Preliminary orbits with line-of-sight correction for LEO satellites observed with radar

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Abstract

We propose a method to account for the Earth oblateness effect in preliminary orbit determination of satellites in low orbits with radar observations. This method is an improvement of the one described in Gronchi et al (2015b), which uses a pure Keplerian dynamical model. Since the effect of the Earth oblateness is strong at low altitudes, its inclusion in the model can sensibly improve the initial orbit, giving a better starting guess for differential corrections and increasing the chances to obtain their convergence. The input set consists of two tracks of radar observations, each one composed of at least 4 observations taken during the same pass of the satellite. A single observation gives the topocentric position of the satellite, where the range is very accurate, while the line of sight direction is poorly determined. From these data we can compute by a polynomial fit the values of the range and range rate at the mean epochs of the two tracks. In order to obtain a preliminary orbit we wish to compute the angular velocity, which is the rate of change of the line of sight. In the same spirit of Gronchi et al (2015b), we also wish to correct the values of the angular measurements, so that they fit the selected dynamical model if the same holds for the radial distance and velocity. The selected model is a perturbed Keplerian dynamics, where the only perturbation included is the secular effect of the J_2 term of the geopotential.

1 Introduction

The growth in the number of space debris orbiting the Earth has increased the interest for the studies of new orbit computation methods, e.g. Farnocchia et al (2010), Gronchi et al (2015a), and of the dynamical properties of Earth satellites, e.g. Celletti and Galeș (2018), Daquin et al (2016), Rosengren and Scheeres (2013). Correlating short arcs of observations that belong to the same object and initial orbit determination (IOD) are of crucial importance for surveillance of the current population of space debris. In the case of optical measurements this problem has been addressed by many authors using different techniques (see for example Siminski et al, 2014, and references therein). On the other hand, only a few methods have been proposed for the case of radar observations

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(Vananti et al, 2017). In this paper we investigate an IOD method that is conceived to compute orbits of Earth satellites at low altitudes (LEO) with radar observations.

Let us assume that each radar measurement at epoch t is composed by a precise value of the range ρ (with standard deviation in the order of meters) and poorly determined values of the topocentric right ascension α and declination δ (with standard deviation for example of 0.2 degrees). The available data are radar tracks of the form

$$(t_i, \rho_i, \alpha_i, \delta_i), \qquad i = 1, \dots, m, \tag{1}$$

where $\Delta t = t_{i+1} - t_i$ is usually a few seconds and $m \geq 4$. Given a radar track we can derive the vector

$$(\bar{t}, \bar{\alpha}, \bar{\delta}, \rho, \dot{\rho}), \tag{2}$$

where $\bar{\alpha}$, $\bar{\delta}$ are the mean values while ρ, $\dot{\rho}$ can be obtained through a cubic fit because the measurements of the range are more precise.

We describe the osculating two-body orbit of the satellite by spherical coordinates (known as *attributable* coordinates)

$$\mathscr{E}_{att} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho}). \tag{3}$$

Therefore, given the data in (2), to compute an orbit we need the values of $\dot{\alpha}$, $\dot{\delta}$, which are the unknowns of our orbit determination problem. We want to correlate two radar attributables at two different epochs corrected for the aberration of light,

$$\tilde{t}_1 = \bar{t}_1 - \rho_1/c, \quad \tilde{t}_2 = \bar{t}_2 - \rho_2/c, \tag{4}$$

where c is the speed of light, to determine the values of $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\delta}_1$, $\dot{\delta}_2$ and compute a preliminary orbit (see Milani and Gronchi, 2010).

Assuming that the motion is Keplerian, Taff and Hall (1977) and more recently Farnocchia et al (2010) proposed to use the conservation of the angular momentum vector and energy to write a polynomial system which is quadratic in the unknowns. This method has been recently improved by Gronchi et al (2015b) by allowing for the correction of the values of $\bar{\alpha}$, $\bar{\delta}$. For this purpose they introduce the quantities $\Delta\alpha$, $\Delta\delta$, which are unknown small deviations from the mean values $\bar{\alpha}$, $\bar{\delta}$:

$$\alpha = \bar{\alpha} + \Delta\alpha, \qquad \delta = \bar{\delta} + \Delta\delta. \tag{5}$$

The deviations $\Delta\alpha$, $\Delta\delta$ are called *infinitesimal angles*. Moreover, in place of the unknowns $\dot{\alpha}$, $\dot{\delta}$ they use the variables

$$\xi = \rho\dot{\alpha}\cos\delta, \qquad \zeta = \rho\dot{\delta}, \tag{6}$$

which are the components of the topocentric velocity of the satellite orthogonal to the line of sight. The orbit at time $\bar{t} - \rho/c$ is completely determined by the *modified attributable* coordinates

$$\mathscr{E}^*_{att} = (\bar{\alpha} + \Delta\alpha, \bar{\delta} + \Delta\delta, \xi, \zeta, \rho, \dot{\rho}). \tag{7}$$

We extend the algorithm introduced in Gronchi et al (2015b), where a two-body approximation is employed, by considering the secular effect of the J_2 term of the geopotential in the dynamical model. The Earth oblateness has been already considered by Farnocchia et al (2010) for the computation of preliminary orbits but without introducing corrections to the angles. Their IOD method is iterative and at each iteration the problem has the same algebraic structure of the unperturbed one.

We want to determine the values of the eight unknowns $(\Delta\alpha_1, \Delta\delta_1, \xi_1, \zeta_1)$ and $(\Delta\alpha_2, \Delta\delta_2, \xi_2, \zeta_2)$ at $\tilde{t}_1, \tilde{t}_2$ from the input data set:

$$(\bar{t}_1, \bar{\alpha}_1, \bar{\delta}_1, \rho_1, \dot{\rho}_1), \quad (\bar{t}_2, \bar{\alpha}_2, \bar{\delta}_2, \rho_2, \dot{\rho}_2), \tag{8}$$

using the Keplerian integrals evolution, the equations of motion projected onto the line of sight, and a suitable version of Lambert's equation.

2 Notation

Let us denote by $\mathbf{e}^\rho$ the unit vector corresponding to the line of sight, and by $\mathbf{q}$ the geocentric position of the observer. Then the geocentric position of the observed body is

$$\mathbf{r} = \mathbf{q} + \rho\mathbf{e}^\rho, \tag{9}$$

where ρ is the range. Using as angular coordinates the topocentric right ascension α and declination δ in an equatorial reference frame (e.g. J2000), we have

$$\mathbf{e}^\rho = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta)^T. \tag{10}$$

We introduce the unit vectors

$$\mathbf{e}^\alpha = (-\sin\alpha, \cos\alpha, 0)^T, \tag{11}$$

$$\mathbf{e}^\delta = (-\sin\delta\cos\alpha, -\sin\delta\sin\alpha, \cos\delta)^T. \tag{12}$$

The set $\{\mathbf{e}^\rho, \mathbf{e}^\alpha, \mathbf{e}^\delta\}$ is an orthonormal system. Denoting by $\dot{\mathbf{r}}$ the geocentric velocity of the satellite, we have

$$\dot{\mathbf{r}} = \xi\mathbf{e}^\alpha + \zeta\mathbf{e}^\delta + (\dot{\rho}\mathbf{e}^\rho + \dot{\mathbf{q}}). \tag{13}$$

We will use the following different sets of coordinates for the orbits:

$$\mathcal{E}_{kep} = (a, e, I, \Omega, \omega, \ell), \tag{14}$$

$$\mathcal{E}_{car} = (x, y, z, \dot{x}, \dot{y}, \dot{z}), \tag{15}$$

$$\mathcal{E}_{att} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho}), \tag{16}$$

$$\mathcal{E}^*_{att} = (\bar{\alpha} + \Delta\alpha, \bar{\delta} + \Delta\delta, \xi, \zeta, \rho, \dot{\rho}), \tag{17}$$

that are respectively Keplerian, Cartesian, attributable and modified attributable coordinates. Note that the Keplerian elements in (14) have their usual meaning and ℓ denotes the mean anomaly.

We also consider the coordinate changes

$$\mathcal{E}_{kep} \xrightarrow{\phi_1} \mathcal{E}_{car}, \qquad \mathcal{E}_{car} \xrightarrow{\phi_2} \mathcal{E}_{att}, \qquad \mathcal{E}_{att} \xrightarrow{\phi_3} \mathcal{E}^*_{att}, \tag{18}$$

and the composite transformation

$$\Phi = \phi_3 \circ \phi_2 \circ \phi_1 \tag{19}$$

from $\mathcal{E}_{kep}$ to $\mathcal{E}^*_{att}$.

3 The equations of motion

Let us consider Newton's equation

$$\ddot{\mathbf{r}} = \nabla U(\mathbf{r}), \tag{20}$$

for the motion of a point mass in the Earth gravity field where the force function U is truncated at the J_2-term, that is

$$U(\mathbf{r}) = \frac{\mu}{r}\Big[1 - J_2\Big(\frac{R_\oplus}{r}\Big)^2 P_2(\sin\delta)\Big]. \tag{21}$$

Here, $r = |\mathbf{r}|$ is the geocentric distance, $R_\oplus$ is the equatorial radius of the Earth and P_2 is the Legendre polynomial of second degree

$$P_2(\sin\delta) = \frac{3}{2}\sin^2\delta - \frac{1}{2} = \frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2}. \tag{22}$$

The problem defined by equation (20) is non-integrable (see Celletti and Negrini, 1981). If we average out the short period term in (20) we obtain an integrable system (see Roy, 2004) given by

$$\begin{cases} \dot{a} = 0, \\ \dot{e} = 0, \\ \dot{I} = 0, \\ \dot{\Omega} = -\dfrac{3}{2}J_2\dfrac{R_\oplus^2}{p^2}\tilde{n}\cos I, \\ \dot{\omega} = \dfrac{3}{4}J_2\dfrac{R_\oplus^2}{p^2}\tilde{n}(4 - 5\sin^2 I), \\ \dot{\ell} = \tilde{n} = n\Big[1 + \dfrac{3}{2}J_2\dfrac{R_\oplus^2}{p^2}\Big(1 - \dfrac{3}{2}\sin^2 I\Big)\sqrt{1-e^2}\Big], \end{cases} \tag{23}$$

with $p = a(1-e^2)$ the parameter of the two-body trajectory and $n = \sqrt{\mu/a^3}$ the mean motion. Note that, in the dynamics defined by (23), the elements a, e, I remain constant while the ascending node Ω, the argument of perigee ω, and the mean anomaly ℓ change uniformly with time. Equations (23) can be written shortly as

$$\dot{\mathscr{E}}_{kep} = \mathbf{X}_{kep}(\mathscr{E}_{kep}). \tag{24}$$

In the following we shall assume that the observed body is moving according to the integrable dynamics defined by equations (23), (24), and we shall call oblateness effect (or J_2 effect) the deviation from the pure Keplerian motion which is defined by these equations.

To solve our problem, we express the equations of motion in terms of the coordinates $\mathscr{E}^*_{att}$. First we write equation (24) in Cartesian coordinates $\mathscr{E}_{car} = (\mathbf{r}, \dot{\mathbf{r}})$. We obtain

$$\dot{\mathscr{E}}_{car} = \mathbf{Y}(\mathscr{E}_{car}), \tag{25}$$

where

$$\mathbf{Y} = \Big(\frac{\partial \phi_1}{\partial \mathscr{E}_{kep}}\mathbf{X}_{kep}\Big) \circ \phi_1^{-1} \tag{26}$$

is the transformed vector field. From the expression above we obtain the acceleration $\ddot{\mathbf{r}}$ as a function of $\mathscr{E}_{car}$ along the solutions of (25):

$$\ddot{\mathbf{r}} = \Big(\frac{\partial \dot{\mathbf{r}}}{\partial \mathscr{E}_{kep}}\mathbf{X}_{kep}\Big) \circ \phi_1^{-1} =: \tilde{\mathbf{y}}(\mathbf{r}, \dot{\mathbf{r}}). \tag{27}$$

As done in Gronchi et al (2015b) for the pure Keplerian dynamics, we project the perturbed equation of motion (27) along the line of sight $\mathbf{e}^\rho$ and obtain the equation

$$\mathcal{K} = 0, \tag{28}$$

with

$$\mathcal{K} = (\ddot{\mathbf{r}} - \tilde{\mathbf{y}}) \cdot \mathbf{e}^\rho = \ddot{\rho} - \rho\eta^2 + \ddot{\mathbf{q}} \cdot \mathbf{e}^\rho - \tilde{\mathbf{y}} \cdot \mathbf{e}^\rho, \tag{29}$$

where $\eta = \sqrt{\dot{\alpha}^2 \cos^2 \delta + \dot{\delta}^2}$ is the proper motion.

Equation (29) can be expressed as a function of the unknown variables $(\Delta\alpha, \Delta\delta, \xi, \zeta)$ using the expressions of $\mathbf{r}, \dot{\mathbf{r}}$ given in (9), (13).

4 The J_2 effect on the two-body integrals

We recall the expressions of the conserved quantities in the Keplerian dynamics, i.e. the angular momentum $\mathbf{c}$, the energy $\mathcal{E}$ and the Laplace-Lenz vector $\mathbf{L}$, as a function of $\mathbf{r}, \dot{\mathbf{r}}$. These quantities can be read as functions of the attributable coordinates $\mathscr{E}_{att}$ using (9), (13), and

$$|\dot{\mathbf{r}}|^2 = \xi^2 + \zeta^2 + 2\dot{\mathbf{q}} \cdot \mathbf{e}^\alpha \xi + 2\dot{\mathbf{q}} \cdot \mathbf{e}^\delta \zeta + |\dot{\rho}\mathbf{e}^\rho + \dot{\mathbf{q}}|^2, \tag{30}$$

$$\dot{\mathbf{r}} \cdot \mathbf{r} = \mathbf{q} \cdot \mathbf{e}^\alpha \xi + \mathbf{q} \cdot \mathbf{e}^\delta \zeta + (\dot{\rho}\mathbf{e}^\rho + \dot{\mathbf{q}}) \cdot \mathbf{r}. \tag{31}$$

We have

$$\mathbf{c} = \mathbf{A}\xi + \mathbf{B}\zeta + \mathbf{C}, \tag{32}$$

$$\mathcal{E} = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}, \tag{33}$$

$$\mu\mathbf{L} = \dot{\mathbf{r}} \times \mathbf{c} - \mu\frac{\mathbf{r}}{|\mathbf{r}|} = \Big(|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}\Big)\mathbf{r} - (\dot{\mathbf{r}} \cdot \mathbf{r})\dot{\mathbf{r}}, \tag{34}$$

where

$$\mathbf{A} = \mathbf{r} \times \mathbf{e}^\alpha, \quad \mathbf{B} = \mathbf{r} \times \mathbf{e}^\delta, \quad \mathbf{C} = \mathbf{r} \times \dot{\mathbf{q}} + \dot{\rho}\,\mathbf{q} \times \mathbf{e}^\rho. \tag{35}$$

Including the J_2 effect in the dynamics the angular momentum and the Laplace-Lenz vectors are not conserved anymore. However, the following relations hold:

$$R_c\mathbf{c}_1 = \mathbf{c}_2, \tag{36}$$

$$\mathcal{E}_1 = \mathcal{E}_2, \tag{37}$$

$$R_L\mathbf{L}_1 = \mathbf{L}_2, \tag{38}$$

where

$$R_c = R_{\hat{\mathbf{z}}}(\Delta\Omega), \qquad R_L = R_{\hat{\mathbf{c}}_2}(\omega_1 + \Delta\omega)\, R_{\hat{\mathbf{z}}}(\Delta\Omega)\, R_{\hat{\mathbf{c}}_1}(-\omega_1). \tag{39}$$

Here we denote by $R_{\hat{\mathbf{v}}}(\varphi)$ the rotation matrix defined by the rotation of an angle φ around the axis of the unit vector $\hat{\mathbf{v}}$. Then, the unit vectors $\hat{\mathbf{z}}$, $\hat{\mathbf{c}}_i$, $i = 1, 2$, are given by

$$\hat{\mathbf{z}} = (0,\ 0,\ 1)^T, \tag{40}$$

$$\hat{\mathbf{c}}_i = (\sin\Omega_i \sin I_i,\ -\cos\Omega_i \sin I_i,\ \cos I_i)^T. \tag{41}$$

Moreover, using equations (23) the angular variations $\Delta\Omega$ and $\Delta\omega$ are obtained as

$$\Delta\Omega = \dot{\Omega}_1(\tilde{t}_2 - \tilde{t}_1), \qquad \Delta\omega = \dot{\omega}_1(\tilde{t}_2 - \tilde{t}_1), \tag{42}$$

where $\tilde{t}_1$, $\tilde{t}_2$ are the epochs corrected by aberration. We display the J_2 effect on the two-body integrals in Figure 1. *Remark* We can also write

$$\Delta\Omega = \Omega_2 - \Omega_1, \qquad \Delta\omega = \omega_2 - \omega_1, \tag{43}$$

and

$$R_L = R_2 R_1^T, \tag{44}$$

with

$$R_i = R_{\hat{\mathbf{c}}_i}(\omega_i)\, R_{\hat{\mathbf{z}}}(\Omega_i), \qquad i = 1, 2. \tag{45}$$

5 Lambert's theorem with the J_2 effect

Let us denote by $\mathcal{L}$ the expression defining Lambert's equation. In the dynamics given by (23), the mean motion evolves linearly, thus Lambert's equation can be written as

$$\mathcal{L} = \tilde{n}(\tilde{t}_1 - \tilde{t}_2) + (\beta - \sin\beta) - (\gamma - \sin\gamma) + 2k\pi = 0, \tag{46}$$

with $\tilde{n}$ given by the last equation in (23). Moreover, $k \in \mathbb{N}$ is the number of revolutions in the time interval $[\tilde{t}_1, \tilde{t}_2]$. The angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r_1 + r_2 + d_L}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r_1 + r_2 - d_L}{4a}, \tag{47}$$

where $0 \le \beta - \gamma \le 2\pi$ and r_1, r_2 are the distances from the center of force. In (47) the distance

$$d_L = |\tilde{R}\mathbf{r}_1 - \mathbf{r}_2| \tag{48}$$

is the length of the chord joining the two positions of the body at epochs $\tilde{t}_1$, $\tilde{t}_2$ after rotating the osculating ellipse at epoch $\tilde{t}_1$ so that it overlaps with the osculating ellipse at epoch $\tilde{t}_2$. The rotation $\tilde{R}$ is given explicitly by

$$\tilde{R} = \tilde{R}_2 \tilde{R}_1^T, \tag{49}$$

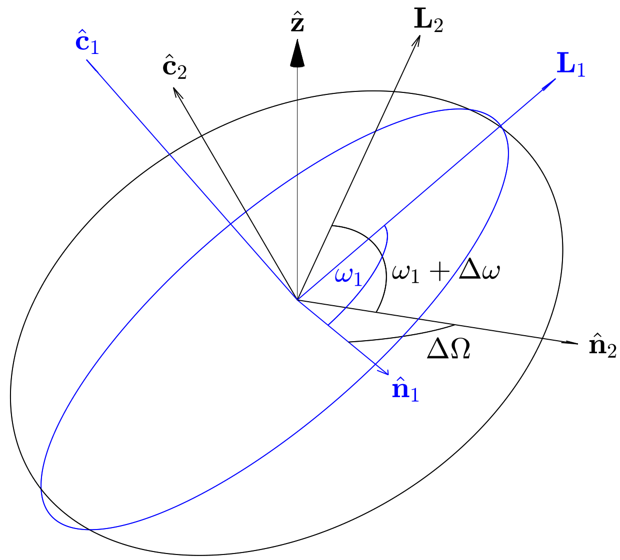

Figure 1: According to the secular effect of the J_2 term (see Equations (23)) the shape of the conic and its inclination remain unchanged between two epochs $\tilde{t}_1$, $\tilde{t}_2$. The directions of the angular momentum ($\hat{\mathbf{c}}$), line of nodes ($\hat{\mathbf{n}}$) and Laplace-Lenz vector ($\mathbf{L}$), by contrast, are rotated due to the secular variations $\Delta\Omega$, $\Delta\omega$ accumulated by Ω, ω during the time interval $\tilde{t}_2 - \tilde{t}_1$.

with $\tilde{R}_1$ and $\tilde{R}_2$ the transformations from the selected equatorial reference frame to the orbital reference frame at the epochs $\tilde{t}_1$ and $\tilde{t}_2$, respectively:

$$\tilde{R}_1 = R_{\hat{\mathbf{c}}_1}(\omega_1)\, R_{\hat{\mathbf{n}}_1}(I_1)\, R_{\hat{\mathbf{z}}}(\Omega_1), \tag{50}$$

$$\tilde{R}_2 = R_{\hat{\mathbf{c}}_2}(\omega_1 + \Delta\omega)\, R_{\hat{\mathbf{n}}_2}(I_2)\, R_{\hat{\mathbf{z}}}(\Omega_1 + \Delta\Omega), \tag{51}$$

where

$$\hat{\mathbf{n}}_i = (\cos\Omega_i,\ \sin\Omega_i,\ 0)^T, \quad i = 1, 2, \tag{52}$$

are the directions of the lines of nodes. For a fixed number of revolutions k we have 4 different choices for the pairs (β, γ) (see Appendix A1 in Gronchi et al, 2015b, for the details).

6 Linkage

We wish to link two sets of radar data of the form (2), with mean epochs $\bar{t}_i$, $i = 1, 2$, and compute one (or more) preliminary orbits. In the following we use labels 1, 2 for the quantities introduced in the previous sections, according to the epoch. Moreover, let us define $\boldsymbol{v}_2 = \mathbf{e}^{\rho}_2 \times \mathbf{q}_2$.

Taking into account the J_2 effect we consider the following system

$$(R_c\mathbf{c}_1 - \mathbf{c}_2,\ \mathcal{E}_1 - \mathcal{E}_2,\ \mathcal{K}_1,\ \mathcal{K}_2,\ (R_L\mathbf{L}_1 - \mathbf{L}_2)\cdot\boldsymbol{v}_2,\ \mathcal{L}) = \mathbf{0} \tag{53}$$

of 8 equations in the 8 unknowns $(\mathbf{X},\ \boldsymbol{\Delta})$, with

$$\mathbf{X} = (\xi_1,\ \zeta_1,\ \xi_2,\ \zeta_2), \qquad \boldsymbol{\Delta} = (\Delta\alpha_1,\ \Delta\delta_1,\ \Delta\alpha_2,\ \Delta\delta_2). \tag{54}$$

Note that the unknowns are divided into two sets so that $\boldsymbol{\Delta}$ is the vector of infinitesimal angles.

In Gronchi et al. (2015), because the motion is assumed Keplerian, $\mathbf{X}(\boldsymbol{\Delta})$ is obtained explicitly from the conservation of the angular momentum and energy. The remaining equations are solved for $\boldsymbol{\Delta}$ using an iterative method. In our method we also separate system (53) into two subsystems which can be solved by a double-iterative scheme. We search for solutions of equation

$$\mathcal{G}(\boldsymbol{\Delta}) = \mathbf{G}(\mathbf{X}(\boldsymbol{\Delta}), \boldsymbol{\Delta}) = \mathbf{0}, \tag{55}$$

where

$$\mathbf{G} = (\mathcal{K}_1,\ \mathcal{K}_2,\ (R_L\mathbf{L}_1 - \mathbf{L}_2)\cdot\boldsymbol{v}_2,\ \mathcal{L}) \tag{56}$$

and $\mathbf{X}(\boldsymbol{\Delta})$ is implicitly defined by the relation

$$\mathbf{J}(\mathbf{X}, \boldsymbol{\Delta}) = \mathbf{0}, \tag{57}$$

where

$$\mathbf{J} = (R_c\mathbf{c}_1 - \mathbf{c}_2,\ \mathcal{E}_1 - \mathcal{E}_2). \tag{58}$$

Newton-Raphson method is used to compute $\boldsymbol{\Delta}$ from the iterative formula

$$\boldsymbol{\Delta}_{h+1} = \boldsymbol{\Delta}_h - \Big[\frac{\partial\mathcal{G}}{\partial\boldsymbol{\Delta}}(\boldsymbol{\Delta}_h)\Big]^{-1}\mathcal{G}(\boldsymbol{\Delta}_h), \qquad \boldsymbol{\Delta}_0 = \mathbf{0}. \tag{59}$$

Here, taking advantage of the assumed smallness of the solutions $\mathbf{\Delta}$, we consider $\mathbf{\Delta} = \mathbf{0}$ as starting guess. At each iteration for $\mathbf{\Delta}$ we apply the Newton-Raphson method to obtain $\mathbf{X}(\mathbf{\Delta})$ from system (57). Precisely, for $\mathbf{\Delta} = \mathbf{\Delta}_h$ we compute $\mathbf{X}^{(h)} = \mathbf{X}(\mathbf{\Delta}_h)$ from the iterative formula

$$\mathbf{X}_{j+1} = \mathbf{X}_j - \Big[\frac{\partial \mathcal{J}}{\partial \mathbf{X}}(\mathbf{X}_j)\Big]^{-1} \mathcal{J}(\mathbf{X}_j), \tag{60}$$

where

$$\mathcal{J}(\mathbf{X}) = \mathbf{J}(\mathbf{X}, \mathbf{\Delta}_h). \tag{61}$$

For $h = 0$ the starting guess $\mathbf{X}_0$ is computed from the interpolated values of $\delta, \dot{\alpha}, \dot{\delta}, \rho$ through equations (6), while for $h > 0$ we set $\mathbf{X}_0 = \mathbf{X}^{(h-1)}$.

Remark Equations (57) are not polynomial in $\mathbf{X}$, unlike the corresponding equations in Gronchi et al (2015b).

7 Computing X, Δ

The algorithm to compute the vectors $\mathbf{X}$, $\mathbf{\Delta}$ that satisfy Equations (53) consists of two nested Newton-Raphson methods. Starting from $\mathbf{\Delta}_0 = \mathbf{0}$, we determine the vector $\mathbf{X}^{(0)}$ such that $\mathbf{J}(\mathbf{X}^{(0)}, \mathbf{\Delta}_0) = \mathbf{0}$, by applying the Newton-Raphson formula (60), wherein $\mathcal{J}(\mathbf{X}) = \mathbf{J}(\mathbf{X}, \mathbf{\Delta}_0)$. Then, after computing the number of revolutions k required in Lambert's equation (46), by

$$k = \left\lfloor \frac{n(\tilde{t}_2 - \tilde{t}_1)}{2\pi} \right\rfloor, \tag{62}$$

where n is the mean motion and $\lfloor x \rfloor$ denotes the integer part of x, we make the first iteration of the outer Newton-Raphson method through Equation (59), wherein $\mathcal{G}(\mathbf{\Delta}_0) = \mathbf{G}(\mathbf{X}^{(0)}, \mathbf{\Delta}_0)$. The iterations in $\mathbf{\Delta}$ are carried out until for some $h \geq 1$ the magnitude of the difference $\mathbf{\Delta}_h - \mathbf{\Delta}_{h-1}$ is smaller than a suitable tolerance. Finally, $\mathbf{X}^{(h)}$ is obtained by the iterative formula (60), and the solution of (53) is given by the pair of vectors $\mathbf{X}^{(h)}, \mathbf{\Delta}_h$.

In equation (55) the components of the vector $\mathbf{G}(\mathbf{X}, \mathbf{\Delta})$ are similar to the ones of the corresponding vector in Gronchi et al (2015b). However, the following differences occur:

i) in place of the angular momentum conservation law we have equation (36);

ii) in $\mathcal{K}$ at epochs $\tilde{t}_1, \tilde{t}_2$, the term $\tilde{\mathbf{y}} \cdot \mathbf{e}^\rho$ replaces the radial component of the Keplerian force, i.e. $-\mu \mathbf{r} \cdot \mathbf{e}^\rho / |\mathbf{r}|^3$;

iii) in place of the Laplace-Lenz conservation law we have equation (38);

iv) in $\mathcal{L}$ the quantity $\tilde{n}$ takes a different expression from the mean motion n, coming from the dynamical model (23). Moreover, the length of the chord is computed in a different way, see (48).

To search for the values of $\mathbf{\Delta}$ that solve Equation (55) we have to compute the first derivatives of $\mathcal{G}(\mathbf{\Delta})$ with respect to $\mathbf{\Delta}$, appearing in (59), that is

$$\frac{\partial \mathcal{G}}{\partial \mathbf{\Delta}}(\mathbf{\Delta}_h) = \frac{\partial \mathbf{G}}{\partial \mathbf{X}}(\mathbf{X}^{(h)}, \mathbf{\Delta}_h) \frac{\partial \mathbf{X}}{\partial \mathbf{\Delta}}(\mathbf{\Delta}_h) + \frac{\partial \mathbf{G}}{\partial \mathbf{\Delta}}(\mathbf{X}^{(h)}, \mathbf{\Delta}_h). \tag{63}$$

From the implicit function theorem applied to equation (57), we have

$$\frac{\partial \mathbf{X}}{\partial \mathbf{\Delta}}(\mathbf{\Delta}) = -\Big[\frac{\partial \mathbf{J}}{\partial \mathbf{X}}(\mathbf{X}, \mathbf{\Delta})\Big]^{-1}\frac{\partial \mathbf{J}}{\partial \mathbf{\Delta}}(\mathbf{X}, \mathbf{\Delta}). \tag{64}$$

Since these computations are similar to the ones reported in Gronchi et al (2015b, Sect. 7), we describe below only the differences coming from the adopted dynamical model.

7.1 The derivatives of $R_c\mathbf{c}_1 - \mathbf{c}_2$, $\mathcal{E}_1 - \mathcal{E}_2$

The derivatives of $R_c\mathbf{c}_1 - \mathbf{c}_2$ with respect to a component x of the vectors $\mathbf{X}$, $\mathbf{\Delta}$ can be written as

$$\frac{\partial(R_c\mathbf{c}_1 - \mathbf{c}_2)}{\partial x} = \frac{\partial R_c}{\partial x}\mathbf{c}_1 + R_c\frac{\partial \mathbf{c}_1}{\partial x} - \frac{\partial \mathbf{c}_2}{\partial x}. \tag{65}$$

We have

$$\frac{\partial R_c}{\partial x} = \begin{bmatrix} -\sin\Delta\Omega & -\cos\Delta\Omega & 0 \\ \cos\Delta\Omega & -\sin\Delta\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{\partial \Delta\Omega}{\partial x}, \tag{66}$$

where

$$\frac{\partial \Delta\Omega}{\partial x} = \frac{\partial \dot{\Omega}_1}{\partial x}\Delta t, \tag{67}$$

and the expressions of $\frac{\partial \dot{\Omega}_1}{\partial x}$ are reported in Appendix C.

Considering the angular momentum we get

$$\frac{\partial \mathbf{c}_i}{\partial \xi_i} = \mathbf{A}_i, \qquad \frac{\partial \mathbf{c}_i}{\partial \zeta_i} = \mathbf{B}_i, \qquad i = 1, 2. \tag{68}$$

The derivatives with respect to $\mathbf{\Delta}$ are computed through the intermediate variables $\mathbf{e}_i^\rho$, $\mathbf{e}_i^\alpha$, $\mathbf{e}_i^\delta$, $i = 1, 2$. After introducing the vector

$$\mathbf{E}_i = \begin{pmatrix} \mathbf{e}_i^\rho \\ \mathbf{e}_i^\alpha \\ \mathbf{e}_i^\delta \end{pmatrix}, \tag{69}$$

we can write

$$\frac{\partial \mathbf{c}_i}{\partial \mathbf{E}_i} = \frac{\partial \mathbf{A}_i}{\partial \mathbf{E}_i}\xi_i + \frac{\partial \mathbf{B}_i}{\partial \mathbf{E}_i}\zeta_i + \frac{\partial \mathbf{C}_i}{\partial \mathbf{E}_i}, \tag{70}$$

with

$$\frac{\partial \mathbf{A}_i}{\partial \mathbf{E}_i} = (O_3,\ S(\mathbf{q}_i),\ \rho_i I_3), \tag{71}$$

$$\frac{\partial \mathbf{B}_i}{\partial \mathbf{E}_i} = (O_3,\ -\rho_i I_3,\ S(\mathbf{q}_i)), \tag{72}$$

$$\frac{\partial \mathbf{C}_i}{\partial \mathbf{E}_i} = (-\rho_i S(\dot{\mathbf{q}}_i) + \dot{\rho}_i S(\mathbf{q}_i),\ O_3,\ O_3), \tag{73}$$

where O_3, I_3 denote the 3×3 zero and identity matrix, respectively. Note that $S(\mathbf{a})$ is the skew-symmetric matrix associated to a vector $\mathbf{a} = (a_1, a_2, a_3)^T$ by

$$S(\mathbf{a})\,\mathbf{y} = \mathbf{a} \times \mathbf{y}, \qquad \forall \mathbf{y} \in \mathbb{R}^3, \tag{74}$$

that is

$$\mathbb{R}^3 \ni \mathbf{a} \mapsto S(\mathbf{a}) \stackrel{def}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{75}$$

Concerning the energy we have[1]

$$\frac{\partial \mathcal{E}_i}{\partial \xi_i} = \xi_i + \dot{\mathbf{q}}_i \cdot \mathbf{e}_i^{\alpha}, \qquad \frac{\partial \mathcal{E}_i}{\partial \zeta_i} = \zeta_i + \dot{\mathbf{q}}_i \cdot \mathbf{e}_i^{\delta}, \tag{76}$$

$$\frac{\partial \mathcal{E}_i}{\partial \mathbf{E}_i} = \Big(\dot{\rho}_i \dot{\mathbf{q}}_i + \mu \rho_i \frac{\mathbf{q}_i}{r_i^3},\ \xi_i \dot{\mathbf{q}}_i,\ \zeta_i \dot{\mathbf{q}}_i\Big). \tag{77}$$

Finally, the derivatives $\frac{\partial \mathbf{E}_i}{\partial(\Delta\alpha_i, \Delta\delta_i)}$, $i = 1, 2$, can be found in Gronchi et al (2015b, Sect. 7.2).

7.2 The derivatives of $(R_L \mathbf{L}_1 - \mathbf{L}_2) \cdot \boldsymbol{v}_2$, $\mathcal{L}$

The derivative of R_L with respect to a component x of the vectors $\mathbf{X}$, $\boldsymbol{\Delta}$ is obtained from equation (44) as

$$\frac{\partial R_L}{\partial x} = \frac{\partial R_2}{\partial x} R_1^T - R_2 R_1^T \frac{\partial R_1}{\partial x} R_1^T, \tag{78}$$

where we have used that R_1 is an orthogonal matrix. A similar expression can be written for $\tilde{R}$ starting from (49). The rotation matrices in (39), (50), (51) are represented by means of Euler-Rodrigues formula (see Gallego and Yezzi, 2015)

$$R_{\hat{\mathbf{v}}}(\varphi) = I_3 + \sin\varphi\, S(\hat{\mathbf{v}}) + (1 - \cos\varphi) S^2(\hat{\mathbf{v}}), \tag{79}$$

where I_3 is the identity matrix and $S(\hat{\mathbf{v}})$ is the skew-symmetric matrix associated to the unit vector $\hat{\mathbf{v}} = (v_1, v_2, v_3)^T$. Then, we have

$$\frac{\partial R_{\hat{\mathbf{v}}}(\varphi)}{\partial x} = \sum_{k=1}^{3} \frac{\partial R_{\hat{\mathbf{v}}}(\varphi)}{\partial v_k} \frac{\partial v_k}{\partial x} + \frac{\partial R_{\hat{\mathbf{v}}}(\varphi)}{\partial \varphi} \frac{\partial \varphi}{\partial x}, \tag{80}$$

where

$$\frac{\partial R_{\hat{\mathbf{v}}}(\varphi)}{\partial v_k} = \sin\varphi \frac{\partial S(\hat{\mathbf{v}})}{\partial v_k} + (1 - \cos\varphi)\Big(\frac{\partial S(\hat{\mathbf{v}})}{\partial v_k} S(\hat{\mathbf{v}}) + S(\hat{\mathbf{v}}) \frac{\partial S(\hat{\mathbf{v}})}{\partial v_k}\Big) \tag{81}$$

and

$$\frac{\partial R_{\hat{\mathbf{v}}}(\varphi)}{\partial \varphi} = \cos\varphi\, S(\hat{\mathbf{v}}) + \sin\varphi\, S^2(\hat{\mathbf{v}}). \tag{82}$$

[1] There is a typo in the equation for $\frac{\partial \mathcal{E}_1}{\partial \mathbf{X}}$ reported in Gronchi et al (2015b, Sect. 7.1)

The derivatives of $\hat{\mathbf{v}}$ are obtained from (41), (52) and using $\frac{\partial \Phi^{-1}}{\partial x}$, which is given in Appendix B. Note that

$$\frac{\partial \Omega_2}{\partial x} = \frac{\partial \Omega_1}{\partial x} + \frac{\partial \dot{\Omega}_1}{\partial x}(\tilde{t}_2 - \tilde{t}_1), \tag{83}$$

$$\frac{\partial \omega_2}{\partial x} = \frac{\partial \omega_1}{\partial x} + \frac{\partial \dot{\omega}_1}{\partial x}(\tilde{t}_2 - \tilde{t}_1), \tag{84}$$

and $\frac{\partial(\dot{\Omega}_1, \dot{\omega}_1)}{\partial x}$ are reported in Appendix C.

Regarding Laplace-Lenz conservation law, the derivatives of $\mathbf{L}_1 \cdot \boldsymbol{v}_2$, $\mathbf{L}_2 \cdot \boldsymbol{v}_2$ are provided in Gronchi et al (2015b, Sections 7.1, 7.3).

For Lambert's equation we have

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \tilde{n}_1}{\partial x}(\tilde{t}_1 - \tilde{t}_2) + \frac{\partial(\beta - \sin\beta)}{\partial x} - \frac{\partial(\gamma - \sin\gamma)}{\partial x}. \tag{85}$$

The derivatives of $\tilde{n}_1$ are computed from (23) as shown in Appendix C. Moreover,[2]

$$\frac{\partial(\beta - \sin\beta)}{\partial x} = \pm 2\sqrt{\frac{\Gamma_+}{1 - \Gamma_+}}\frac{\partial \Gamma_+}{\partial x}, \tag{86}$$

$$\frac{\partial(\gamma - \sin\gamma)}{\partial x} = \pm 2\sqrt{\frac{\Gamma_-}{1 - \Gamma_-}}\frac{\partial \Gamma_-}{\partial x}, \tag{87}$$

where the positive sign holds for $0 < \beta, \gamma < \pi$. The quantities $\Gamma_\pm = (\Gamma_+, \Gamma_-)$ are defined as in Gronchi et al (2015b, Sect. 7.1), where d is replaced by d_L, so that

$$\frac{\partial \Gamma_\pm}{\partial \mathbf{X}} = -\frac{r_1 + r_2 \pm d_L}{2\mu}\frac{\partial \mathcal{E}_1}{\partial \mathbf{X}} \mp \frac{\mathcal{E}_1}{2\mu}\frac{\partial d_L}{\partial \mathbf{X}}, \tag{88}$$

$$\frac{\partial \Gamma_\pm}{\partial \boldsymbol{\Delta}} = -\frac{r_1 + r_2 \pm d_L}{2\mu}\frac{\partial \mathcal{E}_1}{\partial \boldsymbol{\Delta}} - \frac{\mathcal{E}_1}{2\mu}\frac{\partial(r_1 + r_2 \pm d_L)}{\partial \boldsymbol{\Delta}}, \tag{89}$$

where

$$\frac{\partial r_i}{\partial(\Delta\alpha_i, \Delta\delta_i)} = \frac{\rho_i}{r_i}(\cos\delta_i \mathbf{q}_i \cdot \mathbf{e}_i^\alpha,\ \mathbf{q}_i \cdot \mathbf{e}_i^\delta), \tag{90}$$

$$\frac{\partial d_L}{\partial x} = \frac{1}{d_L}\Big(\frac{\partial \tilde{R}}{\partial x}\mathbf{r}_1 + \tilde{R}\frac{\partial \mathbf{r}_1}{\partial x} - \frac{\partial \mathbf{r}_2}{\partial x}\Big) \cdot (\tilde{R}\mathbf{r}_1 - \mathbf{r}_2), \tag{91}$$

and

$$\frac{\partial \mathbf{r}_i}{\partial(\Delta\alpha_i, \Delta\delta_i)} = \rho_i(\cos\delta_i \mathbf{e}_i^\alpha,\ \mathbf{e}_i^\delta). \tag{92}$$

7.3 The derivatives of $\mathcal{K}_1, \mathcal{K}_2$

The value of $\ddot{\rho}$ is required in the equation of motion (94) at the two epochs $\tilde{t}_1, \tilde{t}_2$. The quantity $\ddot{\rho}$ is regarded as a constant whose value is updated by means of equation (94)

[2] There is a typo in the corresponding formulae in Gronchi et al (2015b, Sections 7.1, 7.3).

at each iteration of Newton-Raphson method for computing $\boldsymbol{\Delta}$. Note that, in this way the values taken by $\mathcal{K}_1, \mathcal{K}_2$ are always identically 0.

Since $\mathcal{K}$ depends on quantities that are referred to the same epoch, we will drop the subscript i.

We need to compute:

$$\frac{\partial \mathcal{K}}{\partial \mathscr{E}^*_{att}} = \frac{\partial(\ddot{\mathbf{r}} \cdot \mathbf{e}^\rho)}{\partial \mathscr{E}^*_{att}} - \frac{\partial(\tilde{\mathbf{y}} \cdot \mathbf{e}^\rho)}{\partial \mathscr{E}^*_{att}}. \tag{93}$$

From the equation

$$\ddot{\mathbf{r}} \cdot \mathbf{e}^\rho = \ddot{\rho} - \rho\eta^2 + \ddot{\mathbf{q}} \cdot \mathbf{e}^\rho \tag{94}$$

we obtain

$$\frac{\partial(\ddot{\mathbf{r}} \cdot \mathbf{e}^\rho)}{\partial \mathscr{E}^*_{att}} = \Big(\ddot{\mathbf{q}} \cdot \frac{\partial \mathbf{e}^\rho}{\partial \Delta\alpha}, \ \ddot{\mathbf{q}} \cdot \frac{\partial \mathbf{e}^\rho}{\partial \Delta\delta}, -\frac{2\xi}{\rho}, -\frac{2\zeta}{\rho}, \eta^2, \ 0\Big), \tag{95}$$

with

$$\frac{\partial \mathbf{e}^\rho}{\partial \Delta\alpha} = \frac{\partial \mathbf{e}^\rho}{\partial \alpha} = \mathbf{e}^\alpha \cos\delta, \qquad \frac{\partial \mathbf{e}^\rho}{\partial \Delta\delta} = \frac{\partial \mathbf{e}^\rho}{\partial \delta} = \mathbf{e}^\delta. \tag{96}$$

In (96) we made a little abuse of notation: $\mathbf{e}^\rho$ stands for both a function of (α, δ) and $(\Delta\alpha, \Delta\delta)$.
Then, we introduce $\mathbf{y}^*$, that is the vector $\tilde{\mathbf{y}}$ (see 27) as a function of the coordinates $\mathscr{E}^*_{att}$:

$$\mathbf{y}^* = \tilde{\mathbf{y}} \circ \phi_2^{-1} \circ \phi_3^{-1} = \Big(\frac{\partial \dot{\mathbf{r}}}{\partial \mathscr{E}^*_{att}} \mathbf{X}_{kep}\Big) \circ \Phi^{-1}. \tag{97}$$

Denoting by $x_{(k)}$ the k-th component of x (where x can be here either a vector or a map) we can write

$$\mathbf{y}^*_{(k)} = \Big(\frac{\partial \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} \mathbf{X}_{kep}\Big) \circ \Phi^{-1}, \quad k = 1, 2, 3. \tag{98}$$

Their derivatives are given by

$$\frac{\partial \mathbf{y}^*_{(k)}}{\partial \mathscr{E}^*_{att}} = \left[\frac{\partial}{\partial \mathscr{E}_{kep}} \Big(\frac{\partial \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} \mathbf{X}_{kep}\Big)\right] \circ \Phi^{-1} \frac{\partial \Phi^{-1}}{\partial \mathscr{E}^*_{att}}, \tag{99}$$

where

$$\frac{\partial}{\partial \mathscr{E}_{kep}} \Big(\frac{\partial \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} \mathbf{X}_{kep}\Big) = \frac{\partial^2 \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} \mathbf{X}_{kep} + \frac{\partial \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} \frac{\partial \mathbf{X}_{kep}}{\partial \mathscr{E}_{kep}}, \tag{100}$$

with

$$\frac{\partial \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}_{kep}} = \frac{\partial \phi_{1(k+3)}}{\partial \mathscr{E}_{kep}}, \qquad \frac{\partial^2 \dot{\mathbf{r}}_{(k)}}{\partial \mathscr{E}^2_{kep}} = \frac{\partial^2 \phi_{1(k+3)}}{\partial \mathscr{E}^2_{kep}}, \tag{101}$$

and

$$\frac{\partial \mathbf{X}_{kep}}{\partial \mathscr{E}_{kep}} = \begin{bmatrix} O_3 & O_3 \\ \frac{\partial(\dot{\Omega}, \dot{\omega}, \dot{\ell})}{\partial(a, e, I)} & O_3 \end{bmatrix}. \tag{102}$$

The expressions of

$$\frac{\partial \phi_1}{\partial \mathscr{E}_{kep}}, \ \frac{\partial^2 \phi_1}{\partial \mathscr{E}^2_{kep}}, \ \frac{\partial \Phi^{-1}}{\partial \mathscr{E}^*_{att}}, \ \frac{\partial \mathbf{X}_{kep}}{\partial \mathscr{E}_{kep}}$$

are reported in Appendices A, B, C.

Table 1: Keplerian elements at epoch 54127.1553819 MJD for object 1 and 54127.2991319 MJD for object 2. The values of a, e, I are exact, the others are approximated. Distances are expressed in km, angles in degrees.

obj.	a	e	I	Ω	ω	ℓ
1	7818.10	0.0658	65.81	213.92	356.70	202.25
2	7396.00	0.0341	26.88	255.49	357.13	198.67

Table 2: Standard deviation (rms) of the errors added to the radar tracks.

	α, δ (deg)	ρ (m)
Case 1	0.20	1
Case 2	0.20	10
Case 3	0.15	1
Case 4	0.15	10

8 Numerical tests

We show some numerical tests with two simulated objects whose orbital elements at some epoch are defined in Table 1. For the selected orbits the perturbation due to the J_2 is dominant if we assume a small area-to-mass ratio of the two objects (see Montenbruck and Gill, 2000, Figure 3.1). Moreover, the J_2 effect will be stronger for object 2 because of the smaller values of the inclination and semi-major axis.

A two-body propagation with the J_2 effect (equations 23) is used to generate pairs of radar tracks at epochs $\tilde{t}_1$, $\tilde{t}_2$ of 4 observations each taken at time intervals of 10 s, see (1). Then, we add to ρ, α, δ a Gaussian error with zero mean and the standard deviation (rms) shown in Table 2. In Cases 1, 3 a small error is added to ρ,[3] while in Cases 2, 4 a significant noise affects both the angles and the range. For each object we consider two pairs of radar tracks, separated by a different number of revolutions k. The interpolated data that we get after adding the noise to the simulated observations are given in Tables 3, 4 for object 1 and Tables 5, 6 for object 2. Note that also the values of $\dot{\alpha}$, $\dot{\delta}$ are shown because they are needed to initialize the unknown variables ξ, ζ.

Tables 7, 8 report the absolute errors in each orbital element of objects 1, 2 at epoch $\tilde{t}_1$. For both objects the new method, here referred to as IA-J_2, is able to correct the errors in α, δ and to recover the Keplerian elements of the known orbits with a satisfactory level of accuracy. Note that the performance of the new method is only slightly affected by the increase of the noise level in ρ (Cases 2, 4).

We have also compared IA-J_2 to the method IAQ proposed in Gronchi et al (2015b) which does not take into account the effect of the J_2 term of the geopotential. The

[3]Note that, even if the rms of ρ was 0, the interpolated values of ρ, $\dot{\rho}$ at time $\tilde{t}_1$ would not be exact in general.

Table 3: Data interpolated from two radar tracks of object 1 at epochs $\bar{t}_1$ = 54127.1553820 MJD and $\bar{t}_2$ = 54127.5824653 MJD, using two different noise levels of Table 2. The number of revolutions k, as defined in Equation (62), is also reported.

Object 1, $k = 5$

Epoch	Case	$\bar{\alpha}$ (deg)	$\bar{\delta}$ (deg)	$\dot{\alpha}$ (deg/s)	$\dot{\delta}$ (deg/s)	ρ (km)	$\dot{\rho}$ (km/s)
$\bar{t}_1$	1	40.71190	−4.34972	0.06933	−0.17402	1965.89061	−1.26651
	2	40.71190	−4.34972	0.06933	−0.17402	1965.88651	−1.26557
$\bar{t}_2$	1	243.02897	−79.04074	−0.02136	0.16045	1875.99129	−4.84869
	2	243.02897	−79.04074	−0.02136	0.16045	1876.00141	−4.85139

Table 4: Same as in Table 3 for two radar tracks at $\bar{t}_1$ = 54127.5824653 MJD and $\bar{t}_2$ = 54128.6241320 MJD.

Object 1, $k = 13$

Epoch	Case	$\bar{\alpha}$ (deg)	$\bar{\delta}$ (deg)	$\dot{\alpha}$ (deg/s)	$\dot{\delta}$ (deg/s)	ρ (km)	$\dot{\rho}$ (km/s)
$\bar{t}_1$	3	242.95755	−79.11215	−0.01356	0.16825	1875.98971	−4.84829
	4	242.95755	−79.11215	−0.01356	0.16825	1875.98562	−4.84735
$\bar{t}_2$	3	204.34455	53.03103	0.11860	0.10845	2061.14383	5.59005
	4	204.34455	53.03103	0.11860	0.10845	2061.15395	5.58735

Table 5: Data interpolated from two radar tracks of object 2 at epochs $\bar{t}_1$ = 54127.2991320 MJD and $\bar{t}_2$ = 54127.3828126 MJD, using two different noise levels of Table 2. The number of revolutions k, as defined in Equation (62), is also reported.

Object 2, $k = 1$

Epoch	Case	$\bar{\alpha}$ (deg)	$\bar{\delta}$ (deg)	$\dot{\alpha}$ (deg/s)	$\dot{\delta}$ (deg/s)	ρ (km)	$\dot{\rho}$ (km/s)
$\bar{t}_1$	1	72.88079	34.57579	0.20066	−0.04293	1942.23386	−3.40688
	2	72.88079	34.57579	0.20066	−0.04293	1942.22977	−3.40593
$\bar{t}_2$	1	193.80043	−31.71774	0.13680	0.03447	2057.69443	5.00361
	2	193.80043	−31.71774	0.13680	0.03447	2057.70455	5.00091

Table 6: Same as in Table 6 for two radar tracks at $\bar{t}_1 = 54127.6906251$ MJD and $\bar{t}_2 = 54128.3300348$ MJD.

Object 2, $k = 8$

Epoch	Case	$\bar{\alpha}$ (deg)	$\bar{\delta}$ (deg)	$\dot{\alpha}$ (deg/s)	$\dot{\delta}$ (deg/s)	ρ (km)	$\dot{\rho}$ (km/s)
$\bar{t}_1$	3	162.41862	−0.70695	0.12744	0.07974	1914.59715	−5.09806
	4	162.41862	−0.70695	0.12744	0.07974	1914.59305	−5.09712
$\bar{t}_2$	3	169.53245	−20.90422	0.14196	−0.01396	2013.14698	4.73488
	4	169.53245	−20.90422	0.14196	−0.01396	2013.15710	4.73218

advantage of IA-J_2 over IAQ becomes evident when the time interval between two radar tracks increases. Table 8 shows that by taking two radar tracks of object 1 separated by 13 revolutions, the method IAQ does not find a good orbit, while IA-J_2 is able to determine very accurate values of the orbital elements. Also, IAQ does not work with two radar tracks of object 2 separated by 8 revolutions, while IA-J_2 keeps the errors small. Finally, the corrections to the angles α, δ computed by the method IA-J_2 are shown in Tables 9, 10.

9 Conclusions

We propose a new method to compute preliminary orbits of Earth satellites taking into account the Earth oblateness. The method attempts to link together two radar tracks, which may be separated by several revolutions. It consists in solving system (53) by a double iterative scheme to determine the corrections of α, δ and the angular velocity. Numerical tests show that the method works also in presence of a significant noise level on the range and the angles. Future work will be to include the effect of the atmospheric drag and perform large-scale tests on LEO objects with real observations.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Table 7: Difference (in absolute value) between the computed and true orbital elements of objects 1, 2 at epoch $\tilde{t}_1$ for the interpolated data of Tables 3, 5. The new method is compared to the method IAQ proposed in Gronchi et al (2015b). Distances are expressed in km, angles in degrees.

	Object 1, $k = 5$		Object 2, $k = 1$		Case
	IAQ	IA-J_2	IAQ	IA-J_2	
δa	6.1476	1.1441	5.0957	1.7662	1
	171.4962	1.8259	10.3163	1.3280	2
δe	1.79×10^{-4}	4.43×10^{-5}	4.57×10^{-4}	4.30×10^{-4}	1
	1.48×10^{-2}	7.21×10^{-5}	2.04×10^{-3}	2.75×10^{-4}	2
δI	1.8416	0.4283	0.9966	0.3384	1
	0.8353	0.7366	0.1597	0.2694	2
$\delta\Omega$	0.0677	0.2408	1.1446	0.4020	1
	3.1663	0.4204	0.1827	0.3051	2
$\delta\omega$	3.7614	1.1999	0.8886	1.6806	1
	6.2622	1.8956	6.1047	1.3220	2
$\delta\ell$	4.3016	1.2948	0.0334	1.4552	1
	7.9420	2.0320	6.8222	1.1558	2

Table 8: Same as in Table 7 for the interpolated data of Tables 4, 6. The method IAQ does not work for object 2 when we take two radar tracks separated by 8 revolutions.

	Object 1, $k = 13$		Object 2, $k = 8$	Case
	IAQ	IA-J_2	IA-J_2	
δa	502.4785	0.0105	0.1615	3
	502.4096	0.0116	0.1378	4
δe	4.32×10^{-3}	7.08×10^{-5}	2.00×10^{-4}	3
	4.76×10^{-3}	1.44×10^{-4}	1.50×10^{-4}	4
δI	6.2022	0.0977	0.9485	3
	6.2071	0.0830	0.6724	4
$\delta\Omega$	5.1509	0.0469	0.5746	3
	5.1487	0.0454	0.3900	4
$\delta\omega$	129.2975	0.0203	4.7886	3
	129.3858	0.0388	3.5505	4
$\delta\ell$	239.9665	0.0124	4.2105	3
	239.8206	0.0095	3.1483	4

Table 9: Infinitesimal angles (deg) found by the proposed method using the radar tracks of Tables 3, 4 for object 1.

	Case 1	Case 2	Case 3	Case 4
$\Delta\alpha_1$	−1.31448	−2.30805	0.25737	0.08057
$\Delta\delta_1$	−0.37447	−0.69638	0.08058	0.07575
$\Delta\alpha_2$	0.71856	1.23111	−0.23547	−0.24423
$\Delta\delta_2$	0.32881	0.58535	−0.02666	−0.04653

Table 10: Infinitesimal angles (deg) found by the proposed method using the radar tracks of Tables 5, 6 for object 2.

	Case 1	Case 2	Case 3	Case 4
$\Delta\alpha_1$	0.42261	0.31791	−0.01612	−0.00648
$\Delta\delta_1$	0.35088	0.26697	−1.54546	−1.12770
$\Delta\alpha_2$	0.30596	0.24228	0.03328	0.00800
$\Delta\delta_2$	−0.82680	−0.67820	−3.45550	−2.44161

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Appendix A

The Jacobian matrix of the Cartesian coordinates $\mathscr{E}_{car}$ with respect to the Keplerian elements $\mathscr{E}_{kep}$ (see 14, 15) can be obtained from Table 2 in Broucke (1970) by setting $t = 0$ in the expressions of $\frac{\partial \phi_1}{\partial a}$ and noting that $\frac{\partial \phi_1}{\partial \ell} = \frac{\partial \phi_1}{\partial M_0}$, where M_0 denotes the mean anomaly at epoch in Broucke (1970).

Let us adopt here and in Appendix B the same notation explained in Section 7.3 to refer to the component of a vector and a map. The derivatives $\frac{\partial^2 \phi_{1(i)}}{\partial \mathscr{E}_{kep}^2}$, $i = 4, 5, 6$, are given by:

$$
\begin{aligned}
&\frac{\partial^2 \phi_{1(i)}}{\partial \mathscr{E}_{kep} \partial a} = -\frac{1}{2a}\Big(-\frac{3\dot{\mathbf{r}}_{(k)}}{2a}, \frac{\partial \phi_{1(i)}}{\partial e}, \frac{\partial \phi_{1(i)}}{\partial I}, \frac{\partial \phi_{1(i)}}{\partial \Omega}, \frac{\partial \phi_{1(i)}}{\partial \omega}, \frac{\partial \phi_{1(i)}}{\partial \ell}\Big),\\
&\frac{\partial}{\partial(a,e,\ell)}\frac{\partial \phi_{1(i)}}{\partial e} = \frac{\partial \dot{L}}{\partial(a,e,\ell)}\mathbf{P}_{(k)} + \frac{\partial \dot{M}}{\partial(a,e,\ell)}\mathbf{Q}_{(k)},\\
&\frac{\partial}{\partial(I,\Omega,\omega)}\frac{\partial \phi_{1,(i)}}{\partial e} = \dot{L}\frac{\partial \mathbf{P}_{(k)}}{\partial(I,\Omega,\omega)} + \dot{M}\frac{\partial \mathbf{Q}_{(k)}}{\partial(I,\Omega,\omega)},\\
&\frac{\partial}{\partial(a,e,\ell)}\frac{\partial \phi_{1(i)}}{\partial I} = \Big(\frac{\partial \dot{X}}{\partial(a,e,\ell)}\sin\omega + \frac{\partial \dot{Y}}{\partial(a,e,\ell)}\cos\omega\Big)\mathbf{R}_{(k)},\\
&\frac{\partial}{\partial(I,\Omega)}\frac{\partial \phi_{1(i)}}{\partial I} = (\dot{X}\sin\omega + \dot{Y}\cos\omega)\frac{\partial \mathbf{R}_{(k)}}{\partial(I,\Omega)},\\
&\frac{\partial^2 \phi_{1(i)}}{\partial \omega \partial I} = (\dot{X}\cos\omega - \dot{Y}\sin\omega)\mathbf{R}_{(k)},\\
&\frac{\partial}{\partial \mathscr{E}_{kep}}\frac{\partial \phi_{1(4)}}{\partial \Omega} = -\frac{\partial \phi_{1(5)}}{\partial \mathscr{E}_{kep}},\\
&\frac{\partial}{\partial \mathscr{E}_{kep}}\frac{\partial \phi_{1(5)}}{\partial \Omega} = \frac{\partial \phi_{1(4)}}{\partial \mathscr{E}_{kep}},\\
&\frac{\partial}{\partial \mathscr{E}_{kep}}\frac{\partial \phi_{1(6)}}{\partial \Omega} = 0,\\
&\frac{\partial}{\partial(a,e,\ell)}\frac{\partial \phi_{1(i)}}{\partial \omega} = \frac{\partial \dot{X}}{\partial(a,e,\ell)}\mathbf{Q}_{(k)} - \frac{\partial \dot{Y}}{\partial(a,e,\ell)}\mathbf{P}_{(k)},\\
&\frac{\partial}{\partial(I,\Omega,\omega)}\frac{\partial \phi_{1(i)}}{\partial \omega} = \dot{X}\frac{\partial \mathbf{Q}_{(k)}}{\partial(I,\Omega,\omega)} - \dot{Y}\frac{\partial \mathbf{P}_{(k)}}{\partial(I,\Omega,\omega)},\\
&\frac{\partial \phi^2_{1(i)}}{\partial a \partial \ell} = n\frac{a^2}{r^3}\Big(\frac{3}{2}\mathbf{r}_{(k)} - a\frac{\partial \phi_{1(k)}}{\partial a}\Big),\\
&\frac{\partial}{\partial(e,\ell)}\frac{\partial \phi_{1(i)}}{\partial \ell} = n\frac{a^3}{r^3}\Big(3\frac{\mathbf{r}_{(k)}}{r}\frac{\partial r}{\partial(e,\ell)} - \frac{\partial \phi_{1(k)}}{\partial(e,\ell)}\Big),\\
&\frac{\partial}{\partial(I,\Omega,\omega)}\frac{\partial \phi_{1(i)}}{\partial \ell} = -n\frac{a^3}{r^3}\frac{\partial \phi_{1(k)}}{\partial(I,\Omega,\omega)},
\end{aligned}
$$

where $k = i - 3$, and the quantities $\dot{X}$, $\dot{Y}$, $\dot{L}$, $\dot{M}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are defined in Broucke (1970).

We have

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial I} &= (\mathbf{P}_3 \sin\Omega, -\mathbf{P}_3 \cos\Omega, \sin\omega\cos I)^T,\\
\frac{\partial \mathbf{P}}{\partial \Omega} &= (-\mathbf{P}_2, \mathbf{P}_1, 0)^T,\\
\frac{\partial \mathbf{P}}{\partial \omega} &= \mathbf{Q},\\
\frac{\partial \mathbf{Q}}{\partial I} &= (\mathbf{Q}_3 \sin\Omega, -\mathbf{Q}_3 \cos\Omega, \cos\omega\cos I)^T,\\
\frac{\partial \mathbf{Q}}{\partial \Omega} &= (-\mathbf{Q}_2, \mathbf{Q}_1, 0)^T,\\
\frac{\partial \mathbf{Q}}{\partial \omega} &= -\mathbf{P},\\
\frac{\partial \mathbf{R}}{\partial I} &= (\sin\Omega\cos I,\ -\cos\Omega\cos I,\ -\sin I)^T,\\
\frac{\partial \mathbf{R}}{\partial \Omega} &= (\cos\Omega\sin I,\ \sin\Omega\sin I,\ 0)^T,\\
\frac{\partial \mathbf{R}}{\partial \omega} &= \mathbf{0},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \dot{L}}{\partial a} &= -\frac{\dot{L}}{2a}, \qquad \frac{\partial \dot{M}}{\partial a} = -\frac{\dot{M}}{2a},\\
\frac{\partial \dot{L}}{\partial e} &= n\frac{a^4}{r^4}(2r+a)\sin^3 E - \frac{3\dot{L}}{r}\frac{\partial r}{\partial e},\\
\frac{\partial \dot{M}}{\partial e} &= \dot{M}\Big(\frac{e}{1-e^2} - \frac{3}{r}\frac{\partial r}{\partial e}\Big) + \frac{n}{\sqrt{1-e^2}}\frac{a^4}{r^3}\Big[2e - 3\cos E + (2+\frac{a}{r})\cos^3 E + \frac{a}{r}(e - 2\cos E)\Big],\\
\frac{\partial \dot{L}}{\partial \ell} &= n\frac{a^4}{r^4}\Big[2r\sin^2 E + (e - 2\cos E + e\cos^2 E)\Big(a\cos E - 3\frac{\partial r}{\partial \ell}\Big)\Big],\\
\frac{\partial \dot{M}}{\partial \ell} &= \frac{n}{\sqrt{1-e^2}}\frac{a^5}{r^4}\sin E\Big[e - 4\cos E + 3e\cos^2 E - \frac{3ae}{r}(e^2 - 1 - e\cos E + 2\cos^2 E - e\cos^3 E)\Big].
\end{aligned}
$$

Finally, the derivatives of $\dot{X}$, $\dot{Y}$, r that appear in the previous expressions can be found in Broucke (1970, Table 1).

Appendix B

Let us introduce the coordinate change from $\mathscr{E}^*_{att}$ to $\mathscr{E}_{car}$ as the composite transformation

$$\psi = \phi_2^{-1} \circ \phi_3^{-1}.$$

Then we have

$$\frac{\partial \Phi^{-1}}{\partial \mathscr{E}^*_{att}} = \Big(\frac{\partial \phi_1}{\partial \mathscr{E}_{kep}}\Big)^{-1} \frac{\partial \psi}{\partial \mathscr{E}^*_{att}},$$

where $(k = 1, 2, 3)$

$$\frac{\partial \psi_{(k)}}{\partial \mathscr{E}^*_{att}} = (\rho \mathbf{e}^{\alpha}_{(k)} \cos\delta,\ \rho \mathbf{e}^{\delta}_{(k)},\ 0,\ 0, \mathbf{e}^{\rho}_{(k)},\ 0)^T,$$
$$\frac{\partial \psi_{(k+3)}}{\partial \mathscr{E}^*_{att}} = (\xi \mathbf{e}^{\perp}_{(k)} + \mathbf{e}^{\alpha}_{(k)}(\dot\rho\cos\delta - \zeta\sin\delta), \dot\rho \mathbf{e}^{\delta}_{(k)} - \zeta \mathbf{e}^{\rho}_{(k)},\ \mathbf{e}^{\alpha}_{(k)},\ \mathbf{e}^{\delta}_{(k)},\ 0,\ \mathbf{e}^{\rho}_{(k)})^T,$$

with

$$\mathbf{e}^{\perp} = (-\cos\alpha,\ -\sin\alpha,\ 0)^T.$$

Appendix C

We can write

$$\frac{\partial(\mathbf{X}_{kep} \circ \Phi^{-1})}{\partial \mathscr{E}^*_{att}} = \frac{\partial \mathbf{X}_{kep}}{\partial \mathscr{E}_{kep}} \frac{\partial \Phi^{-1}}{\partial \mathscr{E}^*_{att}},$$

where

$$\frac{\partial \dot\Omega}{\partial a} = -\frac{3}{2} J_2 \frac{R_\oplus^2}{p^2} \cos I \Big(\frac{\partial \tilde n}{\partial a} - \frac{2\tilde n}{a} \Big),$$
$$\frac{\partial \dot\Omega}{\partial e} = -\frac{3}{2} J_2 \frac{R_\oplus^2}{p^2} \cos I \Big(\frac{\partial \tilde n}{\partial e} + \frac{4\tilde n e}{1-e^2} \Big),$$
$$\frac{\partial \dot\Omega}{\partial I} = -\frac{3}{2} J_2 \frac{R_\oplus^2}{p^2} \Big(\frac{\partial \tilde n}{\partial I} \cos I - \tilde n \sin I \Big),$$
$$\frac{\partial \dot\omega}{\partial a} = \frac{3}{4} J_2 \frac{R_\oplus^2}{p^2} (4 - 5\sin^2 I) \Big(\frac{\partial \tilde n}{\partial a} - \frac{2\tilde n}{a} \Big),$$
$$\frac{\partial \dot\omega}{\partial e} = \frac{3}{4} J_2 \frac{R_\oplus^2}{p^2} (4 - 5\sin^2 I) \Big(\frac{\partial \tilde n}{\partial e} + \frac{4\tilde n e}{1-e^2} \Big),$$
$$\frac{\partial \dot\omega}{\partial I} = \frac{3}{4} J_2 \frac{R_\oplus^2}{p^2} \Big[\frac{\partial \tilde n}{\partial I} (4 - 5\sin^2 I) - 5\sin(2I)\tilde n \Big],$$
$$\frac{\partial \tilde n}{\partial a} = \frac{\partial n}{\partial a} - \frac{21}{4} J_2 \frac{R_\oplus^2}{p^2} \frac{n}{a} \Big(1 - \frac{3}{2}\sin^2 I \Big) \sqrt{1-e^2},$$
$$\frac{\partial \tilde n}{\partial e} = \frac{9}{2} J_2 \frac{R_\oplus^2}{p^2} \Big(1 - \frac{3}{2}\sin^2 I \Big) \frac{ne}{\sqrt{1-e^2}},$$
$$\frac{\partial \tilde n}{\partial I} = -\frac{9}{2} J_2 \frac{R_\oplus^2}{p^2} n \sin I \cos I \sqrt{1-e^2}.$$