Approximate Eigen Solutions of D.K.P. and Klein-Gordon Equations with Hellmann Potential

C. A. Onate[1,*], J. O. Ojonubah[2], A. Adeoti[3], E. J. Eweh[4] and M. Ugboja[2]
[1]*Physics Section, Kogi State Teaching Service Commission, Lokoja, (E.C.S.S. Uchuchu) Nigeria*
[2]*Mathematics Department, Federal College of Education, Okene, Nigeria*
[3]*Physics Department, University of Ilorin, Nigeria*
[4]*Physics Department, Federal University of Agriculture, Makurdi, Nigeria*

By using a suitable approximation scheme for orbital centrifugal term, we have studied the analytical solutions of both the D.K.P. equation and the Klein-Gordon equation with Hellmann potential in the framework of super-symmetric approach. In order to test the accuracy of our results, we obtained the non-relativistic limit of the energy equation for the Klein-Gordon equation with potential V . We also obtained the solution of the Schrödinger equation via the Formula method recently proposed by Falaye et al. We numerically obtained energy eigenvalues and compared our result with the results of other methods. The behavior of energy in the first, second and third state with the screening parameter are studied graphically.

1. Introduction

There has been a growing interest in investigating the analytical solutions of the wave equation for some physical potential models. This is due to the fact that the analytical solutions encompass all the necessary information for the quantum system.

Motivated by two particles system interacting through a combination of similar potentials, we attempt to investigate the attractive Coulomb potential and Yukawa potential known as Hellmann potential under the D.K.P. equation and the Klein-Gordon equation. The Hellmann potential is given as

$$V(r)=-\frac{a}{r}+\frac{be^{-\delta r}}{r} \qquad (1)$$

This potential has received considerable interest for several years. In the Hellmann potential (Eqn. (1)), parameters a and b characterize the strength of Coulomb potential and Yukawa potential, respectively, δ is the screening parameter and r is the distance between the two particles. The Hellmann potential found its applications in the field of atomic and condensed matter Physics like electron-core [1,2], electron-ion inner-shell ionization problem, alkali hydride molecules, and solid state Physics.

The D.K.P. equation describes spin-0 and spin-1 particles and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons [3]. It is a direct generalization of Dirac particles of integer spin in which one replaces the gamma matrices by beta matrices, but satisfying a more complicated algebra as D.K.P. algebra [3-8].

The Klein-Gordon equation for a free particle has simple plane wave solution [9]. This equation is similar to but cannot be straight forwardly interpreted as the Schrödinger equation for a quantum state because it is second-order in time and do not admit a positive definite conserved probability density. However, Alhaidari et al. [10] pointed out that one can produced a non-relativistic limit with potential function V(r) instead of 2V(r) from the relativistic Klein-Gordon equation under the condition of equal vector and scalar potentials (V(r) and S(r)) [11]. Generally, the Klein-Gordon equation does describe the quantum amplitude for finding a point particle in various places.

The purpose of this work is three-fold. First, we solve the D.K.P. equation with the Hellmann potential and obtained the approximate solution by applying a suitable approximation scheme to centrifugal barrier. It is understood that the solution of D.K.P. is limited to some potentials due to the square of potential term (U^2) in the equation. Second, we obtain Eigen solutions of the Klein-Gordon equation whose potential function in the non-relativistic limit is V [12]. Third, we obtain the solution of the Schrödinger equation with Hellmann potential using Formula method.

Our work is organized as follows. In Sec. 2, we obtain eigen-solutions of D.K.P. and Klein-Gordon

*onateca12@gmx.us

equations. In Sec. 3, we obtain the non-relativistic limit, the solution of the Schrödinger equation via the Formula method and relevant numerical results. In Sec. 4, we give the concluding remark.

2. Bound State Solutions of D.K.P. Equation

The D.K.P. equation with energy E, total angular momentum centrifugal term $\frac{J(J+1)}{r^2}$ and the mass of the particle m is given as [13]

$$\left[\frac{d^2}{dr^2}-\frac{J(J+1)}{r^2}-m^2+(E-U_\nu^0)^2\right]F_{nJ}(r)=0 \quad (2)$$

Due to the total angular momentum centrifugal term, Eqn. (2) cannot be solved approximately for $J\neq 0$ state. Therefore, we shall employ the Pakeris approximation and Greene-Aldrich approximation [14,15] in order to deal with this centrifugal barrier. It is found that the following approximation

$$\frac{1}{r^2}\approx\frac{\delta^2}{(1-e^{-\delta r})^2} \quad (3)$$

is a good approximation to centrifugal term $\frac{\ell(\ell+1)}{r^2}$. Substituting Eqns. (1) and (3) into Eqn. (2), we have

$$\frac{d^2F_{nJ}(r)}{dr^2}+\left[\lambda_1+\frac{\lambda_2 e^{-\delta r}}{1-e^{-\delta r}}+\frac{\lambda_3 e^{-\delta r}}{(1-e^{-\delta r})^2}\right]F_{nJ}(r)=0 \quad (4)$$

Where,

$$\lambda_1=E_{nJ}^2-m^2-J(J+1)\delta^2+a\delta(a\delta+2E_{nJ})$$

$$\lambda_2=\delta(a-b)\left[\delta(a+b)+2E_{nJ}\right]-J(J+1)\delta^2$$

$$\lambda_3=b\delta^2(b-2a)-J(J+1)\delta^2$$

In order to solve Eqn. (4) using the methodologies of super-symmetric quantum mechanics and shape invariant technique [16,17,18], the super-potential $W(r)$ must be the first ingredient we should search for. Our super-potential is given by

$$W(r)=\alpha+\beta(1-e^{-\delta r})^{-1} \quad (5)$$

The above super-potential is related to the ground state wave function as [19,20]

$$F_{n,J}(r)=\exp\left(-\int W(r)dr\right) \quad (6)$$

In this bound state solution, the radial part of the wave function must satisfy the conditions that $F_{n,J}(r)/r$ becomes zero as $r\to\infty$ and $F_{n,J}(r)/r$ is finite at $r=0$. Relating Eqn. (4) to a non-linear Riccati equation of the form [21,22]

$$\frac{d^2F_{nJ}(r)}{dr^2}=W^2(r)-\frac{dW(r)}{dr} \quad (7)$$

the following identities are obtained

$$\alpha^2=m^2-E_{nJ}^2+J(J+1)\delta^2-a\delta(a\delta+2E_{nJ}) \quad (8)$$

$$\beta=-b(b\delta-2a)+(J+1)\delta \quad (9)$$

$$\alpha=\frac{-\delta(a-b)\left[\delta(a+b)+2E_{nJ}\right]+J(J+1)\delta^2-\beta^2}{2\beta} \quad (10)$$

With the help of the super-potential function of Eqn. (5), a pair of super-symmetric partner potentials $U_\pm(r)=W^2(r)\pm\frac{dW(r)}{dr}$ can be constructed as

$$U_+(r)=\alpha^2+2\alpha\beta(1-e^{-\delta r})^{-1}+\beta^2(1+e^{-\delta r})(1-e^{-\delta r})^{-2}-\beta(\beta+\delta)(1+e^{-\delta r})^{-2}e^{-\delta r} \quad (11)$$

$$U_-(r)=\alpha^2+2\alpha\beta(1-e^{-\delta r})^{-1}+\beta^2(1+e^{-\delta r})(1-e^{-\delta r})^{-2}-\beta(\beta+\delta)(1+e^{-\delta r})^{-2}e^{-\delta r} \quad (12)$$

Eqns. (11) and (12) are related by the following formula

$$U\ (r,a_0)=U_-(r,a_0)+R(a_1) \quad (13)$$

Where, $a_0=\beta$ and a_1 is a function of a_0 i.e., $a_1=f(a_0)=a_0-\delta$ and the residual term $R(a_0)$ is independent of the variable, r [29].

$R(a_1)=\left(\frac{-\lambda_2-a_0^2}{2a_0}\right)^2-\left(\frac{-\lambda_2-a_1^2}{2a_1}\right)^2$. For shape invariant potential $U_-(r)$, its energy spectrum can approximately be determined by employing the following super-symmetric WKB quantization condition [23] given as

$$\int_{r_L}^{r_R}\sqrt{\lambda_{1nJ}^{(-)}-W^2(r)dr}=n\pi\ , \quad n=0,1,2,\ldots \tag{14}$$

Where, r_R and r_L are two turning points that can be obtained by the equation $\lambda_{1nJ}^{(-)}-W^2(r)=0$. With the super-symmetric quantum mechanics and the standard WKB approximation method, Comtet et al. proposed the lowest order super-symmetric WKB quantization condition, which can determine the energy spectra for the shape-invariant potentials. Considering the super-potential function $W(r)$ of Eqn. (5), the super-symmetric WKB quantization condition Eqn. (14) can further be written as follows:

$$\int_{r_L}^{r_R}\sqrt{\lambda_{1nJ}^{(-)}-\left[\left(\frac{\lambda_3}{2\delta}-\frac{\delta\lambda_2}{2\lambda_3}\right)^2-\frac{\lambda_3^2}{\delta^2(1-e^{-\delta r})}\left(1-\frac{1}{1-e^{-\delta r}}\right)-\frac{\lambda_2}{1-e^{-\delta r}}\right]^2}\,dr=n\pi \tag{15}$$

From the shape formalism, the following equation is deduced from Eqns. (12) and (13)

$$\alpha^2=\left(\frac{-\lambda_1-a_n^2}{2a_n}\right)^2 \tag{16}$$

Where, $a_n=a_0-n\delta$ and $a_n=\beta$ as $\beta\to\beta-\delta$. Substituting the values of a^2,λ_2, and a_n into Eqn. (16), the approximate energy equation for the DKP can be obtained as

$$(M+E_{nJ})(M-E_{nJ})+\delta[J(J+1)\delta-a(a\delta+2E_{nJ})]$$

$$=\left[\frac{(a-b)[\delta(a+b)+2E_{nJ}]-J(J+1)\delta-[b(b\delta-2a)+n+J+1]^2}{2[b(b\delta-2a)+n+J+1]}\right]^2 \tag{17}$$

In order to obtain the wave function, we define a variable of the form $s=e^{-\delta r}$ and substitute it into Eqn. (4) to have

$$\frac{d^2F_{nJ}(s)}{ds^2}+\frac{1}{s}\frac{dF_{nJ}(s)}{ds}+\frac{\gamma+\rho s+\eta s^2}{s^2(1-s)^2}F_{nJ}(s)=0 \tag{18}$$

Where,

$$\gamma=\frac{E_{nJ}^2-M^2}{\delta^2}+a\left(a+\frac{2E_{nJ}}{\delta}\right)-J(J+1)$$

$$\rho=-a(a+2b)-\frac{2E_{nJ}}{\delta}(a+b)-\frac{2}{\delta^2}(E_{nJ}^2-M^2)$$

and

$$\eta=\frac{E_{nJ}^2-M^2}{\delta^2}+\frac{2bE_{nJ}}{\delta}+b^2$$

Analyzing the asymptotic behavior of Eqn. (18) at origin and at infinity ($r\to 0(s\to 1)$ and $r\to\infty(s\to 0)$), Eqn. (18) has a solution

$$F_{nJ}(s)=s^{\varepsilon}(1-s)^{\xi}U_{nJ}(s) \tag{19}$$

Where, $\varepsilon=\sqrt{\frac{E_{nJ}^2-M^2}{\delta^2}+a\left(a+\frac{2E_{nJ}}{\delta}\right)-J(J+1)}$, and, $\xi=\sqrt{J(J+1)\delta-b(b\delta-2a)}$. With ε and ξ, Eqn. (19) is written in the form:

$$U''(s)+U'(s)\left[\frac{(2\varepsilon+1)-s(2\varepsilon+\xi+1)}{s(1-s)}-U(s)\left[\frac{(2\varepsilon+\xi)^2+\eta}{s(1-s)}\right]\right]=0 \quad (20)$$

which, gives the total radial wave function as

$$F_{nJ}(s)=N_{nJ}e^{-\delta r\varepsilon}(1-e^{-\delta r})^{\xi}U_{nJ}(-n,n+2(\varepsilon+\xi),$$

$$2\varepsilon+1,e^{-\delta r}) \quad (21)$$

2.1. Bound state solutions of the Klein-Gordon equation

The Klein-Gordon equation with scalar and vector potential is written as

$$\left[\frac{d^2}{dr^2}-(E-V(r))^2+\left(M+S(r)\right)^2-\frac{\ell(\ell+1)}{r^2}\right]R_{n\ell}(r)=0 \quad (22a)$$

In order to obtain the non-relativistic limit and to compare our results with the results of other method, we write the Klein-Gordon equation whose non-relativistic limit has potential *V* instead of 2*V*. Therefore, Eqn. (22a) is re-written as

$$\left[\frac{d^2}{dr^2}-\left(E-\frac{1}{2}V(r)\right)^2+\left(M+\frac{1}{2}S(r)\right)^2-\frac{\ell(\ell+1)}{r^2}\right]R_{n\ell}(r)=0 \quad (22b)$$

Substituting potential (Eqn. (1)) and approximation (Eq. (3)) into Eqn. (22b) and using the same procedure to obtain energy equation for DKP equation, the energy equation for the Klein-Gordon equation is obtained as

$$(M-E_{n\ell})(M+E_{n\ell})+a\delta(M+E_{n\ell})-\ell(\ell+1)\delta^2$$

$$=\left[\frac{\frac{(M+E_{n\ell})(a-b)}{\delta}-\ell(\ell+1)-(n+\ell+1)^2}{2(n+\ell+1)}\right]^2 \quad (23)$$

3. Non-relativistic Limit

In this section, we consider the non-relativistic limit of Eqn. (23). Considering a transformation of the form: $M+E_{n\ell}=\frac{2\mu}{\hbar^2}$ and $M-E_{n\ell}=-En\ell$ and substitute it into Eqn. (23), we have the non-relativistic energy equation as

$$E_{n\ell}=\frac{\ell(\ell+1)\delta^2\hbar^2}{2\mu}-a\delta$$

$$-\frac{\delta^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu(a-b)}{\hbar^2\delta}-\ell(\ell+1)-(n+\ell+1)^2}{2(n+\ell+1)}\right]^2 \quad (24)$$

and, the corresponding wave function as

$$R_{n\ell}(Z)=Nz^e(1-z)^d\,{}_2F_1(-n,n+2e+2d;2e+1,z) \quad (25)$$

3.1. Bound state solution of Schrödinger equation using formula method

In this section, we obtain a solution of the Schrödinger equation with the Hellmann potential via the formula method recently proposed by Falaye et al. [24]. Given the Schrödinger-like equation, including the centrifugal barrier, in the presence of any physical potential in the form

$$\psi''(s)+\frac{k_1-k_2s}{s(1-s)}\psi'(s)+\frac{As^2+Bs+C}{s^2(1-s)^2}\psi(s)=0 \quad (26)$$

The energy eigen-values and the corresponding wave function can now be obtain via the following

$$k_5^2 - \left[\frac{k_4^2 - k_5^2 - \left[\frac{1-2n}{2} - \frac{1}{2k_3} \left(k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right]^2}{2\left[\frac{1-2n}{2} - \frac{1}{2k_3} \left(k_2 - \sqrt{(k_3 - k_2)^2 - 4A} \right) \right]} \right]^2 = 0 \quad (27)$$

$$\psi(s) = N_{n,\ell} s^{k_4} (1 - k_2 s)^{k_5} F_1 \left(-n, n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1; 2k_4 + k_1, k_3 s \right) \quad (28)$$

respectively, where

$$k_4 = \frac{(1-k_1) + \sqrt{(1-k_1)^2 - 4C}}{2}$$

$$k_5 = \frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} + \sqrt{\left(\frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} \right)^2 - \frac{A}{k_3^2} - \frac{B}{k_2} - C} \quad (29)$$

Given the radial Schrödinger equation

$$\frac{d^2 R_{n,\ell}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left(E_{n\ell} - V_{eff}(r) \right) R_{n\ell}(r) = 0 \quad (30)$$

Where

$$V_{eff}(r) = -\frac{a}{r} + \frac{be^{-\delta r}}{r} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} \quad (31)$$

Substituting Eqns. (3) and (29) into Eqn. (27) and introducing an appropriate change of variables $r \to z$ through the mapping function, $z = e^{-\delta r}$, we obtain

$$\frac{d^2 R_{n\ell}(z)}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{dR_{n\ell}(z)}{dz} + \left[\frac{(\tau_1 - \varepsilon_{n\ell}^2 - \tau_2) + (2\varepsilon_{n\ell}^2 + \tau_3 - 2\tau_1)z + (-\varepsilon_{n\ell}^2 - \tau_3)z^2}{z^2(1-z)^2} \right] \quad (32)$$

Where, $\varepsilon_{n\ell}^2 = \sqrt{-\frac{2\mu E_{n\ell}}{\delta^2 \hbar^2}}$, $\tau_1 = \frac{2\mu a}{\delta\hbar^2}$, $\tau_2 = \ell(\ell+1)$, $\tau_3 = \frac{2\mu(a-b)}{\delta\hbar^2}$. Comparing Eqn. (24) with Eqn. (30), we have the following: $k_1 = k_2 = k_3 = 1$, $k_4 = \sqrt{\tau_2 - \tau_1 + \varepsilon_{n\ell}}$, $k_5 = \frac{1}{2}\left(1 + \sqrt{(1+2\ell)^2 + \tau_1} \right)$. Substituting the values of the above parameters into Eqn. (25), we have energy equation as

$$E_{n\ell} = \frac{\delta^2 \hbar^2}{2\mu} \left[\ell(\ell+1) - \frac{2\mu a}{\delta\hbar^2} - \left(\frac{\frac{2\mu(a-b)}{\delta\hbar^2} - \ell(\ell+1)}{2(n+\ell+1)} - \frac{1}{2}(n+\ell+1) \right)^2 \right] \quad (33)$$

The corresponding wave function is obtained using Eqn. (28)

$$R_{n\ell}(z) = N_{n\ell} z^{\sqrt{\tau_2 - \tau_1 + \varepsilon_{n\ell}}} (1-z)^{\frac{1}{2}\left(1 + \sqrt{(1+2\ell)^2 + \tau_1} \right)} F_1(-n, n + 2(\sqrt{\tau_2 - \tau_1 + \varepsilon_{n\ell}}) + \frac{1}{2}(1 + \sqrt{(1+2\ell)^2 + \tau_1}); 2(\sqrt{\tau_2 - \tau_1 + \varepsilon_{n\ell}}) + 1, z) \quad (34)$$

4. Discussion

In Table 1, we show the energy eigen-values of the Hellmann potential with D.K.P. equation for various values of n and J. In Table 2, we show energy eigen-values of the Hellmann potential for the Klein-Gordon energy equation for 1s, 2s, 2p, 3s, 3p, 3d, 4s, 4p, 4d and 4f states with $b=1$ and $a=2b$. In Table 3, we report the energy eigen-values in the non-relativistic limit for Hellmann potential and compare these results with the result obtained using the Nikiforov-Uvarov method and the Amplitude Phase method.

To see the similarity between the two potential strengths, we plot the variation of energy with *a* and *b* in Fig. 1(a) and 1(b), respectively. In Fig. 2, we plot energy of the Hellmann potential with the screening parameter.

Table 1: Energy spectrum ($\pm E_{nJ}$ in MeV) of the D.K P. equation with $b=-1, a=-2b, \delta=6.5\times10^{-1}$ and m = 938MeV.

n	J	$-E_{nJ}$	E_{nJ}
0	0	867.4439796	862.5842894
	1	845.4685995	840.8191045
	2	812.8365099	808.4892471
	3	762.3703438	758.4671998
1	0	845.4681948	840.8186998
	1	812.8357314	808.4884686
	2	762.3692485	758.4661045
	3	681.2617671	678.0129981
2	0	812.8353419	808.4880791
	1	762.3685183	758.4653743
	2	681.2606920	678.0120192
	3	548.2163523	545.8827081
3	0	762.3681532	758.4650092
	1	681.2600392	678.0113664
	2	548.2155644	545.8819202
	3	337.3671953	336.0793087

Table 2: Energy spectrum for the Klein-Gordon Equation with $b=1, a=2b, \delta=6.5\times10^{-1}$ and M = 938MeV.

State	$E_{n\ell}$	$E_{n\ell}$
1s	-937.9998670	-640.8498056
2s	-937.9994681	-131.7728669
2p	-937.9983544	-131.2790568
3s	-937.9988029	242.8675761
3p	-937.9977628	243.1887170
3d	-937.9953298	243.8306458
4s	-937.9978718	472.2251748
4p	-937.9968574	472.4405866
4d	-937.9946301	472.8712117
4f	-937.9907937	473.5166539

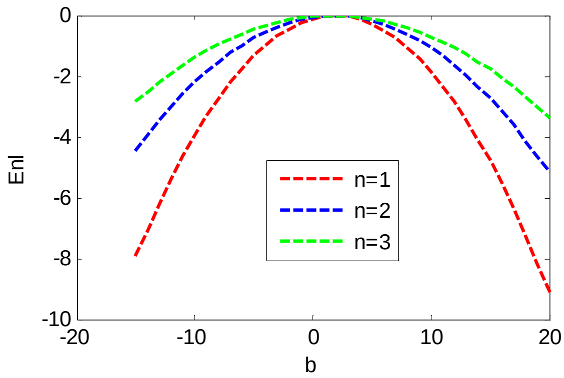

Fig.1(a): Variation of energy with the potential strength *b* for $\hbar=\ell=2\mu=1, \delta=0.01$ and $a=4\mu$.

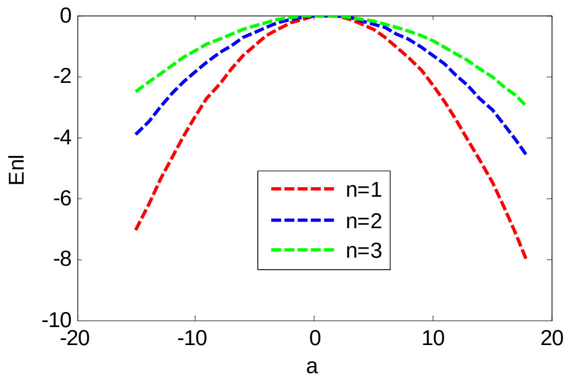

Fig.1(b): Variation of energy with the potential strength *a* for $\hbar=\ell=2\mu=1, \delta=0.01$ and $b=2\mu$.

Table 3: Comparison of energy spectrum from SUSY, Nikiforov-Uvarov (N.U) and Amplitude Phase method with $\hbar = b = 2\mu = 1$ and $a = 4\mu$.

n	ℓ	State	δ	Present	N.U	A.P
0	0	1s	0.001 0.005 0.010	-0.251 500 -0.257 506 -0.265 025	-0.251 500 -0.257 506 -0.265 025	-0.250 969 -0.254 933 -0.259 823
1	0	2s	0.001 0.005 0.010	-0.064 001 -0.070 025 -0.077 600	-0.064 001 -0.070 025 -0.077 600	-0.063 243 -0.067 106 -0.071 689
0	1	2p	0.001 0.005 0.010	-0.063 750 -0.068 756 -0.075 025	-0.064 000 -0.070 000 -0.077 500	-0.063 495 -0.067 377 -0.072 020
2	0	3s	0.001 0.005 0.010	-0.029 280 -0.035 334 -0.043 003	-0.029 280 -0.035 334 -0.043 003	-0.028 283 -0.031 993 -0.036 142
1	1	3p	0.001 0.005 0.010	-0.029 169 -0.034 756 -0.041 803	-0.029 279 -0.035 309 -0.042 903	-0.028 765 -0.032 480 -0.036 142
0	2	3d	0.001 0.005 0.010	-0.028 945 -0.033 617 -0.039 469	-0.029 388 -0.035 817 -0.043 825	-0.028 767 -0.032 526 -0.036 613
3	0	4s	0.001 0.005 0.010	-0.017 129 -0.023 225 -0.031 025	-0.029 280 -0.035 334 -0.043 003	-0.016 601 -0.020 077 -0.023 551
2	1	4p	0.001 0.005 0.010	-0.017 066 -0.022 889 -0.030 306	-0.017 128 -0.023 200 -0.030 925	-0.016 602 -0.020 098 -0.023 641
1	2	4d	0.001 0.005 0.010	-0.016 939 -0.022 227 -0.028 906	-0.017 189 -0.023 464 -0.031 356	-0.016 604 -0.020 098 -0.023 641
0	3	4f	0.001 0.005 0.010	-0.016 750 -0.021 257 -0.026 900	-0.017 311 -0.024 024 -0.032 356	-0.016 607 -0.020 142 -0.024 056

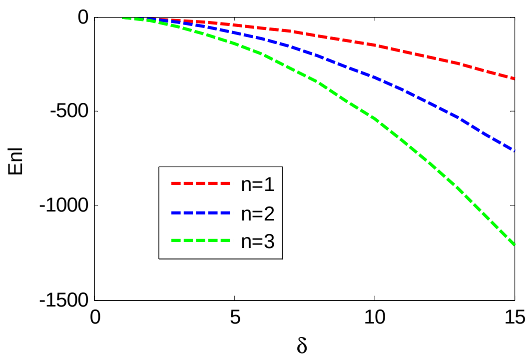

Fig.2: Variation of energy of the Hellmann potential with the screening parameter δ for $\ddot{y} = \ell = 2\mu = b = 1$ and $a = 2b$.

5. Conclusion

In this work, we reported approximate solutions of the D.K.P. and Klein-Gordon equation with Hellmann potential. It is seen from Tables 2(a) and 2(b) that numerical results by the present method are in excellent agreement with the previous results of other two methods. We deduced from the numerical results in Table 2 that the energy is less negative as the angular quantum number increases. It is also deduced in Fig. 1 that the two potential strengths have the same effect on the energy bahvior. It is deduced that Eqn. (24) is as Eqn. (33). This confirms the accuracy of our method and results.

References

[1] P. Combas, *Die Statistische Theorie des Atoms und ihre Anwendungen* (Springer, Berlin, 1949) p. 304.

[2] J. Callaway, Phys. Rev. **112**, 322 (1958).

[3] M. Hamzavi and S.M. Ikhdair, Few-Body Syst. DOI 10.1007/s00601-012-0452-9.

[4] C. Clark et al., Phys. Rev. Lett. **55**, 592 (1985).
[5] G. Kalbermann, Phys. Rev. C **3**, 2240 (1986).
[6] R. E. Kozack, Phys. Rev. C **40**, 2181 (1989).
[7] Y. Nedjadi and R. C. Barrett, J. Phys. G **19**, 87 (1993).
[8] M. Nowakowski, Phys. Lett. A **244**, 329 (1998).
[9] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill Co., Dover Publisher, 2006).
[10] A. D. Alhaidari, H. Bahlouli and A. Al-Hasan, Phys. Lett. A **349**, 87 (2006).
[11] C. S. Jia, T. Chen and S. He, Phys. Lett. A DOI 10.1016/j.physleta.2013.01.016.
[12] D. Aysen and Y. Fevziye, J. Chem. Phys. **120**, 074108 (2007).
[13] R. J. Duffin, Phys. Rev. **54** (1938).
[14] C. L. Pekeris, Phys. Rev. **45**, 98 (1934).
[15] R. Greene and C Aldrich, Phys. Rev. A **14**, 2363 (1976).
[16] C. A. Onate, K. J. Oyewumi and B. J. Falaye, Afr. Rev. Phys. **8**, 0020 (2013).
[17] H. Hassanabadi, E. Maghsoodi and S. Zarrinkamar, Eur. Phys. J. Plus **127**, 31 (2012).
[18] C. A. Onate, K. J. Oyewumi and B. J. Falaye, Few-Body Syst. **55**, 61 (2014).
[19] S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, Ann. Phys. **325**, 2522 (2010).
[20] C. A. Onate, SOP T.T. Phys. **1**, 118 (2014).
[21] H. Hassanabadi, S. Zarrinkamar and H. Rahimov, Commun. Theor. **56**, 423 (2011).
[22] C. A. Onate, Afr. Rev. Phys. **8**, 0046 (2013).
[23] C. A. Onate, Afr. Rev. Phys. **8**, 0057 (2013).
[24] B. J. Falaye, S. M. Ikhdair and M. Hamzavi, arxiv:1408.3523v2 [math-ph] 24 Aug. 2014.

Received: 8 October, 2014
Accepted: 4 February, 2015