An economic MPC formulation with offset-free asymptotic performance

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Abstract: This paper proposes a novel formulation of economic MPC for nonlinear discrete-time systems that is able to drive the closed-loop system to the (unknown) optimal equilibrium, despite the presence of plant/model mismatch. The proposed algorithm takes advantage of: (i) an augmented system model which includes integrating disturbance states as commonly used in offset-free tracking MPC; (ii) a modifier-adaptation strategy to correct the asymptotic equilibrium reached by the closed-loop system. It is shown that, whenever convergence occurs, the reached equilibrium is the true optimal one achievable by the plant. An example of a CSTR is used to show the superior performance with respect to conventional economic MPC and a previously proposed offset-free MPC still based on a tracking cost. The implementation of this offset-free economic MPC requires knowledge of plant input-output steady-state map gradient, which is generally not available. To this aim a simple linear identification procedure is explored numerically for the CSTR example, showing that convergence to a neighborhood of the optimal equilibrium is possible.

Keywords: Economic MPC, RTO, Offset-free Control, Modifier-Adaptation, Uncertainties.

1. INTRODUCTION

Model predictive control (MPC) algorithms are wide-spread in the process industries given their ability to control large-scale, multivariable, constrained (linear and nonlinear) processes (Qin and Badgwell, 2000, 2003). Typical industrial formulations of MPC are implemented within a hierarchical architecture (Scattolini, 2009), so that they indirectly achieve economic optimization goals by tracking setpoints computed by an upper steady-state economic optimization layer, usually referred to as Real-Time Optimization (RTO).

Model uncertainties and disturbances can cause performance degradation in both layers, such as reaching suboptimal equilibria, violating constraints and in worst cases even instability. This motivated extensive research studies, which often found practical implementations with rigorous theoretical guarantees. In RTO, several approaches have been proposed so that, upon convergence, the reached equilibrium is an optimal point for the true unknown plant (Marchetti et al., 2009; Francois and Bonvin, 2013; Navia et al., 2015). One key aspect of so-called "modifier-adaption" methods is that they require knowledge of the true plant gradient, which is typically not available and this prompted for the search of possible alternative routes (Costello et al., 2016). In MPC, offset-free control action is obtained by augmenting the system model with so-called "disturbance", which integrates at each decision time the prediction error until this is eliminated. In this way, the model prediction is asymptotically correct and the MPC tracks admissible setpoints without offset (Pannocchia et al., 2015). This disturbance may take the form of a simple output bias or more elaborated, and effective, state and input disturbance, applicable to both linear and nonlinear MPC algorithms (Pannocchia et al., 2015; Morari and Maeder, 2012). Other offset-free MPC implementations are based on velocity models, which are shown to be still equivalent to particular disturbance models (Pannocchia, 2015).

Recently, the hierarchical separation between economic optimization (RTO) and constrained tracking (MPC) has been questioned by formulations in which a single dynamic optimization layer exists, running at the usual MPC rates, in which an economic cost is minimized. These formulations are now referred to as Economic MPC (Rawlings et al., 2012; Ellis et al., 2014). The key aspect of economic MPC, opposed to standard tracking MPC, is that the cost function is not necessarily positive definite around the equilibrium, so that in the transient it may be more convenient to operate away from the equilibrium (Rawlings et al., 2012). Depending on various factors (system, cost, horizon) asymptotically stable behavior may be optimal, but in some cases even oscillating or chaotic regimes can outperform the steady-state cost on average (Angeli et al., 2012). Closed-loop convergence analysis is now better understood thanks to dissipativity arguments and turnpike properties (Grüne and Müller, 2016; Faulwasser and Bonvin, 2017).

The existing economic MPC algorithms can cope with some kind of bounded perturbations and preserve stability properties (Bayer et al., 2014), but in the presence of plant/model mismatch, the closed-loop system may converge to an equilibrium that is not the most economically optimal one. In some cases, a remedy to achieve the optimal equilibrium is to augment the system model with offset-free disturbances (Pannocchia et al., 2015). Other approaches are based on multi-model linear offset-free formulations (Alvarez and Odloak, 2012; Ferramosca et al., 2017). In a recent work (Vaccari and Pannocchia, 2017) a new offset-free MPC formulation was proposed in which an economic steady-state modifier-adaption strategy was coupled with offset-free augmented models. This method allows the closed-loop system to reach the optimal equilibrium despite plant/model mismatch, but convergence is guaranteed only when the finite-horizon optimal control problem uses a tracking cost.

In the present paper, the work in (Vaccari and Pannocchia, 2017) is extended to achieve a fully economic, offset-free MPC, by including a suitable modifier term into the optimal control problem so that the reached equilibrium is the true optimum for the plant. A second contribution of this paper is to explore by simulation a simple, easily implementable strategy for plant steady-state input/output map gradient estimation from data. The rest of this paper is organized as follows. The problem statement and related material is presented in Section 2, while the proposed method is described in Section 3. A detailed simulation study is reported in Section 4. The main achievements are recapped in Section 5 along with possible future work.

2. PROBLEM STATEMENT AND RELATED WORKS

2.1 Plant, cost and constraints

In this paper we consider that the controlled plant is given by the following discrete-time nonlinear system:

$$\begin{aligned} x^+ &= f_p(x,u) \\ y &= h_p(x) \end{aligned} \tag{1}$$

in which $x \in \mathbb{R}^{n_x}$ is the current state, $x^+ \in \mathbb{R}^{n_x}$ is the successor state, $u \in \mathbb{R}^{n_u}$ is the current input, and $y \in \mathbb{R}^{n_y}$ is the current output. The functions $f_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $h_p : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ are not known precisely but are assumed (at least) continuously differentiable. Unless differently specified, the following assumption is used throughout this paper.

Assumption 1. The output y is measured at each decision time k, and its measured value is denoted by $y(k)$.

A triple (x_s, u_s, y_s) is defined as an equilibrium for the plant (1) if it satisfies:

$$\begin{aligned} x_s &= f_p(x_s,u_s) \\ y_s &= h_p(x_s) \end{aligned} \tag{2}$$

Input and output constraints in the following form should be fulfilled:

$$u(k) \in \mathbb{U}, \qquad y(k) \in \mathbb{Y}, \qquad \forall k \in \mathbb{N} \tag{3}$$

in which $\mathbb{Y} \subset \mathbb{R}^{n_y}$ and $\mathbb{U} \subset \mathbb{R}^{n_u}$ are given, compact sets.

We consider that the controlled system is economically optimized when a given cost function $\ell(y,u)$ is minimized, in which $\ell : \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is assumed continuously differentiable. Thus, the optimal equilibrium for the plant (1) is defined as:

$$(x_s^0, u_s^0, y_s^0) = \arg\min_{(x,u,y)} \ell(y,u) \tag{4a}$$

subject to

$$x = f_p(x,u) \tag{4b}$$

$$y = h_p(x) \tag{4c}$$

$$y \in \mathbb{Y}, \qquad u \in \mathbb{U} \tag{4d}$$

We make the following assumption.

Assumption 2. The triple (x_s^0, u_s^0, y_s^0) is the unique KKT point of (4).

Remark 3. Given that the plant (1) is not precisely known, the optimal solution (x_s^0, u_s^0, y_s^0) is unknown.

2.2 Nominal model and standard economic MPC formulation

In order to design an MPC for controlling plant (1), a nominal model is known, and has the following form:

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x) \end{aligned} \tag{5}$$

in which the functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ are continuously differentiable.

Let $\hat{x}$ denote the current state (estimate) of the nominal model (5). A standard economic MPC algorithm, using the plant model (5), is based on the repeated solution of the following finite-horizon optimal control problem (FHOCP):

$$\mathbb{P}_N(\hat{x}) : \qquad \min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell(y_i, u_i) \tag{6a}$$

subject to:

$$x_0 = \hat{x} \tag{6b}$$

$$x_{i+1} = f(x_i, u_i), \qquad y_i = h(x_i) \tag{6c}$$

$$x_N = x_s \tag{6d}$$

$$y_i \in \mathbb{Y}, \qquad u_i \in \mathbb{U} \tag{6e}$$

in which N is a positive integer, $\mathbf{x} = (x_0, x_1, \ldots, x_N)$, and $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1})$ are state and input sequences. For any given initial state $\hat{x}$, we denote the optimal solution of $\mathbb{P}_N(\hat{x})$ as $(\mathbf{x}^*, \mathbf{u}^*)(\hat{x})$, and hence the feedback control law is given by:

$$\kappa_N(\hat{x}) = u_0^*(\hat{x}) \tag{7}$$

Thus, the closed-loop system has the following dynamics:

$$\begin{aligned} x^+ &= f_p(x, \kappa_N(\hat{x})) \\ y &= h_p(x) \end{aligned} \tag{8}$$

In conventional MPC algorithms $\ell(\cdot)$ is a tracking cost, i.e. it is nonnegative everywhere and zero only at the equilibrium. Then, when the model (5) is perfect, under standard assumptions one can show that the equilibrium is an asymptotically stable fixed point of the closed-loop system. On the other hand, if $\ell(\cdot)$ is a generic cost, there exist non-equilibrium triples $(x, u, y = h(x))$ at which $\ell(y,u) < \ell(y_s^0, u_s^0)$, and this fact may induce complex asymptotic closed-loop behaviors other than asymptotic stability of the equilibrium. Establishing conditions under which the equilibrium of the closed-loop system, with economic MPC, is asymptotically stable has been an active research domain. A Lyapunov candidate for terminally constrained problems was found in (Diehl et al., 2011), and then the concept of dissipativity arose as the most natural one to establish stability (Angeli et al., 2012; Müller et al., 2015). Another approach to prove asymptotic stability is by analyzing the so-called turnpike property (Faulwasser and Bonvin, 2017), which can also be related to dissipativity (Grüne and Müller, 2016).

The situation becomes even more complex when mismatch exists between the model (5) and the actual plant (1). This plant/model mismatch may lead to closed-loop instability and, even when stability is preserved, the closed-loop system does not necessarily converge to the correct optimal equilibrium. However, in the context of tracking MPC, so-called "offset-free" formulations ensure that if the closed-loop convergences to an equilibrium, then the output tracks a given output setpoint. The goal of this work is to design an economic MPC algorithm such that, if the closed-loop system reaches an equilibrium, this corresponds to the most profitable equilibrium for the true plant.

2.3 State observer and state-feedback case

Since the state x may not be measurable, in general it is necessary to use an observer. Many different options can be considered, ranging from simple linear, static observers to nonlinear, time-varying and possibly constrained observers such as the so-called Moving Horizon Estimators (MHE). For simplicity of exposition, we restrict our attention to a static observer in the form:

$$\hat{x}(k|k) = \hat{x}(k|k-1) + \kappa(y(k) - h(\hat{x}(k|k-1))) \quad (9)$$

in which $\hat{x}(k|k)$ represents the filtered estimate of the current state, $\hat{x}(k|k-1)$ the prediction of the current state made at the previous decision time, using the model (5), i.e

$$\hat{x}(k|k-1) = f(\hat{x}(k-1|k-1), u(k-1)) \quad (10)$$

and $\kappa : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$. The argument of $\kappa(\cdot)$ in (9) is referred to as the prediction error:

$$e(k) = y(k) - h(\hat{x}(k|k-1)) \quad (11)$$

which represents the difference between the actual measured output and its predicted value using the information available at the previous decision time. Such an error may arise due to plant/model mismatch, as well as measurement noise.

Remark 4. The function $\kappa(\cdot)$ is assumed continuous and satisfying the condition $\kappa(0) = 0$, i.e. the state prediction is not updated when the prediction error is zero.

Remark 5. If the state is measurable, i.e. $h_p(x) = h(x) = x$, the general formulation (9)-(10) can be recovered by simply defining the observer function as $\kappa(e) = e$ which leads to $\hat{x}(k|k) = x(k)$. This is also known as a deadbeat observer.

3. PROPOSED METHOD

We here present a novel economic MPC that allows the closed-loop system to converge to the true optimal equilibrium (y_s^0, u_s^0), without knowing such an optimal equilibrium. We denote such a controller as offset-free economic MPC.

3.1 Offset-free augmented system and observer

The first "ingredient" of the proposed offset-free economic MPC is an augmented model, which ensures that the prediction error ultimately goes to zero, i.e. $\lim_{k\to\infty} e(k) = 0$ independently of the true plant dynamics. Many different disturbance models and observers can be used (Pannocchia et al., 2015), but in order to streamline the presentation we restrict our attention to linear disturbance models. Thus, the augmented system is:

$$\begin{aligned} x^+ &= f(x,u) + B_d d \\ d^+ &= d \\ y &= h(x) + C_d d \end{aligned} \quad (12)$$

in which $d \in \mathbb{R}^{n_y}$ is the so-called "disturbance", whereas $B_d \in \mathbb{R}^{n_x \times n_y}$ and $C_d \in \mathbb{R}^{n_y \times n_y}$ are user-defined matrices shaping the effect of such disturbance on the state and output maps.

Remark 6. As detailed in (Pannocchia et al., 2015), the number of disturbances should be equal to the number of outputs.

The state and disturbance are simultaneously estimated using the output measurement by an augmented observer:

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + \kappa_x(e(k)) \\ \hat{d}(k|k) &= \hat{d}(k|k-1) + \kappa_d(e(k)) \end{aligned} \quad (13)$$

in which the predicted state, disturbance and output are computed from the augmented model (12), i.e.:

$$\begin{aligned} \hat{x}(k|k-1) &= f(\hat{x}(k-1|k-1), u(k-1)) + B_d \hat{d}(k-1|k-1) \\ \hat{d}(k|k-1) &= \hat{d}(k-1|k-1) \\ \hat{y}(k|k-1) &= h(\hat{x}(k|k-1)) + C_d \hat{d}(k|k-1) \end{aligned} \quad (14)$$

and the prediction error is given by:

$$\begin{aligned} e(k) &= y(k) - \hat{y}(k|k-1) \\ &= y(k) - h(\hat{x}(k|k-1)) - C_d \hat{d}(k|k-1) \end{aligned} \quad (15)$$

We make the following assumption on the augmented observer.

Assumption 7. The functions $\kappa_x : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ and $\kappa_d : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ are continuous, $\kappa_x(0) = 0$, and $\kappa_d(e) = 0$ if and only if $e = 0$. Moreover, the observer formed by (13)-(14) is nominally asymptotically stable.

Remark 8. The observer nominal stability implies detectability of the augmented system (12).

Remark 9. If the state is measurable, $h_p(x) = h(x) = x$, a state disturbance model can be used by choosing $B_d = I$ and $C_d = 0$ coupled with a linear deadbeat observer $\kappa_x(e) = e$ and $\kappa_d(e) = e$. This leads to the following deadbeat filtered state and disturbance

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + (x(k) - \hat{x}(k|k-1) = x(k) \\ \hat{d}(k|k) &= \hat{d}(k|k-1) + (x(k) - \hat{x}(k|k-1) \\ &= x(k) - f(x(k-1), u(k-1)) \end{aligned} \quad (16)$$

That is, the model state is realigned to the true plant state at each decision time, and the disturbance is equal to the so-called innovation $x(k) - f(x(k-1), u(k-1))$. From (12), with $B_d = I$, it follows that such a disturbance is added to the successor state prediction as a bias.

3.2 Steady-state target optimization with modifier adaptation

Given the current disturbance estimate $\hat{d}(k|k)$, as in offset-free tracking MPC algorithms, we compute an equilibrium target using the augmented model (12). However, in order to have this equilibrium target converging to the true optimal equilibrium, we need to add a first-order modifier. The idea of introducing a modifier is borrowed from the RTO literature (Marchetti et al., 2009). Specifically, the target problem solved at each decision time is the following:

$$(x_s^*(k), u_s^*(k), y_s^*(k)) = \arg\min_{(x,u,y)} \ell(y,u) \quad (17a)$$

subject to

$$x = f(x,u) + B_d \hat{d}(k|k) \quad (17b)$$

$$y = h(x) + C_d \hat{d}(k|k) + \Lambda(k)(u - u_s^*(k-1)) \quad (17c)$$

$$y \in \mathbb{Y}, \qquad u \in \mathbb{U} \quad (17d)$$

in which $\hat{d}(k|k)$ is the current disturbance estimate, $\Lambda(k) \in \mathbb{R}^{n_y \times n_u}$ is the current modifier matrix, later defined, and $u_s^*(k-1)$ is the previous input target. The modifier matrix is initialized as $\Lambda(0) = 0$, and updated at each decision time as follows:

$$\Lambda(k+1) = \phi\Lambda(k) + (1-\phi)(DG_p(u_s^*(k)) - DG(u_s^*(k), \hat{d}(k|k))) \quad (18)$$

in which:

- $G_p : \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ is the plant steady-state input-to-output map, i.e. the solution $y_s(u_s)$ of (2), and $DG_p : \mathbb{R}^{n_u} \to \mathbb{R}^{n_y \times n_u}$ is its Jacobian;
- $G : \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is the model steady-state map, i.e. the solution $y_s(u_s, \hat{d}(k|k))$ of (17b), and $DG : \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_y \times n_u}$ is its Jacobian with respect to the first argument;
- $\phi \in (0,1]$ is the first-order filter constant.

By KKT matching (Marchetti et al., 2009), it is shown in (Vaccari and Pannocchia, 2017) that:

$$\lim_{k\to\infty} y_s^*(k) = y_s^0, \qquad \lim_{k\to\infty} u_s^*(k) = u_s^0 \quad (19)$$

provided that $\lim_{k\to\infty} u(k) = \lim_{k\to\infty} u_s^*(k)$.

3.3 FHOCP with modifier adaptation

Given the current state and disturbance estimates $(\hat{x}(k|k), \hat{d}(k|k))$ and the current input target $u_s(k)$, we solve an FHOCP using the augmented model (12). Moreover, as in the target optimization (17) we introduce in the FHOCP a first-order modifier to ensure convergence towards the target. In details, the FHOCP solved at each decision time is:

$$\mathbb{P}_N(\hat{x}(k|k), \hat{d}(k|k), u_s^*(k)) : \qquad \min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell(y_i, u_i) \tag{20a}$$

subject to:

$$x_0 = \hat{x}(k|k) \tag{20b}$$

$$x_{i+1} = f(x_i, u_i) + B_d \hat{d}(k|k) \tag{20c}$$

$$y_i = h(x_i) + C_d \hat{d}(k|k) + \Lambda(k)(u_i - u_s^*(k)) \tag{20d}$$

$$x_N = x_s \tag{20e}$$

$$y_i \in \mathbb{Y}, \qquad u_i \in \mathbb{U} \tag{20f}$$

Dropping the time argument for notational purposes, we denote the optimal solution of this problem as $(\mathbf{x}^*, \mathbf{u}^*)(\hat{x}, \hat{d}, u_s^*)$. Thus, the input implemented in closed-loop is:

$$\kappa_N(\hat{x}, \hat{d}, u_s^*) = u_0^*(\hat{x}, \hat{d}, u_s^*) \tag{21}$$

It is important to remark that the key issues in modifier-adaptation methods is how to obtain the plant gradient using only input-output measurements. To this aim several gradient estimation techniques have been proposed (Marchetti et al., 2016). An alternative estimation based on linear system identification technique is sketched in Section 4.4.

3.4 Summary

The proposed economic MPC is summarized in Algorithm 1.

Algorithm 1. Offset-free economic MPC

1: **Data**: output measurement $y(k)$, state and disturbance prediction $(\hat{x}(k|k-1), \hat{d}(k|k-1))$, modifier matrix $\Lambda(k)$ and previous target $u_s^*(k-1)$.
2: Evaluate the prediction error from (15), and update state and disturbance estimates from (13) to obtain $(\hat{x}(k|k), \hat{d}(k|k))$.
3: Solve the target optimization problem (17) to obtain the input target $u_s^*(k)$.
4: Solve the FHOCP (20) to obtain the input $u(k) = \kappa_N(\hat{x}(k|k), \hat{d}(k|k), u_s^*(k))$ as in (21).
5: Inject the input $u(k)$ into the plant (1).
6: Update the modifier matrix from (18).
7: Update time index $k \leftarrow k+1$ and predict successor state and disturbance from (14).

We can state the main result of this work, which can be proved by KKT matching techniques as in (Marchetti et al., 2009; Vaccari and Pannocchia, 2017), given the previous considerations on the prediction error going to zero, and the convergence properties of economic MPC with terminal constraint (Diehl et al., 2011).

Proposition 10. Assume that problems (17) and (20) remain feasible at all times, and that the closed-loop system:

$$\begin{aligned} x^+ &= f_p(x, \kappa_N(\hat{x}, \hat{d}, u_s^*)) \\ y &= h_p(x) \end{aligned} \tag{22}$$

reaches an equilibrium with input $\lim_{k\to\infty} u(k) = u_\infty$ and output $\lim_{k\to\infty} y(k) = y_\infty$. It follows that the reached equilibrium is the optimal one for the plant (1), i.e.:

$$y_\infty = \lim_{k\to\infty} y_s^*(k) = y_s^0, \qquad u_\infty = \lim_{k\to\infty} u_s^*(k) = u_s^0 \tag{23}$$

4. APPLICATION TO A CSTR EXAMPLE

4.1 Continuous-time dynamics and cost

As illustrative example, we consider an isothermal Continuous-Stirred Tank Reactor (CSTR) in which two consecutive reactions occur: A $\overset{k_1}{\to}$ B $\overset{k_2}{\to}$ C. The system is described by the following continuous-time dynamics:

$$\begin{aligned} \dot{x}_1 &= \frac{u}{V}(c_{A0} - x_1) - k_1 x_1 \\ \dot{x}_2 &= \frac{u}{V}(-x_2) + k_1 x_1 - k_2 x_2 \end{aligned} \tag{24}$$

in which x_1 and x_2 are the molar concentrations of A and B in the reactor, respectively; k_1 and k_2 are the two kinetic constants; u is the feed flow-rate (which is the manipulated input, and is assumed equal to the outlet flow-rate); V is the reactor volume; c_{A0} is the inlet concentration of A. We assume that both states (x_1, x_2) are measurable. The system parameters for the true plant are as follows:

$$k_1 = 1.0\,\text{min}^{-1},\ k_2 = 0.05\,\text{min}^{-1},$$
$$c_{A0} = 1.0\,\text{kmol/m}^3,\ V = 1.0\,\text{kmol/m}^3 \tag{25}$$

The following cost represents the economics of the system (expenditure for raw material - revenue from product):

$$\ell_c(x,u) = \alpha u c_{A0} - \beta u x_2 \tag{26}$$

in which $\alpha = 1.0$ €/kmol and $\beta = 4.0$ €/kmol are the prices of reactant and product, respectively. State and input constraints are as follows:

$$0 \le x_1 \le 1\,\text{kmol/m}^3,\ 0 \le x_2 \le 1\,\text{kmol/m}^3,\ 0 \le u \le 2\,\text{m}^3/\text{min} \tag{27}$$

The optimal equilibrium for the plant, which can be obtained by solving the following target problem:

$$\min_{x_1,x_2,u} \alpha u c_{A0} - \beta u x_2 \tag{28a}$$

subject to (27) and

$$\frac{u}{V}(c_{A0} - x_1) - k_1 x_1 = 0 \tag{28b}$$

$$\frac{u}{V}(-x_2) + k_1 x_1 - k_2 x_2 = 0 \tag{28c}$$

is found to be:

$$u_s^0 = 1.04298,\ x_s^0 = \begin{bmatrix} 0.51052 \\ 0.46709 \end{bmatrix},\ \ell_c(x_s^0, u_s^0) = -0.90568$$

4.2 Discretized system and cost

Given a sampling time h, the system (24) and the cost (26) can be discretized using the backward Euler scheme to obtain:

$$\begin{aligned} x_1^+ &= \frac{x_1 + \frac{u}{V} c_{A0} h}{1 + k_1 h + \frac{u}{V} h} \\ x_2^+ &= \frac{x_2 + k_1 h \left(\frac{x_1 + \frac{u}{V} c_{A0} h}{1 + k_1 h + \frac{u}{V} h} \right)}{1 + k_2 h + \frac{u}{V} h} \end{aligned} \tag{29}$$

$$\ell(x,u) = (\alpha u c_{A0} - \beta u x_2^+) h \tag{30}$$

From now on, both plant and model are given by the discretized model (29) in which $h = 1$ min, but the MPC model uses incorrect parameters as later detailed.

4.3 *Closed-loop performance comparison*

In this section we evaluate the closed-loop performance of three controllers, which use the same, incorrect model. Specifically, the model used by the controllers is still given by (29) but the known kinetic parameters are: $k_1 = 0.9\,\text{min}^{-1}$, $k_2 = 0.0\,\text{min}^{-1}$. From a kinetic point of view the model ignores the consecutive reaction B $\rightarrow$ C and underestimates the rate of the first reaction A $\rightarrow$ B. The considered three controllers are:

- MPC0: standard economic MPC using the nominal model.
- MPC1: offset-free tracking MPC with modified target calculation as described in Section 3.2. The FHOCP uses a conventional (quadratic) tracking cost $\ell_T(x,u,x_s,u_s) = \|x - x_s\|_Q^2 + \|u - u_s\|_R^2$, with $Q = I$, $R = 10^{-4}$.
- MPC2: offset-free economic MPC, as summarized by Algorithm 1 in Section 3.

The closed-loop behavior of states and input vs. time is reported for all three controllers in Fig. 1, starting from the initial condition $x_1(0) = x_2(0) = 0$. The same figure also shows the optimal equilibrium value for each state and for the input, which are obviously not known to the controllers given that they use an incorrect model. From these results, we notice

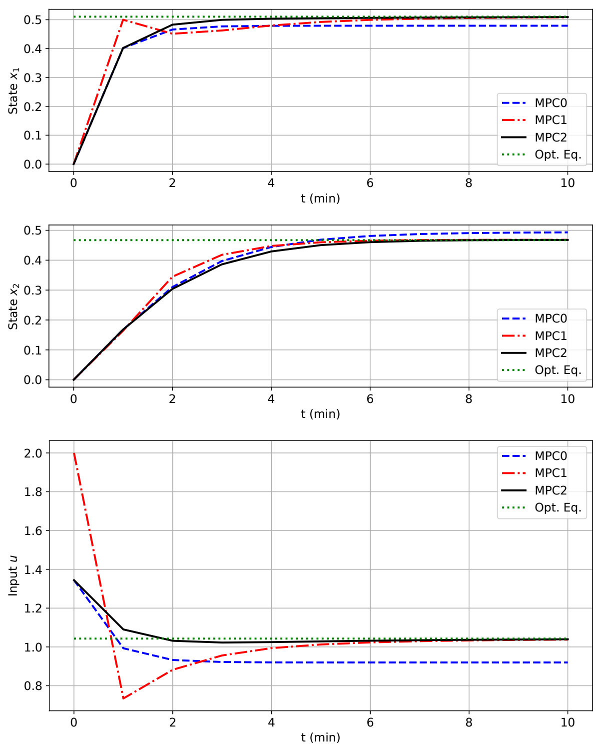

Fig. 1. Comparative closed-loop results using the three controllers: states (top and middle) and input (bottom).

that the system controlled by the standard economic MPC reaches an incorrect equilibrium, whereas both MPC1 and MPC2 reach the correct equilibrium. However, MPC1 has a suboptimal transient behavior since it is designed to be a tracking controller. To further clarify this aspect, we report in Fig. 2 the comparison of the closed-loop economic cost $\ell(x,u)$ obtained with the three controllers. Initially, the tracking controller (MPC1) has higher economic cost than the two economic controllers, which have similar behavior. Then, as the closed-loop system approaches the equilibrium, the nominal

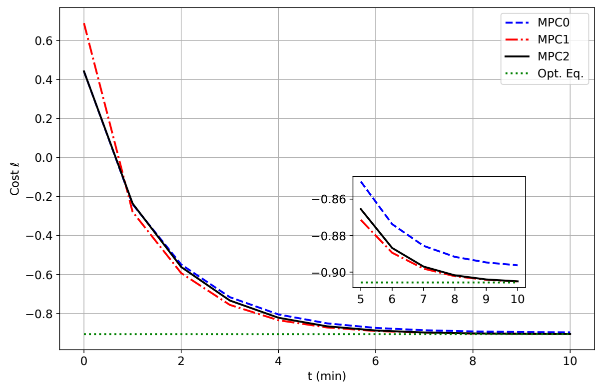

Fig. 2. Comparative closed-loop results using the three controllers: economic cost.

economic controller (MPC0) shows a bias with respect to the optimal steady-state cost (see the zoomed box), reached instead by both MPC1 and MPC2. Overall the cumulative cost, in the considered time window, is: -7.159 for MPC0, -7.143 for MPC1 and -7.270 for MPC2. It is therefore confirmed that MPC2 shows superior transient and steady performance than the other controllers.

4.4 *Gradient estimation from data*

In the previous simulation, MPC1 and MPC2 were designed under the assumption that the gradient of the input-to-output steady-state map of the plant, $DG_p(\cdot)$, is measurable. In practical situations, only input and output (state in this example) measurements are available. Thus, one practical way to estimate $DG_p(\cdot)$ is to perform linear system identification to obtain a local linear model, from which $DG_p(\cdot)$ is readily computed.

We explore this simple path in the following simulation, using the proposed Algorithm 1. While the controller is running we collect the sequence of inputs and states, $\{u_j, x_j\}_{j=k-N_{id}}^{j=k}$, over a moving horizon window of $N_{id} = 40$ sampling times. Then, similarly to a subspace method, at each sampling time we compute state-space matrices (A,B) from a least-square regression aimed at minimizing the one-step ahead prediction $x_{j+1} - (Ax_j + Bu_j)$ over the moving horizon window. Consequently, we define $DG_p(\cdot) = (I - A)^{-1}B$. To avoid numerical instability, the identification is performed only if the variance of states and inputs is larger than a threshold. This controller is denoted by MPC2id, and the obtained closed-loop results are reported in Fig. 3 also including standard economic MPC0 for comparison. From these results, we notice that during the first time period $[0, 20]$ the proposed controller still improves significantly the performance despite the presence of some small oscillations in the input which are due to the numerical estimate of $DG_p(\cdot)$. To further improve the quality of gradient estimation, during the second time window $[20, 40]$ a random signal with variance 0.01 is superimposed on the closed-loop input. In this way, the identified local model matrices (A,B) are a better approximation of the nonlinear plant dynamics and force the plant input to nearly converge to the true unknown optimal value.

5. CONCLUSIONS

We have presented in this paper a novel economic MPC algorithm that is able to achieve the optimal asymptotic performance, i.e. to make the closed-loop system converge to the most

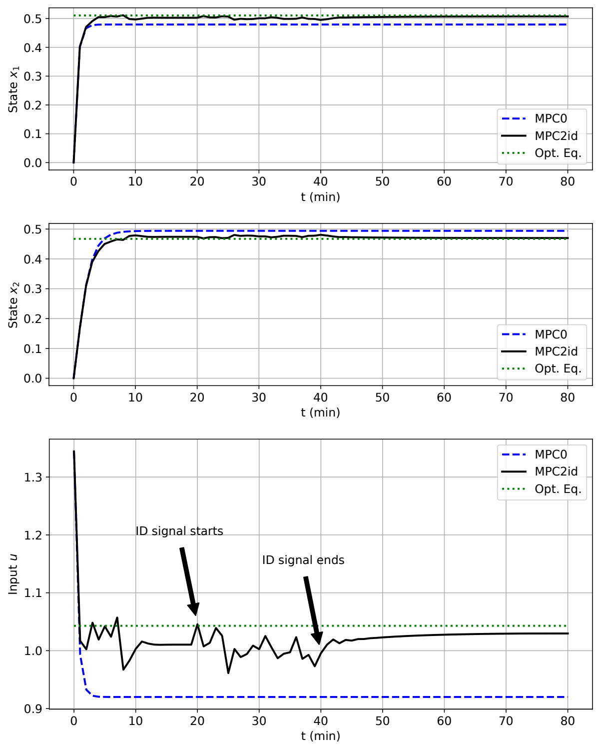

Fig. 3. Comparative closed-loop results using standard economic MPC and proposed offset-free economic MPC with plant gradient estimated from data: states (top and middle) and input (bottom). From time 20 to 40, an additional random signal is superimposed for identification purposes.

profitable equilibrium, in spite of possible plant/model mismatch. The proposed MPC design merges the idea of modifier-adaptation from the Real-Time Optimization field with the disturbance models used in offset-free tracking MPC. Using a numerical example of a CSTR in which the MPC model has some errors in the kinetic parameters, we have shown how the proposed controller achieves the optimal equilibrium whereas conventional economic MPC leaves an offset.

In this paper, we have assumed that convergence to some equilibrium occurs in closed-loop and that recursive feasibility is preserved. Future research should be devoted to eliminating these assumptions and showing under which circumstances convergence and recursive feasibility is guaranteed. Another interesting area of research is related to exploring and comparing different strategies for plant gradient estimation.

REFERENCES

Alvarez, L.A. and Odloak, D. (2012). Optimization and control of a continuous polymerization reactor. *Brazilian J. of Chem. Engineering*, 29(4), 807–820.

Angeli, D., Amrit, R., and Rawlings, J.B. (2012). On average performance and stability of economic model predictive control. *IEEE Trans. Auto. Contr.*, 57(7), 1615–1626.

Bayer, F.A., Müller, M.A., and Allgöwer, F. (2014). Tube-based robust economic model predictive control. *J. Proc. Contr.*, 24(8), 1237 – 1246.

Costello, S., François, G., and Bonvin, D. (2016). A directional modifier-adaptation algorithm for real-time optimization. *J. Proc. Contr.*, 39, 64–76.

Diehl, M., Amrit, R., and Rawlings, J.B. (2011). A Lyapunov function for economic optimizing model predictive control. *IEEE Trans. Auto. Contr.*, 56(3), 703–707.

Ellis, M., Durand, H., and Christofides, P.D. (2014). A tutorial review of economic model predictive control methods. *J. Proc. Contr.*, 24(8), 1156 – 1178.

Faulwasser, T. and Bonvin, D. (2017). Exact turnpike properties and economic NMPC. *Europ. J. Contr.*, 35, 34–41.

Ferramosca, A., González, A.H., and Limon, D. (2017). Offset-free multi-model economic model predictive control for changing economic criterion. *Journal of Process Control*, 54, 1 – 13.

Francois, G. and Bonvin, D. (2013). Measurement-based real-time optimization of chemical processes. In S. Pushpavanam (ed.), *Control and Optimisation of Process Systems*, volume 43 of *Advances in Chemical Engineering*, 1 – 50. Academic Press.

Grüne, L. and Müller, M.A. (2016). On the relation between strict dissipativity and turnpike properties. *Syst. Contr. Lett.*, 90, 45–53.

Marchetti, A.G., Francois, G., Faulwasser, T., and Bonvin (2016). Modifier adaptation for real-time optimization: Methods and applications. *Processes*, 4(55).

Marchetti, A.G., Chachuat, B., and Bonvin, D. (2009). Modifier-adaptation methodology for real-time optimization. *Ind. Eng. Chem. Res.*, 48(13), 6022–6033.

Morari, M. and Maeder, U. (2012). Nonlinear offset-free model predictive control. *Automatica*, 48, 2059–2067.

Müller, M.A., Angeli, D., and Allgöwer, F. (2015). On necessity and robustness of dissipativity in economic model predictive control. *IEEE Trans. Auto. Contr.*, 60(6), 1671–1676.

Navia, D., Briceño, L., Gutiérrez, G., and De Prada, C. (2015). Modifier-adaptation methodology for real-time optimization reformulated as a nested optimization problem. *Ind. Eng. Chem. Res.*, 54(48), 12054–12071.

Pannocchia, G., Gabiccini, M., and Artoni, A. (2015). Offset-free MPC explained: novelties, subtleties, and applications. In *Proceedings of* 5th IFAC Conference on Nonlinear Model Predictive Control, 342–351. Seville (Spain).

Pannocchia, G. (2015). Offset-free tracking MPC: A tutorial review and comparison of different formulations. In *European Control Conference (ECC)*, 527–532.

Qin, S.J. and Badgwell, T.A. (2003). A survey of industrial model predictive control technology. *Contr. Eng. Pract.*, 11, 733–764.

Qin, S.J. and Badgwell, T.A. (2000). An overview of nonlinear model predictive control applications. In Z.A. Allgöwer F. (ed.), *Nonlinear Model Predictive Control*, volume 26, 369–392. Birkhäuser (Basel).

Rawlings, J.B., Angeli, D., and Bates, C.N. (2012). Fundamentals of economic model predictive control. In *51st IEEE CDC*, 3851–3861.

Scattolini, R. (2009). A survey on hierarchical and distributed model predictive control. *J. Proc. Contr.*, 19(723–731).

Vaccari, M. and Pannocchia, G. (2017). A modifier-adaptation strategy towards offset-free economic MPC. *Processes*, 5 (1)(2). doi:10.3390/pr5010002.