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The Influence of Dihedral Angle Error Stability on Beam Deviation for Hollow Retro-Reflectors

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***Abstract*—Retro-reflectors consist of three reflective optical surfaces, which are oriented to reflect the input beam by 180°. For retro-reflector components, it is common to specify an angular beam deviation tolerance, or rather the deviation from the exact 180° return direction. Precision-aligned retro-reflectors provide 180° beam deviation with tolerances on the order of an arcsecond. It is well known that the performance of the retro-reflector depends on the ability to precisely orient the reflective surfaces at mutually perpendicular angles. Precision assembly is therefore critical to ensure highly accurate beam deviation. The dihedral angle errors, and hence the reflected beam deviation, can be measured for the retro-reflector after fabrication, typically by using interferometric techniques. Yet, what is not commonly reported for a fabricated retro-reflector is the stability of the angular beam deviation. For instance, thermo-mechanical effects of the components will contribute to variations in the return beam direction. While the actual stability is design-specific one can develop a mathematical representation for the expected change in the reflected beam direction as a function of the variation in the dihedral angle errors. Presented here is a mathematical formulation for a hollow retro-reflector's beam deviation as a function of the dihedral angle error stability.**

I. Introduction

A retro-reflector is an optical device which is designed to return a beam of light back along the direction from which it originated. A lateral transfer retro-reflector is an optical device that not only changes the direction of the beam by 180°, but also shifts the beam laterally. In both devices, the return beam is parallel to the original incident beam. Common geometry for a retro-reflector is that of a corner-cube, where three optically flat reflective surfaces are oriented in a mutually perpendicular fashion. A hollow retro-reflector refers to a retro-reflector device, which has been constructed out of three optical elements, as opposed to a solid prism. For lateral transfer distances that are large compared to the size of the input beam, it is often advantageous to use a hollow lateral transfer retro-reflector, which is typically designed as a truncated version of a corner-cube style retro-reflector.

For retro-reflector applications, the requirements for the retro-reflector usually involve the beam direction deviation from the idealized 180°. Initial publications on the accuracy of retro-reflectors addressed the accuracy of the return direction from a retro-reflector due to non-idealities in the device [1], [2]. The work was motivated, for example, by satellite ranging experiments and lunar laser ranging experiments [3], [4]. For such applications, the ability for the retro-reflector to return the incident beam at identically 180° from the incoming direction is critical. As such, mathematical derivations pertaining to the accuracy of the retro-reflector as a function of the dihedral angle errors was developed [2]. Yet the publications at that time did not include the mathematical model for the stability of the beam direction due to variations in the incident beam direction or for changes in the dihedral angle errors of the retro-reflecting device. The dihedral angle errors were assumed to be constant for the device. Yet, in actuality, thermo-mechanical effects of the device components will contribute to variations in the return beam direction. For example, thermal environment variations, material thermal gradients, mechanical mount design and mechanical joint construction, including bonding methods and bond line thickness will all affect the stability of the retro-reflector design, and hence the stability of the return beam direction. If the retro-reflector is used to monitor a beam direction, then the stability of the return beam direction is more critical than the actual return direction value. While the actual stability is design-specific and is related to the mechanical and thermal stability of the device, one can develop a mathematical representation for the expected stability in the reflected beam direction as a function of the variation in the dihedral angle errors. This work presents a model for the expected beam deviation stability, by using the work initially published by Chandler [2] as a foundation.

II. Error due to mechanical changes

This section will consider changes to the geometry of the retro-reflector and the corresponding error generated in the reflected beam direction. The fundamental equations for a retro-reflector are completely dependent upon the geometry, specifically the angles between the reflective surfaces. Changes in the geometry, or mechanical stability of these angles, will dictate the performance of the retro-reflector.

A. Fundamental retro-reflector equations

The direction of a ray before and after reflection off of a single mirror surface is given by:

$$\bar{V}' = \bar{V} - 2\left(\bar{V} \cdot \hat{n}\right)\hat{n} \tag{1}$$

where $\bar{V}$ is vector defining the original incident ray, $\bar{V}'$ is the resulting direction vector, and $\hat{n}$ is the unit normal vector defining the surface of reflection. Application of the equation three times for each surface of the retro-reflector results in the direction of the reflected beam. Chandler [2] applies the formula three times recursively and then applies small deviations by which the angles between the mirrors exceed right angles. Chandler's formula (valid to first order approximation, for three nearly perpendicular reflecting surfaces) is:

$$\bar{t} = \hat{q} + 2\hat{q} \times \left(\alpha\hat{a} - \beta\hat{b} + \gamma\hat{c}\right) \tag{2}$$

where $-\bar{t}$ is the final reflected direction, $\hat{q}$ is the vector of the original beam direction, α β, γ, are the small angles by which the angles between the three reflecting surfaces exceed right angles (dihedral angle error). The unit vectors, $\hat{a}$, $\hat{b}$, $\hat{c}$, define the normal to each surface of the three planes in the three bounce sequence, taken in a right hand sense (ray first bounces off of surface defined by $\hat{a}$, then off of the surface defined by $\hat{b}$ and finally off of the surface defined by $\hat{c}$). In Chandler's equation, "The normals may be strictly perpendicular; that is, they do not need to include the small deviations caused by the dihedral-angle offsets." [4]. Thus, if one defines an orthonormal coordinate system $\left(\hat{i}, \hat{j}, \hat{k}\right)$, the normals to the planes may be simply approximated as $\hat{a} = \hat{i}, \hat{b} = \hat{j}, \hat{c} = \hat{k}$. For simplification, we define the combined dihedral angle error $\bar{\epsilon}$ term as

$$\bar{\epsilon} = \alpha\hat{a} - \beta\hat{b} + \gamma\hat{c} \tag{3}$$

$$\bar{\epsilon} \approx \alpha\hat{i} - \beta\hat{j} + \gamma\hat{k} \tag{4}$$

such that Equation 2 becomes:

$$\bar{t} = \hat{q} + 2\hat{q} \times \bar{\epsilon} \tag{5}$$

From Equation 5, the expected relation is observed, whereby the direction of the reflected beam is equal to the negative of the original incident beam direction plus an error term, which is a function of the original incident angle.

$$\bar{t} = \hat{q} + \mathcal{E} \tag{6}$$

$$\mathcal{E} = 2\hat{q} \times \bar{\epsilon} \tag{7}$$

B. Reflected ray direction change due to change in incident ray direction

It is desired to determine the sensitivity, or the resulting error in the final reflected beam direction as a function of a change in the incident ray vector $\hat{q}$. For a perfect retro-reflector, the output ray angle will exactly match the negative of the input ray angle. It is not possible to manufacture the retro-reflector with exactly perpendicular reflective faces, which leads to a non-zero error term. As shown in Equation 7, the error, $\mathcal{E}$, is a function of the incident ray direction. If the incoming ray direction is changing (due to for example an induced change due to a steering mirror), what is the change in the reflected ray direction due to an incremental change in the incoming ray direction. This variation is the retro-reflector non-linearity in the reflected beam direction. Thus, it is desired to determine $\partial\bar{t}/\partial\hat{q}$, which is the Jacobian. Assume that the vector defining the incident ray is defined in terms of the orthonormal coordinate system as:

$$\hat{q} = \eta\hat{i} + \xi\hat{j} + \nu\hat{k} \tag{8}$$

To compute the Jacobian, consider the change in the reflected direction vector $\bar{t}$ with respect to a change in first vector component, the $\hat{i}$ direction, of the incident ray. Applying the chain rule to the cross product term, the partial derivative becomes:

$$\frac{\partial\bar{t}}{\partial\eta} = \frac{\partial\hat{q}}{\partial\eta} + 2\frac{\partial\hat{q}}{\partial\eta} \times \bar{\epsilon} + 2\hat{q} \times \frac{\partial\bar{\epsilon}}{\partial\eta} \tag{9}$$

Since $\bar{\epsilon}$ is not a function of η, the partial derivative $\partial\bar{\epsilon}/\partial\eta$ is equal to zero, resulting in:

$$\frac{\partial\bar{t}}{\partial\eta} = \frac{\partial\hat{q}}{\partial\eta} + 2\frac{\partial\hat{q}}{\partial\eta} \times \bar{\epsilon} \tag{10}$$

Recalling the definition for the incident ray vector, Equation 8, the partial derivative of the incident ray vector is simply the orthonormal basis vector, $\partial\hat{q}/\partial\eta = \hat{i}$. One then obtains:

$$\frac{\partial\bar{t}}{\partial\eta} = \hat{i} + 2\hat{i} \times \bar{\epsilon} \tag{11}$$

Substituting the representation for $\bar{\epsilon}$ in terms of the $\hat{i}, \hat{j}, \hat{k}$, coordinate system, Equation 4, and applying the cross product, one obtains:

$$\frac{\partial\bar{t}}{\partial\eta} = \hat{i} - 2\gamma\hat{j} - 2\beta\hat{k} \tag{12}$$

The sensitivity to the other two components $\hat{j}$ and $\hat{k}$ are computed in a similar fashion resulting in:

$$\frac{\partial\bar{t}}{\partial\xi} = 2\gamma\hat{i} + \hat{j} - 2\alpha\hat{k} \tag{13}$$

$$\frac{\partial\bar{t}}{\partial\nu} = 2\beta\hat{i} + 2\alpha\hat{j} + \hat{k} \tag{14}$$

For completeness, the Jacobian matrix is therefore:

$$\frac{\partial\bar{t}}{\partial\hat{q}} = \left[\frac{\partial\bar{t}}{\partial\eta}, \frac{\partial\bar{t}}{\partial\xi}, \frac{\partial\bar{t}}{\partial\nu}\right] = \begin{bmatrix} 1 & 2\gamma & 2\beta \\ -2\gamma & 1 & 2\alpha \\ -2\beta & -2\alpha & 1 \end{bmatrix} \tag{15}$$

Thus, the rate of change of the outgoing direction vector, as a function of a change in the incoming direction is unity, plus two times the errors in the dihedral angles:

$$\frac{\partial\bar{t}}{\partial\hat{q}} = [\mathrm{I}] + \begin{bmatrix} 0 & 2\gamma & 2\beta \\ -2\gamma & 0 & 2\alpha \\ -2\beta & -2\alpha & 0 \end{bmatrix} \tag{16}$$

By inspection of Equation 16 it is seen that with a change in a component of the incident ray vector, (η, ξ, ν), the errors in the orthogonality of the reflecting surfaces is represented in the other two orthonormal components.

C. Change in reflected ray direction due to dihedral angle changes

The dihedral angle errors can be measured for corner cube retro-reflectors using interferometry measurement techniques as described by Thomas [5] and Zygo [6]. Lateral transfer retro-reflectors can also be characterized using interferometric techniques as described in Martin et al. [7]. After initial measurement and calibration of the retro-reflector device, it is necessary to understand how the reflected ray direction changes with a change in the dihedral angle errors. The rate of change of the reflected beam as a function of a change in the dihedral angle errors, α, β, γ is the Jacobian, $\left[\frac{\partial\bar{t}}{\partial\alpha}, \frac{\partial\bar{t}}{\partial\beta}, \frac{\partial\bar{t}}{\partial\gamma}\right]$.

To compute the Jacobian, the partial derivatives of Equation 5 are readily computed:

$$\frac{\partial \bar{t}}{\partial \alpha} = \frac{\partial \hat{q}}{\partial \alpha} + 2\frac{\partial \hat{q}}{\partial \alpha} \times \bar{\epsilon} + 2\hat{q} \times \frac{\partial \bar{\epsilon}}{\partial \alpha} \tag{17}$$

Since the incident ray vector, $\hat{q}$, is not a function of α, the partial derivatives of $\hat{q}$ with respect to α are equal to zero resulting in:

$$\frac{\partial \bar{t}}{\partial \alpha} = 2\hat{q} \times \frac{\partial \bar{\epsilon}}{\partial \alpha} \tag{18}$$

The partial derivative $\partial\bar{\epsilon}/\partial\alpha$ is simply the orthonormal basis vector, $\hat{\imath}$. Inserting the expression for the incident ray vector, Equation 8, and carrying out the cross product, one obtains:

$$\frac{\partial \bar{t}}{\partial \alpha} = 2\nu\hat{\jmath} - 2\xi\hat{k} \tag{19}$$

In a similar fashion, the sensitivity to the other two dihedral angle errors is also found:

$$\frac{\partial \bar{t}}{\partial \beta} = 2\nu\hat{\imath} - 2\eta\hat{k} \tag{20}$$

$$\frac{\partial \bar{t}}{\partial \gamma} = 2\xi\hat{\imath} - 2\eta\hat{\jmath} \tag{21}$$

For completeness, the Jacobian with respect to a change in the dihedral error components is given by:

$$\frac{\partial \bar{t}}{\partial \bar{\epsilon}} = \left[\frac{\partial \bar{t}}{\partial \alpha}, \frac{\partial \bar{t}}{\partial \beta}, \frac{\partial \bar{t}}{\partial \gamma}\right] = \begin{bmatrix} 0 & 2\nu & 2\xi \\ 2\nu & 0 & -2\eta \\ -2\xi & -2\eta & 0 \end{bmatrix} \tag{22}$$

It is interesting to note that the Jacobian for a change in the dihedral angle is only a function of the incident ray vector components, η, ξ, ν. Thus, depending on the application, it may be possible to reduce the dependency on a change in dihedral angle error by choice of the incident ray direction. Work by Karube [8] also noted the existence of an optimal incident ray direction, where the influence of angular deviations may be eliminated.

III. Error due to thermal changes

By using the equations for the error due to mechanical variations, one can estimate an approximate error from thermal changes which result in a geometrical variation. To estimate a change in the dihedral angle due to thermal bending, one can approximate the hollow retro-reflector as three plates. The physical model of the system may be represented as either a thin plate with bending in two dimensions or a simplified model as a beam in pure bending. To gain an intuitive understanding of the physical system, the simplified model of a beam in pure bending will be discussed here. It is understood that this simplified model will not exactly match the actual physical system, which exhibits bending in two dimensions.

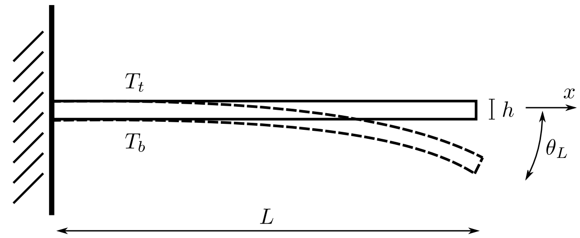


Fig. 1. Standard beam theory model for estimating the angular deviation (slope) due to a temperature differential.

A. Beam in bending model

A simplified model for the thermal deformation is that of a one-dimensional beam in bending with one end fixed and the other end free to move. The fixed end represents the seam forming the dihedral angle. For simplicity, the plates are assumed to be connected only at the seam forming the dihedral angle. This represents a worst case bound on the change in the dihedral angle. An applied thermal load, ΔT, will result in a change in length ΔL, of the beam with length L, resulting in a thermal strain:

$$\varepsilon = \frac{\Delta L}{L} = \alpha_T \Delta T \tag{23}$$

where α_T is the coefficient of thermal expansion for the material. Note that the subscript T is used to denote that the coefficient of thermal expansion is a function of temperature, and also to help avoid confusion with the dihedral angle error α. From structural mechanics and beam bending theory, the radius of curvature of a beam, ρ, with thickness h is given in terms of the strain, ε, as: [9]

$$\frac{1}{\rho} = \frac{\varepsilon}{h} \tag{24}$$

The angular deviation, $\mathrm{d}\theta$, at position x along the beam is given by:

$$\mathrm{d}\theta = \frac{1}{\rho}\mathrm{d}x \tag{25}$$

If one assumes a linear temperature profile in the optical plate, then the difference in the strain at the top surface and the bottom surface is

$$\varepsilon_t - \varepsilon_b = \alpha_T T_t - \alpha_T T_b = \alpha_T \Delta T \tag{26}$$

Thus, the resulting angular deviation in terms of the differential thermal strain is given by:

$$\mathrm{d}\theta = \mathrm{d}x\frac{\alpha_T \Delta T}{h} \tag{27}$$

Evaluating at the length of the beam, L, the angular deviation is (Figure 1):

$$\theta_L = \frac{L\alpha_T \Delta T}{h} \tag{28}$$

The angular deviation, θ_L, can then be inserted into the equations to represent a change in the dihedral angle errors.

IV. Numerical Examples

For numerical estimates of the dihedral angle error and the resulting error in beam deviation, some assumed parameters of the retro-reflector are necessary. Mechanical properties such as the accuracy of the retro-reflector and the incident ray angle direction must be assumed. Lateral transfer hollow retro reflectors can be fabricated with sub-arcsecond beam deviation (total error after a three-bounce reflection) and better than $\lambda/10$ wavefront error (λ = 632.8 nm) [10]. The beam deviation error is the magnitude of the total error, $\|\mathcal{E}\|$, given by Equation 7. In order to generate a value for the dihedral angle errors, we assume the incident ray direction vector is a unit vector such that the elements are equal, i.e. $\eta = \xi = \nu$. This also corresponds to a ray which is parallel to the axis of symmetry for a corner-cube retro-reflector. Thus, for $\hat{q}$ to be a unit vector, $\eta = 1/\sqrt{3}$. Also, for simplicity, one can assume that the dihedral angle errors, α, β, γ are also identical. Given these assumptions, the magnitude of the error term, $\|\mathcal{E}\|$, is easily shown to be:

$$\|\mathcal{E}\| = 4\sqrt{2}\,\alpha\,\eta \tag{29}$$

If the total beam deviation angle is equal to 4.8 urad, (1 arcsec), the dihedral angle errors, for an incident ray direction vector of $\hat{q} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$, would be equal to:

$$\alpha = \frac{\|\mathcal{E}\|}{4\sqrt{2}\,\eta} = 1.5\,\mathrm{urad} \tag{30}$$

For the numerical calculation examples, the dihedral angle errors will therefore be assumed to be $\alpha = \beta = \gamma = 1.5$ urad. Additionally, it will be assumed that the initial incident ray direction vector is a unit vector, such that the elements are equal, i.e. $\eta = \xi = \nu$. It is important to note, that the values of the computed errors will be different depending on the values used for the incident ray direction.

A. Change in incident ray direction

A change in the incident ray direction may occur either by a controlled change, such as through a steering mirror, or through an unknown change due to for instance thermal drift of the optical mounts in the optical system prior to the retro-reflector. Recall the expressions for the sensitivity in the reflected ray direction vector with respect to a change in incident ray direction, $\hat{q}$, Equations 12, 13, and 14. Rewriting the partial derivative as an incremental change from the nominal one obtains:

$$\frac{\partial \bar{t}}{\partial \eta} = \hat{i} - 2\gamma\hat{j} - 2\beta\hat{k}$$

$$\Delta\bar{t}_\eta = \Delta\eta\hat{i} - 2\Delta\eta\left(\gamma\hat{j} + \beta\hat{k}\right) \tag{31}$$

By inspection of Equation 31 it is seen that the error term component consists of the product of two likely small numbers, namely the dihedral angle errors and the change in the incident direction. As a result, the error contribution is likely small compared to the change in the incident ray direction. As an example, consider a change in incident ray direction of the first orthogonal component, η. Since $\hat{q}$ is a unit vector, the components, η, ξ, ν are the direction cosines. Thus, for an angular change, θ_i, relative to the $\hat{i}$ axis, the relationship for η is given by $\cos\theta_i = \eta$. Taking the derivative, one has the relationship between a change in η due to a change in θ_i.

$$\frac{\partial \eta}{\partial \theta_i} = -\sin\theta_i \tag{32}$$

$$\Delta\eta = -\Delta\theta_i \sin\theta_i \tag{33}$$

Substituting the expression for $\Delta\eta$, the error portion of Equation 31, becomes:

$$-2\Delta\eta\left(\gamma\hat{j} + \beta\hat{k}\right) = 2\Delta\theta_i \sin\theta_i\left(\gamma\hat{j} + \beta\hat{k}\right) \tag{34}$$

Assume that $\alpha = \beta = \gamma$, and taking the norm of the error portion of the vector, one obtains:

$$\left\|2\Delta\theta_i \sin\theta_i\left(\gamma\hat{j} + \beta\hat{k}\right)\right\| = 2\sqrt{2}\beta\Delta\theta_i \sin\theta_i = \mathrm{E}_\eta \tag{35}$$

For a numerical value, consider the case where the incident ray angle, θ_i, deviates by 100 urad and also assume that $\alpha = \beta = \gamma = 1.5$ urad. For $\eta = \xi = \nu = \frac{1}{\sqrt{3}}$, and noting that $\theta_i = \cos^{-1}\eta$, the magnitude of the error, E_η, is found to be approximately 0.35 nrad, which is small compared to the 100 urad change in the incident ray angle.

B. Rotation of retro-reflector

For a perfect retro-reflector, the stability of the retro-reflector orientation will not have an effect on the output ray direction vector. Yet, for a retro-reflector with dihedral angle errors, it is readily seen from Equation 2, that the stability of the retro-reflector mounting will affect the output ray direction vector. For this example, the incident ray direction is assumed constant. Thus, the change in the incident ray direction, $\hat{q}$ in Equation 2 is now a result of a change in orientation of the retro-reflector. The analysis for the error generated due to a rotation of the retro-reflector is therefore identical to that of holding the retro-reflector fixed and allowing the incident ray to deviate from the nominal condition. The numerical analysis presented in Section IV-A is therefore directly applicable. For a slightly different numerical example, consider the case where specifications on the stability of the retro-reflector mounting must be specified. To determine the maximum allowable angular rotation of the retro-reflector, one must first establish the maximum allowable error in the outgoing reflected beam direction. Assume, for example, a maximum allowable error in the reflected ray direction to be on the order of 1 nrad. Consistent with the assumptions in Section IV-A, ($\alpha = \beta = \gamma = 1.5$ urad, and $\eta = \xi = \nu = \frac{1}{\sqrt{3}}$, where $\theta_i = \cos^{-1}\eta$), one can apply Equation 35, solving for the change in the incident ray direction, or equivalently the rotational change of the retro-reflector orientation.

$$\mathrm{E}_\eta = 2\sqrt{2}\beta\Delta\theta_i \sin\theta_i$$

$$\Delta\theta_i = \mathrm{E}_\eta / \left(2\sqrt{2}\beta \sin\theta_i\right) \tag{36}$$

Substitution of the numerical values yields a rotational stability value of $\Delta\theta_i$ equal to 0.29 mrad for an error, E_η, of 1 nrad.

C. Change in dihedral angles

Returning to the expressions for the sensitivity in the reflected ray direction vector with respect to a change in the dihedral angle errors, Equations 19, 20 and 21, one can quickly compute the sensitivity for the assumed incident ray direction with $\eta = \xi = \nu$ and when $\alpha = \beta = \gamma$. Rewriting the partial derivative as an incremental change from the nominal one obtains:

$$\frac{\Delta \bar{t}_\alpha}{\Delta \alpha} = \frac{\partial \bar{t}}{\partial \alpha} = 2\nu \hat{j} - 2\xi \hat{k}$$
$$\Delta \bar{t}_\alpha = 2\Delta\alpha \left(\nu \hat{j} - \xi \hat{k}\right) \tag{37}$$

The total magnitude of the change is the norm of the vector:

$$\|\Delta \bar{t}_\alpha\| = 2\sqrt{2}\eta\Delta\alpha \tag{38}$$
$$\|\Delta \bar{t}_\alpha\| = \frac{2\sqrt{6}}{3}\Delta\alpha \tag{39}$$

and similar for the change in the other two dihedral angle errors:

$$\|\Delta \bar{t}_\beta\| = 2\sqrt{2}\eta\Delta\beta \tag{40}$$
$$\|\Delta \bar{t}_\gamma\| = 2\sqrt{2}\eta\Delta\gamma \tag{41}$$

A likely cause for the dihedral angles to change after initial calibration is due to a thermal gradient across the thickness of the reflecting surface within the retro-reflector. Equation 28 can be used to approximate the amount of this dihedral angle change. Equating the dihedral angle error change, $\Delta\alpha$, with the beam angular deviation, θ_L, one obtains:

$$\Delta \bar{t}_\alpha = 2\Delta\alpha \left(\nu \hat{j} - \xi \hat{k}\right)$$
$$\Delta \bar{t}_\alpha = 2\theta_L \left(\nu \hat{j} - \xi \hat{k}\right) \tag{42}$$
$$\Delta \bar{t}_\alpha = 2\frac{L\alpha_T \Delta T}{h} \left(\nu \hat{j} - \xi \hat{k}\right) \tag{43}$$

To obtain a numerical estimate, a geometry as well as thermal and material properties must be assumed. For a representative material, a corner-cube retro-reflector is assumed to be composed of Borofloat 33 glass, with a coefficient of thermal expansion equal to $\alpha_{20-300^\circ\mathrm{C}} = 3.25\text{e-}6\,\mathrm{K}^{-1}$. For the temperature gradient within the plate of glass, a value of 0.05 K is assumed. Note that here, a slightly more conservative value is used than that published by Ortiz and Lee [11], where a 0.01 K temperature gradient after 2.5 hours was assumed. For a representative geometry, a retro-reflector with a clear aperture of 50 mm is used. The nominal glass plate thickness is assumed to be on the order of 13 mm. For the angular displacement calculation, the center of the clear aperture and the full range of the clear aperture can be used as the equivalent beam length. Since the retro-reflector has a clear aperture of 50 mm, a length range of 25 mm (center) to 50 mm (farthest extent) is used for a worst case approximation. Using Equation 28, the change in the dihedral angle at $L = 25$ mm and for $L = 50$ mm are equal to $\theta_{L=25} = 0.3\,\mu$rad and $\theta_{L=50} = 0.6\,\mu$rad respectively. For a worst case scenario, these values are assumed for the dihedral angle error for each of the surfaces, i.e. α, β, γ. Assuming the incident ray direction vector, $\hat{q} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$, the sensitivity, or total magnitude of the change for $\Delta\alpha$ equal to $\theta_{L=25}$ and $\theta_{L=50}$ is provided by Equation 38. The resulting total magnitude of the change, $\|\Delta \bar{t}_\alpha\|$ is 0.5 μrad and 1.0 μrad for $\theta_{L=25}$ and $\theta_{L=50}$ respectively.

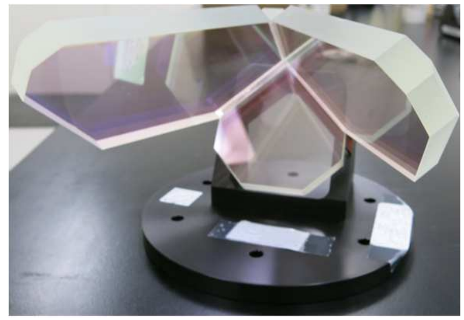
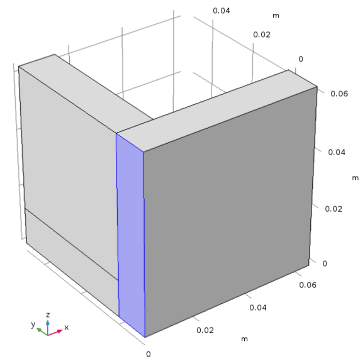


Fig. 2. Photo of retro-reflector hardware. The retro-reflector is a modified corner cube design with a lateral transfer.

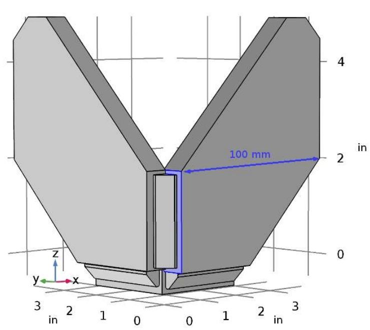
V. Modeling and Simulation

In Section III-A, a simplified beam bending model was described to approximate the deformation of the retro-reflector glass plates due to a thermal gradient. Since the simplified model does not account for additional constraints and bending in two-dimensions, it is known that the simplified model will be in error from the actual observed deformation of the actual corner-cube geometry. In order to get a sense of the validity of the simplified model, a thermal-structural deformation model is simulated in Comsol. For the thermal-structural model, two geometries are considered. First, a hollow retro-reflector is modeled using three solid plates of glass. The clear field of view is 50 mm. Secondly, a solid model geometry representing an actual manufactured lateral transfer hollow retro reflector (a modified corner-cube design) is analyzed. A photo of the hardware for the manufactured hollow retro-reflector is shown in Figure 2. For material properties, both models assume the material is Borofloat 33 glass for the entire geometry. For constraints, a fixed displacement constraint on one edge of the glass plate is used as shown in Figure 3. It is important to note, that the mechanical constraint is chosen to be consistent with the simplified beam bending model described in Section III-A. The actual constraint for a realistic system would need to reflect the corresponding opto-mechanical mount for the retro-reflector. For the thermal boundary conditions, a fixed thermal delta across the thickness of the plates comprising the reflecting surfaces of the corner cube and retro-reflector is applied. Similar to the numerical example in Section IV-C, a temperature differential of 0.05 K is used, where the inside surface of the retro-reflector is warmer than the outside surface.

The resulting angular displacement from the thermal-structural model is shown in Figure 4. From the simulation of the simplified corner cube geometry, Figure 4(a), it is seen that the angular displacement about the z-axis at the far corner edge of the corner cube is on the order of 0.3 urad. At one half the distance to the edge of the corner cube, the angular displacement is on the order of 0.1 urad or less. Recall that

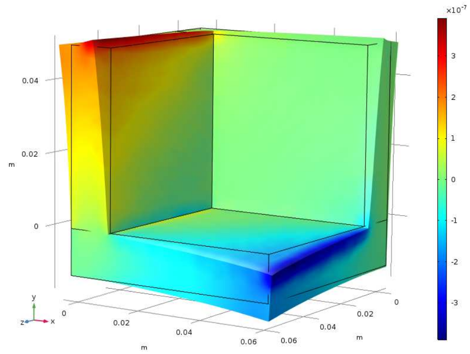


(a) Simplified corner-cube geometry.

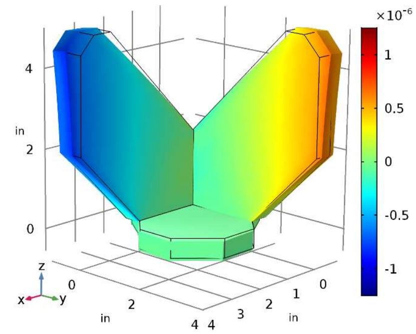


(b) Modified corner-cube retro-reflector.

Fig. 3. Comsol thermal-structural model, mechanical constraint. The model assumes a static, zero displacement at the edge of one glass plate forming the hollow retro-reflector (shaded blue region on back side).



(a) Simplified corner-cube geometry.



(b) Modified corner-cube retro-reflector.

Fig. 4. Comsol thermal-structural models. The simplified geometry does not include an appropriate opto-mechanical mount. The model assumes a static temperature differential across the thickness of the glass plates forming the hollow retro-reflector. The color indicates one half the curl of the displacement about the z-axis, which is the angular displacement around the z-axis.

the computed values in Section IV-C, for the change in the dihedral angle at $L = 25\,\text{mm}$ and for $L = 50\,\text{mm}$ are equal to $\theta_{L=25} = 0.3\,\mu\text{rad}$ and $\theta_{L=50} = 0.6\,\mu\text{rad}$ respectively. The length of $L = 25\,\text{mm}$ corresponds to a location at the center of the corner cube clear aperture and the length of $L = 50\,\text{mm}$ corresponds to the far edge of the corner cube. Thus, the simulation results indicate that the simplified model using a beam in bending is approximately a factor of two to three in a conservative sense as compared to a plate in two-dimensional bending.

Next, the second retro-reflector geometry is analyzed. The thermal-structural angular displacement results for the geometry representing the modified corner-cube hollow retro reflector hardware are shown in Figure 4(b). Since this retro-reflector is not a pure corner cube, there is a portion of the reflecting surface which is unsupported by the other facet of the retro-reflector. From the simulation results in Figure 4(b), it is seen that the majority of the deformation occurs after the central corner-cube portion, where the plate is no longer supported by the adjacent facet. The deformation around the z-axis is observed to be less than approximately $1.0\,\mu\text{rad}$. In order to compare the result to the simplified beam bending model, the beam length is needed. For the modified corner shown in Figure 2, the maximum distance from the z-axis is 100 mm. Using Equation 28, the change in the dihedral angle at $L = 100\,\text{mm}$ is equal to $\theta_{L=100} = 1.25\,\mu\text{rad}$, which is consistent with the simulation results in Figure 4(b). Again, assuming the incident ray direction vector to be $\hat{q} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$, the sensitivity for $\Delta\alpha$ equal to $\theta_{L=100}$ is provided by Equation 38. The resulting magnitude of the change, $\|\Delta\bar{t}_\alpha\|$ is $2.0\,\mu\text{rad}$ for $\theta_{L=100}$. It is important to note that this calculation is for the deformation observed at the maximum normal distance from the seam forming the dihedral angle. The modified corner cube retro-reflector is designed to have a clear aperture of 50 mm. Thus, the actual beam variation across the entire beam width will be less.

VI. Conclusion

A mathematical representation for the change in the beam deviation angle from a retro-reflector as a function of the dihedral angle errors and incident beam direction was presented. Thus, given an estimate for either the expected stability of the dihedral angle errors, or for the expected directional stability of the retro-reflector incident beam, the variation in the beam deviation angle can be estimated to the first order. In addition, a simplified model based on beam theory was developed to estimate the dihedral angle variations of a hollow retro-reflector due to a thermal gradient. The developed mathematical representation, combined with the simplified model can be used to estimate the stability of the retro reflector's beam deviation. For the example presented, a variation in the dihedral angles produces a larger change in the return direction than a variation in the incident beam direction.

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