

Limited Foresight

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Abstract

This thesis is about multi-period problems in which the decision-maker or players cannot see far enough ahead to solve the problem completely.

The thesis considers why it might be that players reason forwards at all, let alone reasoning forwards only finitely far. It shows, using finite automata, that there is a class of problems for which forwards reasoning is more efficient than backwards reasoning. It goes on to use these finite automata to solve for an optimal foresight length.

It then discusses solution concepts, and applies its preferred solution concept to two problems - one macro problem involving a central banker, and one micro problem concerning the decision whether to smoke.

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Chapter 0

Preface

This thesis is about limited foresight. Limited foresight arises in multi-period problems in which the decision-maker or players cannot see far enough ahead to solve the problem completely.

Some decision-problems and games can, of course, be solved completely. Though laborious, it would be tractable in practice as well as possible in theory to solve noughts and crosses completely - for example by backwards induction from an exhaustive search through all legal terminating board positions. Classical game and decision theory treats all problems as if they were like this. Where it is possible in principle to solve a game completely, players are assumed to do so, and issues of tractability are not considered. This is relatively easy to defend. If there is, indeed, some difficulty in searching through possibilities, that could be incorporated into the game through an explicit search cost function. Then, in a sense, we have a different game. Game theory only pretends to offer solutions to the games specified, not to other games not specified.

In this thesis we shall not be disagreeing with the classical position in this sense. But it should be noted that what is proposed here is not straightforward, and it is not even transparent how one should go *about* dealing with a problem which players do not solve completely. Given this, is it worth doing? There are certain classical stylized problems which players do not solve completely. For example, no-one decides what his opening move in chess should be by solving the game through backwards induction on terminating legal board positions. Neither does anyone solve chess totally in any other way (even implicitly). But if limited foresight applied only to chess problems, its study might best be restricted to books on chess theory. However, as we shall see, limited foresight is unlikely to be restricted to such stylized and fantastically complicated situations. One of the themes of this

thesis will be that limited foresight is likely to be an extremely widespread phenomenon. (But do not despair! For we shall also argue that it is a phenomenon which can be modelled plausibly and tractably.)

Chapters 1 and 2 consider the issue of how relevant limited foresight reasoning might be (thus whether it is worth the effort of modelling it) along with the question of how to go about setting up the problem of costing the reasoning process which leads to the limit. We first consider in Chapter 1 why *foresight* is a relevant notion (let alone *limited* foresight). Since we teach our students to solve problems by backwards induction, why are limited foresight agents looking forwards at all?¹ It is tempting to suppose that this is because we are dealing with situations in which it is impossible to reason backwards (perhaps an infinitely repeated sub-game). But is backwards reasoning really denied to us in any setting? And is foresight employed only in such settings? Chess does not involve an infinitely repeated sub-game. So what is going on?

To address this question we consider an extremely simple two-stage problem with two possible actions per stage. The problem is encountered only once (it isn't a repeated setting). The problem is solved completely (so there is no "bounded rationality" in that sense). We find that with two possible stage payoffs which are the same in each round it is more efficient to solve this problem by reasoning from the beginning of the problem to the end (i.e. by forwards reasoning) than from the end of the problem to the beginning. The way we go about showing this is by modelling the reasoning process as a finite automaton.

Finite automata have been used in Game Theory before, for example in modelling strategies played in repeated games². It is important to note that the finite automata employed here are performing a different sort of task. Instead of carrying out play, they are solving for *how* to play. The efficiency criterion we use is analogous to the standard complexity criterion - the fewer the states of the finite automaton, the more efficient the reasoning process. We show that for certain kinds of problem (though not all), the forwards reasoning finite automaton with the fewest states has fewer states than the backwards reasoning finite automaton with the fewest states.

Having shown that agents are likely to solve many even quite simple problems by reasoning forwards, in Chapter 2 we move on to consider the

¹Or even, *are* they reasoning forwards at all? For example, certain Artificial Intelligence "bounded lookahead" algorithms use backwards induction from the horizon of foresight (see Jehiel (1998b)). So, if one believes that backwards reasoning is always more efficient, one might assume that even limited foresight involves backwards reasoning of a sort. But, as we shall see, that is not necessarily so...

²See, for example, Rubinstein (1998)

question of how to model how *far* forwards they will reason. Given the previous discussion, the natural way to proceed seems to be to consider an agent using a finite automaton where each state carries a cost³. The presence of a reasoning cost means that in principle it might no longer be worth solving the problem completely. The form of incompleteness we consider is that in which the agent might not look ahead to the end of the problem. There are, of course, other forms of "bounded" (or, perhaps better, "meta") rationality one could consider at this point, but we leave those to other research.

What we find is that with a cost per state of only $\frac{3}{128}$ of the payoff possibly lost by not looking ahead, it ceases to be worth solving even a very simple two-stage/two-action/two-payoff problem completely. We have not offered any criterion to judge how much a "high" state cost for a reasoning process might be as opposed to a "low" state cost. However, given that our simple problem requires only a seven state automaton to solve completely, and that the number of states required to solve more complicated problems explodes exponentially, we think it reasonable to conjecture that for only slightly more complicated problems than those we consider, the state cost required to justify solving the problem completely (as opposed to looking only boundedly far into it) would become so low that limited foresight reasoning might be expected to be very widespread.

Thus the main contribution of these two chapters is an argument that limited foresight reasoning might be very widespread, because there are many relatively simple problems which are best solved by reasoning forwards and if agents are reasoning forwards, even with relatively low reasoning costs it is likely that an optimizing agent will not find it worth looking right to the end of a problem to solve it. Secondary contributions include the finite automaton approach to addressing the question of how to go about solving a decision-problem, and the offering of a method to endogenize an agent's foresight horizon.

Having satisfied ourselves that limited foresight reasoning is likely to be a sufficiently widespread process that it is worth trying to model, the next question which arises is what solution concept to employ. One possibility is that an agent with limited foresight treats problems which go on for longer as if they continued only to the horizon of foresight. For example, if an agent looks ahead only to the next round, but the problem actually continues for ever, we could assume that, in each round, the decision-maker acts as if the problem finished next round. A solution concept along these lines (which we

³Note that up to this point we have assumed that the agent wishes to use the most parsimonious reasoning process, but that the use of the reasoning process did not actually *cost* anything.

refer to as the "naive" concept) is often employed in the time consistency literature⁴. In contrast, Jehiel (1995) proposed a concept in which players understand that the game will continue beyond their horizon of foresight, but simply lack the ability to see beyond that point. Furthermore, they have learned what their play in such situations is likely to be in the future, and make their decision today based on a correct expectation about the strategy they (as well as other players) will employ in rounds out to the horizon of foresight.

Rubinstein (1998) criticizes both these concepts, and suggests that there is as yet no promising solution concept for limited foresight problems and games. Chapter 3 considers to what extent he is right, and attempts to devise a solution concept which meets some of his concerns.

Rubinstein's main objection to the naive concept is that in a multi-player setting, because of the possibility of time inconsistency, in equilibrium players are assumed to know what other players are going to do in later rounds, but can be wrong about what they themselves will do. How, he asks, can other players know what someone is going to do in a later round if he doesn't even know that himself? We consider this objection well-made and argue that the best type of application for the naive concept is finite one-agent decision-problems in which the agent is unaware of his limited foresight.

Rubinstein's main objection to the Jehiel concept is that players take their own future play as given, even though they must have some control over it. In this sense, the Jehiel concept seems to run against the standard tradition in Game Theory. This is the main issue Chapter 3 aims to address.

First we argue that Rubinstein's objection is slightly misplaced, by proving that there is a solution concept which is formally equivalent to the Jehiel concept in which players do control their actions out to the horizon of foresight, rather than simply taking future play as given. What is taken as given in this alternative but equivalent concept is that certain strategies will lead into time inconsistency, and those strategies are to be ignored.

The idea that players know which strategies lead into time consistency and ignore all such strategies is somewhat problematic, however, as acquiring knowledge of time inconsistency may be demanding and, anyway, it is not always clear that using a time inconsistent strategy will be worse for players than employing a time consistent one. Thus we go on to use the idea of aiming to control one's future play to develop a refinement of the Jehiel concept in which players aim to control play as far ahead as possible, with

⁴Note that the players' inability to look beyond the horizon of foresight can lead them to change their mind about what they intend to do in later rounds and to regret the decisions they have already made. Important references in this literature include Strotz (1956), Pollack (1968), Laibson (1997) and O'Donoghue and Rabin (1999)

as little knowledge of time inconsistency as possible, and take future play as given only from the point ahead beyond which time consistency problems would arise.

The main contributions of Chapter 3 are the proof that the Jehiel concept is formally equivalent to a concept in which agents control their actions over the entire horizon of foresight (rather than just in the current round), and the devising of a refinement for Jehiel's concept.

Having determined our favoured solution concept, the next two chapters offer examples of how to employ it. In the first of these we consider a central banker setting interest rates but possessed of only limited foresight (in the Jehiel sense), operating in an environment with persistence in employment. We argue that this is similar to the situation faced by the Bank of England in the era of inflation targeting where interest rate decisions are justified by appeal to a two-year-ahead rolling forecast for inflation. We solve the bank's decision-problem and find that there will be a lower inflation bias with limited foresight, and that this inflation bias will increase as the horizon of foresight increases. We contrast our limited foresight approach with one in which there is a central banker with a finite horizon.

The other applications chapter is joint with Philippe Jehiel. There we consider the question of why some people smoke when young, then quit when older. We show that this result can be obtained simply by changing the horizon of foresight, without any need for tastes to change or for addiction issues to be involved or for new health information to be discovered. Employing both the standard Jehiel concept and the maximal-control refinement, we show that our result can be obtained with a variety of plausible payoffs from smoking or not smoking, and argue that this approach offers a new insight which might be important for policy⁵.

The thesis then concludes. This research has benefitted from funding by the ESRC under grant number R00429834527. Special thanks are due to my supervisors Philippe Jehiel and Tilman Borger. Thanks also to Marc-Etienne Schlumberger, Peter Postl, Jean Tirole, Nick Rau, Wendy Carlin and Edmund Cargill Thompson for useful comments.

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⁵In this joint work Jehiel provided the idea of studying a smoker problem, aiming to vary the horizon of foresight, and contrasting with time consistency. Lilico provided the contrast with rational addiction, the core example from which the examples in the chapter are derived, and the plausibility analysis of the payoffs. Both authors worked on proofs, structure, and concepts.

Chapter 1

Foundations of Limited Foresight 1: Foundations of Forwards Reasoning

1.1 General Introduction for Chapters 1 & 2

In this chapter and the next we shall attempt to provide rational foundations for limited foresight modelling. We shall show that there are situations in which a rational agent would reason forwards, go on to show that there are situations in which a rational forwards-reasoning agent would not reason forwards to the end of the game, and show how to solve for just how far a rational agent should reason forwards.

There are many situations in life where we seem only to look just so far ahead. Game theoreticians like examples such as chess games, but everyday life produces many less exotic cases. For example, many young people in their twenties will confess to not really having thought much about how they will live after they retire. Supposedly it is a common interview technique to ask people where they see themselves in, say, five years' time, and this is presumably precisely because relatively few of us have any concrete thoughts on such matters.

One issue relating to such situations is how people will behave if they have such limited foresight. For example, Jehiel (1995) introduced a "limited foresight equilibrium" concept. What this means can be illustrated through a simple example¹ (Figure 1).

¹It should be noted immediately that our example illustrates the idea, not strictly the concept - the (n_1, n_2) -equilibrium concept was applied to repeated alternate-move games, not finite decision-problems like those we shall consider.

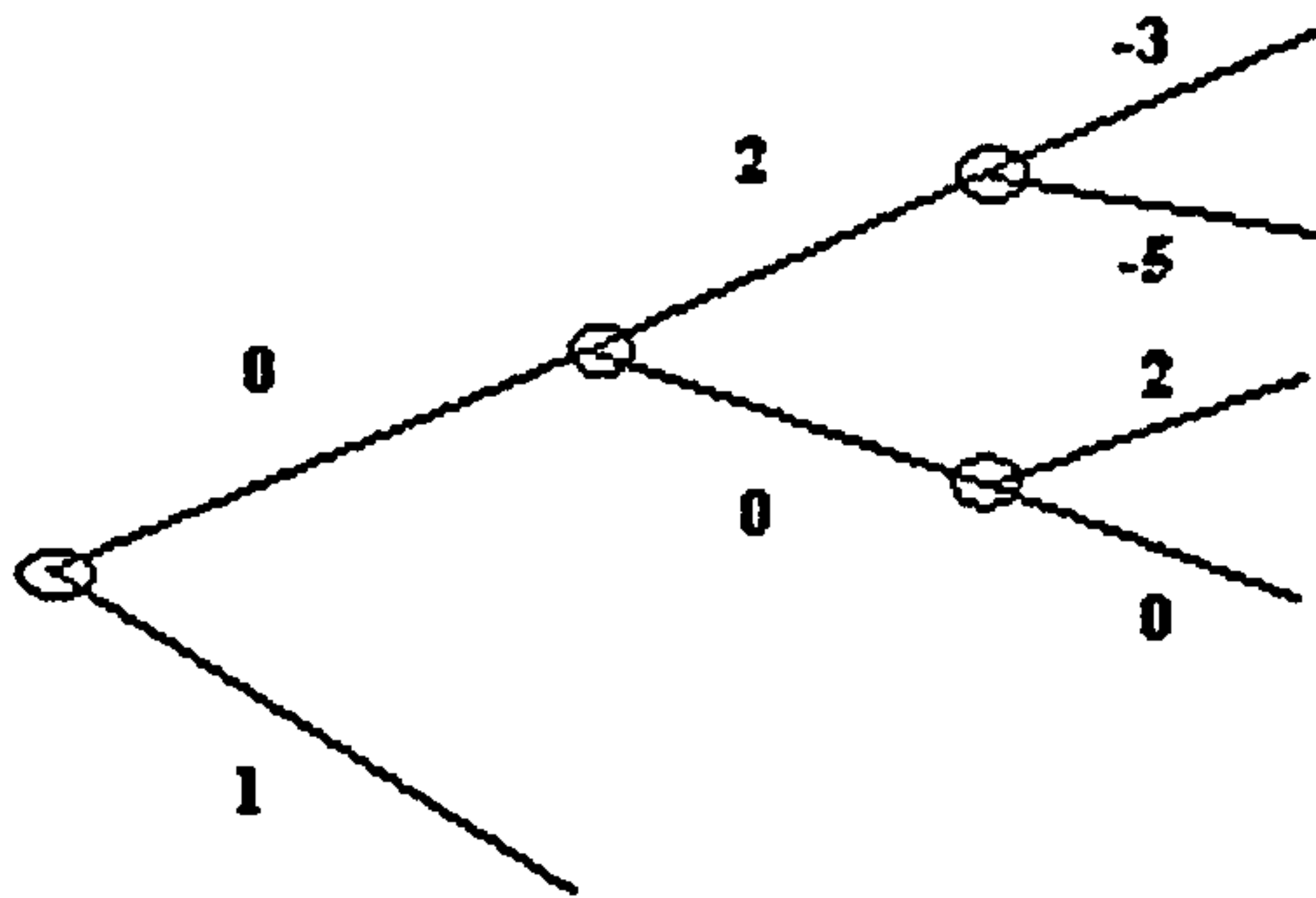


Figure 1.1: A very simple decision-problem

In Figure 1.1 we have a one-person decision problem. Each circle represents a point at which an action is to be taken. The branches of the tree represent those actions, and at the first two decision-points the agent can choose to go up or down in the tree. Branches which have no sub-branches represent terminal nodes. The decision-maker receives payoffs after each action corresponding to the numbers. Suppose that he can see ahead one period. Then from period one he can see that if he goes down to start with he will receive a payoff of 1 and the game will end, while if he chooses to go up then he will be able to choose between a payoff of 2 if he goes up as his second action and 0 if he goes down.

Under the Jehiel equilibrium concept we have in mind an agent who is aware of his limited foresight. He has perhaps played the game many times before, and is aware that, from this situation, if he were to choose to go up at the first decision-point, he would then choose to go down at the second. He is not sure *why* he would do this, i.e. he does have limited foresight. But he has learned that that would be his behaviour, and what he has learned is correct. So at the first decision point he believes (correctly) that he is choosing between going down and receiving a payoff of 1, or going up and receiving a payoff of 0. He goes down, and in the Jehiel concept the equilibrium strategy is (Down, Down, Up, Up). Note that this is different from the sub-game perfect strategy, which is (Up, Down, Up, Up).

Jehiel has shown, in a two-player repeated alternate-move setting, that with such agents who make no guess beyond the forecast horizon, an equilibrium always exists²; that such solutions are cyclical; that the equilibrium forecasts associated with such solutions do not depend on history and that

²in Jehiel (1995), in a repeated alternate-move environment, with two players and limited recall.

the memory capacity of the players has no impact on the set of solutions as long as it is finite; that for generic repeated alternate-move 2×2 games a solution always exists which holds for all n_1, n_2 sufficiently large; that players can sometimes do better with a shorter foresight length; and with agents who *do* make a guess beyond the horizon, he has shown that co-operation can sometimes be the only equilibrium in a repeated Prisoners Dilemma. He has shown that his concept arises out of a learning process with trembles³.

There is another school of models of limited foresight, in which the agents fail to understand that the game continues beyond the horizon they can see.

So, there are models of how people behave given that they have limited foresight, and in Chapter 3 we shall investigate what is the best way to go about it. But first we must address a second, related issue - why people might have limited foresight in the first place - and that is what we shall examine in the following two chapters. In the current chapter we shall show that it is legitimate to consider agents who reason forwards, rather than agents who reason backwards. We shall show that even with players who solve problems completely (identifying the unboundedly Rational solution), it is reasonable to suppose that in certain situations agents will solve problems by looking forwards. Having convinced ourselves that foresight is a relevant component of reasoning, in the following chapter we shall unpack the notion of a *limited horizon*, by showing that an agent with computation costs might find it optimal to adopt a limited foresight horizon, and solving for an optimal foresight horizon of a rational agent facing computation costs.

1.2 Introduction

As is well known, standard Game theoretic equilibrium concepts take Rationality as an axiom (or set of axioms). In finite games we usually teach our students to find such equilibria through backwards induction. In contrast, as we have already mentioned, one class of bounded Rationality models involves players with limited foresight (e.g. Jehiel (1995)), or players who reason forward but face additional costs the further forward they reason (e.g. Gabaix and Laibson (2000b) - see also Gabaix and Laibson (2000a)). This seems a fairly intuitive way to think about Bounded Rationality. After all, no-one plays chess as white by, at the beginning of the game, deciding in which position the black king will be mated and then constructing a pathway of moves from the opening to the mating position⁴.

³Jehiel (1998a)

⁴Such reasoning may, however, be involved in solving chess problems of the "White to play and mate in three" sort. The proof that any chess game could, in principle, be solved

In this chapter we shall consider models in which agents solve the problems completely to find the Rational solution (there is no problem of bounded Rationality as such, in that sense). The question we shall address arises as follows: since standard ideas of how to find Rational solutions involve reasoning backwards from the end of the game⁵, whilst in the Jehiel or Gabaix & Laibson models the boundedness arises because we can only see so far *forwards* in the game, one may feel that there are missing steps between the Rational and boundedly Rational positions. Why are we ever reasoning forwards in the first place, let alone only managing to reason forwards finitely far? It is easy to say "Because it's too hard to reason backwards", but *why* is it too hard? And if the answer to that question is that we have evolved that way, the question remains *why* we have evolved that way? Is there, for example, some advantage to forwards reasoning in reasonably simple problems which then makes it seem natural to us to reason forwards in longer and more complicated problems (like chess games), or is backwards reasoning better for solving simple problems but at some point we switch to forwards reasoning for longer problems? Or is it possible that it is only best to reason forwards in infinitely-repeated games, where backwards reasoning is in some way denied to us?

Camerer et al. (1994) and Johnson et al. (2001) have produced empirical evidence which may suggest that their subjects were reasoning forwards. For example, in Johnson et al. (2001) there are two players who bargain over a pie which shrinks in value over the three periods of the game. The pie is worth \$5 in the first period, then halves in value each subsequent period. If no bargaining solution is reached in the three periods each player receives a zero payoff. Player 1 makes an offer in period 1, which Player 2 either accepts or rejects. Each period, if an offer is rejected the rejecting player makes a counter-offer in the next period. The perfect equilibrium is (approximately) that Player 1 offers \$1.25 in period 1 and is accepted.

In the baseline experiment, the average offer is \$2.11, and offers below \$1.80 are rejected half the time. The experiment is designed so that the experimenters can tell which pie sizes the subjects examine (players are not told in advance what the payoffs are, but they can find them out trivially during the game by clicking computer-screen boxes). By identifying which payoffs players examine, and how long and often they spend considering them, the experimenters attempt to gain insight into whether the players are reasoning forwards or backwards in the game. According to the information-measures

in this way is what is usually referred to as "Zermelo's Theorem" - see Schwalbe & Walker (2001).

⁵For example, see Rosenthal (1981) and Binmore (1989) pp.151-231.

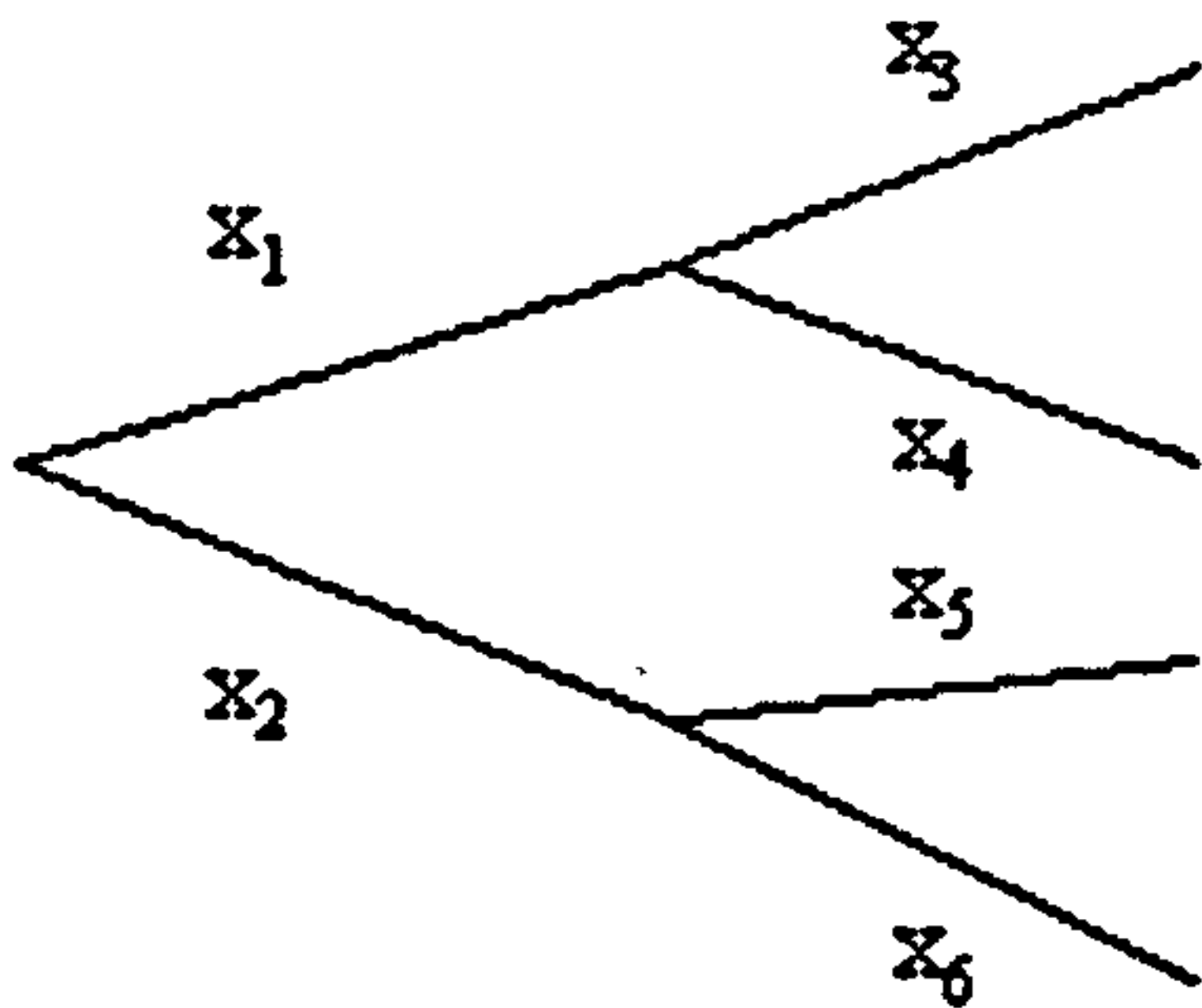


Figure 1.2: A two-stage decision-problem

used in the experiment, players do not tend to look at future rounds then reason backwards, as one might assume. More than that, they often appear to make no use of payoffs in later rounds, not even finding out the value of second- and third-round payoffs in 19% and 10% of the trials respectively. Furthermore, classifying subjects by the degree to which the information criteria suggests they look ahead is strongly predictive of the offers they make.

Though these empirical results are interesting, the fact (if it is a fact) that people do reason forwards is not an explanation of *why* people reason forwards⁶. In this section we argue that, at least in certain situations, fully Rational players *should* reason forwards. Our environment is very different from that discussed in the empirical studies mentioned (in particular we consider decision-problems rather than games). However, we believe that the insights are instructive.

1.3 Framework

We shall attempt to compare the efficiency of forwards and backwards reasoning for solving a very simple two-stage problem, represented by Figure 1.2, and begin by considering the problem of what is the best move to take at the first decision-point.

In the decision-problem in Figure 1.2 we have one player who has to choose between going up or down at the root node (i.e. he chooses an action $a \in A = \{U, D\}$). Having made that choice, he will later be able to choose

⁶Even the title of Johnson et al. (2001), "Detecting Failures of Backwards Induction", suggests that reasoning forwards involves a kind of error, an implication which we do not draw.

to go up or down at a second node. The payoffs he will receive depend on his choices, and are represented as stage payoffs $x_i \in X$ for $i \in I = \{1, \dots, 6\}$. For example, if he goes up, and then up again, he will receive $\pi(UU) = x_1 + x_3$, and so on for the other possible choices. Each of the possible stage payoffs is either 0 or 1, $x_i \in \{0, 1\}$.

We shall consider how efficient it is for him to solve this problem by reasoning forwards as opposed to reasoning backwards. Our tool for comparing (and measuring) "efficiency" in this context will be a special sort of *finite automaton*, which we shall now define.

Definition 1.1 *A finite automaton (in this chapter⁷) will be defined as a 7-tuple, $FA = (Q, q_0, F, \Sigma, \gamma, \delta, \lambda)$ where*

Q is a set of states (q_0, \dots, q_n) for n finite

q_0 is the initial state of the automaton

$(F \subseteq Q) \neq \emptyset$ is a set of 'final states'

$\Sigma = \{0, 1\}$

γ is a mapping from $(Q \setminus F) \rightarrow I$ and is to be interpreted as the member of X to be examined at each non-final state

δ is a mapping from $(Q \setminus F) \times \Sigma \rightarrow Q$ and is to be interpreted as a transition function taking the current state and the value of the examined member of X and determining which state to move to next ($q' = \delta(q, x_{\gamma(q)})$).

λ is a mapping from $F \rightarrow A$ and is to be interpreted as the action to be performed if the automaton terminates at a given final state.

Definition 1.2 *$x \in \Sigma^6$ is a "realisation" of the values of the six variables.*

Definition 1.3 *For a given x and a given FA , a "path" is the unique sequence of states $\hat{q}(x, FA) = q_0, q_m, q_n, \text{etc.}$ of FA which will be reached in order if payoffs are given by X .*

Definition 1.4 *A finite automaton, FA , is "stopping" if and only if, for every $x \in \Sigma^6$, $\hat{q}(x, FA)$ ends with a final state.*

Note that it follows immediately from this definition that no stopping FA can have a "loop" whereby the same state is visited along a path more than once (since there are no non-deterministic transitions).

Definition 1.5 *The members of $\hat{q}(x, FA)$ form the set $\tilde{q}(x, FA)$.*

⁷The automata we consider here are a very special case. For more information on finite automata see Hopcroft and Ullman (1979).

We shall later be counting the states of an FA and be interested primarily in FAs which are state-minimal. Two trivial ways in which an FA might have unnecessary additional states are: if it contains states which are not reached for any realisation of the values of the six variables; or if it contains a state from which there is an input-independent transition. We shall by-pass these trivial cases and call such automata "redundant".

Definition 1.6 *A finite automaton, FA, is "redundant" if, either*

1. $\cup_{x \in \Sigma^6} \tilde{q}(x, FA) \neq Q$, or
2. $\exists q \in (Q \setminus F)$, such that $\delta(q, 1) = \delta(q, 0)$

Definition 1.7 *If an FA is stopping and is not redundant, we shall call it a "regular" FA.*

Hereafter, we shall assume that an FA is regular unless otherwise specified.

Definition 1.8 *A non-final state q of a regular FA will be called a "question" or a "questioning" state. It follows immediately from the definition of a regular FA that all such states involve an input-dependent transition. We shall say that an FA which contains a state q , such that $\gamma(q) = i$, "examines" the value of x_i . We shall call the variables x_i "unknowns" at the initial state of the FA. We shall talk of the action produced by λ as the action "recommended" by the FA.*

Definition 1.9 *We shall call a regular FA "implementing" if it recommends optimal play for all $x \in \Sigma^6$ (where optimal play means an action at the first node which would be part of an optimal strategy for playing the whole game).*

It may not be immediately apparent why the process of examining the value of unknowns in this problem has relevance to the standard problem of finding the best action at the initial decision-point when we already *know* the values of the payoffs. To see the connection it might help to think of the process of "examining" a variable's value as less a matter of discovering something not previously known than a matter of making *use* of the value of a variable. Then a "questioning" state would be an information-based decision within our algorithm for finding the optimal decision. However, we shall stick with our suggested terminology as it seems more natural to regard our finite automaton as a machine which asks questions. Note that our FA has no explicit means of storing information - it has no *memory*.

In the next few sections our finite automata are intended to solve the problem of identifying our player's optimal action at the root node. We shall interpret the number of states of an FA as a measure of efficiency or complexity. For example, if solving a problem one way could be done by an FA with 10 or more states, while a second way of solving the same problem requires only 5 or more states, we shall interpret this as meaning that the second way of solving the problem is more efficient. Clearly the relevance of our approach to the question of the efficiency of different reasoning methods is crucially dependent upon the plausibility of our claim that the number of states required is a good measure of efficiency.

1.3.1 Is the state-counting efficiency criterion a compelling one?

The idea of measuring the complexity of a Finite Automaton by counting its number of states is not an innovation of this thesis⁸, and the idea that efficiency should be regarded as simply the inverse of complexity seems plain. However, in the past Finite Automata have usually been used in Game Theory to model finite strategies in infinitely-repeated games⁹, so that the complexity criterion is a measure of the complexity of a *strategy*. Here the complexity-measure (or, viewed from the opposite direction, the efficiency-measure) is intended to capture the difficulty of *solving* the problem of optimal play, not of implementing that solution in a strategy.

Though one should be aware of this difference, we do not believe that it alters the appropriateness of the state-counting efficiency criterion. Counting the states of a finite automaton is a concrete way to measure the number of steps in a reasoning process. It is not, of course, the only way to do this. If other measures (e.g. the complexity of the transition function) give different results that will be important information about what costs are relevant in assessing how best to go about solving a problem¹⁰.

⁸For example, see Rubinstein (1998) p.150ff.

⁹This has not, of course, been the only use - see, for example, the "learning automata" considered in Fudenberg & Levine (1998).

¹⁰There is, however, no guarantee that a different measure will give a different result. For example, a plausible alternative measure of efficiency, capturing some of the spirit of complexity of the transition function, might be the average path-length to a recommendation (giving an idea of how long, on average, it takes to come up with a result). For the state-minimal forwards reasoning FA we shall later in Figure 1.2 the average path length is 2.75. For the state-minimal backwards reasoning FA in Figure 1.3 the average path-length is 3.25. Whether either of these values could be improved upon by some "length-minimal" FA remains an open question.

1.4. SOLVING FOR WHAT TO DO AT THE FIRST DECISION-POINT²¹

The finite automata we discuss in this chapter solve the problem to find the Rational solution. Armed with concrete measures of complexity, we could modify our problem, attaching costs to our complexity-measures (for example, some cost to each state) and work towards endogenising the degree of Rationality boundedness - for example solving for some optimal foresight horizon¹¹. We believe that this might prove a fruitful way to close the gap between the Rational and Boundedly Rational models.

1.4 Solving for what to do at the first decision-point

1.4.1 A preliminary result

Main Lemma *A state-minimal implementing FA to solve the problem in Figure 1.2 has at least seven states.*

Proof. This proposition is proved in Appendix A. ■

The general method of this proof is as follows. It follows immediately from the definition of an FA that if it is capable of examining five or more variables and of recommending two actions then it must have at least seven states, so we need only consider cases in which fewer than five variables are considered. We show that the fewest variables an implementing FA must be capable of examining is all the variables in one branch (e.g. all of x_1, x_3, x_4) and the first variable in the other branch (e.g. x_2) - i.e. four variables in total, immediately implying at least six states. Then we prove that any implementing FA examining only this set of variables must be capable of examining all of them in turn, and that there must be at least two paths along which all of these variables are examined in turn.¹² Any six-state FA could have only one path along which all the variables are examined in turn, so six states will not be enough.

¹¹See the next chapter

¹²These correspond to the two cases in which $x_1 = x_2$ - where they are both 0 or both 1, and at least one of x_3, x_4 is 0; or alternatively (depending on the FA involved) they correspond to the cases in which $x_3 = x_4 = x_1 = 0$ or one of x_3, x_4 is 1 and $x_1 = 1$. In such cases we may need to examine all four variables to discover whether U or D is (weakly) optimal.

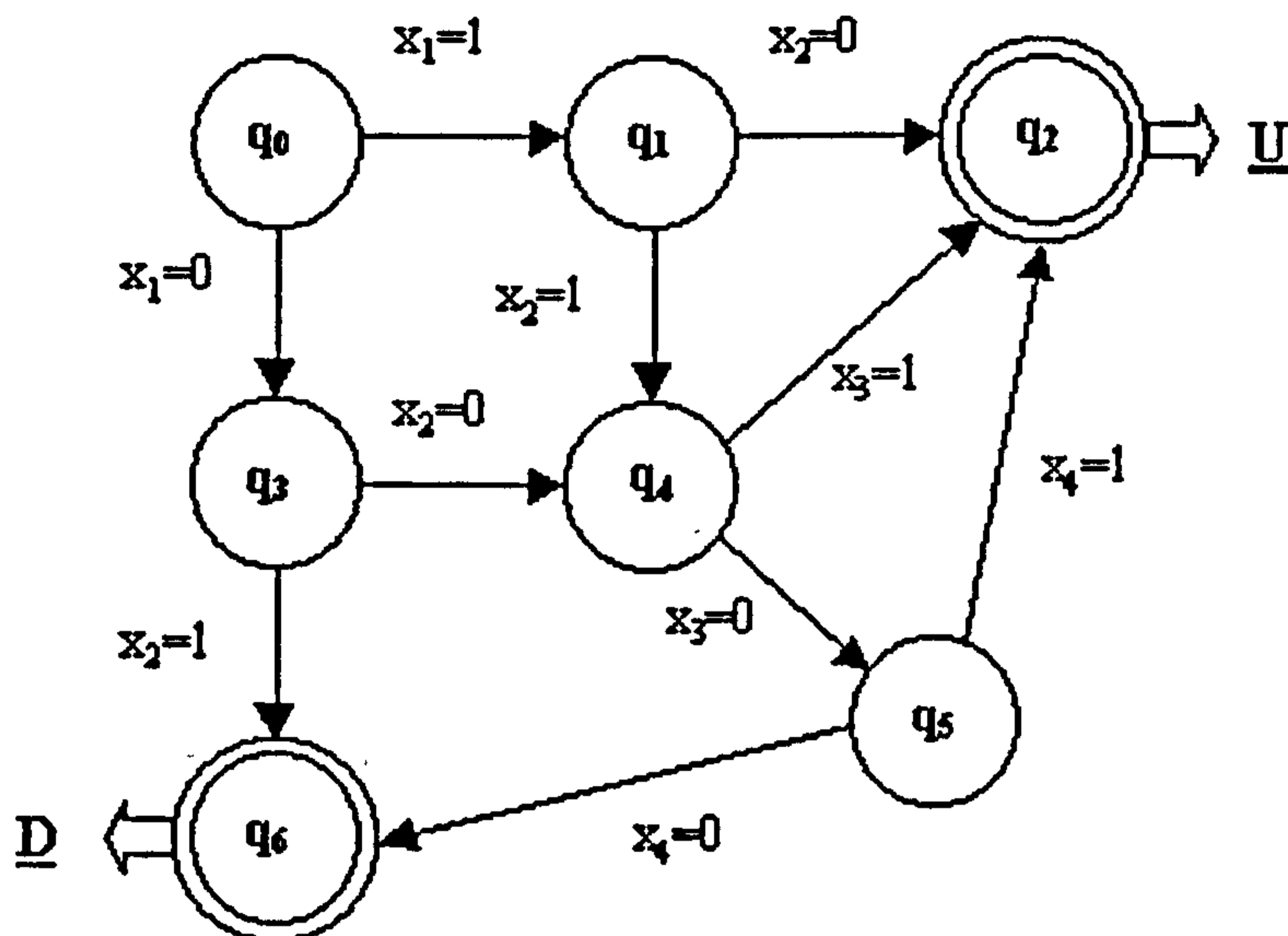


Figure 1.3: A finite automaton

1.5 Forwards Reasoning

Definition 1.10 A *Forwards Reasoning Finite Automaton (FRFA)* for the problem in Figure 1.2 is a Finite Automaton such that, along any path to a final state,

1. if the value of x_3 or x_4 is examined, then the value of x_1 has already been examined earlier along that path;
2. if the value of x_5 or x_6 is examined, then the value of x_2 has already been examined earlier along that path.

Figure 1.3 represents an implementing finite automaton to solve the problem in Figure 1.2. In Figure 1.3 the states are marked $q_0 \dots q_6$, with final states taking a double-circle. Thin arrows show the direction of transition, and are followed if the variable values next to them are realised (so, for example, in the initial state q_0 the FA examines x_1 , and if $x_1 = 0$ we move on to q_3 , while if $x_1 = 1$ we move on to q_1 . Fat arrows represent the actions taken at final states, i.e. $\lambda(q_2) = U$ and $\lambda(q_6) = D$ (so, if the FA terminates at state q_2 the automaton is telling our player to go up at the first decisions point, while if the FA terminates at state q_6 the automaton is telling our player to go down).

The FA in Figure 1.3 works as follows. It examines the values of x_1 and x_2 . If one of these is more than the other, the best action is clear and the FA proceeds immediately to a final state. If $x_1 = x_2$ the FA checks x_3 and x_4 , and if either of these takes value 1, going *Up* is at least as good as going *Down*, but if they are both 0, going *Down* is at least as good as going *Up*.

Inspection suffices to show that this FA is, indeed, implementing. Note that the FA in Figure 1.3 involves forwards reasoning, since the values of x_1 and x_2 are considered first, before moving on to consider x_3 and x_4 .

Proposition 1.1 *A state-minimal implementing FRFA for the problem in Figure 1.2 has seven states and is generally state-minimal among FAs.*

Proof. We can construct an implementing FRFA with 7 states, as illustrated in Figure 1.3. By our Main Lemma this FA must be state-minimal.

■

1.6 Backwards Reasoning

Definition 1.11 *A Backwards Reasoning Finite Automaton (BRFA) for the problem in Figure 1.2 is a Finite Automaton such that, along any path to a final state,*

1. *if the value of x_1 is examined, then either the values of x_3 and x_4 have already been examined earlier along that path, or the value of x_3 has been examined earlier along that path and found to take value 1;*
2. *if the value of x_2 is examined, then either the values of x_5 and x_6 have already been examined earlier along that path, or the value of x_5 has been examined earlier along that path and found to take value 1;*

One might have supposed that the definition of a BRFA would be stricter and simpler, insisting that the values of *all of* x_3, x_4, x_5 and x_6 always be examined before x_1 or x_2 . We have chosen not to define backwards reasoning in this way, because we wanted to allow our BRFA to exploit the same logical shortcuts available to the general state-minimal FA - for example recognising that if $x_3 = 1$ then it is not necessary also to examine x_4 . This presents a more generous and more general test of the relative efficiency of backwards reasoning.

We shall prove that a state-minimal backwards reasoning automaton must have more states than a general state-minimal automaton such as Figure 1.3, and hence that, by our state-counting criterion, backwards-reasoning is not an optimal way to solve the problem in Figure 1.2. It should be noted that this proof does not depend on the result that Figure 1.3 is state-minimal.

Proposition 1.2 *Any implementing state-minimal backwards-reasoning automaton must have at least one more state than a general implementing state-minimal automaton.*

Proof. This is proved in Appendix B. ■

The intuition of this proof is as follows. It follows immediately from the definition of backwards reasoning that any backwards reasoning FA solving the problem in Figure 1.2 must be capable of examining all of x_3, x_4, x_5, x_6 , along with at least one of x_1, x_2 . If it is to have seven states it must examine only one of these, say x_1 . But the general state-minimal FA examines x_1, x_3, x_4 and x_2 , and we can discover the same amount of relevant information by examining x_2 in one question as by examining x_5 and x_6 in two questions. Hence the minimal backwards reasoning FA must use at least one more state than the general state-minimal FA.

Figure 1.4 is an example of a state-minimal BRFA for the problem in Figure 1.2. It works as follows. It checks to see what the best available payoff is at the second stage of the upper branch (i.e. whether the maximum of x_3, x_4 is 1 or 0). If either of these is 1 and x_1 is also 1, then the payoff from the upper branch is 2 and we can safely go *Up*. If both of them are 0 and x_1 is also 0, then the payoff from the upper branch is 0 and we can safely go *Down*. If, however, either $x_1 = 1$ while $\max\{x_3, x_4\} = 0$, or $x_1 = 0$ while $\max\{x_3, x_4\} = 1$, then we need to check the second-stage payoffs from the lower branch. If either of these is 1 we know we get at least 1 by going *Down*, but definitely 1 by going *Up*, so we can safely go *Down*. In contrast, if both the second-stage payoffs from the lower branch are 0 we know we get at most 1 by going *Down*, but definitely 1 by going *Up*, so we can safely go *Up*.

By Proposition 1.2 backwards reasoning is not the most efficient way of solving the problem in Figure 1.2. But Proposition 1.1 showed that the forwards reasoning automaton was indeed state-minimal, and so, by this state-counting criterion, that forwards reasoning is optimal¹³ for solving this problem. Hence we have the following corollary, summarising the results of this section.

Corollary 1.1 *By a state-counting criterion, Forwards Reasoning is more efficient than Backwards Reasoning for solving the problem in Figure 1.2.*

¹³It is very important to note that it is being claimed here that forwards reasoning can be optimal, but not that all optimal reasoning is forwards reasoning. For example, there do exist 7-state FAs to solve the problem in Figure 1.2 which examine the value of x_3 at the initial node. Since such FAs use the variables (x_1, x_2, x_3, x_4) as proved above, they are not backwards reasoning, but they are not forwards reasoning either!

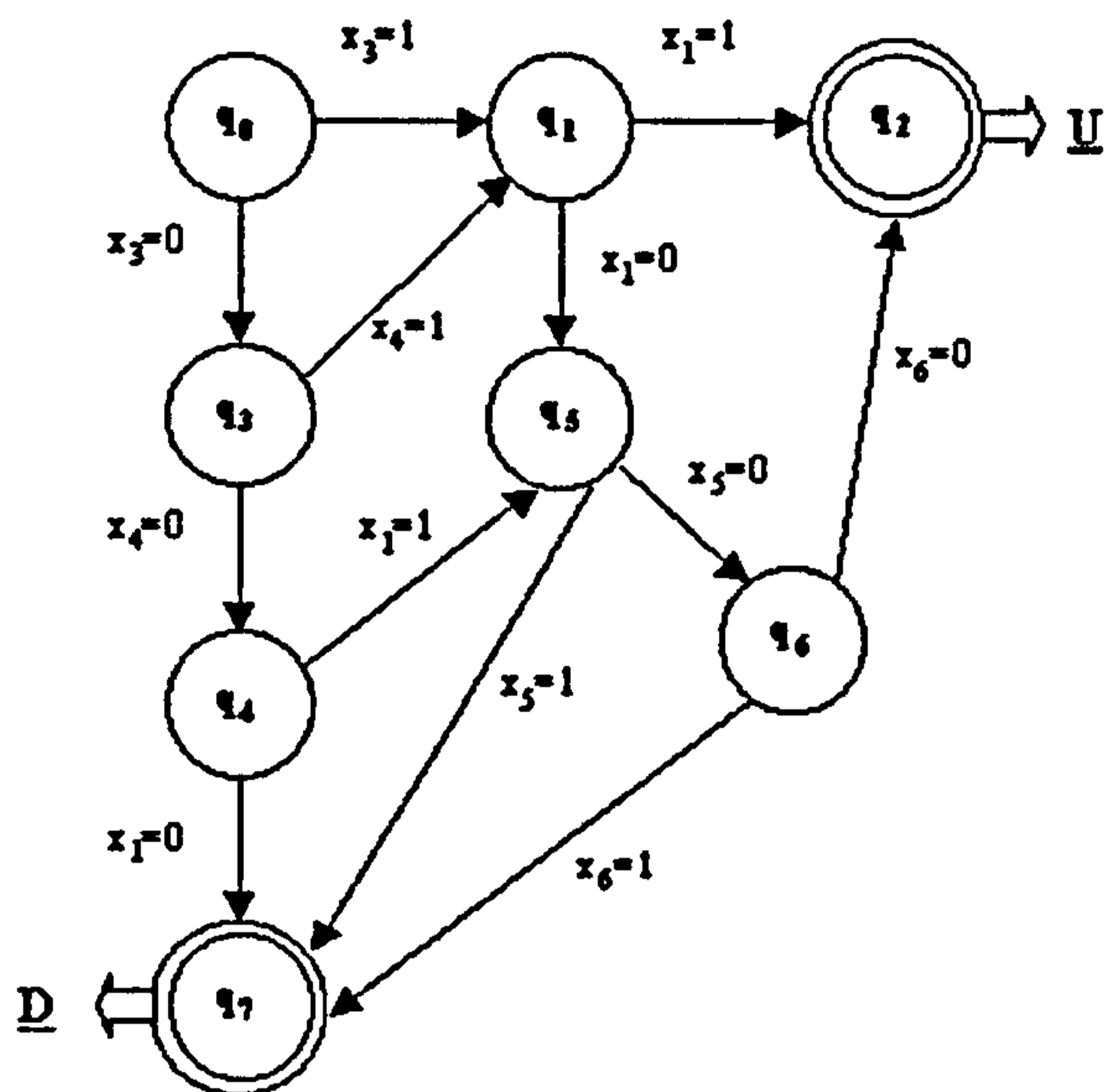


Figure 1.4: A state-minimal BRFA to solve the problem in Figure 1.2

1.7 Discussion: What is it about this problem or about backwards reasoning which leads to the general state-minimal automaton being forwards-reasoning in this case?

What drives our result? Why isn't backwards reasoning the most efficient method in this case? The reason is that backwards reasoning forces us, in this case, to consider more variables than we need to. Since we can find out as much information as we need by looking at the (single) first-stage payoff for the lower branch, it is inefficient to have to consider the (two) second-stage payoffs¹⁴.

Is the advantage of forwards reasoning here dependent on the fact that we are only trying to find out what to do at the beginning of the game? One might guess that if we were trying to work out all the best decisions we would need to make (i.e. work out what we would do at the first node and also what we would do at the decision-point we would reach following our initial choice) then backwards reasoning would do rather better, since backwards

¹⁴It should be clear that if there had been only one second-stage payoff (e.g. if there had been only x_5 but no x_6) then forwards and backwards reasoning would have been equally efficient.

reasoning includes some consideration of what it is best to do at the second node in forming a view as to what it is best to do at the first node.

It may be that this idea drives the popular connection made between *memory* and limited foresight¹⁵. The idea is that perhaps backwards reasoning does better than forwards reasoning if we need only work out once how to play the whole game, then at later decision-points can remember what we worked out was best? But in long games it might be difficult to remember everything that was supposed to be done, so that physical constraints on the available memory-capacity mean we need to go through the process of working out what the best actions are more than once in a game - in which case we might be able to support some kind of limited horizon model with finite memory.

However, as it happens this is not the situation here. Backwards reasoning remains inferior to forwards reasoning when the initial decision is how to play at all reached decision-points, as we now show.

1.8 Solving for all necessary decisions

We shall prove that forward reasoning is more efficient, by the state-counting criterion, than backwards-reasoning in solving the problem of what to do at all reached nodes of the decision-problem illustrated in Figure 1.2. This is not quite the same thing as solving the problem altogether in the traditional sense, since we shall not be solving for what to do at unreached nodes of the problem.

Our proof will proceed as follows. First we shall exhibit an 11-state FRFA which solves this new problem¹⁶. Next we shall modify our definition of a BRFA to fit this new problem. Then we shall prove that no BRFA satisfying this definition could have fewer than 12 states¹⁷. This suffices to prove that the state-minimal FRFA will have fewer states than the state-minimal BRFA.

¹⁵e.g. See Jehiel (1995)

¹⁶We shall leave open the question of whether this is a state-minimal general FA for solving this problem, or even whether it is state-minimal among FRFAs which solve this problem.

¹⁷Similarly, we shall not investigate the question of whether the state-minimal implementing BRFA has only 12 states or more than that.

1.9 An implementing FRFA to solve for play at all reached nodes

Figure 1.5 exhibits an 11-state FRFA solving for the problem of what to play at all reached nodes of the decision-problem in Figure 1.2. Final state output of "UU" means the FA is recommending playing Up at the first node, and then Up again at the second node. Recommendations from other final states are to be interpreted similarly. Inspection suffices to show that this FA is, indeed, forwards reasoning, and is, indeed, implementing. Note that this FA is built upon the FA in Figure 1.3. Once we know what it is best to do at the first node, we check what is best to do at the node we would then reach - a paradigm of forwards reasoning. The other thing to note is that this FRFA uses information about one more variable (namely x_5) than the FA in Figure 1.3.¹⁸

1.10 The definition of Backwards Reasoning in the new problem

Figure 1.6 exhibits a 12-state FA which solves the problem of this section. Inspection shows that it satisfies Definition 1.11. However, note that it works in the same way as the FA in Figure 1.5. First we solve for what to do at the first node, then we reason forwards from there to what to do at the reached node. In particular, in this case we may examine first x_3 , then examine x_1 , then go back to examining x_3 again. This does not conform to our intuitions of what backwards reasoning is about. We should not need to reason backwards, then forwards again. Hence, for the purposes of this section, we replace Definition 1.11 with the following:

¹⁸A by-product here is that it involves fewer states to solve for unreached play from the beginning than to solve for optimal play at the first node, then wait until the next node is reached before deciding what to do there. To do this successfully, we would need to employ the seven-state FA identified in Figure 1.3, then have available two three-state FAs to question x_3 or x_5 for the upper or lower nodes respectively. Seven plus three plus three equals thirteen, which is more than the eleven states required in Figure 1.5.

It is worth noting that waiting until a later round might, however, be more efficient than solving for play at all nodes, including unreached nodes. These issues are investigated further in Appendix C.

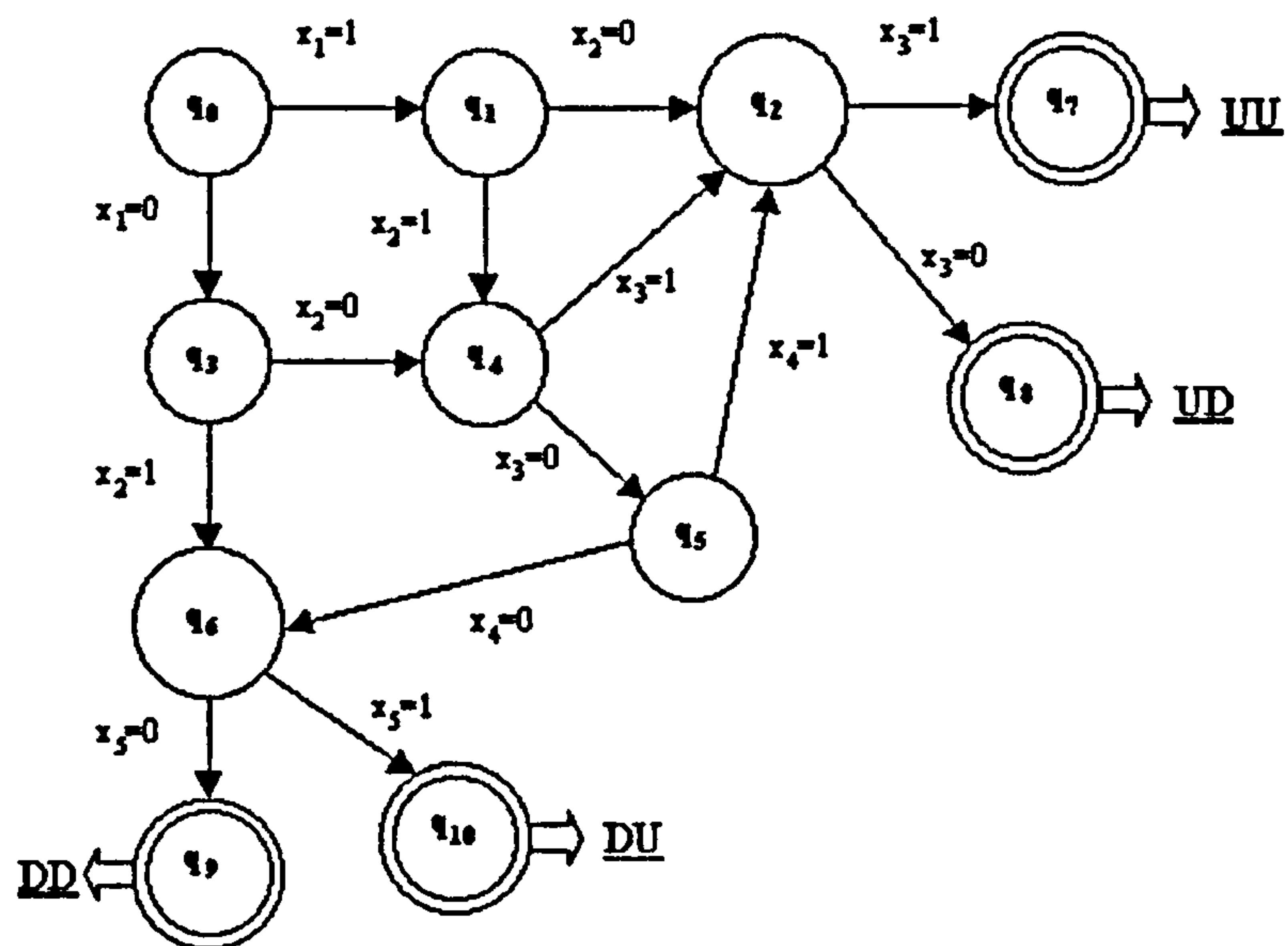


Figure 1.5: A finite automaton to solve the whole game

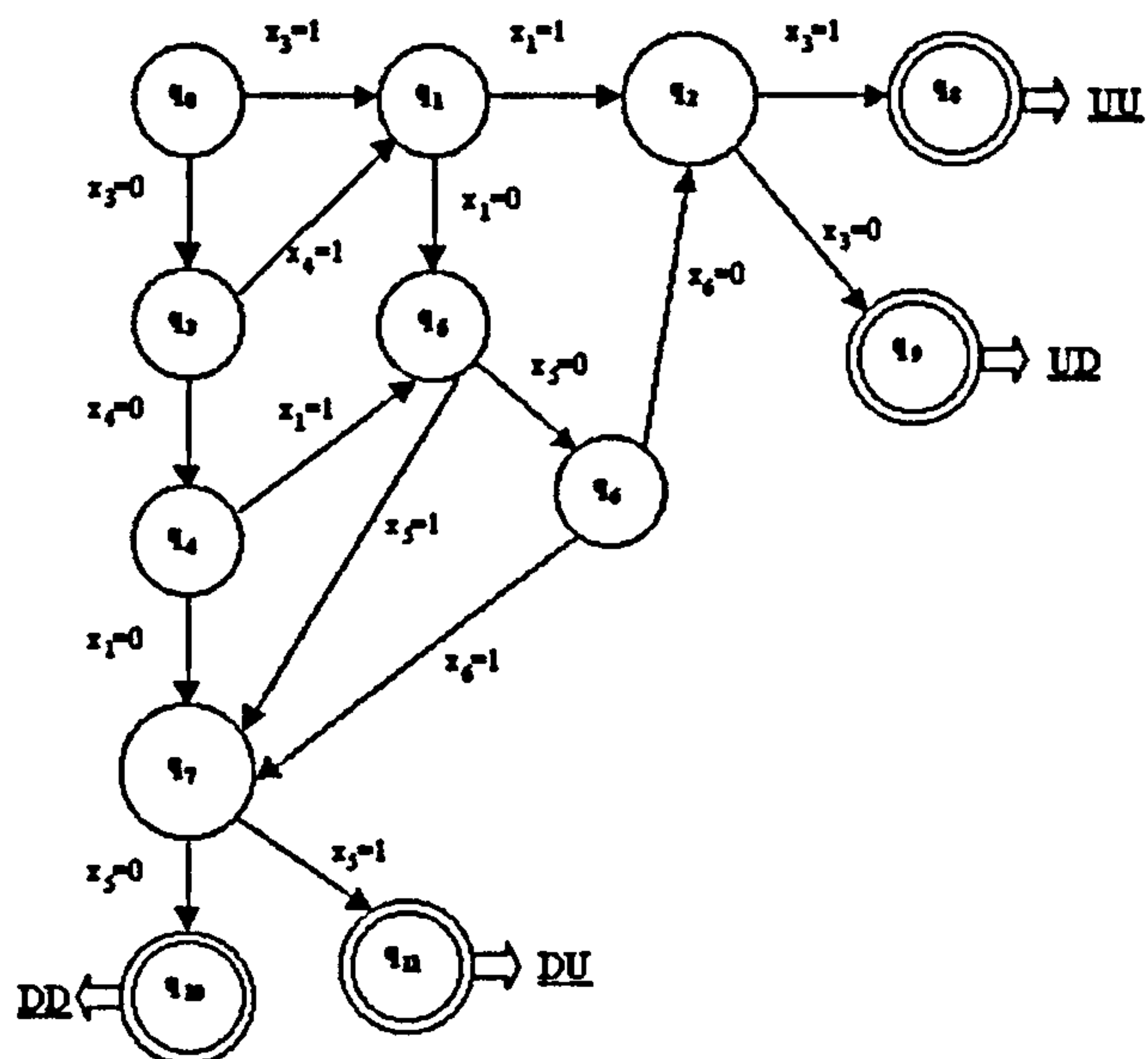


Figure 1.6: A backwards-reasoning finite automaton to solve the whole game

Definition 1.12 *A Backwards Reasoning Finite Automaton (BRFA) for the problem in this section is a Finite Automaton such that, along any path to a final state,*

1. *if the value of x_1 (or x_2) is examined, then either the values of x_3 and x_4 have already been examined earlier along that path, or the value of x_3 has been examined earlier along that path and found to take value 1;*
2. *if the value of x_3 or x_4 is examined, then the value of x_1 has not been examined earlier along that path;*
3. *if the value of x_2 (or x_1) is examined, then either the values of x_5 and x_6 have already been examined earlier along that path, or the value of x_5 has been examined earlier along that path and found to take value 1;*
4. *if the value of x_5 or x_6 is examined, then the value of x_2 has not been examined earlier along that path.*

The bracketed terms in parts 1 and 3 of this definition are merely there to simplify our proof. We conjecture that they can be discarded and the main result still hold. Alternatively, one might feel that this was a better definition of backwards reasoning anyway - namely that no branch of the first node should be examined until optimal play at later branches of the tree has been determined.

1.11 Proof that no implementing BRFA can have as few states as Figure 1.5.

Proposition 1.3 *No BRFA satisfying Definition 1.12 can have fewer than 12 states.*

Proof. Since we need questions at least about the set $\{x_1, x_3, x_4, x_5, x_6\}$ (or a trivially equivalent set, or about all six variables) to identify best play at the first node, we certainly need at least this set to solve for play at all reached nodes. By Definition 1.12, by the point in any path at which a question is asked about x_1 we must already have the information necessary to recommend play at the (x_3, x_4) node and the (x_5, x_6) node. In some cases (e.g. if $x_3 = 1$ and $x_5 = x_6 = 0$) we can exit to a final state straight from information about $\{x_3, x_4, x_5, x_6\}$. However, in cases where $\max\{x_3, x_4\} = \max\{x_5, x_6\}$ we will need to use information about x_1 without losing information about what to play at the later nodes (i.e. a state

questioning x_1 is required). There are four possible combinations of optimal play at the later nodes when $\max\{x_3, x_4\} = \max\{x_5, x_6\}$, all of which might be required for some combination of the six variables, namely that

1. We should play *Up* at the upper node and *Up* at the lower node;
2. We should play *Up* at the upper node and *Down* at the lower node;
3. We should play *Down* at the upper node and *Up* at the lower node;
4. We should play *Down* at the upper node and *Down* at the lower node.

For each of these possible combinations, we will need a separate questioning state on x_1 (or, trivially equivalently, on x_2). This makes four states.

When $\max\{x_3, x_4\} = \max\{x_5, x_6\}$, to resolve which of the four combinations of optimal play of later nodes we are in requires up to four questioning states (checking the values of each of $\{x_3, x_4, x_5, x_6\}$). Since none of these states is examining the value of x_1 , none of them can be any of the four states previously identified questioning x_1 .

There are four final states: *UU, UD, DU, DD*. No final state can question the value of a variable, so all of them must be additional states to the eight previously identified.

Four plus four plus four equals twelve, so the minimum number of states of a BRFA satisfying Definition 1.12 is twelve. ■

Since we can exhibit an 11-state FRFA, while the previous proposition shows that the state-minimal BRFA must have 12 or more states, a corollary follows automatically, namely

Corollary 1.2 *Forwards reasoning is more efficient than backwards reasoning, by the state-counting criterion, for solving the problem of optimal play at all reached nodes of the game.*

1.12 Discussion

Why is forwards reasoning still better? The main reason appears to be that the FRFA is able to "forget" irrelevant information (in the sense of the information not being recoverable from which state the FRFA is in)¹⁹ which

¹⁹We note, once again, that our FA has no memory, and so, strictly speaking, is not capable of either remembering or forgetting anything.

the BRFA is forced to retain. Since it doesn't really matter which of x_5 and x_6 is better if we are going to go Up at the first node anyway, having separate states to tell us which of these is better is inefficient. The BRFA is forced to do this. The FRFA is not.

It is in the nature of the problems studied here that backwards reasoning is more susceptible to the irrelevance trap than forwards reasoning. When there are problems in which there is more danger of irrelevance when reasoning forwards, we should expect that backwards reasoning would prove superior. One very straightforward (and trivial) class of such problems are those in which there is no need to examine the values of the first-stage payoffs at all. For example, suppose we modify our definition of an *FA* such that

$$\begin{aligned} x_1, x_2, x_3, x_4 &\in \{0, 1\} \\ x_5, x_6 &\in \{-1000, +1000\} \\ \Sigma &= \{0, 1, -1000, +1000\} \end{aligned}$$

Then it is clear that we need only examine the values of x_5 and x_6 , and thus do not need to reason forwards. We do not, of course, need to reason *backwards* as such, either!

A less trivial class of cases can be illustrated by Figure 1.7. In this figure we consider a case in which there are three stages, at each of which the decision-maker decides whether to go up or down and thereby receives a stage-payoff. At the first stage the stage payoff is either 1 or 0. At the second stage either 2 or 0. And at the third stage either 4 or 0. It seems clear that, if we are trying to solve for optimal play at the first decision-point, an algorithm along the following (backwards-reasoning) lines must be superior to any forwards-reasoning algorithm:

1. Examine all third-stage payoffs. If all the stage-payoffs of value 4 lie in the upper half of the figure, go *Up* at the first stage. If all the stage-payoffs of value 4 lie in the lower half of the figure, go *Down* at the first stage. Otherwise proceed to step 2.
2. Examine all second-stage payoffs of branches leading to the maximum-valued payoffs identified in step 1. If all the stage-payoffs of value 2 lie in the upper half of the figure, go *Up* at the first stage. If all the stage-payoffs of value 2 lie in the lower half of the figure, go *Down* at the first stage. Otherwise proceed to step 3.
3. Examine the first-stage payoffs, and choose the branch containing the higher payoff.

The reason why forwards-reasoning might prove inferior in this case is that it contains an irrelevance trap. If only one of the final-stage payoffs is

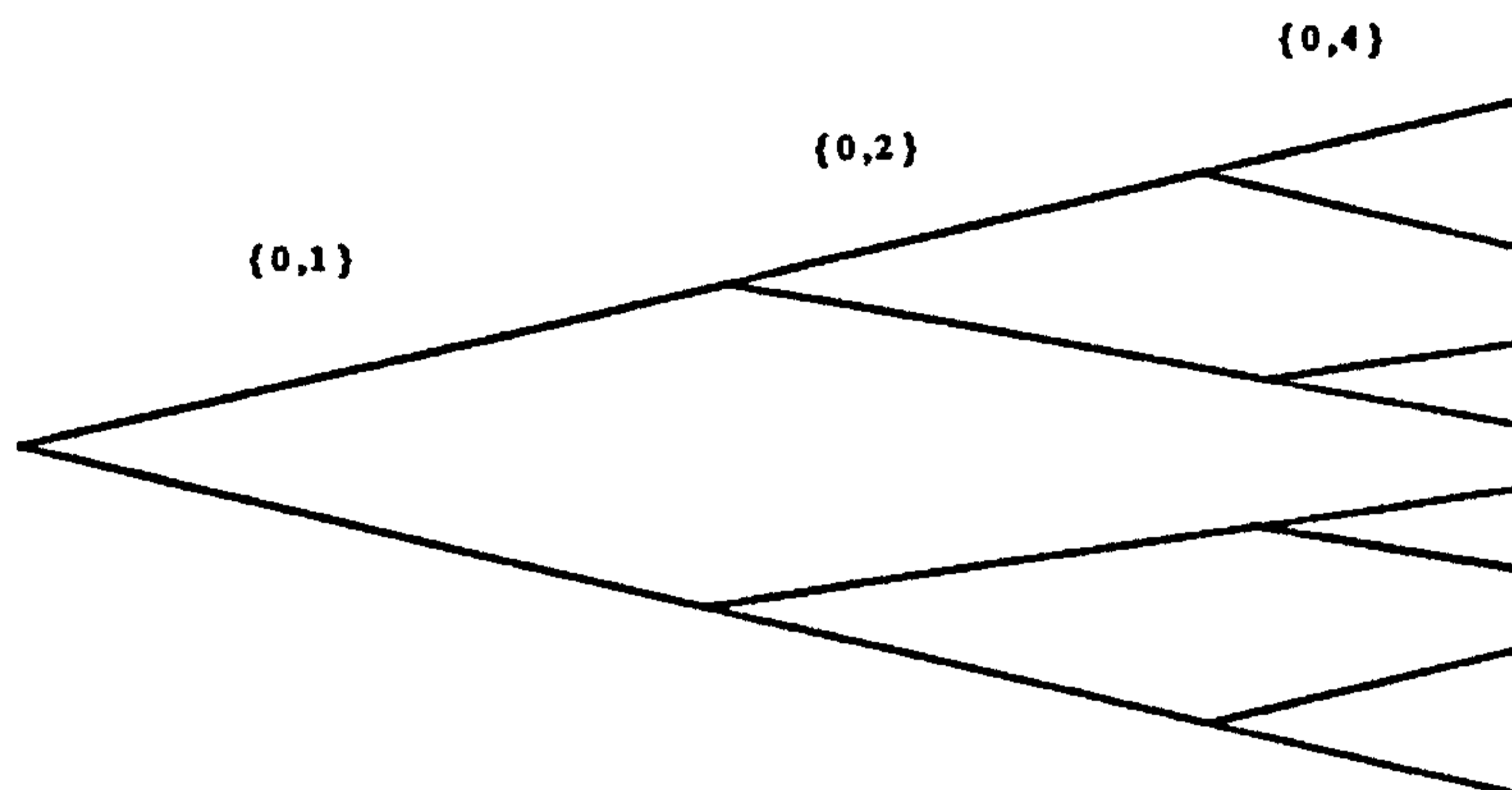


Figure 1.7:

4 there is nothing to be gained by knowing any of the earlier-stage payoffs, just as in Figure 1.2, if $x_1 \neq x_2$ there is nothing to be gained by knowing the value of later payoffs. In general, in such two-branch two-possible-payoff trees, we conjecture that a sufficient condition for backwards reasoning to prove superior to forwards reasoning is that the higher stage payoff from any round be greater than the sum of the higher stage payoffs from all previous rounds. Where the irrelevance lies is crucial to what represents efficient reasoning.

We noted earlier that, for the problem of what to do at the first node, there were 7-state implementing FAs which were neither FRFAs nor BRFAs. It seems plausible that in certain situations neither forwards reasoning nor backwards reasoning might be optimal, but instead some reasoning process in which, perhaps, we reason forwards then backwards then forwards again (say).

However, the point of this section is that there is probably a large class of cases in which backwards reasoning will prove inferior to forwards reasoning because it leads us to solve sub-problems we don't need to solve.

1.13 Conclusion

We have considered the question of whether it is ever best to solve a problem by reasoning forwards as opposed to backwards. We have developed a tool for studying this question - a decision-problem-solving Finite Automaton - and displayed that, at least for certain simple special cases and on the basis of a state-counting criterion of efficiency, forwards reasoning can be more efficient than backwards reasoning.

This would appear to offer a partial explanation of the empirical results of Camerer et al. (1994) and Johnson et al. (2001) that subjects do reason forwards in certain situations (albeit different situations from ours), and some justification for having restrictions on our ability to reason forwards as an element of bounded rationality, thereby closing the explanatory gap identified in the introduction to this section. If some problems are best solved by reasoning forwards, it may be that forwards reasoning would be quite widespread even in the absence of bounds on rationality. However, it should be noted that our interpretation of our second result - namely that it is driven by the need for backwards reasoning to retain more irrelevant information - makes our results here more plausibly applied to decision-problems than to strategic settings. In a decision-problem it seems reasonable to consider only reached nodes (assuming we face no trembling-hand problems). In a multiple-player game this seems less reasonable. Then I would want to know what you might do if I did something different. In a strategic environment there may be much less irrelevance, and a separate proof of the efficacy of forwards reasoning might be required.

Chapter 2

Foundations of Limited Foresight 2: Foundations of a Foresight Horizon

2.1 Introduction

In Chapters 1 & 2 we are investigating the question of why an agent might have limited foresight. In Chapter 1 we have seen that even a Rational agent may reason forwards. In the current chapter we shall move on to investigate the meta-Rationality problem further, and solve for just how far forwards such an agent will reason. We will continue to use the Finite Automata we have examined in the first section. We shall restrict attention to a case in which an agent has a uniform foresight length - he looks the same distance down every path. This is clearly a modelling abstraction. In chess, for example, players do not typically examine every possible move they might play to a length of, say, two moves ahead. Instead they examine only a few plausible first-moves, and perhaps examine some of these two moves ahead, others three, others four - even in these cases only down what appear to be the most likely paths.

We would defend this abstraction in two ways. First of all, it makes our task as modellers simpler and more concrete, and we consider it to be the most obvious first step. Second, it is the same abstraction adopted in the Jehiel environment which is the best developed of the limited foresight models. Nonetheless, we acknowledge that it is a strong assumption.

2.1.1 Thinking about thinking

One might think that, if we are to incorporate cognition costs into our models, we will be inconsistent if we do not also include costs of thinking about thinking, thinking about thinking about thinking, etc.¹. That approach is not followed here.

We introduce costs of thinking into our models because the presence of non-negligible thinking costs appears to be a good way to motivate and calculate a limited foresight horizon. There seems no reason why the costs of thinking about thinking should be the same as the costs of thinking, and it is not at all obvious why costs of thinking about thinking, though perhaps non-zero, should not be negligible. For example, one reason why it might be costly to solve a problem is that the agent may not have faced this particular problem before (or only rarely), and hence may need to engage in costly analysis to solve it². In contrast, the process of solving new problems per se has presumably been engaged in all the agent's life, and by his ancestors and members of his society since the dawn of time. Hence we can reasonably regard the agent as having had forever to work out how best to go about solving new problems, and he may not need to analyse how to do this at all, but rather "just do it" optimally. Hence in the models presented in this section, we shall assume that looking ahead is costly, but that thinking about how far to look ahead bears no relevant costs.

2.2 Endogenising Foresight Length in a Decision-Problem

We shall use the *FAs* introduced in Chapter 1 to solve for the optimal foresight horizon of an agent who has to employ *FAs* to solve the decision-problem in Figure 1.2.

What constitutes foresight in this problem is how far ahead our decision-maker forms expectations over his own play and/or takes account of payoffs he will receive. We shall assume that our players are deciding, each round, what to do that round. For example, if they simply guess what to do at the first node, taking no account of available payoffs at all, we shall call that a foresight horizon of 0. If they decide what to do on the basis of the payoffs consequent only on the decision at the first node - i.e. on x_1 and x_2 - we shall

¹See, for example, Gabaix & Laibson (2000b)

²Presumably a great deal of skill and analysis was employed by Pythagoras in formulating his famous theorem. Rather less is involved in learning it for the first time today, and almost no skill is involved in employing it once it is learnt.

call that a foresight horizon of 1. If their decisions are based on the payoffs available at both stages of the problem, we shall call that a foresight horizon of 2.

2.2.1 State-counting

We shall assume that each state of an FA costs $q \in \mathbb{R}^+$. We shall need to make a further assumption about how players evaluate the worthwhileness of looking ahead. We compare two such assumptions - first, that players aim to minimize their net potential lost payoff³; second that players aim to maximise a form of expected utility.

Minimising net potential lost payoff

We shall assume in this subsection that players choose a foresight length so as to minimise their net potential lost payoff. Such a player's main concern in choosing his foresight length is that he does not make a mistake in play which leads to his not obtaining a high payoff when there was one available. For example, if a player guesses what to do at the opening round without using information about any of the variables involved (i.e. if he has a foresight horizon of 0) he risks ending up with a payoff of 0 from the decision-problem when, by making the optimal choices at each node, he might, if the realisation of the variables were favourable and he made the best choices, have gained a payoff of 2 instead. So the potential lost payoff from a foresight horizon of 0 is 2, and since it costs him nothing to guess at the start, this is also the net potential lost payoff.

If the player wants to base his decision of which action to take at the initial node (i.e. simply whether to go Up or Down) on the payoffs at the first stage (i.e. on a comparison of x_1 and x_2) he will require at least a three-state automaton, as shown in Figure 2.1.⁴

This will guarantee he will forego no more than 1 of payoff (since the worst he could do would be to choose to go Up when $x_3 = x_4 = 0$ but $x_2 = \max(x_5, x_6) = 1$, or go Down when $\max(x_3, x_4) = 1$ but $x_2 = \max(x_5, x_6) = 0$), but it will cost him $3q$ to use it. So his net potential lost payoff is $1 + 3q$.

If he wants to use information about payoffs at both stages of the game, as shown in our Main Lemma, he will require a seven-state FA such as that

³This first assumption is offered merely to illustrate the endogenising concept very simply. We do not pretend it represents a plausible form of rational motivation, and offer no arguments in favour of it.

⁴This is clearly state-minimal because there is one question (the minimum positive information-gathering possible) and two actions between which to recommend.

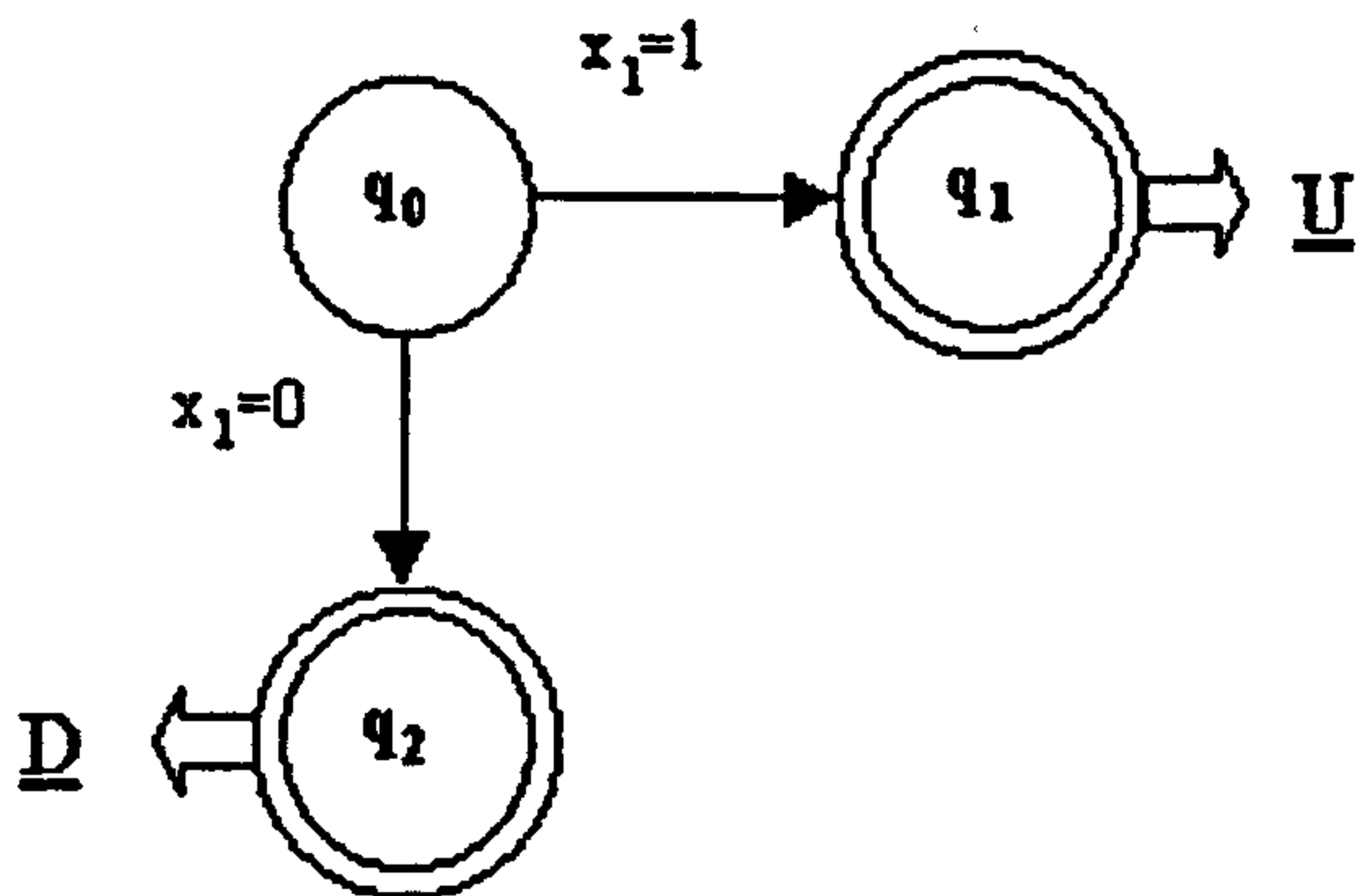


Figure 2.1:

shown again in Figure 2.2.

If he uses this automaton, he will guarantee that he makes no mistake in play, and hence will forego no payoff, but it costs him $7q$ to employ this FA.

The discussion of this section so far is summarised in the following table.

Foresight (Rounds)	Potential lost payoff	Cost	Net potential lost payoff
0	2	0	2
1	1	$3q$	$1 + 3q$
2	0	$7q$	$7q$

Proposition 2.1 *If players are net potential loss minimizers, and $q < \frac{1}{4}$, the optimal foresight horizon is 2 or more, covering the whole game. If $\frac{1}{4} < q < \frac{1}{3}$, the optimal foresight horizon is 1. If $q > \frac{1}{3}$, the optimal horizon is 0 and the player should guess at the opening node.*

Proof. This follows immediately from the information in the table above.

■

Maximising expected net payoff

An alternative assumption is that players choose a foresight length so as to maximise expected net payoff. This is more complicated but still reasonably tractable.

First we need an assumption about the distribution of the x_i . We shall assume there is a 50/50 chance of each of the payoffs being either 1 or 0, and that our decision-maker knows this. He also knows that the payoffs are independently distributed.

We must establish what, in equilibrium, players will do at the second reached node. We shall prove by contradiction that players adopting a

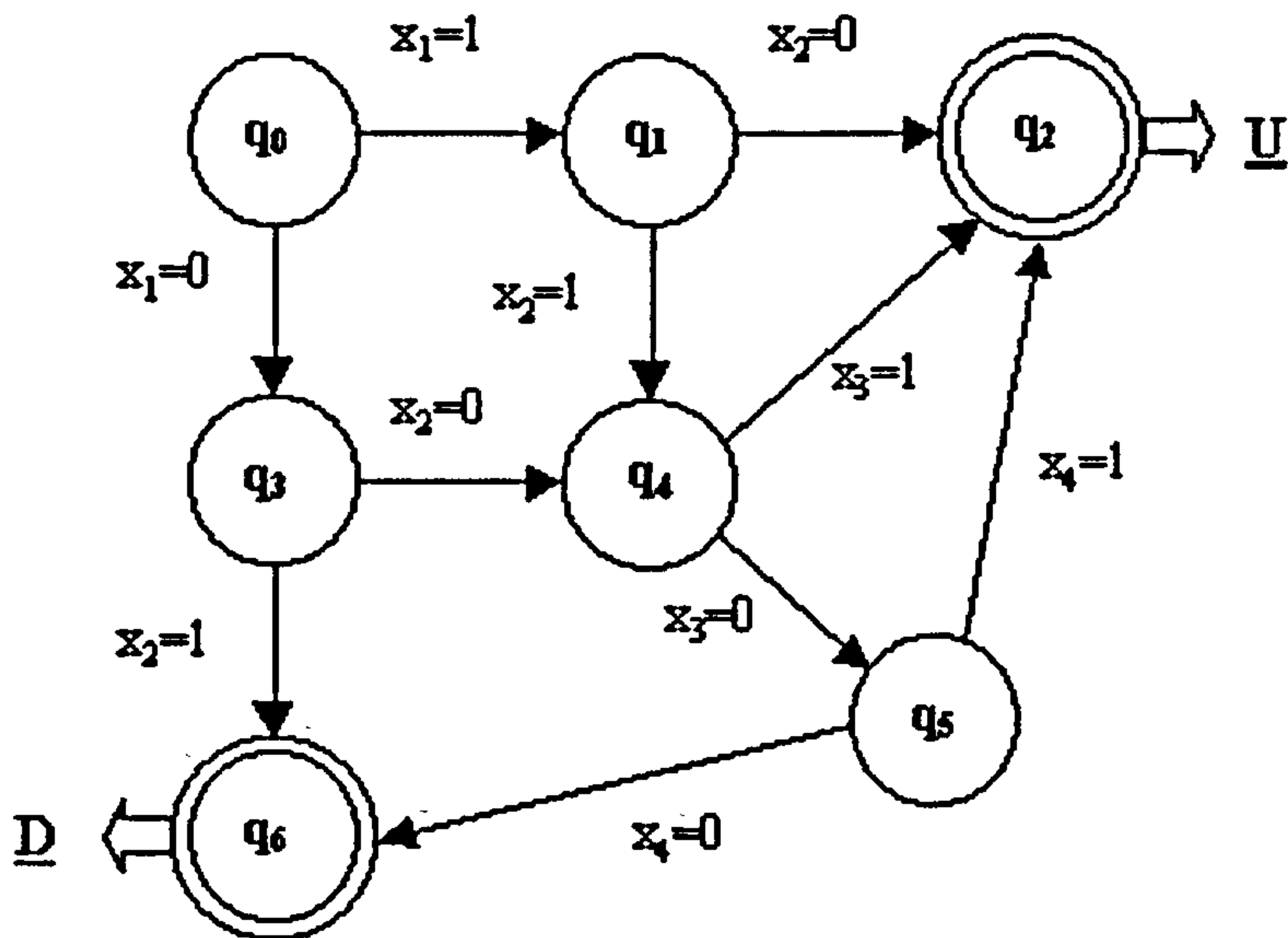


Figure 2.2:

foresight length of 0 assume that they will also adopt a foresight length of 0 at the second node (i.e. if someone is going to guess at the start he is assuming he will guess later as well), except in the special case of $q = \frac{1}{12}$. We shall proceed to show that this leaves the boundary between choosing a foresight length of 1 and a foresight length of 0 unaffected.

Remark 2.1 *It follows immediately from the fact that all of (x_3, x_4, x_5, x_6) are drawn from $\{0, 1\}$ that the expected net payoff from adopting a foresight horizon of 0 at either the upper or lower second-stage nodes is $\frac{1}{2}$.*

Proposition 2.2 *The expected net payoff from adopting a foresight horizon of 1 at either the upper or lower second-stage nodes is $\frac{3}{4} - 3q$.*

Proof. A player with a foresight horizon of 1 must use a three-state FA like Figure 2.2 (questioning x_3 at the upper node or x_5 at the lower node), thereby incurring a cost of $3q$. Use of this FA is sufficient to guarantee playing optimally, thereby securing $\max\{x_3, x_4\}$ at the lower node, or $\max\{x_5, x_6\}$ at the upper node. The expected value of $\max\{x_3, x_4\}$ is $\frac{3}{4}$, hence the expected net payoff is $\frac{3}{4} - 3q$. ■

Corollary 2.1 *It follows from Remark 2.1 and Proposition 2.2 that players will choose a foresight length of 1 at the second stage only if $3q \leq \frac{1}{4}$, i.e. only if $q \leq \frac{1}{12}$.*

Remark 2.2 *A player guessing at both nodes (i.e. adopting a foresight horizon of 0 at each node) will have an expected payoff of $\frac{1}{2} + \frac{1}{2} = 1$.*

Proposition 2.3 *Assuming that players believe they will choose what is best at the second stage (i.e. they will adopt a foresight length of 1 and hence choose $\max\{x_3, x_4\}$ (or, at the upper node, $\max\{x_5, x_6\}$) at the second stage), the expected net payoff from adopting a foresight horizon of 0 at the outset is $1.25 - 3q$.*

Proof. Consider the expected value of $\pi(U) = x_1 + \max\{x_3, x_4\}$. This gives the following:

$$E(\pi(U)) = E(x_1 + \max\{x_3, x_4\}) = \frac{1}{2} + E(\max\{x_3, x_4\}) = 1.25$$

$$\text{By the symmetry of the problem, } E(\pi(D)) = E(\pi(U)) = 1.25$$

At the second stage there will be a cost of $3q$, so the expected net payoff of a foresight horizon of 0 is $1.25 - 3q$. ■

Proposition 2.4 *A player who believes that he will adopt a foresight horizon of 1 at the second stage has an expected net payoff from adopting a foresight horizon of 1 at the first stage of $1.5 - 6q$.*

$$\text{Proof. } E(\max\{x_1, x_2\}) = E(\max\{x_3, x_4\}) = E(\max\{x_5, x_6\}) = \frac{3}{4}$$

$$\frac{3}{4} + \frac{3}{4} = 1.5$$

Each stage of using the three-state FA costs $3q$, so the expected net payoff from a foresight horizon of 1 in this scenario is $1.5 - 6q$. ■

Proposition 2.5 *A player who believes that he will adopt a foresight horizon of 0 at the second stage has an expected net payoff from adopting a foresight horizon of 1 at the first stage of $1.25 - 3q$.*

Proof. Since the expected payoff from his guess at the second stage is $\frac{1}{2}$, as in Remark 2.1, this scenario gives him an expected payoff of $E(\max\{x_1, x_2\}) + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = 1.25$

Using the three-state FA at the first stage costs $3q$, so the expected net payoff from a foresight horizon of 1 in this scenario is $1.25 - 3q$. ■

Proposition 2.6 *A player who believes that he will adopt a foresight horizon of 1 at the second stage has an expected net payoff from adopting a foresight horizon of 2 at the first stage of $1\frac{19}{32} - 10q$.*

Proof. Using the seven-state FA of Figure 2.2 to base the decision on both stage I and stage II payoffs delivers an ex ante expected payoff of

$E(\max\{x_1 + \max\{x_3, x_4\}, x_2 + \max\{x_5, x_6\}\}) = 1\frac{19}{32}$. To see this, note that, for independent $x_i \in \{0, 1\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$,

$$\begin{aligned} Pr(x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0) &= \frac{1}{64} \\ Pr(\max\{x_3, x_4\} = 1) &= \frac{3}{4}, \text{ and } Pr(x_1 = 1) = \frac{1}{2} \\ \Rightarrow Pr(x_1 + \max\{x_3, x_4\} = 2) &= \frac{3}{8} \\ \Rightarrow Pr(x_1 + \max\{x_3, x_4\} \neq 2) &= \frac{5}{8} \\ \Rightarrow Pr(\max\{x_1 + \max\{x_3, x_4\}, x_2 + \max\{x_5, x_6\}\} \neq 2) &= \frac{25}{64} \\ \Rightarrow Pr(\max\{x_1 + \max\{x_3, x_4\}, x_2 + \max\{x_5, x_6\}\} = 2) &= \frac{39}{64} \\ \Rightarrow E(\max\{x_1 + \max\{x_3, x_4\}, x_2 + \max\{x_5, x_6\}\}) &= \frac{1}{64}(0) + \frac{39}{64}(2) + \frac{(64-1-39)}{64}(1) = \\ 1\frac{19}{32} \end{aligned}$$

This costs $7q$, with an additional cost of $3q$ to use the three-state FA at the second stage, so the expected net payoff from a foresight horizon of 2 is $1\frac{19}{32} - 10q$. ■

Proposition 2.7 *Optimal foresight lengths at the first stage are as follows:*

1. For $q > \frac{1}{12}$ we should choose a foresight length of 0
2. For $q < \frac{3}{128}$ we should choose a foresight length of 2
3. For $\frac{1}{12} > q > \frac{3}{128}$ we should choose a foresight length of 1.

Proof. Corollary 2.1 shows that for $q < \frac{1}{12}$, players will adopt a foresight length of 1 at the second stage, hence Proposition 2.7.2 and the lower boundary of Proposition 2.7.3 follow from Propositions 2.4 and 2.6, since those imply that players will choose a foresight length of 2 over 1 at the first stage when foresight length is 1 at the second stage and $4q < \frac{3}{32}$, i.e. when $q < \frac{3}{128} < \frac{1}{12}$. The upper boundary of Proposition 2.7.3 follows from Propositions 2.3 and 2.4, since those imply that agents will choose a foresight length of 1 over 0 at the first stage when $3q < \frac{1}{4}$, i.e. when $q < \frac{1}{12}$. It remains to prove Proposition 2.7.1, because, Corollary 2.1 and Propositions 2.3 and 2.4 also imply that when players believe they will choose a foresight length of 1 at the second stage they will never choose a foresight length of 0 at the first node except in the special case of $q = \frac{1}{12}$. For $q > \frac{1}{12}$, i.e. when the foresight horizon will be 0 at the second stage, Remark 2.2 and Proposition 2.5 imply that we should choose a foresight horizon of 0 at the first stage if $1 > 1.25 - 3q$, i.e. if $q > \frac{1}{12}$, so Proposition 2.7.1 follows. ■

2.3 Discussion

Our main purpose is to illustrate a *method* and build theoretical foundations, rather than to make predictions. However, it is striking quite how low the state-costs, q , have to be (compared with the available payoffs) for an expected-payoff maximiser to justify having a foresight length encompassing the whole game, even in this very simple scenario. If an expected-payoff maximising player facing a potential loss of 1 will prefer this (i.e. a limited foresight length) to guaranteed optimising (i.e. a complete foresight length) for a cost of only $\frac{3}{128}$ per state, even in a case where the highest number of states under consideration is only 7, it seems very plausible to us that in only slightly more complicated problems with much higher total-state requirements, foresight horizons which do not encompass the whole game may be very widespread. If this insight is correct, it illustrates the benefits of seeking foundations.

2.3.1 Alternative forms of cognition cost

Costing states of our automaton may not be the best way to measure the cost of using an automaton. For example, we might prefer to cost the FA's paths rather than its states. Three alternative measures of the cost of the paths of the FA here seem plausible:

1. We could count the total number of paths in the automaton. Since the state-minimal automaton has seven states, of which two are final and the rest questioning, and since every questioning state must have two exiting paths, the ten paths of Figure 2.2 are obviously also path-minimal by this measure. Costing the FA is then trivially analogous to the method shown above.
2. We could use the average path-length between the initial state and the final state reached, averaging over realisations of the variables. The average path length for the FA in Figure 2.2 is 2.75. Whether this is minimal remains an open question.
3. If our agent is an expected payoff maximiser, he could cost only the paths he would actually use for any realisation of the variables, and thereby calculate the net payoff in that way. Quite what criterion one should use for determining which FA the player employs in this case is slightly obscure.

While plausible, it appears to us that, in the absence of any compelling grounds for believing that 'true' cognition costs are more accurately reflected by paths than states, state-counting is sufficient to deliver the insight necessary in this area.

2.4 Conclusion

In this chapter we have used our Finite Automata to determine an optimal foresight length for a decision-maker. We have considered two evaluation-rules for player deciding how far ahead they should look - minimising the net potential lost payoff and maximising the expected net payoff - and illustrated how adopting a limited foresight length can be the fully rational thing to do when solving problems is costly. For our preferred concept (expected net payoff maximisation) the state-cost required to justify an unlimited foresight horizon is so low compared with the potentially available payoffs that we believe it is reasonable to expect that rational players might routinely adopt limited horizons even in relatively simple finite-horizon problems, and that limited-foresight reasoning might be much more wide-spread than is usually thought. Rational players might often be well-advised not to look too far ahead.

2.5 General Conclusion for Chapters 1 & 2

In Chapters 1 & 2 we have been attempting to construct rational foundations for Limited Foresight modelling. We have shown that rational players may want to reason forwards in certain situations, and hence that models involving Foresight are well-justified. We have gone on to show how a forwards-reasoning agent might rationally choose a limited foresight horizon. The insights in both cases could be interpreted as suggesting that limited foresight reasoning might be very widespread, and not merely constrained to special complicated situations like a chess game.

Further useful research in this area would include devising more efficient and elegant ways of identifying the state-minimal FA for solving a given problem; proving that forwards reasoning can be more efficient in strategic settings; showing how to solve for an optimal limited horizon in a strategic setting; extending the results to settings with chance variables and mixed strategies; and investigating decision-problems with richer strategy and payoff spaces.

If limited foresight reasoning is as widespread as we suggest, going to

the trouble of modelling may be worthwhile, provided we can come up with solution-concepts which offer a plausible account of what limited foresight involves and which offer the possibility of modelling limited foresight problems tractably. How to do that is the subject of the next chapter.

Chapter 3

Do we need a new approach to limited foresight?

3.1 Introduction

In our first two chapters we have looked at why people might have limited foresight and argued that limited foresight reasoning is likely to be quite widespread, applying to many different sorts of problem, and hence worth attempting to model. But how should we do it? How effective are the approaches already available in the Game Theoretic literature? Do we, perhaps, need some entirely different approach?¹

In Chapter 7 of Rubinstein (1998), the author compares two models of limited foresight. In both models agents can look ahead a certain number of periods but not right to the end of the game, and in neither case do agents form any guess as to what will happen beyond the foresight horizon. The models differ in that in one case agents only determine their action in the current period and guess what they will do in later periods up to the horizon (with a feature of equilibrium being that these guesses will prove correct) while in the other agents form a plan about their actions across the foresight horizon (which can fall victim to time inconsistency). Rubinstein says that "*the two approaches fall short of capturing the spirit of limited-foresight reasoning.*"². In this chapter we shall consider what he means by this, whether he is right to think these approaches unsatisfactory, and what a different approach might look like.

Our key expression of the first of these notions will be a version of Je-

¹e.g. Perhaps something like the "bounded lookahead" found in the Artificial Intelligence literature? - see Jehiel (1998b)

²Rubinstein (1998), p.134.

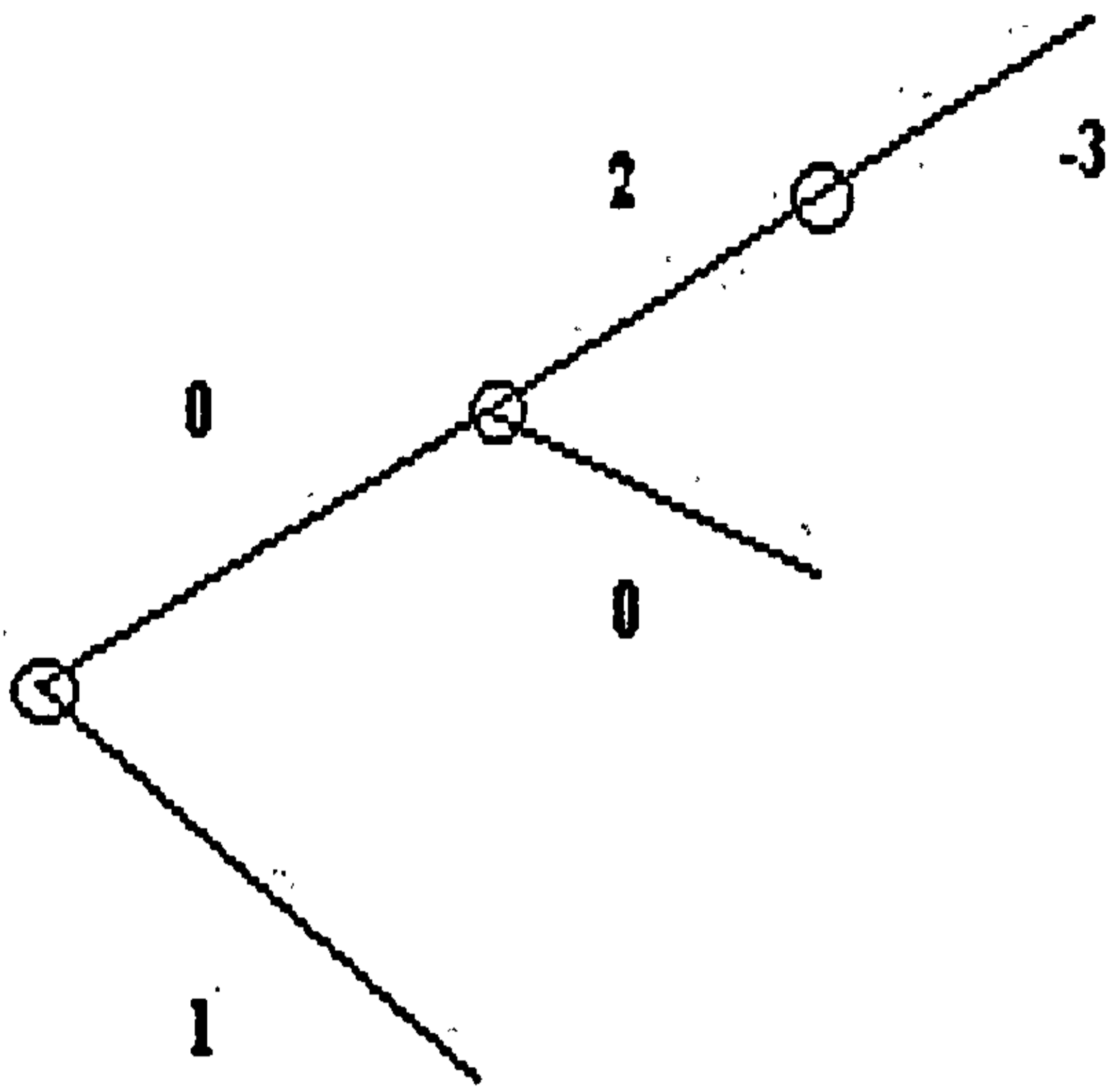


Figure 3.1: A simple decision problem with limited foresight.

hiel's (1995) (n_1, n_2) -equilibrium concept. For the purposes of this chapter, we shall call this the 'Jehiel' concept. Our key expression of the second notion will be an equilibrium concept in the tradition of the time consistency literature following Strotz (1956) and Pollack (1968). We shall call this the 'naive' concept. To illustrate the distinction immediately, consider Figure 3.1.

In Figure 3.1 we have a one-person decision problem. Each circle represents a point at which an action is to be taken. The branches of the tree represent those actions, and at the first two decision-points the agent can choose to go up or down in the tree. Branches which have no sub-branches represent terminal nodes. The decision-maker receives payoffs after each action corresponding to the numbers. Suppose that he can see ahead one period. Then from period one he can see that if he goes down to start with he will receive a payoff of 1 and the game will end, while if he chooses to go up then he will be able to choose between a payoff of 2 if he goes up as his second action and 0 if he goes down.

It is convenient to consider our concepts in reverse order. Under the naive concept, in which he forms a plan across his horizon and acts according to that, he will choose to go up at the first decision-point, anticipating that he would then choose to go up again at the second decision-point. However, having reached the second decision-point, he will then be able to see to the end of the game, and realise that if he goes up at the second decision-point,

he will then have no choice but to go up again at the third action-point and receive an additional payoff of -3. Thus going up will deliver a net payoff of -1, and he will instead opt for going down at the second action-point, receiving the superior payoff of 0. So the naive equilibrium of this game is for the agent to adopt the overall strategy (specifying what he would do at each action point in turn) (Up, Down, Up). In a sense this agent makes a mistake because he fails to take account of his limited foresight.

In contrast, under the Jehiel concept we have in mind an agent who is aware of his limited foresight. He has perhaps played the game many times before, and is aware that, from this situation, if he were to choose to go up at the first decision-point, he would then choose to go down at the second. He is not sure *why* he would do this, i.e. he does have limited foresight. But he has learned that that would be his behaviour, and what he has learned is correct. So at the first decision point he believes (correctly) that he is choosing between going down and receiving a payoff of 1, or going up and receiving a payoff of 0. He goes down, and in the Jehiel concept the equilibrium strategy is (Down, Down, Up).

There is a great deal more that could be said about these concepts, but this example illustrates the key distinction. What Rubinstein says he dislikes about them is that by "*the first approach* [i.e. the Jehiel concept], *a player treats his future behaviour as given, though he can influence it. By the second approach* [i.e. the naive concept], *he treats the other players' plans as known, though he does not know his own moves.*" The main question we shall address is whether there is anything fundamentally unsatisfactory about these two approaches which would not be addressed by refining them (for example, by granting players the ability to guess - perhaps incorrectly - what would happen beyond the horizon).

If these two concepts are flawed or insufficient, that would be a significant conclusion, because a great deal has been done with them. There is an extensive time consistency literature, in which the virtues of pre-commitment³, sophistication and naivete about one's time consistency⁴, and many other issues have been investigated. Jehiel, on the other hand, has shown that with agents who, like here, make no guess beyond the forecast horizon, an (n_1, n_2) -solution always exists⁵; that such solutions are cyclical; that the equilibrium forecasts associated with such solutions do not depend on history and that the memory capacity of the players has no impact on the set of solutions as long as it is finite; that for generic repeated alternate-move 2×2 games a

³Strotz (1956)

⁴O'Donoghue & Rabin (1999)

⁵in Jehiel (1995), in a repeated alternate-move environment, with two players and limited recall.

solution always exists which holds for all n_1, n_2 sufficiently large; that players can sometimes do better with a shorter foresight length; and with agents who *do* make a guess beyond the horizon, he has shown that co-operation can sometimes be the only equilibrium in a repeated Prisoners Dilemma. He has shown that his concept arises out of a learning process with trembles⁶.

Thus it seems worth investigating whether Rubinstein is right, and seeing what he might mean by suggesting that the current approaches are unsatisfactory. We shall start off by formalizing these concepts for our purposes, and see in what ways they are similar and in what ways they differ. We shall go on to suggest a reason why they might, at first seem unsatisfactory, but then address this worry in certain cases, offer a refinement of the Jehiel concept which responds to Rubinstein's critique, and identify the residual class of games for which Rubinstein's suggestion is most convincing.

3.2 Defining the concepts

3.2.1 Notation

Our points will be largely conceptual, and hence for ease of exposition we shall use a restricted class of games. Our results can be generalised quite straightforwardly. We shall consider a maximum of two players⁷ engaged in a game of perfect information with simultaneous moves, chance players and stage payoffs. Player 0 is defined as the chance player. In the first stage of the game players $i \in I = \{1, 2\}$ choose actions from finite choice sets $A_i(h^0)$ where history $h^0 = \emptyset$ and some choice sets may be the singleton "do nothing". At the end of each stage all players observe the stage's action profile $a^t \equiv (a_0^t, a_1^t, a_2^t)$, and receive a payoff $u_i(a^t, h^t)$, where history $h^t \equiv (a^0, \dots, a^{t-1})$. Denote the set of all histories as H . Denote by $A_i(h^t)$ player i 's feasible actions in stage t when the history is h^t . When chance does not do nothing, its action is selected from a probability measure, $f_c(\cdot|h^t)$ on chance's feasible action set, $A_0(h^t)$. Each player's payoff at a terminal history is the sum of the payoffs he collected along the history. In a one-player game player 2 does nothing at every stage and receives zero stage payoffs at every stage⁸.

⁶Jehiel (1998a)

⁷I shall refer to the player whose choices we are considering as 'he' and the other player as 'she', even when we switch which player's choices we are considering.

⁸We shall refer the one-payer game as a "limited-foresight decision-problem". A game which finishes in finitely many rounds, T , will be referred to as a "finite game" and the one-player variant as a "finite limited-foresight decision-problem".

Let $\Delta(A_i(h^t))$ denote the probability distributions on $A_i(h^t)$. A *behaviour strategy* for player i , denoted q_i is an element of the Cartesian product $\times_{h^t \in H} \Delta(A_i(h^t))$. This specifies a probability distribution over actions at each h^t and probabilities for different histories are independent. A profile of behavioural strategies is given by $q = (q_1, q_2)$

Players have perfect recall, and know the decisions they are going to make in the current round. Looking ahead, each player has a *foresight horizon* n_i , over which he is seeking to maximise the sum of his expected payoffs, where these expected payoffs arise out of the expected outcomes of chance moves and behavioural strategies, and obey the von Neumann and Morgenstern expected utility axioms. He takes no account of what happens beyond his foresight horizon, and forms no guess about it.

Denote by $q_{i,t}$ the behavioural strategy of player i in stage t . Define $q_{i,t,v} = (q_{i,t+1}, \dots, q_{i,v})$, and $q_{-i,t,v} = (q_{-i,t+1}, \dots, q_{-i,v})$, so that $q_{i,t,v}$ and $q_{-i,t,v}$ become player i and other player's behavioural strategies between rounds t and v . Denote by $(q_{i,v})_t$ the belief player i has in round t about the behavioural strategy he will follow in round v . Similarly define $(q_{-i,v})_t$ as the belief player i has about what the other player will do. Then $(q_{i,v,w})_t$ and $(q_{-i,v,w})_t$ as expected denote player i 's beliefs at round t about what he and the other player will use as their behavioural strategies between rounds v and w . Define $z_i = t + n_i - 1$. Define $\tilde{q}_{i,t} \equiv (q_{i,t+1,z_i})_t$ and $\tilde{q}_{-i,t} \equiv (q_{-i,t,z_i})_t$. Then denote by $\tilde{\pi}_{i,t}(q_{i,t}, \tilde{q}_{i,t}, \tilde{q}_{-i,t}, h^t)$ player i 's belief at round t about the expected sum of (possibly discounted) payoffs from the current round to his foresight horizon, if he uses the behavioural strategy $\langle q_{i,t}, \tilde{q}_{i,t} \rangle$ (which we shall term player i 's " $\tilde{\sim}$ -strategy") over the next n_i rounds and the other player uses the behavioural strategy $\tilde{q}_{-i,t}$ (which we shall term the other player's " $\tilde{\sim}$ -strategy") over the same period. If there is only one player then $\tilde{q}_{-i,t} = \emptyset$. If the game finishes before round z_i then superfluous elements of $\tilde{q}_{i,t}$ and $\tilde{q}_{-i,t}$ are set empty.

3.2.2 Formalising the two approaches

Armed with our notation, we shall now give formal definitions of the two approaches.

The "Limited Foresight Equilibrium - Jehiel Version" (LFE-J) conditions

The definition given here will be a slight reformulation of Jehiel's 1995 (n_1, n_2) -equilibrium concept, because there his concept was applied to re-



peated alternate-move games with players with limited recall. For our purposes here, we shall call a strategy profile, q , an LFE-J if, and only if, for all players, all histories, all actions, and all time periods,

1. The behavioural strategy of each player in the current period is justified, given the other player's $\tilde{\cdot}$ -strategy, and an expectation as to what each player himself will do in later periods out to the horizon. More precisely

$$q_{i,t} = \arg \max_{q_{i,t}} \tilde{\pi}_{i,t} (q_{i,t}, \tilde{q}_{i,t}, \tilde{q}_{-i,t}, h^t), \forall i, t, h^t$$

2. The strategies are consistent, in the sense that the expectations players form of their own and the other player's behavioural strategies out to the horizon of foresight prove correct. More precisely,

$$\begin{aligned} \tilde{q}_{-i,t} &\equiv (q_{-i,t,z_i})_t = q_{-i,t,z_i}, \text{ and} \\ \tilde{q}_{i,t} &\equiv (q_{i,t+1,z_i})_t = q_{i,t+1,z_i}, \forall i, t, h^t \end{aligned}$$

The "Limited Foresight Equilibrium - Naive Version" (LFE-N) condition

A strategy profile, q , is an LFE-N if, and only if, for all players, all histories, all actions, and all time periods,

- i. The $\tilde{\cdot}$ -strategy of each player is justified, given the other player's $\tilde{\cdot}$ -strategy. More precisely, controlling $q_{i,t}$ and $\tilde{q}_{i,t}$,

$$\tilde{\pi}_{i,t} (q_{i,t}, \tilde{q}_{i,t}, \tilde{q}_{-i,t}, h^t) \geq \tilde{\pi}_{i,t} (q'_{i,t}, \tilde{q}_{i,t}, \tilde{q}_{-i,t}, h^t), \forall i, t, h^t$$

- ii. The expectations players form of the other player's behavioural strategy prove correct, up to the horizon of foresight. More precisely,

$$\tilde{q}_{-i,t} \equiv (q_{-i,t,z_i})_t = q_{-i,t,z_i}, \forall i, t, h^t$$

3.2.3 Initial Results

Our aim in this chapter is to identify a preferred solution concept for limited foresight problems. Two important aspects of this include how the different concepts available in the present literature compare with one another and when these concepts exist and/or are unique. Appendix D exhibits various existence and uniqueness results for the naive and Jehiel concept. Here we focus on comparing them.

Results comparing the naive and Jehiel concepts

Proposition 3.1 *There is no inclusion between the naive and Jehiel concepts.*

Proof. As the game in Figure 3.1 illustrated, the concepts lead to different equilibrium outcomes, not every LFE-N is an LFE-J, and not every LFE-J is an LFE-N. ■

Next we shall prove that if a naive equilibrium happens to be consistent, then it is always also a Jehiel equilibrium.

Definition 3.1 *A "time-consistent" LFE-N⁹ has the following property, in addition to (i.) and (ii.) above:*

$$\text{ii. } (q_{i,t+l,t+k})_t = (q_{i,t+l,t+k})_{t+m}, \forall i, t, \forall m, k \leq n_i - 1, \forall l \leq k$$

Proposition 3.2 *Every time-consistent LFE-N is an LFE-J.*

Proof. In Condition (i.) set $\tilde{q}_{i,t}$ such that $q_{i,t}$ will be maximal. Then Condition (1) is satisfied. In Condition (ii.) set $l = 1$, set $m = k = n_i - 1 \Rightarrow t + m = t + k = z_i$, and note that $(q_{i,t+1,z_i})_{z_i} = q_{i,t+1,z_i}$ (since players cannot be mistaken about what they are going to do in the present round). Then Condition (2) is satisfied. ■

However, it is not true that every LFE-J will be a time-consistent LFE-N, as Figure 3.1 illustrates. There the solitary LFE-J is not an LFE-N at all.

3.2.4 Comparing the two approaches

What Jehiel has in mind in his concept is players who have played the game many times before, and have learned how it works. They recognize that they have only limited foresight, and understand that in later rounds actions that look likely to be what they themselves will do may not turn out that way. The expectations they have about their own play out to the horizon are correct, and they correctly anticipate that they cannot correctly deduce what they will do in later rounds from the information currently available - hence they control only their action in the present round. In Jehiel's formulation an issue arises as to what happens in the long-run if players are never testing their guesses about what would happen if they picked some attractive option which they remember leads to time-inconsistency. Jehiel

⁹We shall refer to this below as a "Limited Foresight Equilibrium - Strategic Control Version", for reasons which will become apparent in Section 3.4.1.

gets around this problem by assuming that the learning process has trembles, so that all possibilities are investigated¹⁰.

Rubinstein criticizes this concept because "*a player treats his future behaviour as given, though he can influence it*". Below we shall prove that, though this is one way of interpreting the Jehiel concept, there is a formally equivalent set of equilibrium conditions which can naturally be interpreted as involving players who form a plan out to the horizon.

The naive concept is fairly straightforward. We have a player who can only see ahead a certain number of periods, and has no means to judge what might happen beyond his horizon. We might think that the best he can do is try to maximise his payoffs and hope that he doesn't fall into traps like that illustrated in Figure 3.1. However, Rubinstein is right to note something slightly awry in the concept, when he says that each player "*treats the other players' plans as known, though he does not know his own moves*". For some reason not explained, in this form of equilibrium in a two-player game each player will always be *correct* in his assessment of what the other player will do, even though he may be *incorrect* about what he himself will do, and even though the other player may be incorrect about what she herself will do! We shall examine this problem further in the next section.

3.3 Problems with the approaches

3.3.1 Problems with the naive concept?

The naive concept seems to have most force when we are thinking of a one-player decision problem. Then we can think of the naive concept as modelling a player who doesn't really understand that the game will continue after his horizon of foresight. Once we move to the two-player game the naive concept seems flawed in many ways. As we have already mentioned, for some reason not explained, in equilibrium players can be incorrect in their assessment of their own strategy out to the horizon, even though they are perfectly correct in their guesses about the other player's strategy. If they are possibly incorrect, why do the players imagine themselves as being able to determine what they do out to the horizon? Why isn't it that they pick a strategy for the current round, guessing what they might do later? But if that were the case, where would such guesses come from? The naive concept isn't, supposedly, motivated by the sort of learning considerations present in the Jehiel model.

¹⁰Jehiel (1998a)

Why should players be correct about the other player's strategy? She isn't even necessarily correct about it herself! Where does the information come from that makes players more informed about other players than about themselves?

Furthermore, are we really content to call something an 'equilibrium' if players change their minds and are likely to feel later that they made mistakes even given the information they had at the time? Certainly not changing one's mind and not regretting one's play are not fundamental features of an equilibrium, and if the concept were otherwise robust their lack might be irrelevant, but mind-changing and regret do make less attractive an equilibrium notion which has already been shown to have fundamental weaknesses.

3.3.2 Problems with the Jehiel concept?

We have already mentioned that Rubinstein criticized the Jehiel approach because it takes future behaviour as given, even though players can influence it. In standard game theoretic models we usually conceive of players forming a plan of play which optimizes over the entire game, specifying what it is best to do at every decision-point. In Rubinstein's interpretation of the Jehiel concept (which is certainly the usual interpretation) players seem unaware of their influence over their own play in later rounds. It is almost as the Jehiel approach involved multiple selves, each regarding moves in later rounds as being done by someone else over whose decisions the current self has no control. This seems unsatisfactory and contrary to the usual spirit of Game Theory.

However, we can construct a formally equivalent limited-foresight equilibrium concept in which players do control their actions out to the horizon of foresight¹¹. In this concept players optimize over time consistent strategies. That is, once again we could imagine that we have players who have learned how the game works. They know that certain strategies lead to time inconsistency for reasons they aren't quite sure of (this uncertainty representing the essence of their limited foresight). Thus they know that apparent payoffs from strategies which lead to time-inconsistency are incorrect, and they ignore them. From the other possible strategies they form a plan across the entire horizon so as to maximise the sum of their expected payoffs. Formally, this concept, which we shall call a "Limited Foresight Equilibrium -

¹¹The distinction between, in the one case, deciding what to do today based on a correct expectation of what I shall do tomorrow, and, in the second case, deciding what to do both today and tomorrow, may not be entirely obvious. We investigate this question further in Appendix C.

Constrained Maximisation Version" (LFE-CM) can be captured by the following condition:

A strategy profile, q , is an LFE-CM if, and only if, for all players, all histories, all actions, and all time periods,

- a. The $\tilde{\cdot}$ -strategy of each player is justified, given the other player's $\tilde{\cdot}$ -strategy. More precisely

$$\langle q_{i,t}, \tilde{q}_{i,t} \rangle = \arg \max_{q_{i,t}, \tilde{q}_{i,t}} \tilde{\pi}_{i,t}(q_{i,t}, \tilde{q}_{i,t}, \tilde{q}_{-i,t}, h^t), \forall i, t, h^t$$

s.t.

$$\tilde{q}_{-i,t} \equiv (q_{-i,t,z_i})_t = q_{-i,t,z_i}, \text{ and}$$

$$(q_{i,t+l,t+k})_t = (q_{i,t+l,t+k})_{t+m}, \forall m, k \leq n_i - 1, \forall l \leq k$$

$$\forall i, t, h^t$$

Remark 3.1 *Every LFE-CM is a time-consistent LFE-N*

Corollary 3.1 *Every LFE-CM is an LFE-J (by Proposition 3.2)*

Proposition 3.3 *Every LFE-J is an LFE-CM*

Proof. Suppose that Condition (2) holds. Consider $t + 1 \leq s \leq z_i$. Then $(q_{i,t+1,z_i})_t \equiv ((q_{i,t+1})_t, \dots, (q_{i,s})_t, \dots, (q_{i,z_i})_t)$, which, with Condition (2), implies $(q_{i,s})_t = q_{i,s}, \forall s, \forall t$. Refer to this as condition (*). Now consider $l \leq s' \leq k$, so that, by (*), $(q_{i,s'})_t = q_{i,s'}$, and

$$(q_{i,t+l,t+k})_t \equiv ((q_{i,t+l})_t, \dots, (q_{i,s'})_t, \dots, (q_{i,t+k})_t).$$

Hence, $(q_{i,t+l,t+k})_t = q_{i,t+l,t+k}$. Then consider $y \equiv t + m$, so that

$$(q_{i,t+l,t+k})_{t+m} \equiv (q_{i,t+l,t+k})_y \equiv ((q_{i,t+l})_y, \dots, (q_{i,s'})_y, \dots, (q_{i,t+k})_y).$$

By (*), $(q_{i,s'})_y = q_{i,s'}$, so, $(q_{i,t+l,t+k})_{t+m} = q_{i,t+l,t+k} = (q_{i,t+l,t+k})_t$. This holds for all $m, k \leq n_i - 1$, and all $l \leq k$. Additionally, by Condition (2) we have $\tilde{q}_{-i,t} \equiv (q_{-i,t,z_i})_t = q_{-i,t,z_i}$. Thus whenever Condition (2) holds, the constraints of Condition (a) are satisfied.

When the constraints are satisfied, in particular the second constraint is satisfied for $l = 1$ and $k = n_i - 1$. Then $(q_{i,t+1,z_i})_t = \tilde{q}_{i,t} = (q_{i,t+1,z_i})_{t+m} = q_{i,t+1,z_i}$ and the second constraint fixes $\tilde{q}_{i,t}$. Thus the maximisation in Condition (a) reduces to Condition (1) and hence whenever both Condition (1) and Condition (2) are satisfied, then Condition (a) holds. ■

Corollary 3.2 *Since every LFE-CM is an LFE-J and every LFE-J is an LFE-CM, it follows that LFE-CM and LFE-J are formally equivalent.*

Thus Rubinstein's criticism of the Jehiel concept seems slightly misplaced, in that it is not necessary to interpret it in the multiple-selves way. Instead of own future play being taken as given, in this alternative but equivalent concept what is taken as given is that certain strategies will lead into time inconsistency - and those strategies are to be ignored. The idea that players know which strategies lead into time consistency and ignore all such strategies is somewhat problematic, however, as acquiring knowledge of time inconsistency may be demanding and, anyway, it is not always clear that using a time inconsistent strategy will be worse for players than employing a time consistent one (as we shall see below in Section 3.4.1.).¹²

There are also further criticisms which one might make of the Jehiel concept. For example, it seems rather ad hoc to suppose that players can learn perfectly correctly what they have done in the past in periods out to the horizon in situations like the current one, and yet not be able to remember what payoffs followed from the end of the horizon. Jehiel has modelled situations in which players form a guess as to what will follow from the end of the horizon¹³, but it is not clear why the guess at that stage should be imperfectly accurate and yet the guesses as to play over the horizon be perfectly accurate¹⁴.

Furthermore, in finite games like that in Figure 3.1, the learning concept implies that we should conceive of this game as being repeated as a stage-game, perhaps infinitely. Why, then, does the horizon of foresight ever reach the end of the game, rather than encompassing the first few rounds of the next repetition of the stage game? Suppose that the horizon length is three, and that we have some five-round stage game that we are repeating indefinitely and learning about over time. The choices that we make in rounds 4 and 5 of the stage game will impact on the decisions made next time in rounds 2 onwards, and hence may make a difference to play in round 1. But why, if the foresight length is three rounds, can't we see round 1 of the next repetition of the stage game from round 4 of the previous stage game? If the counter to this is that the Jehiel concept is really only to be

¹²In Section 3.4.2 we shall use this idea of controlling future play to develop a refinement of the Jehiel concept in which players aim to control play as far ahead as possible, with as little knowledge of time consistency as possible, and take future play as given only from the point beyond which time consistency problems would arise.

¹³Jehiel (2001)

¹⁴Of course, if the guess about payoffs at the horizon is perfectly accurate then we no longer have a limited foresight game.

applied to infinitely repeated stage games anyway, then perhaps we do need an entirely new approach for finite games with limited foresight?

3.4 Refining the Jehiel Concept

For the games in which the Jehiel concept does represent a convincing general approach (infinitely repeated stage games), there are a number of potentially constructive refinements. As mentioned, Jehiel has already refined his concept by considering players who form a guess at the foresight horizon. In this section we shall investigate a different refinement which responds to Rubinstein's critique.

3.4.1 Motivating time consistency: The "Limited Foresight Equilibrium - Strategic Control Version" (LFE-SC)

Earlier we introduced the concept of a "time-consistent" LFE-N. We now offer a different motivation for such an equilibrium concept, which we shall hereafter refer to as an LFE-SC and treat as a separate equilibrium concept from an LFE-N. We shall see that although this concept meets Rubinstein's concern that agents should recognise their ability to control their own future play, it faces an existence problem.

We want to define a limited foresight equilibrium concept in which we have agents who are otherwise rational, and like any conventional game-player, except that they can only see a certain distance ahead. Like any conventional rational players, they are expected utility maximisers, and choose their strategy across the horizon so as to maximise their expected payoff. Now to find a situation where we are in equilibrium, we look at the set of strategies and expectations that our players have, and choose from among them.

First, to cut down the set, we choose the subset for which each player's expectations about the other player's expectations are correct. This is motivated by the usual Nash equilibrium considerations.

Then, to cut down the set even further (and finally) we choose the subset, from the subset left after the first cut-down, for which players do not change their minds. This is motivated by a certain interpretation of the meaning of a strategy in an extended-form game. If, in round t , an agent forms a strategy including what he is going to do in round $t+k$, we contend that this amounts to the claim that, in round t , he is able to define his behavioural strategy in round $t+k$. If, when he gets to round $t+k$, or in any intervening round, he changes what he is going to do in round $t+k$, then he had never

truly defined it in the first place.¹⁵ If he cannot define his strategy more than in the current round, we are reduced to the LFE-J concept, in which he guesses what he will do in later rounds without claiming to be able to control them from the current round.¹⁶

Definition 3.2 *We shall call a strategy profile, q , an LFE-SC if, and only if, for all players, all histories, all actions, and all time periods, it satisfies conditions (i.), (ii.), and (iia.) above.*

That is to say that, formally, an "LFE-SC" is just another name for a "time-consistent" LFE-N.

It is important to see how an LFE-SC differs from an LFE-CM. In both concept players choose what they shall do across their horizon of foresight. The difference is that in an LFE-CM players choose only from among those available strategies which do not lead to time-inconsistent play, while in an LFE-SC players choose from among all their strategies but are only in an *equilibrium* in cases where the strategies they adopt are not time-inconsistent.

Does it make sense to imagine a player who might choose a strategy he knows to be time-inconsistent? Consider the decision-problem in Figure 3.2.

Moves are given by letters. Stage payoffs are given by numbers in bold.

Proposition 3.4 *Suppose that $n = 2$ (i.e. the player can see one round ahead). Then the decision-problem in Figure 3.2 has no LFE-SC.*

Proof. From the root node, the player will be choosing between playing a , yielding a payoff of 2 and ending the game, or b , yielding a payoff of 1 but permitting the game to continue into the next round, in which he will plan to play a , yielding 2 and a total payoff of 3. So the dominant strategy in the opening round is (ba) . No other strategy can be optimal. But once the second round is reached, he will once again prefer (ba) , which is inconsistent with his previous plan to play a in the second round. Thus the only optimal strategy is not time-consistent, and no LFE-SC can exist.¹⁷ ■

¹⁵We can conceive of the process of defining what a player will do in a later round as being analogous to solving the problem solved by the FA in Figure 1.5 (see Appendix C).

¹⁶We can conceive of this as a situation analogous to solving the problem solved by the FA in Figure 2.2 (see Appendix C).

¹⁷Note that if a decision-problem-horizon is such that the player does not have to play again, then existence of LFE-SC is guaranteed (since no consistency problem can arise), and LFE-SC is in that case equivalent to LFE-J (since both are concerned only with optimizing over the current action).

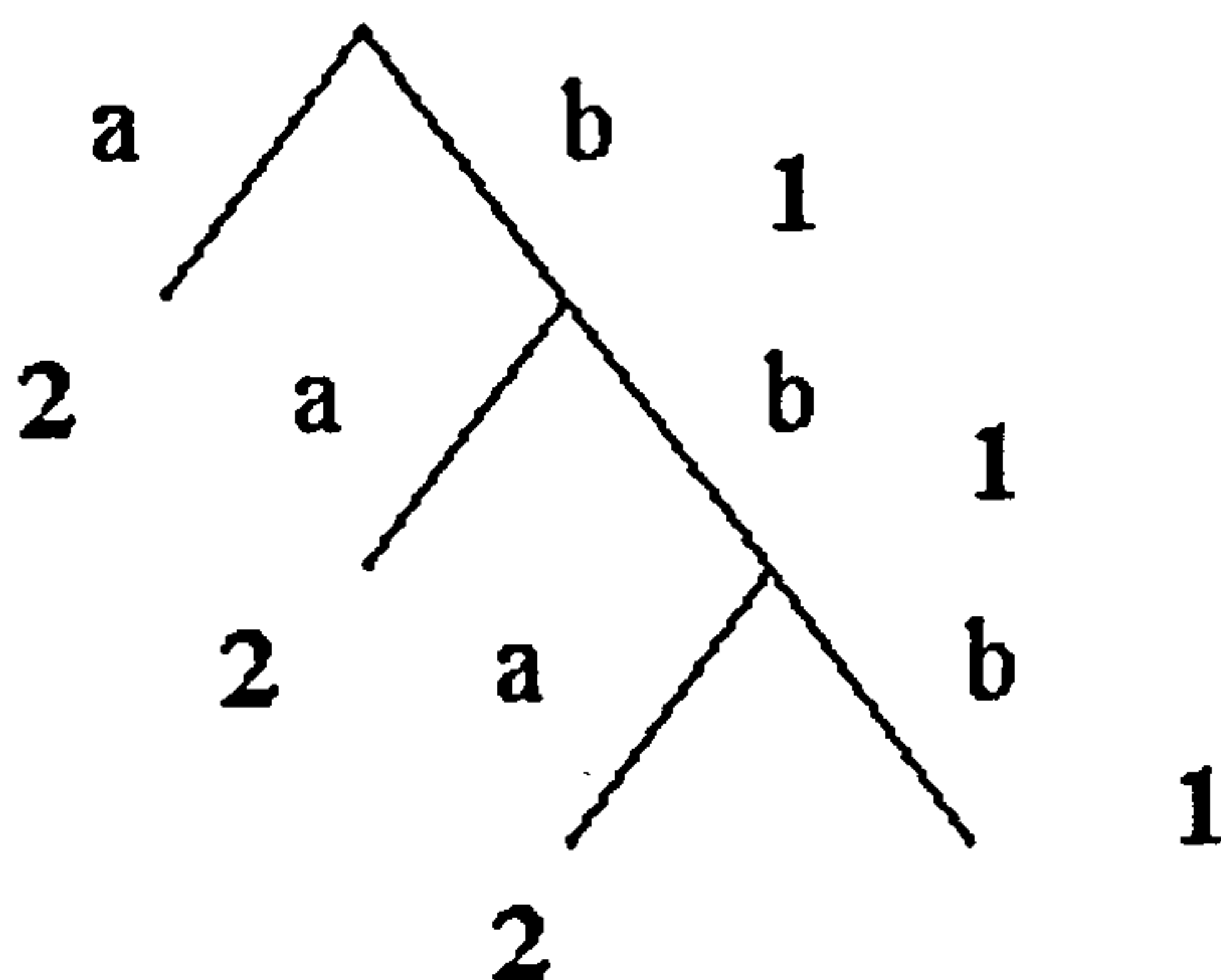


Figure 3.2:

There is, however, an LFE-CM. In that the player chooses (ab) in the opening round and (ba) in the second round. The strategy (ba) in the opening round is time-inconsistent and hence not available in an LFE-CM. This means that the player forced to adopt time-consistent play will finish with a payoff of 2, compared with the payoff of 4 obtained via the (time-inconsistent) dominant strategy (ba) at the opening node. Hence it is not at all clear that the best strategies to adopt will be found among the time-consistent ones, and the LFE-SC concept seems plausible. Nonetheless, it is a potentially serious weakness of the LFE-SC concept that the existence of an equilibrium is not guaranteed. Hence, although we have gone some way to addressing Rubinstein's concerns, we do not yet have a robust equilibrium concept. That is the topic of the next section.

3.4.2 The refinement. The "Limited Foresight Equilibrium - Refined Jehiel Version" (LFE-RJ)

We are now ready to offer a refinement of the Jehiel concept. Our refinement involves cutting down the set of Jehiel equilibria (i.e. producing a rule to select from that set) by assuming that when players have available time-consistent plans they choose these. We assume that players' foresight horizon n_i is broken down into the period over which they plan what to do, p_i (which we shall refer to as the "planning horizon"), and the period over which they merely have exogenous expectations of what they will do, $n_i - p_i$. For the

special case of $p_i = 1$ LFE-RJ and LFE-J will be equivalent, and, again, for the special case of $p_i = n_i$ LFE-RJ and LFE-SC are equivalent. We shall refer to the strategy the player adopts out to his planning horizon as his "p-strategy". The precise equilibrium conditions will be as follows:

We shall call a strategy profile, q , an LFE-RJ if, and only if, for all players, all histories, all actions, and all time periods,

- A. The p -strategy of each player in the current period is justified, given the other player's $\tilde{\cdot}$ -strategy, and an expectation as to what each player himself will do in later periods out to the foresight horizon. More precisely, controlling $(q_{i,t,t+p_i-1})_t$,

$$\tilde{\pi}_{i,t}((q_{i,t,t+p_i-1})_t, (q_{i,t+p_i,z_i})_t, \tilde{q}_{-i,t}, h^t) \geq \tilde{\pi}_{i,t}((q_{i,t,t+p_i-1})'_t, (q_{i,t+p_i,z_i})_t, \tilde{q}_{-i,t}, h^t), \forall i, t, h^t$$

- B. The strategies are consistent, in the sense that the expectations players form of their own and the other player's behavioural strategies out to the horizon of foresight prove correct. More precisely,

$$\tilde{q}_{-i,t} \equiv (q_{-i,t,z_i})_t = q_{-i,t,z_i}, \text{ and} \\ (q_{i,t+p_i,z_i})_t = q_{i,t+p_i,z_i}, \forall i, t$$

- C. Players do not change their minds over the planning horizon about what they intend to do. More precisely,

$$(q_{i,t+l,t+k})_t = (q_{i,t+l,t+k})_{t+m}, \forall i, t, \forall m, k \leq p_i - 1, \forall l \leq k$$

- D. p_i takes the maximum value sufficient to fulfill conditions A. to C. above, $\forall i, t$

Since LFE-RJ is equivalent to LFE-J in the case that $p_i = 1$, Condition D. is sufficient to guarantee existence of an LFE-RJ whenever an LFE-J exists¹⁸. Note also that, when an LFE-RJ includes $p_i > 1$, these will also be LFE-Js, as shown in Proposition 3.2.

Players in an LFE-RJ are trying to control the future as far ahead as they can¹⁹, and trying to form some guess as to what they will do beyond that point, out to some horizon beyond which they cannot see at all. This seems a very intuitive way to think about what players facing limited foresight actually do. We accept that we are really guessing even what we ourselves

¹⁸Compare this with the case of LFE-SC, for which existence is not always guaranteed, as, for example, in the case of the game in Figure 3.2 above.

¹⁹Appendix C argues that it is in their interests to do this.

are likely to do beyond some point ahead, but for at least some period ahead we expect to be able to decide now what we will do.

This refinement seems to respond fully to Rubinstein's criticism of the Jehiel concept. Players recognise their capacity to control their future play, face no artificial restrictions on their strategy space (the restrictions are all in the equilibrium conditions) and existence is assured.

Chapter 5 (joint with Philippe Jehiel) provides and investigates an example LFE-RJ.

3.5 Conclusion

We have investigated two approaches to modelling limited foresight. In one approach, players form an optimal plan of play over the foresight horizon, but can fall victim to time inconsistency. This seems most attractive as an equilibrium notion in one-player decision problems with players who are unaware that their foresight is limited. In multiple-player contexts it produces the serious problem that, for no obvious reason, players appear to know more about the future actions of other players than about their own future actions.

In the other approach, players form a guess about what they will do out to the foresight horizon, and optimize over their current round behaviour only. The suggestion that this runs contrary to the spirit of standard game theoretic models in which players think of themselves as controlling their play in later periods has been shown to be misguided, as this approach is shown to be formally equivalent to an approach in which players optimise plans over time-consistent strategies. The concept seems most convincing if applied to infinitely repeated stage games with learning. In finite games it raises problems of interpretation of the learning claim. In either context the concept raises problems of why play should be remembered perfectly but subsequent payoffs imperfectly. However, in general the Jehiel concept seems to raise fewer problems than the naive concept when applied to multiple-player infinite games. We have offered a refinement of the concept in which players attempt to control their actions over a planning horizon and thereafter form an expectation of what they will do out to a later foresight horizon.

This leaves finite multiple-player games for which neither approach really seems satisfactory. As Rubinstein says, modelling limited foresight in these games remains a significant challenge.²⁰

²⁰Perhaps, at least in contexts where learning might be incomplete, the bounded lookahead approach of the Artificial Intelligence literature might be fruitful in such cases? - see Jehiel (1998b)

Nonetheless, given that many practical problems involve essentially infinitely-repeated stage games in which the players are aware that the game will continue indefinitely beyond the horizon of foresight, the Jehiel concept and our refinement offer us an approach which may be fruitful for many important problems. In the next two chapters we apply these concepts to two such problems.

Chapter 4

Applications 1: Limited Foresight Monetary Policy Games

4.1 Introduction

In the first three chapters we have established that modelling limited foresight is worthwhile and identified for ourselves usable solution concepts. In the next two chapters we shall apply our concepts to solve economically interesting problems. Chapter 5 uses both the Jehiel and Refined Jehiel concepts to address the question of why some people take up smoking when young then give up when older. But first, in the current chapter we shall apply a variant of the Jehiel concept to the case of a Central Banker carrying out monetary policy. The main contribution of this chapter will be to apply the Jehiel concept in a monetary policy setting and show what difference the foresight of the central banker might make to the effectiveness of monetary policy.

We should emphasize immediately the difference between a central banker with limited foresight in an infinite horizon problem and a central banker facing a finite horizon decision. A central banker facing a finite horizon knows (or at least believes) that the world will not continue after the end of the horizon, and that what he will do tomorrow will not incorporate vision into periods further ahead than we can see today. In contrast, our central banker and economic agents in the economy in general recognize that the world will continue after the end of the central banker's foresight horizon. They also understand that, next period, the central banker will be able to see further into our current future than we can today, and that this may affect his

decision-making. What difference this makes to the central banker's decision will be illustrated below. It is worth noting that if a central banker believes his horizon is finite (e.g. if he believes that he will retire next period and doesn't care what will happen after that) but is *mistaken*, the finite horizon solution we shall illustrate here will be like the "naive" limited foresight solutions we investigated in Chapter 3. From our previous discussion we should expect that the Jehiel solution will be different - and it is. We note also from our previous argument that our prior is that the Jehiel solution is probably to be preferred.

The Bank of England targets inflation on a rolling two-year-ahead basis. This structure offers many of the apparent features of a Jehiel setting with an asymmetry in horizon whereby the private sector may have Rational Expectations but the Bank has a two-year limited foresight horizon. If we take the Svensson (1997c) interpretation of inflation targeting, whereby the (limited horizon) inflation forecast is what is targeted, the Jehiel setting appears an even stronger candidate.

The current two-year Bank of England horizon appears to arise largely out of a technical judgement as to how far ahead it is feasible to forecast with any pretence to accuracy. An interesting question is whether, in the future, if forecasting techniques improve and a longer horizon is possible, we should expect the Bank's decision-making to improve or deteriorate.

For our macroeconomic setting, we shall use an environment with persistence in employment, as introduced in Lockwood & Philippopoulos (1994), and explored in Svensson (1997a). We shall start by considering the Lockwood & Philippopoulos (1994) game, and show that even when both the Union and the Bank have limited foresight, the strategy of the Union player in this game is fully characterized by his expectation of inflation, with such expectations being Rational in equilibrium. This will allow us to treat the problem as a decision-problem for a limited-foresight central bank, and we move on to reproduce Svensson's basic inflation-targeting-without-commitment result, then show how the result changes when the central banker's foresight is limited.

As we shall see, the inflation-bias is less under limited foresight than under infinite foresight (perhaps partially explaining the apparent absence of an inflation-bias in what Svensson (1999) calls "real world monetary policy"). We shall examine how the result changes if the central banker's foresight length is increased. We shall find that, perhaps unexpectedly, the inflation-bias increases with an increase in foresight length.

We shall then contrast a limited foresight banker in an infinite setting with a finite horizon banker (which we interpret as modelling a central banker on a fixed-term contract). We shall see that a limited foresight banker exhibits

a higher inflation bias than a finite horizon banker.

4.2 The Game

In the Lockwood & Philippopoulos (1994) game, there are two players, a Union and a Bank, who interact over an infinite number of periods $t = 1, 2, \dots$. The Union sets nominal wage W_t in any time period t . The central bank picks the price-level P_t in any time period t . The level of employment in period t depends on the real wage as $L_t = (\frac{W_t}{P_t})^{-\alpha}$ or, in logs,

$$\hat{l}_t = \alpha(p_t - w_t) \quad (4.1)$$

where

$$\hat{l}_t = \ln(L_t)$$

$$p_t = \ln(P_t)$$

$$w_t = \ln(W_t)$$

Persistence in employment arises because of the insider-outsider nature of Union preferences - in particular its favouring of those already in employment. The Union has an employment target, L_t^u which is the geometric mean of those insiders who have been recently employed, L_{t-1} , and the total labour force N . So $L_t^u = L_{t-1}^\rho N^{1-\rho}$ where $0 \leq \rho \leq 1$, or, in logs,

$$\hat{l}_t^u = \rho \hat{l}_{t-1} + (1 - \rho)n \quad (4.2)$$

for $t = 1, \dots$ and initial employment, \hat{l}_0 , predetermined.

The Union aims to get as close as possible to its employment target each period, so its preferences can be described by

$$-\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} (\hat{l}_t^u - \hat{l}_t)^2 \quad (4.3)$$

where $0 < \beta < 1$ is the discount factor.

The central Bank has an employment target \hat{l}^* and an inflation target of π^* . Its payoffs are given by

$$-\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \left(\lambda (\hat{l}_t - \hat{l}^*)^2 + (\pi_t - \pi^*)^2 \right) \quad (4.4)$$

where $\pi_t = p_t - p_{t-1}$ is the inflation rate, and $\lambda > 0$ is the relative weight that the Bank ascribes to the employment target.

The order of events is: the Union chooses a nominal wage level, w_t , then the Bank chooses a price-level, p_t . Thus policy is discretionary.

Lockwood & Phillipopoulos (1994) derive the linear Markov-perfect equilibria of this game¹. Our purpose in constructing this game is to display that in a Jehiel limited foresight equilibrium of this game, the Union will exhibit Rational Expectations, and to show that a limited foresight version of the game can be represented as a decision-problem for a limited foresight banker facing a Phillips curve with persistence in employment and a Rational Expectations condition.

Jehiel's (1995) (n_1, n_2) -equilibrium concept was applied to repeated alternate-move 2×2 games. However, Chapter 3 has generalised the Jehiel concept, and we shall use the "LFE-J" conditions as specified there. Call the foresight length z_i for players $i = (\text{Union}, \text{Bank})$. Assume that $z_{\text{Union}} = z_{\text{Bank}} = z$.² For our purposes, in a Jehiel equilibrium, each player makes a decision in period t based on his expectations about his own play from periods $t+1$ out to the horizon of foresight, period $t+z-1$, and on the other players' play between periods t and $t+z-1$. In equilibrium, this decision will be optimal for each player within his available action-set, and all expectations of future play will prove correct.

Now consider the Union's play in the infinite-horizon setting. It aims to minimize the deviance of actual employment from the target. Since actual employment is given by $\hat{l}_t = \alpha(p_t - w_t)$, this will be achieved by setting w_t such that $\hat{l}_t^u = \alpha(p_t^e - w_t)$, where p_t^e is the Union's expectation of the price level in period t - i.e. its expectation of the play of the Bank player. In equilibrium this must be correct, so that $p_t^e = p_t$ and hence $\pi_t^e = \pi_t$ (where π_t^e , as expected, represents the Union's expectation of the inflation rate). Since the Union player always achieves his global maximum utility, his expectation of inflation fully characterises his play and defines for us an employment level.

Next consider the limited foresight game. Since, for any period t , in a Jehiel equilibrium the expectations of play between t and $t+z-1$ must be correct, it must again be the case that $p_t^e = p_t$ and hence $\pi_t^e = \pi_t$ for all t . Once again, the Union player achieves a global maximum utility, and his expectation of inflation fully characterises his play.

Since $\hat{l}_t = \alpha(p_t - w_t)$ and $\hat{l}_t^u = \alpha(p_t^e - w_t)$ we have

¹i.e. a sub-game perfect equilibrium in which punishment strategies are excluded by restricting the agents' actions to depend on the past only through the state variable, l_{t-1} .

²We could perhaps interpret this as meaning that the Union gets its forecast of future inflation rates from the same, bank-independent, government agency that supplies the bank.

$$\widehat{l}_t - \widehat{l}_t^u = \alpha (\pi_t - \pi_t^e)$$

Hence

$$\widehat{l}_t - n = (\widehat{l}_t - \widehat{l}_t^u) + (\widehat{l}_t^u - n) = \alpha (\pi_t - \pi_t^e) + \rho (\widehat{l}_{t-1} - n)$$

Since $\widehat{l}_t = \ln(L_t)$ and $n = \ln(N)$, $\widehat{l}_t - n = \ln(\frac{L_t}{N})$, i.e. the proportion of actual employment in full employment. Call this l_t . Then we have a Phillips curve with persistence in employment,

$$l_t = \rho l_{t-1} + \alpha (\pi_t - \pi_t^e) \quad (4.5)$$

Hence the problem can be reduced, in the limited foresight as in the infinite foresight case, to a decision-problem for the Bank where it maximises its payoffs subject to a Phillips curve with persistence and a Rational Expectations condition.

Before solving this, we shall use Svensson's framework to incorporate a stochastic element.

4.3 The Stochastic Version

Our stochastic treatment of inflation targeting under persistence follows Svensson (1997a) - and we shall re-parameterize. The short-run Phillips curve becomes

$$l_t = \rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \epsilon_t \quad (4.6)$$

where l_t is now the (log of) the share of actual employment in full employment (the "employment rate") in period t , α and ρ are constants ($\alpha > 0$ and $0 \leq \rho < 1$), $\pi_t = p_t - p_{t-1}$ is the (log of the gross) inflation rate, p_t is the (log) price level, $\pi_{t|t-1}$ denotes inflation expectations in period $t-1$ of the inflation rate in period t , and ϵ_t is an iid. temporary supply shock with mean zero and variance σ^2 .

From our conclusions above, we can assume that the Union has Rational expectations:

$$\pi_{t|t-1} = E_{t-1} \pi_t \quad (4.7)$$

where E_{t-1} denotes expectations conditional upon information available in period $t - 1$, which includes the realisation of all variables up to and including period $t - 1$, as well as the constant parameters of the model.

These two equations represent the constraints facing the Bank. Its objectives are given by an intertemporal loss function

$$E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} B_{\tau} \right]$$

with the period loss function

$$B_t = \frac{1}{2} [(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2]$$

where $\lambda > 0$ is the relative weight on employment stabilization. π^* is the inflation target (say, the Bank of England's 2.5%) and l^* is some socially-desirable employment rate. Since inflation-targeting authorities like the Bank of England usually have some target band (e.g. the $\pm 1\%$ band around the Bank of England's 2.5% target), or some (perhaps implicit) timescale over which to achieve the inflation target, it seems reasonable to regard the Central Bank as having dual objectives (as reflected in our period loss function) so long as it remains within the target band, and this also fits within the spirit of regarding inflation targeting as a framework of constrained discretion³.

For simplicity, we assume that the Central Bank has perfect control over the inflation rate π_t . It sets the inflation rate in each period after having observed the current supply shock ϵ_t . Although the current supply shock is observed both by the Central Bank and the Union, the assumption behind our Phillips curve (that some prices or wages are set in advance and predetermined by previous expectations) makes monetary policy effective.

4.4 The infinite horizon, infinite foresight case

Still following Svensson (1997a), we write the decision-problem of the central bank as

$$V(l_{t-1}) = E_{t-1} \min_{\pi_t} \left\{ \frac{1}{2} [(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2] + \beta \widehat{V}(l_t) \right\} \quad (4.8)$$

where the minimisation in period t is subject to the Phillips curve and takes inflationary expectations $\pi_{t|t-1}$ as given. The central bank does not

³See Bernanke et al. (1999)

internalize the effect of its decisions on inflation expectations, but it does take into account that changes in the current employment rate will affect current Union expectations of future inflation (which are incorporated in $V(l_t)$). The indirect loss function can be written as

$$V(l_{t-1}) = \gamma_0 + \gamma_1 l_{t-1} + \frac{1}{2} \gamma_2 l_{t-1}^2$$

Svensson (1997a) derives the following equilibrium⁴ decision-rule in the case of discretion:

$$\pi_t = a - b\varepsilon_t - cl_{t-1}$$

where the constants are given by

$$a = \pi^* + \lambda\alpha l^* - \beta\alpha\gamma_1 = \pi^* + \frac{\lambda\alpha l^*}{1 - \beta\rho - \beta\alpha c}$$

$$b = \frac{(\lambda + \beta\gamma_2)\alpha}{1 + (\lambda + \beta\gamma_2)\alpha^2} = \frac{\lambda\alpha + \beta\alpha c^2}{1 + \lambda\alpha^2 - \beta\rho^2 + \beta\alpha^2 c^2}$$

$$c = \frac{1}{2\alpha\beta\rho} \left[1 - \beta\rho^2 - \sqrt{(1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2} \right] \geq 0$$

Both l_{t-1} and ε_t here are state-dependent variables. Under an unlimited foresight there will be a non-state dependent inflation bias⁵ of

$$a - \pi^* = \frac{\lambda\alpha l^*}{1 - \beta\rho - \beta\alpha c} \quad (4.9)$$

4.5 The limited foresight case

To explore the limited foresight case, we shall initially assume that the Central Banker can look ahead only two periods. He has an expectation of his own play (which he takes as given) in the following period, and uses that and the expected loss from the two periods within the foresight horizon in deciding what to do each period. In a Jehiel equilibrium the Central Bank's expectation of his own play out to the foresight horizon will prove to be correct. The Union still has Rational expectations, as shown above.

⁴That is to say, a Markov-perfect equilibrium where trigger strategies are not allowed and actions depend on history only via the lagged state variable, l_{t-1} .

⁵referred to by Svensson (1997a) as an "average inflation bias"

Thus, in any period t the Central Bank decision problem is given by

$$\min_{\pi_t} \widehat{V}_t = \frac{1}{2} [(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2] + \beta E_t \left(\frac{1}{2} [(\pi_{t+1}^B - \pi^*)^2 + \lambda (l_{t+1} - l^*)^2] \right) \quad (4.10)$$

(where π_{t+1}^B is the Central Bank's expectation of its own play in period $t+1$, and arises out of a decision-rule for period $t+1$ which is taken as given in period t)

subject to the Phillips curve and the Rational expectations conditions.

The decision-rule used in $t+1$ is taken as given in period t . As in the infinite-horizon case, we shall take it that the optimal decision-rule will depend on past history only through l_{t-1} . Thus we postulate a form

$$\pi_t^B = \widehat{a}^B - \widehat{b}^B \varepsilon_t - \widehat{c}^B l_{t-1}$$

$$\frac{\partial \pi_{t+1}^B}{\partial l_t} = -\widehat{c}^B \implies (\text{from the Phillips curve}) \frac{\partial \pi_{t+1}^B}{\partial \pi_t} = -\widehat{c}^B \alpha$$

This problem has the first-order condition

$$\begin{aligned} & (\pi_t - \pi^*) + \lambda (l_t - l^*) \frac{\partial l_t}{\partial \pi_t} + \beta E_t \left(\lambda (l_{t+1} - l^*) \frac{\partial l_{t+1}}{\partial \pi_t} + (\pi_{t+1}^B - \pi^*) \frac{\partial \pi_{t+1}^B}{\partial \pi_t} \right) \\ = & \pi_t - \pi^* + \lambda \alpha (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t - l^*) \\ & + \beta (\lambda \rho \alpha (\rho^2 l_{t-1} + \rho \alpha (\pi_t - \pi_{t|t-1}) + \rho \varepsilon_t - l^*) - \widehat{c}^B \alpha (\widehat{a}^B - \widehat{c}^B l_t - \pi^*)) \\ = & \pi_t - \pi^* + \lambda \alpha (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t - l^*) \\ & + \beta \lambda \rho \alpha (\rho^2 l_{t-1} + \rho \alpha (\pi_t - \pi_{t|t-1}) + \rho \varepsilon_t - l^*) \\ & - \beta \widehat{c}^B \alpha (\widehat{a}^B - \widehat{c}^B (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t) - \pi^*) \\ = & 0 \end{aligned}$$

Taking expectations over the first-order condition, and adding our Rational expectations condition, so as to derive $\pi_{t|t-1}$, we obtain

$$\pi_{t|t-1} = E_{t-1}(\pi_t) = (1 - \beta \widehat{c}^B \alpha) \pi^* - (\lambda \alpha \rho + \beta \lambda \alpha \rho^3 + \beta (\widehat{c}^B)^2 \alpha \rho) l_{t-1} + \lambda \alpha (1 + \beta \rho) l^*$$

Putting this back into our first-order condition, we obtain

$$(\pi_t - \pi_{t|t-1})(1 + \lambda\alpha^2(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha^2) + (\lambda\alpha(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha)\varepsilon_t = 0$$

Hence

$$\pi_t = (1 - \beta\widehat{c}^B\alpha)\pi^* + \beta\widehat{c}^B\alpha\widehat{a}^B + \lambda\alpha(1 + \beta\rho)l^* - \left(\lambda\alpha\rho(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha\rho\right)l_{t-1} \\ - \frac{\lambda\alpha(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha}{1 + \lambda\alpha^2(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha^2}\varepsilon_t$$

This leaves the unknowns \widehat{a}^B and \widehat{c}^B . However, we know that π_t takes the form

$$\pi_t = \widehat{a}^B - \widehat{b}^B\varepsilon_t - \widehat{c}^B l_{t-1}$$

Hence

$$\widehat{a}^B = (1 - \beta\widehat{c}^B\alpha)\pi^* + \beta\widehat{c}^B\alpha\widehat{a}^B + \lambda\alpha(1 + \beta\rho)l^*$$

whence

$$\widehat{a}^B = \pi^* + \frac{\lambda\alpha(1 + \beta\rho)}{1 - \beta\alpha\widehat{c}^B}l^*$$

(cf. a above.)

Similarly,

$$\widehat{c}^B = \left(\lambda\alpha\rho(1 + \beta\rho^2) + \beta(\widehat{c}^B)^2\alpha\rho\right)$$

Whence

$$\widehat{c}^B = \frac{1}{2\alpha\beta\rho} \left(1 - \sqrt{1 - 4\beta\lambda\alpha^2\rho^2(1 + \beta\rho^2)}\right)$$

(taking the negative root for consistency with Svensson. cf. c above.)

Finally,

$$\widehat{b}^B = \frac{\lambda\alpha(1 + \beta\rho^2) + \beta\alpha(\widehat{c}^B)^2}{1 + \lambda\alpha^2(1 + \beta\rho^2) + \beta\alpha^2(\widehat{c}^B)^2}$$

(cf. b above.)

Now the average inflation-bias is given by

$$\frac{\lambda\alpha(1 + \beta\rho)}{1 - \beta\alpha\widehat{c}^B} l^*$$

To complete the characterisation of equilibrium, we need to state the Central Bank's expectations of its own play.

$$\pi_{t+1}^b = \widehat{a}^B - \widehat{b}^B \varepsilon_{t+1} - \widehat{c}^B \left(\rho l_{t-1} + \alpha(\widehat{a}^B - \widehat{b}^B \varepsilon_t - \widehat{c}^B l_{t-1} - \pi_{t|t-1}) + \varepsilon_t \right)$$

This then completes the Jehiel equilibrium.

In Appendix E it is proved that, whenever the Rational inflation bias is positive, the average inflation bias is less in the limited foresight case, i.e.

$$\text{Rational } \pi\text{-bias} > \pi\text{-bias under limited foresight}$$

4.5.1 Simulations

We have seen the effect of limited foresight on the *average* inflation bias, but it may not be clear what is the effect on the pattern. To illustrate that effect, simulations were performed over a hundred periods with shocks of standard deviation 0.1. Figures 4.1 to 4.4 illustrate the effect for the case in which variables values are as in the following table⁶:

α	1
λ	1
π^*	0
l^*	0.1
l_0	0

Figures 4.1 and 4.2 take the case of $\beta = \rho = 0.5$.

Note that inflation is always lower in the limited foresight case. Unemployment is almost unaffected by the limited foresight. As one would

⁶Note that these numbers have been chosen arbitrarily for illustrative purposes, and involve no claim of empirical relevance.

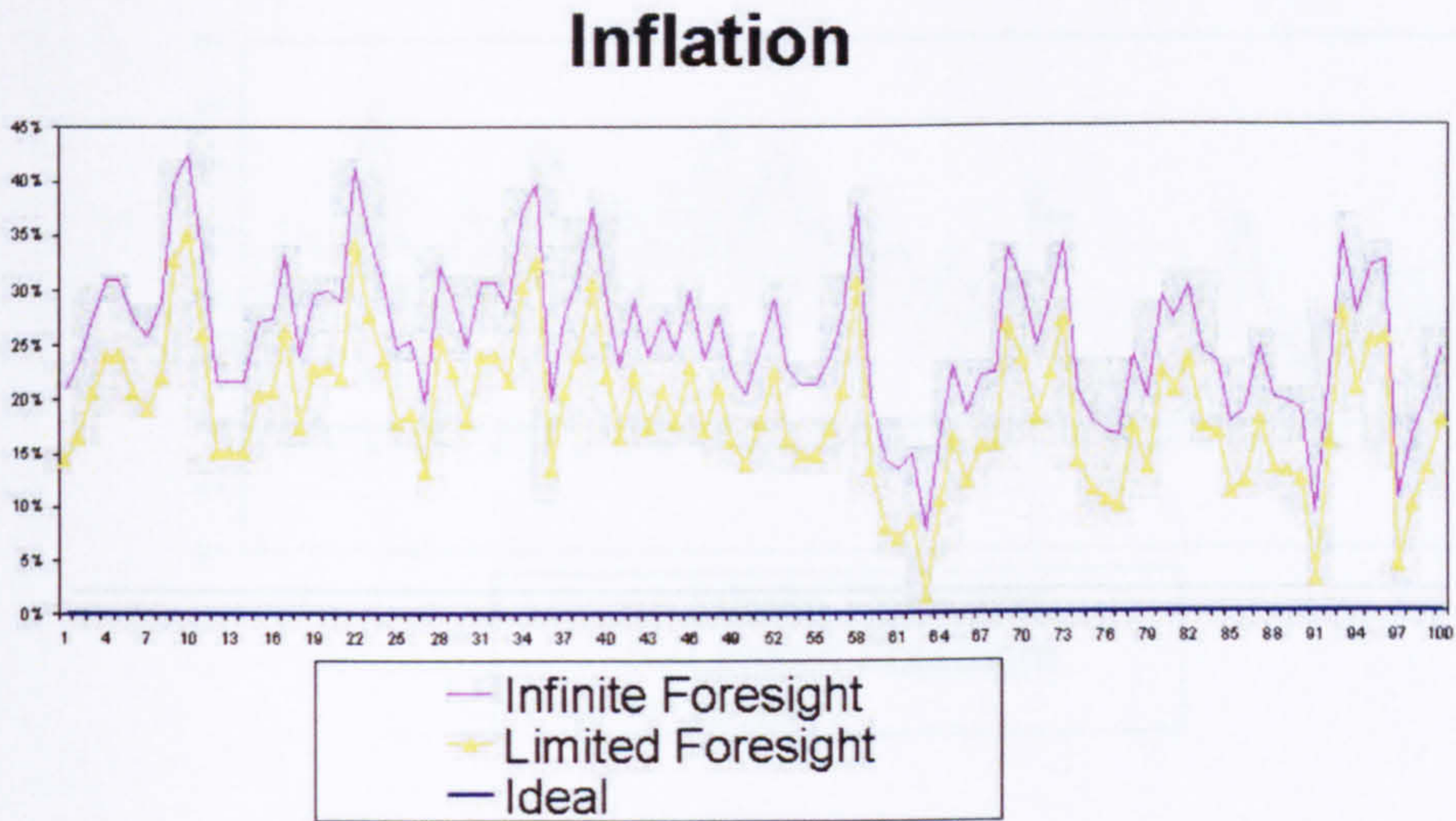


Figure 4.1:

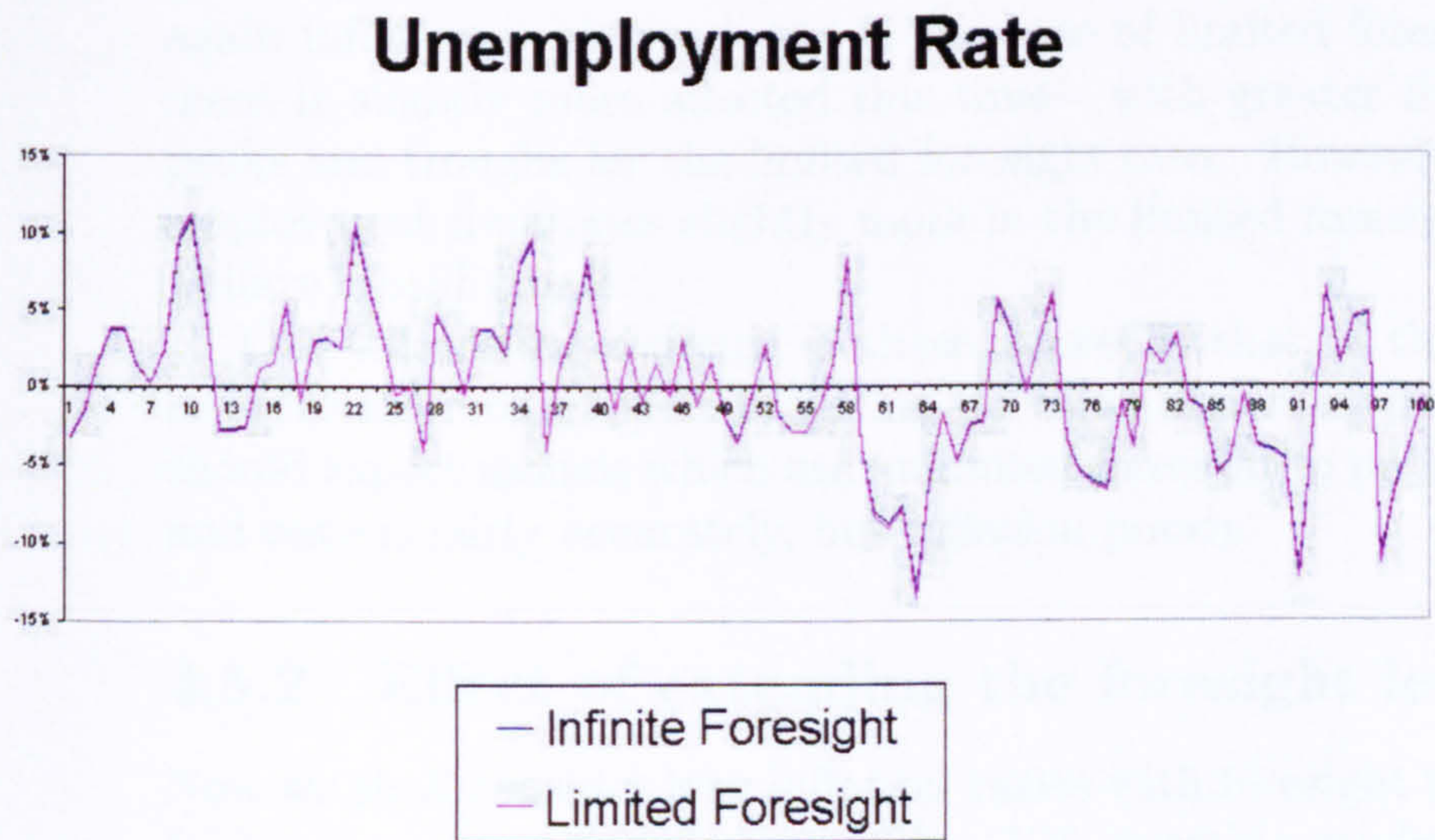


Figure 4.2:

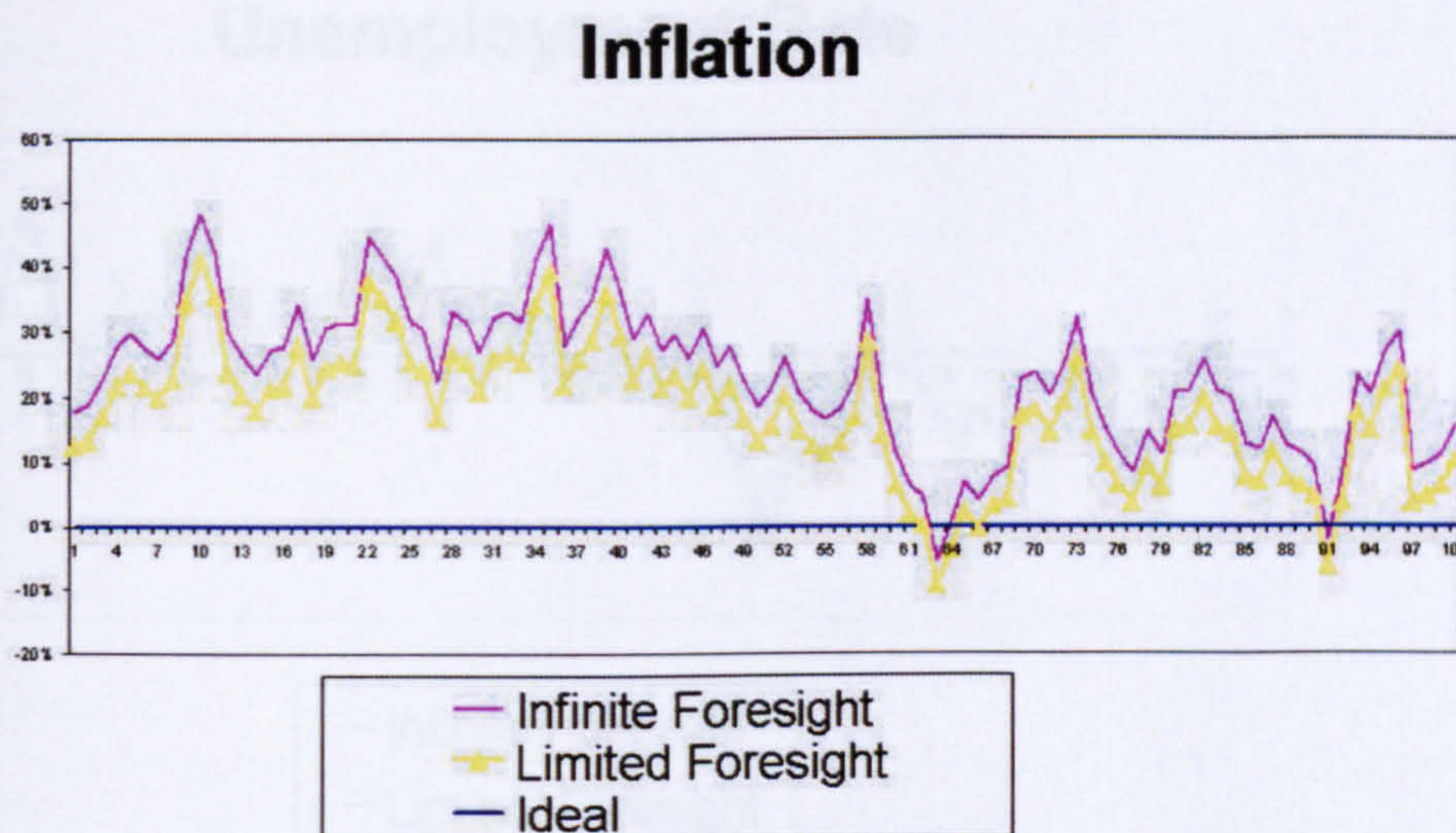


Figure 4.3:

expect, this results in limited foresight improving welfare (the aggregate loss is reduced).

Figures 4.3 and 4.4 illustrate the case where $\beta = 0.25$ and $\rho = 0.8$ (i.e. greater persistence in output, and rather higher discounting). In this case again inflation is always lower in the case of limited foresight. Unemployment is slightly more affected this time - with greater fluctuation between peaks and troughs for the limited foresight case. However, even though unemployment fluctuates slightly more in the limited foresight case, aggregate welfare is still greater.

One interesting message of these figures is that, if the limited foresight interpretation of inflation targeting (or other monetary policy) is correct, we should expect models which use unlimited foresight to predict unemployment and output fairly accurately, but inflation poorly.

4.5.2 Effect of extending the foresight length

Now we shall consider how inflation varies with foresight length. This has a fairly clear economic interpretation. If forecasting models were to improve so that, for example, the Bank of England started to base its interest-rate policy on a three-year projection for inflation rather than merely a two-year projection, what effect would that have on inflation.

For a foresight length of three (i.e. adding one round) we obtain an

Unemployment Rate

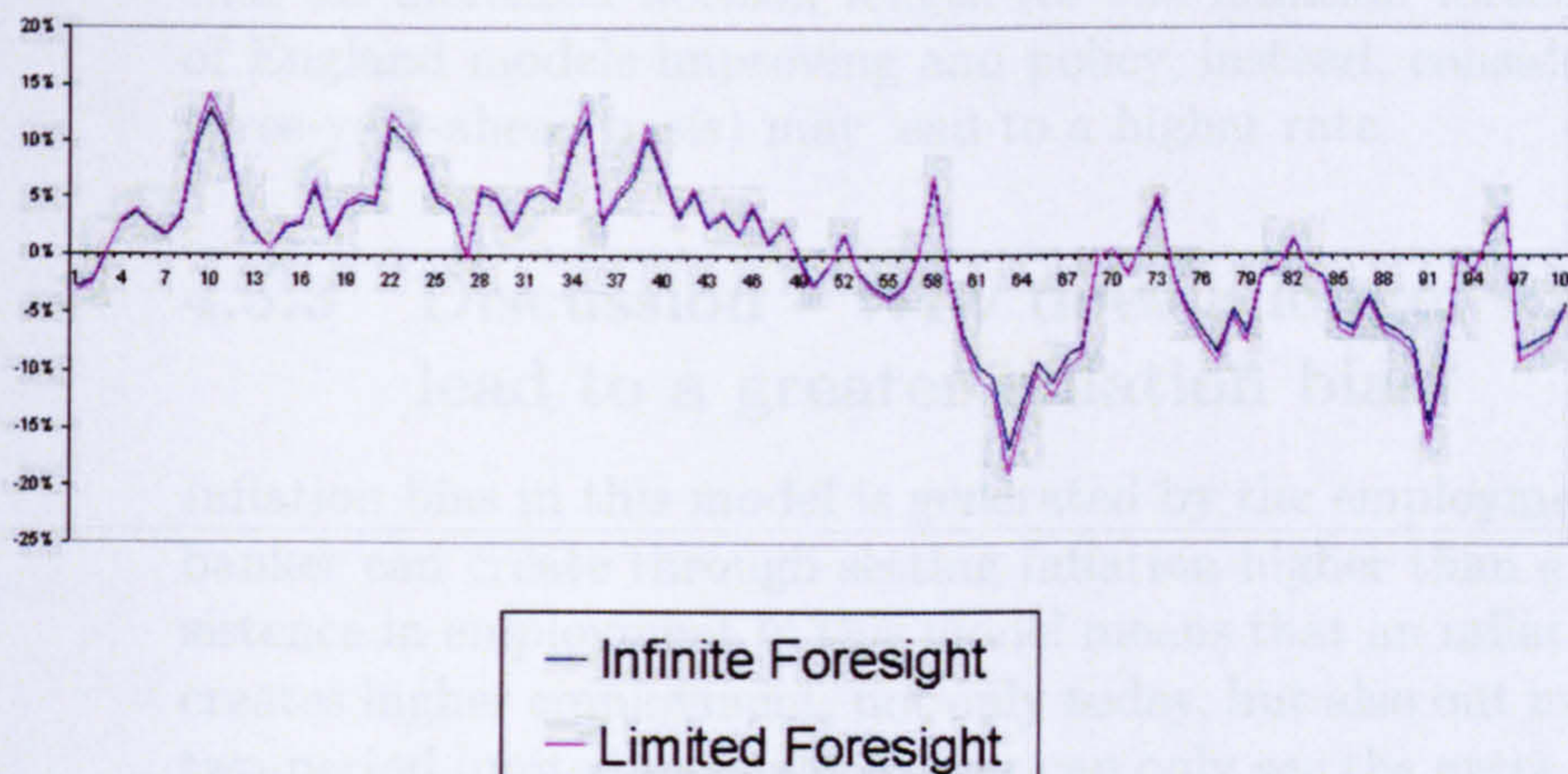


Figure 4.4:

average inflation bias of

$$\frac{\lambda\alpha(1 + \beta\rho + \beta^2\rho^2)l^*}{1 - \beta(\hat{c}_{n=3}^B)\alpha(1 + \beta\rho)} > 0$$

with

$$\hat{c}_{n=3}^B = \frac{1}{2\alpha\beta\rho} \left(\frac{1}{1 + \beta\rho^2} - \sqrt{\left(\frac{1}{1 + \beta\rho^2}\right)^2 - 4\beta\lambda\alpha^2\rho^2 \frac{(1 + \beta\rho^2 + \beta^2\rho^4)}{1 + \beta\rho^2}} \right)$$

Both the inflation bias and $\hat{c}_{n=k}^B$ tend towards the infinite horizon case as k tends towards infinity.

As expected, $\hat{c}_{n=3}^B > \hat{c}^B$. Hence

$$\frac{\lambda\alpha(1 + \beta\rho + \beta^2\rho^2)l^*}{1 - \beta(\hat{c}_{n=3}^B)\alpha(1 + \beta\rho)} > \frac{\lambda\alpha(1 + \beta\rho + \beta^2\rho^2)l^*}{1 - \beta\alpha\hat{c}^B(1 + \beta\rho)}$$

But

$$\frac{\lambda\alpha(1 + \beta\rho + \beta^2\rho^2)l^*}{1 - \beta\alpha\hat{c}^B(1 + \beta\rho)} > \frac{\lambda\alpha(1 + \beta\rho)l^*}{1 - \beta\alpha\hat{c}^B}$$

Hence, perhaps paradoxically, increasing the foresight length tends to lead to a greater average inflation bias. The implication of this result is that an increased horizon length for the inflation forecast (e.g. the Bank of England models improving and policy, instead, considering inflation on a three-year-ahead basis) may lead to a higher rate.

4.5.3 Discussion - Why does a longer foresight length lead to a greater inflation bias?

Inflation bias in this model is generated by the employment gain the central banker can create through setting inflation higher than expected. The persistence in employment in this model means that an inflationary shock today creates higher employment, not only today, but also out into the future. The two-period limited foresight banker can only see the extra employment in the current period and the next period. If we extend his horizon, he can see more of the employment gain from an inflation shock, so creating an inflation shock would give him a greater gain, hence the expectations equilibrium level of inflation is higher.

4.6 The Finite Horizon Case

Next we shall illustrate that a central banker with limited foresight is importantly different from a central banker with a finite horizon. We shall model a central banker with a horizon one period ahead. Perhaps we could interpret this as a central banker who will retire after two periods, and who does not care what happens to the economy after that.

Now our Central Bank faces the decision-problem in any period t given by:

$$\min_{\pi_t, \pi_{t+1}} \tilde{V}_t = \frac{1}{2} [(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2] + \beta E_t \left(\frac{1}{2} [(\pi_{t+1} - \pi^*)^2 + \lambda (l_{t+1} - l^*)^2] \right) \quad (4.11)$$

Solve this problem by backwards induction. First consider the Central Bank's problem in round $t + 1$. This will be

$$\min_{\pi_{t+1}} \tilde{V}_{t+1} = \frac{1}{2} [(\pi_{t+1} - \pi^*)^2 + \lambda (l_{t+1} - l^*)^2] \quad (4.12)$$

subject to the Phillips Curve and the Rational Expectations conditions. From the first-order condition of this problem we obtain

$$(\pi_{t+1} - \pi^*) + \lambda (l_{t+1} - l^*) \frac{\partial l_{t+1}}{\partial \pi_t} = 0$$

whence we derive

$$\begin{aligned} \pi_{t+1} &= \pi^* - \lambda \alpha \rho l_t + \lambda \alpha l^* - \frac{\lambda \alpha \varepsilon_{t+1}}{1 + \lambda \alpha^2} \\ &= \pi^* - \lambda \alpha \rho (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t) + \lambda \alpha l^* - \frac{\lambda \alpha \varepsilon_{t+1}}{1 + \lambda \alpha^2} \end{aligned}$$

This then generates π_{t+1} as a function of π_t for our round t decision-problem. That then becomes

$$\min_{\pi_t} \tilde{V}_t = \frac{1}{2} [(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2] + \beta E_t \left(\frac{1}{2} [(\pi_{t+1} - \pi^*)^2 + \lambda (l_{t+1} - l^*)^2] \right)$$

subject to the Phillips Curve, the Rational Expectations condition, and the π_{t+1} condition.

This problem has the first-order condition

$$\begin{aligned} &(\pi_t - \pi^*) + \lambda (l_t - l^*) \frac{\partial l_t}{\partial \pi_t} + \beta E_t \left((\pi_{t+1} - \pi^*) \frac{\partial \pi_{t+1}}{\partial \pi_t} + \lambda (l_{t+1} - l^*) \frac{\partial l_{t+1}}{\partial \pi_t} \right) \\ &= \pi_t - \pi^* + \lambda \alpha (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t - l^*) \\ &\quad + \beta \left(\begin{array}{l} (-\lambda \alpha \rho (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t) + \lambda \alpha l^*) (-\lambda \alpha^2 \rho) \\ + \lambda (\rho (\rho l_{t-1} + \alpha (\pi_t - \pi_{t|t-1}) + \varepsilon_t) - l^*) (\rho \alpha) \end{array} \right) \\ &= \pi_t - \pi^* \\ &\quad + (\lambda \alpha \rho (1 + \beta \rho^2 (1 + \lambda \alpha^2))) l_{t-1} \\ &\quad - (\lambda \alpha (1 + \beta \rho (1 + \lambda \alpha^2))) l^* \\ &\quad + (\lambda \alpha^2 (1 + \beta \rho^2 (1 + \lambda \alpha^2))) (\pi_t - \pi_{t|t-1}) \\ &\quad + (\lambda \alpha (1 + \beta \rho^2 (1 + \lambda \alpha^2))) \varepsilon_t \\ &= 0 \end{aligned}$$

Hence

$$\pi_{t|t-1} = E_{t-1}(\pi_t) = \pi^* - (\lambda \alpha \rho (1 + \beta \rho^2 (1 + \lambda \alpha^2))) l_{t-1} + (\lambda \alpha (1 + \beta \rho^2 (1 + \lambda \alpha^2))) l^*$$

and

$$(\pi_t - \pi_{t|t-1})(1 + \lambda\alpha^2(1 + \beta\rho^2(1 + \lambda\alpha^2))) + (\lambda\alpha(1 + \beta\rho^2(1 + \lambda\alpha^2)))\varepsilon_t = 0$$

so

$$\pi_t = \pi^* - (\lambda\alpha\rho(1 + \beta\rho^2(1 + \lambda\alpha^2)))l_{t-1} + (\lambda\alpha(1 + \beta\rho(1 + \lambda\alpha^2)))l^* - \frac{\lambda\alpha(1 + \beta\rho^2(1 + \lambda\alpha^2))}{1 + \lambda\alpha^2(1 + \beta\rho^2(1 + \lambda\alpha^2))}\varepsilon_t$$

so in this case, the non-state dependent inflation bias in round t is given by

$$\lambda\alpha(1 + \beta\rho(1 + \lambda\alpha^2))l^*$$

and in round $t + 1$ by

$$\lambda\alpha l^*$$

4.6.1 Comparing the inflation bias in the limited foresight and finite horizon cases

In Appendix E it is proved that the inflation bias in the limited foresight case is greater than that in the finite horizon case, i.e.

Limited Foresight Bias > Finite Horizon Bias

Notice that the limited foresight banker problem is stationary (up to the state variable), whereas the finite horizon Banker problem evolves through time.

We should also note that increasing the central banker's term of office leads to an increase in inflation in period t . If we could hire a new central banker every round, then in this model the inflation bias would be stable at $\lambda\alpha l^*$. By extending the term to two periods, the period t bias increases to $\lambda\alpha(1 + \beta\rho(1 + \lambda\alpha^2))l^* > \lambda\alpha l^*$ (since $\beta, \rho, \lambda, \alpha > 0$).

Since a finite horizon banker has a lower inflation bias than a limited foresight banker, and a limited foresight banker a lower bias than an infinite horizon banker, we can conclude that a finite horizon banker will also exhibit a lower inflation bias (in this setting with persistence in employment) than an infinite-horizon banker.

4.7 Conclusion

In this chapter we have compared the Lockwod & Philippopoulos (1994) and Svensson (1997a) results for a central banker setting inflation, under an environment of persistence in employment, those which arise when a central banker has limited foresight in an infinite setting and those arising for a central banker with a finite horizon. Our main purpose has been to show how to use the Jehiel limited foresight concept in a monetary policy setting.

Our main results have been that under limited foresight private-sector agents will still exhibit Rational Expectations, and that the equilibrium inflation-bias for a limited foresight central banker will be lower than that for a banker with infinite-foresight. This may partially explain the lack of observed inflation bias. We have shown that increasing the foresight length (perhaps by extending inflation forecasts) would be expected to lead to a higher inflation bias. The intuition here is as follows: inflation bias arises in this model because of the potential gain in employment through creating surprise inflation. The persistence in the model means that this potential gain in employment decays gradually away forever. But a limited foresight banker can only see a truncated section of this infinite series. Hence the potential gain for the limited foresight agent is higher, the longer his foresight horizon, and hence so is his exhibited inflation bias in equilibrium.

We have also illustrated how a limited foresight banker differs from a central banker with a finite horizon, and that the inflation bias for the limited foresight banker will be higher.

Possible extensions of this work would consider volatility in inflation and extend the analysis to price-level targeting authorities (as in Svensson (1997b) & (1999)).

Chapter 5

Applications 2: Smoking today and stopping tomorrow - a limited foresight perspective

Philippe Jehiel & Andrew Lilico

5.1 Introduction

In this final chapter we shall consider why some people decide to take up smoking when young, despite knowing the dangers, then, when older, give up - often at considerable effort and after several attempts. In our model the central issue will relate to a change in horizon of foresight. We shall imagine that when people are young they foresee the consequences of their acts only a few years ahead, and form no view of what might happen after that. Then, as they grow older (although they still do not have perfect foresight) they start to foresee a little further ahead than before.

This change in horizon of foresight alone, without any need to assume that preferences change, is sufficient to generate a change in behaviour.

The chapter is organized as follows: First we detail the setting, and then define for the purposes of this chapter what precisely we mean by "limited foresight". Next we use our setup to generate examples in which the decision of whether to smoke or not to smoke is sensitive to the horizon of foresight. Then we discuss the implications and relevance of these examples.

5.2 Setup

A decision-maker faces infinitely-many stages, with a decision to make at each stage. His decision in each period t consists in a choice of action $a_t \in A$ where A is a finite choice set. His preferences are captured by

$$\sum_{t=0}^{\infty} \delta^t U(a_t, \omega_t)$$

where $\omega_t \in \Omega$ is a state variable, $U(a_t, \omega_t)$ is the flow of payoff derived by the agent in period t , and δ is the discount factor between periods.

The state evolves according to a deterministic process mapping the period t profile of state ω_t and action a_t onto the period $t + 1$ state ω_{t+1} (i.e., $(\omega_t, a_t) \rightarrow \omega_{t+1}$). Thus, the state at period $t + k$ can be written as a function $\omega_{t+k}(\omega, a^0, a^1, \dots, a^{k-1})$ where $\omega_t = \omega$ denotes the period t state and $a_{t+l} = a^l$ the period $t + l$ action.

We note that in a standard paradigm with perfect foresight there is no "time inconsistency" problem - that is to say, once a plan of what to do in the future is formed, when the decision-maker gets there he doesn't change his mind. What he thought would be the best thing to do in the future continues to be the best thing to do once the future becomes the present. This is due to the fact that discounting takes an exponential form so that the marginal rate of substitution (in terms of overall preference) between an increase of utility in period t and an increase of utility in period $t + k$ does not change with the time period t . This is a generalization of a result first noted by Strotz (1956).

5.3 The limited foresight approach

We turn now to describing the decision making of an agent with limited foresight. We first review the basic concepts and then offer some preliminary results. It will be convenient for us to use slightly different notation in this chapter from previously, so we shall re-state definitions and some basic results.

5.3.1 Concepts

In a limited foresight problem, the decision-maker can see ahead only n periods (including the current period), and forms no view as to what happens

beyond the horizon of foresight. The following definition (in the spirit of Jehiel 1995) describes how an agent with limited foresight behaves¹.

Definition 5.1 A stream of actions $\{a_t(\omega)\}_{t,\omega}$ is a "limited foresight n -equilibrium" (or "LFE- n ") if and only if², $\forall t, \omega$,

$$a_t(\omega) = \arg \max_a \left[U(a, \omega) + \sum_{k=1}^{n-1} \delta^k U(\hat{a}^k, \omega^k) \right]$$

where $\forall k \leq n-1$,³

$$\begin{aligned} \hat{a}^k &= a_{t+k}(\omega^k) \\ \omega^k &= \omega_{t+k}(\omega, a, \hat{a}^1, \dots, \hat{a}^{k-1}) \end{aligned}$$

The LFE- n solution concept is, of course, simply a one-player version of the LFE-J of Chapter 3 with a state variable and the foresight horizon identified explicitly. This will be convenient in this chapter as our focus here is to consider what happens as the foresight horizon is changed in one-player decision-problems for which the state is important.

Definition 5.2 A stream of actions $\{a_t(\omega)\}_{t,\omega}$ is a " p -controlled n -equilibrium" (or "LFE- (p, n) ") if and only if, $\forall t, \omega$

$$A. (a_t, \dots, a_{t+p-1}) = \arg \max_{a^0, \dots, a^{p-1}} \left[\sum_{k=0}^{p-1} \delta^k U(a^k, \omega^k) + \sum_{k=p}^{n-1} \delta^k U(\hat{a}^k, \omega^k) \right]$$

$$B. \hat{a}^k = a_{t+k}(\omega^k) \text{ for } p-1 < k \leq n-1 \text{ and}$$

$$C. a^k = a_{t+k}(\omega^k) \text{ for } k \leq p-1 \text{ where for all } k,$$

$$\begin{aligned} \omega^k &= \omega_{t+k}(\omega, b^0, \dots, b^{k-1}) \text{ and} \\ b^k &= \begin{cases} a^k & \text{for } k \leq p \\ \hat{a}^k & \text{for } k > p \end{cases} \end{aligned}$$

¹Jehiel (2001) extends the treatment to consider problems in which a guess - modelled as a source of randomness - is formed at the horizon. Such problems will not concern us here.

²We restrict attention to deterministic action schemes that may only depend on the time period t and the current state ω , hence $a_t(\omega)$. Following Jehiel (1995), it can be shown that this is without loss of generality, as long as the agent's choice of action is deterministic and cannot (directly) depend on actions that took place more than N periods earlier (i.e., as long as the agent has a bounded memory).

³Note that ω^k depends on a for $k \geq 1$, so \hat{a}^k depends on a too.

In this case we divide the foresight horizon into a period over which the decision-maker plans what to do, p (which we shall refer to as his "planning horizon"), and the period over which he merely has exogenous expectations of what he will do, $n - p$.

The interpretation of Definition 5.2 is as follows. Condition A means that in every period t and in all states ω , the agent chooses an optimal plan a^0, \dots, a^{p-1} over his planning horizon given his expectations $\hat{a}^p, \dots, \hat{a}^{n-1}$ about what will happen next within his horizon of foresight. Condition B expresses the idea that expectations $\hat{a}^p, \dots, \hat{a}^{n-1}$ are correct, while condition C expresses the idea that within his planning horizon the agent does not change his mind. That is, when the agent reaches period $t + k$, he finds it optimal to do what he had planned to do at this period k periods earlier, i.e. at period t ⁴.

When the agent has perfect foresight ($n = +\infty$), it is readily verified that an LFE- $(p, +\infty)$ exists and that it coincides with the standard perfect foresight optimal plan. This is a simple adaptation/generalization of Strotz (1956)'s result (it is due to the exponential character of the discounting). It is also immediate to see that a limited foresight equilibrium LFE- n corresponds to a p -controlled n -equilibrium or LFE- (p, n) with a planning horizon of $p = 1$ and that any LFE- (p, n) is also a LFE- n .

The next definition considers those LFE- (p, n) which have a maximal planning horizon p .⁵

Definition 5.3 *A stream of actions $\{a_t(\omega)\}_{t,\omega}$ is a "best-controlled n -equilibrium" (or "LFEB- n ") if and only if it is an LFE- (p, n) and there is no LFE- (p', n) with $p' > p$.*

The motivation for a best-controlled n -equilibrium is that an agent may find it desirable to feel he has as much control over his planning scheme as possible subject to the constraint that he does not change his mind (relative to plans made earlier)⁶.

⁴Rubinstein (1998) suggests as an alternative to Jehiel (1995)'s approach a concept in which the planning horizon p coincides with the horizon of foresight n and condition C is dropped. But, this in general would result in the decision maker adopting decisions based on plans that are not followed afterwards, which sounds undesirable (see also Jehiel (1998) and Chapter 3 of this thesis for further discussion of this point).

⁵See Chapter 3 for the first proposing of this definition. A "best-controlled n -equilibrium" is simply a one-player variant of what we have referred to in Chapter 3 as an "LFE-RJ", with a state variable and the foresight horizon identified explicitly.

⁶Clearly, a best-controlled n -equilibrium is an LFE- n equilibrium and it can thus be viewed as a refinement of it.

5.3.2 Preliminary Results

Proposition 5.1 *There always exists an LFE- n . All LFE- n are cyclical - i.e.*

$$\exists k \text{ s.t. } \forall \omega, t, a_{t+k}(\omega) = a_t(\omega)$$

Proof. (Sketch) Start from any profile of n -expectations, i.e. $f_0(\omega, a) \in A^n$. For any ω , $a_{f_0}^*(\omega)$ is a best-action given f . Construct f_{-1} as follows:

For any ω , if a is chosen, ω' arises next. Take the sequence of actions generated by $a_{f_0}^*(\omega')$ and $f_0(\omega', a_{f_0}^*(\omega'))$. Take the truncation of these to the first n ones. Define this to be $f_{-1}(\omega, a)$. Define recursively $f_{-k}(\omega, a)$. At some point, because everything is finite, f_{-k} will correspond to $f_{-k'}$, $k \neq k'$. This allows us to show existence and the cyclical nature of limited foresight plans. N.B. this is analagous to Jehiel (1995). ■

It is instructive to observe that LFE- (p, n) need not exist when $n \geq 2$ and $p \geq 2$. For example, in the decision tree of Figure 1 there is no LFE- $(2, 2)$. At the first node, the optimal 2-plan is (Up, Up), but upon reaching the next node, the agent would choose Down and not Up violating the constraint that he should not change his mind within his planning horizon (Constraint C of Definition 2). Despite the possible inexistence of LFE- (p, n) , we can infer from Proposition 1 that an LFEB- n always exists. This is because LFE-1 and LFE- $(1, 1)$ are equivalent concepts.

5.4 To smoke, or not to smoke? That is the question...

Now we shall employ the concepts we have introduced to address the question of why some people might smoke when young, then give up when older. In so doing we shall interpret a younger (resp. older) person as one with shorter (resp. longer) horizon of foresight.

We shall make the following simplifying assumptions:

A1) $A = \{D, S\}$: there are two actions, which we shall interpret as Don't Smoke and Smoke.

A2) $\omega_t = (a_{t-1}a_{t-2}\dots a_{t-m})$: the state variable is defined to be the m previous period actions.

A3) $\exists \bar{\delta}$ such that, for all $\delta \geq \bar{\delta}$, there is one perfect foresight optimal plan, which is $a_t(\omega) = D, \forall \omega, t$.

In the sequel we shall analyze how the plan of an agent who has limited foresight varies with his horizon of foresight. A3 tells us that an agent with

perfect foresight chooses D in all states ω and in all periods. In light of the smoker's problem, A3 sounds plausible to us.

Let us focus on the sustainability or otherwise of the following two stationary plans.

Plan D : $a_t(\omega) = D, \forall \omega, t$

Plan S : $a_t(\omega) = S, \forall \omega, t$

We are interested in whether or not there exists $\bar{\delta}$ such that for all $\delta \geq \bar{\delta}$ plan a , $a = D$ or S , is sustainable as LFE- n . This is equivalent⁷ to checking the sustainability of these plans as LFE- n in the special case $\delta = 1$. From here on we shall assume that $\delta = 1$.

5.4.1 When the horizon of foresight is long enough

We first show that when the horizon of foresight n is strictly greater than the hindsight dependence m , the only possible plan is D .

Proposition 5.2 *For all $n \geq m + 1$, Plan D (but not S) is an LFE- n .*

This result follows from the observation that LFE- n are not affected by the horizon of foresight n as long as $n \geq m + 1$:

Lemma 1 *For all $n \geq m + 1$, Plan a is an LFE- n if and only if it is an LFE- $(m + 1)$.*

And only plan D can be an LFE- $(m + 1)$:

Lemma 2 *Plan D (but not S) is an LFE- $(m + 1)$.*

5.4.2 When the horizon of foresight is short

It remains to analyze what happens when the horizon of foresight is smaller than $m + 1$. We shall focus attention on proving that, if the horizon of foresight is not too long, there are a number of scenarios in which Plan S would be optimal, and some in which Plan D would not be optimal. We offer the interpretation that those who smoke may face decision-problems like those below.

In the sequel, we shall specialize to the case of $m = 2$.

⁷up to indifferences (that we assume are not here) in the limit as δ goes to 1.

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When Plan S (but not D) is a limited foresight equilibrium:

Consider the following payoffs (reading states horizontally and this-period actions vertically):

$U(a, \omega)$	DD	DS	SD	SS
S	0	8	11	4
D	10	4	3	0

Table 1

This system of payoffs guarantees that if the agent is patient enough (δ close to 1), the optimal perfect foresight plan is to never smoke, i.e. Plan D .⁸ Hence, Assumption A3 is satisfied.

We next observe that Plan D is not an LFE-2. This is because, for example,⁹

$$U(D, SD) + U(D, DS) < U(S, SD) + U(D, SS).$$

Thus, anticipating he will not smoke in the next period, the agent would be better off smoking today if yesterday he smoked and the day before he did not. Thus plan D is not an LFE-2.

Finally, Plan S is an LFE-2. This is because the following system of inequalities is satisfied (checking all possible states):¹⁰

- 1) $2U(S, SS) > U(D, SS) + U(S, DS)$
- 2) $U(S, DS) + U(S, SD) > U(D, DS) + U(S, DD)$
- 3) $U(S, DD) + U(S, SD) > U(D, DD) + U(S, DD)$
- 4) $U(S, SD) + U(S, SS) > U(D, SD) + U(S, DS)$

Hence, whatever the state, the agent finds it optimal to choose S today if he anticipates he will choose S tomorrow.

Remark: In the above example, Plan D is an LFE-3, but Plan S is not (see Proposition 2). Additionally, Plan S is an LFE-2, but Plan D is not (see

⁸To see this, observe that obtaining the highest payoff $U(S, SD) = 11$ would require a bad experience associated with $U(S, SS) = 9$ or $U(S, DS) = 8$. It is thus not worthwhile.

⁹ $3 + 4 = 7 < 11 = 11 + 0$

¹⁰1) $2 * 9 = 18 > 8 = 0 + 8$

2) $8 + 11 = 19 > 4 = 4 + 0$

3) $0 + 11 = 11 > 10 = 10 + 0$

4) $11 + 9 = 20 > 11 = 3 + 8$

above calculations). Thus if decision-makers faced such preferences at a time when their foresight horizon was relatively short (2) they would smoke. If subsequently (and unexpectedly) their foresight horizon later became longer they would switch to not smoking. We shall discuss this further below.

When Plan S is a better-controlled plan than Plan D :

Another type of scenario in which people might smoke is one in which, although both Plans would be equilibria if we took our future behaviour as exogenous, if we aim to control what we do in the future as far ahead as possible (as in the LFEB- n concept), there are advantages to smoking.

Consider the following payoffs:

$U(a, \omega)$	DD	DS	SD	SS
S	4	5	14	7
D	8	10	6	0

Table 2

Again, it can be checked that Plan D is the only optimal perfect foresight plan.

Also, both Plans D and S are LFE-2.

For Plan S this is because¹¹

- 1) $2U(D, DD) \geq U(S, DD) + U(D, SD)$
- 2) $U(D, SD) + U(D, DS) \geq U(S, SD) + U(D, SS)$
- 3) $U(D, SS) + U(D, DS) \geq U(S, SS) + U(D, SS)$
- 4) $U(D, DS) + U(D, DD) \geq U(S, DS) + U(D, SD)$

For Plan D this is because¹²

- 5) $2U(S, SS) > U(D, SS) + U(S, DS)$
- 6) $U(S, DS) + U(S, SD) > U(D, DS) + U(S, DD)$

¹¹1) $2 * 7 = 14 > 5 = 0 + 5$

2) $5 + 14 = 19 > 12 = 10 + 2$

3) $2 + 14 = 16 > 10 = 8 + 2$

4) $14 + 7 = 21 > 11 = 6 + 5$

¹²5) $2 * 8 = 16 > 8 = 2 + 6$

6) $6 + 10 = 16 > 14 = 14 + 0$

7) $0 + 10 = 10 > 7 = 7 + 0$

8) $10 + 8 = 18 > 11 = 5 + 6$

- 7) $U(S, DD) + U(S, SD) > U(D, DD) + U(S, DD)$
 8) $U(S, SD) + U(S, SS) > U(D, SD) + U(S, DS)$

Although both Plans D and S are LFE-2s, only Plan S is an LFEB-2. We know that with $p = 1$ both Plans are LFE-(1, 2)s, since an LFE-(1, 2) is formally equivalent to an LFE-2. Thus for Plan S to be an LFEB-2 but Plan D not, Plan S must be an LFE-(2, 2) but Plan D not. For Plan S to be an LFE-(2, 2) it should be that for all states the best 2-plan is to Smoke in the current and next period. This requires the following inequalities in addition to those required for LFE-2:

- 9) $2U(S, SS) \geq U(D, SS) + U(D, DS)$
 10) $2U(S, SS) \geq U(S, SS) + U(D, SS)$
 11) $U(S, DS) + U(S, SD) \geq U(D, DS) + U(D, DD)$
 12) $U(S, DS) + U(S, SD) \geq U(S, DS) + U(D, SD)$
 13) $U(S, DD) + U(S, SD) \geq U(D, DD) + U(D, DD)$
 14) $U(S, DD) + U(S, SD) \geq U(S, DD) + U(D, SD)$
 15) $U(S, SD) + U(S, SS) \geq U(D, SD) + U(D, DS)$
 16) $U(S, SD) + U(S, SS) \geq U(S, SD) + U(D, SS)$

It is easily verified that these all hold.

In contrast, the equivalent inequalities for Plan D do not all hold. In particular, we have¹³:

$$U(D, SD) + U(D, DS) < U(S, SD) + U(S, SS)$$

Thus, when the state is $\omega = SD$, the best 2-plan is not DD , since it is dominated by SS . So Plan is not an LFE-(2, 2) and Plan S is the sole LFEB-2.

Once again, if decision-makers faced such preferences at a time when their foresight horizon was relatively short (2) they might smoke, then if their foresight horizon later became longer they might again switch to not smoking.

5.5 A tale about the smoker's problem

In Section 5.4 we have seen that it is possible to switch the decision of whether to smoke or not smoke simply by changing the horizon of foresight. We propose to interpret this in terms of an (unanticipated) change in foresight

¹³ $6 + 10 = 16 < 21 = 14 + 7$

over the life-cycle. We suggest that younger people may have less ability to look ahead into the future than older people, and that this offers some insight into why significant numbers of young people take up smoking, only to give it up in middle-life.

Let us think for a moment how relevant our examples are to the problem of why people smoke. How plausible are our payoffs - does our result depend on some very particular arrangement of payoffs which is unlikely to exist in reality?

Think first of the payoffs shown in Table 1. There the least desirable things to be doing in any period are either to start smoking having had a history of not smoking, or to stop smoking having smoked ($U(S, DD) = U(D, SS) = 0$). Anyone who has turned green after smoking his first cigarette or struggled to stop smoking will attest to the plausibility of this.

The highest payoff comes from smoking having just recently started ($U(S, SD) = 11$). Presumably, since many people go on to be smokers after enduring the unpleasantness of the first cigarette, there is probably a period between just starting and when smoking becomes routine in which smoking is very enjoyable.

Not smoking in any one period having not smoked previously ($U(D, DD) = 10$) is not so attractive as the period just after one has begun smoking (which presumably must be right otherwise who would smoke?), but it is preferable to smoking once smoking has become routine ($U(S, DS) = 8$; $U(S, SS) = 9$). As non-smokers ourselves, the authors have little difficulty in believing this.

Taking up smoking again after one has only recently stopped ($U(S, DS) = 8$) is more attractive than sticking to not smoking ($U(D, DS) = 4$). Perhaps this is why it is hard to give up?

Similarly, having started smoking, even though it was very unpleasant, it is pretty unattractive to go back to being a non-smoker ($U(D, SD) = 3$), while smoking is very enjoyable at this point ($U(S, SD) = 11$).

In the view of these authors, this set of payoffs seems highly intuitive - indeed compelling. And payoffs of this sort lead to the result that people with a shorter foresight horizon will smoke, while those with a longer foresight horizon (perhaps those who are older and wiser?) will not.

However, we note that the result is not dependent on the ordering of the payoffs so far discussed. In the payoffs shown in Table 2 the ordering is importantly different. In that case, for example, sticking to not smoking having recently given up ($U(D, DS) = 10$) is more attractive than going back to smoking ($U(S, DS) = 5$). Similarly, not continuing with smoking having started ($U(D, SD) = 6$) is higher up the ranking and closer to being a regular smoker ($U(S, SS) = 7$) than in the previous example. An agent with these preferences would appear to find it easier to give up when he

wants to. It should hardly be surprising that an agent of this sort might be swayed between smoking and not smoking by the added perspective of a longer foresight horizon.

5.6 Conclusion

In this chapter we have offered one interpretation of why people start smoking only to give it up later: as people grow older they gain a longer-term perspective which changes the balance of advantage away from smoking and towards not smoking. We contend that the preferences required to obtain this result are interpreted plausibly in terms of scenarios in which people start smoking only to give it up later.

We note that in our model there is no change in information (about, say the real impact of smoking). It isn't that as they grow older people discover that smoking really *is* unhealthy - it's not just a lie old people tell you. In our model, as people grow older they get to understand better how they will react in the future and this in turn induces a change in behavior.

Note also that our model does not rely on any change in tastes (nor in terms of the perception of an end¹⁴). One important alternative approach to these questions is to regard the decision about whether to smoke as involving a time inconsistency problem. Perhaps when young people want, like the rock star, to "live fast and die young", but once they have done the living fast the dying young seems less attractive (see Laibson (1997) and more recently Harris and Laibson (2001) for an analysis of the life cycle consumption generated by such preferences).

A third approach which is, again, different from ours, follows in the Stigler and Becker (1977) tradition on "rational addiction", whereby people employ "appreciation capital" which is affected by consuming.

We do not suggest that time-consistency or appreciation capital are not important elements of an explanation about smoking. But we do suggest that our approach offers a different insight which might also be important. For example, if a longer-term perspective is likely to result in people switching away from smoking, that may make a difference to the focus and value of government information programmes about smoking. Perhaps it isn't that young people lack *information* that smoking will damage health. Perhaps, rather, it is that young people lack *wisdom*, in some sense.

O'Donoghue and Rabin (1999) also make distinctions between sophisticated and naive agents in the context of preferences with non-exponential

¹⁴That might be a relevant issue too: when a cancer becomes likely, one sees the end approaching and the smoking attitude may be affected.

discounting. In their approach, a sophisticated agent is one who is aware of his limited control capabilities and of the nature of his preferences. A naive agent is one who behaves as if he could stick to his plan afterwards, which in reality he cannot. O'Donoghue and Rabin (2001) also consider the idea of partial sophistication modelled as a convex combination between the two extreme modes of behavior outlined above. But, the O'Donoghue and Rabin approach to partial sophistication and the limited foresight approach are very different and have very different policy implications in terms of how to improve the well-being of agents.¹⁵

5.7 Proofs appendix

5.7.1 Proof of Lemma 1:

Proof. Assume, without loss of generality, that $a = D$ (we could simply reverse symbols if $a = S$).

If Plan D is an LFE- $(m + 1)$ then the following 2^m inequalities must hold (2^m because there is one inequality for each possible state at time t , and there are 2^m different possible states):

$$1) (m+1)U(D, D...D) \geq U(S, D...D) + U(D, SD...D) + \dots + U(D, D...DS)$$

$$2) U(D, SD...D) + \dots + U(D, D...D) \geq U(S, SD...D) + \dots + U(D, D...DS)$$

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$$2^m) U(D, S...S) + \dots + U(D, D...D) \geq U(S, S...S) + \dots + U(D, D...DS)$$

These inequalities state that, for every possible state at t , it is better to play D this period than to play S , on the assumption that D will be played thereafter.

If Plan D is an LFE- n there will again be 2^m inequalities (only 2^m because there are still only 2^m possible states, even though there is now greater foresight). Now the sum of the payoffs from playing D this period (the left-hand side) for any state at t would differ from the inequalities above by the presence of an additional $n - m - 1$ added terms of $U(D, D...D)$. But the sum of payoffs from playing S this period (the right hand side) would, likewise, differ only by the presence of the same additional $n - m - 1$ added terms of $U(D, D...D)$.

Thus, for example, inequality (2) would be modified to read

$$2') U(D, SD...D) + \dots + U(D, D...D) + (n - m - 1)U(D, D...D) \geq$$

¹⁵One simple reason is that in our approach the underlying preferences are standard preferences with exponential discounting.

$$U(S, SD\dots D) + \dots + U(D, D\dots DS) + (n - m - 1)U(D, D\dots D)$$

but this simplifies back to equation (2). Hence the required inequalities are equivalent, and whenever they are satisfied Plan D will be both an LFE- $(m + 1)$ and an LFE- n , and whenever they are not all satisfied, Plan D will be neither an LFE- $(m + 1)$ nor a LFE- n . ■

5.7.2 Proof of Lemma 2:

Proof. If Plan D is optimal with perfect foresight, then the infinite stream of payoffs received from playing D every period must be greater than that from playing S this period then playing D thereafter, for any state. But after period $t + m$, if D is being played from period $t + 1$, all subsequent payoffs will be $U(D, D\dots D)$, regardless of whether D or S is played in period t since there are only m elements in the state variable. That means that when Plan D is optimal with perfect foresight, the 2^m inequalities of the proof of Lemma 1 must hold, which is a sufficient condition for Plan D to be an LFE- $(m + 1)$. ■

5.7.3 Remark on Proposition 5.2

Note that since (by assumption) Plan S is not optimal with perfect foresight, it cannot be true that a set of 2^m inequalities equivalent to those in the proof of Lemma 1 hold. Hence Plan S is not an LFE- $(m + 1)$. By Lemma 1 this implies that Plan S is not an LFE- n for any $n > m + 1$ either. This should help to verify Proposition 5.2.

Chapter 6

Conclusion

In this thesis we have considered various problems in limited foresight. We have considered why people might reason forwards, and how to go about modelling how far forwards they might look. We have investigated what is an appropriate solution concept for agents with limited foresight, and we have looked at two examples of how to employ our limited foresight equilibrium concepts to address real-world problems.

A number of points emerge from this research. Limited foresight reasoning may be applicable to a wide number and variety of even relatively simple decision problems and games. It may be employed both in individual behavioural settings usually considered in microeconomics (such as our smoker problem) and in macroeconomic settings (for which, in general, bounded rationality models have not so far achieved any great penetration).

We have argued that we do have the tools to model limited foresight in many settings. Jehiel's limited foresight concept, as we have refined it here, can be employed to provide tractable equilibrium solutions to many problems, providing interesting and sometimes unexpected new insights into problems. It is very important in bounded (or meta-) rationality modelling that the solutions offered should provide the possibility of modelling problems tractably. Otherwise one runs the risk of seeking spurious "realism" at the expense of losing the ability to gain insights through abstraction. Bounded rationality models which claim to represent "true" human behaviour but which cannot provide tractable solutions to problems are of little practical value. We have illustrated that this is not the case with Jehiel's limited foresight equilibrium concept. We can use it.

Clearly there are other possible limited foresight approaches which may also be tractable and which may ultimately prove more accurate than that we have pursued here. Though we have mentioned chess often in this thesis

to drive the intuition of what limited foresight means, chess players do not, of course, use our sort of limited foresight at all. Chess players look different depths along different pathways in the game, using some kind of analogy-based reasoning to determine which pathways are worth exploring in more detail and which should be largely ignored. There is research in this area which may yet prove fruitful¹. However, it seems to this author that any equilibrium concept arising out of this line of thought will inevitably be much more complicated than the Jehiel concept employed here, and that it will be many years before this area will yield an approach which is of practical value.

Deviating less from the approach here, Jehiel (2001) modifies the Jehiel concept here by assuming that agents make a guess at the horizon of foresight (rather than simply being utterly ignorant, as here). This line of thought may be promising, but it seems that the main insights from limited foresight reasoning can be delivered in the simpler setting employed here, with considerably less modelling complexity.

The issue of how far ahead the agents in our models should look is still moot. Our first two chapters offered the nucleus of an approach which might ultimately form the basis of deriving a model of how far ahead agents look. Alternatively, perhaps experimental research might be useful here. In the meantime, calibration may be the appropriate method to employ for practical predictive models with limited foresight.

There is much other research into other forms of bounded rationality. Though this thesis has been about limited foresight, this author would like to note in closing that he does not believe in bounded rationality as such - usually conceived as the claim that decision-makers are not fully rational. The rational agent provides an important modelling discipline which guides research along productive pathways and forces researchers to come up with rational answers where behaviour is not as our initial intuitions might predict. If we assume too easily that agents are not rational we may fail to learn important lessons and our research may become woolly - rather as if bounded rationality became Game Theory's version of a conspiracy theory whereby everything is "explained" by some grand cover-up².

That said, the idea that agents may face reasoning costs which they respond to (rationally) by not solving problems completely seems completely in keeping with orthodox, disciplined research. Given the complexity of mod-

¹Consider, for example, Jehiel (2002), Gabaix and Laibson (2000a,2000b)

²For example, economists might have responded to observations that demand for potatoes increased as the price rose by just saying "Ah, well. Agents aren't fully rational." But then we would have been robbed of important insights into the drivers of demand. And so on with many other cases. The danger is that answers in terms of bounded rational can be too easy to come up with, and consequently unproductive...

elling a reasoning cost problem explicitly, this (rational) incomplete solution of problems may be best absorbed through models in which reasoning costs are ignored but agents behave *as if* they were less than fully rational. The challenge is to provide models of this sort which are tractable and which offer improved insight into practical problems. Limited foresight does just that.

Andrew Lilico, UCL October 2002

Appendix A

Proof of Main Lemma

It follows from the definition of an implementing FA that any implementing FA capable of examining five or more variables and recommending two actions must have at least seven states. Hence it will suffice to prove our result to consider cases in which an implementing FA examines fewer than five variables.

Proposition A.1 *Any implementing FA examines the values of at least four of the six variables, i.e. $\#\gamma(Q \setminus F) \geq 4$.*

Proof. The cases where only one or two variables are examined are trivial. Consider the case where γ can take a maximum of three different values, and assume, for convenience, that the FA examines all three of these variables along any path to any recommendation (this is the maximal information case so proof here will apply *a fortiori* to cases when this assumption is not met).

Write the expected payoffs as

$$\pi(U) = x_1 + \max\{x_3, x_4\}$$

$$\pi(D) = x_2 + \max\{x_5, x_6\}$$

Then at the point of making the recommendation the three examined variables (call them l, m, n) can be arranged in two different ways (only two by symmetry): All of them could be in one of $\pi(U), \pi(D)$; or two in one and one in the other. By symmetry, without loss of generality we can use $\pi(U)$ as the fully identified payoff in the first case, so $\pi(U) = l + \max\{m, n\}$. Then consider $\pi(U) = 1$. Use $k(\pi(a))$ to denote the set of possible values of π when the FA recommends action a . Here we have $k(\pi(D)) = \{0, 1, 2\}$ and this is not enough to identify the optimal action (choosing U will be worse than choosing D if $\pi(D) = 2$, and better than D if $\pi(D) = 0$).

In the second case, by symmetry, without loss of generality we can put the single identified variable (which we shall denote l) in $\pi(U)$ and the two

identified variables in $\pi(D)$. For $\pi(U)$ we then have two different sub-cases (only two by symmetry):

$$\pi(U) = l + \max\{x_3, x_4\}$$

or

$$\pi(U) = x_1 + \max\{l, x_4\}$$

Similarly, for $\pi(D)$ we have two different sub-cases:

$$\pi(D) = m + \max\{n, x_6\}$$

or

$$\pi(D) = x_2 + \max\{m, n\}$$

Now consider $l = m = n = 0$. Then $k(\pi(D)) = \{0, 1\}$, and $k(\pi(D)) = \{0, 1\}$ for the first $\pi(U)$ sub-case or $k(\pi(D)) = \{0, 1, 2\}$ for the second $\pi(U)$ sub-case. In either case this is not enough to identify optimal play.

Since an FA cannot recommend optimal play for all circumstances with only three variables examined, *a fortiori* no FA can recommend optimal play for all circumstances with only the *possibility* of examining three variables - i.e. if $\#\gamma(Q \setminus F) \leq 3$. ■

Proposition A.2 *In an implementing FA, if only four of x_1, \dots, x_6 can be examined two of these variables must be x_1 and x_2 , and the others must be either the pair x_3, x_4 or the pair x_5, x_6 .*

Proof. By symmetry, there are four types of case where this condition is not met: when x_1, x_3, x_4, x_5 can be examined, when x_1, x_3, x_5, x_6 can be examined, when x_1, x_2, x_3, x_5 can be examined, and when x_3, x_4, x_5, x_6 can be examined. In all cases consider fully-examining FAs. For this first case consider $\pi(U) = 1, x_5 = 0 \implies k(\pi(D)) = \{0, 1, 2\}$. For the second case consider $x_1 = x_3 = x_5 = x_6 = 0 \implies k(\pi(U)) = \{0, 1\}, k(\pi(D)) = \{0, 1\}$. For the third case consider $x_1 = x_2 = x_3 = x_5 = 0 \implies k(\pi(U)) = \{0, 1\}, k(\pi(D)) = \{0, 1\}$. For the fourth case consider $x_3 = x_4 = x_5 = x_6 = 0 \implies k(\pi(U)) = \{0, 1\}, k(\pi(D)) = \{0, 1\}$. ■

Corollary A.1 *It follows from Proposition A.2 that an implementing FA for the problem in Figure 1.2, for which only, at most, four of the variables can be examined will recommend one of the actions for 10 of the sixteen possible combinations of the four variables, and the other action for 6 of those combinations, as illustrated in the following table for the case where x_1, x_2, x_3, x_4 are examined:*

x_1	x_2	x_3	x_4	$k(\pi(U))$	$k(\pi(D))$	<i>Optimal Action</i>
0	0	0	0	{0}	{0, 1}	<i>D</i>
0	0	0	1	{1}	{0, 1}	<i>U</i>
0	0	1	0	{1}	{0, 1}	<i>U</i>
0	1	0	0	{0}	{1, 2}	<i>D</i>
1	0	0	0	{1}	{0, 1}	<i>U</i>
0	0	1	1	{1}	{0, 1}	<i>U</i>
0	1	0	1	{1}	{1, 2}	<i>D</i>
1	0	0	1	{2}	{0, 1}	<i>U</i>
0	1	1	0	{1}	{1, 2}	<i>D</i>
1	0	1	0	{2}	{0, 1}	<i>U</i>
1	1	0	0	{1}	{1, 2}	<i>D</i>
0	1	1	1	{1}	{1, 2}	<i>D</i>
1	0	1	1	{2}	{0, 1}	<i>U</i>
1	1	0	1	{2}	{1, 2}	<i>U</i>
1	1	1	0	{2}	{1, 2}	<i>U</i>
1	1	1	1	{2}	{1, 2}	<i>U</i>

We have shown that any six-state implementing FA would have to be capable of examining four variables. However, it is not yet clear whether some of these variables are examined for some realisations of X and others for different realisations without there being any realisation of the variables for which all the variables are examined in turn (e.g. sometimes only x_1 and x_2 , and sometimes x_1, x_3, x_4 , but never all of x_1, x_2, x_3, x_4). We now show that, if the FA is to be implementing, for some realisation of the variables all four variables will have to be examined in turn.

Definition A.1 *An N -path through a finite automaton FA is a sequence $\hat{q}(x, FA)$ containing N members which are not final states.*

Proposition A.3 *Any 6-state FA which is implementing for the problem in Fig. 2 must contain at least one 4 - path.*

Proof. We shall prove that $\gamma(q_0) \notin \{1, 2, 3, 4\}$ (i.e. that there is no first question) for an implementing FA if it has at most 3 - paths. Without a first question there cannot exist an implementing FA.

We have already proved that any implementing 6-state automaton would need to be a 4-question automaton, investigating the values of (x_1, x_2, x_3, x_4) (or equivalently (x_1, x_2, x_5, x_6)). Thus the first question could be about the value of x_1, x_2, x_3 , or x_4 . We shall consider these possibilities in turn.

1. Consider the case in which $\gamma(q_0) = 1$ (i.e. in which the first question asked is about x_1). Then there must be two exit branches from q_0 , one for $x_1 = 1$ and one for $x_1 = 0$. We shall focus attention on the $x_1 = 0$ branch.

(a) Suppose that the $x_1 = 0$ branch leads to a final state F . Suppose that $\lambda(F) = D$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 1, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. Next suppose that $\lambda(F) = U$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_1 = 0$ branch cannot lead to any final state F .

(b) Now suppose, instead, that on $x_1 = 0$, a state q for which $\gamma(q) = 2$ is reached (i.e. the x_2 investigation is reached). Then consider the $x_2 = 0$ branch out of this state. If this leads to a final state F for which $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 1, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. If this leads to a final state F for which $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Thus the $x_2 = 0$ branch could not lead to a final state.

Since we have already asked an x_1 question and an x_2 question, and only have one question available for each variable, if the $x_2 = 0$ branch does not lead to a final state it must lead to a state investigating one of x_3 or x_4 . Suppose that the $x_2 = 0$ branch leads to a state q for which $\gamma(q) = 3$ (i.e. we ask the x_3 question). This, in turn, must have two exiting branches, one for $x_3 = 0$ and one for $x_3 = 1$. Since the maximum path-length is three, and we have now asked three questions, neither of these exiting branches can lead to the state q for which $\gamma(q) = 4$. Since loops are also forbidden, it must be the case that each of these exiting branches leads to a final state. Consider the final state F reached on $x_3 = 0$. If $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. If $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. Since every case

for which the $x_2 = 0$ branch leads to a state q for which $\gamma(q) = 3$ leads to a contradiction, it cannot be the case that the $x_2 = 0$ branch leads to a state q for which $\gamma(q) = 3$. By symmetry, this applies also to the case in which the $x_2 = 0$ branch leads to a state q for which $\gamma(q) = 4$. But this means that for all possible termini of the $x_2 = 0$ branch, where the x_2 question is reached from the $x_1 = 0$ branch when $\gamma(q_0) = 1$, there is a contradiction. Hence there cannot be an $x_2 = 0$ branch, where the x_2 question is reached from the $x_1 = 0$ branch when $\gamma(q_0) = 1$. Hence the x_2 question cannot be reached from the $x_1 = 0$ branch when $\gamma(q_0) = 1$.

- (c) Next suppose that the $x_1 = 0$ branch leads to state q for which $\gamma(q) = 3$ (i.e. that we reach the x_3 question second if $x_1 = 0$). Now consider the $x_3 = 0$ branch exiting state q . Suppose this leads to some final state F . Suppose that $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. If this leads to a final state F for which $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 0, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. Thus the $x_3 = 0$ branch could not lead to a final state.

Suppose that the $x_3 = 0$ branch leads to some state q for which $\gamma(q) = 4$ (i.e. that we reach the x_4 question third if $x_1 = 0$ and $x_3 = 0$). Since the longest path is a 3-path, each branch out of this q must lead to a final state. Consider the $x_4 = 1$ branch, and call the final state it reaches state F . If $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. If $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 0, 1, 1, 1)$, and hence be non-implementing, which would be a contradiction. Thus the $x_4 = 1$ branch could not lead to a final state, but it must, so we have a contradiction. Hence the $x_3 = 0$ branch cannot lead to some state q for which $\gamma(q) = 4$, given that $\gamma(q_0) = 1$ and the $x_1 = 0$ branch leads to state q_2 for which $\gamma(q_2) = 3$.

Suppose, instead, that the $x_3 = 0$ branch leads to some state q for which $\gamma(q) = 2$ (i.e. that we reach the x_2 question third if $x_1 = 0$ and $x_3 = 0$). Since the longest path is a 3-path, each branch out of this q must lead to a final state. Now consider the $x_2 = 0$ branch, and call the final state it reaches F . If $\lambda(F) = D$, the FA

will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. If $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_3 = 0$ cannot lead to the x_2 question. But since it cannot lead to a final state or the x_4 question either, and since looping back on itself or to x_1 is not possible for an implementing FA, there cannot be any $x_3 = 0$ branch following $x_1 = 0$ in a 6-state automaton with at most 3-paths.

- (d) By symmetry, we could reverse the labels on x_3 and x_4 without loss of generality. Hence, since $x_1 = 0$ cannot lead to the x_3 question, it cannot lead to the x_4 question either.

But since the $x_1 = 0$ branch cannot lead to a final state, and cannot lead back onto itself by the no-looping condition, and cannot lead to any of x_2, x_3, x_4 it cannot lead anywhere. Hence there cannot be any $x_1 = 0$ branch leading out of q_0 in such an FA, hence $\gamma(q_0) \neq 1$.

This completes case 1.

2. Consider the case in which $\gamma(q_0) = 2$ (i.e. in which the first question asked is about x_2). Then there must be two exit branches from q_0 , one for $x_2 = 1$ and one for $x_2 = 0$. We shall focus attention on the $x_2 = 0$ branch.

- (a) Suppose that the $x_2 = 0$ branch leads to a final state F . Suppose that $\lambda(F) = D$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 1, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. Next suppose that $\lambda(F) = U$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_2 = 0$ branch cannot lead to any final state F .

- (b) Now suppose, instead, that on $x_2 = 0$, a state q for which $\gamma(q) = 1$ is reached (i.e. the x_1 investigation is reached). Then, if we focus attention on the sub-case in which $x_1 = 0$ we have exactly the same case as in (1.b.) above, in which the first two questions have shown that $x_1 = x_2 = 0$, and exactly the same arguments show that this leads to a contradiction.

- (c) Next suppose that the $x_2 = 0$ branch leads to state q for which $\gamma(q) = 3$ (i.e. that we reach the x_3 question second if $x_2 = 0$). Now consider the $x_3 = 0$ branch exiting state q . Suppose this leads to some final state F . Suppose that $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. If this leads to a final state F for which $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Thus the $x_3 = 0$ branch could not lead to a final state.

Suppose that the $x_3 = 0$ branch leads to some state q for which $\gamma(q) = 4$ (i.e. that we reach the x_4 question third if $x_2 = 0$ and $x_3 = 0$). Since the longest path is a 3-path, each branch out of this q must lead to a final state. Consider the $x_4 = 0$ branch, and call the final state it reaches state F . If $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. If $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Thus the $x_4 = 0$ branch could not lead to a final state, but it must, so we have a contradiction. Hence the $x_3 = 0$ branch cannot lead to some state q for which $\gamma(q) = 4$, given that $\gamma(q_0) = 1$ and the $x_2 = 0$ branch leads to state q_2 for which $\gamma(q_2) = 3$.

Suppose, instead, that the $x_3 = 0$ branch leads to some state q for which $\gamma(q) = 1$ (i.e. that we reach the x_1 question third if $x_2 = 0$ and $x_3 = 0$). Since the longest path is a 3-path, each branch out of this q must lead to a final state. Now consider the $x_1 = 0$ branch, and call the final state it reaches F . If $\lambda(F) = D$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. If $\lambda(F) = U$, the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_3 = 0$ cannot lead to the x_1 question. But since it cannot lead to a final state or the x_4 question either, and since looping back on itself or to x_2 is not possible for an implementing FA, there cannot be any $x_3 = 0$ branch following $x_2 = 0$ in a 6-state automaton with at most 3-paths.

- (d) By symmetry, we could reverse the labels on x_3 and x_4 without loss of generality. Hence, since $x_2 = 0$ cannot lead to the x_3 question, it cannot lead to the x_4 question either.

But since the $x_2 = 0$ branch cannot lead to a final state, and cannot lead back onto itself by the no-looping condition, and cannot lead to any of x_1, x_3, x_4 it cannot lead anywhere. Hence there cannot be any $x_2 = 0$ branch leading out of q_0 in such an FA, hence $\gamma(q_0) \neq 2$.

This completes case 2.

3. Consider the case in which $\gamma(q_0) = 3$ (i.e. in which the first question asked is about x_3). Then there must be two exit branches from q_0 , one for $x_3 = 1$ and one for $x_3 = 0$. We shall focus attention on the $x_3 = 0$ branch.

- (a) Suppose that the $x_3 = 0$ branch leads to a final state F . Suppose that $\lambda(F) = D$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 1, 0, 0)$, and hence be non-implementing, which would be a contradiction. Next suppose that $\lambda(F) = U$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_3 = 0$ branch cannot lead to any final state F .
- (b) Suppose that the $x_3 = 0$ branch leads to state q for which $\gamma(q) = 1$ (i.e. that we reach the x_1 question second if $x_3 = 0$). Then if we consider the $x_1 = 0$ branch exiting q we have the same situation as in (1.c.) above, and the same arguments lead to a contradiction. Hence the $x_3 = 0$ branch cannot lead to the x_1 question second.
- (c) Suppose that the $x_3 = 0$ branch leads to state q for which $\gamma(q) = 2$ (i.e. that we reach the x_2 question second if $x_3 = 0$). Then if we consider the $x_2 = 0$ branch exiting q we have the same situation as in (2.c.) above, and the same arguments lead to a contradiction. Hence the $x_3 = 0$ branch cannot lead to the x_2 question second.
- (d) Suppose that the $x_3 = 0$ branch leads to state q for which $\gamma(q) = 4$. Suppose that the $x_4 = 0$ branch exiting q leads to some state q_2 for which $\gamma(q_2) = 2$. Then if we consider the case $x_2 = 0$ we have a sub-case considered in (2.c.) above, and the same arguments lead to a contradiction. Suppose instead that the $x_4 = 0$ branch exiting q leads to some state q_2 for which $\gamma(q_2) = 1$. Then, since the longest path is a 3-path, both branches exiting from q_2

must lead to final states. Call the state that $x_1 = 1$ leads to F . Suppose that $\lambda(F) = D$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. Next suppose that $\lambda(F) = U$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_1 = 1$ branch cannot lead to any final state F , but it must, so we have a contradiction. Hence it cannot be that the $x_4 = 0$ branch exiting q leads to some state q_2 for which $\gamma(q_2) = 1$.

We have considered x_1 and x_2 , and by the no-looping condition the $x_4 = 0$ branch could not lead onto the x_4 or x_3 questions, so the only remaining options are final states. Suppose that the $x_4 = 0$ branch leads to some final state F . Suppose that $\lambda(F) = D$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 0)$, and hence be non-implementing, which would be a contradiction. Next suppose that $\lambda(F) = U$. Then the FA will recommend a non-optimal action for $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 0, 0, 1, 1)$, and hence be non-implementing, which would be a contradiction. Hence $x_4 = 0$ branch cannot lead to any final state F , but it must, so we have a contradiction. Hence we cannot have an $x_4 = 0$ branch exiting q . Hence, if the $x_3 = 0$ branch exiting the initial state leads to state q , $\gamma(q) \neq 4$.

But if the $x_3 = 0$ branch exiting the initial state cannot lead to any final state, or to any of x_1, x_2, x_4 it cannot lead anywhere. Hence there cannot be any $x_3 = 0$ branch. Hence $\gamma(q_0) \neq 3$.

This completes case 3.

4. By symmetry, we could swap the labels of x_3 and x_4 without loss of generality. Hence, since $\gamma(q_0) \neq 3$, we know also that $\gamma(q_0) \neq 4$.

This completes case 4.

But if $\gamma(q_0) \notin \{1, 2, 3, 4\}$, since these are the only possible values of $\gamma(q_0)$, and since the initial state of an implementing FA cannot be a final state, that means that there is no initial state. Hence there is no such implementing FA with at most 3 – *paths*. ■

We now show that no six-state implementing FA could have more than one path along which all four variables are examined, but that no implementing

FA could have an odd number of such paths. Since a six-state implementing FA would have to have exactly one such path and yet not an odd number of such paths, we conclude by contradiction that a six-state FA cannot be implementing.

Proposition A.4 *There is no implementing FA for the problem in Figure 1.2 which has an odd number of 4 – paths, and hence no implementing FA with only 6 states.*

Proof. The problem in Figure 1.2 involves two possible actions. By the definitions of an FA and a question, any six-state automaton capable of recommending either of two actions can ask at most four questions. Let us assume there exists such a FA. It must take a form like that in Figure A.1, where the circles marked Q_1 to Q_4 represent non-terminal states, the double circles marked F_1 and F_2 represent final states and R_1 to R_3 represent dummy states, by which we mean that we allow R_1 to R_3 to be any of the six states and do not specify which (so each of $R_k \in \{Q_1, \dots, Q_4, F_1, F_2\}$). By Proposition A.3, if the FA has only six states and is implementing then having at most 3 – paths is not enough. Since the maximum path-length with only six variables is 4, if an implementing 6-state FA exists, it must contain at least one pathway leading through all four variables. Since we are considering regular FAs, loops are not allowed so the branches from Q_4 cannot lead to prior states in this path, and hence must lead to the two final states. Hence any six-state automaton asking four questions is captured by this stylised representation.

Now consider R_1 . We cannot have $R_1 = Q_1$, by the stopping condition, and we cannot have $R_1 = Q_2$, or else the move to state Q_2 is input-independent, and the first question has not truly been asked (which is excluded by non-redundancy). Hence we have $R_1 \in \{Q_3, Q_4, F_1, F_2\}$. Similarly, by the stopping and non-redundancy conditions, $R_2 \in \{Q_4, F_1, F_2\}$ and $R_3 \in \{F_1, F_2\}$. This means that, apart from the single path from the initial state to final states along four branches (thus asking four questions), the longest other path is of length three or fewer (and hence asks at most three questions).

There are sixteen possible combinations of the values of the four variables. A four-questions path identifies one of these sixteen combinations exactly. An n -question path reduces the combination space to one of $\frac{16}{2^n}$ combinations. So for $n \leq 3$ the combination space is reduced to an even number of combinations at a final state. Since the total number of combinations reaching each final state is one odd number plus some even numbers,

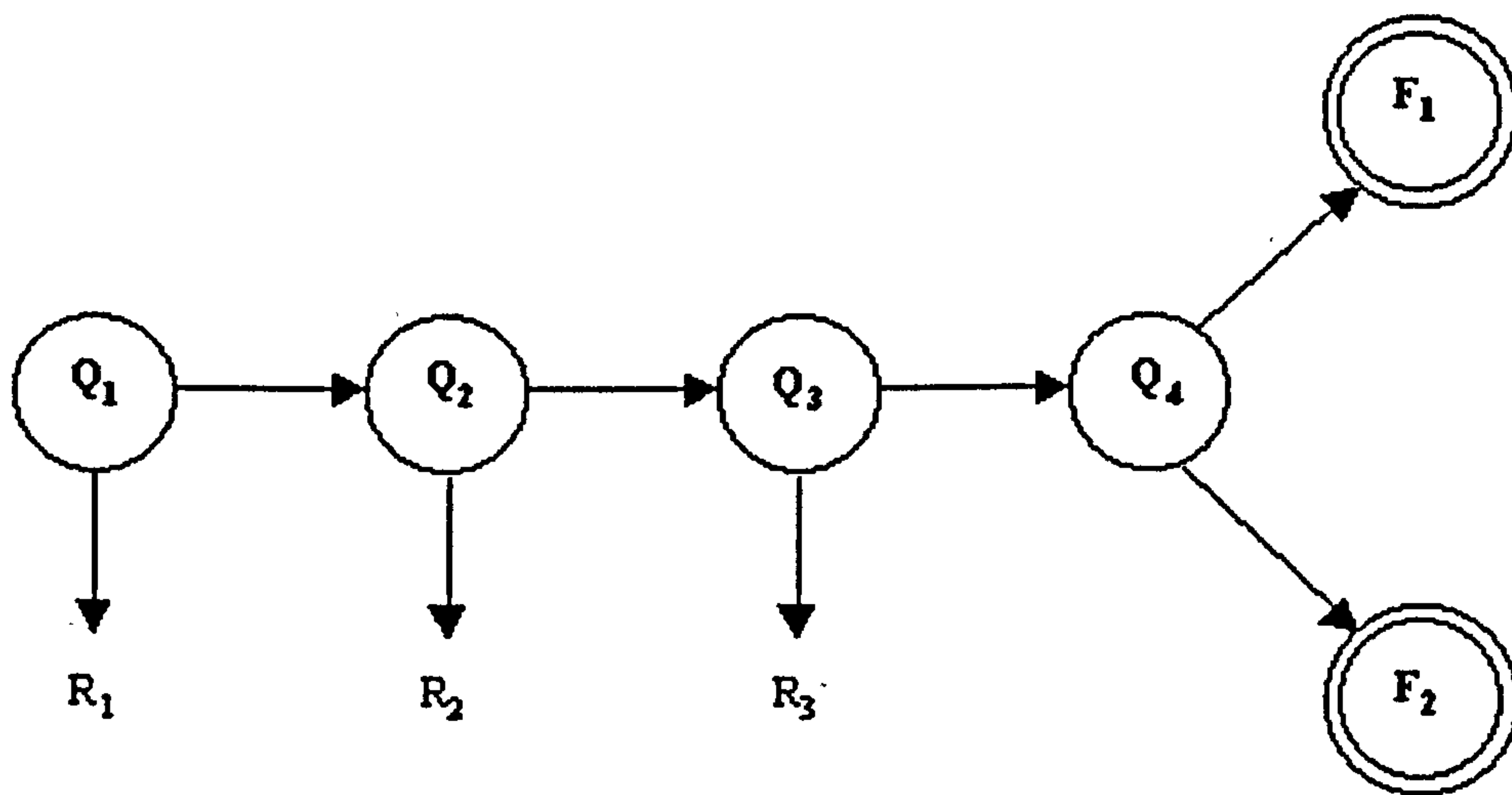


Figure A.1: A stylized representation of a FA which asks four questions

the total number of combinations at each final state of such an automaton must be odd.

However, by Corollary A.1 we know that the number of combinations at each final state is even, which generates a contradiction. ■

Since it follows from Proposition A.1 and the definition of an FA that any implementing FA must have at least six states, and from Proposition A.4 that six states cannot be enough, the Main Lemma follows immediately.

Appendix B

Proof of Proposition 1.2

Proposition B.1 *Any implementing BRFA must contain states examining the values of either $(x_1, x_3, x_4, x_5, x_6)$, or $(x_2, x_3, x_4, x_5, x_6)$, or $(x_1, x_2, x_3, x_4, x_5, x_6)$.*

Proof. It follows from the definition of a BRFA that if x_1 is examined, then so must be x_3 and x_4 , and similarly that if x_2 is examined, then so must be x_5 and x_6 . This means that the proposition identifies all possible BRFA five-variable combinations and the single six-variable combination. We know from Proposition A.1 that no three-or-fewer-variables combination can be implementing. It remains only to exclude all four-variable combinations which do not include both x_1 and x_2 , which is done in Proposition A.2. ■

Proposition B.2 *A state-minimal FA for solving the problem in Figure 1.2, asking questions only about the set of variables $\{x_1, x_3, x_4, x_5, x_6\}$ must contain at least one more state than a state-minimal FA for solving the problem in Figure 1.2, asking questions only about the set of variables $\{x_1, x_2, x_3, x_4\}$.¹*

Proof. Each state q can be associated with a set of tuples of possible values of $\pi(U)$ and $\pi(D)$. Where there is only one path leading to a state there is only one tuple of possible values of each of $\pi(U)$ and $\pi(D)$. Where there is more than one path there may be more than one tuple. For example, the initial state q_0 can be associated with sets of possible values of $\pi(U)$ and $\pi(D)$ (where we shall term the tuple of sets $K(q_0)$) as follows:

$$k(\pi(U), q_0) = \{0, 1, 2\}$$

$$k(\pi(D), q_0) = \{0, 1, 2\}$$

$$K(q_0) = (k(\pi(U), q_0), k(\pi(D), q_0)) = (\{0, 1, 2\}, \{0, 1, 2\})$$

As another example, consider state q_4 of Figure 1.3. Here we would have

¹NB. The proposition applies to *general* FAs, and makes no mention of either BRFAs or FRFAs.

$$K(q_4) \in \{(\{1, 2\}, \{1, 2\}), (\{0, 1\}, \{0, 1\})\}$$

At a final state F of an implementing automaton, we are able to recommend an action a because

$$\min \hat{\pi}(a) \in k(\pi(a), F) \geq \max \hat{\pi}(\sim a) \in k(\pi(\sim a), F)$$

For example, following the path q_0, q_1, q_2 in Figure 1.3, on reaching q_2 we have

$$k(\pi(U), q_2) = \{1, 2\}$$

$$k(\pi(D), q_2) = \{0, 1\}$$

so that

$$\min \hat{\pi}(U) \in k(\pi(U), q_2) = 1 \geq \max \hat{\pi}(D) \in k(\pi(D), q_2) = 1$$

On the other hand, if we had followed the path q_0, q_1, q_4, q_5, q_2 we would have had

$$k(\pi(U), q_2) = \{1\}$$

$$k(\pi(D), q_2) = \{1, 2\}$$

so that

$$\min \hat{\pi}(D) \in k(\pi(D), q_2) = 1 \geq \max \hat{\pi}(U) \in k(\pi(U), q_2) = 1$$

Now consider a path between two states \hat{q} and \tilde{q} along which there is a transition from

$$k(\pi(D), \hat{q}) = \{0, 1, 2\}$$

to

$$k(\pi(D), \tilde{q}) = \{0, 1\}$$

or

$$k(\pi(D), \tilde{q}) = \{1, 2\}$$

It is clear that the presence of such a transition is necessary for an automaton to be implementing (consider any situation in which $\pi(U) = 1$). Next note that this transition can be made with one question about x_2 but requires two questions about x_5 and x_6 , and that no other information relevant to a decision between U and D at the first node can be provided by any question about any of x_2, x_5, x_6 . Thus however many states, say S , is required for the state-minimal automaton using questions only about (x_1, x_2, x_3, x_4) , an automaton using questions only about $(x_1, x_3, x_4, x_5, x_6)$ will require at least one more state. ■

Now we are ready to prove Proposition 1.2, as follows.

Proof. Proposition B.1 shows that any implementing backwards-reasoning automaton must contain states examining the values of either $(x_1, x_3, x_4, x_5, x_6)$, or $(x_2, x_3, x_4, x_5, x_6)$, or $(x_1, x_2, x_3, x_4, x_5, x_6)$. Proposition B.2 shows that automata asking about only $(x_1, x_3, x_4, x_5, x_6)$ cannot be state-minimal. By symmetry, this proof also applies to automata asking about only $(x_1, x_3, x_4, x_5, x_6)$. The set of questions $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ can identify $\pi(U)$ exactly for us

(i.e. we can reach some state q for which $k(\pi(D), q)$ is a singleton). Hence Proposition B.2 does not show that a state-minimal automaton asking questions about (x_1, \dots, x_6) will have more states than a state-minimal automaton asking questions only about (x_1, x_2, x_3, x_4) . However, Figure 1.3 exhibits an implementing automaton with only seven states, whilst the definition of regularity guarantees that any implementing automaton asking questions about all of (x_1, \dots, x_6) must have at least eight states. Hence a state-minimal backwards-reasoning automaton must have at least one more state than a general implementing state-minimal automaton. ■

Appendix C

Deciding and anticipating

In this appendix we shall investigate further the distinction between deciding what to do today on the basis of a correct expectation about tomorrow and deciding what to do both today and tomorrow.

Our Main Lemma shows that Figure 1.3 is a state-minimal FA for solving the problem of optimal play at the opening node of Figure 1.2.

Note that this FA tells our decision-maker what to do at the opening node based on the *expectation* that he will play optimally at the second reached node. It does not, however, tell him what to play at that node. He is taking his play at the second node as given, even though he will be able to control it - just as in the standard formulation of the Jehiel Limited Foresight concept.

Now consider a FA which tells us what to do at both the first and second reached nodes. It follows immediately from our Main Lemma that such a FA must have at least nine states (since it requires four final states rather than the two of Figure 1.3) - and, although it is not contended that nine states is enough, that is sufficient to show that a structurally different FA will be required for this problem than for the one solved in Figure 1.3 (an example of a FA which solves this problem is exhibited in Figure 1.3). That is to say, the problem of deciding what it is best to do at the opening node based on a *correct expectation* of what will be done at later nodes (the problem solved in Figure 1.3) is a conceptually different problem from that of deciding what it is best to do throughout the foresight horizon (the problem solved in Figure 1.5).

The other thing to notice is that there may be efficiency advantages to determining play from the beginning of the problem rather than waiting until a decision is necessary. Consider a player who decides what to do at the initial node of the problem in Figure 1.2, and waits until he reaches a subsequent node to decide what to do there. At the initial node his state-minimal FA is Figure 1.3 - seven states. Suppose optimal play at the opening

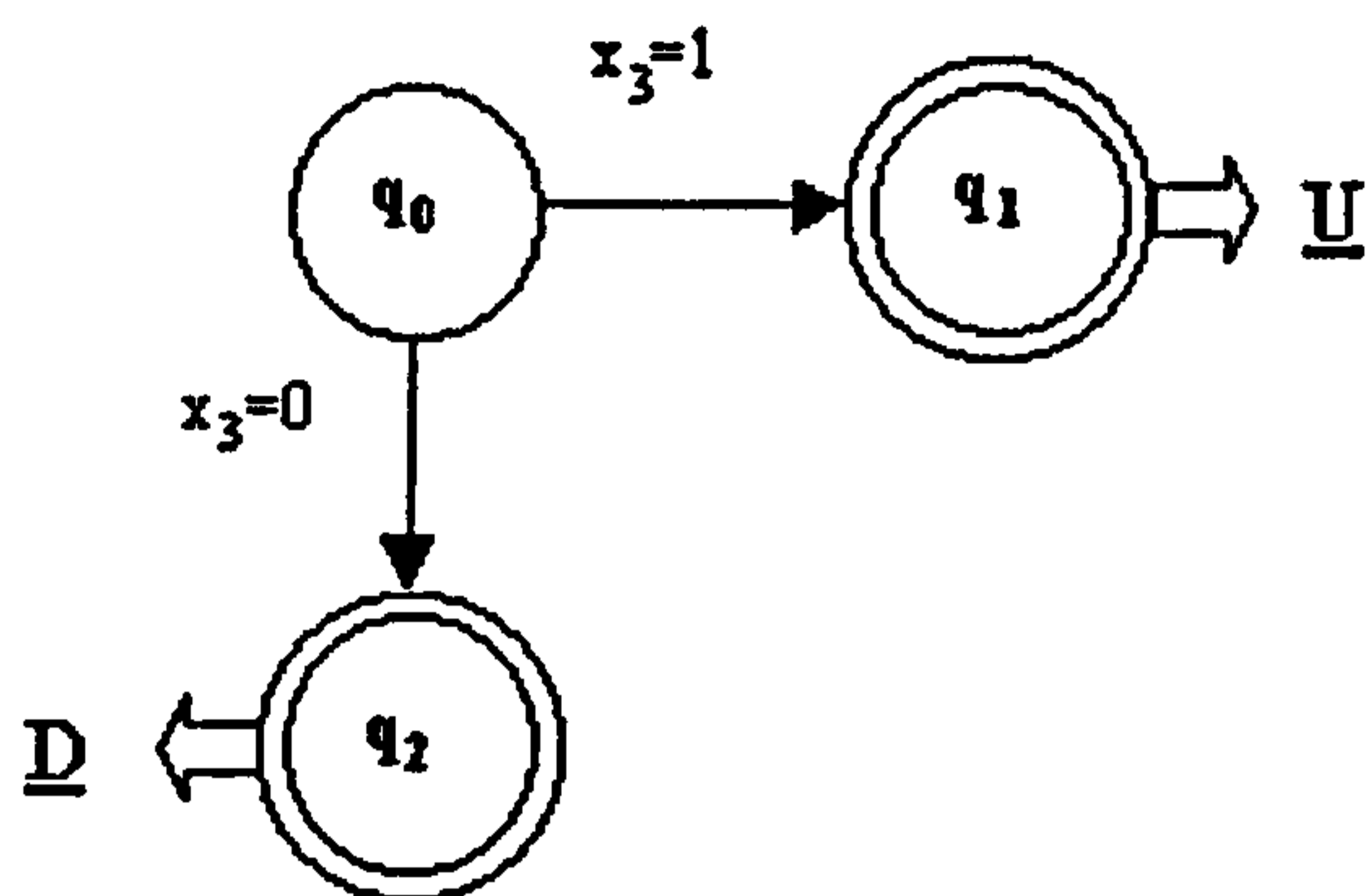


Figure C.1:

node is Up. Then at the second node he will have to choose the better of $\{x_3, x_4\}$, a task for which a state-minimal FA clearly has three states and looks like Figure C.1.

If Down had been best at the initial node, then he would have reached the lower right node, and had to use another three-state FA like Figure C.1, except questioning x_5 (say). Therefore, a player deciding optimal play for the game in two steps will need to have available three automata totalling thirteen states ($7 + 3 + 3$). If he decides optimal play at the beginning using Figure 1.5 he will need only eleven (or possibly fewer) states - more efficient¹. Hence it seems reasonable to suppose that players will try to control what they do in the future as far ahead as they can achieve with certainty.

¹Of course, there may be advantages to waiting until later rounds if "complexity" is measured by the highest number of states of any single required FA - here seven for the step-by-step solution compared with eleven for the whole-game solution.

Appendix D

Existence and uniqueness proofs for the LFE-N and LFE-J concepts

Remark D.1 *There is no existence issue for LFE-N as all such problems are simply overlaying finite horizon games, for which existence is well-established.*

Jehiel (1995) has provided an existence proof of wide application for the (n_1, n_2) -equilibrium concept. Here it is convenient to exhibit a few results for the slightly broader LFE-J concept.

Proposition D.1 *An LFE-J exists in all finite limited-foresight decision-problems without a chance player and in which payoffs do not depend on history (i.e. u_1 is independent of h^t).*

Proof. Denote $n \equiv n_1$. Cases $n = 0$ or $n = 1$ are trivial (since there is no foresight in the $n = 0$ case everything is an equilibrium, while in the $n = 1$ case the equilibrium consists simply of adopting the optimal one-shot action in any period). Assume $n > 1$. Prove by construction. Consider a problem with T periods to go. Construct an LFE-J by induction on periods from period T . For the base step, in period T set $q_{1,T}^* = \arg \max_a u_1(a^T)$. Then for $T + 1 - n \leq t < T$ set $(q_{1,T}) = q_{1,T}^*$. For the induction step, for $\tilde{q}_{1,t}$ already constructed, set $q_{1,t}^* = \arg \max_{q_{1,t}} \tilde{\pi}_{i,t}(q_{i,t}, \tilde{q}_{i,t})$. Denote by q_1^* the profile $q_{1,t}^*, \forall t$. q_1^* is an LFE-J by construction, hence an LFE-J exists. ■

Proposition D.2 *An LFE-J exists in all infinitely-repeated limited-foresight decision-problems in which stage payoffs are independent of history.*

Proof. Consider the following strategy/expectations pair. In each round play the behaviour-strategy maximising the one-shot stage payoff, expecting that in all subsequent rounds out to the horizon of foresight play will again be that maximising the one-shot stage payoff. This is clearly an LFE-J. ■

Having illustrated that an LFE-J exists for many interesting problems¹, we shall now illustrate that (unsurprisingly, but comfortingly) sometimes there is only one LFE-J, but on other occasions there are many. The possibility of multiple LFE-Js will be important later when we use Rubinstein's critique to construct a solution-concept which is a *refinement* of LFE-J.

An important class of solutions to infinitely-repeated decision-problems are those in which players always perform the same action in each sub-game - stationary equilibria. We shall now prove that in pure strategies, under certain conditions, there will be no more than one stationary LFE-J².

Proposition D.3 *With only one player, no chance player, two actions ($a \in \{X, Y\}$), only pure strategies permitted, no discounting, no history-dependence of strategies in equilibrium (so q is independent of h^t), and u dependent on history only as far back as the horizon of foresight, n , there is no more than one LFE-J except in special cases where the payoffs to playing X every period and playing Y every period are the same. In particular, the following two strategies cannot both be equilibria:*

Strategy X: *Play X every round, expecting play of X in all future rounds*

Strategy Y: *Play Y every round, expecting play of Y in all future rounds*

Proof. State payoffs in the form $U(b, V_1 \dots V_{n-1})$, where b is the action this period, V_1 is the action last period, V_2 is the action two period ago, etc., and $b, V_s \in \{X, Y\}, \forall s$.

Call the reference period at which the decision must be take period t .

Assume first that the payoffs to playing X every period are not equal to those of playing Y every period (i.e. rule out the trivial case):

$$0) U(X, X \dots X) \neq U(Y, Y \dots Y)$$

¹We exhibit a sketch of a more general existence proof for decision-problems in Chapter 5.

²Note that the LFE-J concept violates the Vieille & Weibull (2002) "weakly increasing patience" sufficiency condition for uniqueness in decision problems, since here at the horizon of foresight patience suddenly decreases (the decision-maker becomes, as it were, infinitely impatient).

For Strategy X to be a Jehiel equilibrium, we require that the combined payoffs over the foresight horizon must be greater from playing X today than from playing Y, and this must be true for every history. In particular it must be true for the history where Y has been played in every period up to t :³

$$\begin{aligned} 1) & U(X, Y...Y) + U(X, XY...Y) + \dots + U(X, X...Y) + U(X, X...X) \\ & \geq U(Y, Y...Y) + U(X, Y...Y) + U(X, XY...Y) + \dots + U(X, X...Y) \end{aligned}$$

Similarly, for Strategy Y to be a Jehiel equilibrium we require that the combined payoffs over the foresight horizon must be greater from playing X today than from playing Y for the history where X has been played in every period up to t :

$$\begin{aligned} 2) & U(Y, X...X) + U(Y, YX...X) + \dots + U(Y, Y...X) + U(Y, Y...Y) \\ & \geq U(X, X...X) + U(Y, X...X) + U(Y, YX...X) + \dots + U(Y, Y...X) \end{aligned}$$

Simplifying (1) gives

$$3) U(X, X...X) \geq U(Y, Y...Y)$$

Simplifying (2) gives

$$4) U(Y, Y...Y) \geq U(X, X...X)$$

(3) & (4) together imply

$$5) U(Y, Y...Y) = U(X, X...X)$$

But (5) is a flat contradiction of inequality (0)

Hence we reject the assumption. Strategy X & Strategy Y cannot both be Jehiel equilibria. Since the only strategies under consideration are those in which play is the same every period and expectations of future play are history-independent, these are the only possible equilibria. Hence there cannot be more than one such LFE-J in this case. ■

Remark D.2 *It should be noted that the Proposition D.3 does not show that an LFE-J of this form exists. It merely shows that if there is such an LFE-J, there is no more than one.*

Remark D.3 *In Proposition D.3 the requirement that stage payoffs depend only on history of length no greater than the horizon of foresight is vital. For example, consider the following:*

Take a foresight length of $n = 2$, and a hindsight dependence of 3 (i.e. $U = U(b, V_1V_2)$).

Now consider the following payoffs.

³Note that even for the case when history up to t had consisted of a continuous sequence of X, for the consistency condition of the LFE-J to be fulfilled, Condition (1) would still be necessary, since Strategy X requires that X is played in future rounds out to the horizon of foresight, *whatever is done today*. In particular, off the equilibrium path it could not be best for a play of Y today followed by a future continuous stream of Y to be superior to playing X at any point.

$$U(Y, YX) = 12$$

$$U(X, XY) = 10$$

$$U(Y, YY) = 9$$

$$U(X, XX) = 7$$

$$U(X, YX) = 6$$

$$U(Y, XY) = 5$$

$$U(X, YY) = 3$$

$$U(Y, XX) = 2$$

For Strategy X (i.e. play X every time, expecting X to be played in the future regardless of what is played today and regardless of history) to be an LFE- J , the following need to be true (checking all relevant histories):

- 1) $2U(X, XX) \geq U(Y, XX) + U(X, YX)$
- 2) $U(X, YX) + U(X, XY) \geq U(Y, YX) + U(X, YY)$
- 3) $U(X, YY) + U(X, XY) \geq U(Y, YY) + U(X, YY)$
- 4) $U(X, XY) + U(X, XX) \geq U(Y, XY) + U(X, YX)$

In this case these inequalities imply

- 1) $2 * 7 = 14 > 2 + 6 = 8$
- 2) $6 + 10 = 16 > 12 + 3 = 15$
- 3) $3 + 10 = 13 > 9 + 3 = 12$
- 4) $10 + 7 = 17 > 5 + 6 = 11$

Similarly, for Strategy Y to be an LFE- J , we need the following to be true:

- 5) $2U(Y, YY) \geq U(X, YY) + U(Y, XY)$
- 6) $U(Y, XY) + U(Y, YX) \geq U(X, XY) + U(Y, XX)$
- 7) $U(Y, XX) + U(Y, YX) \geq U(X, XX) + U(Y, XX)$
- 8) $U(Y, YX) + U(Y, YY) \geq U(X, YX) + U(Y, XY)$

which implies here

- 5) $2 * 9 = 18 > 3 + 5 = 8$
- 6) $5 + 12 = 17 > 10 + 2 = 12$
- 7) $2 + 12 = 14 > 7 + 2 = 9$
- 8) $12 + 9 = 21 > 6 + 5 = 11$

Hence both Strategy X and Strategy Y are LFE- J s in this case. This suffices to prove that the previous restriction we imposed (namely that the hindsight length was no greater than the foresight length) was vital.

Appendix E

Proofs about rankings of inflation biases

E.1 Proof that the average inflation bias is less in the limited foresight case than the infinite horizon case

First prove that $c > \widehat{c}^B$. Resolve:

$$\begin{aligned}
 \frac{1}{2\alpha\beta\rho} \left[1 - \beta\rho^2 - \sqrt{(1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2} \right] &\stackrel{c \Leftrightarrow}{\Leftrightarrow} \frac{1}{2\alpha\beta\rho} \left(1 - \sqrt{1 - 4\beta\lambda\alpha^2\rho^2(1 + \beta\rho^2)} \right) \\
 &\stackrel{\widehat{c}^B}{\Leftrightarrow} \frac{\beta\rho^2 + \sqrt{(1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2}}{\beta^2\rho^4 + (1 - \beta\rho^2)^2 - 4\beta\lambda\alpha^2\rho^2} \\
 &\stackrel{\Leftrightarrow}{\Leftrightarrow} \frac{1 - 4\beta\lambda\alpha^2\rho^2 - 4\beta^2\lambda\alpha^2\rho^4}{+2\beta\rho^2\sqrt{(1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2}} \\
 &\stackrel{\Leftrightarrow}{\Leftrightarrow} \frac{2\beta\rho^2(1 - \beta\rho^2 - 2\beta\lambda\alpha^2\rho)}{2\beta\rho^2\sqrt{(1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2}} \\
 (1 - \beta\rho^2)^2 + 4\beta^2\lambda^2\alpha^4\rho^2 - 4\lambda\alpha^2\beta\rho^2 + 4\beta^2\lambda\alpha^2\rho^3 &\stackrel{\Leftrightarrow}{\Leftrightarrow} (1 - \beta\rho^2)^2 - 4\lambda\alpha^2\beta\rho^2
 \end{aligned}$$

But $4\beta^2\lambda^2\alpha^4\rho^2 + 4\beta^2\lambda\alpha^2\rho^3 > 0$
since all coefficients are positive

Therefore
 $c > \widehat{c}^B$

Now

$$\begin{aligned}
 c &> \widehat{c}^B \Rightarrow 1 - \beta\rho - \beta\alpha c < 1 - \beta\rho - \beta\alpha\widehat{c}^B \\
 \text{and hence, since } 1 - \beta\rho - \beta\alpha c &> 0, \\
 \frac{\lambda\alpha l^*}{1 - \beta\rho - \beta\alpha c} &> \frac{\lambda\alpha l^*}{1 - \beta\rho - \beta\alpha\widehat{c}^B} \\
 &\text{but} \\
 \frac{\lambda\alpha l^*}{1 - \beta\rho - \beta\alpha\widehat{c}^B} &> \frac{\lambda\alpha(1 + \beta\rho)l^*}{1 - \beta\alpha\widehat{c}^B}
 \end{aligned}$$

Hence, whenever the Rational inflation bias is positive,

$$\text{Rational } \pi\text{-bias} > \pi\text{-bias under limited foresight}$$

E.2 Proof that the average inflation bias is greater in the limited foresight case than the finite horizon case

Remember that the limited foresight non-state dependent inflation bias is given by $\frac{\lambda\alpha(1+\beta\rho)l^*}{1-\beta\alpha\widehat{c}^B}$. Clearly the round $t + 1$ inflation bias will be lower for the finite horizon Central Banker, since $\beta\rho > 0$ and $0 < \beta\alpha\widehat{c}^B < 1$ implies $\frac{(1+\beta\rho)}{1-\beta\alpha\widehat{c}^B} > 1$. In the final round before he retires the finite horizon Banker is like a limited foresight Banker with a foresight length of 1 (caring only about today). Thus he will have a lower inflation bias than a limited foresight Banker with a foresight length of 2, for the same reasons as above.

In round t the limited horizon inflation bias is once again lower, as follows:

E.2. PROOF THAT THE AVERAGE INFLATION BIAS IS GREATER...123

Limited Foresight Bias \Leftrightarrow	Finite Horizon Bias
$\frac{\lambda\alpha(1+\beta\rho)l^*}{1-\beta\alpha\hat{c}^B} \Leftrightarrow$	$\lambda\alpha(1+\beta\rho(1+\lambda\alpha^2))l^*$
$1+\beta\rho \Leftrightarrow$	$1+\beta\rho+\beta\rho\lambda\alpha^2-\beta\alpha\hat{c}^B-\beta^2\rho\alpha\hat{c}^B-\beta^2\rho\lambda\alpha^3\hat{c}^B$
$\hat{c}^B(1+\beta\rho+\beta\rho\lambda\alpha^2) \Leftrightarrow$	$\rho\lambda\alpha$
$1-\sqrt{1-4\beta\lambda\alpha^2\rho^2(1+\beta\rho^2)} \Leftrightarrow$	$\frac{2\beta\rho^2\lambda\alpha^2}{(1+\beta\rho+\beta\rho\lambda\alpha^2)}$
$1-\frac{2\beta\rho^2\lambda\alpha^2}{(1+\beta\rho+\beta\rho\lambda\alpha^2)} \Leftrightarrow$	$\sqrt{1-4\beta\lambda\alpha^2\rho^2(1+\beta\rho^2)}$
$-4\beta\rho^2\lambda\alpha^2(1+\beta\rho+\beta\rho\lambda\alpha^2)+4\beta^2\rho^4\lambda^2\alpha^4 \Leftrightarrow$	$-4\beta\lambda\alpha^2\rho^2(1+\beta\rho^2)(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
$-(1+\beta\rho+\beta\rho\lambda\alpha^2)+\beta\rho^2\lambda\alpha^2 \Leftrightarrow$	$-(1+\beta\rho^2)(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
$-1-\beta\rho-\beta\rho\lambda\alpha^2+\beta\rho^2\lambda\alpha^2 \Leftrightarrow$	$-\beta\rho^2(1+\beta\rho+\beta\rho\lambda\alpha^2)^2-(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
$-1-\beta\rho-\beta\rho\lambda\alpha^2+\beta\rho^2\lambda\alpha^2 \Leftrightarrow$	$-\beta\rho^2(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
$\beta\rho^2\lambda\alpha^2 \Leftrightarrow$	$-\left(\begin{array}{l} 1+2\beta\rho+\beta^2\rho^2 \\ +2(1+\beta\rho)\beta\rho\lambda\alpha^2+\beta^2\rho^2\lambda^2\alpha^4 \end{array}\right)$
$0 >$	$-\beta\rho^2(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
	$-\left(\begin{array}{l} \beta\rho+\beta^2\rho^2 \\ +(1+2\beta\rho)\beta\rho\lambda\alpha^2+\beta^2\rho^2\lambda^2\alpha^4 \end{array}\right)$
	$-\rho(1+\beta\rho+\beta\rho\lambda\alpha^2)^2$
	$-(1+\beta\rho+\lambda\alpha^2+2\beta\rho\lambda\alpha^2+\beta\rho\lambda^2\alpha^4)$
	$-\rho\lambda\alpha^2$
	(since all coefficients are positive)

Therefore
Limited Foresight Bias $>$

Finite Horizon Bias

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