in it.*
- *The* macro block height *(*MBH*) of a derivation tree is the maximal number of macro blocks from its root to its leaves.*

So we can take any $G3i^M/IG^r$-sequent-tree and divide its paths into several sections, called *macro blocks*, each of these macro blocks is separated by the application of a critical rule, namely $\supset r$ or $\neg r$.[6] To indicate a certain macro block of a path in a sequent tree, we refer to its level in the path, the *macro level*. The lowest macro block has level 0.

[6]The concept of macro blocks has similarities to *layers* defined by Egly and Schmitt [44]. However, they follow another purpose and are defined in a different way: layers contain whole sub-deductions and therefore several branches at once. Further, a layer requires that sequents having at most one formula in the succedent may only occur in the leaves of the layer, while the remaining sequents contain more formulas in the succedent.

A macro block can be empty, i.e., contain no rule application. This happens for example, if two critical rules are applied consecutively. Figure 3.10 shows how a G3i^M-proof-tree is divided into macro blocks. The double lines indicate the borders of the four blocks (we ignore the dashed line for now). The applications of the critical rules are not part of the macro blocks, whereas the axiom rule is part of the uppermost one.

Now it is possible to create so-called *micro blocks* within the macro blocks of the sequent tree.

Definition 3.4 (Micro Block)**.**

- *A* micro block *is a section of a macro block that contains at least one rule application, but all rule applications are non-critical and independent of the other rule applications of the micro block. The first rule application that depends on the application of another rule in some micro block is the start of a new one.*
- *A micro-block is* closed *iff a closing rule is applied in it.*
- *The* micro block height *(mbh) of a derivation tree is the maximal number of micro blocks from its root to its leaves.*

A macro block consists of zero or more micro blocks, each consisting of non-critical rule applications. The lowest micro block of a macro block starts with its lowest rule application and ends when the macro block ends or when a rule is applied that depends on any of the rule applications in the micro block.

When we consider again the proof tree of Figure 3.10, we see that the lowest macro block contains no micro block, as there is no non-critical rule application. The second and the third one each contains one micro block. The top macro block consists of two micro blocks (the start of the second one is indicated by a dashed line), as the application of $\neg$l depends on that of $\wedge$l.

Now we have a structure we will make use of to obtain the desired form of the proof tree that corresponds to the structure of a dialogue. When a macro block ends, the proponents react to a critical attack. The micro blocks are to be used to simulate the round structure of the game.

In $G3I^{M}$ and IG^{r}, the contraction and weakening rules are missing. However, both are still *admissible* as we show next (we first look at weakening, contraction is considered a bit later).

Lemma 3.1 (Admissibility of Weakening ($G3I^{M}$, IG^{r})[7]). *For all* φ, Γ, Δ*:*

1. *If* $\Vdash_n \Gamma \Rightarrow \Delta$ *then* $\Vdash_n \Gamma, \varphi \Rightarrow \Delta$.
2. *If* $\Vdash_n \Gamma \Rightarrow \Delta$ *then* $\Vdash_n \Gamma \Rightarrow \varphi, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of rule applications is preserved.

Proof by induction on n. Assume that $\Vdash_n \Gamma \Rightarrow \Delta$.

Base Case: $n = 1$ — The only applied rule must be ax or $\bot ax$. So the same formula appears in both sequents or $\bot$ is an element of Γ. In both cases, adding a formula φ to one of the sides does not change this fact and no rule application is added.

Inductive Step: Assume that $\Vdash_{n+1} \Gamma \Rightarrow \Delta$. We have to consider all kinds of rule applications in the root sequent of the derivation. We look at two examples.

- Assume we have the root sequent $s = A \supset B, \Gamma \Rightarrow \Delta$. Then we have some derivation tree t of height $n + 1$:

$$\dfrac{\dfrac{— t_1 —}{\Gamma, A \supset B \Rightarrow A, \Delta} \qquad \dfrac{— t_2 —}{\Gamma, B \Rightarrow \Delta}}{\Gamma, A \supset B \Rightarrow \Delta} \supset l$$

  We can now weaken the sequents above s by hypotheses, so $\Vdash_n \Gamma, A \supset B, \varphi \Rightarrow A, \Delta$ and $\Vdash_n \Gamma, B, \varphi \Rightarrow \Delta$. We can also add the φ to

[7] The lemma is adapted from [39] and [142].

the succedents, the argument is almost the same. The $\supset l$-rule can then be applied as well, by adding the φ to the antecedent (or succedent respectively) of the root sequent.
Therefore, we have $\Vdash_{n+1} \Gamma, A \supset B, \varphi \Rightarrow \Delta$. As no extra rule applications are necessary in t_1 or t_2, there is also no extra rule necessary in the whole derivation.

- Assume we have $\Vdash_{n+1} \Gamma \Rightarrow A \supset B, \Delta$ and the application of a critical rule in the root sequent:

$$\dfrac{\dfrac{— t' —}{\Gamma, A \Rightarrow B}}{\Gamma \Rightarrow A \supset B, \Delta} \supset r$$

  Adding a φ to the succedent of the root sequent does not change anything, as it is removed again with the application of $\supset r$. Therefore, $\Vdash_{n+1} \Gamma \Rightarrow A \supset B, \varphi, \Delta$. When we add φ to the antecedent, we have to apply the hypothesis again on the sequent above.

□

Lemma 3.2 (Inversion (G3I^{M})[8]). *For all* Δ, Γ *and formulas* A *and* B*:*

1. *If* $\Vdash^!_n \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\Vdash^!_n \Gamma, A, B \Rightarrow \Delta$.
2. *If* $\Vdash^!_n \Gamma \Rightarrow A \vee B, \Delta$ *then* $\Vdash^!_n \Gamma \Rightarrow A, B, \Delta$.
3. *If* $\Vdash^!_n \Gamma, A \vee B \Rightarrow \Delta$ *then* $\Vdash^!_n \Gamma, A \Rightarrow \Delta$ *and* $\Vdash^!_n \Gamma, B \Rightarrow \Delta$.
4. *If* $\Vdash^!_n \Gamma \Rightarrow A \wedge B, \Delta$ *then* $\Vdash^!_n \Gamma \Rightarrow A, \Delta$ *and* $\Vdash^!_n \Gamma \Rightarrow B, \Delta$.
5. *If* $\Vdash^!_n \Gamma, A \supset B \Rightarrow \Delta$ *then* $\Vdash^!_n \Gamma, A \supset B \Rightarrow A, \Delta$ *and* $\Vdash^!_n \Gamma, B \Rightarrow \Delta$.
6. *If* $\Vdash^!_n \Gamma, \neg A \Rightarrow \Delta$ *then* $\Vdash^!_n \Gamma, \neg A \Rightarrow A, \Delta$.

[8] The lemma and the proof are adapted from [142] and extended; see also Lemma 3.14 by Dragalin [39].

The derivations in the conclusions do not contain further rule applications in all paths than the derivations in the conditions. The order of rule applications is preserved.

Proof by induction on n. We consider one of the cases (5). The others are treated similarly.

Base Case: $n = 1$
Consider a sequent $s = \Gamma, A \supset B \Rightarrow \Delta$. A closing rule must be applied on this. As closing rules are only applied on atoms, we conclude that either the same prime formula appears in Γ and Δ, or a $\bot$ occurs in Γ. In both cases, the same closing rule can be applied in both sequents $\Gamma, A \supset B \Rightarrow A, \Delta$ and $\Gamma, B \Rightarrow \Delta$. No further applications are necessary.

Inductive Step: For a derivation tree t of height $n + 1$, we consider the root sequent $s = \Gamma, A \supset B \Rightarrow \Delta$. If a rule is applied on $A \supset B$ in s, we have:

$$\dfrac{\dfrac{— t_1 —}{\Gamma, A \supset B \Rightarrow A, \Delta} \quad \dfrac{— t_2 —}{\Gamma, B \Rightarrow \Delta}}{\Gamma, A \supset B \Rightarrow \Delta} \supset l$$

There are derivations with a maximal height of n for both sequents $\Gamma, A \supset B \Rightarrow A, \Delta$ and $\Gamma, B \Rightarrow \Delta$. No rule applications are added to t_1 or t_2.

However, if $A \supset B$ is not the principal formula in s, we must have a closer look and perform a case analysis on the rule that is applied there. We consider two of them. The others are similar:

- $\wedge r$: We have some $C \wedge D$ in the succedent:

  $$\dfrac{\dfrac{— t_1 —}{\Gamma, A \supset B \Rightarrow C, \Delta} \quad \dfrac{— t_2 —}{\Gamma, A \supset B \Rightarrow D, \Delta}}{\Gamma, A \supset B \Rightarrow C \wedge D, \Delta} \wedge r$$

  The sequents starting directly above the root sequent have a maximal height of n, so by hypothesis we have

- $\Vdash^{I}_{n} \Gamma, A \supset B \Rightarrow A, C, \Delta$
- $\Vdash^{I}_{n} \Gamma, B \Rightarrow C, \Delta$
- $\Vdash^{I}_{n} \Gamma, A \supset B \Rightarrow A, D, \Delta$
- $\Vdash^{I}_{n} \Gamma, B \Rightarrow D, \Delta$

without adding extra rule applications in t_1 or t_2. This is quite fine, because from the first and the third derivation we can conclude $\Vdash^{I}_{n+1} \Gamma, A \supset B \Rightarrow C \wedge D, A, \Delta$ and from the second and forth $\Vdash^{I}_{n+1} \Gamma, B \Rightarrow C \wedge D, \Delta$, simply by using rule $\wedge r$ on $C \wedge D$ each time. As this $\wedge r$ had been removed before we applied the hypothesis, we can now add it again and we conclude that no extra rule applications are necessary for the new derivations.

- $\supset r$: a critical rule has to be treated a little bit differently.

$$\dfrac{\dfrac{— t_1 —}{\Gamma, A \supset B, C \Rightarrow D}}{\Gamma, A \supset B \Rightarrow C \supset D, \Delta} \supset r$$

We have to show that $\Vdash^{I}_{n+1} \Gamma, A \supset B \Rightarrow A, C \supset D, \Delta$ and $\Vdash^{I}_{n+1} \Gamma, B \Rightarrow C \supset D, \Delta$. The second statement can again be shown by the hypothesis applied on the premise. Then we get $\Vdash^{I}_{n} \Gamma, B, C \Rightarrow D$, and with rule $\supset r$ we have $\Vdash^{I}_{n+1} \Gamma, B \Rightarrow C \supset D, \Delta$. No extra rule applications are necessary.

The other case is even easier, as from $\Vdash^{I}_{n+1} \Gamma, A \supset B \Rightarrow C \supset D, \Delta$ we get directly $\Vdash^{I}_{n+1} \Gamma, A \supset B \Rightarrow A, C \supset D, \Delta$ by *weakening* (without adding more rule applications, Lemma 3.1).

□

Lemma 3.3 (Admissibility of Contraction ($G3\text{I}^M$)[9]). *For all φ, Γ, Δ:*

1. *If* $\Vdash^{\mathsf{I}}_{n} \Gamma, \varphi, \varphi \Rightarrow \Delta$ *then* $\Vdash^{\mathsf{I}}_{n} \Gamma, \varphi \Rightarrow \Delta$.
2. *If* $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow \varphi, \varphi, \Delta$ *then* $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow \varphi, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of rule applications is preserved.

Proof by induction on n.

Base Case: $n = 1$ — A closing rule must be used. If it is applied on φ, then this can also be done if it occurs only once in the left or right part. Otherwise, the closing is independent of φ and can therefore be achieved directly.

Inductive Step: We consider a $G3\text{I}^M$-tree of height n with root sequent $s = \Gamma, \varphi, \varphi \Rightarrow \Delta$ or $s' = \Gamma \Rightarrow \varphi, \varphi, \Delta$, respectively. If **no** rule is applied on φ in the root sequent, we apply the hypothesis on the tree(s) starting above the root sequent.

We have a detailed look on three cases for φ. The others can be dealt with in a similar way.

- $\varphi = A \wedge B$, $\Vdash^{\mathsf{I}}_{n+1} \Gamma, A \wedge B, A \wedge B \Rightarrow \Delta$
  As a rule is applied on φ, we have the sequent $\Gamma, A, B, A \wedge B \Rightarrow \Delta$ above for which there must be a derivation with a maximal height of n. By *inversion* (Lemma 3.2), there is also a $G3\text{I}^M$-tree for $\Gamma, A, B, A, B \Rightarrow \Delta$ with a maximal height of n, without further rule applications. We use the hypothesis twice to show that $\Vdash^{\mathsf{I}}_{n} \Gamma, A, B \Rightarrow \Delta$. Therefore we have $\Vdash^{\mathsf{I}}_{n+1} \Gamma, A \wedge B \Rightarrow \Delta$ after adding the omitted application of $\wedge l$ again. No extra rule is applied.

[9]The lemma and proof are again adapted from Troelstra and Schwichtenberg [142] and extended, *dp-admissibility of contraction*; see also Lemma 3.1.5 by Dragalin [39].

- $\varphi = A \supset B, \quad \Vdash^{I}_{n+1} \Gamma, A \supset B, A \supset B \Rightarrow \Delta$
  Above the root sequent we have the sequents
  $s_1 = \Gamma, A \supset B, A \supset B \Rightarrow A, \Delta$ and $s_2 = \Gamma, B, A \supset B \Rightarrow \Delta$.

  The case of s_1 is easy, because it has a derivation height of n, and we can therefore apply the hypothesis directly on $A \supset B$, which results in $\Vdash^{I}_{n} \Gamma, A \supset B \Rightarrow A, \Delta$ without using further rule applications.

  For s_2 we use the inversion again (Lemma 3.2) to derive that $\Vdash^{I}_{n} \Gamma, B, B, \Rightarrow \Delta$. By hypothesis applied on B we then also have $\Vdash^{I}_{n} \Gamma, B \Rightarrow \Delta$. With rule $\supset l$ we combine both to $\Vdash^{I}_{n+1} \Gamma, A \supset B \Rightarrow \Delta$.

- $\varphi = A \supset B, \quad \Vdash^{I}_{n+1} \Gamma \Rightarrow A \supset B, A \supset B, \Delta$
  Here, we consider the application of the *critical rule* $\supset r$ on one $A \supset B$. In the premise, we have sequent s'_1 such that $s'_1 = \Gamma, A \Rightarrow B$. The other implication vanishes and it is irrelevant, because there must be a derivation for s'_1 in n steps. Therefore $\Vdash^{I}_{n+1} \Gamma \Rightarrow A \supset B, \Delta$ by rule $\supset r$.

□

There are now special cases for which we make further observations. First of all, for any propositional formula φ, we introduce a multi-set Υ_φ called *disjunctive derivable set of* φ:

Definition 3.5 (Disjunctive Derivable Set)**.** *Let* φ *be a propositional formula. Then the multi-set* Υ *is a* disjunctive derivable set of φ, *written* $d^{\vee}(\Upsilon, \varphi)$, *iff for any formulas* A *and* B*:*

- $\Upsilon = \{\varphi\}$, *or*
- $\varphi = A \wedge B$ *and* $d^{\vee}(\Upsilon, A)$ *or* $d^{\vee}(\Upsilon, B)$, *or*
- $\varphi = A \vee B$ *and* $\Upsilon = \Upsilon_A, \Upsilon_B$ *such that* $d^{\vee}(\Upsilon_A, A)$ *and* $d^{\vee}(\Upsilon_B, B)$.

Lemma 3.4 (Generalized Right-Inversion)**.** *For any* Γ, Δ, A, *and* n*:*
if $\Vdash^{I}_{n} \Gamma \Rightarrow A, \Delta$ *then for all* Υ_A *such that* $d^{\vee}(\Upsilon_A, A)$, *we have* $\Vdash^{I}_{n} \Gamma \Rightarrow \Upsilon_A, \Delta$

without using further rule applications than for the derivations in the premises. The order of rule applications is preserved.

Proof. Assume $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow A, \Delta$. By structural induction on A:

Base Case: A is an atom. Then $\Upsilon_A = \{A\}$. This case is trivial.

Inductive Step: For any formulas A_0 and A_1:

- $A = A_0 \vee A_1$. Then there are two possibilities for Υ_A:
  - $\Upsilon_A = \{A\}$: trivial.
  - $\Upsilon_A = \Upsilon_{A_0}, \Upsilon_{A_1}$ with $d^{\vee}(\Upsilon_{A_0}, A_0)$ and $d^{\vee}(\Upsilon_{A_1}, A_1)$:
    By inversion (Lemma 3.2), $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow A_0, A_1, \Delta$ without further rule applications. Then, by hypothesis, $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow \Upsilon_{A_0}, \Upsilon_{A_1}, \Delta$.
- $A = A_0 \wedge A_1$. Then there are three possibilities for Υ_A:
  - $\Upsilon_A = \{A\}$: trivial.
  - $\Upsilon_A = \Upsilon_{A_0}$ for $d^{\vee}(\Upsilon_{A_0}, A_0)$:
    By inversion (Lemma 3.2), $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow A_0, \Delta$.
    By hypothesis, $\Vdash^{\mathsf{I}}_{n} \Gamma \Rightarrow \Upsilon_{A_0}, \Delta$.
  - $\Upsilon_A = \Upsilon_{A_1}$ for $d^{\vee}(\Upsilon_{A_1}, A_1)$: same procedure.
- For any other A we have $\Upsilon_A = \{A\}$, so we already finished.

In none of these cases, additional rule applications are necessary. □

Lemma 3.5 (Dispensability of Duplication Rules (G3I$^{\mathrm{M}}$))**.** *For all* Γ, Δ, A, n, *and all* Υ_A *such that* $d^{\vee}(\Upsilon_A, A)$*:*

- *if* $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Upsilon_A, \Delta$, *then* $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Upsilon_A, \Delta$ *without any rule applications of* $\supset$l *on* $A \supset B$ *in the macro blocks of level 0.*

- *if* $\Vdash^{\mathsf{I}}_{n} \Gamma, \neg A \Rightarrow \Upsilon_A, \Delta$*, then* $\Vdash^{\mathsf{I}}_{n} \Gamma, \neg A \Rightarrow \Upsilon_A, \Delta$ *without any rule applications of* $\neg\mathsf{l}$ *on* $\neg A$ *in the macro blocks of level 0.*

For the transformation, no further rule applications are necessary in any of the paths. The order of rule applications is preserved.

Proof by induction on n*.*

Base Case: $n = 1$ — There is only one closing rule, so the rule application is not necessary.

Inductive Step: We assume $\Vdash^{\mathsf{I}}_{n+1} \Gamma, A \supset B \Rightarrow \Upsilon_A, \Delta$. The case for the negation is very similar, so it is omitted here. We consider the sequent $\Gamma, A \supset B \Rightarrow \Upsilon_A, \Delta$ to be the root of some derivation tree t'.

- If the lowest application is critical, it is automatically the end of the macro block and no transformation is necessary as the target requirement is already fulfilled.

- Else, if neither $A \supset B$, nor a formula of Υ_A are the principals in this sequent, we simply apply the hypothesis on the tree(s) starting above and we are done.

- If a non-critical formula γ of Υ_A is the principal in this sequent, we have $\Vdash^{\mathsf{I}}_{n+1} \Gamma, A \supset B \Rightarrow \gamma, \Upsilon'_A, \Delta$ with $\Upsilon_A = \gamma, \Upsilon'_A$.
  Then, due to the application on γ, we have active formulas Δ_γ in the premise, i.e., $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Delta_\gamma, \Upsilon'_A, \Delta$.
  Then by weakening (Lemma 3.1) $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Delta_\gamma, \gamma, \Upsilon'_A, \Delta$ which is the same as $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Delta_\gamma, \Upsilon_A, \Delta$. Here we can apply the induction hypothesis and everything is fine.

- If $A \supset B$ is the principal, then we have $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow A, \Upsilon_A, \Delta$. By Lemma 3.4, we obtain $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Upsilon_A, \Upsilon_A, \Delta$. Then contraction (Lemma 3.3) can be applied on all formulas of Υ_A, so we get $\Vdash^{\mathsf{I}}_{n} \Gamma, A \supset B \Rightarrow \Upsilon_A, \Delta$. Finally, we apply the hypothesis and realize that this derivation is possible without the application of $\supset l$ on $A \supset B$ in the macro blocks of level 0.

□

Let us consider a $G3I^M$-sequent-tree t that consists of some macro blocks. Assume that a rule is applied in the same macro block of t on the same formula instance more than once. The first of these applications might be useful, but the others are *redundant* and can be omitted. To prove this, we first show that the macro blocks of level 0 in every derivation can be made *redundance-free*.

Lemma 3.6 (Rule Application Redundancy in Root Macro Blocks ($G3I^M$)). *Let* t *be a valid* $G3I^M$*-proof-tree. Then* t *can be transformed to a closed* $G3I^M$*-proof-tree* t' *such that for all macro blocks* M *of level* 0 *in* t', M *is redundance-free. No further rule applications are needed and the application orders are preserved.*

Proof. Normally, we would have to consider all non-critical rules that can be applied in a macro block. But in fact, when $\wedge l$, $\vee l$, $\wedge r$ or $\vee r$ is applied on some formula, then this principal vanishes from the premises. In case it appears again later, the application on this refers to another formula instance. The closing rules ax and $\bot ax$ are obviously also not interesting for this proof, so everything left are the *duplicating rules* $\supset l$ and $\neg l$. The latter can be seen as a special case of the former one. The arguments are very similar and we concentrate on $\supset l$.

Consider the derivation $\Vdash^{I}_{n} \Gamma \Rightarrow \Delta$ of t. We prove the lemma by induction on n.

Base Case: $n = 1$ — Only a closing rule is applied, so there is nothing to do (no redundancy).

Inductive Step: We want to show that $\Vdash^{I}_{n+1} \Gamma \Rightarrow \Delta$ can be made redundance-free in macro-level 0. Let $\Gamma^R \subseteq \Gamma$ be the set of formulas on which we have redundant rule applications in any macro block of level 0 in t.

- If the lowest rule application in t is a critical, nothing needs to be done, as the lowest macro block is already redundance-free.
- If it is non-critical and done on a formula $\psi \in \Gamma \setminus \Gamma^R$ or $\psi \in \Delta$, we can simply apply the hypothesis on the sequents above and everything is fine (we do not leave the macro block).
- Otherwise, the rule is applied on a formula $\gamma \in \Gamma^R$. We assume that γ is an implication $A \supset B$ (as stated before, negations are handled similarly). So for $\Gamma' = \Gamma \setminus \{\gamma\}$: $\Vdash^{\mathsf{I}}_n \Gamma', A \supset B \Rightarrow A, \Delta$ and $\Vdash^{\mathsf{I}}_n \Gamma', B \Rightarrow \Delta$. Both derivations can be made redundance-free in macro-level 0 by hypothesis, so the latter is no problem. Concerning the former one, we conclude that $A \supset B$ is the only possible formula which can cause a redundancy in macro-level 0. Further, as $d^{\vee}(\{A\}, A)$, we can use Lemma 3.5 and enforce that there is no application of $\supset l$ on this $A \supset B$ in all macro blocks of level 0. As no extra rule is needed to be applied there, this will also cause no extra $\supset l$-applications. So $\Vdash^{\mathsf{I}}_{n+1} \Gamma, A \supset B \Rightarrow \Delta$ can be built without redundant rule applications on $A \supset B$.

□

Lemma 3.7 (Rule Application Redundancy in G3i^M-Trees)**.** *Let* t *be a valid* G3i^M*-proof-tree. Then* t *can be transformed to a closed* G3i^M*-proof-tree* t' *such that all macro blocks of* t' *are redundance-free.*

Proof. This can now be done quite easily with an induction on the macro block height (*MBH*) h of t:

Base Case: $h = 1$ — We simply apply Lemma 3.6 and are done.

Inductive Step: Again, we apply Lemma 3.6 to make all macro blocks of macro level 0 redundance-free and obtain t'. Then we apply the hypothesis on all trees that start in macro-level 1 of t' and everything is done. □

Definition 3.6 (Macro Block Saturation [139])**.** *Let* M *be a macro block of an* $G3I^M$*-proof-tree* t*. Then* M *is* macro-saturated *iff rules are applied on all formula instances of* M *which are neither critical nor atomic.*

What we want to achieve is that in every macro block we apply as many non-critical rules as possible; duplication rules are of course applied only once on the same principal instance. For the dialogue this means that the proponents postpone reactions to critical attacks as long as possible.

Lemma 3.8 (Macro Block Saturation in $G3I^M$-Proof-Trees [139])**.** *Let* t *be a* $G3I^M$*-proof-tree. Then* t *can be transformed to be* macro-saturated *in such a way that it is still a valid proof tree and without increasing its macro block height (MBH).*

Proof. By induction on the MBH h of t:

Base Case: $h = 1$ — All macro blocks are closed.

We consider each sequent $\Gamma_i \Rightarrow \Delta_i$ in which a closing rule is applied. Then all non-atomic and non-critical formulas φ are collected from these and for each we perform an *induction on the structure* of this φ to show that it can be saturated until its atoms are reached or critical rules are needed to go further. We define $\Gamma_i' =_{df} \Gamma_i \setminus \{\varphi\}$. As an example we let φ be $A \wedge B$. Then $\Vdash^{!}_{n} \Gamma_i', A \wedge B \Rightarrow \Delta_i$. By inversion (Lemma 3.2), $\Vdash^{!}_{n} \Gamma_i', A, B \Rightarrow \Delta_i$, i.e., adding an application of $\wedge l$ on $A \wedge B$ does not harm the proof. The (inner) induction hypothesis can be applied on A and B to saturate the premise sequent again until we reach only atoms or critical formulas. Once this is done, we continue with the next φ which is not saturated in the leaf.

Inductive Step: Increasing the *MBH* means that there is at least one critical rule application in at least one path of t. In general, t has the following structure [139]:

$$
\dfrac{
\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ & — t_1 — & \end{array}}{\Gamma_1 \Rightarrow \Delta_1}\, c_1
\quad
\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ & — t_2 — & \end{array}}{\Gamma_2 \Rightarrow \Delta_2}\, c_2
\quad \cdots \quad
\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ & — t_n — & \end{array}}{\Gamma_n \Rightarrow \Delta_n}\, c_n
\\
\vdots\ \vdots \qquad\qquad \vdots\ \vdots\ \vdots \qquad\qquad \vdots\ \vdots\ \vdots
}{\Gamma_R \Rightarrow \Delta_R}
$$

The sequents $\Gamma_i \Rightarrow \Delta_i$ $(1 \leqslant i \leqslant n)$ are the upper borders of the macro blocks starting in the root sequent $\Gamma_R \Rightarrow \Delta_R$, i.e., the macro blocks of macro-level 0. The critical (or closing) rules applied there are labelled with c_i. We can use the hypothesis to saturate t_1 to t_n.

For each of the border sequents $\Gamma_i \Rightarrow \Delta_i$, we now do the same as in the root case: we collect the non-critical non-atomic formulas γ of $\Gamma_i \Rightarrow \Delta_i$ and perform a structural induction on each γ. We make use of the inversion property which adds a non-critical rule application to the border sequent. The rest is done by the (internal) induction hypothesis which adds further non-critical and non-closing applications as long as this is possible. Note that by the inversion lemma (Lemma 3.2) no critical rules are added and the existing ones stay on their paths. So the *MBH* remains the same.

Once we saturated one branch starting in $\Gamma_R \Rightarrow \Delta_R$, we take the next border sequent and so on, until all border sequents are saturated. Then we apply the (outer) hypothesis to update the macro saturation in all macro blocks starting above. □

Figure 3.11 shows how the macro blocks of the derivation we saw before are saturated. The underlined rule names indicate the applications that are added. Now as the macro blocks are saturated, our next goal is to move the rule applications to obtain the round structure of an MPID.

A *rule swap* is the exchange of two rule applications at two succeeding positions of the sequent tree. A *rule movement* consists of multiple swaps which means that the application is moved step-wise towards the root of the tree. Note that a rule movement possibly causes a modification of other paths.

$$
\begin{array}{cl}
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A, C & ax \\
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow C & \neg l \\
\hline
\neg A \wedge B, \neg(A \supset C), A \Rightarrow C & \wedge l \\
\hline\hline
\neg A \wedge B, \neg(A \supset C) \Rightarrow A \supset C & \supset r \\
\hline
\neg A \wedge B, \neg(A \supset C) \Rightarrow & \neg l \\
\hline\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \neg r \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \vee r \\
\hline\hline
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) & \supset r
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{cl}
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A \supset C, A, C & ax \\
\hline
\neg A, C, \neg(A \supset C), A \Rightarrow A, C & \underline{\neg l} \\
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow C & \neg l \\
\hline\hline
\neg A, B, \neg(A \supset C) \Rightarrow A, A \supset C & \supset r \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow A \supset C & \underline{\neg l} \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow & \neg l \\
\hline\hline
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C, D & \neg r \\
\hline
\neg A, B \Rightarrow \neg\neg(A \supset C), C, D & \underline{\neg l} \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C, D & \underline{\wedge l} \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \underline{\vee r} \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \vee r \\
\hline\hline
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) & \supset r
\end{array}
$$

Figure 3.11: Example of a macro saturation (c.f. [139])

Lemma 3.9 (Independent Rule Movement in Macro-Saturated G3I^{M}-trees [139]). *Let* t *be a* macro-saturated *and* redundance-free *sequent tree with an MBH of* 1, *and* $a_l = (r, \hat{\varphi})$ *a rule application at level* l *in* t. *Suppose that for all rule applications* a_m *with* $0 \leqslant m < l$, a_l *does not depend on* a_m, *i.e.,* $a_l \not\prec a_m$. *Then* t *can be transformed to* t' *for which*

- a_l *is moved downwards to the root, and*
- *the root sequent and the leaf sequents are the same as in* t, *and*
- *no rule application in* t *is added and none is removed, and*
- *the mbh and the MBH are the same as in* t, *and*
- t' *is also macro-saturated and redundance-free.*

Proof. By induction on the total number of rule applications n in t:

Base Case: For $n = 1$ the movement is trivial, as there is no movement.

Inductive Step: We have to consider all combinations of non-closing, non-critical rules and show that it is possible to move down the corresponding rule application to the root. The relevant rules are $\vee r$, $\wedge r$, $\vee l$, $\wedge l$, $\supset l$ and $\neg l$. We discuss three cases here, the others are very similar.

1. We combine $\vee r$ with $\vee r$ (this is one of the very simple cases). Consider the following sequent tree, with r applied in some sequent of level l:

$$
\dfrac{\dfrac{\dfrac{\dots}{\Gamma_{l+1} \Rightarrow C, D, \Delta_{l+1}}}{\Gamma_l \Rightarrow C \vee D, \Delta_l}\vee r \qquad \dots \qquad \dots}{\dfrac{\dfrac{- u -}{\Gamma_1 \Rightarrow A, B, C \vee D, \Delta_1}}{\Gamma_0 \Rightarrow A \vee B, C \vee D, \Delta_0}\vee r}
$$

   The letter u stands for a section of t. The indexes of Γ and Δ indicate the level in t. Obviously, $\Gamma_0 = \Gamma_1$, $\Delta_0 = \Delta_1$, $\Gamma_l = \Gamma_{l+1}$ and $\Delta_l = \Delta_{l+1}$. Due to the rule application independence of $\vee r$ in level l, $C \vee D$ must also be part of the sequents below level l.

Let us omit the root sequent of t for a moment. This tree is still macro-saturated. So by hypothesis, we can pull down this rule application to the root, modifying u in between to u'. After adding the original root of t afterwards, we obtain the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\ldots \quad \ldots \quad \ldots}{— u' —}}{\Gamma_2 \Rightarrow A, B, C, D, \Delta_2}}{\Gamma_1 \Rightarrow A, B, C \vee D, \Delta_1}\vee r}{\Gamma_0 \Rightarrow A \vee B, C \vee D, \Delta_0}\vee r$$

Here, $\Gamma_2 = \Gamma_1 = \Gamma_0$ and $\Delta_2 = \Delta_1 = \Delta_0$. Neither the leaf sequents nor the *mbh* or *MBH* have changed. Now we can simply exchange the two rule applications on the bottom and the tree is still valid:

$$\dfrac{\dfrac{\dfrac{\dfrac{\ldots \quad \ldots \quad \ldots}{— u' —}}{\Gamma_2 \Rightarrow A, B, C, D, \Delta_2}}{\Gamma_1 \Rightarrow A \vee B, C, D, \Delta_1}\vee r}{\Gamma_0 \Rightarrow A \vee B, C \vee D, \Delta_0}\vee r$$

Everything above level 2 is not touched and therefore the leaves are not changed. The *MBH* remains one, as no critical rule is added. Also the *mbh* does not change by swapping two independent rule applications. Macro-saturation is therefore also preserved.

2. And now for something more complicated: we combine the rules $\vee r$ and $\vee l$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots}{\Gamma_{l+1} \Rightarrow C, D, \Delta_{l+1}}}{\Gamma_l \Rightarrow C \vee D, \Delta_l}\vee r \quad \ldots \quad \ldots}{— u —}}{\Gamma_1, A \Rightarrow C \vee D, \Delta_1} \qquad \dfrac{— t^{\#} —}{\Gamma_1, B \Rightarrow C \vee D, \Delta_1}?}{\Gamma_0, A \vee B \Rightarrow C \vee D, \Delta_0}\vee l$$

The tree $t^{\#}$, which is a result of an arbitrary rule application (indicated as '?'), is unknown. Again, $\Gamma_0 = \Gamma_1$, $\Delta_0 = \Delta_1$, $\Gamma_l = \Gamma_{l+1}$, and $\Delta_l = \Delta_{l+1}$.

Because of the rule application independence, $C \vee D$ of level l must also be present in all levels below. When we omit the root sequent of t we obtain two independent sequent trees, t_1 and t_2, which are both macro-saturated. And because they are macro-saturated, there must be an application of $\vee r$ on $C \vee D$ in $t^{\#}$ (which is embedded in t_2). Therefore, we can apply the hypothesis both on t_1 and t_2, which moves the $\vee r$ down to the root respectively. Afterwards, we can put the resulting t_1' and t_2' together by adding the sequent $\Gamma_0, A \vee B \Rightarrow C \vee D, \Delta_0$; again with the application we omitted before. We obtain a tree which looks thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots \quad \dots \quad \dots}{-\, u' \,-}}{\Gamma_2, A \Rightarrow C, D, \Delta_2}}{\Gamma_1, A \Rightarrow C \vee D, \Delta_1}\vee r \qquad \dfrac{\dfrac{-\, t^{\#'} \,-}{\Gamma_2, B \Rightarrow C, D, \Delta_2}}{\Gamma_1, B \Rightarrow C \vee D, \Delta_1}\vee r}{\Gamma_0, A \vee B \Rightarrow C \vee D, \Delta_0}\vee l$$

All displayed Γ's and Δ's are equal. The desired properties are preserved (hypothesis). Now we simply swap the applications of $\vee r$ with the one of the root ($\vee l$) and we get this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots \quad \dots \quad \dots}{-\, u' \,-}}{\Gamma_2, A \Rightarrow C, D, \Delta_2} \qquad \dfrac{-\, t^{\#'} \,-}{\Gamma_2, B \Rightarrow C, D, \Delta_2}}{\Gamma_1, A \vee B \Rightarrow C, D, \Delta_1}\vee l}{\Gamma_0, A \vee B \Rightarrow C \vee D, \Delta_0}\vee r$$

The sub-trees starting in level 2 are not changed and therefore the leaves are not modified. The macro saturation and the block heights also remain the same, as only two rules in the lowest micro block are exchanged independently.

3. Now the last case we discuss here: the combination of $\supset l$ and $\supset l$. The procedure is almost the same: first we omit the root sequent $\Gamma_0, A \supset B, C \supset D \Rightarrow \Delta_0$ of t, and obtain two different sequent trees t_1 and t_2. In both, there occurs an application of $\supset l$ on $C \supset D$:

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma_{l+1}, A \supset B, C \supset D \Rightarrow C, \Delta_{l+1}} \qquad \dfrac{\dots}{\Gamma_{l+1}, A \supset B, D \Rightarrow \Delta_{l+1}}}{\Gamma_l, A \supset B, C \supset D \Rightarrow \Delta_l}\supset l \qquad \dots}{\dfrac{-\, u_1 \,-}{\Gamma_1, A \supset B, C \supset D \Rightarrow A, \Delta_1}} \qquad t_1$$

$$t_2 \qquad \cfrac{\cfrac{\cfrac{\dots}{\Gamma_{m+1}, C \supset D \Rightarrow C, \Delta_{m+1}} \quad \cfrac{\dots}{\Gamma_{m+1}, D \Rightarrow \Delta_{m+1}}}{\Gamma_m, C \supset D \Rightarrow \Delta_m}\supset l \quad \dots}{\cfrac{\text{— } u_2 \text{ —}}{\Gamma_1, C \supset D, B \Rightarrow \Delta_1}}$$

The tree t_2 is no problem: as t is macro-saturated, t_2 must also be macro-saturated, so we can apply the hypothesis to pull the application of $\supset l$ on $C \supset D$ down to its root.

However, as t was redundance-free, we will not find an application on $A \supset B$ in t_1, which means that it is *not* macro-saturated. Nevertheless, because t_1 is redundance-free and there are no critical rule applications in t_1 (the *MBH* is 1), we can modify t_1 *temporarily*. We simply drop $A \supset B$ from all antecedents Γ_i of t_1. We call the result t_1^-. This again must be macro-saturated and redundance-free, so we can apply the hypothesis to pull down the application of $\supset l$ on $C \supset D$ to the root. Once this is done, we add $A \supset B$ to all antecedents of the resulting $t_1^{-\prime}$ and obtain t_1'.

Now we put t_1' and t_2' on the root sequent of t:

$$\cfrac{\cfrac{\cfrac{\text{— } u_1' \text{ —}}{\Gamma_2, A \supset B, C \supset D \Rightarrow A, C, \Delta_2} \quad \cfrac{\text{— } u_1'' \text{ —}}{\Gamma_2, A \supset B, D \Rightarrow A, \Delta_2}}{\Gamma_1, A \supset B, C \supset D \Rightarrow A, \Delta_1}\supset l \quad \cfrac{\cfrac{\text{— } u_2' \text{ —}}{\Gamma_2, C \supset D, B \Rightarrow C, \Delta_2} \quad \cfrac{\text{— } u_2'' \text{ —}}{\Gamma_2, D, B \Rightarrow \Delta_2}}{\Gamma_1, C \supset D, B \Rightarrow \Delta_1}\supset l}{\Gamma_0, A \supset B, C \supset D \Rightarrow \Delta_0}\supset l$$

Then the exchange is again easy.

$$\cfrac{\cfrac{\cfrac{\text{— } u_1' \text{ —}}{\Gamma_2, A \supset B, C \supset D \Rightarrow A, C, \Delta_2} \quad \cfrac{\text{— } u_2' \text{ —}}{\Gamma_2, C \supset D, B \Rightarrow C, \Delta_2}}{\Gamma_1, A \supset B, C \supset D \Rightarrow C, \Delta_1}\supset l \quad \cfrac{\cfrac{\text{— } u_1'' \text{ —}}{\Gamma_2, A \supset B, D \Rightarrow A, \Delta_2} \quad \cfrac{\text{— } u_2'' \text{ —}}{\Gamma_2, B, D \Rightarrow \Delta_2}}{\Gamma_1, A \supset B, D \Rightarrow \Delta_1}\supset l}{\Gamma_0, A \supset B, C \supset D \Rightarrow \Delta_0}\supset l$$

The leaves are reordered but actually not changed. As the whole parts u_1', u_1'', u_2', and u_2'' were macro-saturated and redundance-free before the swap, they are still redundance-free and macro-saturated now. For the same reason, the *mbh* does not change. No critical rule is involved, so the *MBH* is still 1.

□

So far, we know that we have some possibilities to move rule applications from top to bottom. This allows us to start the *normalization* of a G3i^{M}-sequent-tree. By Lemma 3.9 it is possible to move independent rule applications of one micro block into another micro block below. This makes it possible to *saturate micro blocks* as well.

Definition 3.7 (Micro Block Saturation [139])**.** *A* micro block *with* $\Gamma \Rightarrow \Delta$ *as root sequent is* saturated *iff non-critical and non-closing rules are applied on all formulas* γ *of* Γ *and* δ *of* Δ *for which* γ *is not atomic and* δ *is neither atomic nor critical.*
A sequent tree *is* micro-saturated *iff all the micro blocks on all of its paths are saturated.*

To check saturation of a *micro block*, we have to look at the non-atomic formulas in its root sequent and find out whether rules are applied in the micro block on all these formulas (as long as the applications are independent and non-critical).

Lemma 3.10 (Micro Block Saturation in G3i^{M}-Proof-Trees [139])**.** *Let* t *be a* G3i^{M}*-proof-tree of* micro block height h*. Then* t *can be transformed so that it is* micro-saturated*, but still closed, and without increasing its micro block height* h*.*

Proof. As preparation, we make t *redundance-free* (Lemma 3.7) and *macro-saturate* it afterwards (Lemma 3.8). We perform an induction on the *micro block height* (*mbh*) h of this prepared t.

Base Case: If $h = 1$ then there is one micro block in each path of t. For each of these micro blocks, the enveloping macro block is saturated. Therefore there is a rule application on each formula of the micro block's root sequent. This means that the micro block is already saturated.

Inductive Step: Consider the proof tree t of *mbh* $h+1$.

$$\dfrac{\begin{matrix} \dfrac{\begin{matrix}\vdots & \vdots & \vdots \\ & \text{— } t_1 \text{ —} & \end{matrix}}{\Gamma_1 \Rightarrow \Delta_1} & \dfrac{\begin{matrix}\vdots & \vdots & \vdots \\ & \text{— } t_2 \text{ —} & \end{matrix}}{\Gamma_2 \Rightarrow \Delta_2} & & \dfrac{\begin{matrix}\vdots & \vdots & \vdots \\ & \text{— } t_n \text{ —} & \end{matrix}}{\Gamma_n \Rightarrow \Delta_n} \\ \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & \cdots & \vdots \quad \vdots \quad \vdots \end{matrix}}{\Gamma_R \Rightarrow \Delta_R}$$

The sequents $\Gamma_i \Rightarrow \Delta_i$ for $1 \leqslant i \leqslant n$ now mark the end of the micro blocks that start in the root sequent $\Gamma_R \Rightarrow \Delta_R$. We call this section of the tree s_{mb}. The trees t_1 to t_n have a maximum *mbh* of h (at least one has an *mbh* of h, some of them might be empty). By hypothesis, they can be micro-saturated without increasing their *mbh*. For all these separated border sequents $\Gamma_i \Rightarrow \Delta_i$ we do the following:

We consider all formulas γ of Γ_R which still occur in Γ_i without a rule application on γ below. Such a γ causes the non-saturation of the micro block. So for each of the γ's we do the following:

As the enveloping macro block is saturated, there must be a rule application on γ in the same macro block, but in some micro block above the one we are just considering. However, as this application is independent of all rule applications above $\Gamma_i \Rightarrow \Delta_i$, it can be moved to the level of the border sequent (Lemma 3.9).

Here is an example:

$$\dfrac{\begin{matrix} & \dfrac{\text{— } t_i \text{ —}}{\Gamma_i, A \vee B \Rightarrow \Delta_i} & \\ \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \end{matrix}}{\Gamma_R, A \vee B \Rightarrow \Delta_R} \quad \Longrightarrow \quad \dfrac{\begin{matrix} & \dfrac{\dfrac{\text{— } t_i' \text{ —}}{\Gamma_i, A \Rightarrow \Delta_i} \quad \dfrac{\text{— } t_i'' \text{ —}}{\Gamma_i, B \Rightarrow \Delta_i}}{\Gamma_i, A \vee B \Rightarrow \Delta_i} \vee l & \\ \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \end{matrix}}{\Gamma_R, A \vee B \Rightarrow \Delta_R}$$

Then γ is saturated in the micro block and we can proceed with the next γ. It is however possible that the number of parallel micro blocks in level 0 is increased, as it happens in the shown example. In this case we need to cope with the remaining γ's in both branches. This is no problem, as the transformation does not create new non-saturated formulas in Γ_R.

Anyway, once every antecedent formula in question is saturated, we can do the same with all non-critical and non-atomic δ's of Δ_R for which there is no application below $\Gamma_i \Rightarrow \Delta_i$. Finally, the micro block is saturated. We apply the hypothesis again to saturate all micro blocks above the border sequent. □

Figure 3.12 shows the micro saturation of the macro-saturated G3i^M-proof we considered before. The dashed lines show the borders of micro blocks within macro blocks. Note that the saturation reduces the number of micro blocks in the second macro block (read from bottom to top).

From Normalized G3i^m-Proofs to DiaSeqI-Proofs

From now on, we call sections of G3i^M-sequent-trees, for which each path has a *micro block height* of 1, *single micro sections*. If all the micro blocks of a single micro section are saturated, it is a *saturated single micro section*.

Let us pause for a moment and have a look at the things we have achieved so far. We can take any closed G3i^M-proof-tree t and make it *redundance-free*, i.e., we can remove repetitive rule applications on duplication formulas within the same macro block. The resulting proof tree t' is still closed (Lemma 3.7). Next, we *macro-saturate* the tree to t'' (Lemma 3.8). Then we have the non-critical and non-redundant rule applications on non-atomic formulas in macro blocks as far as possible (Definition 3.6). We take t'' and *micro-saturate* it (Lemma 3.10). This is how we obtain t'''. So, on every non-critical and non-atomic formula of the root sequent of each micro block, a rule is applied within this micro block (Definition 3.7).

Why have we done all that? Well, now we can move the rule applications in all *single block sequent sections* of t''' the way we want. We simply need

$$
\begin{array}{c l}
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A \supset C, A, C & ax \\
\hline
\neg A, C, \neg(A \supset C), A \Rightarrow A, C & \neg l \\
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow C & \neg l \\
\hline\hline
\neg A, B, \neg(A \supset C) \Rightarrow A, A \supset C & \supset r \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow A \supset C & \neg l \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow & \neg l \\
\hline\hline
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C, D & \neg r \\
\hdashline
\neg A, B \Rightarrow \neg\neg(A \supset C), C, D & \neg l \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C, D & \wedge l \\
\hdashline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \vee r \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \vee r \\
\hline\hline
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) & \supset r
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{c l}
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A \supset C, A, C & ax \\
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow A, C & \neg l \\
\hline
\neg A, B, \neg(A \supset C), A \Rightarrow C & \neg l \\
\hline\hline
\neg A, B, \neg(A \supset C) \Rightarrow A, A \supset C & \supset r \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow A \supset C & \neg l \\
\hline
\neg A, B, \neg(A \supset C) \Rightarrow & \neg l \\
\hline\hline
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C, D & \neg r \\
\hline
\neg A, B \Rightarrow \neg\neg(A \supset C), C, D & \neg l \\
\hdashline
\neg A, B \Rightarrow \neg\neg(A \supset C), C \vee D & \vee r \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \wedge l \\
\hline
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \vee r \\
\hline\hline
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) & \supset r
\end{array}
$$

Figure 3.12: Example of a micro saturation (c.f. [139])

to pull down the applications of the rules we want to have at the bottom, then pull down those of the rules we want to have above and so on. The sequents of the leaves do not change (Lemma 3.9), so there is nothing to worry about. For the translation to DiaSeqI (which will follow soon), we want the applications of right-hand side rules before the applications of the left-hand side rules in every micro block. This corresponds to the *defences* performed by the P-agents, followed by the *defences* by O.

Let us determine the required order of rule applications in all single block sequent sections of t''' as follows (on all paths read from bottom to top):

$$\neg l \quad \supset l \quad \wedge l \quad \vee l \quad \wedge r \quad \vee r \quad .$$

The problem is that if the dialogue does not start with a critical attack performed by O, the order is different within the lowest macro blocks, so we have to make a difference between the lowest macro blocks and everything above. For the transformation we do not distinguish the different proponent labels and therefore we also omit the addressees attached to the o. As mentioned before, this does not make a difference as they are only used to assign the communication partners to each other. We also ignore the blocking with the tilde ($\tilde{o}$) as this issue is solved because our G3i^M-proofs are considered to be redundance-free (Lemma 3.7).

Lemma 3.11 (Macro-Block Transformation (G3i^M))**.** *Every redundance-free, macro-saturated, and micro-saturated* G3i^M*-proof with root sequent* $\Gamma_R \Rightarrow \Delta_R$ *can be transformed to a* DiaSeqI*-tree which is closed and which has a sequent* $\overline{\Phi_R} \vdash_O \Psi_R$ *as root, where* $\overline{\Phi_R}$ *and* Ψ_R *contain the formulas of* Γ_R *and* Δ_R *respectively, augmented with labels (left-hand formulas with* o*, right-hand with* p*), and all non-atomic formulas of* $\overline{\Phi_R}$ *are marked with an attacked-bar* ($\overline{o}$)*.*

Note that the resulting tree is closed but not a completely valid DiaSeqI-proof-tree, as the root sequent has a special form which is not desired (the O-labels of non-atomic formulas are all marked as attacked). However we need this intermediate DiaSeqI-tree which we call d.

Proof by induction on the MBH h *of* t.

Base Case: $h = 1$ — If there is only one macro block then there is no critical rule application in t. We perform another induction on the *mbh* j of t.

Base Case: $j = 1$ — As t is closed, there must be at least one micro block including a closing rule. Because of the independence of all rule applications in the micro block, the closing rule can be applied immediately in the root sequent and the rest is obsolete. Therefore, we construct d with the root sequent $\overline{\Phi_R} \vdash_O \Psi_R$ and append all possible triggers, defences and attacks by O, followed by an application of c_O and PN. Finally, we add the ipse dixisti remark ($P!!$) which must be possible as the atom was attacked by O before and it must be present in Φ_R because of the rule independence within the same micro block. In the case that t is closed because of a $\perp$ in Γ_R, absurdum dixisti ($P?\perp$) is stated by a P (remember that only non-atomic formulas in $\overline{\Phi_R}$ are marked as attacked in the root).

Inductive Step: We take the lowest saturated single micro section of t and use Lemma 3.9 to reorder the rule applications ($\neg l$, $\supset l$, $\wedge l$, $\vee l$, $\wedge r$, $\vee r$). Our t has the following form:

$$\dfrac{\dfrac{— t_1 —}{\Gamma_1 \Rightarrow \Delta_1} r_1 \quad \cdots\cdots \quad \dfrac{— t_n —}{\Gamma_n \Rightarrow \Delta_n} r_n}{\Gamma_R \Rightarrow \Delta_R}$$

We assume that r_i are the rules leading to the next micro blocks above and $\Gamma_i \Rightarrow \Delta_i$ are the last sequents of the single micro blocks starting in $\Gamma_R \Rightarrow \Delta_R$. We take the root sequent $\overline{\Phi_R} \vdash_O \Psi_R$ as in the base case and add all possible trigger rules for O (corresponding to $\neg l$ and $\supset l$) and the O-defending rules (corresponding to the remaining left rules). Then we append all possible O-attacks followed by c_O and PN in each branch (there is no critical application in t). Afterwards, the P-defences follow which correspond to right-hand rule applications of t. Finally, we add the proponents' attacks and c_P to d. All branches of the height which correspond to one micro block are then done: every non-atomic formula on the O-side is attacked and every formula on the P-side defended (no critical rule, so no delay), so we obtain again the form $\overline{\Phi_i} \vdash_O \Psi_i$. We apply the (inner) induction hypothesis to cope with t_1 to t_n accordingly.

Inductive Step: Now we have at least one critical rule application in t. Again, we perform the induction on the *mbh* j of the lowest saturated macro block section of t,[10] to show that the macro block can be translated to an equivalent block in DiaSeqI:

Base Case: $j = 0$ — If the *mbh* of the section is zero, then it consists of a single empty macro block and the first rule application in t is critical. Due to the saturation of the blocks, there is nothing to defend for O in this micro block. Also note that only one branch can be involved. We start d with our usual $\overline{\Phi_R} \vdash_O \Psi_R$ and add all possible attacks, including the critical one, performed by O, followed by a c_O. Then we add the critical move by P (corresponding to $\supset r$ or $\neg r$), all possible attacks which can then be performed by P, and finally c_P.

Inductive Step: We do the same as in the outer base case. In the saturated micro block section the desired order of rule applications is enforced (same as in base case above) and we obtain the corresponding structure of O-triggers, O-defences, and O-attacks followed by non-critical P-defences, P-attacks, and c_P. We then apply the (inner) induction hypothesis on all branches to transform further micro blocks.

Once the macro block is transformed, we simply apply the hypothesis on the G3i^M-trees starting with the macro blocks in the next macro block level. □

[10] The section of all saturated macro blocks of t with a macro block level of 0.

Theorem 3.2 (DiaSeqI Completeness)**.** *Every closed* G3i^{M}*-proof-tree can be transformed to a* DiaSeqI*-proof-tree.*

Proof. Let t with *MBH* h be the G3i^{M}-proof we want to transform and d the target DiaSeqI-poof-tree. We assume that t is *redundance-free*, *macro-saturated*, and *micro-saturated*. If it is not, we apply our Lemmas 3.7, 3.8, and 3.10. We say that $\Gamma_R \Rightarrow \Delta_R$ is the root sequent of t. We consider two cases:

- $h = 1$ — If there is only one macro-block in each path then there is no critical rule application in t. We perform an induction on the *mbh* j of t, similarly as in the proof of Lemma 3.11:

  Base Case: $j = 1$ — We construct d with a root sequent $\Phi_R \vdash_O \Psi_R$ where Φ_R and Ψ_R correspond to the formulas of Γ and Δ, augmented with labels o and p, respectively. We append all possible attacks by O, followed by an application of c_O and PN. Finally, we add the ipse dixisti remark (P!!) which must be possible as the atom was attacked before by O and it must be present in Φ_R because of the rule independence within the same micro block.

  Inductive Step: We take the lowest saturated single micro section of t and use Lemma 3.9 to reorder the rule applications, but this time in another order ($\vee r$, $\wedge r$, $\neg l$, $\supset l$, $\wedge l$, $\vee l$). Then, as in the proof of Lemma 3.11, we simply construct the root sequent $\Phi_R \vdash_O \Psi_R$ like in the base case and add all possible attacks that can be performed by O. We append c_O and PN in each branch (there is no critical application in t). Afterwards, the P-defences (which correspond to right-hand rule applications of t) and the P-attacks follow. Then, in all non-closed branches (in the closed ones, P defended with ipse dixisti), we add c_P to d, then the trigger-rules for O (corresponding to $\neg l$ and $\supset l$) and O's defending rules (corresponding to the remaining left rules). We apply the (inner) induction hypothesis to cope with all micro blocks above.

- $h > 1$ — for the lowest macro-saturated section of t,[11] we do almost the same as for the case $h = 0$. Again, we perform the induction on the *mbh* j of this section, to show that the macro block can be translated to an equivalent block in DiaSeqI reaching a P-agent's reaction to a critical attack:

  Base Case: $j = 0$ — If the *mbh* of the section is zero then it consists of a single empty macro block and the first rule application in t is critical. Note that only one branch can be involved. We start d with our usual $\Phi_R \vdash_O \Psi_R$ and add all attacks, including the critical one, performed by O, followed by a c_O. Then we add the critical move by P (corresponding to $\supset$r or $\neg$r) and all possible attacks which can then be performed by P. There is nothing remaining to defend for P as other agents (if available) were deactivated. We then append c_P.

  Inductive Step: In the saturated micro block section the desired order of rule applications is enforced (same as in the case of $h = 0$) and we obtain the corresponding structure of O-attacks followed by non-critical P-defences, P-attacks, O-triggers, and O-defences. If the micro block contains a closing rule, we do the same as in the case of $h = 0$.

  Now, in the leaves of our current d, all non-atomic statements by O are attacked while the P-statements are not. So we take the sections of t starting in macro block level 1 and apply Lemma 3.11. This is how we obtain the desired structure for d.

□

To complete our continuous example we first need to perform the rearrangement of the rule applications within the saturated micro blocks. As there is no micro block in the lowest macro block, only the rule order of Lemma 3.11

[11] The section of all saturated macro blocks of t which have a macro block level of 0.

$$
\begin{array}{cl}
\hline & ax \\
\neg A, B, \neg(A \supset C), A \Rightarrow A \supset C, A, C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C), A \Rightarrow A, C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C), A \Rightarrow C & \\
\hline\hline & \supset r \\
\neg A, B, \neg(A \supset C) \Rightarrow A, A \supset C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C) \Rightarrow A \supset C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C) \Rightarrow & \\
\hline\hline & \neg r \\
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C, D & \\
\hline & \neg l \\
\neg A, B \Rightarrow \neg\neg(A \supset C), C, D & \\
\hdashline & \vee r \\
\neg A, B \Rightarrow \neg\neg(A \supset C), C \vee D & \\
\hline & \wedge l \\
\neg A \wedge B \Rightarrow \neg\neg(A \supset C), C \vee D & \\
\hline & \vee r \\
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \\
\hline\hline & \supset r \\
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) &
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{cl}
\hline & ax \\
\neg A, B, \neg(A \supset C), A \Rightarrow A \supset C, A, C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C), A \Rightarrow A, C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C), A \Rightarrow C & \\
\hline\hline & \supset r \\
\neg A, B, \neg(A \supset C) \Rightarrow A, A \supset C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C) \Rightarrow A \supset C & \\
\hline & \neg l \\
\neg A, B, \neg(A \supset C) \Rightarrow & \\
\hline\hline & \neg r \\
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C, D & \\
\hline & \vee r \\
\neg A, B \Rightarrow A, \neg\neg(A \supset C), C \vee D & \\
\hdashline & \neg l \\
\neg A, B \Rightarrow \neg\neg(A \supset C), C \vee D & \\
\hline & \vee r \\
\neg A, B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \\
\hline & \wedge l \\
\neg A \wedge B \Rightarrow \neg\neg(A \supset C) \vee (C \vee D) & \\
\hline\hline & \supset r \\
\Rightarrow (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D)) &
\end{array}
$$

Figure 3.13: Changing the order of rule applications (c.f. [139])

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\widetilde{o} : \neg A,\ o_{p2} : B,\ \widetilde{o} : \neg(A \supset C),\ o_{p8} : A\ \vdash_{PD}\ \overline{p0} : C,\ \overline{p6} : A,\ \overline{p7} : A \supset C}\ \text{PN},\underline{\text{P !!}}}{\overline{o_{p6}} : \neg A,\ o_{p2} : B,\ \overline{o_{p7}} : \neg(A \supset C),\ o_{p8} : A\ \vdash_O\ p0 : C}\ \underline{O{*}\neg}(\times 2), O?a(\times 2), O?{\supset}, c_O}{\widetilde{o} : \neg A,\ o_{p2} : B,\ \widetilde{o} : \neg(A \supset C)\ \vdash_{PD}\ \overline{p5} : A,\ \overline{p0} : A \supset C}\ \underline{P!{\supset}}, P?\neg(\times 2), c_P}{\overline{o_{p5}} : \neg A,\ o_{p2} : B,\ \overline{o_{p0}} : \neg(A \supset C)\ \vdash_O}\ \underline{O{*}\neg}(\times 2), O?a, O?{\supset}, c_O}{\widetilde{o} : \neg A,\ o_{p2} : B\ \vdash_{PD}\ \overline{p0} : \neg\neg(A \supset C),\ \overline{p3} : C,\ \overline{p4} : D,\ \overline{p1} : A}\ \underline{P{*}\neg}, P?\neg(\times 2), c_P}{\widetilde{o} : \neg A,\ o_{p2} : B\ \vdash_O\ \overline{p0} : \neg\neg(A \supset C),\ p3 : C,\ p4 : D,\ \overline{p1} : A}\ O?a(\times 2), c_O}{\widetilde{o} : \neg A,\ o_{p2} : B\ \vdash_{PN}\ \overline{p0} : \neg\neg(A \supset C),\ \overline{p3} : C \vee D,\ \overline{p1} : A}\ \underline{P!{\vee}}, c_P}{o_{p1} : \neg A,\ o_{p2} : B\ \vdash_{PN}\ p0 : \neg\neg(A \supset C),\ p3 : C \vee D}\ P?\neg, c_P, \underline{O{*}\neg}, O?a, O?{\vee}, c_O, \text{PN}}{o_{p1} : \neg A,\ o_{p2} : B\ \vdash_{PN}\ \overline{p0} : \neg\neg(A \supset C) \vee (C \vee D)}\ \underline{P!{\vee}}}{\overline{o_{p1}}^L : \neg A \wedge B,\ \overline{o_{p2}}^R : \neg A \wedge B\ \vdash_O\ p0 : \neg\neg(A \supset C) \vee (C \vee D)}\ \underline{O!{\wedge}}(\times 2), O?{\vee}, c_O, \text{PN}}{o_{p1} : \neg A \wedge B\ \vdash_{PN}\ p0 : \neg\neg(A \supset C) \vee (C \vee D)}\ P?{\wedge}, c_P}{\vdash_{PD}\ \overline{p0} : (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D))}\ \underline{P!{\supset}}}{\vdash_O\ p0 : (\neg A \wedge B) \supset (\neg\neg(A \supset C) \vee (C \vee D))}\ O?{\supset}, c_O$$

Figure 3.14: Constructed DiaSeqI-proof from micro-saturated G3i^M-proof (c.f. [139])

is relevant. The old and the new G3i^M-proof are shown in Figure 3.13. The resulting DiaSeqI-proof is displayed in Figure 3.14. The underlined rule names correspond to those of the G3i^M-tree (now usually defences and triggers). Doubled and dashed lines indicating the previous borders of macro and micro blocks are adopted for illustrating purposes.

3.3.2 Completeness of DiaSeqI$^+$

Now we adjust the completeness proof for DiaSeqI to make it fit for DiaSeqI$^+$. Our reference calculus is IG (Chapter 2.1.4). In the end, we only consider *regular* IG-*derivations* (IG^r-derivation) [33] with the restrictions we discussed on page 39. As preparation we interpret these informal meta-rules as follows:

Definition 3.8 (IG^r-derivations, c.f. [33]). *In every* IG^r *(regular* IG*) proof tree* t *the following restrictions are obeyed for all formulas* ϕ *and* ψ*:*

1. *If there is an application* $a_l = (\supset r, \hat{\phi})$ *in* t*, then there is* no $a_m = (\supset r, \hat{\psi})$ *on the same path such that* $m < l$ *and* $\phi = \psi$.
2. *If there is an application* $a_l = (a.f, \hat{\phi})$ *in* t*, then there must also be an application* $a_m = (\supset r, \hat{\psi})$ *on the same path such that* $m < l$ *and* $\phi = \psi$.
3. *If there are two rule applications* $a_l = (\supset l, \hat{\phi})$ *and* $a_n = (\supset l, \hat{\psi})$ *with* $\phi = \psi$ *and* $n < l$ *in the same path of* t*, then there must also be a* critical *application* a_m *in the same path, such that* $n < m < l$.

We divide the non-critical rules into two parts. The *conventional* rules are $\wedge l$, $\vee l$, $\supset l$, $\wedge r$, and $\vee r$. Their applications are also called *conventional*. The *a fortiori* rule and its application are now called *unconventional*. The rule $\supset r$ is the only *critical rule* we consider now. For the sake of simplicity, we omit negations (as they can be interpreted in terms of implications) and therefore ignore the conventional rule $\neg l$ and the critical $\neg r$.

In the following, we also refer to IG-derivations which are not completely *regular*, but adapt only some of the corresponding properties:

Definition 3.9 (Semi-Regular IG-Derivation). *Let* t *be an* IG*-proof-tree. Then* t *is* semi-regular *iff the following holds in each of its paths: if there is an application* $a_l = (\supset r, \hat{\phi})$*, then there is no other application* $a_m = (\supset r, \hat{\psi})$ *in the same path such that* $\phi = \psi$.

Weakening (Lemma 3.1) is also admissible in IG (no matter whether the derivation is regular or not), i.e., the proof works without problems when we take rule *a.f* into account. By contrast, *inversion* and *contraction* are a bit more problematic. We do not discuss the details regarding contraction here, as they are not relevant for our purpose.

Lemma 3.12 (Inversion (IG)[12]). *For all* Δ, Γ *and formulas* A *and* B*:*

1. *If* $\Vdash^{IG}_n \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\Vdash^{IG}_n \Gamma, A, B \Rightarrow \Delta$.
2. *If* $\Vdash^{IG}_n \Gamma \Rightarrow A \vee B, \Delta$ *then* $\Vdash^{IG}_n \Gamma \Rightarrow A, B, \Delta$.
3. *If* $\Vdash^{IG}_n \Gamma, A \vee B \Rightarrow \Delta$ *then* $\Vdash^{IG}_n \Gamma, A \Rightarrow \Delta$ *and* $\Vdash^{IG}_n \Gamma, B \Rightarrow \Delta$.
4. *If* $\Vdash^{IG}_n \Gamma \Rightarrow A \wedge B, \Delta$ *then* $\Vdash^{IG}_n \Gamma \Rightarrow A, \Delta$ *and* $\Vdash^{IG}_n \Gamma \Rightarrow B, \Delta$.
5. *If* $\Vdash^{IG}_n \Gamma, A \supset B \Rightarrow \Delta$ *then* $\Vdash^{IG}_n \Gamma, A \supset B \Rightarrow A, \Delta$ *and* $\Vdash^{IG}_n \Gamma, B \Rightarrow \Delta$.
6. *If* $\Vdash^{IG}_n \Gamma \Rightarrow A \supset B, \Delta$ *without any application of* $\supset r$ *on some* $A \supset B$*, then* $\Vdash^{IG}_n \Gamma \Rightarrow B, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of rule applications is preserved.

Proof by induction on n. The new feature compared to the proof of Lemma 3.2 (p. 119) is item number 6 which states that *a fortiori* is invertible under certain circumstances. Therefore, we discuss only this case here.

Base Case: $n = 1$ — Trivial as a closing rule can only be applied on atoms, so this does not depend on $A \supset B$ and also not on the corresponding B.

Inductive Step: We have $\Vdash^{IG}_{n+1} \Gamma \Rightarrow A \supset B, \Delta$ without an application of $\supset r$ on $A \supset B$. We have a look at the lowest rule application of the derivation tree.

- If it is a *conventional* rule application, we can simply apply the hypothesis on the premise(s) and add the rule application afterwards again.
- If it as an application of *a fortiori* on some $C \supset D$ of Δ, we can also apply the hypothesis on the premise and add the *a fortiori* again.

[12]see also Lemma 3.2 and [39, 142]; the lemma applies for all IG-derivations, not only regular ones.

- If it as an application of *a fortiori* on $A \supset B$, we are already done, as $\Vdash_n^{\mathsf{IG}} \Gamma \Rightarrow B, \Delta$ and therefore also $\Vdash_{n+1}^{\mathsf{IG}} \Gamma \Rightarrow B, \Delta$.
- If it is a *critical* rule application, then on some implication $C \supset D$ of Δ. We then have $\Vdash_n^{\mathsf{IG}} \Gamma, C \Rightarrow D$. So the derivation does not have anything to do with $A \supset B$, therefore we can replace it directly by B in the root sequent.

In all cases, no further rule applications are necessary and the order of applications is preserved. □

It is not necessary to care about redundancy as we did for $\mathrm{G3I}^{\mathrm{M}}$, because IG^{r}-proofs are by definition redundance-free (see Definition 3.8: number 3, p. 146).

Macro-Saturation

Because the restriction that *a fortiori* may only be applied above corresponding applications of $\supset$r in regular derivations, we have to adjust the definition of macro saturation a little bit:

Definition 3.10 (Macro Block Saturation (IG))**.** *Let* M *be a macro block of an* IG*-tree* t*. Then* M *is* macro-saturated *iff conventional rules are applied on all formula instances in all sequents of* M*, wherever syntactically possible. Further, for all implication formulas* φ *which occur in a succedent of a sequent in* M*: if there is an application of* $\supset$r *on some instance of* φ *below* M*, then there must be an application of a fortiori on* φ *in* M*.*
An IG*-proof-tree is macro-saturated iff all its macro blocks are macro-saturated.*

However, this is not everything: the problem is that in the macro saturation as presented in the proof of Lemma 3.8 the saturation happens by using the induction hypothesis from the leaves to the root of the derivation tree. In IG^{r}-derivations we cannot have applications of *a fortiori* in the macro blocks of macro level 0. Therefore, we introduce further definitions for macro saturation.

Definition 3.11 (Macro Block Conventional Saturation (IG))**.** *Let* M *be a macro block of an* IG*-tree* t*. Then* M *is* conventionally macro-saturated *iff conventional rules are applied on all formula instances in all sequents of* M*. An* IG*-proof-tree* t *is* conventionally macro-saturated *iff all its macro blocks are conventionally saturated.*

The conventional macro block saturation corresponds to the macro block saturation of G3i^M-trees (Definition 3.6, p. 128):

Lemma 3.13 (Macro Block Conventional Saturation of IG-Proof-Trees)**.** *Let* t *be an* IG*-proof-tree. Then* t *can be transformed and made* conventionally macro-saturated*, preserving validity of the proof tree and without increasing its macro block height (MBH). No critical or unconventional rule application is added or removed in any path.*

Proof. The proof of Lemma 3.8 (p. 128) works here without changes. Applications of *a fortiori* are not added and not removed. □

Definition 3.12 (Macro Block Semi-Saturation (IG))**.** *Let* S *be a set of propositional formulas which are implications. An* IG*-tree* t *is* macro-semi-saturated with respect to S *iff all its macro blocks are conventionally saturated and for each* $\varphi \in S$*: if there is* no *critical rule application on a* φ *anywhere in* t*, then in all macro blocks in which* φ *occurs in the succedent of a sequent, there is an application of a fortiori on* φ *in this macro block.*

Note that a semi-macro-saturated IG-tree is usually not regular, as we have the usage of *a fortiori*-rules without any application of $\supset$r.

Lemma 3.14 (Macro Block Semi-Saturation of IG-Proof-Trees)**.** *Let* t *be an* IG*-proof-tree and* S *be an arbitrary finite set of implication formulas. Then* t *can be transformed and made* semi-macro-saturated with respect to S*, preserving validity of the proof tree and without increasing its macro block height (MBH).*

Proof. We prove this the same way as Lemmas 3.8 and 3.13. The difference is that implications in succedents must also be taken into account when there is no corresponding application of $\supset$r in the path of t for all formulas of S. However, this is no problem due to our new *inversion* (Lemma 3.12). $\square$

Lemma 3.15 (Full Macro Block Saturation of Semi-Regular IG-Proof-Trees)**.** *Let* t *be the tree of a semi-regular* IG*-derivation (Definition 3.9). Then* t *can be macro-saturated (Definition 3.10) preserving validity and without increasing its* MBH.

Proof. As preparation, the semi-regular t is *conventionally macro-saturated* (Lemma 3.13). The proof is performed by induction on the *MBH* h of the prepared t.

Base Case: $h = 1$ — There is no critical rule application in t, so it is sufficient to saturate it *conventionally*, which is already done. Then it is also fully saturated.

Inductive Step: The procedure is the same as usual: we increase h by extending the tree at the bottom:

$$\begin{array}{c}
\dfrac{\text{— } t_1 \text{ —}}{\Gamma_1 \Rightarrow \Delta_1}\, c_1 \qquad \dots \qquad \dfrac{\text{— } t_n \text{ —}}{\Gamma_n \Rightarrow \Delta_n}\, c_n \\
\vdots \;\; \vdots \qquad\quad \vdots \;\; \vdots \;\; \vdots \qquad\quad \vdots \;\; \vdots \\
\hline
\Gamma_R \Rightarrow \Delta_R
\end{array}$$

As we have seen now for several times, the sequents $\Gamma_i \Rightarrow \Delta_i$ mark the borders to the next macro blocks starting at macro block level 1. By hypothesis, all of the derivations t_i can be macro-saturated completely. The rule c_i is applied on some implication $A \supset B$ of Δ_i. There is no other application of c_i on an occurrence of $A \supset B$ in t_i. So, t_i is fully macro-saturated, but not yet with respect to $A \supset B$. So we set $S = \{A \supset B\}$ and apply Lemma 3.14 on t_i.

Once we have done this with all derivations t_1 to t_n, we are already finished, because the complete tree was conventionally saturated as preparation and no critical rule occurs in the macro blocks starting in $\Gamma_R \Rightarrow \Delta_R$. $\square$

Micro-Saturation

Before we achieve the *micro-saturation*, we need a fourth form of macro block saturation.

Definition 3.13 (Macro Block Cross-Saturation (IG))**.** *Let* S *be a set of propositional formulas which are implications. An* IG*-tree* t *is* macro-cross-saturated with respect to S *iff all its macro blocks are conventionally saturated and for each* $\varphi \in S$ *and macro block* M *in* t*: if* φ *occurs in some succedent in* M*, then there is an application of* a fortiori *on* φ *in all macro blocks that start in the same sequent as* M *and in which* φ *occurs.*

The difference to *macro-semi-saturation* (Definition 3.12) is that in the latter *a fortiori* must be applied on all implications in succedents *only if* there is no application of $\supset$r on the same formula in the whole tree. The *cross-saturation* enforces that *a fortiori* is applied on all occurrences of the implication in all parallel macro blocks.

The rule movement as described in Lemma 3.9 (p. 131) is quite easy in macro-cross-saturated IG-trees. We reformulate Lemma 3.9 as follows:

Lemma 3.16 (Independent Rule Movement in Macro-Cross-Saturated IG-trees)**.** *Let* S *be a set of formulas which are implications and* t *be an* IG*-sequent-tree which is* macro-cross-saturated *with respect to* S*, and which has an MBH of* 1*. Further, let* $a_l = (r, \varphi)$ *be a conventional rule application or an application with* $\varphi \in S$*. Suppose that for all rule applications* a_m *with* $0 \leqslant m < l$*,* a_l *does not depend on* a_m*, i.e.,* $a_l \not\prec a_m$*. Then* t *can be transformed to* t' *for which*

- a_l *is moved downwards to the root, and*
- *the root sequent and the leaf sequents are the same as in* t*, and*
- *no rule application is added and none is removed, and*
- *the mbh and the MBH are the same as in* t*, and*
- t' *is also macro-cross-saturated and redundance-free.*

Proof. The proof works the same way as the one for Lemma 3.9. We simply take the *a fortiori*-rule into account as well. If a_l is an application of *a fortiori*, the corresponding formula must be in S and therefore in all other branches, so we can pull it down each time. The swap at the bottom is possible with all combinations of non-critical rule applications. □

Now the *micro-saturation*: here, some adjustments are necessary as well.

Definition 3.14 (Micro Block Semi-Saturation (IG)). *Let* $\mathfrak{m}$ *be a micro block of an* IG*-proof-tree* t *with* $\Gamma \Rightarrow \Delta$ *as root sequent of* $\mathfrak{m}$*. Then* $\mathfrak{m}$ *is* semi-micro-saturated *with respect to* S *iff conventional rules are applied on all formulas* γ *of* Γ *and* δ *of* Δ *when syntactically possible, and* a fortiori *is applied on all implication formulas in* $\Delta \cap S$*.*
An IG*-tree is semi-micro-saturated iff all its micro blocks are semi-saturated.*

Definition 3.15 (Micro Block Saturation (IG)). *Let* $\mathfrak{m}$ *be a micro block of an* IG*-proof-tree* t *with* $\Gamma \Rightarrow \Delta$ *as root sequent of* $\mathfrak{m}$*. Then* $\mathfrak{m}$ *is* micro-saturated *iff conventional rules are applied on all formulas* γ *of* Γ *and* δ *of* Δ *whenever syntactically possible, and* a fortiori *is applied on all implication formulas* $\delta' \in \Delta$*, for which there is an application of* $\supset r$ *on some instance of* δ' *below* $\mathfrak{m}$*.*

We first enforce a semi-saturation in IG-proof-trees which are already macro-saturated and macro-cross-saturated.

Lemma 3.17 (Micro Block Semi-Saturation in Macro-Saturated IG-Proof-Trees). *Let* S *be a set of formulas which are implications and* t *be an* IG*-proof-tree which is macro-saturated, macro-cross-saturated with respect to* S*, and redundance-free. Then* t *can be transformed to* t' *so that* t' *is an* IG*-proof-tree which is* micro-semi-saturated *with respect to* S*, which has the same* mbh *and* MBH *as* t*, and which is still macro-saturated and redundance-free. In all paths and macro blocks of* t' *the same rule applications occur as in the paths and macro blocks of* t*.*

Proof by induction on the mbh h *of* t.

Base Case: $h = 1$ — Each path of t contains one micro block. The enveloping macro block is saturated and macro-cross-saturated with respect to S. Therefore, there is nothing to be done.

Inductive Step: We consider t with an *mbh* of $h + 1$:

$$
\begin{array}{c}
\begin{array}{ccccc}
\dfrac{\vdots \quad \vdots \quad \vdots \atop \text{— } t_1 \text{ —}}{\Gamma_1 \Rightarrow \Delta_1} & \dfrac{\vdots \quad \vdots \quad \vdots \atop \text{— } t_2 \text{ —}}{\Gamma_2 \Rightarrow \Delta_2} & & & \dfrac{\vdots \quad \vdots \quad \vdots \atop \text{— } t_n \text{ —}}{\Gamma_n \Rightarrow \Delta_n} \\
\vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & & \cdots & \vdots \quad \vdots \quad \vdots
\end{array} \\
\hline
\Gamma_R \Rightarrow \Delta_R
\end{array}
$$

Again, the sequents $\Gamma_i \Rightarrow \Delta_i$ for $1 \leqslant i \leqslant n$ mark the end of the micro blocks that start in the root sequent $\Gamma_R \Rightarrow \Delta_R$. We call this section of the tree s_{mb}. For each of the border sequents $\Gamma_i \Rightarrow \Delta_i$ we do the following:

We collect all non-atomic formulas γ of Γ_R and δ of Δ_R which are still present in Γ_i and Δ_i, with the additional restriction that all δ which are implications must also be in S, the others are ignored. We concentrate on these formulas of Δ_R which are still present in Δ_i and which are also in S. The other cases are handled as in the proof of Lemma 3.10.

As the enveloping macro block is macro-cross-saturated with respect to S, the δ occurs in the macro block(s) above $\Gamma_i \Rightarrow \Delta_i$. We take t_i and prune it at its borders to the next macro blocks. In this resulting tree, the application of *a fortiori* on δ must be independent of all applications below and therefore we can pull it down to the root. Then the micro block of t that ended in $\Gamma_i \Rightarrow \Delta_i$ is extended by an application of *a fortiori* on δ which is now saturated in this micro block. As no critical application is involved, the *MBH* does not change. Further, because all rule applications are still present in all the macro blocks and none is added, the tree is still macro-saturated and redundance-free. We proceed with the next unsaturated formula accordingly.

Finally, we apply the hypothesis to micro-semi-saturate everything above the new border and we are done. □

Lemma 3.18 (Full Micro Block Saturation of Semi-Regular IG-Proof-Trees)**.** *Let* t *be the tree of a macro-saturated, redundance-free, and semi-regular* IG*-derivation. Then* t *can be micro-saturated preserving validity and macro saturation, and without increasing its* MBH*. The rule applications in the various macro blocks remain the same, i.e., none is added and none is removed.*

Proof. The structure of the proof is very similar to the one of Lemma 3.15: we perform an induction on the *MBH* h of t.

Base Case: $h = 1$ — There is no critical rule application in t and it is macro-cross-saturated with respect to $S = \emptyset$. It is sufficient to *micro-semi-saturate* t with respect to $S = \emptyset$ (Lemma 3.17). Then it is also already fully saturated.

Inductive Step: h is increased in the usual way. The sequent $\Gamma_i \Rightarrow \Delta_i$ is the border to the next macro block starting in macro level 1 (see proof of Lemma 3.15).

First of all, we micro-semi-saturate the whole t with respect to $S = \emptyset$. Note that t is macro-cross-saturated with respect to $\emptyset$. Macro-saturation is preserved. The macro blocks of level 0 are then completely micro-saturated.

By hypothesis, all of the derivations t_i can be micro-saturated completely, which we do. The rule c_i is applied on some implication $A \supset B$ of Δ_i, for which t_i is not necessarily micro-saturated. Macro-saturation is preserved.

Because the whole t is semi-regular (Definition 3.9), there is no other application of c_i on an occurrence of $A \supset B$ in t_i. As t is still (fully) macro-saturated, in all macro blocks of t_i in which $A \supset B$ occurs in the succedent, there is an application of *a fortiori*. Further, because all these blocks are saturated, they are also *macro-cross-saturated* with respect to $\{A \supset B\}$. We use Lemma 3.17 to micro-semi-saturate t_i with respect to $S = \{A \supset B\}$. The micro-semi-saturation with respect to the other implication formulas is preserved, as well as the macro-saturation. The macro blocks that start in the root of t are already finished (we did this at the beginning of the inductive step).

□

From Normalized IG^r-Proofs to DiaSeqI$^+$-Proofs

It is almost accomplished! Our aim is to transform an IG^r-proof (regular IG-proof) into a DiaSeqI$^+$-derivation. IG^r-proofs are by definition redundance-free and it is helpful that every regular IG-proof is also semi-regular, therefore we can apply Lemmas 3.15 and 3.18 (full macro-saturation and full micro-saturation) without any problem and the result is still regular. So what needs to be done is the resorting within the micro blocks.

Technically, an intermediate dialogical calculus is required which corresponds to semi-regular IG-trees, because it is again not possible to perform the induction on the *MBH* of IG^r-proofs directly. However, in the end, things work the same way as presented in Lemma 3.11 (p. 139). The *a fortiori*-rule simply corresponds to P's alternative defence P!$\supset^+$ against attacks on implications. It is a non-critical move and therefore the defence is assigned to the PN-phase. In the lowest macro blocks of the semi-regular IG-proof-tree t, in each micro block, the rules shall be ordered as (read from bottom to top)

$$\vee r \quad \wedge r \quad a.f \quad \neg l \quad \supset l \quad \wedge l \quad \vee l$$

because O starts with her attacks. In the macro blocks starting in macro level 1, we use the order

$$\neg l \quad \supset l \quad \wedge l \quad \vee l \quad \wedge r \quad \vee r \quad a.f.$$

Theorem 3.3 (DiaSeqI$^+$ Completeness)**.** *Every closed IG^r-proof tree can be transformed to a* DiaSeqI$^+$*-proof tree.*

Proof. Actually, we need a stronger assertion which transforms *semi-regular* IG-proofs to proofs of an intermediate "semi-regular" DiaSeqI, which allows proponents to defend in an *a fortiori*-way also without defending critically against a corresponding attack on an implication.

We do not discuss the proof here, as it works the same way as the proofs of Lemma 3.11 (p. 139) and Theorem 3.2 (p. 142). Note that Lemma 3.16, as the new version of Lemma 3.9, refers to macro-semi-saturated and macro-cross-saturated proofs. It can however be used easily to move rule applications within sections of the IG^r-tree with an *MBH* of 1. □

3.3.3 Soundness

Soundness is actually shown quite easily, as it is not a big problem to transform DiaSeqI-proofs to $G3I^M$-proofs. The same holds for $DiaSeqI^+$ and IG^r.

Theorem 3.4 (DiaSeqI Soundness ([138, 139])**.** *Every* DiaSeqI*-proof-tree can be transformed to a* $G3I^M$*-proof-tree.*

Proof. We only sketch the proof here as the procedure is quite obvious.

The $G3I^M$-proof is the result of removing the announcer labels off the signed formulas and replacing the turnstiles $\vdash_O$, $\vdash_{PN}$, and $\vdash_{PD}$ by the standard $\Rightarrow$. The rules that change the phases (c_O, PN, c_P) can therefore be omitted, as well as the attacking moves as they only add the bars as sign of being attacked. The only exception is rule O?∧ where a branching is involved, but more on this later. So the moves which are relevant for the transformation are the defences and triggers. These have the same behaviour as the rules of $G3I^M$. We simply replace the O-defences and O-triggers by left-hand rules, and the P-defences and P-triggers by right-hand rules. The moves P!! and P?⊥ are translated to *ax* and ⊥l, respectively.

The only remaining problems are O?∧ and the corresponding defences P!L and P!R. However, this is not difficult to solve either. Within a phase, moves (or rule applications) can be exchanged almost arbitrarily. There are exceptions regarding trigger rules but their movement is not necessary for our purposes. An application of $\mathsf{O?}\wedge$ that occurs in some O-phase can always be moved directly below the c_{O} that ends the phase, so the following form is obtained:[13]

$$\dfrac{\dfrac{\vdots\quad\vdots\quad\vdots}{\Phi \vdash_{\mathsf{O}} \overline{p}^{L} : A \wedge B, \overline{\Psi}}\,c_{\mathsf{O}} \qquad \dfrac{\vdots\quad\vdots\quad\vdots}{\Phi \vdash_{\mathsf{O}} \overline{p}^{R} : A \wedge B, \overline{\Psi}}\,c_{\mathsf{O}}}{\Phi \vdash_{\mathsf{O}} p : A \wedge B, \overline{\Psi}}\,\mathsf{O?}\wedge$$

$$\vdots\quad\vdots\quad\vdots$$

All statements of $\overline{\Psi}$ are and none of Φ is attacked. Now we consider three different cases:

1. In both branches p defends the $\wedge$-attack in the next PN-phase.
   Then in each of the branches there must be an application of PN above the c_{O} and there is no reaction to a critical attack in the decide phase. Like $\mathsf{O?}\wedge$ before, the defence can be moved freely in the proponents' normal phase. We simply move it directly above the application on PN. Then we obtain the following form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\quad\vdots\quad\vdots}{\Phi \vdash_{\mathsf{PN}} p : A, \overline{\Psi}}}{\Phi \vdash_{\mathsf{PN}} \overline{p}^{L} : A \wedge B, \overline{\Psi}}\,\mathsf{P!L}}{\Phi \vdash_{\mathsf{PD}} \overline{p}^{L} : A \wedge B, \overline{\Psi}}\,\mathsf{PN}}{\Phi \vdash_{\mathsf{O}} \overline{p}^{L} : A \wedge B, \overline{\Psi}}\,c_{\mathsf{O}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots\quad\vdots\quad\vdots}{\Phi \vdash_{\mathsf{PN}} p : B, \overline{\Psi}}}{\Phi \vdash_{\mathsf{PN}} \overline{p}^{R} : A \wedge B, \overline{\Psi}}\,\mathsf{P!R}}{\Phi \vdash_{\mathsf{PD}} \overline{p}^{R} : A \wedge B, \overline{\Psi}}\,\mathsf{PN}}{\Phi \vdash_{\mathsf{O}} \overline{p}^{R} : A \wedge B, \overline{\Psi}}\,c_{\mathsf{O}}}{\Phi \vdash_{\mathsf{O}} p : A \wedge B, \overline{\Psi}}\,\mathsf{O?}\wedge$$

$$\vdots\quad\vdots\quad\vdots$$

   These DiaSeqI-rule-applications of the section displayed here can simply be translated to an application of $\wedge r$ in $\mathrm{G3I}^{\mathrm{M}}$.

[13] We do not give details about the movement here, but the procedure is similar to the movement of rule applications in $\mathrm{G3I}^{\mathrm{M}}$.

2. Only in one of these branches p defends against the $\wedge$-attack. Then there are two possibilities for the other branch.

   a) There is an occurrence of P!! or P?$\bot$ in the next PN-phase.

   b) A reaction to a critical attack follows in the proponents' decide phase.

   In both cases, in the branch in which P performs the $\wedge$-attack, we can simply move the application of P!L or P!R directly above PN:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \vdots\ \vdots}{\Phi \vdash_{PN} p : A, \overline{\Psi}}}{\Phi \vdash_{PN} \overline{p}^L : A \wedge B, \overline{\Psi}}\,\mathsf{P!L}}{\Phi \vdash_{PD} \overline{p}^L : A \wedge B, \overline{\Psi}}\,\mathsf{PN}}{\Phi \vdash_{O} \overline{p}^L : A \wedge B, \overline{\Psi}}\,c_O \qquad \dfrac{\dfrac{\vdots\ \vdots\ \vdots}{\Phi \vdash_{PD} \overline{p}^R : A \wedge B, \overline{\Psi}}}{\Phi \vdash_{O} \overline{p}^R : A \wedge B, \overline{\Psi}}\,c_O}{\Phi \vdash_{O} p : A \wedge B, \overline{\Psi}}\,\mathsf{O?}\wedge$$
$$\vdots\ \vdots\ \vdots$$

   We translate this section to $\wedge$r of G3I^M. Although (in the shown example) p : B does not occur in the right branch, this is no problem, as the closure is completely independent of B, as well as of $A \wedge B$: rules *ax*/P!! can only be applied on prime formulas and in case of $\supset$r/P!$\supset$ or $\neg$r/P$*\neg$, B vanishes anyway.

3. In both of these branches p does not defend against the $\wedge$-attack. There is also no problem to translate O's attack to an application of $\wedge$r in the G3I^M-tree for the same reasons as described in the previous case.

□

Soundness for DiaSeqI$^+$ is shown in the same way, but of course we translate the DiaSeqI$^+$-trees to IG^r-trees. The rule P!$\supset^+$ then corresponds to *a fortiori* of IG/IG^r. The meta-restrictions of DiaSeqI$^+$ already correspond to the ones of regular IG-derivations, as discussed in Section 3.2.2.

Theorem 3.5 (DiaSeqI$^+$ Soundness)**.** *Every* DiaSeqI$^+$*-proof tree can be transformed to a regular* IG*-proof-tree.*

3.4 Summary and Comparison to other Systems

The purpose of our multi-proponent system is to have a game-theoretic decision procedure for intuitionistic propositional logic. Instead of most of the standard dialogical approaches there are not only two players in MPID, namely one opponent and one proponent, but instead the initial proponent stating the hypothesis is supported by further proponent agents. Together they fight a single opponent. In this chapter we discussed this approach in detail and considered an extension that enforces termination of the dialogues in all cases. The informal game rules were implemented in the formal dialogical sequent systems DiaSeqI and DiaSeqI^{+}. These were then used to show that MPID can be used as a sound and complete decision procedure for intuitionistic propositional logic. In the following we compare the multi-proponent intuitionistic dialogues and the calculi DiaSeqI and DiaSeqI^{+} with different sequent calculi (especially those using a focus) and dialogical systems.

3.4.1 Comparison to other Dialectic Systems

Concerning the different properties of the dialectics presented in Chapter 2.3, one can say that MPID uses a variant of the E-rule because O is always obliged to react to all moves of the previous round. Like in the constructive dialectics by Barth and Krabbe, we make use of the *ipse dixisti* remark. A difference is that in our case this remark is part of the particle rules (!!), but this is not a profound change. The same thing is done with the *absurdum dixisti* remark. We interpret it as a $\bot$-attack against which one cannot defend and which cannot be counter-attacked. However, the result is the same. Allowing the proponents to state prime formulas that have not yet been stated by O, and using a variant of the E-rule, makes it easier to show soundness and completeness of MPID as calculus.

We achieve *termination* without the usage of ranks because these are problematic with respect to proof searches (see Chapter 2.3.4). By contrast, we restrict the proponents' possibilities to repeat moves. A similar technique was already proposed by Barth and Krabbe in terms of structural rules [10]. In our attempt, we tried to simplify the complex rules by introducing a non-critical defence against attacks on implications. The idea is adapted from Corsi's *a fortiori* rule of sequent system IG [32, 33] (Chapter 2.1.4).

In the dialogue sequents by Barth and Krabbe [10], every rule application corresponds exactly to one move performed by a player (see Chapter 2.3.3). By contrast, in DiaSeqI and DiaSeqI$^+$ we have additional rules, which are needed for scheduling, e.g., c_O, PN, and c_P. There are also the trigger rules O$*\neg$, O$*\supset$, and P$*\neg$ which are necessary to add formulas to the sequents in the right phase. One could criticize DiaSeqI because its rule applications do not correspond exactly to the single moves of the dialogue tableau. However, the proponents' decision process when confronted with a critical attack, and the fact that there are now many proponents, makes it harder and actually unnecessary to put too much importance on this issue.

Now let us turn to the multi-proponent dialectics MPD^{FM} by Fermüller and Ciabattoni (Chapter 2.3.5). They implement proof systems for different intermediate propositional logics, whereas MPID is currently restricted to intuitionistic logic. So we compare MPID only to the intuitionistic variant of MPD^{FM}. These are the main differences:

1. In MPD^{FM} any P-agent can perform a *fork*-move anytime before performing an *internal move*. In MPID a P-agent forks *automatically* whenever he has the possibility to perform different moves. This reduces the flexibility, but on the other hand uncontrolled/inexpedient forking is prevented.

2. In MPD^{FM} O commits herself towards the different proponents with different concessions, i.e, concessions towards a proponent P_1 are only visible and usable for P_1 and not for any other agent P_2. In MPID

O's concessions towards any P-agent are globally available to all the proponents, as long as the addressed proponent is not deactivated in the same round.

3. MPD^{FM} does not implement the concept of critical attacks, as there are several concurrent dialogues and in each only one proponent acts. In MPID, as O's concessions are shared among all P-agents, an *isolation* is necessary which deactivates proponents in certain cases.

4. The proponents in MPID have a *global strategy*, i.e., they have to decide *together* in the decision phase which agent shall carry on and which are deactivated. In MPD^{FM} the P-agents are completely independent of each other once they are introduced.[14]

Note that MPD^{FM} and MPID pursue different aims. MPD^{FM} provides calculi for different intermediate logics and therefore gives the proponents the ability to merge again, i.e., to *share* their information. MPID is created only for intuitionistic logic (although it can easily be modified to deal with classical logic as well). We later extend it to cope with modal logics in which merging does not play a role and the MPID-metaphor of isolation fits well.

3.4.2 Comparison to Intuitionistic Sequent Calculi

Technically, MPD^{FM} follows the idea of single-conclusion hypersequents, while MPID (with DiaSeqI and DiaSeqI^{+}) is related to ordinary multi-conclusion sequent systems. With respect to termination, DiaSeqI^{+} clearly adapts the properties of regular derivation in IG which are also partly implemented in DiaSeqI.

It is quite obvious that DiaSeqI and DiaSeqI^{+} enforce a scheduling in the proof, i.e., one is less free to choose different rule applications as in

[14] This is not the case when *super-intuitionistic* (logics less restrictive than intuitionistic) logics are considered, as then *merging* becomes relevant.

standard sequent calculi. This shows similarities to focusing sequent systems as discussed in Chapter 2.1.5.[15] The following comparison refers to systems LJQ* (Figure 2.8, p. 43) and LJF (Figure 2.9, p. 47). System LJQ′ is the single-conclusion variant of LJQ* and therefore, for our purposes, not as relevant as the latter.

Both LJQ* and LJF use different symbols to distinguish different kinds of *phases*: in LJQ* these are two different arrows ($\Rightarrow$ and $\rightarrow$) and in LJF there are three different forms of sequents ($\longrightarrow$, $\xrightarrow{A}$, and $-_{A}\rightarrow$). In DiaSeqI we use the three different turnstiles ($\vdash_{\mathsf{O}}$, $\vdash_{\mathsf{PN}}$, and $\vdash_{\mathsf{PD}}$) for the scheduling mechanism.

In both the focus systems and DiaSeqI, decisions have to be made at certain points of the proving process, which are significant for success. In LJQ* it is the choice of the formula which shall be put into the *stoup* with rule Der. In LJF it is the selection of the focus with rules $F\mathsf{l}$ and $F\mathsf{r}$.

However, as mentioned before, relevant decisions are also made in LJF when rules $\wedge^{-}\mathsf{l}$ or $\vee\mathsf{r}$ are used. By contrast, in DiaSeqI there are actually no other choices than in the PD-phase. The rest is deterministic. Nevertheless, different choices might lead to other successful proofs, i.e., P-winning strategies. A P-agent's defence against a critical attack ($\mathsf{P}!\supset$) can be postponed to one of the next rounds with PN. For example, consider the sections of DiaSeqI-derivations shown in Figure 3.15. Both have the same initial situation. In the left dialogue, the defence against the critical attack is postponed, in the right one it is tackled immediately. Both P-strategies lead to the same result, although the left one contains unnecessary moves. In Figure 2.10 (p. 49) we see a similar situation in LJF: two derivations for the same sequent, one of them with redundant rule applications. So in both systems, unnecessary rule applications are possible.

[15] For this comparison the same features of DiaSeqI and DiaSeqI^{+} are relevant. Therefore we do not mention every time that we also compare DiaSeqI^{+} with the focusing sequent systems.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{PN}} \overline{p1} : A}\,\mathsf{P!!}}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{O}} \overline{p1} : A}\,c_{\mathsf{O}}, \mathsf{PN}}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{O}} p1 : A}\,\mathsf{O?}a}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{PN}} p1 : A}\,c_{\mathsf{P}}}{o_{p0} : A \vdash_{\mathsf{PD}} \overline{p1} : B \supset A,\ \overline{p2} : C}\,\mathsf{P!}\supset}{o_{p0} : A \vdash_{\mathsf{O}} \overline{p1} : B \supset A,\ \overline{p2} : C}\,c_{\mathsf{O}}}{o_{p0} : A \vdash_{\mathsf{O}} \overline{p1} : B \supset A,\ p2 : C}\,\mathsf{O?}a}{o_{p0} : A \vdash_{\mathsf{PN}} \overline{p1} : B \supset A,\ p2 : C}\,c_{\mathsf{P}}}{o_{p0} : A \vdash_{\mathsf{PN}} \overline{p1} : B \supset A,\ \overline{p2}^{L} : C \wedge D}\,\mathsf{P!L}}{\dfrac{o_{p0} : A \vdash_{\mathsf{PD}} \overline{p1} : B \supset A,\ \overline{p2}^{L} : C \wedge D}{\vdots \quad \vdots \quad \vdots}}\,\mathsf{PN}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{PN}} \overline{p1} : A}\,\mathsf{P!!}}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{O}} \overline{p1} : A}\,c_{\mathsf{O}}, \mathsf{PN}}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{O}} p1 : A}\,\mathsf{O?}a}{o_{p0} : A, o_{p3} : B \vdash_{\mathsf{PN}} p1 : A}\,c_{\mathsf{P}}}{\dfrac{o_{p0} : A \vdash_{\mathsf{PD}} \overline{p1} : B \supset A,\ \overline{p2}^{L} : C \wedge D}{\vdots \quad \vdots \quad \vdots}}\,\mathsf{P!}\supset$$

Figure 3.15: Section of DiaSeqI-proof containing irrelevant moves

The scheduling mechanism in DiaSeqI works differently than in LJQ*/LJF, as the round structure does not exist in the latter ones: in DiaSeqI, rules have to be applied on the left side, one on each formula, then on the right side and so on. By contrast, once the focus is set in LJQ* or LJF, a sequence of rule applications is performed on a formula and its active formulas, until the focus is released. One can say that in DiaSeqI the scheduling is *parallel*, while in the traditional focusing sequent systems it is *sequential*. Note that this parallelisation could be used to implement DiaSeqI in a concurrent way. This is left as future work.

The parallel scheduling forces us to perform rule applications which might not be helpful. For example, in the DiaSeqI-sequent

$$o_{p0} : (A \wedge D) \wedge B,\ o_{p_2} : C \supset A\ \vdash_{\mathsf{PN}}\ \overline{p1} : B$$

agent $p2$ is forced to attack $C \supset A$ and $p0$ is going to introduce a colleague so that both $A \wedge D$ and B can be demanded. Many of such moves are not helpful for the proponents to win, as here they only need the B. However, this *breadth-first search* is actually a reasonable strategy when trying to find a proof. Heuristics can be applied to restrict moves, e.g., by looking at the subformulas and thereby checking whether it makes actually sense to introduce further P-agents in certain situations.

A big advantage of MPID compared to focus calculi is its flexibility. It is easy to translate non-invertible rules into proponent moves and put these into the decision phase of the round cycle. Other logics can be adapted in a straightforward way, as we do with modal logics in the following chapter. In LJF the different polarities add an extra complexity which have an impact on different rules. The lack of ordinary focused sequent systems for modal logic can be seen as an indication that such systems cannot be established in a direct way.

4 Multi-Proponent Dialogues for Modal Logic

In this chapter we investigate a multi-proponent approach for modal logics, in particular for S4 and its constructive counterpart CS4. We call these dialectics *multi-proponent modal dialectics (S4/CS4)* (MPMD/S4 and MPMD/CS4 for short). Alternative attempts for modal dialogues have been discussed in Chapter 2.4. The system presented here[1] is the first multi-proponent approach for modal logic. Further, to my current knowledge, the first completely formal adequateness proof for a dialogical system for the modal logic S4, and the first for a dialogical system that implements a *constructive modal logic*, are presented here.

In Section 4.1 we have a look at the game rules for the multi-proponent S4-approach and the corresponding sequent system DiaSeqS4. These are based on the rules of MPID and on DiaSeqI, respectively. In 4.2 we modify the rules to obtain a system for CS4. Proofs for soundness and completeness of both sequent-style calculi are presented in Section 4.3. The chapter closes with a short summary and a comparison to other dialectic systems and sequent calculi.

[1] I already introduced MPMD/S4 in [139].

Assert	$\Box A$	$\Diamond A$
Attack	$?_{\Box}$	$?_{\Diamond}$
Defence	A	A

Figure 4.1: Particle Rules for modal operators

4.1 Multi-Proponent Dialogues for S4

4.1.1 Game Rules

Semantically, we interpret the $\Box$-modality in a similar way as Krabbe (see Chapter 2.4.1) but without different strictness levels. As there is no obvious dialogical interpretation for the $\Diamond$-modality and because of the duality between $\Box$ and $\Diamond$ in classical and normal modal logics, we propose two different sets of structural rules: one *simplified* that does not cope with the $\Diamond$-operator, and one *complete*. The complete one is necessary, as later, when we establish a system for CS4, the duality of $\Box$ and $\Diamond$ is given up and therefore the $\Diamond$ must be handled as an independent logical operator.

Particle Rules

As particle rule for the $\Box$ we adapt that proposed by Lorenzen (Chapter 2.4.1) and adapt a $\Diamond$-rule accordingly. The modal particle rules are shown in Figure 4.1 and are simply added to those of MPID (Figure 3.1, p. 100). There is actually nothing special or surprising about these rules. Like Lorenzen [102] and almost all of his successors, we define the properties about the modalities in the structural rules.

Simplified Structural Rules

The simplified structural rules for MPMD/S4 without the $\Diamond$-operator were already presented in [139]. They are based on those of MPID (p. 100), but implication and negation are not critical operators anymore. Instead, critical attacks are only attacks on $\Box$-statements. Like in Krabbe's attempts [83, 86] O may *withdraw* a commitment when a P-agent defends against a $\Box$-attack. We distinguish *weak* commitments and *strong* ones: *strong commitments* are assertions by O prefixed by a $\Box$. By contrast, *weak commitments* are all other concessions by O.

The rules I1 to I4 and I8 of MPID (pp. 100, 101) are adapted without any changes. Rules I5 to I7 are replaced by the following [139]:

S4-5 Only O is allowed to attack *prime formulas*. P-agents may defend against these attacks only if O has stated the prime formula herself towards a P-agent who is not *deactivated* in the same round and if O has not *withdrawn* it.

S4-6 Attacks on $\Box$ are considered to be *critical attacks*. Other attacks are *non-critical*.

S4-7 Whenever a P-agent reacts to a *critical attack*, all other active proponent agents are immediately *deactivated*, i.e., excluded from the rest of the dialogue run. O may immediately *withdraw* all *weak commitments* she made before in same the run.

As we see, these rules are very close to those of MPID. The set of critical attacks is changed and the concept of withdrawals is introduced and connected to these attacks. Deactivating is also a bit stricter, as the concerned agents are excluded completely .

Complete Structural Rules

The structural rules which involve the $\Diamond$ are more complex and considered separately because there is currently no dialogical interpretation for the $\Diamond$ that fits to the interpretation of the $\Box$. We now call assertions starting with a $\Diamond$ *hypothetical assertions*. Hypothetical assertions stated by the opponent are accordingly called *hypothetical concessions*. We choose these terms as they give an idea of how the ◇-operator can be interpreted semantically in a dialogical setting.[2] We do not claim that this is the best choice but rather a possible suggestion, as our focus lies on proof theory and not on philosophical issues or argumentation theory. Attacks against hypothetical assertions are called *co-critical attacks*.

For the complete structural rules for MPMD/S4 we adapt again rules I1, I2 and I4 of MPID and rule S4-5 of the simplified rules. We have to adjust rules I3, S4-6, S4-7, and I8 as follows:

S4-3* If possible, all players are obliged to perform moves. A P-agent may postpone a move until succeeding rounds if he is forced to react to a *critical attack* or to perform a *co-critical attack*. Whenever a P-agent has several possibilities of how to react to an O-move, new P-agents are introduced to take out these remaining possibilities.
When a proponent agent states a hypothetical assertion, he is *protected* and *stunned*, i.e., he stays active but does not perform moves anymore. Instead, reactions to the (co-critical) attacks towards him are performed by new colleagues.[3]

S4-6* Attacks on □ are considered to be *critical attacks*. Attacks on ◇ are considered to be *co-critical attacks*. Other attacks are *non-critical*.

S4-7* Whenever a P-agent reacts to a *critical attack*, all other non-protected proponent agents are immediately *deactivated*, i.e., excluded from the rest of the dialogue run. The remaining proponent agents miss their turn in the round. O may immediately *withdraw* all *weak commitments* she made before in the same run.

[2]In the *alethic* interpretation of the modal operators, □ corresponds to *necessarily* and ◇ to *possibly* which we interpret here as *hypothetically*.

[3]In this variant of MPMD/S4, a P-agent can be either stunned and protected, or neither stunned nor protected at the same time. For CS4 (Section 4.2.2) we need to distinguish these properties.

	O	P0	P1	P2
1		$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$		
2	$[\mathbf{?}, 1]^0$ ~~$\Diamond A \supset \Box B$~~	$[\mathbf{!}, 2]$ $\Box(A \supset B)$	$[\mathbf{?}, 2]$ $\Diamond A$	
3	$[\mathbf{?}, 2]^0$ $\mathbf{?}_{\Box}$ $[\mathbf{?}, 2]^1$ $\mathbf{?}_{\Diamond}$	$[\mathbf{!}, 3]$ $A \supset B$	*stunned* *protected*	
4	$[\mathbf{?}, 3]^0$ A $[\mathbf{?}, 2]^1$ $\mathbf{?}_{\Diamond}$	$[\mathbf{!}, 4]$ B		 $[\mathbf{!}, 4]$ A
5	$[\mathbf{?}, 4]^0$ $\mathbf{?}_{B}$ $[\mathbf{?}, 4]^2$ $\mathbf{?}_{A}$	— —		 $[\mathbf{!}, 5]$ **!!**

Figure 4.2: An example of an MPMD/S4-Dialogue

Whenever the opponent reacts to a *co-critical attack*, all non-protected P-agents are immediately *deactivated*, i.e., excluded from the rest of the dialogue run. O may immediately *withdraw* all *weak commitments* she made before in the same run.

S4-8* A P-agent may repeat critical attacks on the same assertion only after some P-agent reacted to a critical attack performed by O or O reacted to a co-critical attack.
The opponent may repeat an attack against the same protected agent only after some P-agent reacted to a critical attack performed by O or O reacted to a co-critical attack.
Other repetitions are not allowed.

With the new concepts of *hypothetical assertions*, *co-critical attacks*, *protected* and *stunned agents*, these rules are much more complex than the simplified ones for which the $\Diamond$-operator is not regarded. Figure 4.2 shows a dialogue where the hypothesis is the **IK5**-axiom which is valid in classical K and therefore also in S4. Only one of O's strategies is displayed (she counter-attacks P1 in row 3).[4] Actually, a new P-agent should defend here with A while P1 is stunned and protected, but this agent is deactivated immediately as P0

[4]The other run is won by the proponents as well.

defends against a critical attack. O may then attack P1 again (row 4). This is defended by P2 who eventually finished the game with *ipse dixisti*. Note that O withdraws her implication of row 2 after P0 defends in row 3, although this is not necessary as the proponents may only repeat critical attacks after defending against a critical attack, and implications are not critically here.

4.1.2 The System DiaSeqS4

As we did in Section 3.2 for intuitionistic propositional logic, we now translate the rules from above into a sequent system we call DiaSeqS4. To make this as general as possible, we include the $\diamond$-operator and refer to the complete set of structural rules, although it is not really relevant from the dialogical point of view.[5]

The rules are displayed in Figure 4.3. Only the rules which differ from those of DiaSeqI (Figure 3.5, p. 107) are shown. Note that P!$\supset$ and P$*\neg$ are not critical anymore and therefore are now part of the *normal phase*. Further, duplication is not necessary for these connectives, so O$*\supset$ and O$*\neg$ are changed (the proponents' attacks on implications and negations are not critical anymore).

Attacks by O on $\Box$ and $\diamond$ are straightforward. The delicate issues are P-agents' defences of $\Box$ and attacks on $\diamond$, as these result in a change of the world in the sense of Kripke. By contrast to DiaSeqI, these moves do not lead to the normal phase, but instead end the party's moves and let O start the next turn (see Figure 4.4). If a P-agent defends with P!$\Box$, the other players are deactivated and O withdraws her weak commitments (Φ^{δ}). When P attacks with P?$\diamond$, O defends directly.

[5]DiaSeqS4 was already presented in [139], but in a variant in which the *simplified* structural rules are implemented, i.e., the $\diamond$-operator is not considered there, which also makes the sequent-rules simpler.

o-rules

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : A \supset B}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : A \supset B}\ \mathsf{O?}\!\supset \qquad \frac{\Phi \vdash_{\mathsf{O}} \Psi,\ p : A \qquad o_p : B,\ \Phi \vdash_{\mathsf{O}} \Psi}{\overline{o_p} : A \supset B,\ \Phi \vdash_{\mathsf{O}} \Psi}\ \mathsf{O}{*}\!\supset$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \neg A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \neg A}\ \mathsf{O?}\neg \qquad \frac{\Phi \vdash_{\mathsf{O}} \Psi,\ p : A}{\overline{o_p} : \neg A,\ \Phi \vdash_{\mathsf{O}} \Psi}\ \mathsf{O}{*}\neg$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \Box A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \Box A}\ \mathsf{O?}\Box \qquad \frac{\Phi,\ \widetilde{o} : \Box A,\ o_p : A \vdash_{\mathsf{O}} \Psi}{\Phi,\ \overline{o_p} : \Box A,\ \vdash_{\mathsf{O}} \Psi}\ \mathsf{O!}\Box$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \Diamond A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \Diamond A}\ \mathsf{O?}\Diamond \qquad \frac{\Phi^{\delta},\ o_p : A \vdash_{\mathsf{O}} \Psi^{\delta}}{\Phi,\ \overline{o_p} : \Diamond A,\ \vdash_{\mathsf{O}} \Psi}\ \mathsf{O!}\Diamond$$

p-rules – decide phase

$$\frac{\Phi^{\delta} \vdash_{\mathsf{O}} p : A,\ \Psi^{\delta}}{\Phi \vdash_{\mathsf{PD}} \overline{p} : \Box A,\ \Psi}\ \mathsf{P!}\Box \qquad \frac{\overline{o_p} : \Diamond A,\ \Phi \vdash_{\mathsf{O}} \Psi}{o_p : \Diamond A,\ \Phi \vdash_{\mathsf{PD}} \Psi}\ \mathsf{P?}\Diamond$$

p-rules – normal phase

$$\frac{\overline{o_p} : \Box A,\ \Phi \vdash_{\mathsf{PN}} \Psi}{o_p : \Box A,\ \Phi \vdash_{\mathsf{PN}} \Psi}\ \mathsf{P?}\Box \qquad \frac{\Phi \vdash_{\mathsf{PN}} \Psi,\ \widetilde{p} : \Diamond A,\ q : A}{\Phi \vdash_{\mathsf{PN}} \Psi,\ \overline{p} : \Diamond A}\ \mathsf{P!}\Diamond$$

$$\frac{o_q : A,\ \Phi \vdash_{\mathsf{PN}} \Psi,\ p : B}{\Phi \vdash_{\mathsf{PN}} \Psi,\ \overline{p} : A \supset B}\ \mathsf{P!}\!\supset \qquad \frac{o_p : A,\ \Phi \vdash_{\mathsf{PN}} \Psi}{\Phi \vdash_{\mathsf{PN}} \Psi,\ \overline{p} : \neg A}\ \mathsf{P}{*}\neg$$

$$\Phi^{\delta} =_{df} \{o_r : \Box\varphi \mid r \in \mathit{Propos}, o_r : \Box\varphi \in \Phi\} \cup \{\overline{o_r} : \Box\varphi \mid r \in \mathit{Propos}, \overline{o_r} : \Box\varphi \in \Phi\} \cup \{o_s : \varphi \mid \widetilde{o} : \varphi \in \Phi\}$$

every s is a new P-agent.

$$\Psi^{\delta} =_{df} \{r : \Diamond\varphi \mid r \in \mathit{Propos}, \widetilde{r} : \Diamond\varphi \in \Psi \text{ or } r : \Diamond\varphi \in \Psi\} \cup \{\overline{r} : \Diamond\varphi \mid r \in \mathit{Propos}, \overline{r} : \Diamond\varphi \in \Psi\}$$

q is a new P-agent in rules $\mathsf{P!}\Diamond$ and $\mathsf{P!}\supset$.

Figure 4.3: Rules of DiaSeqS4

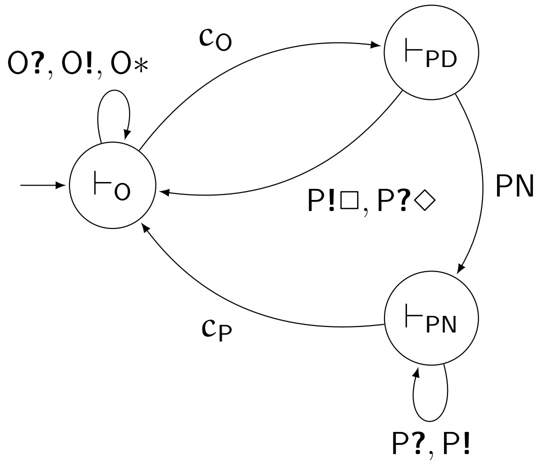


Figure 4.4: The phases in DiaSeqS4

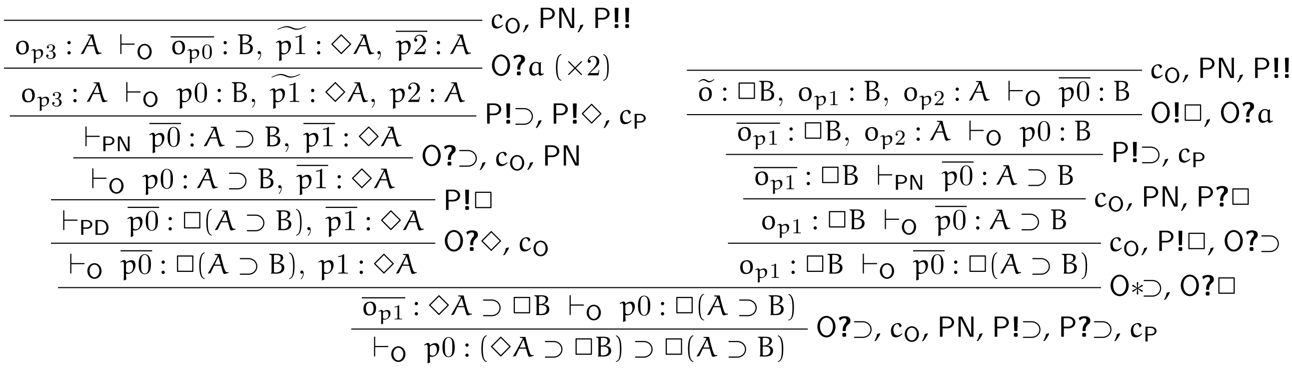


Figure 4.5: An example of a DiaSeqS4-proof

The duplication rules are now O!□ and P!◇ which correspond to □l and ◇r of G3S4 (Section 2.2.1). Note that the tilde is now also a possible marker on the P-side. Therefore, we redefine the earlier definitions of Φ and Ψ as follows:

$$\Phi \subseteq \{o_p : \varphi,\ \overline{o_p} : \varphi,\ \overline{o_p}^L : \varphi,\ \overline{o_p}^R : \varphi,\ \widetilde{o_p} : \varphi \mid p \in \mathit{Propos}, \varphi \in \mathit{Form}\} \quad (4.1)$$

$$\Psi \subseteq \{p : \varphi,\ \overline{p} : \varphi,\ \overline{p}^L : \varphi,\ \overline{p}^R : \varphi,\ \widetilde{p} : \varphi \mid p \in \mathit{Propos}, \varphi \in \mathit{Form}\} \quad (4.2)$$

Figure 4.5 shows a DiaSeqS4-proof for the **IK5**-axiom. Note that there are small differences to the tableau of Figure 4.2. The first corresponds to the left path: when p0 defends $\Box(A \supset B)$ with P!□, p1 stays attacked and the defence is postponed to the next round. If we transferred the structural rules directly, it would have been necessary to remove the bar and let O attack p1 again in the next round. In the end, it does not make a difference. Another point is that O states A towards a new proponent p3 who does not appear in the tableau. This is because of a technical issue and only becomes relevant when O's statement is counter-attacked.

Note that in this chapter, we only refer to *mono-modal* S4 and CS4. *Multi-modal* variants in which the elements of the modal signature are independent of each other, can be used here as well. This is simply done by replacing $\Box$ and $\Diamond$ by $[l]$ and $\langle l \rangle$ in the corresponding rules O?$\Box$, O?$\Diamond$, O!$\Box$, O!$\Diamond$, P?$\Box$, P?$\Diamond$, P!$\Box$, and P!$\Diamond$ (including Φ^δ and Ψ^δ).

4.2 Dialogues for Constructive S4

In this section we combine the features of MPID and DiaSeqI with those of MPMD/S4 and DiaSeqS4, to obtain a dialectic system for the constructive modal logic CS4. The structural rules of theses dialogues are very complex and it is easier to start with DiaSeqS4 and modify it referring to the special properties of G3iCS4^M to obtain the adequate dialogical reasoning procedure DiaSeqCS4. Once this is established, we derive the informal structural rules.

4.2.1 The System DiaSeqCS4

The sequent version of multi-proponent CS4-dialogues is adapted from DiaSeqS4 and G3iCS4^M. The latter is investigated in Chapter 2.2.2. The rules of DiaSeqCS4 are displayed in Figure 4.6. The rules of $\wedge$, $\vee$, and $\bot$ are the same as in both DiaSeqI and DiaSeqS4, and therefore omitted in the figure. The same holds for the rules c_O, c_P, and PN. There are no surprises on the O-side: O$*\supset$, O$*\neg$, and O!$\Box$ keep the attacked formula in Φ, as $\supset$l, $\neg$l, and $\Box$l of G3iCS4^M are also duplication rules. As one might assume,

o-rules

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : A \supset B}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : A \supset B}\ \mathsf{O?{\supset}} \qquad \frac{\widetilde{o_p} : A \supset B,\ \Phi \vdash_{\mathsf{O}} \Psi,\ p : A \qquad o_p : B,\ \Phi \vdash_{\mathsf{O}} \Psi}{\overline{o_p} : A \supset B,\ \Phi \vdash_{\mathsf{O}} \Psi}\ \mathsf{O{*}{\supset}}$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \neg A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \neg A}\ \mathsf{O?{\neg}} \qquad \frac{\widetilde{o_p} : \neg A,\ \Phi \vdash_{\mathsf{O}} \Psi,\ p : A}{\overline{o_p} : \neg A,\ \Phi \vdash_{\mathsf{O}} \Psi}\ \mathsf{O{*}{\neg}}$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \Box A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \Box A}\ \mathsf{O?{\Box}} \qquad \frac{\Phi,\ \widetilde{o} : \Box A,\ o_p : A \vdash_{\mathsf{O}} \Psi}{\Phi,\ \overline{o_p} : \Box A,\ \vdash_{\mathsf{O}} \Psi}\ \mathsf{O!{\Box}}$$

$$\frac{\Phi \vdash_{\mathsf{O}} \Psi,\ \overline{p} : \Diamond A}{\Phi \vdash_{\mathsf{O}} \Psi,\ p : \Diamond A}\ \mathsf{O?{\Diamond}} \qquad \frac{\Phi^{\epsilon},\ o_p : A \vdash_{\mathsf{O}} \Psi^{\epsilon}}{\Phi,\ \overline{o_p} : \Diamond A \vdash_{\mathsf{O}} \Psi}\ \mathsf{O!{\Diamond}}$$

p-rules – decide phase

$$\frac{o_q : A,\ \Phi^{\delta} \vdash_{\mathsf{O}} p : B}{\Phi \vdash_{\mathsf{PD}} \Psi,\ \overline{p} : A \supset B}\ \mathsf{P!{\supset}} \qquad \frac{o_p : A,\ \Phi^{\delta} \vdash_{\mathsf{O}} \emptyset}{\Phi \vdash_{\mathsf{PD}} \Psi,\ \overline{p} : \neg A}\ \mathsf{P{*}{\neg}}$$

$$\frac{\Phi^{\epsilon} \vdash_{\mathsf{O}} p : A}{\Phi \vdash_{\mathsf{PD}} \overline{p} : \Box A,\ \Psi}\ \mathsf{P!{\Box}} \qquad \frac{\overline{o_p} : \Diamond A,\ \Phi \vdash_{\mathsf{O}} \Psi \qquad \Psi^{\epsilon} \neq \emptyset}{o_p : \Diamond A,\ \Phi \vdash_{\mathsf{PD}} \Psi}\ \mathsf{P?{\Diamond}}$$

$$\frac{\Phi^{\delta} \vdash_{\mathsf{O}} p : A}{\Phi \vdash_{\mathsf{PD}} \Psi,\ \overline{p} : \Diamond A}\ \mathsf{P!{\Diamond}}$$

p-rules – normal phase

$$\frac{\overline{o_p} : \Box A,\ \Phi \vdash_{\mathsf{PN}} \Psi}{o_p : \Box A,\ \Phi \vdash_{\mathsf{PN}} \Psi}\ \mathsf{P?{\Box}}$$

$$\Phi^{\delta} =_{df} (\Phi \setminus \{\widetilde{o_p} : \varphi \mid p \in \mathit{Propos}\}) \cup \{o_s : \varphi \mid \widetilde{o_p} : \varphi \in \Phi, p \in \mathit{Propos}\},$$

$$\Phi^{\epsilon} =_{df} \{o_s : \Box\varphi \mid \widetilde{o} : \Box\varphi \in \Phi\} \cup \{o_p : \Box\varphi \mid p \in \mathit{Propos}, o_p : \Box\varphi \in \Phi\} \cup \{\overline{o_p} : \Box\varphi \mid p \in \mathit{Propos}, \overline{o_p} : \Box\varphi \in \Phi\}$$

every s is a new P-agent.

$$\Psi^{\epsilon} =_{df} \{r : \Diamond\varphi \mid r \in \mathit{Propos}, \widetilde{r} : \Diamond\varphi \in \Psi \text{ or } r : \Diamond\varphi \in \Psi\} \cup \{\overline{r} : \Diamond\varphi \mid r \in \mathit{Propos}, \overline{r} : \Diamond\varphi \in \Psi\}$$

Figure 4.6: Rules of DiaSeqCS4

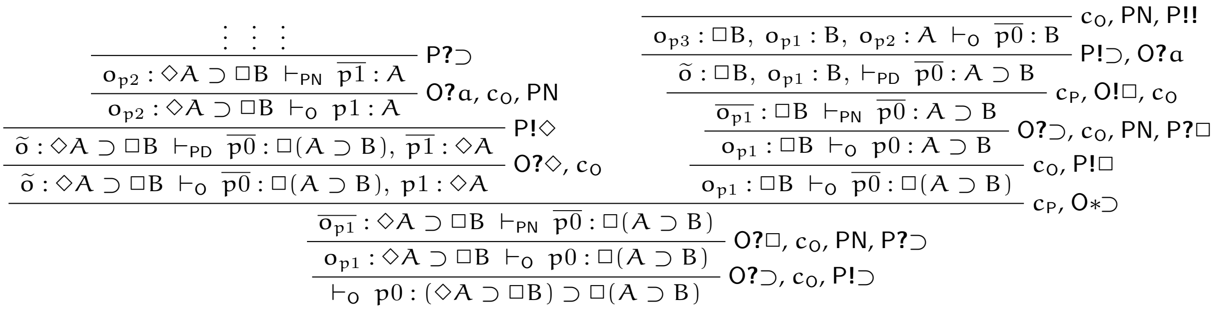


Figure 4.7: The proponents losing in DiaSeqCS4

P!⊃ and P∗¬ are critical as in DiaSeqI, and P!□ and P?◇ as in DiaSeqS4. Additionally, P!◇ counts as critical. In P?◇ it is required that some active P-agent's last assertion was a ◇-formula (corresponds to rule ◇l in G3iCS4^{M}). By contrast to DiaSeqS4, all reactions to a critical attack remove all the other P-agents.

Figure 4.7 shows the **IK5**-example again, but this time in DiaSeqCS4. In the left branch, as P!◇ performed by p1 is a reaction to a critical attack, p0 is deactivated, which makes it impossible for the proponents to win the game. Actually, the run will never terminate, as the proponents are allowed to attack O's implication again and again. Note that if p0 had defended instead of p1, then p1 would have been deactivated as well. There is no protection as compared to MPMD/S4.[6]

4.2.2 Game Rules

It is now time to have a look at the structural rules of MPMD/CS4. As particle rules we use the same set as that of MPMD/S4.[7] In CS4 the modal operators □ and ◇ are independent of each other. Therefore, a simplified version of structural rules as in MPMD/S4 is not possible, so the structural rules for MPMD/CS4 are based on the complete version of MPMD/S4.

[6]There is some kind of protection which is however weaker than in MPMD/S4. More on this will be discussed in the following section.

[7]See Figures 3.1 and 4.1 on pages 100 and 166.

The rules are only given here for the sake of completeness. We are not interested in the philosophical interpretation or whether they make sense from an argumentation-theoretic perspective. For this work the proof-theoretical consequences of the scheduling are more important, which are however more apparent in sequent-style counterpart.

The complete set of informal structural rules for MPMD/CS4 are as follows:

CS4-1 At the beginning of a dialogue, O states *initial concessions* and a single P-agent states the *hypothesis*.

CS4-2 A *round* consists of a sequence of moves by O, followed by moves of all active P-agents. A *dialogue run* is a sequence of such rounds. The first round starts after the assertion of the hypothesis.

CS4-3 If possible, all players are obliged to perform moves. A P-agent may postpone a move until succeeding rounds if he is forced to react to a *critical attack* or to perform a *co-critical attack*. Statements by O made with the critical attack are only conceded as soon as the critically attacked agent reacts. Whenever a P-agent has several possibilities of how to react to an O-move, new P-agents are introduced to take out these remaining possibilities. When a proponent agent states a hypothetical assertion, he is *protected*.[8]

CS4-4 A dialogue is *won* by the proponents iff the opponent cannot react to all of the proponents' moves of the previous round. The opponent wins iff no P-agent can react to any of O's statements of the same round (either with an attack or a defence).

CS4-5 Only O is allowed to attack *prime formulas*. P-agents may defend against these attacks only if O has stated the prime formula herself towards a P-agent who is not *deactivated* in the same round and if O has not *withdrawn* it.

[8]Note that the protection is necessary as in O!$\Diamond$ these agents are kept active while the others are deactivated.

CS4-6 Attacks on $\Box$, $\Diamond$, implications, and negations are considered to be *critical attacks*. Attacks on $\Box$ and $\Diamond$ are *strongly critical*. Attacks on $\Diamond$ are (additionally) considered to be *co-critical attacks*. Other attacks are *non-critical*.[9]

CS4-7a Whenever a P-agent reacts to a *critical attack*, all other active proponent agents are immediately *deactivated*, i.e., they may not perform defences or counter-attacks. The remaining proponent agents miss their turn in the round. If the critical attack was strong, the deactivated agents are excluded completely from the rest of the dialogue run and O may immediately *withdraw* all *weak commitments* she made before in the same run.
Whenever the opponent reacts to a *co-critical attack*, all non-protected proponent agents are immediately *deactivated* and excluded from the rest of the dialogue run. O may immediately *withdraw* all *weak commitments* she made before in the same run.[10]

CS4-7b A P-agent may perform a *co-critical attack* only if there are *active protected* P-agents.[11]

CS4-8 A P-agent may repeat critical attacks on the same, not yet withdrawn assertion, only after any P-agent reacted to a critical attack performed by O.
A P-agent may also repeat strongly critical attacks after O reacted to a co-critical attack.
Other repetitions are not allowed.

In total, these structural rules are quite complex. The concept of *stunned P-agents* is given up, instead we distinguish *strongly critical attacks* and *critical*

[9]Here, we consider attacks on $\Diamond$ as critical as well. However, it is necessary to distinguish attacks on modal operators (strongly critical) and those on the critical operators in the sense of IPL (weakly critical).

[10]The new part of CS4-7a is the result of O!$\Diamond$, where non-protected P-agents are deactivated and weak concessions are withdrawn. When a P-agent reacts to a critical attack, O may only withdraw assertions if the attack was strongly critical.

[11]Rule CS4-7b corresponds to the restriction that Ψ^{ϵ} must not be empty in P?$\Diamond$.

attacking type	MPID	MPMD/S4 (simple)	MPMD/S4 (compl.)	MPMD/CS4
(weak) critical attacks	$A \supset B, \neg A$	$\Box A$	$\Box A$	$A \supset B, \neg A$
strong critical attacks				$\Box A, \Diamond A$
co-critical attacks			$\Diamond A$	$\Diamond A$

Table 4.1: Critical and co-critical attacks in the various systems

attacks. A new restriction for the proponents is introduced in terms of the new rule CS4-7b. This is necessary so that they do not have a winning strategy for $\neg\Diamond\bot$ which is not a theorem of CS4.

Because of the complexity of the rules, the most important differences between the presented dialogical systems are summarized in Tables 4.1 and 4.2. To simplify some issues we introduce the term *event*. An event is triggered when some P reacts to a critical attack. For MPMD/CS4 we need to distinguish *strong* and *weak* events.

Although the tables express again the complexity of the rules, they also reveal opportunities which have not yet been discovered, i.e., a combination of the various rule features might lead us to new proof systems for different logics. This offers a new perspective, as new possibilities become visible, which were not apparent when looking at axioms, Kripke semantics or sequent rules. For example, one can change the set of weak critical attacks and remove the negation, but keep the implication. One could also add operators to the set of co-critical attacks or change the conditions for stunned or protected proponent-agents.

Besides this, we can also add further features. In the presented multi-proponent systems all other P-agents are deactivated when one reacts to a critical attack. It might be possible that the remaining P-agent is allowed to *save* another one for the next rounds in certain circumstances.

Property	MPID	MPMD/S4 (simple)	MPMD/S4 (compl.)	MPMD/CS4
(weak) event occurs when	P reacts to critical attack	P reacts to critical attack	P reacts to critical attack or O reacts to co-critical attack	P reacts to weak critical attack
strong event occurs when				P reacts to strong critical attack or O reacts to co-critical attack
P-agents deactivated (kind of deactivation indicated in brackets)	all others when agent reacts to critical attack (no defences or counter-attacks)	all others when agent reacts to critical attack (excluded for rest of game)	all others who are not protected when P-agent causes event; all non-protected P-agents when O causes event (excluded for rest of game)	all others when P-agent causes event (weak: no defences or counter-attacks, strong: excluded completely); all non-protected P-agents when O causes event (excluded)
O withdraws weak commitments		event occurs	event occurs	strong event occurs
requirement for P to perform co-critical attacks				active P-agent exists whose last statement is $\Diamond A$
P may postpone	reaction to critical attack	reaction to critical attack	reaction to critical attack and co-critical attack	reaction to critical attack and co-critical attack
protected P-agents			last statement $\Diamond A$	last statement $\Diamond A$
stunned P-agents			protected P-agents	
O repeats attacks	never	never	on stunned agents after event	never
P repeats	critical attacks after event	critical attacks after event	critical attacks after event	critical attacks after event

Table 4.2: Comparison of Structural Rules of multi-proponent systems

4.3 Adequateness of Multi-Proponent S4-Dialogues

This section deals with soundness and completeness of DiaSeqS4 and DiaSeqCS4. The proofs are based on these for DiaSeqI (Chapter 3.3). We simply adjust some definitions and the proofs are extended in a straightforward manner.

4.3.1 Soundness

Soundness of DiaSeqS4 and DiaSeqCS4 is shown easily by transforming the derivations of the systems to the corresponding sequent systems G3S4 and G3iCS4^{M}. For both dialogical calculi, the transformation works exactly as for DiaSeqI as described in Chapter 3.3.3. The only issue that causes a bit of rearrangement are again O's attacks on conjunctions, but we deal with them the same way as before.

Theorem 4.1 (DiaSeqS4/DiaSeqCS4 Soundness)**.** *Every* DiaSeqS4*-proof-tree can be transformed to a* G3S4*-proof. Every* DiaSeqCS4*-proof-tree can be transformed to a* G3iCS4^{M}*-proof.*

4.3.2 Completeness of DiaSeqS4

Our reference sequent system is G3S4 as presented in Chapter 2.2.1. The rules are shown in Figure 2.12 (p. 56). The definitions of Chapter 3.3.1 (Section "Terminology") are adapted directly. Most of the lemmas and their proofs can be reused, as well as the definitions of macro blocks, micro blocks, and dependencies.

Normalization of G3S4-Proofs

We start with the lemmas which lead us to non-redundancy. To avoid confusion, we write $\Vdash_n^{S4}$ to express deducibility in G3S4.

Lemma 4.1 (Admissibility of Weakening (G3S4)). *For all* φ, Γ, Δ*:*

1. *If* $\Vdash_n^{S4} \Gamma \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma, \varphi \Rightarrow \Delta$.
2. *If* $\Vdash_n^{S4} \Gamma \Rightarrow \Delta$ *then* $\Vdash_n \Gamma \Rightarrow \varphi, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of the rule applications is preserved.

Proof. Straightforward: the base case is exactly the same as in the proof of Lemma 3.1. For the inductive step, we have to consider the rules $\Box l$, $\Diamond l$, $\Box r$, and $\Diamond r$ as well. $\Box l$ and $\Diamond r$ are trivial, whereas $\Box r$ and $\Diamond l$ can technically be treated similarly as the critical rule $\supset r$ of $G3I^M$, which is not critical here. □

Lemma 4.2 (Inversion (G3S4)). *For all* Δ, Γ *and formulas* A *and* B*:*

1. *If* $\Vdash_n^{S4} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma, A, B \Rightarrow \Delta$.
2. *If* $\Vdash_n^{S4} \Gamma \Rightarrow A \vee B, \Delta$ *then* $\Vdash_n^{S4} \Gamma \Rightarrow A, B, \Delta$.
3. *If* $\Vdash_n^{S4} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma, A \Rightarrow \Delta$ *and* $\Vdash_n^{S4} \Gamma, B \Rightarrow \Delta$.
4. *If* $\Vdash_n^{S4} \Gamma \Rightarrow A \wedge B, \Delta$ *then* $\Vdash_n^{S4} \Gamma \Rightarrow A, \Delta$ *and* $\Vdash_n^{S4} \Gamma \Rightarrow B, \Delta$.
5. *If* $\Vdash_n^{S4} \Gamma, A \supset B \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma \Rightarrow A, \Delta$ *and* $\Vdash_n^{S4} \Gamma, B \Rightarrow \Delta$.
6. *If* $\Vdash_n^{S4} \Gamma \Rightarrow A \supset B, \Delta$ *then* $\Vdash_n^{S4} \Gamma, A \Rightarrow B, \Delta$.
7. *If* $\Vdash_n^{S4} \Gamma, \neg A \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma, \neg A \Rightarrow A, \Delta$.
8. *If* $\Vdash_n^{S4} \Gamma \Rightarrow \neg A, \Delta$ *then* $\Vdash_n^{S4} \Gamma, A \Rightarrow \Delta$.
9. *If* $\Vdash_n^{S4} \Gamma, \Box A \Rightarrow \Delta$ *then* $\Vdash_n^{S4} \Gamma, \Box A, A \Rightarrow \Delta$.
10. *If* $\Vdash_n^{S4} \Gamma \Rightarrow \Diamond A, \Delta$ *then* $\Vdash_n^{S4} \Gamma \Rightarrow \Diamond A, A, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of rule applications is preserved.

Proof. Items 5 to 10 are different to those of Lemma 3.2. Items 5 to 8 are standard (c.f. Troelstra and Schwichtenberg [142], duplication is not necessary here). Items 9 and 10 follow directly by weakening (Lemma 4.1). □

Note that the *world-changing* rules $\Box r$ and $\Diamond l$ are the only non-invertible/ critical ones.

Lemma 4.3 (Admissibility of Contraction (G3S4)). *For all* φ, Γ, Δ*:*

1. *If* $\Vdash^{S4}_{n} \Gamma, \varphi, \varphi \Rightarrow \Delta$ *then* $\Vdash^{S4}_{n} \Gamma, \varphi \Rightarrow \Delta$.
2. *If* $\Vdash^{S4}_{n} \Gamma \Rightarrow \varphi, \varphi, \Delta$ *then* $\Vdash^{S4}_{n} \Gamma \Rightarrow \varphi, \Delta$.

The derivations in the conclusions do not contain further rule applications than the derivations in the conditions. The order of rule applications is preserved.

Proof by induction on n. Works the same way as the proof of Lemma 3.3. Nevertheless, here is an example.

Assume that $\Vdash^{S4}_{n+1} \Gamma', \Box\Gamma, \Diamond A, \Diamond A \Rightarrow \Delta', \Diamond\Delta$.

- If the lowest rule application a_0 of the derivation tree is non-critical and refers to a formula of Γ' or Δ', we can use the hypothesis on the sequent above and use a_0 to obtain the deduction with only one $\Diamond A$.
- If it is an application of $\Diamond l$ on $\Diamond A$, we have $\Vdash^{S4}_{n} \Box\Gamma, A \Rightarrow \Diamond\Delta$. The same result is obtained when it is applied in $\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Delta', \Diamond\Delta$.
- We deal with $\Box r$ as lowest rule application the same way.

□

The *disjunctive derivable set* (Definition 3.5, p. 123) is not sufficient to show that applications of duplication rules can be omitted when used more than once in the same macro block. We need a more general structure.

Definition 4.1 (Non-Critically Derivable Set (G3S4))**.** *Let* φ *be a formula of propositional modal logic. The pair* (χ^+, χ^-) *is a* left-derivable *pair of multi-sets for* φ*, written* $d^+(\chi^+, \chi^-, \varphi)$ *iff for any* A *and* B*:*

- $\chi^+ = \{\varphi\}$ *and* $\chi^- = \emptyset$ *or*
- $\varphi = A \wedge B$ *and there are* χ^+_A*,* χ^+_B*,* χ^-_A*, and* χ^-_B *such that* $\chi^+ = \chi^+_A, \chi^+_B$ *and* $\chi^- = \chi^-_A, \chi^-_B$ *with* $d^+(\chi^+_A, \chi^-_A, A)$ *and* $d^+(\chi^+_B, \chi^-_B, B)$ *or*
- $\varphi = A \vee B$ *and either* $d^+(\chi^+, \chi^-, A)$ *or* $d^+(\chi^+, \chi^-, B)$ *or*
- $\varphi = A \supset B$ *and either* $d^-(\chi^+, \chi^-, A)$ *or* $d^+(\chi^+, \chi^-, B)$ *or*
- $\varphi = \neg A$ *and* $d^-(\chi^+, \chi^-, A)$ *or*
- $\varphi = \Box A$ *and there are* χ^+_A *and* χ^-_A *such that* $\chi^+ = \Box A, \chi^+_A$ *and* $\chi^-_A = \chi^-$ *with* $d^+(\chi^+_A, \chi^-_A, A)$.

The pair (χ^+, χ^-) *is a* right-derivable *pair of multi-sets for* φ*, written* $d^-(\chi^+, \chi^-, \varphi)$ *iff for any* A *and* B*:*

- $\chi^+ = \emptyset$ *and* $\chi^- = \{\varphi\}$ *or*
- $\varphi = A \wedge B$ *and either* $d^-(\chi^+, \chi^-, A)$ *or* $d^-(\chi^+, \chi^-, B)$ *or*
- $\varphi = A \vee B$ *and there are* χ^+_A*,* χ^+_B*,* χ^-_A*, and* χ^-_B *such that* $\chi^+ = \chi^+_A, \chi^+_B$ *and* $\chi^- = \chi^-_A, \chi^-_B$ *with* $d^-(\chi^+_A, \chi^-_A, A)$ *and* $d^-(\chi^+_B, \chi^-_B, B)$ *or*
- $\varphi = A \supset B$ *and there are* χ^+_A*,* χ^+_B*,* χ^-_A*, and* χ^-_B *such that* $\chi^+ = \chi^+_A, \chi^+_B$ *and* $\chi^- = \chi^-_A, \chi^-_B$ *with* $d^+(\chi^+_A, \chi^-_A, A)$ *and* $d^-(\chi^+_B, \chi^-_B, B)$ *or*
- $\varphi = \neg A$ *and* $d^+(\chi^+, \chi^-, A)$ *or*
- $\varphi = \Diamond A$ *and there are* χ^+_A *and* χ^-_A *such that* $\chi^+_A = \chi^+$ *and* $\chi^- = \Diamond A, \chi^-_A$ *with* $d^-(\chi^+_A, \chi^-_A, A)$.

Lemma 4.4 (Generalized Inversion (G3S4)). *For any* Γ, Δ, A, *and* n*:*

- *If* $\Vdash_n^{S4} \Gamma, A \Rightarrow \Delta$ *then for all* χ_A^+ *and* χ_A^- *such that* $d^+(\chi_A^+, \chi_A^-, A)$*:* $\Vdash_n^{S4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Delta$.
- *If* $\Vdash_n^{S4} \Gamma \Rightarrow A, \Delta$ *then for all* χ_A^+ *and* χ_A^- *such that* $d^-(\chi_A^+, \chi_A^-, A)$*:* $\Vdash_n^{S4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Delta$.

It is not necessary to use further rule applications than for the derivations in the premises. The order of rule applications is preserved.

Proof by induction on the structure of φ*.*

Base Case: φ is an atom P. Assume we have $\Vdash_n^{S4} \Gamma, P \Rightarrow \Delta$. Then by Definition 4.1, the only possible assignment for χ_P^+ and χ_P^- with $d^+ = (\chi_P^+, \chi_P^-, P)$ are $\chi_P^+ = \{P\}$ and $\chi_P^- = \emptyset$. So we automatically obtain $\Vdash_n^{S4} \Gamma, \chi_P^+ \Rightarrow \chi_P^-, \Delta$ without doing anything. The same procedure applies for $\Vdash_n^{S4} \Gamma \Rightarrow P, \Delta$.

Inductive Step: We discuss only two cases here.

- $\varphi = \Box A$ in $\Vdash_n^{S4} \Gamma, \Box A \Rightarrow \Delta$.

  We consider arbitrary χ_φ^+ and χ_φ^- such that $d^+(\chi_\varphi^+, \chi_\varphi^-, \Box A)$. By Definition 4.1, there are two possibilities.

  1. $\chi_\varphi^+ = \{\Box A\}$ and $\chi_\varphi^- = \emptyset$. This case is dealt with as in the base case.
  2. There are χ_A^+ and χ_A^- such that $\chi_\varphi^+ = \Box A, \chi_A^+$, $\chi_\varphi^- = \chi_A^-$, and $d^+(\chi_A^+, \chi_A^-, A)$.
     By inversion (Lemma 4.2), we know $\Vdash_n^{S4} \Gamma, \Box A, A \Rightarrow \Delta$. Then by hypothesis, we also have $\Vdash_n^{S4} \Gamma, \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ for all χ_A^+ and χ_A^- such that $d^+(\chi_A^+, \chi_A^-, A)$. So we select the χ_A^+ and χ_A^- from above and obtain $\Vdash_n^{S4} \Gamma, \chi_\varphi^+ \Rightarrow \chi_\varphi^-, \Delta$.

- $\varphi = A \supset B$ in $\Vdash_n^{S4} \Gamma, A \supset B \Rightarrow \Delta$. There are again two possibilities:

  1. $\chi_\varphi^+ = \{A \supset B\}$ and $\chi_\varphi^- = \emptyset$.
     This case is dealt with as in the base case.

  2. Either $d^-(\chi_\varphi^+, \chi_\varphi^-, A)$ or $d^+(\chi_\varphi^+, \chi_\varphi^-, B)$.

     – Case 1: by inversion we get $\Vdash_n^{S4} \Gamma \Rightarrow A, \Delta$.
       Then by hypothesis: $\Vdash_n^{S4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ for all χ_A^+ and χ_A^- such that $d^-(\chi_A^+, \chi_A^-, A)$. Then we simply set $\chi_A^+ = \chi_\varphi^+$ and $\chi_A^- = \chi_\varphi^-$ and are done.

     – Case 2: by inversion we get $\Vdash_n^{S4} \Gamma, B \Rightarrow \Delta$.
       Then by hypothesis: $\Vdash_n^{S4} \Gamma, \chi_B^+ \Rightarrow \chi_B^-, \Delta$ for all χ_B^+ and χ_B^- such that $d^+(\chi_B^+, \chi_B^-, B)$. Then we simply set $\chi_B^+ = \chi_\varphi^+$ and $\chi_B^- = \chi_\varphi^-$ and are done.

□

Lemma 4.5 (Dispensability of Duplication Rules (G3S4)). *For all Γ, Δ, A, n, and all χ_A^+, χ_A^-:*

- *If $d^+(\chi_A^+, \chi_A^-, A)$ and $\Vdash_n^{S4} \Gamma, \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ then $\Vdash_n^{S4} \Gamma, \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ without any rule applications on $\Box A$ in macro blocks of level 0.*

- *If $d^-(\chi_A^+, \chi_A^-, A)$ and $\Vdash_n^{S4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Diamond A, \Delta$ then $\Vdash_n^{S4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Diamond A, \Delta$ without any rule applications on $\Diamond A$ in macro blocks of level 0.*

For the transformation, no further rule applications are necessary on any of the paths. The order of rule applications is preserved.

Proof by induction on n.

Base Case: $n = 1$ (trivial)

Inductive Step: We only consider the first part with $\Box A$ in the antecedent. The other part with $\Diamond A$ in the succedent works accordingly.

We assume to have $\Vdash^{S4}_{n+1} \Gamma, \Box A, \chi^+_A \Rightarrow \chi^-_A, \Delta$ for arbitrary χ^+_A, χ^-_A such that $d^+(\chi^+_A, \chi^-_A, A)$. Assume that there is a rule application of $\Box l$ on $\Box A$ somewhere in macro block level 0 in the derivation (otherwise there would not be anything to do). We consider the rule application in the root of the derivation tree (which must be non-critical).

- Assume that neither $\Box A$, nor a formula of χ^+_A or χ^-_A is principal of this application. This cannot be a critical rule, so we can simply apply the hypothesis on the premise(s) and append the application afterwards.

- Assume that a formula $\gamma \in \chi^+_A$ is principal.
  Therefore, $\Vdash^{S4}_{n+1} \Gamma, \gamma, \chi^+_\alpha, \Box A \Rightarrow \chi^-_A, \Delta$ with $\chi^+_A = \gamma, \chi^+_\alpha$.
  Then $\Vdash^{S4}_{n} \Gamma, \chi^+_\alpha, \Gamma_\gamma, \Box A \Rightarrow \chi^-_A, \Delta_\gamma, \Delta$ with Γ_γ and Δ_γ containing the active formulas of the rule application.
  By weakening we get $\Vdash^{S4}_{n} \Gamma, \chi^+_\alpha, \gamma, \Gamma_\gamma, \Box A \Rightarrow \chi^-_A, \Delta_\gamma, \Delta$ which corresponds to $\Vdash^{S4}_{n} \Gamma, \chi^+_A, \Gamma_\gamma, \Box A \Rightarrow \chi^-_A, \Delta_\gamma, \Delta$. Now we can apply the hypothesis and append the omitted rule application again.

- The case where a $\delta \in \chi^-_A$ is principal is handled almost the very same way.

- Finally, if $\Box A$ is the principal formula, then $\Vdash^{S4}_{n} \Gamma, \Box A, A, \chi^+_A \Rightarrow \chi^-_A, \Delta$.
  By *generalized inversion* (Lemma 4.4) we get
  $\Vdash^{S4}_{n} \Gamma, \Box A, \chi^+_A, \chi^+_A \Rightarrow \chi^-_A, \chi^-_A, \Delta$. Then *contraction* on both sides leads us to $\Vdash^{S4}_{n} \Gamma, \Box A, \chi^+_A \Rightarrow \chi^-_A, \Delta$ on which we can apply the hypothesis and that's it.

□

Lemma 4.6 (Rule Application Redundancy in Root Macro Blocks (G3S4)). *Let* t *be a valid* G3S4*-proof-tree. Then* t *can be transformed to a closed* G3S4*-proof-tree* t' *such that for all macro blocks* M *of level* 0 *in* t', M *is redundance-free. No further rule applications are needed and the orders are preserved.*

Proof. Consider t as the tree of derivation $\Vdash_n^{S4} \Gamma \Rightarrow \Delta$. The lemma is proven by induction on n.

Base Case: $n = 1$ (trivial)

Inductive Step: For $\Vdash_{n+1}^{S4} \Gamma \Rightarrow \Delta$ we want to show that there is an equivalent derivation which is redundance-free in the macro blocks of level 0. We only need to look at rules $\Box l$ and $\Diamond r$, as these are the duplication rules of G3S4.

Consider the sets $\Gamma^R \subseteq \Gamma$ and $\Delta^R \subseteq \Delta$ which contain all $\Box$-formulas of Γ and $\Diamond$-formulas of Δ respectively, on which a redundant rule application occurs somewhere in t in macro block level 0. Now we check the lowest rule application a_0 of t (i.e., in the root). This application is not critical (otherwise nothing needs to be done).

- If a_0 is an application on a formula $\gamma \in \Gamma \setminus \Gamma^R$ or $\delta \in \Delta \setminus \Delta^R$, then we simply apply the hypothesis on the premise(s) of a_0 and append a_0 again below.
- If a_0 is an application on a $\gamma \in \Gamma^R$ then it is an application of $\Box l$ on some $\gamma = \Box A$. We define $\Gamma' =_{df} \Gamma \setminus \{\Box A\}$. Then we have $\Vdash_n^{S4} \Gamma', \Box A, A \Rightarrow \Delta$. The hypothesis makes this derivation redundance-free in macro level 0.

  Now we define the multi-sets $\chi_A^+ =_{df} \{A\}$ and $\chi_A^- =_{df} \emptyset$. Obviously, $d^+(\chi_A^+, \chi_A^-, A)$. Then $\Vdash_n^{S4} \Gamma', \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$, and by dispensability (Lemma 4.5) also without the usage of applications of $\Box l$ on $\Box A$ or additional rule applications. Therefore, $\Vdash_{n+1}^{S4} \Gamma', \Box A \Rightarrow \Delta$ without redundant rule application in macro level 0.
- The case for a_0 being an application on a $\delta \in \Delta^R$ is solved accordingly.

$\Box$

Lemma 3.7 and its proof (p. 127) can now be adapted directly so that every G3S4-proof-tree can be made redundance-free.

Lemma 4.7 (Rule Application Redundancy in G3S4-Trees)**.** *Let* t *be a valid* G3S4*-proof-tree. Then* t *can be transformed to a closed* G3S4*-proof-tree* t' *such that all macro blocks of* t' *are redundance-free.*

Proof. Omitted due to similarity to proof of Lemma 3.7. □

The next steps are the *macro-saturation* (Definition 3.6, p. 128) and the *micro-saturation* (Definition 3.7, p. 135): it is quite nice that Lemmas 3.8, 3.9, and 3.10 (pp. 128, 131, 135) and their proofs can also be adapted directly when taking the new non-redundancy and the rules of G3S4 into account. As the proofs work the same way, they are not discussed here.

Lemma 4.8 (Macro Block Saturation in G3S4-Proof-Trees)**.** *Let* t *be a* G3S4*-proof-tree. Then* t *can be transformed to be* macro-saturated *in such a way that it is still a valid proof tree and without increasing its macro block height (MBH).*

Lemma 4.9 (Independent Rule Movement in Macro-Saturated G3S4-Trees)**.** *Let* t *be a* macro-saturated *and* redundance-free *sequent tree with an MBH of* 1*, and* $a_l = (r, \hat{\varphi})$ *a rule application at level* l *in* t*. Suppose that for all rule applications* a_m *with* $0 \leqslant m < l$*,* a_l *does not depend on* a_m*, i.e.,* $a_l \not\prec a_m$*. Then* t *can be transformed to* t' *for which*

- a_l *is moved downwards to the root, and*
- *the root sequent and the leaf sequents are the same as in* t*, and*
- *no rule application in* t *is added and none is removed, and*
- *the mbh and the MBH are the same as in* t*, and*
- t' *is also macro-saturated and redundance-free.*

Lemma 4.10 (Micro Block Saturation in G3S4-Proof-Trees)**.** *Let* t *be a* G3S4*-proof-tree of* micro block height h*. Then* t *can be transformed so that it is* micro-saturated*, but still closed and without increasing its micro block height* h*.*

From Normalized G3S4-Proofs to DiaSeqS4-Proofs

So again, after making an arbitrary G3S4-proof *redundance-free* and after *macro-saturating* and *micro-saturating* it, we can turn to the final part and rearrange the order of rules in the single block sequent sections. For the macro blocks that start in macro level 1, we use the following order (from bottom to top):

$$\wedge r \quad \vee r \quad \supset r \quad \neg r \quad \Diamond r \quad \Box l \quad \neg l \quad \supset l \quad \wedge l \quad \vee l$$

Again, the macro blocks are separated at points where the proponents react to a critical attack performed by O, or where one of them performs a co-critical attack. The micro blocks are used to establish the dialogical round structure. Things are now a bit different than for DiaSeqI, as now, after a P-agent reacted to a critical attack or performed a co-critical attack in the decision phase, it is O's turn, which was not the case in DiaSeqI (compare Figure 3.6, p. 109, with Figure 4.4, p. 172).

For the sake of simplicity, the different proponent labels are not distinguished. The blocking with the tilde ($\tilde{o}$) is also ignored, because our G3S4-proofs are considered to be redundance-free (Lemma 4.7).

Lemma 4.11 (Macro-Block Transformation (G3S4))**.** *Every redundance-free, macro-saturated, and micro-saturated* G3S4*-proof with root sequent* $\Gamma_R \Rightarrow \Delta_R$ *can be transformed to a* DiaSeqS4*-tree which is closed and which has sequent* $\Phi_R \vdash_O \Psi'_R$ *as root, where* Φ_R *and* Ψ'_R *contain the formulas of* Γ_R *and* Δ_R *augmented with labels (left-hand formulas with* o*, right-hand with* p*). Any subset of* Ψ'_R *may be marked with attacked-bars.*

Proof. By induction on the *MBH* h of t. The argument is very similar to the proofs of Lemma 3.11 and Theorem 3.2. We call the target DiaSeqS4-tree d.

Base Case: $h = 0$ — We perform an inner induction on the *mbh* j of t.

Base Case: $j = 1$ — Either Φ_R contains a $\perp$ or Ψ'_R a prime formula which causes a closure. In both cases, O can perform her attacks as far as necessary, followed by c_O and PN, and the *ipse dixisti* or *absurdum dixisti*, respectively.

Inductive Step: We reorder the rule application of the lowest micro block as determined above. We take the root $\Phi_R \vdash_O \Psi'_R$ and

- append all possible attacks which can be performed by O on top,
- append c_O and PN,
- append all defences/triggers which can be performed by the P-agents (correspond to $\wedge r$, $\vee r$, $\supset r$, $\neg r$, and $\Diamond r$ of the G3S4-micro block),
- append all attacks which can be performed by the P-agents,
- append c_P, and
- append all defences/triggers by O (correspond to $\Box l$, $\neg l$, $\supset l$, $\wedge l$, and $\vee l$ of the G3S4-micro block).

Afterwards, we apply the hypothesis on all trees starting in the next micro block level.

Inductive Step: There is at least one critical rule application in t. Same procedure: let us do an induction on the *mbh* j of t:

Base Case: $j = 0$ — The lowest rule application is a critical one. We take the root $\Phi_R \vdash_O \Psi'_R$, add all possible attacks by O, followed by rule c_O. If the critical rule application is $\Diamond l$, we add P?$\Diamond$, followed by O!$\Diamond$. Else, if it is $\Box r$, we simply add P!$\Box$. The resulting sequents have the desired form to continue the transformation.

Inductive Step: This is almost the same as in the (outer) base case. We simply translate a round starting with O's attacks and ending with O's defences. Then we apply the (inner) hypothesis.

Once the lowest macro blocks are translated, we can apply the (outer) hypothesis to perform the others. □

Theorem 4.2. *Every closed* G3S4*-proof-tree can be transformed to a* DiaSeqS4*-proof-tree.*

Proof. This follows directly from Lemma 4.11 as special case. □

4.3.3 Completeness of DiaSeqCS4

The transformation of G3iCS4^{M}-derivations to DiaSeqCS4-derivations is left. Many parts of the previous section can be adapted directly. Others need to be modified slightly. In general, we refer to the lemmas of the previous section and skip the proofs unless they vary substantially. Modified definitions are given in detail. For the deducibility (with a maximal height of n) in G3iCS4^{M} we write $\Vdash_n^{\mathsf{CS4}}$.

First, *weakening* (c.f. Lemma 4.1) is admissible in G3iCS4^{M} (see also Lemma 2.1, p. 60). The *inversion* lemma needs to be modified, as fewer of the rules are invertible, i.e., more rules are critical. These are $\supset r$, $\neg r$, $\Box r$, $\Diamond r$, and $\Diamond l$.

Lemma 4.12 (Inversion (G3iCS4^{M}))**.** *For all* Δ*,* Γ *and formulas* A *and* B*:*

1. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma, A, B \Rightarrow \Delta$.

2. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma \Rightarrow A \vee B, \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma \Rightarrow A, B, \Delta$.

3. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma, A \Rightarrow \Delta$ *and* $\Vdash_n^{\mathsf{CS4}} \Gamma, B \Rightarrow \Delta$.

4. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma \Rightarrow A \wedge B, \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma \Rightarrow A, \Delta$ *and* $\Vdash_n^{\mathsf{CS4}} \Gamma \Rightarrow B, \Delta$.

5. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma, A \supset B \Rightarrow \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma, A \supset B \Rightarrow A, \Delta$ *and* $\Vdash_n^{\mathsf{CS4}} \Gamma, B \Rightarrow \Delta$.

6. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma, \neg A \Rightarrow \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma, \neg A \Rightarrow A, \Delta$.

7. *If* $\Vdash_n^{\mathsf{CS4}} \Gamma, \Box A \Rightarrow \Delta$ *then* $\Vdash_n^{\mathsf{CS4}} \Gamma, \Box A, A \Rightarrow \Delta$.

The derivations in the conclusions do not contain further rule applications in all paths than the derivations in the conditions. The order of rule applications is preserved.

Proof. For this, the critical rules must of course also be taken into account when they are applied as lowest rules. However, the procedure is not different to that of the G3S4-version (Lemma 4.2). □

Contraction (c.f. Lemma 4.3) follows from inversion. As (compared to G3S4) inversion is restricted, we also have to cut down the *non-critically derivable set*.[12]

Definition 4.2 (Non-Critically Derivable Set (G3ɪCS4$^{\mathrm{M}}$)). *Let* φ *be a formula of propositional modal logic. The pair* (χ^+, χ^-) *is a* left-derivable *pair of multi-sets for* φ*, written* $d^+(\chi^+, \chi^-, \varphi)$ *iff for any* A *and* B*:*

- $\chi^+ = \{\varphi\}$ *and* $\chi^- = \emptyset$ *or*
- $\varphi = A \wedge B$ *and there are* χ_A^+*,* χ_B^+*,* χ_A^-*, and* χ_B^- *such that* $\chi^+ = \chi_A^+, \chi_B^+$ *and* $\chi^- = \chi_A^-, \chi_B^-$ *with* $d^+(\chi_A^+, \chi_A^-, A)$ *and* $d^+(\chi_B^+, \chi_B^-, B)$ *or*
- $\varphi = A \vee B$ *and either* $d^+(\chi^+, \chi^-, A)$ *or* $d^+(\chi^+, \chi^-, B)$ *or*
- $\varphi = A \supset B$ *and either there is a* χ_A^+ *such that* $\chi^+ = A \supset B, \chi_A^+$ *and* $d^-(\chi_A^+, \chi^-, A)$*, or* $d^+(\chi^+, \chi^-, B)$ *or*
- $\varphi = \neg A$ *and there is a* χ_A^+ *such that* $\chi^+ = \neg A, \chi_A^+$ *and* $d^-(\chi_A^+, \chi^-, A)$ *or*
- $\varphi = \Box A$ *and there are* χ_A^+ *and* χ_A^- *such that* $\chi^+ = \Box A, \chi_A^+$ *and* $\chi_A^- = \chi^-$ *with* $d^+(\chi_A^+, \chi_A^-, A)$.

[12] The *generalized inversion* only holds for non-critical formulas, as with critical rule applications, formulas are dropped from the sequents.

The pair (χ^+, χ^-) *is a* right-derivable *pair of multi-sets for* φ, *written* $d^-(\chi^+, \chi^-, \varphi)$ *iff for any* A *and* B*:*

- $\chi^+ = \emptyset$ *and* $\chi^- = \{\varphi\}$ *or*
- $\varphi = A \wedge B$ *and either* $d^-(\chi^+, \chi^-, A)$ *or* $d^-(\chi^+, \chi^-, B)$ *or*
- $\varphi = A \vee B$ *and there are* χ_A^+, χ_B^+, χ_A^-, *and* χ_B^- *such that* $\chi^+ = \chi_A^+, \chi_B^+$ *and* $\chi^- = \chi_A^-, \chi_B^-$ *with* $d^-(\chi_A^+, \chi_A^-, A)$ *and* $d^-(\chi_B^+, \chi_B^-, B)$.

With this adjusted definition, the *generalized inversion* can be shown for G3ICS4[M]. The proof is easy and omitted here, as this is a restricted version of Lemma 4.4 (Generalized Inversion (G3S4)).

Lemma 4.13 (Generalized Inversion (G3ICS4[M])). *For any* Γ, Δ, A, *and* n*:*

- *if* $\Vdash_n^{CS4} \Gamma, A \Rightarrow \Delta$ *then for all* χ_A^+ *and* χ_A^- *such that* $d^+(\chi_A^+, \chi_A^-, A)$*:* $\Vdash_n^{CS4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Delta$.
- *if* $\Vdash_n^{CS4} \Gamma \Rightarrow A, \Delta$ *then for all* χ_A^+ *and* χ_A^- *such that* $d^-(\chi_A^+, \chi_A^-, A)$*:* $\Vdash_n^{CS4} \Gamma, \chi_A^+ \Rightarrow \chi_A^-, \Delta$.

It is not necessary to use further rule applications than for the derivations in the premises. The order of rule applications is preserved.

This is then used to show the *dispensability of duplication rules*.

Lemma 4.14 (Dispensability of Duplication Rules (G3ICS4[M])). *For all* Γ, Δ, A, n, *and all* χ_A^+, χ_A^-*:*

- *if* $d^-(\chi_A^+, \chi_A^-, A)$ *and* $\Vdash_n^{CS4} \Gamma, A \supset B, \chi_A^+ \Rightarrow \chi_A^-, \Delta$, *then* $\Vdash_n^{CS4} \Gamma, A \supset B, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ *without any rule applications on* $A \supset B$ *in macro blocks of level 0.*
- *if* $d^-(\chi_A^+, \chi_A^-, A)$ *and* $\Vdash_n^{CS4} \Gamma, \neg A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$, *then* $\Vdash_n^{CS4} \Gamma, \neg A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ *without any rule applications on* $\neg A$ *in macro blocks of level 0.*

- *if* $d^+(\chi_A^+, \chi_A^-, A)$ *and* $\Vdash_n^{CS4} \Gamma, \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$, *then* $\Vdash_n^{CS4} \Gamma, \Box A, \chi_A^+ \Rightarrow \chi_A^-, \Delta$ *without any rule applications on* $\Box A$ *in macro blocks of level 0.*

For the transformation, no further rule applications are necessary on any of the paths. The order of rule applications is preserved.

Proof. For details see the proof of Lemma 4.5, p. 185. □

As for G3S4, we can now use this to show that any G3ICS4^M-tree can be made *redundance-free*.[13] The rest, i.e., the *macro block saturation*, the *rule movement*, and the *micro block saturation* are no problem and work as shown before. Of course, the new issues with the G3ICS4^M-rules need to be checked, but actually, the differences are so small that they are not worth to be discussed in detail at this point. Also the reordering and transformation process works exactly as for DiaSeqS4. The main difference is that we have additional critical rules, but these are independent of the structure of the micro blocks within the macro blocks.

Theorem 4.3. *Every closed* G3ICS4^M*-proof-tree can be transformed to a* DiaSeqCS4*-proof-tree.*

4.4 Summary and Comparison to other Systems

In this chapter, multi-proponent dialogical calculi for the modal systems S4 and CS4 have been investigated. For both, informal game rules and sequent-style implementations have been presented. The structural rules for MPMD/S4 come in two variants, one simple one, where the only modal operator of the language is □, and one complete one which also involves ◇. The latter is much more complex and the foundation for the structural rules for MPMD/CS4. In these, distinctions between (weak and strong) critical at-

[13] Again, first the macro blocks of macro block level 0, and then globally.

tacks and co-critical attacks, as well as between active, deactivated, stunned and protected P-agents are necessary.

The rules of the sequent variants DiaSeqS4 and DiaSeqCS4 are clear and, compared to the informal structural rules, easy to use. The dialogical sequent systems work like DiaSeqI of Chapter 3. Soundness and completeness of DiaSeqS4 and DiaSeqCS4 have been shown by extending the adequateness proofs for DiaSeqI. MPMD/CS4 is the first dialogical system for a constructive (or intuitionistic) modal logic for which a formal adequateness proof is provided.

As our systems do not make use of explicit contexts and only a single player is part of the opponent party, the greatest similarities can be found in the approach by Krabbe [83, 86] (see Chapter 2.4.1). Like him, we allow the proponent(s) to state prime formulas and the *ipse dixisti* remark. The distinction of strong and weak commitments is also adapted, as well as O's ability to *withdraw* concessions.

However, the $\Diamond$-operator is not involved in Krabbe's work and, as mentioned, only one proponent agent plays in these games. Krabbe also added the significance level as label to the $\Box$, which we simplify to the mono-modal version.

When we compare DiaSeqS4 and DiaSeqCS4 with different modal sequent systems (discussed in Chapter 2.2), we detect similarities to *ordinary sequent systems*, in particular G3S4 and G3iCS4$^{\mathrm{M}}$ (Sections 2.2.1 and 2.2.2). In these, formulas are *dropped* from the sequents whenever non-invertible rules are applied, which corresponds to the critical moves in DiaSeqS4/DiaSeqCS4. We refrain from using labels attached to formulas that indicate a *context* or a *Kripke world* as done in G3S4$^{\mathrm{l}}$ (Chapter 2.2.4) which is strongly related to tableau calculi and reproduces Kripke models explicitly. We also do not apply techniques of *hypersequents* or *nested sequents* in which these structures are built more implicitly but are obviously still present (Chapter 2.2.5).

The enforced round structure of the dialogical sequents has again similarities to the principle of focus calculi. Modal focus calculi which are presented in the literature are, to my current knowledge, only those which are mentioned in Chapter 2.2.6. One of them [113] is based on first-order logic and works on the explicit relations established by labels, the others are working on nested sequents [24, 25, 94]. There is currently no focusing system for ordinary modal sequents which are more related to the multi-proponent dialogical sequents. The reason is probably that they are considered to be less flexible than the other approaches. Nevertheless, as ordinary sequents are very fundamental and there has been a lot of work on them since Gentzen, they are still worth to be considered. Further differences of multi-proponent dialogical sequents and focus sequents were already discussed in Chapter 3.4.2 and are not repeated here.

5 Conclusion – Results, Open Problems, Future Work

In this last chapter the key aspects of this work are summarised. Open problems of MPID and MPMD and their sequent style interpretations are discussed and different directions for future work are suggested.

5.1 Results

In this work we have investigated an account to proof search in modal logic (in particular for the modal systems S4 and CS4) and intuitionistic propositional logic. The focus was set on propositional modal logic due to its relevance and popularity in computer science. Most of the usual modal logics are decidable, a big advantage compared to first-order predicate logic which is on the other hand more expressive.

The presented approach is based on dialogical games in which a proponent tries to defend the validity of a formula towards an opponent. In our attempt further proponent-agents are introduced for support. This has two effects. First, our calculus enforces a normalization of proofs, as the order of steps/moves is restricted due to the game setting. From the game-theoretic perspective this also implies that agents' decisions which are significant for winning or losing are not taken individually with every move, but instead as a collective agreement of which the consequence usually is that some players leave the game. This new kind of dialogues allows the proponent players to work separately and in parallel until a collective decision is made.

Soundness and completeness of the presented dialogical systems has been shown by implementing the rules in terms of sequent calculi and by showing their correspondence to known ordinary sequent systems. As by-product a new cut-free ordinary multi-conclusion sequent system G3ICS4^{M} was presented for which soundness and completeness were also shown due to its equivalence to system G3CS4.

The multi-proponent dialogical systems were compared to different sequent calculi, especially systems that make use of the focusing technique (also leading to normalization of proofs), and to different dialogical systems. Gentzen sequent systems were also related to dialogues to emphasize common features.

With this thesis I intend to present a foundation for a new proving technique based both on multi-conclusion sequent systems and dialogues. There are many open problems and much space for future work concerning different problem areas. Some of them are discussed in the following.

5.2 Efficiency

MPID and MPMD with their sequent-style interpretations provide reasoning techniques for IPL and modal logics S4 and CS4. The systems work with a round-based scheduling technique that forces the parties to perform moves whenever it is their turn. As we have seen in particular in the completeness proof of DiaSeqI, some of these moves do not help in the proof searching process as they are irrelevant.[1] So one improvement would be to avoid such irrelevant moves, possibly by considering methods which are able to predict

[1]This becomes obvious in the *macro block saturation* (Chapter 3.3.1). In many cases it is stronger than actually necessary, e.g., shortcuts are possible when a P-agent reacts to a critical attack earlier.

that a move is unnecessary. This would strongly increase the efficiency of searching for derivations.

Now, as the decision phase of DiaSeqI/DiaSeqS4/DiaSeqCS4-proofs is the significant part for the proponents' success, it is also possible to concentrate on this phase and establish heuristics that help the P-agents to decide how to behave in these situations, i.e., decide which of the proponents is the one to react to a critical move or perform a co-critical attack.

Concerning implementation we have already pointed out that the proponents, as long as they are not situated in the decision phase, are working independently of each other, everyone only concentrating on what the opponent states towards them. So each of the P-agents can work in parallel with O until the next decide phase is reached. The dialectic reasoning procedures can therefore be implemented in a concurrent fashion where each P-agent corresponds to a single process/thread which is *synchronized* with the others in the decision phase.

It seems to be unlikely that the time complexity of DiaSeqI is better than that of $G3i^M$, or the complexity of DiaSeqS4/DiaSeqCS4 is better than G3S4/ $G3iCS4^M$. However, it is one aim of the work to provide a foundation in terms of the scheduling mechanism which can be the basis of more efficient systems.

5.3 Termination

Termination is guaranteed in our multi-proponent system DiaSeqI^+ for IPL based on the results by Corsi and Tassi [33] (Chapter 2.1.4). For systems DiaSeqS4 and DiaSeqCS4 termination is not guaranteed as S4 and CS4 are more complex than IPL. To fill this gap one might adapt the ideas by Heuerding et al. [68] (sequent system $KS4^{SU}$) or Howe [72, 73] who also provides a system for the $\Box$-fragment of CS4 (Chapter 2.2.3). Another approach would be to transfer the structural game rules proposed by Krabbe [86] (see rule **(F2^{S4})** in Chapter 2.4.1). However, it is probably harder to find a formal adequateness proof for a system that includes this rather informal restriction.

An easier way to solve the problem of termination guarantee would be to introduce *ranks* as Clerbout [29, 30] does for his modal dialectics (Chapter 2.4.3). As already mentioned, the idea of ranks is quite attractive from the game-theoretic point of view, but facing the problems of proof search it is rather inappropriate because it would be necessary to guess the correct ranks beforehand.

5.4 Intermediate Systems and Extensions

In this work multi-proponent systems were introduced only for IPL and modal systems S4 and CS4. We have not considered classical propositional logic as a variant of MPID/DiaSeqI without critical attacks. Obviously, the sequent calculus G3c serves as counterpart to show adequateness.

In Chapter 2.3.5 we discussed variants of multi-proponent dialectics (MPD^{FM}) by Fermüller and Ciabattoni for the propositional language. These are not only used as reasoning procedures for intuitionistic and classical logic, but also for intermediate logics like the *Gödel-Dummett logic*. The adequate reasoning for the different semantics is accomplished by giving the proponents different abilities of how to *merge*, i.e., share their information [51, 49]. It would be interesting to find a set of rules in our multi-proponent setting (MPID) that achieves the same. For example, it is thinkable to relax the restriction that *all* other proponent agents are deactivated if one reacts to a critical attack performed by O. Instead, one might allow two or three proponents to continue the play.

Accordingly, it is possible to modify the informal structural rules for MPMD/S4 and MPMD/CS4: both modal systems S4 and CS4 have common features, but CS4 is much more restrictive, i.e., has less theorems. When we have a look at Table 4.2 (p. 179) again, we see that many modifications are possible. For example, one could change the set of moves causing weak/strong events and the effects, introduce further possibilities for P-protections or O-withdrawals, or change the restrictions for P-agents to perform co-critical attacks. In this way intermediate or even more restrictive systems than CS4 can be obtained.

It seems that DiaSeqS4 can be modified easily to work also for other modal systems like K, KT, or K4, for which there exist cut-free, ordinary sequent systems. The proposed technique of macro and micro saturation should be applicable there and also for the constructive variants.

As already mentioned in Chapter 2.2.1, no ordinary sequent systems for the IK-family can be found in the literature. The reason seems to be the **IK5** axiom. In labelled sequents its derivation requires to travel forth and back between Kripke worlds which is not possible once the formulas of the previously visited world are dropped (see example on p. 57). Because of this, our transformation technique to dialogue sequents cannot be applied to obtain dialectic calculi for IK, IKT, IS4...

Labelled sequents can be simulated with dialogues using contexts like in the work of Rückert and Rahman (Chapter 2.4.3) or as multi-opponent dialogues as proposed by Van Dun (Chapter 2.4.2). An interesting approach would be to combine our multi-proponent system for IPL (MPID/DiaSeqI) with Van Dun's attempt to obtain multi-player (both multi-proponent *and* multi-opponent) dialectics for intuitionistic modal logics. Each O-agent then represents a Kripke world on the modal level while P-agents can be deactivated to enforce intuitionism. Restrictions for the proponents must then also be established to prevent them from changing their communication partners arbitrarily.

Blackburn [15] once proposed an attempt to use dialogues as reasoning procedure for *hybrid logic*. This also fits quite well to the idea of multi-opponent dialogues: just consider the statement $@_i\phi$ which simply means, if asserted by some O-agent j, that her colleague, O-agent i, is ready to commit assertion ϕ. Or if asserted by a P-agent and attacked by an O-agent, it results in a change to another communication partner on the O-side.

Another idea is to build a reasoning system for *public announcement logic* (PAL), a variant of dynamic logic that goes back to Plaza [119]. Van Dun's system might also be extended for this: as soon as new information is suddenly available, the argumentation partners have to react to this somehow. In a tableau calculus proposed by De Vuyst [37], using a rule for a public announcement formula $[!\varphi]\psi$[2] means that one has to decide for every Kripke world which is represented in the tableau, whether the statement φ holds or not, and then remove the worlds in which φ does not hold. This fits our game semantics very well, where agents have to decide whether they commit themselves to the truth of an announced formula or not. However, things get quite complex very quickly, as it can be seen in the rules of sequent systems for PAL, e.g. as in [105].

5.5 Further Alternations and Applications

Original attempts by Lorenzen and Lorenz aimed at reasoning procedures for intuitionistic first-order logic which was not the main subject here, as we concentrated on propositional modal logic. The first-order language is also interesting with respect to multi-proponent dialogues. Different proponent-agents would be allowed to substitute bound variables with different terms/objects.

A philosophical disadvantage of our sequential interpretation of dialogues is that the moves are not mapped to sequent rules one by one. For example,

[2] If φ is announced publicly then this results in ψ.

DiaSeqI (and also DiaSeqS4 and DiaSeqCS4) contains *trigger rules* ($O{*}\neg$, $O{*}\supset$, $P{*}\neg$) which do not correspond to dialogical moves but are instead applied *automatically* when possible or necessary. The same holds for the changing rules c_O, c_P and PN. To solve this issue, one could rebuild the dialogical sequent calculi to obtain systems in the style of Barth and Krabbe [10] (Chapter 2.3.3) which interpret dialogical moves literally. A multi-proponent variant does not yet exist in this style. Concerning game-theoretic calculi it would also be interesting to examine multi-proponent *material dialogues* or *Hintikka games* (see Chapter 2.3.6) for constructive modal logic in order to do reasoning in predefined models.

In the literature, dialogical logic is rarely used for applications which are neither philosophical nor proof-theoretical. However, there are argumentation-theoretic approaches as researched by Barth and Krabbe [10] and Walton and Krabbe [147]. In the work by Fach [45] dialogues are used as foundation to build tutor systems for software usage.[3] Alama [3] suggests to use dialogues to implement a theorem prover. With the multi-proponent approach, a new concept with new semantics is introduced that could lead to ideas for further applications. The usage of Lorenzen dialogues for communication of agents is currently not present in the literature.

[3]The tutor/system has some kind of contract with the user. The system helps to solve a task, e.g., by showing the next step, if the user has achieved the requirements for solving it. Fach relates this to some features of intuitionistic logic.

Bibliography

[1] Samson Abramsky. A compositional game semantics for multi-agent logics of partial information. In Johan van Benthem, Dov Gabbay, and Benedikt Löwe, editors, *Interactive Logic*, pages 11–47. Amsterdam University Press, 2007.

[2] Jesse Alama. Toward a structure theory for Lorenzen dialogue games. Published online, 2013. URL `http://arxiv.org/abs/1311.1917`.

[3] Jesse Alama. Dialogues for proof search. In Christoph Benzmüller and Jens Otten, editors, *ARQNL 2014. Automated Reasoning in Quantified Non-Classical Logics*, volume 33 of *EPiC Series in Computing*, pages 65–70, 2015.

[4] Natasha Alechina, Michael Mendler, Valeria de Paiva, and Eike Ritter. Categorical and Kripke Semantics for Constructive S4 Modal Logic. In Laurent Fribourg, editor, *Computer Science Logic (CSL 2001)*, volume 2142 of *Lecture Notes in Computer Science*, pages 292–307. Springer, 2001.

[5] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[6] Ryuta Arisaka, Anupam Das, and Lutz Straßburger. On nested sequents for constructive modal logics. *Logical Methods in Computer Science*, 11(3:7), 2015. doi: 10.2168/LMCS-11(3:7)2015. URL `http://arxiv.org/pdf/1505.06896`.

[7] Arnon Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3):225–248, 1991. ISSN 1573-7470.

[8] Franz Baader and Carsten Lutz. Description logic. In Patrick Blackburn, Johan van Benthem, and Frank Wolter, editors, *Handbook of Modal Logic*. Elsevier, 2007.

[9] Roberta Ballarin. Modern origins of modal logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition, 2017. URL `https://plato.stanford.edu/archives/sum2017/entries/logic-modal-origins/`.

[10] Else M. Barth and Erik C. Krabbe. *From Axiom to Dialogue: A philosophical study of logics and argumentation*. Walter de Gruyter, 1982.

[11] Evert W. Beth. Semantic entailment and formal derivability. *Mededelingen der Koninklijke Nederlandse Akademie van Wetenschappen*, 18(13):309–342, 1955.

[12] Gavin M. Bierman and Valeria C. V. de Paiva. Intuitionistic necessity revisited. In *Proceedings of Logic at Work Conference*, Amsterdam, 1992.

[13] Gavin M. Bierman and Valeria C. V. de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65(3):383–416, 2000. ISSN 1572-8730. doi: 10.1023/A:1005291931660.

[14] Patrick Blackburn. Internalizing labelled deduction. *Journal of Logic and Computation*, 10(1):137–168, 2000.

[15] Patrick Blackburn. Modal logic as dialogical logic. *Synthese*, 127:57–93, 2001.

[16] Andreas Blass. A game semantics for linear logic. *Annals of Pure and Applied Logic*, 56(1–3):183–220, 1992.

[17] Torben Braüner. *Hybrid Logic and its Proof-Theory*. Springer, 2011.

[18] Luitzen E. J. Brouwer. *Over de Grondslagen der Wiskunde*. PhD thesis, University of Amsterdam, 1907.

[19] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.

[20] Michael Burrows, Martín Abadi, and Roger M. Needham. A logic of authentication. In *Proceedings of the Royal Society of London. Series A, Mathematical, Physical and Engineering Sciences*, volume 426, pages 233–271. The Royal Society, 1989.

[21] Claudio Cerrato. Modal sequents. In Heinrich Wansing, editor, *Proof Theory of Modal Logic*. Kluwer Academic Publishers, 1996.

[22] Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin. Canonical sequent proofs via multi-focusing. In Giorgio Ausiello, Juhani Karhumäki, Giancarlo Mauri, and Luke Ong, editors, *Fifth Ifip International Conference On Theoretical Computer Science – Tcs 2008*, pages 383–396, Boston, MA, 2008. Springer US. ISBN 978-0-387-09680-3. doi: 10.1007/978-0-387-09680-3_26.

[23] Kaustuv Chaudhuri, Stefan Hetzl, and Dale Miller. A systematic approach to canonicity in the classical sequent calculus. In *21st EACSL Annual Conference on Computer Science Logic*, volume 16, pages 183–197, 2012.

[24] Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger. Modular Focused Proof Systems for Intuitionistic Modal Logics. In *1st International Conference on Formal Structures for Computation and Deduction*, volume 52 of *Leibniz International Proceedings in Informatics*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. doi: 10.4230/LIPIcs.FSCD.2016.16. URL `http://drops.dagstuhl.de/opus/volltexte/2016/5994/`.

[25] Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger. Focused and synthetic nested sequents. In Bart Jacobs and Christof Löding, editors, *Foundations of Software Science and Computation Structures: 19th International Conference, FOSSACS 2016*, pages 390–407. Springer Berlin Heidelberg, 2016.

[26] Brian F. Chellas. *Modal logic: an introduction*. Cambridge University Press, 1980.

[27] Agata Ciabattoni and Mauro Ferrari. Hypersequent calculi for some intermediate logics with bounded Kripke models. *Journal of Logic and Computation*, 11(2):283–294, 2001.

[28] Agata Ciabattoni, Dov M. Gabbay, and Nicola Olivetti. Cut-free proof systems for logics of weak excluded middle. *Soft Computing*, 2(4):147–158, 1999.

[29] Nicolas Clerbout. Context-shifting in formal dialogues of argumentation. In Dan Gabriel Sîmbotan and Gherasim Ovidiu, editors, *Limits of knowledge society*, pages 27–38, 2012. Proceedings of the International Conference Iaşi, 6-9 October 2010, Iaşi – România.

[30] Nicolas Clerbout. *Étude sur quelques sémantiques dialogiques : concepts fondamentaux et éléments de metathéorie*. PhD thesis, Leiden University, 2013.

[31] Nicolas Clerbout. First-order dialogical games and tableaux. *Journal of Philosophical Logic*, pages 1–17, 2013. ISSN 0022-3611. doi: 10.1007/s10992-013-9289-z.

[32] Giovanna Corsi. The a fortiori rule: The key to reach termination in intuitionistic logic. In Eduardo Ballo and Miriam Franchella, editors, *Logic and Philosophy in Italy, Some trends and perspectives; essays in honor of Corrado Mangione on his 75th birthday*, pages 26–47. Polimetrica, Milano, 2006.

[33] Giovanna Corsi and Gabriele Tassi. Intuitionistic logic freed of all metarules. *The Journal of Symbolic Logic*, 72:1204–1218, 12 2007. ISSN 1943-5886.

[34] Haskell B. Curry. A theory of formal deducibility, 1950. Notre Dame Mathematical Lectures, Number 6.

[35] Haskell B. Curry. The elimination theorem when modality is present. *The Journal of Symbolic Logic*, 17(4):249–265, Dec 1952.

[36] Rowan Davies and Frank Pfenning. A modal analysis of staged computation. *Jornal of the ACM*, 48(3):555–604, 2001.

[37] Jonas De Vuyst. *Dynamic Tableaux for Dynamic Modal Logics*. PhD thesis, Vrije Universiteit Brussel, November 2013.

[38] Albert G. Dragalin. *Matematičeskij intuicionizm, Vvedenie v teoriju dokazatel'stv*. Nauka, 1979.

[39] Albert G. Dragalin. *Mathematical Intuitionism, Introduction to Proof Theory*, volume 67 of *Translations of Mathematical Monographs*. American Mathematical Society, 1988. Translated from the Russian [38] by E. Mendelson.

[40] Roy Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *The Journal of Symbolic Logic*, 57(3):795–807, 1992.

[41] Roy Dyckhoff. Intuitionistic Decision Procedures Since Gentzen. In Reinhard Kahle, Thomas Strahm, and Thomas Studer, editors, *Advances in Proof Theory*, pages 245–267. Birkhäuser, 2016.

[42] Roy Dyckhoff and Stéphane Lengrand. LJQ: A strongly focused calculus for intuitionistic logic. In *Logical Approaches to Computational Barriers: Second Conference on Computability in Europe, CiE 2006, Swansea, UK, June 30–July 5, 2006. Proceedings*, pages 173–185. Springer Berlin Heidelberg, 2006. ISBN 978-3-540-35468-0.

[43] Uwe Egly and Stephan Schmitt. Intuitionistic proof transformations and their application to constructive program synthesis. In Jacques Calmet and Jan Plaza, editors, *Artificial intelligence and symbolic computation : proceedings / International Conference AISC '98*, pages 132–144. Springer, 1998.

[44] Uwe Egly and Stephan Schmitt. On intuitionistic proof transformations, their complexity, and application to constructive program synthesis. *Fundamenta Informaticae*, 39(1-2):59–83, 1999.

[45] Peter W. Fach. *Dialogspiele in der Mensch-Computer-Interaktion : eine Fallstudie am Beispiel von Online-Hilfen*. Waxmann, 1997.

[46] Solomon Feferman, editor. *Kurt Gödel: Collected Works: Volume I: Publications 1929–1936*. Clarendon Press, 1986.

[47] Walter Felscher. Dialogues, strategies, and intuitionistic provability. *Annals of Pure and Applied Logic*, 28(3):217–254, 1985.

[48] Walter Felscher. Dialogues as a foundation for intuitionistic logic. In Dov Gabbay and Franz Guenther, editors, *Handbook of Philosophical Logic*, volume III: Alternatives in Classical Logic, pages 341–372. D. Reidel Publishing Company, 1986.

[49] Christian G. Fermüller. Parallel dialogue games and hypersequents for intermediate logics. In *Automated Reasoning with Analytic Tableaux and Related Methods : International Conference, TABLEAUX 2003, Rome, Italy, September 2003. Proceedings*, pages 48–64. Springer Berlin Heidelberg, 2003.

[50] Christian G. Fermüller. Dialogue games for many-valued logics—an overview. *Studia Logica*, 90(1):43–68, 2008.

[51] Christian G. Fermüller and Agata Ciabattoni. From Intuitionistic Logic to Gödel-Dummett Logic via Parallel Dialogue Games. In *Proceedings of the 33rd IEEE International Symposium on Multiple-Valued Logic*, pages 188–193, 2003.

[52] Christian G. Fermüller and Norbert Preining. A dialogue game for intuitionistic fuzzy logic based on comparisons of degrees of truth. In *Proceedings of InTech'03 (Fourth International Conference on Intelligent Technologies)*, 2003.

[53] Mauro Ferrari, Camillo Fiorentini, and Guido Fiorino. Contraction-free linear depth sequent calculi for intuitionistic propositional logic with the subformula property and minimal depth counter-models. *Journal of Automated Reasoning*, 51(2):129–149, 2013.

[54] Camillo Fiorentini. Terminating sequent calculi for proving and refuting formulas in S4. *Journal of Logic and Computation*, 25(1):179, 2015. doi: 10.1093/logcom/exs053.

[55] Gisèle Fischer-Servi. Semantics for a class of intuitionistic modal calculi. *Italian Studies in the Philosophy of Science*, pages 59–72, 1980.

[56] Frederic B. Fitch. *Symbolic Logic – An Introduction*. The Ronald Press Company, 1952.

[57] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Springer-Science+Business Media Dordrecht, 1983.

[58] Melvin Fitting. Nested sequents for intuitionistic logics. *Notre Dame J. Formal Logic*, 55(1):41–61, 2014. doi: 10.1215/00294527-2377869.

[59] Didier Galmiche, Dominique Larchey-Wendling, and Joseph Vidal-Rosset. Some remarks on relations between proofs and games. In *Construction – Festschrift for Gerhard Heinzmann*, pages 339–357. College Publications, 2010.

[60] Gerhard Gentzen. Untersuchungen über das logische Schließen. I. *Mathematische Zeitschrift*, 39:176–210, 1935.

[61] Jean-Yves Girard. A new constructive logic: classic logic. *Mathematical Structures in Computer Science*, 1(3):255–296, 1991.

[62] Jean-Yves Girard. On the unity of logic. *Annals of Pure and Applied Logic*, 59(3):201–217, 1993. ISSN 0168-0072. doi: 10.1016/0168-0072(93)90093-S.

[63] Kurt Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. Ergebnisse eines mathematischen Kolloquiums 4 39–40, 1933. Reprinted/translated: "An interpretation of the intuitionistic propositional calculus" in [46].

[64] Carsten Grefe. *Fischer Servi's Intuitionistic Modal Logic and Its Extensions*. PhD thesis, Freie Universität Berlin, 1997.

[65] Gerrit Haas. Hypothesendialoge, Konstruktiver Sequenzenkalkül und die Rechtfertigung von Dialograhmenregeln. In *Theorie des wissenschaftlichen Argumentierens*. Suhrkamp Verlag, 1980.

[66] Hugo Herbelin. A Lambda-Calculus Structure Isomorphic to Gentzen-Style Sequent Calculus Structure. In *Selected Papers from the 8th International Workshop on Computer Science Logic*, pages 61–75, London, UK, 1994. Springer-Verlag. ISBN 3-540-60017-5.

[67] Hugo Herbelin. *Séquents qu'on calcule: de l'interprétation du calcul des séquents comme calcul de lambda-termes et comme calcul de stratégies gagnantes*. PhD thesis, Université Paris-Diderot-Paris VII, 1995.

[68] Alain Heuerding, Michael Seyfried, and Heinrich Zimmermann. Efficient loop-check for backward proof search in some non-classical propositional logics. In *International Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, pages 210–225. Springer, 1996.

[69] Arend Heyting. *Mathematische Grundlagenforschung – Intuitionismus – Beweistheorie*, volume 3 (4) of *Ergebnisse der Mathematik und ihre Grenzgebiete*. Springer, 1934. Reprinted 1974.

[70] Jaakko Hintikka. *Logic, language-games and information: Kantian themes in the philosophy of logic*. Oxford Univ Press, 1973.

[71] Gerard J. Holzmann. *The spin model checker*. Addison-Wesley, 4th edition, 2008.

[72] Jacob M. Howe. Two loop detection mechanisms: A comparison. In Didier Galmiche, editor, *Automated Reasoning with Analytic Tableaux and Related Methods: International Conference, TABLEAUX'97*, pages 188–200. Springer Berlin Heidelberg, 1997.

[73] Jacob M. Howe. *Proof search issues in some non-classical logics*. PhD thesis, University of St Andrews, 1998.

[74] Jörg Hudelmaier. Bounds for cut elimination in intuitionistic propositional logic. *Archive for Mathematical Logic*, 31(5):331–353, 1992.

[75] Jörg Hudelmaier. Improved decision procedures for the modal logics K, T and S4. In *Computer Science Logic*, pages 320–334. Springer, 1996.

[76] Rüdiger Inhetveen. Ein konstruktiver Weg zur Semantik der "möglichen Welten". In E.M. Barth and J.L. Martens, editors, *Argumentation – approaches to theory formation – containing the contributions to*

the Groningen Conference on the Theory of Argumentation, October 1978, 1982.

[77] Wilhelm Kamlah and Paul Lorenzen. *Logische Propädeutik Oder Vorschule des Vernünftigen Redens*, chapter VI 2. Nicht-empirische Wahrheit. Bibliographisches Institut, 1967. Excerpt reprinted in [103].

[78] Stig Kanger. *Provability in Logic*. University of Stockholm, 1957.

[79] Oiva Ketonen. Untersuchungen zum Prädikatenkalkül. *Journal of Symbolic Logic*, 10(4):127–130, 1945.

[80] Walther Kindt. *Eine abstrakte Theorie von Dialogspielen*. PhD thesis, Albert-Ludwig-Universität Freiburg, 1972.

[81] Stephen C. Kleene. *Introduction to Metamathematics*. North Holland, 1952.

[82] Erik C.W. Krabbe. The adequacy of material dialogue-games. *Notre Dame Journal of Formal Logic*, 19(3):321–330, 1978.

[83] Erik C.W. Krabbe. *Studies in Dialogical Logic*. PhD thesis, Rijksuniversiteit te Groningen, 1982.

[84] Erik C.W. Krabbe. Noncumulative dialectical models and formal dialectics. *Journal of Philosophical Logic*, 14(2):129–168, 1985.

[85] Erik C.W. Krabbe. Formal systems of dialogue rules. *Synthese*, 63(3): 295–328, 1985.

[86] Erik C.W. Krabbe. A theory of modal dialectics. *Journal of Philosophical Logic*, 15(2):191–217, 1986. ISSN 1573-0433.

[87] Saul A. Kripke. Semantical Analysis of Modal Logic I Normal Modal Propositional Calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[88] Roman Kuznets, Sonia Marin, and Lutz Straßburger. Justification logic for constructive modal logic. In *MLA 2017 – 7th Workshop on Intuitionistic Modal Logic and Applications*, Toulouse, France, 2017. URL `https://hal.inria.fr/hal-01614707/`. Downloaded in February 2018.

[89] Richard E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM journal on computing*, 6(3): 467–480, 1977.

[90] Ori Lahav. From frame properties to hypersequent rules in modal logics. In *Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 408–417, 2013.

[91] Daniel Leivant. On the proof theory of the modal logic for arithmetic provability. *The Journal of Symbolic Logic*, 46(3):531–538, 1981.

[92] Björn Lellmann. Linear nested sequents, 2-sequents and hypersequents. In Hans De Nivelle, editor, *Automated Reasoning with Analytic Tableaux and Related Methods: 24th International Conference*, pages 135–150. Springer International Publishing, 2015.

[93] Björn Lellmann and Dirk Pattinson. Constructing cut free sequent systems with context restrictions based on classical or intuitionistic logic. In Kamal Lodaya, editor, *Logic and Its Applications: 5th Indian Conference, ICLA 2013, Chennai, India, January 10-12, 2013. Proceedings*, pages 148–160, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg. ISBN 978-3-642-36039-8. doi: 10.1007/978-3-642-36039-8_14.

[94] Björn Lellmann and Elaine Pimentel. Proof search in nested sequent calculi. In Martin Davis, Ansgar Fehnker, Annabelle McIver, and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning: 20th International Conference*, pages 558–574. Springer Berlin Heidelberg, 2015.

[95] Clarence I. Lewis and Cooper H. Langford. *Symbolic Logic*. Dover Publications, second edition, 1932, 1959.

[96] Chuck Liang and Dale Miller. Focusing and polarization in intuitionistic logic. In *Computer Science Logic*, pages 451–465. Springer, 2007.

[97] Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46): 4747–4768, 2009.

[98] Kuno Lorenz. *Arithmetik und Logik als Spiele*. PhD thesis, Christian-Albrechts-Universität zu Kiel, 1961. Excerpts reprinted in [103].

[99] Kuno Lorenz. *Dialogspiele als semantische Grundlage von Logikkalkülen*. Springer, 1968. Reprinted in [103].

[100] Paul Lorenzen. Logik und Agon. In *Atti del XII Congresso Internazionale di Filosofia*, volume 4, pages 187–194, 1958. Reprinted in [103].

[101] Paul Lorenzen. Ein dialogisches Konstruktivitätskriterium. In *Infinistic Methods. Proceeding of the Symposium on Foundations of Mathematics*, pages 193–200, 1961. Reprinted in [103].

[102] Paul Lorenzen. *Normative Logic and Ethics*. Bibliographisches Institut, 1969.

[103] Paul Lorenzen and Kuno Lorenz. *Dialogische Logik*. Wissenschaftliche Buchgesellschaft, 1978.

[104] Shôji Maehara. Eine Darstellung der intuitionistischen Logik in der klassischen. *Nagoya mathematical journal*, 7:45–64, 1954.

[105] Paolo Maffezioli and Sara Negri. A gentzen-style analysis of public announcement logic. In *Proceedings of the International Workshop on Logic and Philosophy of Knowledge, Communication and Action*, pages 293–313, 2010.

[106] Allen L. Mann, Gabriel Sandu, and Merlijn Sevenster. *Independence-Friendly Logic – A Game-Theoretic Approach*. London Mathematical Society lecture note series. Cambridge Univ. Press, 2011.

[107] Sonia Marin and Lutz Straßburger. Label-free modular systems for classical and intuitionistic modal logics. In Rajev Goré, Barteld Kooi, and Agi Kurucz, editors, *Advances in Modal Logic*, volume 10, 2014.

[108] Andrea Masini. 2-sequent calculus: Intuitionism and natural deduction. *Journal of Logic and Computation*, 3(5):533–562, 1993.

[109] Gregor Mayer. *Die Logik im Deutschen Konstruktivismus – Die Rolle formaler Systeme im Wissenschaftsaufbau der Erlanger und Konstanzer Schule*. PhD thesis, Ludwig-Maximilians-Universität München, 1981.

[110] Michael Mendler and Valeria de Paiva. Constructive CK for contexts. *Context Representation and Reasoning (CRR-2005)*, 13, 2005.

[111] Michael Mendler and Stephan Scheele. Towards constructive DL for abstraction and refinement. *Journal of Automated Reasoning*, 44:207–243, 2009.

[112] Michael Mendler and Stephan Scheele. Cut-free gentzen calculus for multimodal CK. *Information and Computation*, 209(12):1465–1490, 2011. ISSN 0890-5401.

[113] Dale Miller and Marco Volpe. Focused labeled proof systems for modal logic. In *Logic for Programming, Artificial Intelligence, and Reasoning, 20th International Conference*, 2015.

[114] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51(1–2):125–157, 1991. ISSN 0168-0072.

[115] Grigori Mints. Indexed systems of sequents and cut-elimination. *Journal of Philosophical Logic*, 26(6):671–696, 1997.

[116] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5):507–544, 2005.

[117] Masao Ohnishi and Kazuo Matsumoto. Gentzen Method in Modal Calculi. *Osaka Mathematical Journal*, 9:113–130, 1957.

[118] Masao Ohnishi and Kazuo Matsumoto. Gentzen Method in Modal Calculi II. *Osaka Mathematical Journal*, 11:115–120, 1959.

[119] Jan Plaza. Logics of public communications. In M.L. Emric et al., editor, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[120] Gordon Plotkin and Colin Stirling. A framework for intuitionistic modal logics. In J. Y. Halpern, editor, *Theoretical Aspects of Reasoning about Knowledge*, pages 399–406, 1986.

[121] Amir Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science*, 1977.

[122] Francesca Poggiolesi. The method of tree-hypersequents for modal propositional logic. In David Makinson, Jacek Malinowski, and Heinrich Wansing, editors, *Towards Mathematical Philosophy: Papers from the Studia Logica conference Trends in Logic IV*, pages 31–51. Springer Netherlands, 2009.

[123] Francesca Poggiolesi. *Gentzen Calculi for Modal Propositional Logic*. Trends in Logic 32. Springer, 2011.

[124] Garrel Pottinger. Uniform, cut-free formulations of T, S_4, and S_5. *Journal of Symbolic Logic*, 48(3):900, 1983. Abstract.

[125] Arthur N. Prior. *Time and Modality: Being the John Locke Lectures for 1955–6 delivered in the University of Oxford*. Oxford at the Clarendon Press, 1957.

[126] Shahid Rahman. *Über Dialoge, Protologische Kategorien und andere Seltenheiten*. Europäische Hochschulschriften. Peter Lang, 1993.

[127] Shahid Rahman and Laurent Keiff. On how to be a dialogician. In Daniel Vanderveken, editor, *Logic, Thought and Action*, volume 2 of *Logic, Epistemology, and the Unity of Science*, pages 359–408. Springer Netherlands, 2005. ISBN 978-1-4020-2616-4. doi: 10.1007/1-4020-3167-X_17.

[128] Shahid Rahman and Helge Rückert. Dialogische Logik und Relevanz. Technical report, Universität des Saarlandes, Dec 1998.

[129] Manuel Rebuschi. Implicit versus Explicit Knowledge in Dialogical Logic. In Ondrej Majer, Ahti-Veikko Pietarinen, and Tulenheimo Tero, editors, *Games: Unifying Logic, Language, and Philosophy*, pages 229–246. Springer, 2009.

[130] Helge Rückert and Shahid Rahman. Dialogische Modallogik (für T, B, S4 und S5). Technical report, Universität des Saarlandes, Nov 1998.

[131] Esa Saarinen. Dialogue Semantics versus Game-Theoretic Semantics. *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, 1978(2):41–59, 1978.

[132] Stephan Scheele. *Model and Proof Theory of Constructive ALC – Constructive Description Logics*. PhD thesis, Otto-Friedrich Universität Bamberg, 2015.

[133] Klaus Schild. A correspondence theory for terminological logics: preliminary report. Technical report, Technische Universität Berlin, 1991.

[134] Robert J. Simmons. Structural focalization. *ACM Transactions on Computational Logic (TOCL)*, 15(3):21, 2014.

[135] Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.

[136] Wolfgang Stegmüller. Remarks on the Completeness of Logical Systems Relative to the Validity-Concept of P. Lorenzen and K. Lorenz. *Notre Dame Journal of Formal Logic*, 5(2):81–112, 1964.

[137] Martin Sticht. The Design and Implementation of a Game-Theoretic Decision Procedure for the constructive Description Logic $c\mathcal{ALC}$. Otto-Friedrich-Universität Bamberg, Diploma thesis, 2011.

[138] Martin Sticht. Multi-Agent Dialogue Games and Dialogue Sequents for Proof Search and Scheduling. In Camillo Fiorentini and Alberto Momigliano, editors, *Proceedings of the 31st Italian Conference on Computational Logic*, 2016. URL `http://ceur-ws.org/Vol-1645/paper_20.pdf`.

[139] Martin Sticht. Multi-Agent Dialogues and Dialogue Sequents for Proof Search and Scheduling in Intuitionistic Logic and Modal Logic S4. *Fundamenta Informaticae*, 161(1-2):191–218, 2018. doi: 10.3233/FI-2018-1700.

[140] Martin Sticht. A Game-Theoretic Interpretation for the Constructive Modal Logic CK. In Jean-Yves Béziau, Alexandre Costa-Leite, and Itala M.L. D'Ottaviano, editors, *Aftermath of the Logical Paradise*, volume 81. Coleção CLE, in press. Manuscript submitted in 2013.

[141] Lutz Straßburger. Cut Elimination in Nested Sequents for Intuitionistic Modal Logics. In Frank Pfenning, editor, *Foundations of Software Science and Computation Structures: 16th International Conference*, pages 209–224. Springer Berlin Heidelberg, 2013.

[142] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge Univ. Press, second edition, 2000.

[143] Rolf D. Valerius. *Die Logik von Rahmen- und Stopregeln in Lorenzen-Spielen*. PhD thesis, Universität Stuttgart, 1990.

[144] Frank Van Dun. On the modes of opposition in the formal dialogues of P. Lorenzen. *Logique et analyse*, 15:103–136, 1972.

[145] Luca Viganò. *Labelled non-classical logics*. Kluwer Academic Publishers, 2000.

[146] Nikolai N. Vorob'ev. A new algorithm for derivability in the constructive propositional calculus. *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, 52:193–225, 1958. English translation in "American Mathematical Society Translations" series 2, vol 94 (1970), pp. 37–71.

[147] Douglas N. Walton and Erik C.W. Krabbe. *Commitment in dialogue: basic concepts of interpersonal reasoning*. State University of New York Press, 1995.

[148] Heinrich Wansing. Sequent systems for modal logics. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, pages 61–145. Kluwer, Dodrecht, 2002.

[149] Duminda Wijesekera. Constructive Modal Logics I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.

Index

University
of Bamberg
Press

In computer science, modal logics play an important role in various areas, e.g., to model knowledge structures or ontologies. Intuitionistic or constructive propositional logic can be considered as a special kind of modal logic. Constructive modal logics describe a restrictive family of modal systems. To prove validity of a statement formalized in such logics, various reasoning procedures (calculi) have been investigated. When searching for proofs, a normalization of deductions in a calculus is desired. A new attempt to find a normalized calculus leads to dialogical logic, a game-theoretic reasoning technique. Usually, two players, one proponent and one opponent, argue about an assertion, expressed as a formula and stated by the proponent at the beginning of the play. The dialogical approach is very flexible as the game rules can be adjusted easily. It turns out that introducing further proponent-players who fight one opponent in a round-based setting leads to a normalization of proofs. Ordinary sequent systems can easily be transferred into the dialectic setting to achieve that normalization. Further, the round-based scheduling induces a method to parallelize the reasoning process. Modifying the game rules makes it possible to construct new intermediate or even more restrictive logics.

In this work, multi-proponent dialogical systems are introduced and adequateness proofs for rules that implement intuitionistic propositional logic and the modal systems S4 and CS4, are provided. Similarities and differences to various sequent systems and dialogical approaches are investigated.

eISBN: 978-3-86309-600-7

www.uni-bamberg.de/ubp