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On stabilizability of strict additive convex processes with arbitrary stability domains

Işıl Öner[1] and Kanat Camlibel[2]

Abstract— This paper studies stabilizability problem with respect to an arbitrary stability domain for strict closed additive convex processes. The main results state necessary and sufficient conditions in terms of the eigenstructure of the dual process. The results presented in this paper are stronger than those of existing in the literature in two respects: (i) they are valid for arbitrary stability domains and (ii) they guarantee existence of Bohl-type stable trajectories. We also demonstrate the application of the main results by streamlining them for linear systems subject to conic input constraints.

Index Terms— Differential inclusion (34A60), Set-valued and variational analysis (49J53), Set-valued maps (54C60), Problems involving relations other than differential equations (49K21)

I. INTRODUCTION

Ever since introduced by [11,12], convex processes have been a subject of study in different contexts from different angles. Roughly speaking, convex processes are generalizations of linear maps with the distinguishing property that they enjoy certain closure property with respect to conic combinations rather than linear combinations as in the case of linear maps. The interest in convex processes mainly stems from their fine mathematical structure as well as from the applications they are encountered in.

One line of research on convex processes focuses on their mathematical properties from various angles with an eye towards applications in optimization and control. Example of such work include studies of normed convex processes in [4], norm duality between convex processes and their duals in [5], and estimation of eigenvalue sets in [1].

Another line of research more akin to what this paper studies focuses on differential inclusions ([2,9,15]) with convex processes with an eye towards systems and control theory. Such differential inclusions arise sometimes due to the intrinsic conic constraints (see e.g. [10, Ex. 2.1]) or sometimes as approximations of set-valued maps (see e.g. [15, Sec. 2.2] or [6]). Examples of work this line of research include controllability of strict convex processes in the seminal paper [3], controllability of nonstrict convex processes in [13], and Lyapunov theory in [8].

In this paper we study yet another system-theoretic problem for differential inclusions with convex processes: stabilizability with respect to a given stability domain. From application point of view, one is often interested in not only the ordinary stability but also with the stability with a prescribed rate of convergence and/or the stability by avoiding certain frequencies. Such practical scenarios are the foremost motivation for the current paper. At the same time, the study of stabilizability with respect to a given stability domain sheds a stronger light on the spectral structure of convex processess and hence is of interest in its own.

Stabilizability problem for differential inclusions with convex processes have already been addressed in the literature. For instance, the papers [7,14] addressed the stabilizability[1] problem and presented spectral conditions for a given convex process to be stabilizable. In particular, [14] (see also [15, Sec. 8.3]) provides necessary and sufficient conditions for exponential stabilizability of strict convex processes whereas [7] presents a set of necessary and a set of sufficient conditions for the nonstrict case. In addition, [7] provides interesting results on the eigenstructure of convex processes.

Being highly inspired by the earlier work [3,7,14], this paper deals with differential inclusions based on strict additive convex processes and investigates the stabilizability problem with respect to a given stability domain. This problem significantly differs from the ordinary stabilizability problem studied in [7,14] and requires a different approach. Still, the necessary and sufficient conditions that are stated in the main results very much resemble those presented in [14]. However, the conditions we present are stronger than those of [14] in two respects for additive convex processes. They are valid for *arbitrary stability domains* as well as they guarantee stabilizability within *Bohl-type* trajectories.

The organization of the paper is as follows. In Section II, we recall basic definitions and results of convex processes. This will be followed by the introduction of the stabilizability problem with respect to a given stability domain in Section III. The main results will be presented in Section IV. These results are applied to linear systems with conic input constraints in Section V. Finally, the paper closes with conclusions in Section VI.

II. PRELIMINARIES

Throughout the paper we use standard mathematical notation. In what follows we quickly review some notions and results for the sake of completeness.

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[1]These papers use the terminology of 'weakly asymptotically stable' for what we call stabilizability in this paper.

A. Cones and polar cones

We say that a set $C \subseteq \mathbb{R}^n$ is a *cone* if $\alpha x \in C$ whenever $x \in C$ and $\alpha \geqslant 0$. For a nonempty set S, we define its polar cone by

$$S^- := \{y \mid x^T y \leqslant 0 \text{ for all } x \in S\}.$$

Clearly, S^- is always a closed convex cone.

B. Convex processes

Let $H : \mathbb{R}^r \rightrightarrows \mathbb{R}^h$ be a set-valued mapping, that is $H(x) \subseteq \mathbb{R}^h$ is a set for all $x \in \mathbb{R}^r$. The *domain* and *graph* of H are defined, respectively, by

$$\text{dom } H = \{x \in \mathbb{R}^r \mid H(x) \neq \varnothing\}$$
$$\text{gr } H = \{(x,y) \in \mathbb{R}^r \times \mathbb{R}^h \mid y \in H(x)\}.$$

The inverse H^{-1} of H is the set-valued map from $\mathbb{R}^h$ to $\mathbb{R}^r$ defined by

$$(\eta, \xi) \in \text{gr}(H^{-1}) \iff (\xi, \eta) \in \text{gr}(H).$$

We say that a set-valued mapping $H : \mathbb{R}^r \rightrightarrows \mathbb{R}^h$ is

- *strict* if $\text{dom } H = \mathbb{R}^r$,
- *convex* if its graph is convex,
- *closed* if its graph is closed,
- *a process* if its graph is a cone,

Given a set-valued mapping $H : \mathbb{R}^r \rightrightarrows \mathbb{R}^r$, the dual convex process $H^- : \mathbb{R}^r \rightrightarrows \mathbb{R}^r$ is defined by

$$(q,p) \in \text{gr } H^- \iff q^T y \leqslant p^T x \quad \text{for all } (x,y) \in \text{gr } H.$$

An immediate consequence of this definition is the following relationship between the graphs of H and H^-:

$$\text{gr } H^- = \begin{bmatrix} 0_{r \times r} & I_r \\ -I_r & 0_{r \times r} \end{bmatrix} (\text{gr } H)^-. \quad (1)$$

Clearly, H^- is a closed convex process for every set-valued mapping H.

In this paper, we are particularly interested in additive convex processes.

Definition 1 [16] A convex process $H : \mathbb{R}^r \rightrightarrows \mathbb{R}^r$ is said to be an *additive process* if for any $x_1, x_2 \in \text{dom } H$

$$H(x_1 + x_2) = H(x_1) + H(x_2) \quad (2)$$

Throughout the paper, we are interested in convex processes. A detailed treatment of convex processes and their duals can be found in [15, Sec. 2.6]. To be self-contained, we recall the following facts. If H is a strict closed convex process, then it follows from [15, Lem. 2.11 and Thm. 2.12] that $\text{dom } H^- = (H(0))^-$ and hence $\text{dom } H^-$ is a closed convex cone. As such, $-\text{dom } H^- \cap \text{dom } H^-$ is a subspace. Moreover, the restriction of H^- to the subspace $-\text{dom } H^- \cap \text{dom } H^-$ is a linear map [15, Thm. 2.12]. We denote this linear map by $L(H^-)$. Let $J(H^-) \subseteq -\text{dom } H^- \cap \text{dom } H^-$ be the largest subspace that is invariant under $L(H^-)$. We denote the restriction of $L(H^-)$ to $J(H^-)$ by $L_{J(H^-)}$.

We say that a real number λ is an *eigenvalue* of the set valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, if there exists a nonzero vector $x \in \mathbb{R}^n$ such that $\lambda x \in H(x)$. Such a vector x is called an *eigenvector* of H corresponding to λ. The *spectrum*, that is the set of eigenvalues, of H will be denoted by $\sigma(H)$.

C. Bohl functions

A function $x : \mathbb{R} \to \mathbb{R}^n$ is said to be a *Bohl function* if there exist a monic polynomial $p(\zeta) = \zeta^r + p_{r-1}\zeta^{r-1} + \cdots + p_1\zeta + p_0$ such that

$$p(\frac{d}{dt})x := \frac{d^r x}{dt^r} + p_{r-1}\frac{d^{r-1}x}{dt^{r-1}} + p_1 \frac{dx}{dt} + p_0 x = 0. \quad (3)$$

We denote the set of all Bohl functions from $\mathbb{R}$ to $\mathbb{R}^n$ by $\mathcal{B}_n$. If (3) holds for a Bohl function $x \in C^\infty$, we call p an *annihilator polynomial* of x. Every Bohl function x has infinitely many annihilator polynomials. Nevertheless, there is a unique monic polynomial, called *minimal polynomial* of x, such that it divides any other annihilator polynomial. The *spectrum* of x is defined as the set of the roots of its minimal polynomial and will be denoted by $\sigma(x)$.

III. $\mathbb{C}_g$-STABILIZABILITY PROBLEM

Consider the differential inclusion

$$\dot{x}(t) \in H(x(t)) \quad (4)$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex process. Let $T > 0$ and $\text{AC} = \text{AC}([0,T], \mathbb{R}^n)$ denote the set of absolutely continuous functions defined from $[0,T]$ to $\mathbb{R}^n$. We define the set of all trajectories satisfying (4) as the behavior of H:

$$\mathfrak{B}_T(H) := \{x \in \text{AC} \mid x \text{ satisfies } (4)$$
$$\text{for almost all } t \geqslant 0\}.$$

Next, we will define various fundamental sets for a given differential inclusion of the form (4). The set of feasible states is defined by

$$\mathcal{X}_T(H) := \{\xi \in \mathbb{R}^n \mid \exists\, x \in \mathfrak{B}_T(H) \text{ with } x(0) = \xi\}. \quad (5)$$

Since H is a convex process, $\mathcal{X}_T(H)$ is always a convex cone. In general, it is not closed even if H is closed. If H is a strict closed convex process, then

$$\mathcal{X}_T(H) = \mathbb{R}^n \quad (6)$$

for all $T > 0$ in view of [15, Cor. 4.1 and Thm. 2.12].

A subset $\mathbb{C}_g$ of $\mathbb{C}$ is said to be *stability domain* if it is closed under complex conjugation and $\mathbb{C}_g \cap \mathbb{R}$ is nonempty. The set of $\mathbb{C}_g$-stabilizable states is defined by

$$\mathcal{S}(H; \mathbb{C}_g) := \{\xi \in \mathbb{R}^n \mid \exists\, x \in \mathfrak{B}_\infty(H) \cap \mathcal{B}_n$$
$$\text{such that } x(0) = \xi \text{ and } \sigma(x) \subseteq \mathbb{C}_g\}.$$

Definition 2 We say that the differential inclusion (4) (or equivalently H) is $\mathbb{C}_g$-*stabilizable* if $\mathcal{X}_\infty(H) \subseteq \mathcal{S}(H; \mathbb{C}_g)$, i.e. if all feasible states are $\mathbb{C}_g$-stabilizable.

From (6), it is clear that a strict convex process H is stabilizable if and only if $\mathcal{S}(H; \mathbb{C}_g) = \mathbb{R}^n$.

The goal of this paper is to present necessary and sufficient conditions for $\mathbb{C}_g$-stabilizability of strict closed convex processes. Before doing so, we summarize the existing results on similar problems.

Stabilizability of strict convex processes have already been studied in the literature. In [15], Smirnov defines a strict closed convex process H as *weakly asymptotically stable* if $\mathcal{S}(H) = \mathbb{R}^n$ where

$$\mathcal{S}(H) := \{\xi \in \mathbb{R}^n \mid \exists x \in \mathfrak{B}_\infty(H)$$
$$\text{such that } x(0) = \xi \text{ and } \lim_{t \to \infty} x(t) = 0\}.$$

To state the main result (Thm. 8.10) of [15], we need a bit of nomenclature. A vector $\xi \in \mathbb{R}^n$ is said to be an *exponentially stabilizable state* of H if there exists $x \in \mathfrak{B}_\infty(H)$ with $x(0) = \xi$ such that for some positive numbers a and γ

$$|x(t)| \leqslant a e^{\gamma t} |\xi| \text{ for all } t \geqslant 0$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. The set of all exponentially stabilizable states of H is denoted by $\mathcal{S}_{\exp}(H)$. Finally, let

$$\mathbb{C}_- := \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$$

and

$$\mathbb{C}_+ := \mathbb{C} \setminus \mathbb{C}_- = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \geqslant 0\}.$$

Necessary and sufficient conditions for H being weakly asymptotically stable are stated next.

Theorem 3 [15, Lem. 8.4 and Thm. 8.10] *Let* $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a strict closed convex process. Then, the following statements are equivalent:*

1) H *is weakly asymptotically stable, that is* $\mathcal{S}(H) = \mathbb{R}^n$.
2) $S_{\exp}(H) = \mathbb{R}^n$.
3) $\sigma(L_{J(H^-)}) \cap \mathbb{C}_+ = \varnothing$ *and* $\sigma(H^-) \cap \mathbb{C}_+ \cap \mathbb{R} = \varnothing$.

The $\mathbb{C}_g$-stabilizability problem we want to address in this paper differs from that of what Smirnov calls weak asymptotical stability in two ways. To begin with, we work with an arbitrary stability domain whereas [15, Thm. 8.10] deals with $\mathbb{C}_-$ in a sense. Also, we require the trajectories not only to be stable but also to be Bohl functions whereas [15, Thm. 8.10] deals with exponentially stabilizable trajectories.

IV. MAIN RESULTS

Let $\mathbb{C}_g$ be a stability domain. Define

$$\mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g$$

and

$$\mathbb{R}_b = \{\lambda \in \mathbb{R} \mid \lambda > \mu \text{ for all } \mu \in \mathbb{C}_g \cap \mathbb{R}\}.$$

The following theorem is the main result of this paper.

Theorem 4 *Let* $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a strict closed additive convex process and* $\mathbb{C}_g$ *be a stability domain such that* $\mathbb{R}_b$ *is closed. Then, the following statements are equivalent:*

1) H *is weakly asymptotically stable, that is* $\mathcal{S}(H; \mathbb{C}_g) = \mathbb{R}^n$.
2) $\sigma(L_{J(H^-)}) \cap \mathbb{C}_b = \varnothing$ *and* $\sigma(H^-) \cap \mathbb{R}_b = \varnothing$.

The proof of this theorem differs from [15, Thm. 8.10] in a fundamental way and cannot be obtained by following the same ideas/tools. Two main ingredients that do not appear in the proof of [15, Thm. 8.10] are the invariance properties of the set $S(H; \mathbb{C}_g)$ and the use of results from asymptotic behavior of almost periodic functions.

Remark 5 In the special case of $\mathbb{C}_g = \mathbb{C}_-$. We have $\mathbb{C}_b = \mathbb{C}_+$ and $\mathbb{R}_b = \mathbb{C}_+ \cap \mathbb{R}$. As such, the condition 3 in Theorem 3 and the condition 2 in Theorem 4 coincide. However, the result of Theorem 4 is still slightly stronger (for additive convex processes) as it proves $\mathcal{S}(H; \mathbb{C}_g) = \mathbb{R}^n$ whereas Theorem 3 proves only $\mathcal{S}(H) = \mathcal{S}_{\exp}(H) = \mathbb{R}^n$.

V. LINEAR SYSTEMS WITH CONIC INPUT CONSTRAINTS

As an application of Theorem 4, consider the continuous-time, linear and time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7a}$$

where the state x is an n-vector, the input u is an m-vector, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$. We assume that B is of full column rank. Suppose that the inputs take values in a closed convex cone $\mathcal{U} \subseteq \mathbb{R}^m$, that is

$$u(t) \in \mathcal{U} \tag{7b}$$

for all $t \geqslant 0$. We assume that the set $\mathcal{U}$ has nonempty interior. The constrained system (7) can be described by the differential inclusion:

$$\dot{x}(t) \in F(x(t))$$

where

$$F(x) := \{Ax + Bu \mid u \in \mathcal{U}\}.$$

Note that

$$\operatorname{gr} F = \begin{bmatrix} I_n & 0_{n \times m} \\ A & B \end{bmatrix} (\mathbb{R}^n \times \mathcal{U}).$$

Clearly, $\operatorname{gr} F$ is a convex cone and $\operatorname{dom} F = \mathbb{R}^n$. Since B is full column rank and $\mathcal{U}$ is a closed convex cone, it follows from [12, Thm. 9.1] both $B\mathcal{U}$ and $\operatorname{gr} F$ are closed convex cones. As such, F is a strict closed convex process. In addition, it follows from [16, Prop. 2] that F is additive.

From (1) and [12, Cor. 16.3.2], we have that

$$\operatorname{gr}(F^-) = \begin{bmatrix} 0_{n \times n} & I_n \\ -I_n & 0_{n \times n} \end{bmatrix} (\operatorname{gr} F)^-$$

$$= \begin{bmatrix} 0_{n \times n} & I_n \\ -I_n & 0_{n \times n} \end{bmatrix} \begin{bmatrix} I_n & 0_{n \times m} \\ A & B \end{bmatrix}^{-T} (\{0\} \times \mathcal{U}^-)$$

$$= \begin{bmatrix} A^T & -I_n \\ B^T & 0_{m \times n} \end{bmatrix}^{-1} (\{0\} \times \mathcal{U}^-)$$

where $M^{-1}S$ denotes the inverse image of S under M, that is $M^{-1}S = \{\xi \mid M\xi \in S\}$. Consequently, we obtain

$$F^-(q) = \{A^T q \mid B^T q \in \mathcal{U}^-\}.$$

Note that $\operatorname{dom} L(F^-) = -F(0)^- \cap F(0)^- = -B^{-T}\mathcal{U}^- \cap B^{-T}\mathcal{U}^- = B^{-T}(-\mathcal{U}^- \cap \mathcal{U}^-) = B^{-T}(\mathcal{U} - \mathcal{U})^- = \{0\}$ since

$\mathcal{U}$ has nonempty interior. This means that $\mathrm{dom}\, L(F^-) = \ker B^T$, $L(F^-)(\xi) = \{A^T\xi \mid \xi \in \ker B^T\}$, and $J(F^-) = \langle \ker B^T \mid A^T \rangle := \ker B^T \cap A^{-1} \ker B^T \cap \cdots \cap A^{-n+1} \ker B^T$.

Therefore, applying Theorem 4 we can obtain the following à la Hautus test for $\mathbb{C}_g$-stabilizability of the system (7).

Corollary 6 *Consider a linear system with conic input constraints of the form* (7). *Assume that B is of full column rank, $\mathcal{U}$ is a closed convex cone with nonempty interior, and $\mathbb{C}_g$ is a stability domain such that $\mathbb{R}_b$ is closed. Then, the system* (7) *is $\mathbb{C}_g$-stabilizable if and only the following implications hold:*

1) $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}_b$, $A^T z = \lambda z$, $B^T z = 0 \implies z = 0$,
2) $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_b$, $A^T z = \lambda z$, $B^T z \in \mathcal{U}^- \implies z = 0$.

VI. Conclusions

This paper provided necessary and sufficient conditions for the stabilizability problem of strict closed additive convex processes with respect to a prespecified stability domain. In addition, we demonstrated the main results by applying them to linear systems with conic input constraints. For additive processes, the results we presented extends the earlier results on stabilizability in two ways as they apply to arbitrary stability domains and to Bohl-type trajectories. Further research directions are dropping the strictness assumption in order to be able to deal with state constraints.

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