Annales Mathematicae et Informaticae
49 (2018) pp. 61–73
DOI: 10.33039/ami.2018.05.005
http://ami.uni-eszterhazy.hu

On hyperbolic k-Pell quaternions sequences*

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Submitted December 30, 2017 — Accepted May 22, 2018

Abstract

In this paper we introduce the hyperbolic k-Pell functions and new classes of quaternions associated with this type of functions are presented. In addition, the Binet formulas, generating functions and some properties of these functions and quaternions sequences are studied.

Keywords: Quaternions, Hyperbolic functions, k-Pell sequence, Binet's identity, Generating functions.

MSC: 11B37, 11R52, 05A15, 11B83.

1. Introduction and background

Fibonacci sequence is one of the sequences of positive integers that has been studied over several years. Such sequence is associated with the well-known golden ratio and there exist several relations of this sequence with different scientific areas with many applications. Both Fibonacci and Lucas sequences are examples of sequences which have been studied by many scientists. One can get more detailed information on these sequences from the research works [2, 13] among others.

The Pell numbers are defined by $P_{n+1} = 2P_n + P_{n-1}$, $n \geq 1$, with initial conditions given by $P_0 = 0$, $P_1 = 1$. This sequence is associated with the silver

*The author is member of the Research Centre CMAT-UTAD and colaborator of the Research Centre CIDTFF. This research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UID/MAT/00013/2013 and UID/CED/00194/2013.

ratio $\delta = 1 + \sqrt{2}$ which has been investigated by several authors and some of its basic properties have been stated in several papers (see, for example, the study of Horadam in [14] and Koshy in [15]). One generalization of the Pell sequence is the k-Pell sequence for any positive real number k. The k-Pell sequence $\{P_{k,n}\}_n$ is recursively defined by

$$P_{k,0} = 0,\ P_{k,1} = 1,\ P_{k,n+1} = 2P_{k,n} + kP_{k,n-1},\ n \geq 1. \tag{1.1}$$

The Binet-style formulae for this sequences is given by

$$P_{k,n} = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}, \tag{1.2}$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation

$$r^2 - 2r - k = 0 \tag{1.3}$$

associated with the above recurrence relation (1.1). Note that $r_1 + r_2 = 2,\ r_1 r_2 = -k$ and $r_1 - r_2 = 2\sqrt{1+k}$. For more details about this sequence see, for example, [7, 8, 9].

The subject of quaternions sequences has been a focus of great research. Now we find in the literature so many different types of sequences of quaternions: Fibonacci quaternions, Lucas quaternions, k-Fibonacci and k-Lucas Generalized quaternions, Pell quaternions, Pell-Lucas quaternions, Modified Pell quaternions, Jacobsthal quaternions, etc., and their generalizations. One can see several recent research papers on this subject from, for example, [1, 3, 4, 5, 6, 16, 17, 22].

It is well-known that a quaternion is defined by

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3,$$

where $q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}$ and $i_1,\ i_2$ and i_3 are complex operators such that

$$\begin{aligned} i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,\\ i_1 i_2 = -i_2 i_1 = i_3,\\ i_2 i_3 = -i_3 i_2 = i_1,\\ i_3 i_1 = -i_1 i_3 = i_2. \end{aligned} \tag{1.4}$$

The conjugate of q is the quaternion

$$q^* = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3$$

and the norm of q is

$$\|q\| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The quaternion was formally introduced by Hamilton in 1843 and some background about this type of hypercomplex numbers can be found for example in [10].

In [21], the authors introduced the Pell quaternions as

$$R_n = P_n + P_{n+1}i_1 + P_{n+2}i_2 + P_{n+3}i_3,$$

where R_n is the nth Pell quaternion and $i_1,\ i_2,\ i_3$ satisfy the rules (1.4). Following this idea, now we introduced the k-Pell quaternions as follows:

Definition 1.1. For any positive real number k, the nth k-Pell quaternion is defined as

$$R_{k,n} = P_{k,n} + P_{k,n+1}i_1 + P_{k,n+2}i_2 + P_{k,n+3}i_3,$$

where $i_1,\ i_2,\ i_3$ satisfy the rules (1.4).

The classical hyperbolic functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Stakhov and Rozin in [19] defined the symmetrical hyperbolic functions and, in particular, they gave all details of symmetrical hyperbolic Fibonacci and symmetrical hyperbolic Lucas functions. Also in [20] the authors have introduced the hyperbolic Fibonacci functions and the hyperbolic Lucas functions. Several research papers on this subject can be found in the literature, see, for example, the works [23, 11, 12, 18], among others.

In the light of all these concepts and information stated before, in this paper we introduce the hyperbolic k-Pell functions and new classes of quaternions associated with this type of functions are presented. In addition, the Binet formulas, generating functions and some properties of these quaternions are studied.

2. The hyperbolic *k*-Pell functions

In this section we introduce the hyperbolic k-Pell functions and some properties of these type of functions are studied.

The hyperbolic Pell functions are defined by

$$sP(x) = \frac{\delta^x - \delta^{-x}}{\delta + \delta^{-1}}$$

and

$$cP(x) = \frac{\delta^x + \delta^{-x}}{\delta + \delta^{-1}},$$

where $\delta = 1 + \sqrt{2}$ is the silver ratio. For more information about this type of functions, see the works [11, 12].

Now, these equalities can naturally be introduced for the case of k-Pell sequence. Based on an analogy between Binet's formula (1.2) and the classical hyperbolic functions we define the hyperbolic k-Pell functions as follows:

Definition 2.1. For any positive real number k and any real number x, the hyperbolic k-Pell functions are defined as

$$sP_k(x) = \frac{(r_1)^x - (r_1)^{-x}}{r_1 + k(r_1)^{-1}}, \tag{2.1}$$

and

$$cP_k(x) = \frac{(r_1)^x + (r_1)^{-x}}{r_1 + k(r_1)^{-1}}, \tag{2.2}$$

since $r_1 + k(r_1)^{-1} = 2\sqrt{1+k}$, for $n \geq 2$, where $r_1 = 1 + \sqrt{1+k}$ is the positive root of the characteristic equation (1.3) associated with the above recurrence relation (1.1).

Note that for the particular case of $k = 1$ we obtain the hyperbolic Pell functions. Next we present the main properties of these functions in a similar way in which similar properties of the Pell hyperbolic functions are usually presented.

Theorem 2.2 (Pythagorean theorem). *Let $sP_k(x)$ and $cP_k(x)$ be two functions of hyperbolic k-Pell functions. For $x \in \mathbb{R}$ and k any positive real number,*

$$\left(cP_k(x)\right)^2 - \left(sP_k(x)\right)^2 = \frac{1}{1+k}.$$

Proof. From the definition of the hyperbolic k-Pell functions (2.1) and (2.2), we have

$$\begin{aligned}
\left(cP_k(x)\right)^2 - \left(sP_k(x)\right)^2 &= \left(\frac{(r_1)^x + (r_1)^{-x}}{r_1 + k(r_1)^{-1}}\right)^2 - \left(\frac{(r_1)^x - (r_1)^{-x}}{r_1 + k(r_1)^{-1}}\right)^2 \\
&= \frac{(r_1)^{2x} + 2\,(r_1)^x\,(r_1)^{-x} + (r_1)^{-2x} - (r_1)^{2x} + 2\,(r_1)^x\,(r_1)^{-x} - (r_1)^{-2x}}{\left(r_1 + k\,(r_1)^{-1}\right)^2} \\
&= \frac{4}{\left(r_1 + k\,(r_1)^{-1}\right)^2} = \frac{4}{\left(2\sqrt{1+k}\right)^2}
\end{aligned}$$

and the result follows. $\square$

Theorem 2.3 (Sum and Difference). *Let $sP_k(x)$ and $cP_k(x)$ be the hyperbolic k-Pell functions. For $x, y \in \mathbb{R}$ and k any positive real number,*

1. $cP_k(x+y) = \sqrt{1+k}\,\left(cP_k(x)cP_k(y) + sP_k(x)sP_k(y)\right);$
2. $cP_k(x-y) = \sqrt{1+k}\,\left(cP_k(x)cP_k(y) - sP_k(x)sP_k(y)\right);$
3. $sP_k(x+y) = \sqrt{1+k}\,\left(sP_k(x)cP_k(y) + cP_k(x)sP_k(y)\right);$
4. $sP_k(x-y) = \sqrt{1+k}\,\left(sP_k(x)cP_k(y) - cP_k(x)sP_k(y)\right).$

Proof. From the definition of the hyperbolic k-Pell functions (2.1) and (2.2), we have

$$\begin{aligned} cP_k(x)cP_k(y)+sP_k(x)sP_k(y) &= \left(\frac{(r_1)^x+(r_1)^{-x}}{r_1+k(r_1)^{-1}}\right)\left(\frac{(r_1)^y+(r_1)^{-y}}{r_1+k(r_1)^{-1}}\right) \\ &\quad +\left(\frac{(r_1)^x-(r_1)^{-x}}{r_1+k(r_1)^{-1}}\right)\left(\frac{(r_1)^y-(r_1)^{-y}}{r_1+k(r_1)^{-1}}\right) \\ &= \frac{2\,(r_1)^{x+y}+2\,(r_1)^{-x-y}}{\left(r_1+k\,(r_1)^{-1}\right)^2} \\ &= \frac{2}{r_1+k\,(r_1)^{-1}}\left(\frac{(r_1)^{x+y}+(r_1)^{-x-y}}{r_1+k\,(r_1)^{-1}}\right) \\ &= \frac{1}{\sqrt{1+k}}cP_k(x+y) \end{aligned}$$

as required. The proofs of the other equalities are similar. □

By setting $x=y$ in these sums equations, we have the following corollary.

Corollary 2.4 (Double argument)**.** *Let $sP_k(x)$ and $cP_k(x)$ be the hyperbolic k-Pell functions. For $x\in\mathbb{R}$ and k any positive real number,*

1. $cP_k(2x)=\sqrt{1+k}\left((cP_k(x))^2+(sP_k(x))^2\right);$
2. $sP_k(2x)=\sqrt{1+k}\,(2sP_k(x)cP_k(x)).$

From Pythagorean theorem and the equalities of double argument we have the following results:

Corollary 2.5 (Half argument)**.** *Let $sP_k(x)$ and $cP_k(x)$ be the hyperbolic k-Pell functions. For $x\in\mathbb{R}$ and k any positive real number,*

1. $(cP_k(x))^2=\frac{1}{2\sqrt{1+k}}\left(cP_k(2x)+\frac{1}{\sqrt{1+k}}\right);$
2. $(sP_k(x))^2=\frac{1}{2\sqrt{1+k}}\left(cP_k(2x)-\frac{1}{\sqrt{1+k}}\right).$

Proof. For the proof of the first identity we use the double argument for the hyperbolic cosine of k-Pell function and the equation of Pythagorean theorem. Hence we have:

$$\begin{aligned} cP_k(2x) &= \sqrt{1+k}\left((cP_k(x))^2+(sP_k(x))^2\right) \\ &= \sqrt{1+k}\left((cP_k(x))^2\right)+\sqrt{1+k}\left((sP_k(x))^2\right) \\ &= \sqrt{1+k}\left((cP_k(x))^2\right)+\sqrt{1+k}\left((cP_k(x))^2-\frac{1}{1+k}\right) \end{aligned}$$

$$= 2\sqrt{1+k}\,(cP_k(x))^2 - \frac{\sqrt{1+k}}{1+k},$$

and the identity required easily follows.

About the second identity we use first the Pythagorean theorem and after we finish the proof with the use of the previous statement in this Corollary. Hence we have:

$$\begin{aligned}(sP_k(x))^2 &= (cP_k(x))^2 - \frac{1}{1+k}\\ &= \frac{1}{2\sqrt{1+k}}\left(cP_k(2x) + \frac{1}{\sqrt{1+k}}\right) - \frac{1}{1+k}\\ &= \frac{1}{2\sqrt{1+k}}cP_k(2x) + \frac{1}{2(1+k)} - \frac{1}{1+k},\end{aligned}$$

and the result follows. □

3. The hyperbolic *k*-Pell quaternions sequences

This section aims to set out the definition of the hyperbolic k-Pell quaternions sequences and some elementary results involving it.

First of all, we define the hyperbolic k-Pell sine and the hyperbolic k-Pell cosine quaternions.

Definition 3.1. The hyperbolic k-Pell sine and the hyperbolic k-Pell cosine quaternions are defined, respectively, by the relations

$$sP_k(x)q = sP_k(x) + sP_k(x+1)i_1 + sP_k(x+2)i_2 + sP_k(x+3)i_3 \tag{3.1}$$

and

$$cP_k(x)q = cP_k(x) + cP_k(x+1)i_1 + cP_k(x+2)i_2 + cP_k(x+3)i_3, \tag{3.2}$$

where x is any real number and $sP_k(x), cP_k(x)$ are the hyperbolic k-Pell functions stated in Definition 2.1.

The next result shows some correlations between these type of quaternions.

Theorem 3.2. *For any $x \in \mathbb{R}$, we have*

1. $sP_k(x+1)q = \left(r_1 - (r_1)^{-1}\right) cP_k(x)q + sP_k(x-1)q;$
2. $cP_k(x+1)q = \left(r_1 - (r_1)^{-1}\right) sP_k(x)q + cP_k(x-1)q;$
3. $sP_k(x+2)q = \left((r_1)^2 + (r_1)^{-2}\right) sP_k(x)q - sP_k(x-2)q;$
4. $cP_k(x+2)q = \left((r_1)^2 + (r_1)^{-2}\right) cP_k(x)q - cP_k(x-2)q;$

Proof. For the first formula we use identities (2.1), (2.2), (3.1) and (3.2). Therefore by the use of the identity (3.1), we have that

$$\begin{aligned}&sP_k(x+1)q - sP_k(x-1)q\\ &= (sP_k(x+1) - sP_k(x-1)) + (sP_k(x+2) - sP_k(x))\, i_1\\ &\quad + (sP_k(x+3) - sP_k(x+1))\, i_2 + (sP_k(x+4) - sP_k(x+2))\, i_3.\end{aligned}$$

Now using the identities (2.1) and (2.2),

$$\begin{aligned}&sP_k(x+1)q - sP_k(x-1)q\\ &= \left(r_1 - (r_1)^{-1}\right) cP_k(x) + \left(r_1 - (r_1)^{-1}\right) cP_k(x+1)i_1\\ &\quad + \left(r_1 - (r_1)^{-1}\right) cP_k(x+2)i_2 + \left(r_1 - (r_1)^{-1}\right) cP_k(x+3)i_3.\end{aligned}$$

Finaly the use of the identity (3.2) gives the result required.

The second formula can be proved in a similar way. The third formula is proved by the use of the identities (2.1) and (3.1) and similarly we can show the last identity of this theorem using (2.2) and (3.2). □

In the next result it is presented the norm of these quaternions.

Theorem 3.3. *For any $x \in \mathbb{R}$, we have*

1. $\|sP_k(x)q\|^2 = \frac{((r_1)^2+1)((r_1)^4+1)\left((r_1+k(r_1)^{-1})sP_k(2x)+(r_1)^{-2x}(1+(r_1)^{-6})\right)-8}{(r_1+k(r_1)^{-1})^2}$;

2. $\|cP_k(x)q\|^2 = \frac{((r_1)^2+1)((r_1)^4+1)\left((r_1+k(r_1)^{-1})cP_k(2x)+(r_1)^{-2x}((r_1)^{-6}-1)\right)+8}{(r_1+k(r_1)^{-1})^2}$.

Proof. We prove the first formula using the definition of the norm of a quaternion and the identity (2.1). We have

$$\begin{aligned}&\|sP_k(x)q\|^2\\ &= (sP_k(x))^2 + (sP_k(x+1))^2 + (sP_k(x+2))^2 + (sP_k(x+3))^2\\ &= \frac{1}{(r_1+k(r_1)^{-1})^2}\left((r_1)^{2x}\left(\frac{(r_1)^8-1}{(r_1)^2-1}\right) + (r_1)^{-2x}\left(\frac{(r_1)^{-8}-1}{(r_1)^{-2}-1}\right) - 8\right)\\ &= \frac{1}{(r_1+k(r_1)^{-1})^2}\left(\left((r_1)^2+1\right)\left((r_1)^4+1\right)\left((r_1)^{2x} + (r_1)^{-6}(r_1)^{-2x}\right) - 8\right)\end{aligned}$$

and the result follows.

The proof of the second identity is similar by the use, once more, of the definition of the norm of a quaternion and identity (2.2). □

Now we introduce a new sequences of quaternions, namely the hyperbolic k-Pell quaternions sequences.

For any positive real number k and n a non negative integer, the hyperbolic k-Pell quaternions sequences can be divided into two types of sequences: the hyperbolic k-Pell sine quaternions denoted by $\{sP_k(x+n)q\}_{n=0}^{\infty}$ and the hyperbolic k-Pell cosine quaternions denoted by $\{cP_k(x+n)q\}_{n=0}^{\infty}$. Having regard to the identities (3.1) and (3.2) we have

$$sP_k(x+n)q = sP_k(x+n)q + \sum_{s=1}^{3} sP_k(x+n+s)qi_s \tag{3.3}$$

and

$$cP_k(x+n)q = cP_k(x+n)q + \sum_{s=1}^{3} cP_k(x+n+s)qi_s \tag{3.4}$$

for the nth term of the hyperbolic k-Pell sine and cosine quaternions sequences respectively. Such sequences are defined recurrently by

$$sP_k(x+n+4)q = \left((r_1)^2 + (r_1)^{-2}\right) sP_k(x+n+2)q - sP_k(x+n)q \tag{3.5}$$

with initial conditions given by $sP_k(x)q$ and $sP_k(x+1)q$, and

$$cP_k(x+n+4)q = \left((r_1)^2 + (r_1)^{-2}\right) cP_k(x+n+2)q - cP_k(x+n)q \tag{3.6}$$

with initial conditions given by $cP_k(x)q$ and $cP_k(x+1)q$, respectively.

4. Generating functions and Binet formulas of these quaternions sequences

Next we shall give the generating functions for the hyperbolic k-Pell quaternions sequences and also we give the Binet-style formulae for these quaternions sequences.

Next, we shall give the generating function for the hyperbolic k-Pell quaternions sequences. We shall write this quaternion sequence as a power series, where each term of the sequence correspond to coefficients of the series. Let us consider the sequences $\{sP_k(x+n)q\}_{n=0}^{\infty}$ and $\{cP_k(x+n)q\}_{n=0}^{\infty}$ of hyperbolic k-Pell sine quaternions and hyperbolic k-Pell cosine quaternions, respectively. We define the respective generating functions as

$$g_s(x,t) = \sum_{n=0}^{\infty} sP_k(x+n)qt^n \tag{4.1}$$

and

$$g_c(x,t) = \sum_{n=0}^{\infty} cP_k(x+n)qt^n \tag{4.2}$$

A new expression of the generating function of these kind of quaternions is given in the following result.

Theorem 4.1. *The generating functions for the hyperbolic k-Pell quaternions sequences are*

$$g_s(x,t)=\frac{sP_k(x)q+sP_k(x+1)qt-sP_k(x-2)qt^2-sP_k(x-1)qt^3}{1-\left((r_1)^2+(r_1)^{-2}\right)t^2+t^4} \tag{4.3}$$

and

$$g_c(x,t)=\frac{cP_k(x)q+cP_k(x+1)qt-cP_k(x-2)qt^2-cP_k(x-1)qt^3}{1-\left((r_1)^2+(r_1)^{-2}\right)t^2+t^4} \tag{4.4}$$

Proof. Using (4.1) we have

$$g_s(x,t)=sP_k(x)q+sP_k(x+1)qt+sP_k(x+2)qt^2+\cdots+sP_k(x+n)qt^n+\cdots$$

Multiplying both sides of previous identity by $-\left((r_1)^2+(r_1)^{-2}\right)t^2$ and t^4, and consider $\left(1-\left((r_1)^2+(r_1)^{-2}\right)t^2+t^4\right)g_s(x,t)$, we obtain the result required take into account the third identity of Theorem 3.2.

A similar way can be used for the proof of (4.4) by taking into account the last identity of Theorem 3.2 when we use $\left(1-\left((r_1)^2+(r_1)^{-2}\right)t^2+t^4\right)g_c(x,t)$. □

The following result gives us the Binet-style formula for the hyperbolic k-Pell quaternions sequences

Theorem 4.2 (The Binet-style formulae)**.** *For the hyperbolic k-Pell sine and the hyperbolic k-Pell cosine quaternions, the binet formulae are given, respectively, by*

$$sP_k(x+n)q=\frac{A\,(r_1)^{x+n}-B\,(r_1)^{-x-n}}{r_1+k(r_1)^{-1}} \tag{4.5}$$

and

$$cP_k(x+n)q=\frac{A\,(r_1)^{x+n}+B\,(r_1)^{-x-n}}{r_1+k(r_1)^{-1}}, \tag{4.6}$$

where $A=1+r_1i_1+(r_1)^2\,i_2+(r_1)^3\,i_3$ *and* $B=1+(r_1)^{-1}\,i_1+(r_1)^{-2}\,i_2+(r_1)^{-3}\,i_3$.

Proof. For the first formula we use identity (3.1) and equation (2.1). Therefore

$$\begin{aligned}
&sP_k(x+n)q\\
&=sP_k(x+n)+sP_k(x+n+1)i_1+sP_k(x+n+2)i_2+sP_k(x+n+3)i_3\\
&=\frac{1}{r_1+k(r_1)^{-1}}\,(r_1)^{x+n}\left(1+r_1i_1+(r_1)^2\,i_2+(r_1)^3\,i_3\right)\\
&\quad-\frac{1}{r_1+k(r_1)^{-1}}\,(r_1)^{-x-n}\left(1+(r_1)^{-1}\,i_1+(r_1)^{-2}\,i_2+(r_1)^{-3}\,i_3\right),
\end{aligned}$$

as required.

The second Binet's formula can be similarly proved with the use of identity (3.2) and equation (2.2). □

5. More identities involving these sequences

In this section we state some identities related with these type of quaternions sequences. Such identities can be show by the use of Binet's formula of each sequence.

Theorem 5.1 (Catalan's Identity)**.** *For n and r, nonnegative integer numbers, such that $r \leq n$, and for k a positive real number, the Catalan identities*

$$sP_k(x+n+r)qsP_k(x+n-r)q - (sP_k(x+n)q)^2$$

for the hyperbolic k-Pell sine quaternions and

$$cP_k(x+n+r)qcP_k(x+n-r)q - (cP_k(x+n)q)^2$$

for the hyperbolic k-Pell cosine quaternions are given by

$$\frac{1}{4(1+k)}\left(AB\left(1-(r_1)^{2r}\right)+BA\left(1-(r_1)^{-2r}\right)\right) \tag{5.1}$$

and

$$\frac{1}{4(1+k)}\left(-AB\left(1-(r_1)^{2r}\right)-BA\left(1-(r_1)^{-2r}\right)\right), \tag{5.2}$$

respectively, where A and B are the quaternions defined in Theorem 4.2.

Proof. Using the Binet formula of $sP_k(x+n)q$ stated in Theorem 4.2 and the fact that $r_1 + k(r_1)^{-1} = r_1 - r_2 = 2\sqrt{1+k}$, we obtain that

$$\begin{aligned}
&sP_k(x+n+r)qsP_k(x+n-r)q - (sP_k(x+n)q)^2 \\
&= \frac{-AB(r_1)^{2r} - BA(r_1)^{-2r} + AB + BA}{\left(r_1 + k(r_1)^{-1}\right)^2} \\
&= \frac{AB\left(1-(r_1)^{2r}\right)+BA\left(1-(r_1)^{-2r}\right)}{\left(2\sqrt{1+k}\right)^2}
\end{aligned}$$

and the result follows.

With a similar reasoning we prove the other Catalan Identity for the hyperbolic k-Pell cosine quaternions. □

In particular case of $r = 1$ in Catalan's Identity, we obtain the Cassini's Identity for both quaternions sequences which is presented in the following Corollary.

Corollary 5.2 (Cassini's Identity)**.** *For any natural number n and for k a positive real number, the Cassini Identities*

$$sP_k(x+n+1)qsP_k(x+n-1)q - (sP_k(x+n)q)^2$$

for the hyperbolic k-Pell sine quaternions and

$$cP_k(x+n+1)qcP_k(x+n-1)q - (cP_k(x+n)q)^2$$

for the hyperbolic k-Pell cosine quaternions are given by

$$\frac{1}{4(1+k)}\left(AB\left(1-(r_1)^2\right)+BA\left(1-(r_1)^{-2}\right)\right)$$

and

$$\frac{1}{4(1+k)}\left(-AB\left(1-(r_1)^2\right)-BA\left(1-(r_1)^{-2}\right)\right),$$

respectively, where A and B are the quaternions defined in Theorem 4.2.

Once more the next identity is easily proved by the use of the Binet formula of each sequence. In fact we have:

Theorem 5.3 (d'Ocagne's Identity)**.** *For n a nonnegative integer number and m any natural number, if $m > n$, the d'Ocagne Identities*

$$sP_k(x+m)qsP_k(x+n+1)q - sP_k(x+m+1)qsP_k(x+n)$$

for the hyperbolic k-Pell sine quaternions and

$$cP_k(x+m)qcP_k(x+n+1)q - cP_k(x+m+1)qcP_k(x+n)$$

for the hyperbolic k-Pell cosine quaternions are given by

$$\frac{\left(r_1-(r_1)^{-1}\right)\left(AB\,(r_1)^{m-n}-BA\,(r_1)^{-(m-n)}\right)}{4(1+k)}$$

and

$$\frac{\left(r_1-(r_1)^{-1}\right)\left(-AB\,(r_1)^{m-n}+BA\,(r_1)^{-(m-n)}\right)}{4(1+k)}$$

respectively, where A and B are the quaternions defined in Theorem 4.2.

Finally, note that the all three identities for the hyperbolic k-Pell cosine quaternions sequences are symmetrical of the identities of the the hyperbolic k-Pell sine quaternions sequences.

Conclusions

In this paper, we have introduced the hyperbolic k-Pell functions and new classes of quaternions associated with this type of functions are presented, namely, the sequences of hyperbolic k-Pell sine and cosine quaternions defined by a recurrence relation. Some properties involving these sequences, Binet formulas, generating functions and some identities were studied. In the future, we intend to continue the study of these sequences and it is our aim to study the generating matrices and some combinatorial identities.

Acknowledgements. The author would like to thank the referees for their pertinent comments and valuable suggestions, which significantly improve the final version of the manuscript.

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