THE INFLUENCE OF CHINESE MATHEMATICAL ARTS ON SEKI KOWA

by

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ABSTRACT

I will consider the influence of Chinese mathematics on Seki Kôwa. For this purpose, my thesis is constructed in four parts,

introduction,

Ⅰ the studies of editions; *Shu Shu Jiu Zhang* and *Yang Hui Suan Fa*,

Ⅱ the conception and extension of method for making magic squares,

and

Ⅲ the analysis for solving indeterminate equations.

In the introduction, I will explain some similarities between Chinese mathematics in the Song dynasty and Seki Kôwa's works. It will become clear that the latter was influenced by Chinese mathematics.

Then I introduce some former opinions concerning which Chinese mathematical book influenced him. I shall show that two Chinese mathematical books, *Shu Shu Jiu Zhang* and *Yang Hui Suan Fa*, are particularly important.

Some Chinese mathematical books were republished and studied by Japanese mathematicians, but these two books were not accessible to Japanese mathematicians. Thus we must study them for considering questions of influence. I will consider two subjects, the treatment of magic squares in *Yang Hui Suan Fa* in chapter Ⅱ and the method of solving indeterminate equations in *Shu Shu Jiu Zhang* in chapter Ⅲ.

Before considering the contents of these subjects, we must know more about the available versions of these two books in chapter Ⅰ, otherwise we cannot know whether Seki Kôwa could have obtained them.

It seems certain that Seki Kôwa studied the *Yang Hui Suan Fa*, but I cannot know whether he studied the *Shu Shu Jiu Zhang*. However, Seki Kôwa's method of solving indeterminate equations is very similar to that of Qin Jiushao, especially when their methods of changing negative constants into positive are similar. Thus I would like to propose that Seki Kôwa studied the Chinese method of solving indeterminate equations.

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CONTENTS

I : THE STUDY OF EDITIONS

II : THE CONCEPTION AND EXTENSION OF METHOD FOR MAKING MAGIC SQUARE

Ⅲ : THE ANALYSIS FOR SOLVING INDETERMINATE EQUATIONS

INTRODUCTION

(1) Aims of this thesis

The influence of Chinese mathematics was felt in most East Asian countries. In the case of Japan, it was introduced into the country two times. The first time was in the eighth century, when many mathematical arts were introduced and taught in the University, but Japanese mathematicians only imitated the work of Chinese authorities, and its level was limited. The second time was at the end of the sixteenth century—at which point, Japanese mathematicians applied Chinese mathematics, and were to produce brilliant achievements.

It is difficult to make a comparative study of the mathematics of two completely different civilizations, because they do not have the same intellectual background. But Chinese mathematics and Japanese mathematics used the same language, rather than just the same Chinese characters, thus mathematicians could understand mathematical notions easily. That is to say, if Japanese mathematicians had Chinese mathematical books, they could have had the same background as Chinese mathematicians. I wonder whether it is possible to make a comparative study of them using historical method. Instead, through studying Japanese mathematics in the 17th century, we may be able to understand strong and weak points of Chinese mathematics in the 13th century.

I would like to consider the case of Seki Kôwa 關孝和 (1642?-1708), the best mathematician in Japan. He very probably studied Chinese mathematics, but it is difficult to demonstrate precisely how he studied Chinese mathematics from his biography. Because his son-in-law gambled away his post, nobody could hand down Seki Kôwa's biography. Thus we cannot know exactly the nature of his education. Therefore, I will have to try, in this thesis, to consider his education in light of the similarity between his works and Chinese mathematics.

(2) Biographical study of Seki Kōwa

Because nobody could hand down Seki Kôwa's biography, we cannot even fix his birth year exactly. He became revered as a "Sansei" 算聖 (mathematical sage), and the scientific studies of his biograpy was neglected. However, since Mikami Yoshio's 三上義夫 (1875-1950) study (*1), there are some good studies (*2), and his biographical table(*3) is as follows

Seki Kôwa's biography

1637? or 1642?

Born in Fujioka (now Fujioka-shi, Gumma prefecture 群馬縣藤岡市).

Son of Uchiyama Nagaakira 内山永明 (father), and of the daughter (name unknown) of Yuasa Yoemon 湯淺與右衛門. His other names are Sinsuke 新助, Shihyô 子豹 and Jiyûtê 自由亭.

year unknown

Became a son-in-law of Seki Gorôzaemon 關五郎左衛門.

1661

Transcribed the *Yang Hui Suan Fa* 楊輝算法 (Yang Hui's Method of Computation) at Nara 奈良.

1663

Wrote the *Kiku Yōmei Sampō* 規矩要明算法 (Essential Mathematical Methods of Measures)

1672?

Wrote the *Ketsugi-Shō Tō-jutsu* 闕疑抄答術 (Answers and Methods of the *Sampō Ketsugi-Shō*).

1674

wrote the *Huttan Kai Tō-jutsu* 勿憚改答術 (Answers and Methods of the *Sampō Huttan Kai*).

In Dec., published the *Hatsubi Sampō* 發微算法 (Mathematical Methods for Finding Details).

1676?

Became a retainer of Tokugawa Ienobu's 徳川家宣 family.

1678

Became a Kanjō Ginmiyaku 勘定吟味役 (auditor) of Tokugawa Ienobu's family.

1680

In Mar. wrote the *Juji Hatsumē* 授時發明 (Comments of the Works and Days Calendar).

In July, wrote the *Happō Ryakketsu* 八法略訣 (Short Explanations of Eight Items).

1681

In Apr., wrote the *Jujireki-kyō Rissē no Hō* 授時暦經立成之法 (Methods of Manual Tables of the Works and Days Calendar).

1683

In June, wrote the *Shoyaku no Hō* 諸約之法 (Methods of Reduction), *Sandatsu Kempu no Hō* 算脱驗符之法 (Methods of Solving Josephus Question), *Hōjin Ensan no Hō* 方陣圓攅之法 (Methods of Magic Squares and Magic Circles).

In Aug., wrote the *Kaku Hō* 角法並演段圖 (Methods of Angles and Figures of Japanese Algebra).

On 9th Sep., wrote the *Kai Hukudai no Hō* 解伏題之法 (Methods of Solving Secret Questions).

1685

In Aug., wrote the *Kai Indai no Hō* 解隠題之法 (Methods of Solving Conceal ed Questions).

Wrote the *Byōdai Meichi no Hō* 病題明致之法(Methods of Correcting Failures as Questions).

In Nov., wrote the *Kaihō Hompen no Hō* 開方翻變之法 (Overturn Methods of

Solving Higher Degree Equations).

In Dec., wrote the *Daijutsu Bengi no Hō* 題術辨議之法(Methods of Discriminant).

Wrote the *Kai Kendai no Hō* 解見題之法 (Methods of Solving Findable Questions), *Kyūseki* 求積 (Computations of Area and Volume), *Kyūketstu Henkē Sō* 毬闕變形草 (Manuscript of Transformation of Spheres) and *Kaihō Sanshiki* 開方算式 (Formulae of Solving Higher Degree Equations).

1686

In Jan., wrote the *Seki Tēsho* 關訂書 (Seki Kōwa's Amendments).

1697

In May, wrote the *Shiyo Sampō* 四餘算法 (Mathematical Methods of Computing Four Points on the Lunar Orbit).

1699

In Jan., wrote *Temmon Sūgaku Zatcho* 天文數學雜著 (Notes of Astronomy and Mathematics).

1704

In Nov., gave a "Sampô Kyojô" 算法許状 (Licence of Mathematics) to Miyaji Shingorô 宮地新五郎.

In Dec., when Tokugawa Ienobu became Shôgun, Seki Kôwa became a Nando Kumigashira 納戸組頭 (chief treasurer). His salary was 250 Pyō 俵 and Jûnin-buchi 10人扶持 (ten retainers' salary), which was increased to 300 Pyô.

1706

In Nov., retired, and became a member of Kobushin-gumi 小普請組 (lit. small builders group).

1708

On 24th Oct., died from a disease.

1710

The *Taisei Sampō* 大成算法 (Complete Works of Seki Kôwa) was edited.

1712

The *Katsuyō Sampō* 括要算法 (Essential Mathematics) was published.

1714

On 30th Mar., the *Shukuyō Sampō* 宿曜算法 (Mathematical Methods of Constellations).

1724

Seki Kôwa's son-in-law, Shinshichirô 新七郎 became Kôfu Kimban 甲府勤番 (member of Kôfu city office).

1727

Shinshichirô lost his position owing to his gambling activities.

As above, Seki Kôwa's works are quite huge, however, there are few evidences about his education and his personal infomation. Even his exact name is not known to foreign scholars. Now, let us consider this problem.

In Eastern Asian countries, an adult had some names, which were called "Zi" (or "Azana" in Japanese) 字 (alias) or "Hao" (or "Gô" in Japanese) 號 (pen name) and so on. Their real name was used only in the family, "Zi" (alias) was used officialy, thus the real name was not known to unrelated person.

In the case of Seki Kôwa, his alias ("Azana") was Shinsuke 新助, and his styles or pen-names ("Gô") was Shihyô 子豹 and Jiyûtei 自由亭, and real name was Takakazu 孝和. He used his real name when he published books, it was a custom of Japanese mathematicians in that age. Thus readers did not know how "孝和" was read.

It does not mean that the education of readers was inferior. It was a characteristic of Japanese names. Ancient Japan had no written characters of its own; Chinese characters were introduced in the Nara 奈良 period, and not before. Chinese characters have no element of pronunciation, i.e., they are not phonetic signs. Therefore Japanese could read one Chinese character in several ways of pronunciation.

For example; the character of 數 (numbers) is read "Shu" by Chinese. Japanese imitated this sound, but the system of Japanese utterance was not the same as that of the Chinese, thus Japanese spoke with an accent to read this character to "Sū". It is called "On" 音 (Chinese pronunciation) reading. And this character means "number", In Japanese it is read "Kazu", which means "number". This is the translation, it is called "Kun" 訓 (Japanese pronunciation) reading.

Moreover Chinese culture was introduced for many years and from many localities in China, sometimes the pronunciations were different. Thus there are three "On" readings in Japan, basically.

On 音 (Chinese pronunciation)	Kan-on	漢音 (Han dynasty's)	(1)
	Go-on	呉音 (Wu dynasty's)	(2)
	Sô-on	宋音 (Song dynasty's)	(3)
Kun 訓 (Japanese pronunciation)			(4)

Usually, Confucian terms are "Kan-on" reading and Buddhist's terms are "Go-on" or "Sô-on" reading.

The two characters "孝" and "和" can be read in the followings ways:

"孝"	(1) Kô	"和"	(1) Ka
	(2) Kyô		(2) Wa
	(3) -		(3) -
	(4) Takashi (adv.)		(4) Kazofu (v.)
	(Tsukaeru (v.))		(Yawaragu (v.))
			(Nagomu (v.))

Therefore it is possible to read "孝和" in nine ways.

There are some rules to read a set phrase, which is consisted of two Chinese characters. If "On" reading is used for the first character, the same would apply in reading the second character; if "Kun" reading is used for the first character, it is also used in reading the second character. Usually "On" and

"Kun" readings are not used together, but there is no absolute rule. Japanese personal names are often too difficult to read.

In Japan, when there is uncertainty concerning how a name is read it is customary to use the "On" reading to avoid serious mistake. Seki Kôwa became too famous, most Japanese mathematicians knew him only through his mathematical arts. Thus Seki Takakazu became known as Seki Kôwa. Therefore he is called Seki Kôwa in this thesis.

For Japanese mathematicians I give the original pronunciation of the real name if possible, but sometimes I cannot read the name. Thus all nemes of Oriental persons are given in Chinese characters, together with birth year and death year in the text of this thesis.

(3) Similarity between Chinese mathematics and Japanese mathematics

As we considered in section 2, Seki Kôwa's works are in many fields. According to Hirayama Akira's平山諦 (fl.1959) studies (*4), these are classified under 15 categories, as follows:

Seki Kôwa's works

(1) "Bôsho-hô" 傍書法 and "Endan-jutsu" 演段術, Japanese algebra

(2) solution of higher degree equation

(3) properties (e.g., number of solutions) of higher degree equation

(4) infinite series

(5) "Rêyaku-jutsu" 零約術, approximate value of fractions

(6) "Senkan-jutsu" 翦管術, indeterminate equations

(7) "Shôsa-hô" 招差法, the method of interpolation

(8) Obtained Bernoulli numbers by "Ruisai Shôsa-hô" 累裁招差法

(9) computing area of polygons

(10) "Enri" 圓理, principle of the circle

(11) Newton's formula by

"Kyûshô" 窮商 method

$$\beta = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$

(12) computing area of rings

(13) conic curves line

(14) "Hôjin" 方陣, magic squares, and "Enjin" 圓陣, magic circle

(15) "Mamakodate" 繼子立, Josephus question

"Metuke-ji" 目付字, game of finding a Chinese character

Table 1 Hirayama's classification of Seki Kôwa's works

Each of Seki Kōwa's works was a great one, but most of the subjects he took up were not his original ideas, being typical works of Chinese mathematics in the Song 宋 and Yuan 元 dynasties. These are, according to Li Yan's 李儼 (1892-1963) studies[*5], as follows:

a) "Cheng-Chu Ke-Jue" 乘除歌訣 (verses for multiplication and division)

Yang Hui Suan Fa 楊輝算法 (Yang Hui's Method of Computation)

Suan Fa Tong Zong 算法統宗 (Systematic Treatise on Arithmetic)

b) "Cong-Huang-Tu Shuo" 縦横圖説 (magic squares)

Yang Hui Suan Fa (Yang Hui's Method of Computation)

Suan Fa Tong Zong (Systematic Treatise on Arithmetic)

c) "Shu Run" 數論 (numbers theorems)

Shu Shu Jiu Zhang 數書九章 (Mathematical Treatise in Nine Sections)

d) "Ji-Shu Run" 級數論 (series)

Meng Xi Bi Duan 夢溪筆談 (Dream Pool Essays)

Yang Hui Suan Fa (Yang Hui's Method of Computation)

Si Yuan Yu Jian 四元玉鑑 (Precious Mirror of the Four Elements)

e) "Fang-Cheng Run" 方程論 (higher degree equations)

Ce Yuan Hai Jing 測圓海鏡 (Sea Mirror of Circle Measurements)

Shu Shu Jiu Zhang (Mathematical Treatise in Nine Sections)

Suan Xue Qi Meng 算學啓蒙 (Introduction to Mathematical Studies)

f) "He Yuan Shu" 割圓術 (the method of dividing the circle)

Shou Shi Li 授時暦 (Works and Days Calendar)

Table 2

Classification of Chinese mathematics works and important books in the Song and Yuan dynasties

Considering to which categories each Seki Kōwa's work belongs in table 2, we see that only (1) and (14) are Seki Kōwa's original subjects, the others belong to inherited Chinese subjects in the Song and Yuan dynasties, as illustrated in table 3.

categories	Seki's work
a) verse	-
b) magic squares	(14)
c) indeterminate eqs.	(5)(6)(15)
d) series	(4)(7)(8)(11)
e) higher degree eqs.	(2)(3)
f) circles	(9)(10)(12)

Table 3

Similarity between Chinese Mathematics and Seki Kōwa's Works

In table 3, category f) describes one of the most popular subjects in Eastern mathematics. Since the *Jiu Zhang Suan Shu* 九章算術 (Nine Chapters on the Mathematical Arts), most mathematical books described this subject. Thus it is not worth our while considering this origin. While category a) is one of the most important subjects for primary pupils of mathematics, it is not necessary to consider it in this thesis. Therefore we will consider categories b) to e) in the next section.

(4) Opinions of how Chinese mathematics influenced on Seki Kôwa

As we considered above, there is no doubt that Seki Kôwa was influenced by Chinese mathematics. Of course, he, as the other Japanese mathematicians, studied two popular texts, the *Suan Fa Tong Zong* and the *Suan Xue Qi Meng*. But the main works of the former fall into categories a) and b) of table 2, and the latter falls entirely into category e). Moreover, these are only introductions to these subjects. Thus it is not only these two books that influenced Seki Kôwa. We must consider the more important books.

But his biography is unreliable so much so that, we cannot know even his birth year. We have some indirect evidence about which Chinese mathematical arts influenced Seki Kôwa. We will consider this first.

(a) Opinions that Seki Kôwa studied Chinese mathematics

According to chapter 5 of the *Burin Inkenroku* 武林隠見録 (Anecdotes of Mathematicians) by Sai Tô Ya Jin 齋東野人 (18c), written in 1738, Seki Kôwa discovered a difficult mathematics book in Nara 奈良, and studied it.

> There was a Chinese book with Buddhistic books in Nanto 南都 (lit. northern capital, Nara 奈良), but nobody had been able to understand it. That book was not a Buddhistic book, a Confucian book nor a medical book. It was not known what kind of book it was, so it was only mended and given a summer airing. Shinsuke 新助 (Seki Kôwa) knew it, and he guessed that it might be a mathematical book. He took vacations and went to Nanto to borrow it. He stayed there and sat up all night to copy it. Then he brought this hand-copied book back to Edo 江戸 (Tôkyô 東京). He studied it day and night for three years, at last he mastered the secrets. He became the best

mathematician in Japan [*6].

And Aida Yasuaki 會田安明 (1747-1817) criticized the alleged fact that Seki Kôwa had burned this text-book in the *Toyoshima Sankyō Hyōrin* 豐島算經評林 (Toyoshima's Comments about Mathematical Manual), written in 1804:

> Toyoshima Masami 豐島正美 (?-?) said, "Seki Kôwa was a good mathematician but his manner of study was poor. He burned his mathematical text book which he found." It seems probably that he burned the book because he had plagiarized Chinese mathematical methods and then wrote about them as if they were his own work. [*7]

It had been known that Seki Kôwa referred to Chinese mathematical texts since that time. Very probably he had studied some Chinese mathematical books that were not popularly known, and obtained certain ideas for his own works.

Many scholars concluded what Seki Kôwa's text book was. Here I introduce the former opinions and would like to comment on them.

(b) Opinion concerning the *Suan Xue Qi Meng* (Introduction to Mathematical Studies)

In a memorandum which is kept in Mito Shôkôkan 水戸彰考館 (Mito private school), Honda Toshiaki 本多利明 (1744-1821) concluded that the book Seki Kôwa copied was the *Suan Xue Qi Meng*.

> Seki Kôwa was self-educated. First, he studied three mathematicians' books, Imamura Chishô 今村知商 (?-?), Yoshida Mitsuyoshi 吉田光由 (1598-1672) and Takahara Yoshitane 高原吉種 (?-?), and he mastered these mathematicians' strong points. He became the best mathematician.
>
> Then he borrowed *Suan Xue Qi Meng* (Introduction to Mathematical Studies) at Kôfuku-ji 興福寺 (Kôfuku temple) in Nanto, hand-copied it and mastered "Tengen-jutsu" 天元術 ("Tian Yuan Shu" in Chinese, technique of the celestial element). Then he continued to study mathematics, and completed the great works. [*8]

The *Suan Xue Qi Meng* is the introduction to "Tian Yuan Shu" (Technique of the Celestial Element); this view is directed at Seki Kôwa's work in higher degree equations. However, the *Suan Xue Qi Meng* was already republished in 1658 in Japan.—Hisada Gentetsu 久田玄哲 (?-?) translated it to Japanese [*9]. So, of course, Seki Kôwa studied the *Suan Xue Qi Meng*, but it would not have been necessary to go to Nara to find this text book. Mikami Yoshio 三上義夫 (1875-1950) suggested that it was Hisada Gentetsu that found the *Suan Xue Qi Meng* at Tôfuku-ji 東福寺 (Tôfuku temple) in Raku 洛 (Kyôto 京都) [10], not at Kôfuku-ji (Kôfuku temple) in Nara. Moreover the works of higher degree equations had been realized by Sawaguchi Kazuyuki 澤口一之 (?-?) [11]. Therefore, we cannot accept this evidence.

(c) Opinion concerning the *Ce Yuan Hai Jing* (Sea Mirror of Circle Measurement)

Kanô Ryôkichi 狩野亨吉 (1865-1942) also pointed to the work of higher degree equation, but he concluded the textbook in question was the *Ce Yuan Hai Jing* [12], which is the speciality book about "Tian Yuan Shu" (Technique of the Celestial Element).

It, however, was introduced into Japan for the first time in 1726 [13], so Seki Kôwa could not have read it. The fact that it had not been introduced before 1726 is supported by the evidence as follows: the letter which Mukai Motonari 向井元成 (?-?) sent to Hosoi Kôtaku 細井廣澤 (?-?), published in the *Sokuryō Higen* 測量秘言 (Secret Comments of Surveying) by Hosoi Kôtaku, states

> The chapter of Chôken-jutsu 町見術 (Surveying); Some methods were described in the *Ce Yuan Hai Jing Len Rei Shi Shu* 測圓海鏡分類釋術 (Classified Methods of the *Ce Yuan Hai Jing*)[14], *Li Suan Quan Shu* 曆算全書 (Complete Works on Calendar and Mathematics), *Gou Gu Yin Meng* 句股引蒙 (Introduction to Sides of Triangle) and *Shu Du Yan* 數度衍 (Generalisation on Numbers) which are thereby introduced into Japan [15].

This evidence is very reliable because the *Li Suan Quan Shu* was introduced into Japan this same year[16]. Therefore, we can conclude that the *Ce Yuan Hai Jing* was first known to Japanese mathematicians in 1726. Therefore we cannot accept Kanô Ryôkichi's opinion.

(d) Opinion concerning the *Zhui Shu* 綴術 (Bound Methods)

Uchida Itsumi 内田五觀 (1805-1882) told Okamoto Noriyoshi 岡本則録 (1847-1931) that Seki Kōwa's text book was *Zhui Shu*[17], which is Zu Chongzhi's 祖沖之 (429-500) work, however, this work was lost in the Northern Song 北宋 dynasty.

If the *Zhui Shu* was extant in Seki Kōwa's time, it would be the biggest discovery in the history of mathematics in Eastern Asia. A handwritten manuscript entitled *Zhui Shu* is kept at Tôkyô University Library 東京大學圖書館. It is an 1897 copy from a manuscript of Okamoto's collections, so it must be Uchida's " *Zhui Shu* ". It, however, describes series, which was a very popular subject among Japanese mathematicians only after Seki Kôwa developed Japanese algebra. Probably this MS. was forged much after Seki Kôwa's time. Thus we can not agree with Uchida's claim.

(e) Opinions concerning the *Yang Hui Suan Fa* (Yang Hui's Method of Computation)

Fujiwara Shôzaburô 藤原松三郎 (1881-1946) and Shimodaira Kazuo 下平和夫 (b. 1928) concluded that the book Seki kôwa used was *Yang Hui Suan Fa* (18).

There are two pieces of evidence. The first is that Seki Kôwa hand-copied it in his youth (see section Ⅰ-2-a). The other is that the term Seki Kôwa used to refer to his treatment of indeterminate equations is the same as "Senkan-jutsu" 翦管術 (technique of cutting tube) of *Yang Hui Suan Fa*.

Seki Kôwa, however, did not burn the text book he used, rather, he retained it. And the explanation of solving indeterminate equations of this book is too simple to complete Seki Kôwa's works on indeterminate equations (I will discuss in chapter Ⅲ). Thus I think that he used other text books, as well as *Yang Hui Suan Fa*, as sources in his development of method for solving indeterminate equations.

Seki Kôwa did not open his manuscript to other mathematicians, thus it is worth to consider the influence of *Yang Hui Suan Fa* for analysing his original works. *Yang Hui Suan Fa* is one of the best works about magic squares. Magic squares were a very popular subject for Japanese mathematicians, and most of them worked on this subject. Therefore, we will have to consider the influence of *Yang Hui Suan Fa* with respect to magic squares in chapter 2, in order to determinate how Seki Kôwa and Japanese mathematicians influenced the design of magic squares beyond the treatment found in the *Yang Hui Suan Fa* .

(f) New Opinion: Opinion concerning the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections)

We introduced some opinions, but these cannot perfectly explain the influence of Chinese mathematics (see table 4). In particular, these opinions do not explain Seki Kôwa's treatment of indeterminate equations. Some mathematical books we have discussed above describe indeterminate equations, but they are too simple. Therefore we will consider the subject of whether Japanese mathematicians were influenced by Chinese mathematical arts.

Because the best and the only work of indeterminate equations in China is Qin Jiushao's work, we will consider whether Seki Kôwa's text book was *Shu Shu Jiu Zhang*.

categories	Seki's work	Opinions of Seki's text
a) verse	-	(*Suan Fa Tong Zong*)
b) magic squares	(14)	*Yang Hui Suan Fa?*
c) indeterminate eqs.	(5)(6)(15)	–
d) series	(4)(7)(8)(11)	*Zhui Shu?*
e) higher degree eqs.	(2)(3)	*Suan Xue Qi Meng* *Ce Yuan Hai Jing*
f) circles	(9)(10)(12)	(*Jiu Zhang Suan Shu*)

Table 4 Opinions of Seki Kôwa's text

Notes

(*1): Mikami Yoshio, 1932.

(*2): Nihon Gakushiin, 1954, vol.2: 133-46. Hirayama Akira, 1959. Hirayama Akira et al (eds.), 1974.

(*3): See Hirayama Akira et al (eds.), 1974: 14-24.

(*4): See Hirayama Akira, 1959.

(*5): Li Yan, 1937.

(*6): See Nihon Gakushiin, 1954, vol.2: 142-3.

(*7):

(*8): Hosoi Sosogu, 1941: 93.

(*9): Chinese characters are also used in Japanese, so with the help of some "Kaeriten" 返り點 (reading order symbol) and "Okurigana" 送り仮名 (traditional Japanese pronunciation to support reading Chinese), Japanese scholars could understand Chinese. The republished book of *Suan Xue Qi Meng* was published by Hisada Gentetsu and Haji Dôun 土師道雲 (?-?).

(10): Endô Toshisada, 1896; 1981: 73-4, Mikami's comment.

(11): Jôchi Shigeru, 1991.

(12): Kanô Ryôkichi 狩野亨吉. 1902. "Seki Kôwa 200-Nensai Kinen Honchô Sûgaku Tsûzoku Kôen-Shû" 關孝和二百年祭記念本朝數學通俗講演集 (Transcript of Lectures for Popularising Japanese Mathematics, in Remembrance of Seki Kôwa on 200th anniversary of his death) (Nihon Gakushi-in, 1954, vol.2: 143).

(13): Oba Osamu, 1967: 689. Moreover *Ce Yuan Hai Jing Xi Cao* 測圓海鏡細草 (Comments of *Ce Yuan Hai Jing*) was also introduced in 1726 (Oba Osamu, 1967: 689).

(14): It is not *Ce Yuan Hai Jing* (Sea Mirror of Circle Measurement) itself, but it quotes the whole sentence of *Ce Yuan Hai Jing*, so Japanese mathematicians had access to the contents of *Ce Yuan Hai Jing*.

(15): Nihon Gakushiin, 1954, vol.5: 428.

(16): Oba Osamu, 1967: 687.

(17): Nihon Gakushiin, 1954, vol.2: 143.

(18): Nihon Gakushiin, 1954, vol.2: 7 and 17. Shimodaira Kazuo, 1965, vol.1: 188.

I : THE STUDY OF EDITION

(1) The *Shu Shu Jiu Zhang*

The subjects of the text of *Shu Shu Jiu Zhang* 數書九章 (Mathematical Treatise in Nine Sections) has already been studied by several scholars. The first was Qian Baocong 錢寶琮 (1892-1974) (*1), then Libbrecht (fl. 1973) (*2) succeeded in making a diagram showing manuscript traditions. Moreover, Li Di 李迪 (b. 1928) (*3) continued these studies; most problems were solved by these scholars. They solved most of the problems relating to the period before the compilation of *Si Ku Quan Shu* 四庫全書 (Complete Works of the Four Categories, 1789) and to the period after publication of the *Yi Jia Tang Cong Shu* 宜稼堂叢書 (Collection of Yijiatang, 1842). This thesis will deal with three periods divided as below;

(1) before completion of *Si Ku Quan Shu* (1789)

(2) from *Si Ku Quan Shu* to publication of *Yi Jia Tang Cong Shu* (1789-1842)

(3) after publication of *Yi Jia Tang Cong Shu* (1842)

Now, I cannot add to what has already been said about era (1) and era (3), so I will summarize the former studies and answer two remaining problems concerning era (2). After discussing this point, we will consider whether the *Shu Shu Jiu Zhang* was introduced into Japan.

(a) Before completion of the *Si Ku Quan Shu*

The *Shu Shu Jiu Zhang* was written by Qin Jiushao in 1247, but at that time its name was *Shu Shu Da Lue* 數術大略 (literally, Outline of Mathematical Art) [*4], which had nine chapters or *Shu Xue Da Lue* 數學大略 (literally, Outline of Chinese Mathematics) [*5]. The art of printing was already invented in that age, but it was not published.

In 1421, in the Ming 明 dynasty, thousands of books were transported from Nanjing 南京 to Beijing 北京, the new capital. A version of this work was included, then it was kept at Wenyuange 文淵閣 (Wenyuan building) in the Palace. This copy was probably the version copied into the *Yong Le Da Dian* 永楽大典 (Great Encyclopaedia of the Yongle Reign-period, 1403-1408). Its name was changed to *Shu Xue Jiu Zhang* 數學九章 (in literally, Nine Chapters of Chinese Mathematics), which had three volumes [*6] comprising eighteen chapters. But the *Yong Le Da Dian* was the emperor's personal encyclopedia; mathematicians and other scholars could not read it.

In the Wanli 萬暦 period (1573-1620), the war period between China, Korea and Japan (1592-3, 1597-8), Wang Yinglin 王應遴 (1545-1620) [*7] made a manuscript copy from the Wenyange text, then Zhao Qimei 趙琦美 (1563-1624) [*8] recopied it by hand in 1616 [*9]. The name was changed again to the one used today, *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections), which has eighteen chapters. This is one of the most important versions, and is usually called the "Zhao Qimei version".

Qian Zeng 錢曾 (1626-1701) [10] probably had a copy of this version [11], then this book was kept by Zhang Dunren 張敦仁 (1754-1834) [12] [13], but he did not open it to the public, he only communicated it to Shen Xinfei 沈欽裴 (19c), his disciple (see section Ⅰ-1-c).

A poet, Qian Qianyi 錢謙益 (1582-1664) [14], recorded *Shu Shu Jiu Zhang* in his personal book catalogue, and it was a nine chapters version [15]. No further details are known.

(b) <u>From completion of the *Si Ku Quan Shu* to publication of *Yi Jia Tang Cong Shu*</u>

In the Qianlong 乾隆 period (1736-1795), in the Qing 清 dynasty, a big project was executed; the vast collection, *Si Ku Quan Shu* , was edited. Dai Zhen 戴震 (1724-1777) [16] made a manuscript copy of the *Shu Xue Jiu Zhang* from the *Yong Le Da Dian*, which had nine chapters because he conflated two chapters of the *Yong Le Da Dian* version into one new chapter, that is, he reestablished the original structure (chapters). It was usually called "Guan-ben" 館本 (literally, [Si Ku Quan Shu's] official book).

The name and numbers of chapters of the main versions are as below;

version	name		chapters
original	*Shu Shu (or Xue) Da Lue*	數術 (學) 大略	9
Yong Le Da Dian	*Shu Xue Jiu Zhang*	數學九章	18
Zhao Qimei	*Shu Shu Jiu Zhang*	數書九章	18
Guan-ben	*Shu Xue Jiu Zhang*	數學九章	9

The "Guan-ben" was widely current among mathematicians in the Qing dynasty, and "Zhao Qimei version" was also read. It is therefore difficult to know the correct filiation path to follow. The study of this era is much indebted to Li Di. We explain the outline of his conclusion have.

Kong Guangsen 孔廣森 (1752-1786) [17] was from the same town as Dai Zhen, so he obtained "Guan-ben" from him and studied it [18].

Li Huang 李潢 (d. 1811) [19] took part in editing *Si Ku Quan Shu,* so he must have had "Guan-ben". Then Zhang Dunren (1754-1834) obtained this version, and studied "Da Yan Zong Shu Shu"大衍總數術 (The General Dayan Computation). He wrote *Jiu Yi Suan Shu* 求一算術 (Mathematics Searching for One, 1803) based on this version of *Shu Xue Jiu Zhang* [20].

One of the most important persons in restoring our knowledge concerning the filiation of these books was Li Rui 李銳 (1768[21] -1817)(see his biography). He was friendly with Jiao Xun 焦循 (1763-1820)[22] and Wang Lai 汪萊 (1768-1813)[23]. These three mathematicians obtained different versions, and discussed what the true method of *Shu Shu Jiu Zhang* was.

Li Rui obtained a handwritten copy from Gu Qianli 顧千里 (1770-1839)[24], this manuscript was owned by Qin Enfu 秦恩復 (1760-1843)[25][26], but we cannot know which version it was.

Li Rui obtained the "Guan-ben" from Zhang Dunren[27], too.

Moreover, Li Rui obtained one more "Guan-ben" from Qian Daxin 錢大昕 (1728-1804)[28]. Li Rui based his version on this book, and made "Li Rui's manuscript". It is now kept at Seikadō Bunko 静嘉堂文庫 (Seikadō Lib.) in Japan. Qian Daxin's annotation about the question 2 of "Gu Li Kuai Ji"古歷會積 (the accumulated years from the epoch of old almanacks) in chapter 2 (or chapter 1 part 2, because this version is "Guan-ben" so it has nine chapters) was written on 10th Feb. 1798, according to this book. Then Li Rui added his annotation, and made this manuscript. Jiao Xun added comments in red ink. But only two volumes remain of this manuscript, as far as chapter 6 (chapter 3 part 2) p.13, that is, question 2 of "Huan Tian San Ji"環田三積 (square of three loops), which later has been lost.

"Li Rui's manuscript" was used in editing *Yi Jia Tang Cong Shu*, then Lu Xinyuan 陸心源 (1834-1894) had kept it[29].

According to this evidence, we solved two remaining problems concerning which origin of "Li Rui's manuscript" was [30], and which versions Lu Xinyuan had [31].

Wang Xuanling 王萱齡 (?-?) handcopied "Li Rui's manuscript", and the book is now kept at Beijing Tushuguan 北京圖書館 (National Lib.)[32].

When Jiao Xun commented on "Li Rui's manuscript", Jiao Xun probably referred to the other versions. We know that it is one of the "Zhao Qimei versions", as Jiao Xun wrote the foreword, then Li Shengduo李盛鐸 (?-?) obtained it. It is kept at Beijing Daxue Tushuguan 北京大學圖書館 (Beijing Univ. Lib.) now.

Jiao Xun did not only keep the book, but also studied *Shu Shu Jiu Zhang*, then wrote *Da Yan Jiu Yi Shu* 大衍求一術 (The Method of Searching for One of Dayan Rule), which has one chapter, and *Jiu Yi Gu Fa* 求一古法 (The Old Method of Searching for One), which are kept at Beijing Tushuguan (National Lib.).

The other historical materials are fragments.

The version of the *Shu Xue Jiu Zhang* that Sun Xingyan 孫星衍 (1753-1818) kept probably came from the *Yong Le Da Dian*, because it bore the title *Shu Xue Jiu Zhang* and had eighteen chapters (33).

Zhou Zhongfu 周中孚 (1786-1831) also copied the book of "Wenyange" by hand, but the detail is not known (34).

Luo Tengfeng 駱騰鳳 (1770-1841) studied *Shu Shu Jiu Zhang* (35).

(c) After publication of the *Yi Jia Tang Cong Shu*

"Zhao Qimei version", which was treasured by Zhang Dunren, was communicated to his disciple, Shen Xinfei (36), then it was obtained by Song Jingchang 宋景昌 (?-?) (37).

Li Chaoluo 李兆洛 (1769-1841) also retained a "Zhao Qimei version"; Song Jingchang later obtained it (38).

Li Rui's manuscript was communicated to Mao Yuesheng 毛岳 (嶽) 生 (1791-1841), and it was also obtained by Song Jingchang.

That is, Song Jingchang obtained at least three versions and established a definitive version. His manuscript was then published by Yu Songnian 郁松年 (?-?) in 1842. This is the *Yi Jia Tang Cong Shu* version. At the same time, *Shu Shu Jiu Zhang Zha ji* 數書九章札記 (Impressions of Mathematical Treatise in Nine Sections), written by Song Jingchang, was published. It was the first printed version and the best version.

Then Zou Anchang 鄒安鬯 (?-?) corrected it, and published *Gu Jin Suan Xue Cong Shu* 古今算學叢書 (1898).

The *Yi Jia Tang Cong Shu* edition was republished in a movable type printed version in 1936, in both *Cong Shu Ji Cheng Chu Bian* 叢書集成初編 edition and *Guo Xue Ji Ben Cong Shu* 國學基本叢書 by Wang Yunwu 王雲五 (20c). These editions are currently in general circulation.

Moreover, *Si Ku Quan Shu* was published photographically in Taiwan 臺灣 from 1984 to 1988, and is now generally available.

(d) Conclusion to section Ⅰ-1

If Seki Kôwa 關孝和 (1642?-1708) was influenced by *Shu Shu Jiu Zhang*, it must have been introduced into Japan by the end of the seventeenth century [39], very probably during the Korean War. Probably the control of the *Yong le Da Dian* became loosened, so Wang Yinglin would have had the chance to copy it. I think there were not only Wang Yinglin's but also Zhao Qimei's manuscripts in circulation.

But no trace of the "Zhao Qimei version" (or " *Yong Le Da Dian* version" and "Wenyange version") has yet been discovered in Japan. The oldest version in Japan today is the *Yi Jia Tang Cong Shu* version [40]. This version was published in 1842, more than 130 years after Seki Kôwa's death.

In Tôhoku Univ. Lib. 東北大學圖書館, there is a manuscript version of *Shu Shu Jiu Zhang* with eighteen chapters. This indicates a "Zhao Qimei version" or a "Yi Jia Tang Cong Shu version". But, to our regret, we cannot know when it was copied, so we cannot conclude which versions it is. However the evidence described below, dated in 1840, shows the improbability that any versions of *Shu Shu Jiu Zhang* before the *Yi Jia Tang Cong Shu* version were introduced into Japan.

> Qin Jiushao was a skillful (mathematician) and Zu Chongzhi 祖冲之 (429-500) was a meticulous (mathematician), they could be called those who knocked on the door of mystery. [41]

We cannot tell whether Japanese mathematicians obtained *Shu Shu Jiu Zhang* itself, however this material shows that they had some information [42] about Qin Jiushao at least by 1840, two years before the publication of *Yi Jia Tang Cong Shu*.

(2) The *Yang Hui Suan Fa*

The studies of editions of the *Yang Hui Suan Fa* 楊輝算法 (Yang Hui's Method of Computation) began from Mikami Yoshio's 三上義夫 (1875-1950) study [43]. Li Yan 李儼 (1892-1963) discussed *Yang Hui Suan Fa* in his books and articles [44]. Yan Dunjie 嚴敦傑 (d.1989) analysed Yang Hui's books and techonical terms [45]. Kodama Akio's 児玉明人 (fl.1966) works [46] will be often quoted in this thesis. Lam Lay-Yong's 藍麗蓉 (fl.1977) work [47] is one of the best studies in English. However, the study of the editions of the *Yang Hui Suan Fa* is still incomplete. I use these works and newer works [48] and will make diagrams to show the transmissions of various versions in the forms of manuscripts and books.

The *Yang Hui Suan Fa* is the collective name for a collection of three mathematical books written by Yang Hui 楊輝 (fl.1274-5). These are *Cheng Chu Tong Bian Suan Bao* 乘除通變算寶 (Treasure of Multiplication and Division — three chapters, 1274), *Tian Mu Bi Lei Cheng Chu Jie Fa* 田畝比類乘除捷法 (Fast Method of Multiplication and Division in Field —two chapters, 1275) and *Xu Gu Zhai Qi Suan Fa* 續古摘奇算法 (Continuation of Ancient Mathematical Methods for Elucidating the Strange—two chapters, 1275). These books were published during the tumultuous time when the Southern Song 南宋 dynasty was close to collapse, and the original editions are lost.

The most widely circulated editions of these works are those contained in *Zhi Bu Zu Zhai Cong Shu* 知不足齋叢書 (Works of the Library of Zhibuzuzhai) published in 1814 and *Yi Jia Tang Cong Shu* 宜稼堂叢書 (Works of the Library of Yijiatang) published in 1842. But the latter covers 6 volumes of *Yang Hui Suan Fa* 's total 7 volumes and does not cover the first chapter of *Xu Gu Zhai Qi Suan Fa* . And the former covers only half of this chapter.

As a result, there are two versions of *Xu Gu Zhai Qi Suan Fa*, which are the *Yi Jia Tang Cong Shu* edition in the Qindetang 勤德堂 family of edition and the edition the *Zhi Bu Zu Zhai Cong Shu* in the *Yong Le Da Dian* 永楽大典 (Great

Encyclopeadia of the Yongle reign-period) family of edition. Therefore, we can understand which versions Japanese mathematicians used by researching contents of vol.1 of *Xu Gu Zhai Qi Suan Fa*. Let us consider these two versions in this thesis.

(a) Versions of Qindetang press family of edition

In 1378, in the Ming dynasty, a mathematical text was published by Qindetang 勤德堂 at Hangzhou 杭州. This was most likely the first publication of a collection of three books, *Cheng Chu Tong Bian Suan Bao*, *Tian Mu Bi Lei Cheng Chu Jie Fa* and *Xu Gu Zhai Qi Suan Fa*. The book was named *Yang Hui Suan Fa* in reference to the author's name. This edition was later destroyed, but it was republished in Korea nearly half a century later, so we know its contents.

It is not known how *Yang Hui Suan Fa* was introduced into Korea, but the Ming dynasty and Yi 李 dynasty were on friendly terms, so it was probably introduced as a diplomatic gift. However, as trade grew in the Ming dynasty, especially at Hangzhou and in southern China, it was possible that *Yang Hui Suan Fa* was introduced by commercial trade.

In 1392, Yi Sung-gye 李成桂 (reign 1392-1398) founded his kingdom because the influence of Mongolia which had helped Koryo 高麗 had been removed. In the 15th century, Se-jong 世宗 (reign 1456-1468) ascended to the throne. He wanted to develop culture, so many government enterprises were set up with this aim in view. In particular, printing technology using movable copper type was developed; many classical texts were republished in this era. Mathematics was no exception [49]: *Yang Hui Suan Fa* was republished on 25th August, in the 15th year of reign (1433). According to the *Se-jong Sillok* 世宗實録 (History of the Se-jong Reign) of *Yijo Sillok* 李朝實録 (History of Yi Dynasty in Choson)

> The Kyóngsang-do Kamsa 慶尚道監司 (governor of Kyónsang-do) republished one hundred copies of *Yang Hui Suan Fa* (Yanghwi Sanpóp) and sent them to the king; they were distributed to the Chiphónjón 集賢殿 (Advisory Body), Hojo 戸曹 (civil administration), Sóun'gwan 書雲観 (Royal Observatory) and Súpsan'guk 習算局 (school of mathematics). [50]

Kyong-ju 慶州 is one of the most famous places where paper was produced, so many books were printed there.

This edition is the best of *Yang Hui Suan Fa* up to the present; it is still found in Korea, Japan and China.

In Korea itself, it was in the Yi Royal Library 李王家圖書館 (51) before the Second World War. *Yang Hui Suan Fa* was sent to the king, as we have seen, of course it was the Korean edition.

In Japan, four books are confirmed (52).

Two books are preserved in the Library of Tsukuba University 筑波大學圖書館 and one was republished in 1966 by Kodama Akio. These two books were plundered during Bunroku-Keichō no Eki 文祿慶長の役 (the Hideyoshi (豐臣) 秀吉 War) in 1592 and 1597. This is shown by the fact that they have the seal of "Yōanin" 養安院, otherwise known as Dr. Manase Shōrin 曲直瀬正琳 (1565-1611), who was the physician of Ukita Hideie 宇喜多秀家 (1573-1655) (53), Commander of Japanese troops (54).

The other book is to be found in the Sonkeikaku Bunko 尊經閣文庫 (55) which was founded by the Marquis Maeda 前田 family. This book gathers three prefaces of *Yang Hui Suan Fa* moved from original place and inserted before the text. Maeda Toshiie 前田利家 (1538-1599) (56) worked as a supply officer in the Hideyoshi War, so probably it was also introduced during this War.

The text found in the Kunai-chō Syoryō-bu 宮内庁書陵部 (Library of the Imperial Household Agency) is in good condition. At first glance, this book does not look like one published 500 years ago. The first page of this book has the seal of "Kanzanbyōei" 咸山苗裔, "Nam-Kwung Si Hwu" 南宮氏厚 (?-?) and "Saeki-kō Mōri Takasue, Azana Baishō Zōsho Kaku no In" 佐伯侯毛利高標字培松藏書劃之印 (the seal of the collection of the Marquis Saeki, Mōri Takasue, also known as Baisyō (1755-1801)). We do not know even which nationality Nam-Kwung Si Hwu was, because Nam-Kwung (Nangong in Chinese, Nangû in Japanese) is a common surname in Korean, Chinese and Japanese. Then it was obtained by Mōri Takasue. And Mōri Bunko 毛利文庫 (Mōri's collection), were presented to Shōg un Tokugawa Ienari 徳川家齊 (reign. 1787-1837) by Môri Takayasu 毛利高翰 (1795-1852) in 1827.

In China, there was only Mao Jin's 毛晋 (1598-1652) text which will be described below, so there was no complete version in the Qing dynasty.

However, Yang Shoujing 楊守敬 (19c) visited Japan in 1880, and bought the Korean edition (57). Then this book was placed in the Beijing National Library 北京圖書館(58), and now it is in the Library of Taiwan Gugong Bowuyuan 臺灣故宮博物院圖書館. But the condition of the print is much worse than the text of the Kunai-chô Syoryô-bu in Japan, I suspect that this book was republished in Japan at the Edo 江戸 period using the technique of "Okkabuse" 覆っ被せ(59), or that the original copper blocks was brought from Korea, then printed in Japan.

Before Yang Shoujing, there were no Korean editions in China, but only the text hand-copied by Mao Jin and preserved in his library, Jiguke 汲古閣. However, this book lacked chapter 1 of *Xu Gu Zhai Qi Suan Fa*. This book has the curious feature, that the characters of "chapter 2" have been painted over by white ink, so I wonder whether Lu Xinyuan took chapter 2 away intentionally. Moreover this book was not known, so it was not copied into the *Si Ku Quan Shu*. It was later acquired by Lu Xinyuan 陸心源 (19c)(60) at his library of Shiwanjuanlou 十萬卷樓 (the Library of One Hundred Thousand Books), and it now preserved in the Seikado Bunko 静嘉堂文庫 (Seikado Library) in Japan.

In 1814, Huang Pilie 黃丕烈 (1762-1825)(61) found this book in the Jiguke and He Yuanxi 何元錫 (19c)(62), who was a subordinate of Ruan Yuan 阮元 (1764-1849)(see his biography), hand-copied it. However, this book had many errors, so Ruan Yuan extended an invitation to one of the most famous mathematicians in this age; Li Rui 李鋭 (1768-1817)(see his biography) and asked him to correct this text which Huang Pilie found. The result was the Bai Song Yi Chen 百宋一廛(63) version.

Li Rui finished collating on the 10th of October in that year. Then this book probably became the *Yang Shi Suan Fa* 楊氏算法 (Mr. Yang's Method of Computation), one of *Wan Wei Bie Zan* 宛委別藏 (Another Complete Works of *Si Ku*

Quan Shu) series. We must pay attention to the fact that this name is *Yang Shi Suan Fa* not *Yang Hui Suan Fa*. Why did Ruan Yuan, who was the editor of *Wan Wei Bie Zan* choose this name, though he had probably already heard the name *Yang Hui Suan Fa* ?

The *Wan Wei Bie Zan* edition, in which chapter 1 of the *Xu Gu Zhai Qi Suan Fa*is missing is the same as Mao Jing's edition. And Ruan Yuan had already gotten chapter 1 of *Xu Gu Zhai Qi Suan Fa* from the *Yong Le Da Dian* , which will be described in the section Ⅰ-2-b. Therefore, he knew *Wan Wei Bie Zan* 's version was not complete, so that may be why he chose this new name.

This edition was proofread by Song Jingchang 宋景昌 (19c) (64) on 17th July 1842. It was published by Yu Songnian 郁松年 (19c) (65) in May 1842, in the *Yi Jia Tang Cong Shu* which is the most popular edition. This manuscript is preserved in the Zhongguo Kexueyuan Ziran Kexue-shi Yanjiusuo Tushuguan 中國科學院自然科學院研究所圖書館 (the Library of History of Science Institute, Academia Sinica).

Some Korean editions were introduced into Japan, but all of them were collected by the Daimyô 大名 (feudal lords), so Japanese mathematicians had no chance to study them. Seki Kōwa 關孝和 (1642?-1708), however, hand-copied a Korean edition in 1661. This manuscript is no longer extant but Ishiguro Nobuyoshi 石黒信由 (1760-1836) (66), a mathematician of the Seki School 關流, copied it by hand and his manuscript remains at the Kōju Bunko 高樹文庫 at Simminato, Toyama 富山縣新湊, Japan.

This edition corrected the disorder in pagination of *Xu Gu Zhai Qi Suan Fa* in the Korean edition, so it is one of the most important reference editions when correcting the Korean edition, but it contains so many mistakes, that it cannot be said to be the best edition.

Mikami Yoshio copied it by hand and sent to Gakushiin 學士院 (Japanese Academy) and Li Yan. Ogura Kinnosuke 小倉金之助 (1885-1962) also copied Mikami's MS by hand, but it was only chapter 1 of *Xu Gu Zhai Qi Suan Fa,* now it is kept at Waseda University Library 早稲田大學圖書館.

Li Yan proofread it minutely. It remains at the Zhongguo Kexueyuan Ziran Kexue-shi Yanjiusuo Tushuguan [67]. About 1926, Qiu Chongman 裘沖曼 (fl. 1926) [68] inserted three pages of the preface and contents of *Tian Mu Bi Lei Cheng Chu Jie Fa* and *Cheng Chu Tong Bian Suan Bao,* twenty-one pages of the preface and contents and three pages of the text corrected by Seki Kôwa of *Xu Gu Zhai Qi Suan Fa,* into the *Yi Jia Tang Cong Shu* edition, and made a "Bai Na Ben" 百訥本 (lit. hundred patches edition, composite book) which is preserved in the Zhejiang Library 浙江圖書館.

(b) Versions of the *Yong Le Da Dian*

In 1408, the sixth year of the emperor Yongle's reign, he ordered scholars to edit the great encyclopedia, *Yong Le Da Dian,* comprising the astonishing number of twenty-two thousand chapters —even its contents also spanned sixty chapters —collected in seven or eight thousand books. The part on mathematics consists of chapters 16329-16364, thirty-five chapters. *Xu Gu Zhai Qi Suan Fa* was the only material from the *Yang Hui Suan Fa* copied into this collection.

One more copy of this book was made in 1562, but some parts of both books were lost during internal disturbances at the end of the Ming dynasty, Chinese scholars could not restore it to its original state.

In the early Qing dynasty, scholarly work was done using the *Yong Le Da Dian,* the most famous work was editing *Si Ku Quan Shu* in 1782. Thus some parts of the *Yong Le Da Dian* are preserved in the *Si Ku Quan Shu.* But, in the disorder of the end of the Qing dynasty, most of the *Yong Le Da Dian* was lost. Only seven hundred volumes remain in the world. Of the mathematical parts, only chapter 16343 and 16344 remain in the Cambridge University Library.

Before this loss, some of the mathematical parts had been copied out by hand. Parts of first chapter of the *Xu Gu Zhai Qi Suan Fa* are collected in the *Zhi Bu Zu Zhai Cong Shu* series 27, which also has the other two mathematical books.

The *Zhu Jia Suan Fa* 諸家算法 (Records of Mathematical Methods and Prefaces of all Schools, one volume), which had been owned by Mo Youzhi 莫友芝 (1811-1871) was found by Li Yan in 1912. The texts of *Xu Gu Zhai Qi Suan Fa* in *Zhi Bu Zu Zhai Cong Shu* and in the *Zhu Jia Suan Fa* are not complete but they have much significance as historical materials because *Yi Jia Tang Cong Shu* lacks chapter 1 of *Xu Gu Zhai Qi Suan Fa* and the lost material remains in these books.

Zhi Bu Zu Zhai Cong Shu series 27 are from chapter 16350-16264 of *Yong Le Da Dian,* while *Zhu Jia Suan Fa* are from chapter 16361-16364 of it [69].

We will now consider the date of copies of the *Zhu Jia Suan Fa*.

Before the *Si Ku Quan Shu* was compiled in 1782, even high officials had little chance to see *Yong Le Da Dian* under the Qing dynasty's control. Then the project of editing *Si Ku Quan Shu* was started. *Yong Le Da Dian* was used intensively because it was the most important source for collection purposes. Therefore few people had a chance to see the *Yong Le Da Dian* apart from the editors of the *Si Ku Quan Shu* 四庫全書. It was very difficult for even editors to refer to the *Yong Le Da Dian* privately. They could only refer to the part related to their work. *Yang Hui Suan Fa* was not collected into *Si Ku Quan Shu,* thus *Zhu Jia Suan Fa,* which is including *Yang Hui Suan Fa* , probably was not copied at this stage. After edited *Si Ku Quan Shu,* the control of *Yong Le Da Dian* became a little loose. Members of Hanlinyuan 翰林院 became able to gain access to it. Thus the oldest limit of the copying of *Zhu Jia Suan Fa* must be in 1782.

We can guess the latest limit considering the contents of *Zhu Jia Suan Fa.* That is to say, it is the same question as the *Zhi Bu Zu Zhai Cong Shu* series 27. They both are not only the same question, but also list the same order. Thus it is certain that the editors of *Zhi Bu Zu Zhai Cong Shu* referred to *Zhu Jia Suan Fa.* Therefore we ought to conclude that *Zhu Jia Suan Fa* was copied before *Zhi Bu Zu Zhai Cong Shu* was published in 1814.

There is only historical evidence to contend with in the short period from 1782 to 1814; "Yang Hui" section of the *Bu Chou Ren Zhuan* 補疇人傳 (Supplement of Biographies of Mathematicians and Astronomers) on *Yi Jia Tang Cong Shu* edition of *Yang Hui Suan Fa* says:

> In 1810, I (Ruan Yuan) became a scholar of Hanlin 翰林學士, researcher of Wen Ying Guan 文穎館提調[70]. I copied one hundred and more questions of *Yang Hui Zhai Qi* 楊輝摘奇 and (blank of four words) and the others from *Yong Le Da Dian.*

We know of no other person who copied from the *Yong Le Da Dian.* In addition to this, Mo Youzhi, the first owner of *Zhu Jia Suan Fa,* had a connection with Ruan Yuan. Mo Yuchou 莫與儔 (18c), the father of Mo Youzhi, obtained Jinshi 進士 (the doctoral degree) in 1799, and the vice-president of the Hui Shi 會試 (examination which is held by Li-bu 禮部 (National Personel Authority), virtually the final examination) was Ruan Yuan (the president was Zhu Gui 朱珪 (18c)). In Chinese custom, the examiner was regarded as the teacher because he gave candidates their status, making a teacher-student relation. I wonder if Mo Youzhi had a chance to get *Zhu Jia Suan Fa* from Ruan Yuan [71]. Therefore, we have good reason to believe that *Zhu Jia Suan Fa* was copied by Ruan Yuan.

Zhu Jia Suan Fa, which remained at the Zhongguo Kexueyuan Ziran Kexue-shi Yanjiusuo Tushuguan, lacks the record of this book and preface, and contains only fifty-six questions. So is it not possible that this book is vol.2 or 3 (last volume) of the original book? If so, it is in accordance with the statement that Ruan Yuan recorded one hundred questions [72] and no preface. Because if the extract from vol. 16350-16364 of *Yong Le Da Dian,* which is *Zhi Bu Zu Zhai Cong Shu* 's part, comprised one hundred questions, *Zhu Jia Suan Fa* which extracts from vol. 16361-16364 of *Yong Le Da Dian* would comprise fifty-six.

Then *Zhu Jia Suan Fa* was copied by Qiu Chongman and remains at the Zhejiang province Library.

It is no doubt this manuscript of Ruan Yuan which we call *Zhu Jia Suan Fa* was sent to Jiang Fan 江藩 (18c) [73] who was an assistant of Ruan Yuan, and Jiang Fan proofread it. Then it was probably proofread by Ma Yigen 馬以艮 (19c) and became *Zhi Bu Zu Zhai Cong Shu* because the editor of it, Bao Yanbo 鮑延博 (19c) [74] was also on the staff of Ruan Yuan when they [75] edited *Si Ku Wei Shou Shu Mu Ti Yao* 四庫未收書目提要 (Catalogue of Uncollected Books of *Si Ku Quan Shu*).

(c) Conclusion to section Ⅰ-2

The version of *Zhi Bu Zu Zhai Cong Shu* was covered by Ruan Yuan and his staff. All versions in China were created by Ruan Yuan. *Yang Hui Suan Fa* was not collected in *Si Ku Quan Shu*, and would have been doomed to extinction had it not been for the activities of a genius, Ruan Yuan, who obtained "Jin Shi" (the doctoral degree) at only twenty-six years old. Thus we can read *Yang Hui Suan Fa* now.

However, Japanese mathematicians in the Edo period did not use Ruan Yuan's works. They used the Korean editions directly. And I wonder whether *Yang Hui Suan Fa* was republished in this age? The reasons are as follows.

Firstly, some Japanese mathematicians studied it. Nozawa Sadanaga 野澤定長 (17c)(76) referred to *Yang Hui Suan Fa*. He tried to make a 19 degree magic square (77), but he failed, and commented;

> I could not find the method. It was not described in *Yang Hui Suan Fa* (78).

The works of Sawaguchi Kazuyuki 澤口一之 (17c)(79), about higher degree equations were influenced by *Yang Hui Suan Fa* (80).

Seki Kôwa copied *Yang Hui Suan Fa* by hand, so he had not had the original Korean edition itself. Therefore he borrowed *Yang Hui Suan Fa*, but Shôgun's 將軍(81) Library, Momiji-yama Bunko 紅葉山文庫 and Shôgun's school, Shôheikô 昌平黌, did not include it (82) because Tokugawa Ieyasu 徳川家康 (1542-1616)(83), who was the first Shôgun of the Edo period, did not attend the Hideyoshi War. The possibility of Seki Kôwa borrowing from his teacher (84), Sawaguchi Kazuyuki is the strongest.

Secondly, the printing quality of the Korean edition of *Yang Hui Suan Fa* was excellent. However some books are faulty, moreover they had no official seal. I think these books were republished editions.

In any case, it is certain that Seki Kôwa studied *Yang Hui Suan Fa*. But that book was not publicly available, so it was difficult for the other Japanese mathematicians to study it. If Seki Kôwa studied *Shu Shu Jiu Zhang*, the situation would be the same as the case of *Yang Hui Suan Fa*. Therefore, we will consider how Seki Kôwa referred to *Yang Hui Suan Fa* in chapter Ⅱ before considering the influence of *Shu Shu Jiu Zhang* on him. The most important work of *Yang Hui Suan Fa* must be the method of making magic squares, thus we wiil focus on magic squares.

(*1): Qian Baocong, 1964: 60-103.

(*2): Libbrecht, 1973: 35-53.

(*3): Li Di, 1987: 43-58 of Wu Wenjun (ed.), 1987.

(*4): Chen Zhensun 陳振孫 (1190-after 1249), *Zhi Zhai Shu Lu Jie Ti* 直齋書録解題 (Bibliography of Chen Zhensun's (Zhi Zhai) Library), part 3, chapter 12, p.355, "Xiang Wei Lei" 象緯類 (Qian Baocong: 64-65) (Libbrecht, 1973: 37) (Li Di, 1987: 44).

(*5): Zhou Mi 周密 (1232-after 1308), *Gui Xin Za Zhi Xu Ji* 癸辛雜識續集 (Miscellaneous Information from Gui Xin Street; First Addendum), chapter 2 (Qian Baocong, 1964: 64-5)(Libbrecht, 1973: 38)(Li Di, 1987: 44).

(*6): Yang Shiqi 楊士奇 (1365-1441), *Wen Yuan Ge Shu Mu* 文淵閣書目 (Bibliography of Wenyan Building) (1441), chapter 14 (Libbrecht, 1973: 39-41)(Li Di, 1987: 46). Ye Sheng葉盛 (1420-1474), *Lu Zhu Tang Shu Mu*, 菉竹堂書目 (Bibliography of Ye Sheng's Library (Lu Zhu Tang)), chapter 5, p.22B (Li Di, 1987: 47).

(*7): His other name is Yunlin 雲遴 (Zang Lisu, 1921: 153).

(*8): His other names are Yuandu元度 and Qingchangdaoren 清常道人. Zhao Qimei became Xingbu Langzhong 刑部郎中 (secretary of Justice-Ministry). He was a book collector, and wrote *Miao Wang Guan Shu Mu* 脈望館書目 (Catalogue of Miao-Wang-Guan Lib.) (Zang Lisu, 1921: 1413).

(*9): Libbrecht, 1973: 41.

(10): His other names are Shunwan 遵王 and Yeshiweng 也是翁. Qian Zeng was born at Changshu 常熟 (now Suzhou, Jiangsu province 江蘇省蘇州). He was a book collector, and wrote *Ye Shi Yuan Shu Mu* 也是園書目 (Catalogue of Qian Zong's Lib. (Ye-Shi-Yuan)) and *Shu Gu Tang Shu Mu* 述古堂書目 (Catalogue of Qian Zong's Lib. (Shu-Gu-Tang)) (Zang Lisu, 1921: 1617).

(11): Qian Zeng 錢曾 (1629-1701), *Ye Shi Yuan Shu Mu* 也是園書目 (Bibliography of Qian Zeng's Library (Ye Shi Yuan)), chapter 1 (Libbrecht, 1973: 42)(Li

Di, 1987: 48).

(12): His other name is Guyu 古餘. Zhang Dunren was born at Yangcheng 陽城 (now in Shanxi province 山西省陽城). He was a bureaucrat and a mathematician. He obtained Jinshi 進士 (doctoral degree), then became Yunnan-sheng Yanyidao 雲南省鹽驛道 (president of salt monopoly in Yunnan province). Wrote *Kai Fang Bu Ji* 開方補記 (Supplement of Solving Equation) (Zang Lisu, 1921: 955).

(13): Li Di, 1987: 51.

(14): His other names are Shouzhi 受之 and Muzhai 牧齋. Qian Qianyi was born at Changshu 常熟 (now Suzhou, Jiangsu province 江蘇省蘇州). Was a bureaucrat and a book collector. Obtained Dr. degree, and became Libu Shilang 禮部侍郎 (Vice-minister of the National Personnel Authority). He wrote *Jiang Yun Lou Shu Mu* 絳雲楼書目 (Bibliography of Qian Qian Yi's Library (Jiang Yun Lou)) (Zang Lisu, 1921: 1621).

(15): Qian Qianyi 錢謙益, *Jiang Yun Lou Shu Mu* 絳雲楼書目 (Bibliography of Qian Qian Yi's Library (Jiang Yun Lou)), chapter 2 (Li Di, 1987: 48).

(16): His other name is Dongyuan 東原. Dai Zhen Was born at Xiuning 休寧 (now Dongxi, Anhui province 安徽省屯溪). He was a bureaucrat and a scholar. He studied under Jiang Yong 江永 (18c). He obtained Ci-Junshi 賜進士 (honorary doctorate), then became the editor of *Si Ku Quan Shu*. He wrote *Kao Gong Ji Tu* 考工記圖 (Figures of *Kao Gong Ji*), *Gao Gu He Yuan Ji* 句股割圜記 (Records of Triangle and Determining Segment Area), *Ce Suan Sheng Yun Kao* 策算聲韻考 (Studies on Napier's Bones and Verse), *Jiu Zhang Bu Tu* 九章補圖 (Supplement and figures of *Jiu Zhang Suan Shu*), *Gu Li Kao* 古曆考 (Studies on Ancient Calendars), etc. (Zang Lisu, 1921: 1717).

(17): His other names are Zhongzhong 衆仲, Huiyue 撝約 and Xunxuan 顨軒. Kong Guansen was born at Qufu 曲阜 (Shangdong 山東 province). He was a bureaucrat and a mathematician. He studied from Dai Zhen, and obtained Jins hi 進士 (doctoral degree), became Hanlinyuan Jiantao 翰林院檢討 (sub-editor of Hanlinyuan). He wrote *Shao Guan Zheng Fu Shu* 少廣正負術 (Method of How much width by plus and minus) (Zang Lisu, 1921: 46).

(18): Luo Shilin 羅士琳 (d.1853), *Xu Chou Ren Zhuan* 續疇人傳 ((Biographies of Mathematicians and Astronomers, part 2), chapter 48, section "Kong Guansen".

Shao Guang Zheng Fu Shu 少廣正負術 (The Technique of Positive and Negative about Sides), his work, also quoted *Shu Shu Jiu Zhang* (Li Di, 1987: 49).

(19): His other name is Yunmen 雲門. Li Huang was born at Zhongxiang 鍾祥 (now Xiang Yang, Hubei province 湖北省襄陽). He was a bureaucrat and a mathematician, and obtained Jinshi 進士 (doctoral degree), then became Gongbu Zuo-Shilang 工部左侍郎 (Vice-minister of Industry Department). He wrote *Jiu Zhang Suan Shu Xi Cao Tu Shuo* 九章算術細草圖說 (Careful Explanation of Nine Chapters on the Mathematical Arts) (Zang Lisu, 1921: 443).

(20): Zhang Dunren 張敦仁 (1754-1834), *Qiu Yi Suan Shu* 求一算術 (Mathematical Methods of Acuiring One, 1803), preface (Li Di, 1987: 50)

(21): Li Di, 1987: 50. Li Di converted into 1769 on the solar system calendar.

(22): His other name is Litang 理堂. Jiao Xun was born at Jiangsu Ganquan 江蘇甘泉 (now Jiangdu, Jiangsu province 江蘇省江都). He was a mathematician, obtained Juren 擧人 (master's degree). He wrote *Tian Yuan Yi Shi* 天元一釋 (Interpretation of Technique of the Celestial Element), *Kai Fang Tong Shi* 開方通釋 (Interpretation of Solving Higher degree equation)) (Zang Lisu, 1921: 1175).

(23): His other names are Xiaoyin 孝嬰, and Hengzhai 衡齋. Wang Lai was born at She 歙 (now Wuhu, Anhui province 安徽省蕪湖). He was a mathematician, and became Xundao 訓導 (Reader) at Shidai 石埭. He wrote *Heng-Zhai Suan Xue* 衡齋算學 (Wan-Lai's Mathematical Studies) (Zang Lisu, 1921: 480).

(24): Luo Shilin 羅士琳 (d. 1853), *Xu Chou Ren Zhuang* 續疇人傳 (Biographies of Mathematicians and Astronomers, part 2) , chapter 50, 'Li Rui'(Li Di, 1987: 50).

(25): His other names are Jinguang 近光 and Dunfu 敦夫. Qiu Enfu was born at Jiangdu 江都 (now in Jiangsu province 江蘇省). He was a book collector. He obtained Jinshi (doctoral degree), became Hanlinyuan Bianxiu 翰林院編修 (Deputy Editor of Hanlinyuan) (Zang Lisu, 1921: 828).

(26): Gu Qianli 顧千里 (1770-1839), *Si Di Zhai Ji* 思適齋集, chapter 10 (Li Di, 1987: 50).

(27): Li Rui 李鋭, *Guan Miao Ju Ri Ji* 觀妙居日記 (Li Rui's Diary), vol. 2 (Li Di, 1987: 50).

(28): His other names are Xiaozheng 曉徵, Xinmei 辛楣 and Zhuting 竹汀. Qian Daxin was born at Jiading 嘉定 (now in Jiangsu province 江蘇省). He was a bureaucrat and a astronomer. He obtained Jinshi (doctoral degree), then became Hanlinyuan Shao-Linshi 翰林院少詹事 (vice-secretary of Hanlinyusn), Guangdong Tiduxuezheng 廣東提督學政 (Minister of Education at Guangdong province). He wrote *San-Tong Shu Yan* 三統術衍 (Extension of *San Tong Li*), *Si Shi Suo Run Kao* 四史朔閏考 (Studies on the New Moons and Intercalary Months in Four Histories), etc. (Zang Lisu, 1921: 1611).

(29): Lu Xinyuan 陸心源, *Yi Ku Tang Ti Ba* 儀顧堂題跋 (Bibliography of Lu Xinyuan's Library), chapter 8 (Li Di, 1987: 52).

(30): Li Di, 1987: 50-1.

(31): Li Di, 1987: 52.

(32): Bai Shangshu, 1964: 290.

(33): Sun Xingyan 孫星衍, *Sun-Shi Ci Tang Shu Mu Nei Bian* 孫氏祠堂書目内編 (Bibliography of Sun Xingyan; Inner Versions), chapter 3. and *Sun-Shi Ci Tang Shu Mu Xu* 孫氏祠堂書目序 (Bibliography of Sun Xingyan; Preface) (Li Di, 1987: 49).

(34): Zhou Zhongfu 周中孚 (1786-1831), *Zheng Tang Du Shu Ji* 鄭堂讀書記 (Descriptions of Zhou Zhongfu's impressions), chapter 45. (Li Di, 1987: 49-50).

(35): Li Di, 1987: 51

(36): Li Di, 1987: 52.

(37): Libbrecht, 1973: 46.

(38): Li Di, 1987, 1973: 52.

(39): Seki Kôwa's complete work, *Katsuyō Sampō* 括要算法 (Essential Points of Mathematics) was published in 1712, but the editor, Araki Murahide 荒木村英 (1640-1718), wrote the foreword in 1709.

(40): *Yi Jia Tang Cong Shu* version is kept at Kokkai Toshokan 國會圖書館, Kôbunshokan 公文書館, Tôyô Bunko 東洋文庫 and Jimbun Kenkyûjo of Kyôto University 京都大學人文科學研究所, at least.

(41): Tanimatsu Shigeru's 谷松茂 (18-19c) preface of *Tan'i Sampō* 探頤算法 (Searching of Bringing-up in Mathematics, by Kemmochi Masayuki 劍持章行 (1790-1871) (Nihon Gakushiin, 1954, vol.5: 429).

(42): *Chou Ren Zhuan* 疇人傳 (Biographies of Mathematicians and Astronomers) was published in 1799 and there is a section on Qin Jiushao, therefore it can be the source of Qin Jiushao's information. This book was edited in *Wen Xian Lou Cong Shu* 文選樓叢書 (Collection of Wenxianlou) and *Huang Qing Jing Jie* 皇清經解 (Collection of Monographs on Classical Subjects Written During the Qing Dynasty). The former was introduced in 1846, the latter was introduced in 1840 (Oba Osamu, 1967: 453 and 516), the same year that *Tan'i Sampō* (Searching of Bringing-up in Mathematics) was written. We must, however, be attentive to the fact that these books were kept in Momijiyama Bunko 紅葉山文庫 (Lib. of Momijiyama, which was Shôgun's Lib.) and Shôheizaka Gakumonsho 昌平坂學問所 (University of Shôheizaka, which was Shôgun's university). Before retaining any books these libraries, had to examine them with a critical eye on their claims for Christianity. Thus process took months, and sometimes as long as one year, thus it is difficult to believe that Tanimatsu Shigeru read *Huang Qing Jing Jie* in 1840.

(43): Mikami Yoshio, 1932.

(44): Li Yan. 1928. "*Yong Le Da Dian* Suan Shu" 永樂大典算書 (Mathematical Booksin the *Yong Le Da Dian*) in pp.47-53 of vol.2 of Li Yan, 1933. Li Yan. 1930. "Song Yang Hui Suanshu Kao" 宋楊輝算書考 (Studies of Yang Hui's Mathematical Books in the Song Dynasty) in pp. 54-47 of vol.2 of Li Yan, 1933.

(45): Yan Dunjie, 1964.

(46): Kodama Akio, 1966.

(47): Lam Lay-Yong, 1977.

(48): See Guo Shuchun, 1988. Zhang Jiamin, 1988. discussed the *Yang Hui Suan Fa.*

(49): According to 12th October 12th year of Sejong reign (1430) on *Sejong Sillok* 世宗實録 (Veritable Records of the King Sejong Era) of *Yijo Sillok* 李朝實録 (Veritable Records of Yi Dynasty), another mathematical book was republished.

> The King (Sejong) studied the *Kemongsan* 啓蒙算 (Elementary Mathematics) and said to Chǒng Inji 鄭麟趾 (fl.1433, who was Pujehak 副提學 (vice-minister of Educational Department)), "Mathematics is usually not of practical value, but it was founded by sages, so I would like to know mathematics." (vol.3: 267)

The *Kemongsan* (Elementary Mathematics) probably refers to *Suan Xue Qi Meng* (Sanhak Kemong) 算學啓蒙 (Introduction to Mathematical Studies). Therefore it must have been republished near this year, and a Korean edition of *Suan Xue Qi Meng* is still found in only the Tsukuba Univ. Lib. 筑波大學圖書館.

There is a report that the Korean edition of *Xiang Jie Jiu Zhang Suan Fa* 詳解九章算法 (Yang Hui's Comment for Nine Chapters on the Mathematical Arts), republished in 1482 written by Yang Hui in 1261, is kept at the Beijing National Library 北京圖書館 (Dongbei Shifan Daxue (ed), 1987: 567). But the catalogue of Beijing National Library is not arranged yet so we cannot confirm it, and there is no record in *Sejong Sillok* about this edition.

(50): Kuksa P'yǒnch'an Wiwǒnhoe 國史編纂委員會 edition, vol. 3: 501.

(51): Hirayama Akira, 1988: 57.

(52): According to Mikami Yoshio's research (Mikami Yoshio, 1932-5), it remained at Tôkyô Kôtô Shihan Gakkô 東京高等師範學校 (now Tsukuba Univ. 筑波大學), Kunaishô Zushoryô 宮内省圖書寮 (Library of the Imperial Household Agency, now Kunaichô Shoryôbu 宮内庁書陵部) and Naikaku Bunko 内閣文庫 (Library of Cabinet, now Kokuritsu Kôbunshokan 國立公文書館).

Naikaku Bunko's one, however, was lost until Kodama Akio 児玉明人 researched (Kodama Akio, 1966: 7). In that time, two books were kept at Tôkyô Kyôiku Daigaku 東京教育大學 (now Tsukuba Univ.) and one book at Sonkeikaku Bunko 尊經閣文庫.

(53): General of the Azuchi-Momoyama period. His other Japanese name was Ukita Hideie 浮田秀家. He became the son-in-law of Toyotomi Hideyoshi 豐臣秀吉, then one of Gotairô 五大老 (Five Ministers (of Toyotomi Hideyoshi's Cabinet)). In the Hideyoshi War, Ukita Hideie became the commander of Japanese 8th Corps. After Hideyoshi's death, he fought with Tokugawa Ieyasu 徳川家康 in Sekigahara 関が原; however he lost the war, then fell from his position.

(54): Mikami Yoshio, 1932-5.

(55): Kodama Akio, 1966: 10.

(56): General of the Azuchi-Momoyama period. His other name was Inuchiyo 犬千代. He became one of the Five Ministers of Toyotomi Hideyoshi's Cabinet. In the Hideyoshi War, Maeda Toshiie worked in general support at Nagoya 名護屋 (now Chinzeichô 鎮西町, Saga 佐賀 prefecture). The abacus which he used at that time, the oldest one in Japan, remains in his family. In the Sekigahara War, his son supported Tokugawa Ieyasu and they won, so his family became the biggest Daimyô in the Edo period.

(57): On pp.36-9 of chapter 7 of the *Ri-Bei Fang Shu Zhi* 日本訪書志 (Research of Chinese Books in Japan, 1881). Guangwen Shuju 廣文書局 edition, vol.2: 493-9.

(58): Li Yan, 1933, 1954; vol.2: 60.

(59): Okkabuse was one of the popular techniques of republishing in that age. One book was sacrificed to make a newer printing block. Each page was wetted and reversed, then put on a wood block and cut with original paper in place to make printing block. Wet paper was somewhat larger than dry paper, so the republished edition is a little larger than original one.

(60): His other names were Gangfu 剛甫, Qianyuan 潛園 and Zunzhai 存齋. Was born at Guian 歸安 (now Huzhou, ZheJiang province 浙江省湖州) in the Qing dynasty. He obtained Juren 舉人 (master's degree) in Xianfeng 咸豐 reign (1851-1861), then he became FuJian Yanyunshi 福建鹽運使 (Transporter of Salt in Fujian province). Lu Xinyuan collected lots of books; his Song and Yuan dynasties books were preserved in the Pi Song Lou 皕宋樓 (the Library of two hundred books in the Song dynasty), his hand-copied books were preserved in the Shi Wan Quan Lou 十萬卷樓 (the Library of One Hundred Thousand Books), the others were preserved in the Shou Xian Ke 守先閣 (the Library of Keeping Antiquity). Most of them were collected from Yi Jia Tang (see note 57). After his death, most of his books are preserved in the Seikado Bunko in Japan now. He published *Shi Wan Quan Lou Cong Shu* 十萬卷樓叢書 (Complete Works of the Library of One Hundred Thousand Books) in 1879, and wrote *Pi Song Lou Zang Shu Zhi* 皕宋樓藏書志 (Catalogue of the Library of two hundred books in the Song dynasty) and *Yi Gu Tang Ji* 儀顧堂集 (Songs of Yi Gu Tang) (Zang Lisu, 1921: 1114).

(61): His other names were Shaowu 紹武, Raopu 蕘圃, Fuweng 復翁 and Ningsongjushi 佞宋居士. He was born at Wuxian 吳縣 (now Suzhou, Jiansu province 江蘇省蘇州) in the Qing dynasty. Huang Pilie became Fenbu Zhushi 分部主事 (6th grade 正六品). He obtained Juren 舉人 (master's degree) at Qinglong 乾隆 reign (1736-1795). His library's name was Bai Song Yi Chan 百宋一廛. (Zang Lisu, 1921: 1229).

(62): His other names were Menghua 夢華, Jingzhi 敬祉 and Huiyin 嬇隱. He Yuanxiwas born at Qiantang 錢塘 (now Hanzhou, Zhejiang province 浙江蘇省杭州) in the Qing dynasty. He became Zhubu 主簿 (secretary, 9th grade 正九

品) (Zang Lisu, 1921: 285).

(63): However, there were no records about *Yang Hui Suan Fa* in *Bai Song Yi Chan Shu Lu* 百宋一廛書録 (Catalogue of the Library of One Hundred Books in the Song Dynasty) or *Bai Song Yi Chan Fu* 百宋一廛賦 (Songs of the Library of One Hundred Books of the Song Dynasty).

(64): His other name was Junmian 君勉. Song Jingchang was born at Jiangyin 江陰 (now Jiangyin, Jiangsu province 江蘇省蘇州地区江陰縣) at the Qing dynasty. He became Xian Xuesheng 縣學生 (post graduate student). He proofread some mathematical books in the *Yi Jia Tang Cong Shu*. He wrote *Xing Wei Ce Liang* 星緯測量 (celestial surveying). *Qing Shi Gao* 清史稿 (Manuscript of History of the Qing Dynasty), CHSJ, vol.44, p.13416.

(65): His other name was Wanzhi 萬枝, Taifeng 泰豐. Yu Songnian was born at Shang Hai 上海 in the Qing dynasty. He obtained En Gengsheng 恩貢生 (Bachelor emeritus). He founded Yi Jia Tang 宜稼堂 (the Library of good agriculture) (Zang Lisu, 1921: 718).

(66): Mathematician of Seki Kôwa's School. His other names were Tôemon 藤右衛門 and Kôju 高樹. Ishiguro Nobuyoshi was born at Shinminato, Toyama prefecture, Japan 富山縣新湊.

(67): Then a microfilm was made from it, and it was sent to The Needham Research Institute, Cambridge. Using this microfilm, Lam Lay-Yong 藍麗容 completed the English translation of *Yang Hui Suan Fa*.

(68): His other name was Han Xing 翰興. Qiu Chongman was born at Sheng-xian, Zhejiang province 浙江省嵊縣 in the Mingguo 民國 period. He collected three hundred mathematical books, which now remain in the Zhejiang Province Library 浙江省圖書館, in the room known as Shuangxiaoshi 雙嘯室 (Double Cries Room). He wrote "Zhong Guo Suan Xue Shu Mu Hui Bian" 中國算學書目匯編 (Mathematical Catalogue in China) in *Qing Hua Xue Bao* 清華學報 (Journal of Qing Hua Univ.) 1926-1: 43-92.

(69): Yan Dunjie, 1987.

(70): Ruan Yuan, who was the governor of Zhejiang 浙江巡撫 (Governer of Zhejian province), dispatched Liu Fenggao 劉鳳誥 (19c), who was the head of the Education department of Zhejiang province 浙江學政提督 to be president of Ningbo Examination for the master degree 寧波郷試主考 on 20th Aug., 1809. However, Liu Fenggao did not have the award licence, moreover, dishonesty occurred, therefore Ruan Yuan was relegated to the rank of Hanlin Xueshi 翰林學士 (scholar of Hanlin) (Section Gengxu 庚戌 (17th) of 8th month in 1809, Chapter Jiaqing Benqi 嘉慶本紀 (Records of emperor Jiaqing) on *Qing Shi Gao* 清史稿 (Manuscript of History of the Qing Dynasty), ZHSJ vol.3: 597). But, therefore, he had a chance to research the *Yong Le Da Dian*.

(71): Mo Yuchou 莫與儔 (was born in Deshan 獨山 at the Qing dynasty. His other names are Youren 猶人, Jiefu 傑夫. Mo Yuchou became Zunyi Fuxue Jiaoshou 遵義府學教授 (Professor of Zunyi-fu Univ.) (Zang Lisu, 1921: 1027)), and took appointment of Hanlinyuan Shujushi 翰林院庶吉士 (Fellow of Hanlinyuan). So he also had a chance to see the *Yong Le Da Dian* himself, we cannot deny it.

Mo Yuchou collected some books from Yi Jia Tang (the Library of good agriculture), but *Zhu Jia Suan Fa* was not published for *Yi Jia Tang Cong Shu*, but rather in *Zhi Bu Zu Zhai Cong Shu*, so the possibility that he got *Zhu Jia Suan Fa* from Yi Jia Tang was not strong.

(72): Li Rui recorded in the preface of *Yi Jia Tang Cong Shu* (Complete Works of the Library of Yi Jia Tang) edition:

> In 1810, I attended the Shuntian Shi 順天試 (examination at Shuntian (Capital) for the master degree), so I stayed at Beijing. At the house of Li Huang 李潢, I found one hundred mathematical questions which were hand-copied from *Yong Le Da Dian* by Ruan Yuan when he was a researcher of Wenyingguan 文穎館提調. Some questions on the *Yang Hui Zhai Qi* 楊輝摘奇 remain.

questions.

(73): His other names are Zibing 子屏 and Zhengtang 鄭堂. Jian Fan was born at Ganquan 甘泉 (now Yangzhou, Jiangsu province 江蘇省揚州) in the Qing dynasty. He became Lizheng Shuyuan Shangzhang 麗正書院山長 (Master of Lizheng school) when Ruan Yuan was Waian Ducao 淮安督漕 (Minister of Shipping Trade in Waian). Wrote *Han Xue Shi Cheng Ji* 漢學師承記 (Relation between Teachers and Students about Chinese Studies) and others (Zang Lisu, 1921: 276).

(74): His other names are Yiwen 以文 and Luying 淥飲. Bao Yanbo was born at Shexian 歙縣 (now in Anhui province 安徽省歙縣) in the Qing dynasty. He collected lots of books, and sent six hundred books to the emperor when *Si Ku Quan Shu* was edited. He published the *Zhi Bu Zu Zhai Cong Shu* in thirty series, which turned out to be one of the best collection of his time. In Jiaqing 嘉慶 period (1796-1820), he obtained En-juren 恩舉人 (Master honoris causa) when he was eighty-six years old (Zang Lisu, 1921: 276).

(75): The other staff was He Yuanling 何元鈴.

(76): Mathematician of the early Edo period. His other name is Chûbei 忠兵衛. He wrote the *Dōkai-Shō* 童介抄 (Introduction for Pupils) in five chapters in 1664.

(77): It was "Idai" 遺題 (Leaf Question) of the *Sampō Ketsugi Shō* 算法闕疑抄 (Solving Mathematics Questions).

(78): Question 100, chapter 4 of the *Dōkai-Shō* 童介抄 (Introduction for Pupil); the same comment is also in question 99 (Nihon Gakushiin, 1959: vol.1, 333).

(79): Mathematician of the early Edo period. His other names are Saburôzaemon 三郎左衛門, Sôin 宗隱. Sawaguchi Kazuyuki was a student of Hashimoto Masakazu 橋本正數. He lined in Kyoto, and published the *Kokon Sampō Ki* 古今算法記 (Mathematics of All Ages), seven chapters, in 1671.

(80): Jôchi Shigeru, 1991.

(81): Seki Kôwa was a subordinate of the Shôgun 將軍 (征夷大將軍), and his rank was 250 Koku 石 (later became 300 koku). In the Edo period, a Samurai's (or Bushi) 武士 rank was indicated by their salary or territory equivalent of rice. 1 Koku 石 meant about 180 litre. Among the subordinates of the Shôgun, those with ranks of Hatamoto 旗本 and above could see their master directly. The hierachy, indicated by salaries expressed in terms of quantity of rice is shown below:

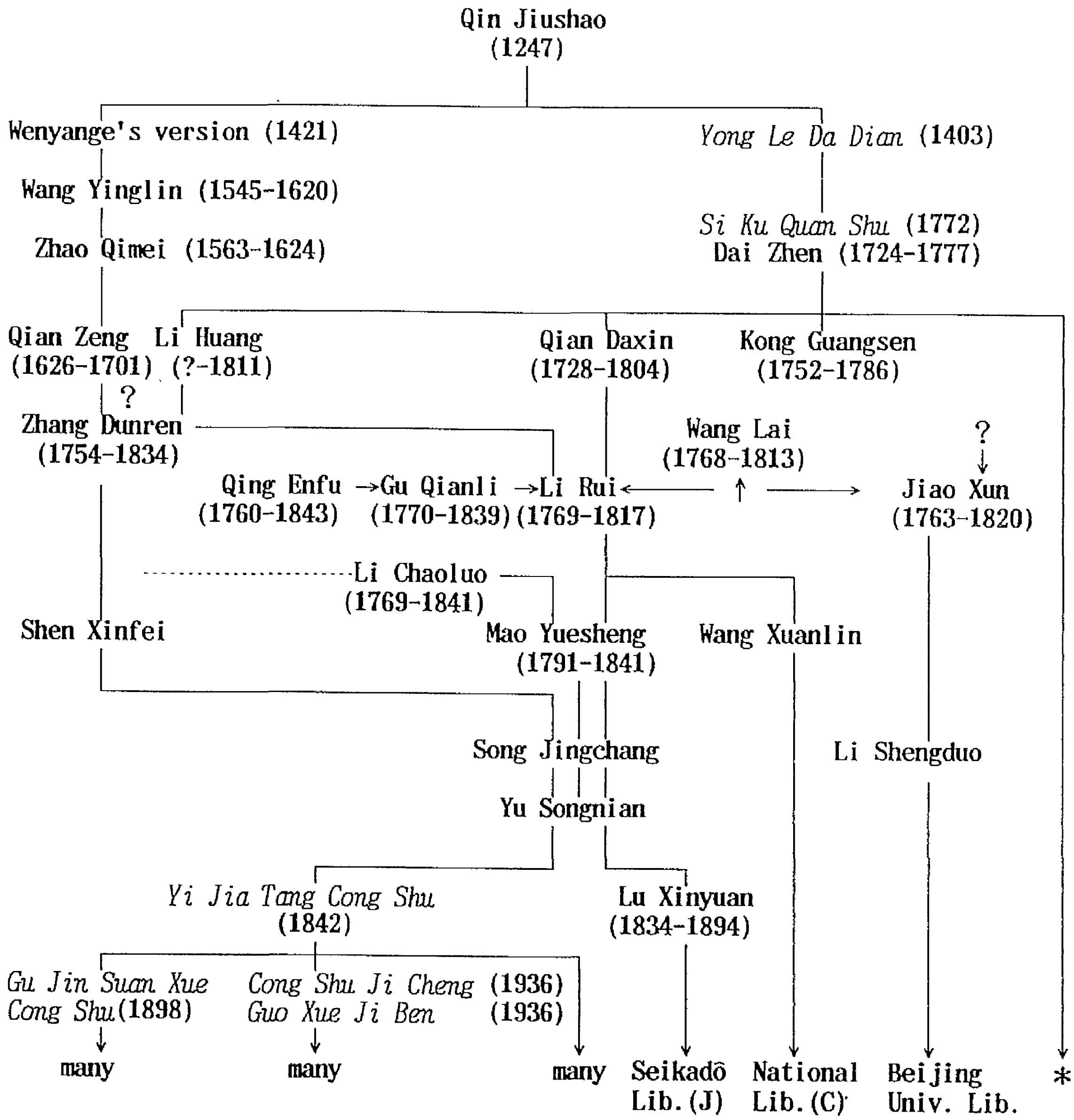
10,000+	Daimyô	大名
3,000- 9,999	Yoriai	寄合
200- 2,999	Hatamoto	旗本
199-	Gokenin	御家人

(82): These books usually remain at the Kokuritsu Kôbunshokan 國立公文書館, but there are no *Yang Hui Suan Fa* , though the work of Mikami Yoshio was found here (see note 45).

(83): First Shôgun of Edo Bakufu 幕府 (Shôgun's Cabinet). His other names are Takechiyo 竹千代, Motonobu 元信 and Motouasu 元康. Tokugawa Ieyasu supported Toyotomi Hideyoshi in unifying Japan, then became the first minister of the Five Ministers. After Toyotomi Hideyoshi's death, he won the Sekigahara War in 1600, then became a Shôgun in 1603.

(84): *Araki Murahide Sadan* 荒木村英茶談 (Talks of Araki Murahide) says that Seki Kôwa was a teacher of Sawaguchi Kazuyuki, but Seki Kôwa solved the question of *Kokon Sampō Ki* 古今算法記 (Mathematics of All Ages) written by Sawaguchi Kazuyuki in 1671, so this is impossible to believe (Nihon Gakushiin, 1959, vol.1: 349). Probably Sawaguchi Kazuyuki was Seki Kôwa's teacher. At least their relation was close.

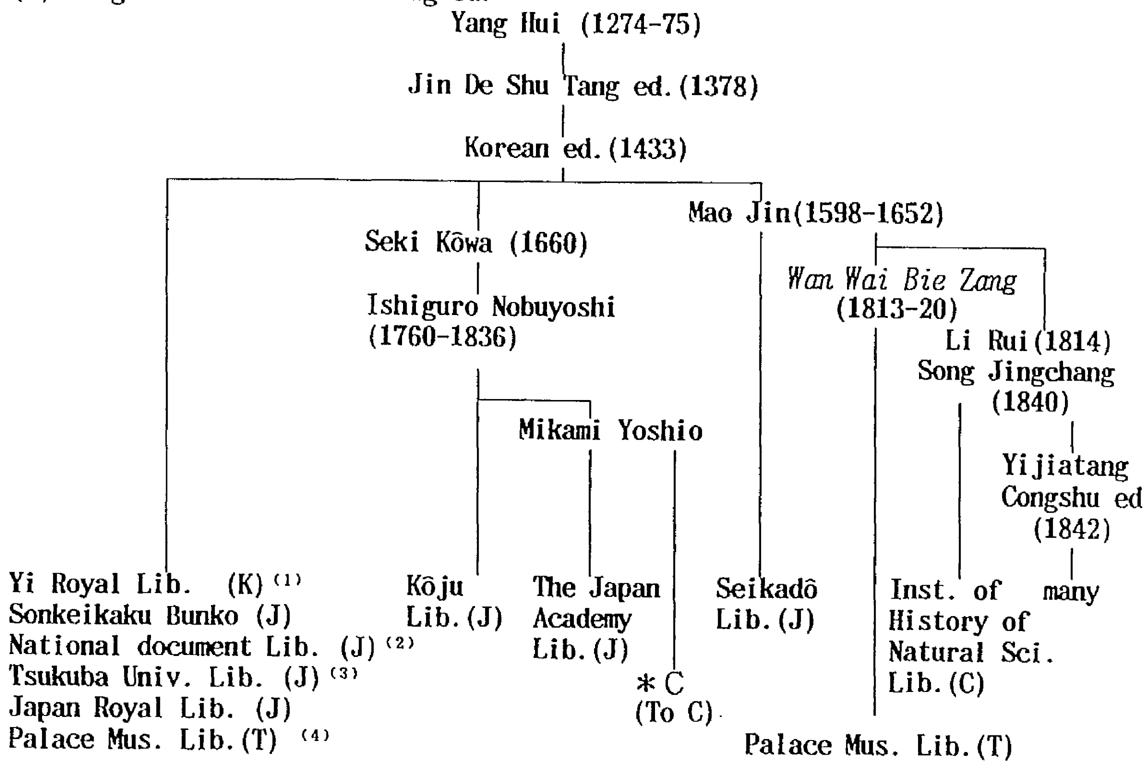
Diagram of manuscript tradition of the *Shu Shu Jiu Zhang*



*.: National Library of China (Beijing)
: Gansu Province Library 甘肅省圖書館 (Lanzhou 蘭州)
: Gugong Bowuyuan (Taiwan)

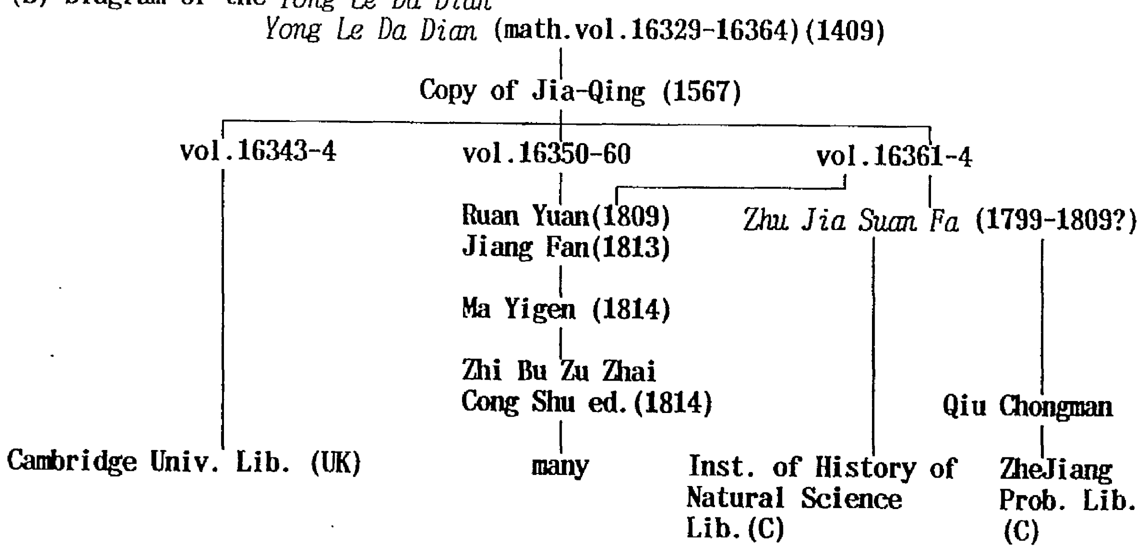
Diagram of manuscript tradition of the *Yang Hui Suan Fa*

(A) Diagram of Jin De Shu Tang ed.

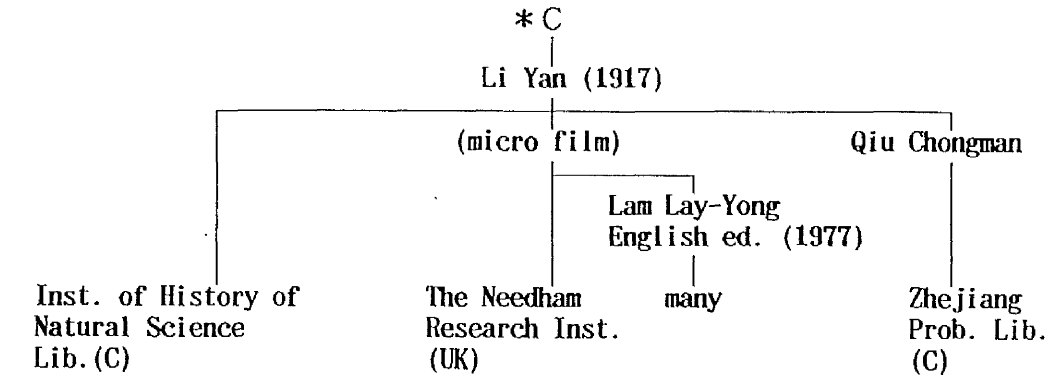


1 Researched by Dr. Hirayama Akira, but now unknown.
2 It was lost since 1966.
3 Two books.
4 From National Lib. at Beijing.

(B) Diagram of the *Yong Le Da Dian*



(C) Diagram of Mikami Yoshio's MS



(D) Main Reprints

original	year	press
(1)Korean ed. (Tsukuba ver.)	1966	Kodama Akio, Tokyo, Japan
(2)Zhi Bu Zu Zhai ed.	1921	Shanghai Gushu Liutongzhu 上海古書流通処
(3) (2)	1980	Chûbun 中文 press, Tokyo, Japan
(4)Yi Jia Tang ed.	1936	SWYSG, Shanghai, China
(5)*Wan Wei Bie Zang*	1981	SWYSG, Taiwan
(6)*Yong Le Da Dian*	1959-	ZHSJ, Beijing, China

1764

Born in Yizheng (now Yizheng district, Yangzhou, Jiang-Su province 江蘇省楊州儀征, grand son of Yutang 玉堂 who was Hunan Canjiang 湖南参將 (Major General of Hunan province).

1786

Obtained the Juren 舉人 (master degree).

1789

On 25th Apr., obtain the Jinshi 進士 (doctoral degree) and was appointed Hanlinyuan Shujishi 翰林院庶吉士 (Fellow of Royal Academy).

1790

Became Hanlinyuan Bianxiu 翰林院編修 (Vice Editor of Royal Academy, 7th grade 正七品).

1791

On 3rd Feb., obtained good result of the examination of Hanlinyuan, became Xiao-Linshi 少詹事 (Associate Director, 4th grade 正四品).

Wrote the *Zhang Heng Tian Xiang Fu* 張衡天象賦 (Verse of astronomer Zhang Han).

On 24th Oct., became Linshi 詹事 (Vice Director, 3rd grade 正三品).

1793

On 23rd June, became Shandong Tiduxuezheng 山東提督學政 (Minister of Educational Department of Shandong province).

1795

On 28th Aug., became Zhejiang Tiduxuezheng 浙江提督學政 (Minister of Educational Department of Zhejiang province).

12th Sep. 1795, became Neike Xueshi 内閣學士 (Deputy Minister of Cabinet, 2nd grade 從二品) and Libu Shilang 禮部侍郎 (Vice Minister of Personal Department).

Started to edit the *Chou Ren Zhuan* 疇人傳 (Bibliography of Mathematicians and Astronomers).

1797

Edited the *Jing Ji Suan Gu* 經籍纂詁 (Dictionary of Old Literature), his chief assistant was Zang Yong 臧庸.

1798

On 16th July, became Bingbu You-shilang 兵部右侍郎 (Vice Minister of Defence Department).

On 18th July, became Libu You-Shilang 禮部右侍郎 (Vice Minister of Personnel Department).

Drafted the preface of the *Zhong Ke Ce Yuan Hai Jing Xi Cao* 重刻測圓海細草 (Commentary of *Ce Yuan Hai Jing*) written by Li Rui 李銳.

1799

Completed the *Chou Ren Zhuan*.

On 18th Jan., became Libu Zuo-Shilang 禮部左侍郎 (First Vice Minister of Personal Department).

On 2nd Mar. 1799, became Hebu Zuo-Shilang 戸部左侍郎 (First Vice Minister of Civil Administration Department).

On 6th Mar. 1799, became Huishi Fu-Kaoguan 會試副考官 (Vice Examiner of doctoral degree), Mo Yuchou 莫與儔 obtained the doctoral degree.

In Sep., became Guozijian Suanxue 國子監算學 (professor of mathematics).

1800

On 8th Jan., became Zhejiang Xunfu 浙江巡撫 (Governor of Zhejiang province, 2nd grade 正二品).

Founded Gujing Jingshe 詁經精舍 (School of Classical Literature), where Wang Chang 王昶 and Sun Xingyan 孫星衍 taught literature, astronomy, geometry and mathematics.

1805

Edited the *Shi-San Jing Jiao Kan Ji* 十三經校勘記 (Study of Thirteen Classics).

On 23rd intercalary 6th month, resigned his post.

1 8 0 6

In Nov., became Henan Anshi 河南按事 (Chancellor of Henan province).

1 8 0 7

Published the *Si Ku Wei Shou Shu Mu Ti Yao* 四庫未收書目提要 (Catalogue of the books not collected in the *Si Ku Quan Shu* (Catalogue of the *Wan Wei Bie Zang* 宛委別藏)), and sent it to the emperor. His assistants were Bao Yanbo鮑延博 and He Yuanxi 何元錫.

On 29th Nov., became Bingbu You-Shilang.

On 16th Dec., became Zhejiang Xunfu.

1 8 0 9

On 20th Aug., dispatched Liu Fenggao 劉鳳誥 who was a Zhejiang Tiduxuezheng (Minister of the Educational department of Zhejiang province) to be a Ningbo Xiangshi Zhukao 寧波鄉試主考 (Chief Examiner of master degrees for the Ningbo Examination). Dishonesty occurred, and so Ruan Yuan was relegated to the rank of Hanlin Xueshi 翰林學士 (Scholar of Royal Academy).

1 8 1 0

Became Hanlinyuan Bianxiu and Guoshiguan Tidiao 國史館提調 (Researcher of Historical Library). Copied questions of mathematics from the *Yong Le Da Dian*.

In Apr., became Shijiang Xueshi 侍講學士 (Associate Director of Hanlinyuan) and Guoshiguan Zongsuan 國史館総纂 (Director of Historical Library).

Edited the *Ru Lin Zhuan* 儒林傳 (Bibliography of Confucians).

In Aug., Li Rui found the manuscript of *Xu Gu Zhai Qi Suan Fa* in Li Huang's 李潢 house.

In Sep., became Rijiangqiguan 日講起官 (Historian of the emperor).

In Nov., became Shijing Jiaokanguan 石經校勘官 (Researcher of Arts on the Stone).

1 8 1 1

In July, became Xiao-Linshi.

On 10th Dec., became Neike Xueshi and Libu Shilang.

1 8 1 2

On 7th May, became Gongbu You-Shilang 工部右侍郎 (Vice Minister of Department of Works).

On 14th Aug., became Caoyun Zongdu 漕運総督 (Minister of Shipping Trade in Huaian 淮安, 2nd grade 正二品).

1 8 1 4

On 12th Mar., became Jiangxi Xunfu 江西巡撫 (Governor of Jiangxi province) and Taizi Shaobao 太子少保 (Vice Instractor of Prince).

Jiang Fan 江藩 proofread the manuscript of *Xu Gu Zhai Qi Suan Fa*. Then Ma Yigen 馬以艮 recalculated the result, published *Zhi Bu Zu Zhai Cong Shu* 知不足齋叢書 collection.

He Yuanxi 何元錫 copied the *Yang Hui Suan Fa* which Mao Jin 毛晋 copied and Huang Pilie 黄丕烈 owned that time, then it became the *Yang Shi Suan Fa* 楊氏算法 of *Wan Wei Bie Zang* 宛委別藏.

On 10th Oct., Li Rui drafted the preface of the *Yang Hui Suan Fa*.

Edited the *Shi-San Jing Chu Shu* 十三經注疏 (Commentary of Thirteen Classics).

1 8 1 6

On intercalary 6th month, became Henan Xunfu 河南巡撫 (Governer of Henan province).

On 7th Nov., became Hu-Guang Zongdu 湖廣総督 (Viceroy of Hezhou and Guangzhou, 2nd grade正二品).

Published the *Shi-San Jing Chu Shu*.

1 8 1 7

On 12th Sep., became Liang-Guang Zongdu 兩廣総督 (Grand Governer of two Guangzhou)

Edited the *Guan Dong Tong Shi* 廣東通志 (Guangdong Gazetteer).

1 8 2 0

Founded Xuehaitang 學海堂 School.

Edited the *Huang Qin Jing Jie* 皇清經解 (Imperial Elucidations of the Classics).

1822

On 9th Feb., became Auhaiguan Jiandu 粵海關監督 (Director of Auhaiguan) and Tiduxuezheng.

1826

On 17th May, became Yun-Gui Zongdu 雲貴総督 (Grand Governer of Yunnan and Guizhou).

1832

On 20th Aug., became Xiebian Da-xueshi 協辨大學士 (Vice Minister of Cabinet, 1st grade 從一品).

1833

On 6th Mar., became Huishi FuKaoguan (Deputy Examiner at the third degree Jinshi Examination).

1835

On 25th Feb., became Tirenge Da-xueshi 體仁閣大學士 (Tirenge Cabinet Minister, 1st grade 正一品).

1836

On 21st Apr., became Dianshi Dujuanguan 殿試讀卷官 (Examiner of the last examination of doctoral degree).

On 5th May, bacame Hanlinguan Shujishi Jiaoxi 翰林館庶吉士教習 (Supervisor for Fellows of Royal Academy).

1838

Retired. Became Taizi Taibao 太子太保 (Grand Guardian of the Crown Prince).

1839

Drafted the preface of the *Suan Xue Qi Meng* 算學啓蒙 (Introduction to Mathematical Studies) in Yangzhou 揚州.

1840

Sun Jinchang 宋景昌 proofread the manuscript of Li Rui's *Yang Hui Suan Fa*.

1842

Published *Yi Jia Tang Cong Shu* version of the *Yang Hui Suan Fa*.

1846

On 29th June, became Taizhuan 太傳 (Grand Tutor).

1849

Died, the posthumous name was Wenda 文達.

Li Rui's biography (see Yan Dunjie, 1990)

1768

On 8th Dec., born in Yuanhe 元和 (now Suzhou, Jiangsu province 江蘇省蘇州市). His other names are Shangzhi 尚之 and Sixiang 四香.

1784

Studied the *Suan Fa Tong Zong* 算法統宗 (Systematic Treatise on Arithmetic) at primary school.

1788

Obtained the Xiucai 秀才 (Bachelor degree), became Shengyuan 生員 (graduate student) of Yuanhe Xianxue 元和縣學 (School of Yuanhe province).

1789

Studied mathematics from Qian Daxin 錢大昕 at Ziyang Shuyuan 紫陽書院 (Ziy School).

1790

Zhao Xun 焦循 sent the *Qun Jing Gong Shi Tu* 羣經宮室圖 (Figures of Palace and Rooms in Many Classics) to Li Rui.

1791

Wrote a postscript of the *San Tong Li Yan Qian* 三統曆衍鈐 (Commentary of the *San Tong Li*, by Qian Daxin).

1795

Was invited to Hangzhou 杭州 by Ruan Yuan 阮元, studied mathematics.

1797

Wrote the *Zhong Ke Ce Yuan Hai Jing Xi Cao* 重刻測圓海細草 (Commentary of *Ce Yuan Hai Jing*).

Commented on the *Yi Gu Yan Duan* 益古演段 (New Steps in Computation).

Studied the *Shu Shu Jiu Zhang* 數書九章 (Mathematical Treatise in Nine Sections).

1798

Wrote the *Hu Shi Suan Shu Xi Cao* 弧矢算術細草 (Commentry of Calculations of Arcs and Segments).

Zhong Ke Ce Yuan Hai Jing Xi Cao was published.

1799

Wrote the *Ri Fa Shuo Yu Qiang Ruo Kao* 日法朔餘強弱考 (Studies of Denominator of Tropical Year).

Chou Ren Zhuan 疇人傳 (Bibliographies of Chinese Mathematicians and Astronomers) was published.

1800

Studied mathematics with Jiao Xun in Hangzhou.

1802

His wife (name unknown) died.

Wrote a postscript of the *Qi Gu Suan Jing Xi Cao* 緝古算經細草 (Commentary of *Qi Gu Suan Jing*, by Zhang Dunren 張敦仁).

1806

Wrote the *Gou Gu Suan Shu Xi Cao* 句股算術細草 (Commentary of Studies of Traiangles).

1807

The *Gou Gu Suan Shu Xi Cao* was published.

1810

Went to Beijing to take the master degree examination (but failed), stayed at Li Huang's 李潢 house and found *Yang Hui Suan Fa* 楊輝算法 (Yang Hui's Method of Computation) there.

1813

Wrote the *Kai Fang Shuo* 開方説 (Theory of Equations of Higher Degree).

1814

The *Yang Hui Suan Fa* (*Zhi Bu Zu Zhai Cong Shu* ed.) was published, and he wrote the postscript.

1816

Studied the *Si Yuan Yu Jian* 四元玉鑑 (Precious Mirror of the Four

Elements).

1817

On 30th June, Died.

1823

Li Shi Suan Xue Yi Shu 李氏算學遺書 (Mathematical Remains of Mr. Li) was published.

II: THE CONCEPTION AND EXTENTION OF METHOD FOR MAKING MAGIC SQUARES

(1) Introduction

(a) Magic squares as "mathematics"

Seki Kôwa and his school's mathematicians studied the *Yang Hui Suan Fa* 楊輝算法 (Yang Hui's Method of Computation). But the other Japanese mathematicians did not have access to it. The *Yang Hui Suan Fa* is the best work for studying magic squares. Therefore, Seki Kôwa's work on magic squares must be influenced by the *Yang Hui Suan Fa*. In this chapter, I will analyse both the works of Yang Hui and Seki Kôwa, and will consider how Seki Kôwa applied the works of Yang Hui to his work.

Magic squares involve arranging integers in a square grid such that the sums of individual rows and columns are the same. Also all integers from 1 to n^2 (for an n degree magic square) are used uniquely. For example, Fig. 1-1 is the most simple magic square, and the order is three, thus we call Fig. 1-1 three degree magic square hereafter. Therefore, Fig. 1-2 is a four degree magic square, which is in the picture "Melancholia" by Dürer (1471-1528) in 1514.

4	9	2
3	5	7
8	1	6

"Luo Shu"

Fig. 1-1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

"Melancholia"

Fig. 1-2

Today, the study of magic squares is not regarded as a subject of mathematics. There is no mathematical school which teaches the subject of magic squares. However many mathematicians in China and Japan had studied it. The study of magic squares was "mathematics" for mathematicians in that age. Japanese mathematicians studied the Chinese method of making magic squares as "normal science" and applied the works from China to their own works. Henceforth I am going to trace the development of studies of magic squares in this chapter, in addition to Seki Kôwa's works.

(b) History of Chinese magic squares

Chinese philosophy was strongly coloured by mathematics. Since the magic square has mysterious mathematical character, so it was connected with philosophy in the Song 宋 dynasty [*1]. The term used was Kongzi's 孔子 (B.C. 551-479), "Luo Shu" 洛書 (the writing of Luo river, three degree (three by three) magic square)(see section II-2-c).

Therefore the magic square was studied in two fields, mathematics and philosophy. I will consider the point of contact between Chinese philosophy and Chinese mathematics as deeply as possible, however the theme of this thesis is a study of mathematics in the Song dynasty and in Japan. I will focus on the magic square as "mathematics". Then we will consider the weak points of Chinese studies by researching Japanese studies.

The magic square was usually thought of as a religious, divine or philosophical matter. There are a few studies from a mathematical point of view. Scientific studies of magic square were started by Andrews (fl. 1908) [*2]; his book was one of the oldest and most complete works. Then Li Yan 李儼 (1892-1963) [*3] using his theories, appraised the magic square as Chinese mathematics. Cammann's (fl.1963) [*4] work is one of the best studies in English. Lam Lay-Yong 藍麗蓉 (fl.1977) [*5] translated the *Yang Hui Suan Fa* into English and discussed magic squares. *Yang Hui Suan Fa* is the best study of magic squares thus some of her works are studies of magic squares. We must also note the works of J. Major (fl.1976) [*6] and Ho Peng-Yoke (b.1926) [*7], since their approach was based on Chinese philosopy, and is of great use in designing magic squares.

In Japan, most historians of Japanese mathematics studied magic squares. Mikami Yoshio 三上義夫 (1875-1950) [*8] concluded various studies of Chinese and Japanese magic squares. Katô Heizaemon 加藤平左エ門 (fl.1956) [*9] composed magic squares in Japan and China, and Abe Gakuhô 阿部楽方 (fl.1983) [10] is a specialist on magic squares. Fujiwara Shôzaburô's 藤原松三郎 (1881-1946)

complete work[11] is also one of the best studies in this field.

I will use these works and reconsider studies of magic squares, then consider how Seki Kôwa studied magic squares from Chinese mathematics.

(2) Before the *Yang Hui Suan Fa*

(a) Characters of "Luo Shu"; three degree magic square

Magic squares are that the sums of individual rows and columns are the same. With only this condition, magic squares are mysterious. Moreover the simplest magic square in China, "Luo Shu" embodied Chinese philosophy. In Zhang Huang's 章潢 (fl. 1562) explanation, chapter 1 of the *Tu Shu Bian* 圖書編 (On Maps and Books of Encyclopeadia), there are two more conditions as follows;

> In "Luo Shu", "Yang" 陽 (odd numbers) are on four sides, and "Yin" 陰 (even numbers) are on four square. Five is the pivotal number, and sums of numbers symmetrical (about the pivot) are ten, excepting the number at the centre. That is to say, one-nine, three-seven, two-eight, four-six, are all sums of verticals and horizontals. According to the idea of "Wu Xing" 五行 (Five Phases), however, this order is the reverse of the Heaven, i.e. one-six of Water conquers two-seven of Fire, two-seven of Fire conquers four-nine of Metal, four-nine of Metal conquers three-eight of Wood, three-eight of Wood conquers five of Soil on centre, moving in a counter-clockwise direction as in Heaven. [12]

That is to say, he investigated three characteristics, these are

(1) Odd numbers are on the edge, even numbers are in the corners,

(2) Sum of symmetrical positions are the same, therefore the sum of rows and columns are the same,

(3) This order is the Principle of Mutual Conquest of the Five Phases [13].

TABLE 1 CHARACTERISTICS OF "LUO SHU" BY ZHANG HUANG

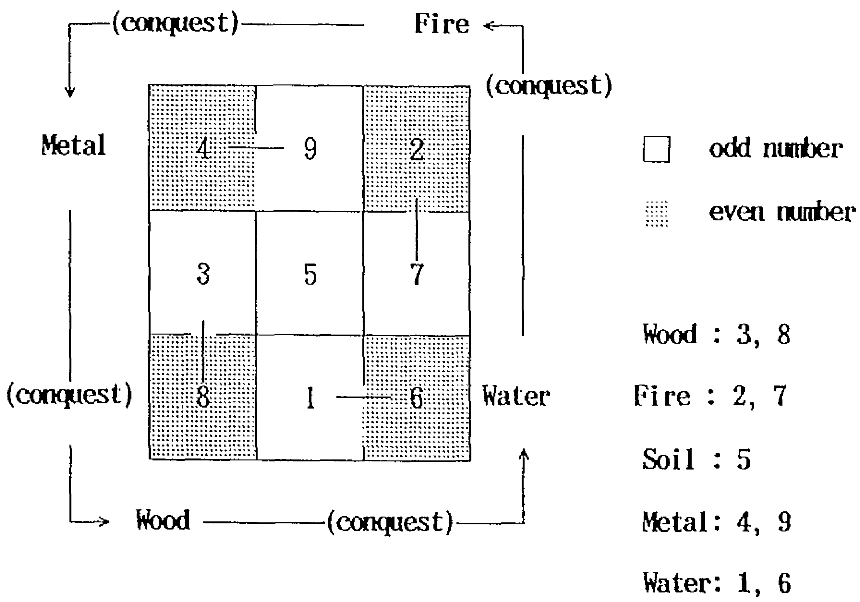


Fig. 2-1

(b) Origin of "Luo Shu"

This order of "Luo Shu" is a counter-clockwise direction as in Heaven, according to the Principle of Mutual Conquest of the Five Phases [14]. Therefore one of the origins of "Luo Shu" came from observing the Heavens.

Another alleged origin is from the ancient social system. According to Zhu Xi 朱熹 (or Zhuzi 朱子 (1130-1200)) [15], this mysterious magic square was designed according to an ancient system of land ownership.

> Zhuzi said, "Ming Tang" 明堂 (Hall of Light, it was a three degree magic square, see section Ⅱ-1-c) was discussed but not with clarity. It has nine rooms according to the law of "Jing Tian Zhi" 井田制 (Well Field System, the nine square system of land ownership in China's early society)" [16].

This opinion may be correct, since "Luo Shu" resembles "Jing Tian Zhi". Thus it was thought that "Luo Shu" symbolises not only Heaven but also Earth. Therefore it was thought that "Luo Shu" embodies Chinese micro-cosmos.

4	9	2
3	5	7
8	1	6

Fig. 2-2 "Luo Shu"

a	b	c
d	░	e
f	g	h

a-h: private field

░ : common field

Fig. 2-3 "Jing Tian Zhi"

(c) "He Tu" and "Luo Shu" [17]

The oldest reference to the magic square in China is in the chapter *Xi Ci Zhuan Shang* 繫辭傳上 (Commentary on the Appended Judgments, part Ⅰ) of the *Yi Jing* 易經 (the Book of Changes). It was written by Kongzi [18], according to the *Shi Ji* 史記 (Records of the Historiographer). But, according to the *Yi Tong Zi Wen* 易童子問 (Pupils Question about "Change") which was written by Ouyang Xiu 歐陽脩 (1007-1072) [19] in the Song dynasty, the chapter of *Xi Ci Zhuan Shang* was not written by Kongzi [20]. It was probably written by Kongzi's students in the Warring States period. Anyhow it was the oldest reference in China. It states:

> The Yellow River (黄) 河 brought forth a map ("Tu" 圖) and Luo 洛 River brought forth a writing ("Shu" 書): the sage took these as models. [21]

We cannot understand what "Tu" and "Shu" were from this material alone. But according to the commentary of Kong Anguo 孔安國 (2-1c B.C.) [22] in the Chapter *Hong Fan Zhuan* 洪範傳 (Section Law of Heaven and Earth) in the *Shang Shu* 尚書 (Historical Classic) as follows:

> Heaven gave King Yu 禹 the writing ("Shu" 書) of nine numbers on the back of a spirit turtle in Luo River. King Yu understood the meaning, he conducted the affairs of state by "Jiu Lei" 九類 (Nine Parts). It became the law of Heaven and Earth. [23]

Therefore we know "Luo Shu" was something to be divided into nine parts [24].

In 1977, the explanation of "Nine" was uncovered. "Taiyi Jiu-Gong Zhan-Pan" 太乙九宮占盤 (The Diviner's Board of Nine Palaces in Heaven) which was made in

B.C. 173 was unearthed from the tomb of Ruyin-hou 汝陰侯 (Marquis Ruyin) of the Western Han 西漢 dynasty, at Shuangudui, Fuyang district, Anhui province 安徽省阜陽縣雙古堆[25]. It is laid out as follows;

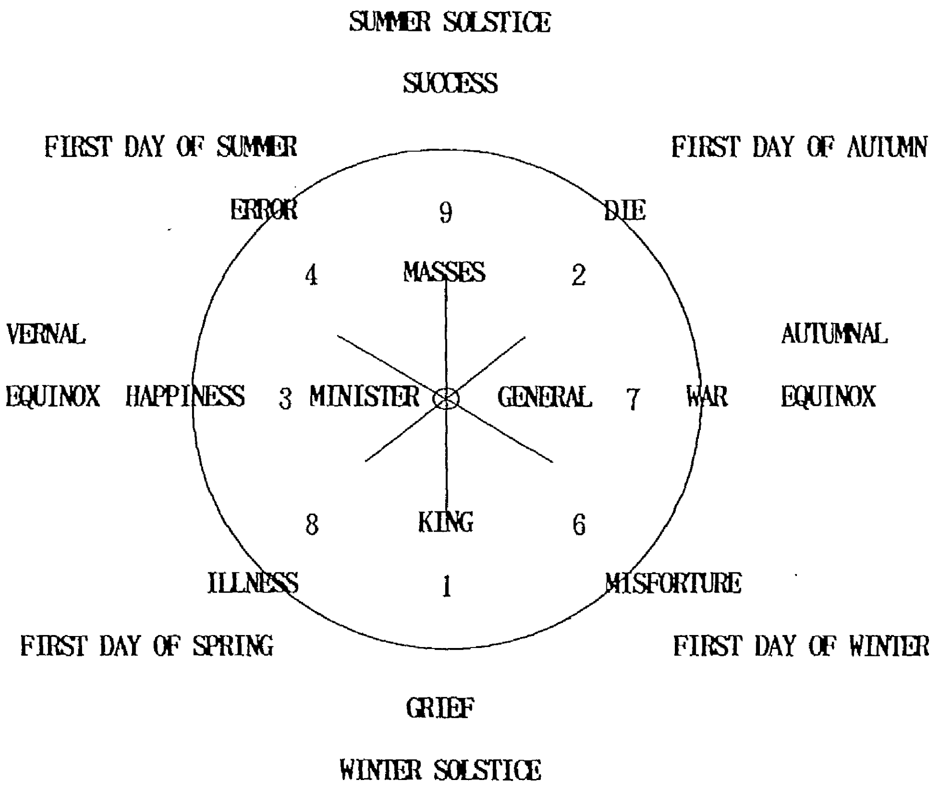


Fig.2-4 "Tai-Yi Jiu-Gong Zhan-Pan"

The numbers on this board were arranged as a magic square.

4	9	2
3		7
8	1	6

4	9	2
3	5	7
8	1	6

"Luo Shu"

Fig. 2-5 "Nine Palaces" and "Luo Shu"

Thus we know that "Luo Shu" was "Nine", and was a three degree magic square.

The first historical material, sec.66 [26] of *Sheng De* 盛德 (Fullness of Power) section, chapter 8 of the *Da Dai Li Ji* 大戴禮記 (Record of Rites Compiled by Dai the Elder)[27] was written as follows:

The "Ming Tang" (Hall of Light) was founded in the old days. It had nine rooms, two-nine-four, seven-five-three, six-one-eight.

4	9	2
3	5	7
8	1	6

Fig.2-6 (the original book does not have figures)

"Ming Tang" was one of the real palaces (28). On the other hand, "Jiu Gong" (Nine Palace) was a ficticious palace in Heaven(29).

Again, in a sub-section "Jiu Gong Suan" 九宮算 of the *Shu Shu Ji Yi* 數術記遺 (Memoir on some Traditions of Mathematical Art)(30), a mathematical text of the public school in the Tang 唐 dynasty, another method is described for composing the magic square. The magic square was likened to the body of man.

> "Jiu Gong" (Nine Palace): to make "Jian" 肩 (shoulders), use two and four, to make "Zu" 足 (legs), use six and eight, on "Zuo" 左 (left) is three, on "You" 右 (right) is seven, put (戴) nine as the head, put (履) one for the shoes, five is in "Zongyang" (the centre).
> Five Phases of numbers are arranged by this rule, Five Phases are described at the previous section. (31)

4	9	2	肩	戴	肩
3	5	7	左	中央	右
8	1	6	足	履	足

Fig.2-7 (the original book does not have figures)

From these evidences, we can conclude that the three degree magic square was called "Jiu Gong" or "Ming Tang" before the Song dynasty. Although the term of

"Luo Shu" had been known, it was not known what "Luo Shu" meant exactly.

We must also note the term of "Jiu Gong" and "Ming Tang". That is, "nine" meant "all" or "the best". I include these evidences to support any proposition that "Luo Shu" had been thought of as the embodiment of the "micro-cosmos".

Then philosophers in the Song dynasty tried to connect the magic square and philosophy, rather than idea of Confucius. So they called the magic square "Luo Shu", the term used by Confucius. It has been clearly called "Luo Shu" since the Song dynasty, but sometimes the three degree magic square has been called the "He Tu" 河圖 (the map in the Yellow River), because there was no established theory.

Firstly Liu Mu 劉牧 (1011-1064)[32] drew the picture of "He Tu" and "Luo Shu". However he thought the three degree magic square was "Ten", "Nine" referred to a magic circle. That is to say, "He Tu" means the magic square. In Chapter 3 of the *Yi Shu Gou Yin Tu* 易數鉤隱圖 (The Hidden Number-Diagrams in the Book of Changes) states;

Section 49: "He Tu" 河圖;

4	9	2
3	5	7
8	1	6

Fig. 2-8 after *SKQS*

Five is main, to make knees, use six and eight, to make shoulders, use two and four, on the left is three, on the right is seven, put nine as the head, put one for the shoes.

Section 53: "Luo Shu" Wu Xing Sheng Shu 洛書五行生數 (Producing Numbers of Five Elements in "Luo Shu");

Section 54: "Luo Shu" Wu Xing Cheng Shu 洛書五行成數 (Produced Numbers of Five Elements in "Luo Shu");

	2	
3	5	4
	1	

Fig. 2-9

	7	
8	10	9
	6	

Fig. 2-10

(*SKQS*)

		7		
		2		
8	3	5	4	9
		1		
		6		

Fig. 2-11

("He Tu" now)

> One is water, two is fire, three is wood, four is metal and five is soil. [33]

He insisted that "Ten", the number at the centre in Fig. 2-10, meant "Luo Shu". Because "Luo Shu" was described after "He Tu" in the *Yi Jing* (the Book of Changes) and "Yang" 陽 precedes "Yin" 陰, thus "Luo Shu" must be "Yin" and "He Tu" must be "Yang". All matters (including ideas) has the character of "Yin" or "Yang". In the case of integral numbers, "Yin" is even and "Yang" is odd. Therefore, "Luo Shu" which is "Yin" must be even, i.e., "Ten". It was natural that he thought "Luo Shu" was "Ten". Liu Mu thought very logically.

However, from the historical evidence given above, it was clear that "Nine" meant a magic square. Zhu Xi criticized Liu Mu's views in chapter 1 of the *Yi Xue Qi Meng* 易學啓蒙 (Introduction to the Study of the Book of Changes);

> Guan Ziming 關子明 (13c) [34] pointed out that the paragraph of "He Tu"; front is seven, rear is six, left is eight, right is nine, the paragraph of "Luo Shu"; front is nine, rear is one, left is three, right is seven, left front is four, right front is two, left rear is eight, right rear is six" [35].

Zhu Xi's opinion [36] was based on the opinions of many earlier scholars, that is, it was a historical view. He argued in detail that "He Tu" means Fig.2-11 and "Luo Shu" means Fig.2-8 which is a magic square.

Because it was thought that these classics were absolutely true in China, Zhu Xi's opinion became the established theory.

(d) Making magic squares and the concept of "Changes"

The philosophical view and the historical view were different. It was proved that magic squares were not a systematic part of Chinese philosophy in early ages. They had been connected with "Luo Shu" since the Song dynasty.

If so, why did the magic square become "Luo Shu"? There was no historical material relevant to this question because philosophers in the Song dynasty imagined it had been an ancient matter.

I wonder whether the reason was that the method of making magic squares was solved in this age, and it was thought of as the concept of "Changes".

The idea of "Yin Yang" "was planned by the changing of day and night, sun and moon. It was the doctrine that all phenomena in Heaven and Earth could be explained by the theory of changing 'Yin' and 'Yang', which were divided into 2, 4, 8, 16 and more, and that they were confronting each other. (37) ."

On the other hand, magic squares are usually made by the exchanging method, as follows.

A square tabular arrangement of numbers in order is called a "natural square". The sums of the lines have regular differences. For example, in Fig. 2-12: the average of nine numbers is five, and one line has three numbers, so the sum would be fifteen. But the lines whose sums are fifteen are only the middle rows and columns. In Fig. 2-13, the sums of the lines would be 65.

1	2	3	-9
4	5	6	0
7	8	9	+9
-3	0	+3	

Fig.2-12

1	2	3	4	5	-50
6	7	8	9	10	-25
11	12	13	14	15	0
16	17	18	19	20	+25
21	22	23	24	25	+50
-10	-5	0	+5	+10	

Fig. 2-13

It seems obvious, therefore, that by interchanging numbers in some regular way, we should be able to obtain a magic square, i.e., a square array of numbers in which rows and columns have a constant sum (see section II-3-b and II-3-d).

Interchanging by a particular theory, we could obtain the mysterious square. I wonder if it embodies the concept of interchanging "Yin" and "Yang". Otherwise the magic square had not been studied deeply during the Song dynasty.

Though the interchanging of "Yin" and "Yang" is a hypothesis, it is a fact that philosophers in the Song dynasty tried to explain the idea using magic squares. For example, Ding Yidong 丁易東 (13c) explained the number of "Da Yan" 大衍 using magic squares. "Da Yan" was the principal concept in the *Yi Jing*, however examining the many theories that were advanced, there was no concensus. The classic reference to "Da Yan" is;

> The number of the total ("Da Yan") is fifty. Of these, forty-nine are used (38).

In Chapter 2 of *Da Yan Suo Yin* 大衍索隱 (Studies about Da Yan) written by Ding Yidong:

> The case of "Luo Shu", the sum of 1 and 9, 2 and 8, 3 and 7, 4 and 6 is 10. The 5, in the centre, was added to itself so it also

becomes 10. Although the name is "Jiu Gong" (Nine Palace), the value is 10.[39]

That is to say, the sums of symmetrical positions are one more than the number of items in the magic square. Therefore the number of "Da Yan" is also one more than the number 49 (see section Ⅱ-3-f).

(3) Concerning *Yang Hui Suan Fa* (Yang Hui's Method of Computation)

(a) Yang Hui's term for magic squares

Although it was published after Zhu Xi had proposed his theory, *Xu Gu Zhai Qi Suan Fa* 續古摘奇算法 (Continuation of Ancient Mathematical Methods for Elucidating the Strange) follows Liu Mu's theory. In other words, the magic square is named "He Tu" and the magic circle is named "Luo Shu" in the *Xu Gu Zhai Qi Suan Fa*.

This mistake is probably not Yang Hui's for the following reasons.

The Korean edition of *Yang Hui Suan Fa* draws the figure of "'Luo Shu'" first and this figure is not a magic square, but a magic circle. Then "'He Tu'", which is the three degree magic square in this edition, is drawn next.

But "He Tu" and "Luo Shu" are fixed in their order. We use the phrase "Tu Shu" 圖書, to means books in modern Chinese and Japanese, but never use "Shu Tu", so the figure of "He Tu" must be drawn before "Luo Shu". The order of two terms in the Korean edition is very strange. Yang Hui very probably understood that the first must be the magic circle, and next is the magic square.

If so, did Yang Hui introduce earlier scholars' opinions? In many other fields, *Yang Hui Suan Fa* collects theories and methods from many other books, thus he again collected the former magic squares. However, the magic squares of the *Yang Hui Suan Fa* are systematic, and are made by one person, i.e., Yang Hui himself (40). Therefore, this possibility is not strong.

Yang Hui's knowledge of the *Yi Jing* (the Book of Changes) is quite detailed; for instance, he explains "Hu Huan Shu" 互換術 (proportion) in question 5, chapter 2 of the *Tian Mu Bi Lei Cheng Chu Jie Fa* 田畝比類乘除捷法 (Practical Rules of Arithmetic for Surveying);

A few examples are now selected and illustrated with detailed diagrams for the benefit of the reader. The others can be easily

> understood by to "continue and go further and add to the situation all their transitions" 引而伸之，觸類而長. Similar examples are far too many and do not require individual discussions. [41]

This quotation, "continue and go further and add to the situation all their transitions", is from section Xi Ci Zhuan Shan of the *Yi Jing* [42]. Yang Hui was educated as a mandarin, and had wide general knowledge.

Moreover, there is a strong possibility that Yang Hui studied *Yi Xue Qi Meng* 易學啓蒙 (Introduction to the Study of the Book of Changes), because Yang Hui used the same term for magic squares as Zhu Xi, i.e., "Zong Heng Tu"縱横圖 (Vertical and Horizontal Figure)

> The vertical and horizontal 縱横 of "Luo Shu" are fifteen [43].

Zhu Xi generalized this characteristic of the "Luo Shu" to the sum of colmuns and rows being the same. Then Yang Hui used this term as the general name of magic squares.

Yang Hui had very probably studied Zhu Xi's works. Therefore I conclude that this mistake is a misprint, probably made by the editors of the Korean version.

Yang Hui describes how to make the three degree magic square ("Luo Shu") and four degree magic square, and then continues to give figures until the ten degree magic square without any explanation. So we will consider Yang Hui's description of how to make the basic magic squares, and then try to explain how he made higher degree magic squares.

(b) Composing the three degree magic square ("Luo Shu")

Yang Hui made a verse for making the "Luo Shu". Making verses was the popular way to discribe mathematical methods in that age, this verse is quoted in the *Suan Fa Tong Zong* 算法統宗 (Systematic Treatise on Arithmetic) in the Ming dynasty. It is;

Arrange the nine numbers diagonally (Fig.3-1).

Interchange the top number and the bottom number (Fig.3-2).

Interchange the left number and the right number (Fig.3-3).

Four numbers on the corner are projected outwards (Fig.3-4)[44].

1	9	9	4 9 2
4 2	4 2	4 2	
7 5 3	7 5 3	3 5 7	3 5 7
8 6	8 6	8 6	
9	1	1	8 1 6
Fig.3-1	Fig.3-2	Fig.3-3	Fig.3-4

This is the "arrangement method"[45], all odd degree magic squares can be made by this method. Therefore he probably suceeded in making all odd degree magic squares. However Yang Hui did not use it generally [46] but only used it in part to make a five degree magic square (see section Ⅱ-3-d). He should have tried other methods, since new magic squares could not keep the characteristic (3) of Zhang Huang, the counter-clockwise order.

Anyhow we should call this "Yang Hui's method".

(c) Composing four degree magic sqdares

Yang Hui explains two methods for making four degree magic squares.

The first, "Zong Shu" 総術 (general method) or "Huan Yi Shu" 換易術 (47) (Interchange Method) is:

Arrange the sixteen numbers in four columns (Fig.3-5)

First interchange the numbers in the four corners; [interchange 1 and 16, 4 and 13.] (Fig.3-6)

Similarly interchange the numbers in the four inner corners; [interchange 6 and 11, 7 and 10.] (Fig.3-7)

The horizontal, vertical and diagonal sums are all 34.

The small numbers are thus balanced by this interchange. This can also be regarded as a general method. (48)

13	9	5	1
14	10	6	2
15	11	7	3
16	12	8	4

Fig.3-5

4	9	5	16
14	10	6	2
15	11	7	4
1	12	8	13

Fig.3-6

4	9	5	16
14	7	11	2
15	6	10	4
1	12	8	13

Fig.3-7

This method is the "symmetrical interchange method" (49) , it can be used for all 4 n degree magic squares. The magic square produced by this method is "Hua Shi-Liu Tu Yin Tu" 花十六圖陰圖 (Yin Flower Sixteen Figure).

The second method, "Qiu Deng Shu" 求等術 (method of finding equal sums) is as follows:

Divide the numbers into two columns [1, 16; 2, 15; 3, 14; 4, 13; 5, 12; 6, 11; 7, 10; 8, 9] so that all pairs of numbers have equal sums [17] (Fig.3-8).

First arrange these numbers into four columns so that the (sum of the horizontal) rows have equal sums [34] (Fig.3-9).

Next, without changing the amount for each row, arrange the numbers in the columns so that all columns have the original sum [34] (Fig.3-10).

Once this rule is fixed there should be no doubt that the required result cannot be obtained. [50]

16	1
15	2
14	3
13	4
12	5
11	6
10	7
9	8

Fig.3-8

12	5	16	1
11	6	15	2
10	7	14	3
9	8	13	4
+8	-8	+24	-24

Fig.3-9

12	5	16	1
6→	11	2←	15
7→	10	3←	14
9	8	13	4

Fig.3-10

However Fig.3-10 is not "Hua Shi-Liu Tu" 花十六圖 (Flower Sixteen Figure). We can make the "Hua Shi-Liu Tu" figure by "Qiu Deng Shu" method, but the sums of the rows, which have already been made equal to thirty-four, are confused again during the process of rearrangement.

The "Huan Yi Shu" method is described before the "Qiu Deng Shu" method. And the "Hua Shi-Liu Tu Yin Tu" figure is made by the "Huan Yi Shu" method. Thus it is doubtful whether the "Hua Shi-Liu Tu" figure is made by the "Qiu Deng Shu" method. But the "Hua Shi-Liu Tu" figure can easily be composed by the improved "Huan Yi Shu" method. Abe Gakuhô explained as follows [51]:

i) Arrange the sixteen numbers in four columns starting from the bottom left (Fig.3-11).

ii) Interchange the left and right halves of figure (Fig.3-12).

iii) Use the "Huan Yi Shu" method (Fig.3-13).

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

Fig. 3-11

15	16	13	14
11	12	9	10
7	8	5	6
3	4	1	2

Fig. 3-12

2	16	13	3
11	5	8	10
7	9	12	6
14	4	1	15

Fig. 3-13

Therefore we conclude that the "Hua Shi-Liu Tu" figure was made by the improved "Huan Yi Shu" method. The magic square by "Qiu Deng Shu" method is not even a "middle level magic square" (52) (see Fig.3-10), so it was not drawn in the *Yang Hui Suan Fa*.

However the "Qiu Deng Shu" method is very useful for higher 4 n degree magic squares, which will be described in the section concerning eight degree magic squares (section II-3-g).

(d) Composing five degree magic squares

First, a five degree magic square in the *Yang Hui Suan Fa* is made using the following procedure [53];

i) Arrange twenty-five items as in Fig. 3-14.

ii) Select the inner three degree square, and arrange it using the "arrangement method" (Fig. 3-15).

iii) Reverse them (Fig. 3-16).

iv) Arrange the other items so that sum of pairs become 26, in the outer stratum (Fig.3-16).

1	2	3	4	5
6	7 —	8—	9	10
11	12—	13—	14	15
16	17—	18—	19	20
21	22	23	24	25

Fig. 3-14

12(4)	19(9)	8(2)
9(3)	13(5)	17(7)
18(8)	7(1)	14(6)

Fig. 3-15

14	7	18
17	13	9
8	19	12

Fig. 3-16

1	23	16	4	21
15	14	7	18	11
24	17	13	9	2
20	8	19	12	6
5	3	10	22	25

Fig. 3-17

This magic square, Fig. 3-17, is "Wu Wu Tu" 五五圖 (Five by Five Figure). This method can be classified as an "arrangement method".

Moreover we must note that Yang Hui arranged items in a stratiform pattern, and the sums of pairs in symmetrical positions (or diagonal positions) are 26, twice the average value [54] .

Yang Hui did not comment on this point, but Cheng Dawai 程大位 (1533-1606) commented clearly in the *Suan Fa Tong Zong*. His term was "Huan Yi Shu" (interchange method) but it was not the same method as *Yang Hui Suan Fa*' s "Huan Yi Shu". That method is more similar to the "Qiu Deng Shu" rather than the "Huan Yi Shu". It is as follows.

First, Put 13 at the center. The circumference consists of three strata. Arrange numbers as follows.

1 – 25, diagonal, outer stratum.
2 – 24, symmetrical, outer stratum.
3 – 23, symmetrical, outer stratum.
4 – 22, symmetrical, outer stratum.
5 – 21, diagonal, outer stratum.
6 – 20, symmetrical, outer stratum.
7 – 19, symmetrical, inner stratum.
8 – 18, diagonal, inner stratum.
9 – 17, symmetrical, inner stratum.
10 – 16, symmetrical, outer stratum.
11 – 15, symmetrical, outer stratum.
12 – 14, diagonal, inner stratum.

5	23	16	4	25	+8
15	14	7	18	11	0
24	17	13	9	2	0
20	8	19	12	6	0
1	3	10	22	21	–8

Fig. 3-18

However, this is not a magic square [55]: the sum of the top row is plus eight, the sum of the bottom row is minus eight[56].

No rule is given for the arrangement method of the perimeter. This was probably done by trial and error, so it would be more difficult for larger squares. However this method became the starting point for studying the magic squares in the Edo period in Japan (see sections II-4). Let us name this method the "stratiform pair method" [57] . This method keeps characteristic (2) of Zhang Huang. However it does not keep character (1), four corners are odd numbers and four sides are even numbers[58].

"Wu Wu Yin Tu" 五五陰圖 (Negative Five by Five Figure) is made with the numbers from 9 to 33. All other magic squares are made by using numbers from 1 upwards, but this is an exception. Of course if each number of a magic square is increased by the same value, a new magic square is created. It is a simple principle, but it is easy to miss, unless care is taken.

If all numbers are decreased by eight, we obtain:

(4)	19	(25)	15	(2)
20	10	5	18	12
(3)	17	(13)	9	(23)
14	8	21	16	6
(24)	11	(1)	7	(22)

Fig. 3-19

So we can notice that the frame of "Wu Wu Yin Tu" figure is "Luo Shu"; the nine numbers with parentheses in Fig.3-19 abide by the order of "Luo Shu". The centre of the bottom row starts from 1, centre position of the magic square is 13 which is the average of all numbers, so the case of "Luo Shu" is 5, the centre of the top row ends in 25, which is the last number of this magic square so "Luo Shu" is 9.

We will call this method in which the frame is made by "Luo Shu" the "Luo Shu frame type method". This method can fix only nine numbers, however it tries to keep characteristic (3) of Zhang Huang, counter clock wise.

Four corners of this magic square are even numbers and four sides are odd numbers [59].

Then the others were arranged using the "stratiform pair method". In this case, some of the positions of symmetry numbers by the "stratiform pair method" are symmetrical by line. For example, 15 and 11, 19 and 7 in Fig.3-19. However there was no general theorem, and Yang Hui arranged the numbers by trial and error, so it is more difficult to make larger magic squares (see section II -3-f).

(e) Composing six degree magic squares [60]

The number six is not prime, so Yang Hui used the principle of "complex magic squares". He divided the magic square under construction into smaller magic squares, then solved these problems.

Six is a multiple of three, which is the degree of "Luo Shu". Therefore Yang Hui divided thirty-six blanks into nine blocks, each block with four blanks. Firstly, he arrange nine blocks as in Fig.3-20 and 3-21, in which each sum of columns and rows 15.

2	3	2	3	2	3
4	1	4	1	4	1
2	3	2	3	2	3
4	1	1	4	4	1
2	3	2	3	2	3
1	4	4	1	1	4
15	15	15	15	15	15

Fig. 3-20

"Liu Liu Tu" 六六圖

(Six by Six Figure)

1	2	4	3	4	1	15
3	4	2	1	2	3	15
1	3	3	4	3	1	15
4	2	1	2	2	4	15
2	3	3	4	1	2	15
4	1	2	1	3	4	15
15	15	15	15	15	15	

Fig. 3-21

"Liu Liu Yin Tu" 六六陰圖

(Yin Six by Six Figure)

And Yang Hui applied "Luo Shu", numbering to each block. For example, the top left block in Fig.3-20 is the position of 4 in "Luo Shu" therefore they become;

2	3	13	22	13= 9× (2-1)+4	22= 9× (3-1)+4
4	1	31	4	31= 9× (4-1)+4	4= 9× (1-1)+4

Other blocks are the same, therefore;

13	22	18	27	11	20
31	4	36	9	29	2
12	21	14	23	16	25
30	3	5	32	34	7
17	26	10	19	15	24
8	35	28	1	6	33

Fig. 3-22

We will called this method of dividing nine blocks and then apply by "Luo Shu" the "Luo Shu block type method"[61]. This method is very useful, but can be used only for 3n degree magic squares.

(f) Composing seven degree magic squares

Seven degree magic squares are named "Yan Shu Tu" 衍數圖 (the Yanshu Number Figure) and "Yan Shu Yin Tu" 衍數陰圖 (the Yin Yanshu Number Figure). These figures require forty-nine items and the number of "(Da) Yan (Zhi) Shu" is forty-nine, thus these are named "Yan Shu Tu" and "Yan Shu Yin Tu". Ding Yidong 丁易東 (13c) gives an interesting explanation as follows:

> Twice the average value of these (see Fig. 3-23) numbers, which is the sum of symmetry numbers, is fifty. However forty-nine numbers are used, which is "Da Yen Zhi Shu" (62).

49 1

48 2

47 3

⋮ ⋮

26 24

25

Fig. 3-23

He connected the method of making magic squares, the "stratiform pair method", and the concept of "Da Yan".

The "Yan Shu Tu" figure is made by the "stratiform pair method", the method of Yang Hui was explained as follows (63) ;

i) Arrange forty-nine numbers as a "natural square" in Fig. 3-24.

ii) Select nine numbers with parentheses (Fig. 3-25).

iii) Using the "arrangement method", arrange them to get Fig.3-26.

iv) Then arrange the circumference using "stratiform pair method" so that the sum of symmetrical pairs becomes 50 (Fig.3-27) and again (Fig.3-28)

49	42	35	28	21	14	7
48	41	34	(17)	20	13	6
47	40	33	26	(19)	12	5
46	(39)	32	(25)	18	(11)	4
45	38	(31)	24	(17)	10	3
44	37	30	(23)	16	9	2
43	36	29	22	15	8	1

Fig. 3-24

		27		
	33		19	
39		25		11
	31		17	
		23		

Fig. 3-25

33	23	19
11	25	39
31	27	17

Fig. 3-26

40	12	14	18	41
37	33	23	19	13
15	11	25	39	35
24	31	27	17	26
9	38	36	32	10

Fig. 3-27

46	8	16	20	29	7	49
3	40	12	14	18	41	47
44	37	33	23	19	13	6
28	15	11	25	39	35	22
5	24	31	27	17	26	45
48	9	38	36	32	10	2
1	42	34	30	21	43	4

Fig. 3-28

The "Yan Shu Yin Tu" figure (Fig. 3-29) is made by the "Luo Shu frame type method" and "stratiform pair method". However all pairs are arranged symmetrically about the centre [64]. Four corners of this magic square are even numbers and four sides are odd numbers, thus this magic square keeps the "Luo Shu" characteristics (1).

(4)	43	40	(49)	16	21	(2)
44	8	33	9	36	15	30
38	19	26	11	27	22	32
(3)	13	5	(25)	45	37	(47)
18	28	23	39	24	31	12
20	35	14	41	17	42	6
(48)	29	34	(1)	10	7	(46)

Fig. 3-29

(g) Composing eight degree magic squares

The eight degree magic squares were named "Yi Shu Tu" 易數圖 (Magic Square of the *Yi Jing* Number Sixty-four) and "Yi Shu Yin Tu" 易數陰圖 (Yin Magic Square of the *Yi Jing* Number Sixty-four). These figures require sixty-four numbers, and sixty-four (2^6) is also the number of "Chong Gua" 重卦 (hexagram) in the *Yi Jing* (the Book of Changes), so the number sixty-four was called "Yi Shu".

These magic squares were made [65] by the "Qiu Deng Shu" method. However Yang Hui rearranged the magic square obtained, since more complex magic squares, are more mysterious. Using the "Qiu Deng Shu" method, we obtain Fig. 3-30 which is already a magic square.

1	64	9	56	17	48	25	40
63→2		55→10		47→18		39→26	
62→3		54→11		46→19		38→27	
4	61	12	53	20	45	28	37
5	60	13	52	21	44	29	36
59→6		51→14		43→22		35→30	
58→7		50→15		42→23		34→31	
8	57	16	49	24	41	32	33
c8	c7	c6	c5	c4	c3	c2	c1

Fig. 3-30

Interchanging columns and then transposing (columns → rows) would still preserve the magic square property. Then reverse four rows.

								direction	columns in Fig. 3-30
57	7	6	60	61	3	2	64	←	c7
16	50	51	13	12	54	55	9	←	c6
24	42	43	21	20	46	47	17	←	c4
33	31	30	36	37	27	26	40	←	c1
25	39	38	28	29	35	34	32	→	c2
48	18	19	45	44	22	23	41	→	c3
56	10	11	53	52	14	15	49	→	c5
1	63	62	4	5	59	58	8	→	c8

Fig. 3-31

Then reverse the right half and left half, to obtain the "Yi Shu Yin Tu".

61	3	2	64	57	7	6	60
12	54	55	9	16	50	51	13
20	46	47	17	24	42	43	21
37	27	26	40	33	31	30	36
29	35	34	32	25	39	38	28
44	22	23	41	48	18	19	45
52	14	15	49	56	10	11	53
5	59	58	8	1	63	62	4

Fig. 3-32

The "Yi Shu Tu" figure is more complex: divide Fig. 3-30 into eight blocks, so that the sum of each block is the same value.

1	64	9	56	17	48	25	40
63	2	55	10	47	18	39	26
62	3	54	11	46	19	38	27
4	61	12	53	20	45	28	37
5	60	13	52	21	44	29	36
59	6	51	14	43	22	35	30
58	7	50	15	42	23	34	31
8	57	16	49	24	41	32	33

①	②	③	④
⑤	⑥	⑦	⑧

Fig. 3-33

Arranging items regularly and making rows by each block, the magic square would be obtained. These patterns are;

1	2	2	1	7	8	8	7
3	4	4	3	5	6	6	5
5	6	6	5	3	4	4	3
7	8	8	7	1	2	2	1
A		B		C		D	

Fig. 3-34

A being the basic order, B is symmetrical by vertical line, C is symmetrical by horizontal line and D is a rotation. Then rows are made from each block as in Fig. 3-34. to produce Fig. 3-35.

									block	pattern
61	4	3	62	2	63	64	1	←	①	A
52	13	14	51	15	50	49	16	←	⑥	C
45	20	19	46	18	47	48	17	←	③	A
36	29	30	35	31	34	33	32	←	⑧	C
5	60	59	6	58	7	8	57	←	⑤	D
12	53	54	11	55	10	9	56	←	②	B
21	44	43	22	42	23	24	41	←	⑦	D
28	37	38	27	39	26	25	40	←	④	B

Fig. 3-35

These figures are quite complex, but their construction is based on the "Qiu Deng Shu" method. The "Qiu Deng Shu" method is the method of making for 2^n degree magic squares.

(h) Composing a nine degree magic square

This magic square whose name is "Jiu Jiu Tu" 九九圖 (Nine by Nine Figure) is made by the "Luo Shu block type method". Nine is three by three, so it is one of the most typical cases when the "Luo Shu block type method" can be used. Yang Hui did not explain how to make a nine degree magic square. However Ding Yidong, who was a contemporary of Yang Hui, made the same figure as the "Jiu Jiu Tu", under the name of "Luo Shu Jiu Shu Cheng" 洛書九數乘 (Nine Times Numbers of "Luo Shu"), and explained it clearly in chapter 2 of the *Da Yan Suo Yin* 大衍索隱 (Studies of Dayan).

> These two figures (he drew two figures but the second is not a magic square) are variations of "Luo Shu". In the first figure (the same as Fig. 3-37): each block has nine items, so there are 81, 9 by 9, items. The order is the same as "Luo Shu". The sum of each column and row is 369. All sums of symmetrical pairs are 82 [66].

Therefore the position of each number is indicated;

9 ×("Luo Shu"'s order in the block − 1) +"Luo Shu"'s order of the block

4	9	2
3	5	7
8	1	6

→

34	79	16
25	43	61
70	7	52

34= 9×(4-1)+7 79= 9×(9-1)+7 16= 9×(2-1)+7

25= 9×(3-1)+7 43= 9×(5-1)+7 61= 9×(7-1)+7

70= 9×(8-1)+7 7= 9×(1-1)+7 52= 9×(6-1)+7

Fig. 3-36

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	28	49	72	9	54	65	10	46
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	3	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

Fig. 3-37 the complete "Jiu Jiu Tu"

(i) Composing a ten degree magic square

This magic square named the "Bai Zi Tu" 百子圖 (One Hundred Figure) is obtained by application of the "Qiu Deng Shu" method [67]. The "Qiu Deng Shu" is the method for making 2^n degree magic squares, but 10 is not 2^n so the method is a little more complex ;

i) Arrange one hundred numbers as Fig.3-38.

ii) Reverse odd horizontal lines, but there are some exceptions which are indicated with parenthesis (Fig.3-39).

100	81	80	61	60	41	40	21	20	1
99	82	79	62	59	42	39	22	19	2
98	83	78	63	58	43	38	23	18	3
97	84	77	64	57	44	37	24	17	4
96	85	76	65	56	45	36	25	16	5
95	86	75	66	55	46	35	26	15	6
94	87	74	67	54	47	34	27	14	7
93	88	73	68	53	48	33	28	13	8
92	89	72	69	52	49	32	29	12	9
91	90	71	70	51	50	31	30	11	10

Fig. 3-38

1	20	21	40	41	60	61	80	81	100
99	82	79	62	59	42	39	22	19	2
3	18	23	38	43	58	63	78	83	98
97	84	77	64	57	44	37	24	17	4
5	16	25	38	45	56	65	76	85	96
95	86	75	66	55	46	35	26	15	6

(14→ 7) (34→27) (54→47) (74→67) (94→87)

(88→93) (68→73) (48→53) (28→33) (8→13)

(12→ 9) (32→29) (52→49) (72→69) (92→89)

91	90	71	70	51	50	31	30	13	10

Fig. 3-39

(j) Conclusion

First of all, let us made a table of Yang Hui's magic squares for the sake of clarity. Yang Hui's six methods for designing magic squares are as follows;

① "Luo Shu" method or Yang Hui's method (arrangement method)

② "Huan Yi Shu" (Interchange method)

③ "Qiu Deng Shu" (Seeking Equality Method)

④ "Luo Shu frame type method"

⑤ "Luo Shu block type method"

⑥ "stratiform pair method"

He used these six methods and made new 12 magic squares, which are;

degree	name of magic square	method	interpreter
3	"Luo Shu"	①	Yang Hui, 1275
4	"Hua Shi-Liu Tu"	②	Yang Hui, 1275
	"Hua Shi-Liu Yin Tu"	②	Abe Gakuhô, 1976
	no figure	③	Yang Hui, 1275
5	"Wu Wu Tu"	①+⑥	Xiong Jisheng, 1955
	"Wu Wu Yin Tu"	⑤+⑥	Li Yan, 1933
6	"Liu Liu Tu"	⑤	Li Yan, 1933
	"Liu Liu Yin Tu"	⑤	Li Yan, 1933
7	"Yan Shu Tu"	①+⑥	Lam Lay-Yong, 1977
	"Yan Shu Yin Tu"	④+⑧	Li Yan, 1933
8	"Yi Shu Tu"	③	Li Yan, 1933
	"Yi Shu Yin Tu"	③	Li Yan, 1933
9	"Jiu Jiu Tu"	⑤	Li Yan, 1933
10	"Bai Zi Tu"	③	Li Yan, 1933

TABLE 3 Magic Square Composition Methods

As shown in table 3, the magic squares of the *Yang Hui Suan Fa* were based on "Luo Shu", and Yang Hui used two methods to apply the "Luo Shu".

One is to decide nine numbers which are in the centre, four corners and the middle of each side first, then to arrange the other numbers. I named it the "Luo Shu frame type method". But this method decides only nine numbers. It would be difficult to make higher degree magic squares because we must decide $n^2 - 9$ numbers by the "stratiform pair method". However Yang Hui could not find the method for arrangement.

The other is to divide the magic square under construction into nine blocks, making nine complex magic squares of "Luo Shu". I named it the "Luo Shu block type method". However this method can be only applied to $3n$ degree magic squares. The numbers of $3n$ include odd numbers and even numbers, and the process of making odd number magic squares is not the same as even number magic squares. But Yang Hui made both magic squares using one method.

Yang Hui based his study of the "Luo Shu" on Chinese philosopy. He retained the characteristics mentioned by Zhang Huang in table 1 when he made larger magic squares. It was a strong point for demonstration but it became a weak point for mathematics.

Now, let us analyse his methods for magic squares of every degree. He solved the case of double-even ($4n$) degree magic squares completely, however the case of oddly-even ($2n$) degree magic squares were not completely solved. Yang Hui adopted the concept of compound magic squares. Therefore he could compose multiple degree magic squares from smaller magic squares. Thus we can make most larger even degree magic squares using his six methods.

On the one hand, Yang Hui did not explain the general method for odd degree magic squares, and so these remained problems for prime number degree magic squares for later ages, since all magic squares can be reduced to magic squares of prime degree.

We must note that he used the "stratiform pair method"; this method influenced later mathematicians strongly, the details will be described in sections , II-4. This method requires a lot of four figure calculations (see Seki Kōwa's method), and therefore difficult to calculate using counting rods. I wonder if Yang Hui used an abacus for its speed in calculation, hence making the "stratiform pair method" more practical. This method is an application of the "Yi Xue" 易學 (studies of the *Yi Jing*), i.e., characteristic (2) of Zhang Huang in table 1.

Yang Hui's passion for making magic squares, however, was not just philosophical, but also because it was mathematically interesting. For example, eight degree magic squares were made via the "Jiu Deng Shu" method, but he rearranged them into more complex forms. If he pursued the philosophical principle, it was enough to explain the "Jiu Deng Shu" method. But he advanced one more step. This was done out of his interest in mathematics.

The *Yang Hui Suan Fa* is one of the best mathematical books for magic squares, but there is no magic square in the *Shu Shu Jiu Zhang*. Both the *Shu Shu Jiu Zhang* and the *Yang Hui Suan Fa* are strongly influenced by traditional Chinese philosophy, the "Yi Xue" (Studies of the *Yi Jing*), thus they had the same philosophy. However, Qin Jiushao looked down on magic squares as little more than fortune-telling, instead of seeing magic squares as important "mathematics" as Yang Hui did. The difference in opinion between the two mathematicians is very interesting. I wonder if one of the reasons is their occupations. Qin Jiushao was a bureaucrat, and Yang Hui was probably a teacher of a private school. I think that making large magic squares was a good advertisement for pupils.

(4) Before Seki Kōwa

(a) Before the Edo period

In Japan, the oldest evidence of magic squares is probably in the *Kuchizusami* 口遊 (Humming) in 970, the three degree magic square[68] is described as follows:

> The verse is, "Two is in 'Kaku' 角 (corner) [69], 'Sa' 左 (left) is three, 'Yû' 右 (right) is seven, Six and eight are 'Soku' 足 (legs), nine is 'Tô' 頭 (head), five is 'Shin' 身 (body), one is 'Bi' 尾 (tail), four is 'I' 維 (corner)." Soil, Water, Metal Fire and Wood, these are "Gyônenyô" 行年曜 (Destiny stars). [70]

4	9	2	"I"	"Tô"	"Kaku"
3	5	7	"Sa"	"Shin"	"Yû"
8	1	6	"Soku"	"Bi"	"Soku"

Fig. 4-1

(the original book does not actually show the magic square)

It is the "Luo Shu", and the describing method is similar to the way of *Shu Shu Ji Yi* 數術記遺 (Memoir on some Traditions of Mathematical Art). There were two describing methods for magic squares. One is in the *Da Dai Li Ji* 大戴禮記 (Record of Rites Compiled by Dai the Elder)(Fig.4-7), which describes only the order of numbers. The other, in the *Shu Shu Ji Yi* (Memoir on Some Traditions of Mathematical Art), describes positions of numbers (see section Ⅱ-2-c) and is called the method of the *Kuchizusami* (Humming).

The author, Minamoto-no Tamenori 源爲憲 (fl.970), suggested this magic square is related to fortune-telling. He suggested the characteristic of the Five Phases, however, the order of the Five Phases he maintained, is not correct. Thus his comprehension of magic squares was not so deep, since he imitated the philosophical aspect of "Luo Shu" erroneously.

There are two three degree magic squares in the *Renchū-shō* 簾中抄 (Records of Court) during 12c and 13c. These are called "Jūgo(15)-date" 十五立 (lit. standing fifteen) because sums of columns and rows are fifteen.

> "Jūgo-date" (standing fifteen); four four seven, eight five two, three six six (Fig.4-2).
>
> Six seven two, one five nine, eight three four (Fig.4-3). (71)

3	8	4
6	5	4
6	2	7

Fig. 4-2

8	1	6
3	5	7
4	9	2

Fig. 4-3

In Fig.4-2, four and six are used two times, thus it is not a magic square by our definition, however the sums are fifteen.

In Fig.4-3, it is the reverse figure of "Luo Shu".

In the Muromachi 室町 period (72), the name of magic squares is also "Jūgo-date". The *Ni Chū Reki* 二中歴 (Two Hand Almanac) which was written between 1444 and 1448 has the following material on magic squares (73):

> "15-date" (Standing fifteen); six seven two, one five nine, eight three four (Fig.4-4).
>
> Another method; four four five, two three six, six seven eight. Arrange three lines, Make nine numbers (Fig.4-5).
>
> Another method in another book; four four seven, eight five two, three six six. Arrange three lines, make nine numbers (Fig.4-6).

2	7	6		5	4	4		7	4	4		4	9	2
9	5	1		6	3	2		2	5	8		3	5	7
4	3	8		8	7	6		6	6	3		8	1	6

Fig.4-4 Fig.4-5 Fig.4-6 Fig.4-7

Da Dai Li Ji

In Fig. 4-4 is a "Luo Shu". Fig. 4-6 is the same as Fig. 4-1. However, Fig. 4-5 does not have even the same sums.

In the same period, in the section of the seventh year of Kanshô 寛正 (1466) of the *Kemmon Zakki* (Note of Experience) is written:

> Fifteen stones are arranged as in the following figure (74).

4	9	2
3	5	7
8	1	6

Fig. 4-8

Although the "Luo Shu" was introduced into Japan, it was a game, and there is no evidence that Japanese mathematicians studied it.

(b) The early Edo period

In the early Edo period, the *Suan Fa Tong Zong* (Systematic Treatise on Arithmetic) was introduced to Japan, so that some of Yang Hui's magic squares became known by Japanese mathematicians. And Japanese mathematicians started to study magic squares.

One of the oldest studies in this age is the *Kigū Hōsū* 奇偶方數 (Odd and Even Squares, or another name is *Narabemono-jutsu* 並物術 (Arranged Matter) about 1653, a manuscript preserved at the Nihon Gakushiin 日本學士院 (Japan Academy)。 The author is unknown, I wonder if it was written by Shimada Sadatsugu 嶋田貞繼 (1608-1680)[75].

The author designed magic squares from 3rd degree to 16th degree. He explained the "arrangement method" and the "Qiu Deng Shu" method, and made the core of larger magic squares. Then he used the "stratiform pair method" to complete larger magic squares.

For example, a five degree magic square was made from the "Luo Shu". He did not describe the details of making it, but the method must have been

ⅰ) Add eight to each item of "Luo Shu" (Fig. 4-10).

ⅱ) Make the outer stratum so that the sum of each pairs is 26 (Fig. 4-11).

4	9	2
3	5	7
8	1	6

"Luo Shu"

Fig.4-9

12	17	10
11	13	15
16	9	14

Fig.4-10

4	19	21	1	20
8	12	17	10	18
23	11	13	15	3
24	16	9	14	2
6	7	5	25	22

Fig.4-11

He sugested some numerical values, these are for n degree magic squares,

terms	formulae
Kyokusû 極數 (extremity numbers)	$2(n-1)$
Ichigyō Shôsû 一行小数 (samll numbers of one row)	$n(n-1)/4$
Ichigyō Tasû 一行多数 (large numbers of one row)	$\frac{\{n(3n-1)+2\}}{4}$
Nigûsû 二隅數 (two corner numbers)	$2n$
Shôsû Kyokusû 少数極數 (small extremity numbers)	$n-1$
Ruisû 累數 (tied numbers)	n^2+1.

Of these numbers, "Kyokusû" and "Ruisû" are important. "Kyokusû" is the number added each item of the core magic square. And "Ruisû" is the sum of pairs.

For greater than eight degree magic squares, he designed only the outmost numbers, for example, the nine degree magic square in Fig. 4-12. A completed nine degree magic square is in Fig. 4-13.

8	78	75	76	81	15	12	14	10
66								16
71								11
69								13
73								9
5								77
2								80
3								79
72	4	7	6	1	67	70	68	74

Fig.4-12

8	78	75	76	81	15	12	14	10
66	57	22	56	55	19	17	61	16
71	54	32	47	49	29	48	28	11
69	18	36	40	45	38	46	64	13
73	58	51	39	41	43	31	24	9
5	20	52	44	37	42	30	62	77
2	59	34	35	33	53	50	23	80
3	21	60	26	27	63	65	25	79
72	4	7	6	1	67	70	68	74

Fig.4-13

His original figure is not a complete magic square, he drew each stratum. The independent strata square (until 16 degree) is

10	24	23	22	21	3	2	15	227	296	237	238	239	244	243	252	
11	6	176	5	183	18	182	26	181	25	180	2	196	9	190	246	
12	174	140	135	126	9	7	21	14	2	13	130	134	139	23	245	
25	13	3	97	15	84	13	9	100	6	83	2	96	142	184	232	
26	178	18	93	61	9	2	14	60	58	53	3	8	127	19	231	
27	189	1	11	54	2	29	33	7	6	34	11	90	144	8	230	
240	12	133	94	55	10	○	○	○	○	27	10	7	12	185	17	
251	173	128	14	1	28	○	○	○	○	9	64	87	17	24	6	
250	193	141	91	6	32	○	○	○	○	5	59	10	4	4	7	
229	11	125	12	8	36	○	○	○	○	1	57	89	20	186	28	
228	194	137	85	13	3	8	4	30	31	35	52	16	8	3	29	
249	22	16	3	62	56	63	51	5	7	12	4	98	129	175	8	
248	187	22	5	86	17	88	92	1	95	18	99	4	133	10	9	
241	20	6	10	19	136	138	124	131	143	132	15	11	5	<u>176</u>	16	176 should
4	7	21	192	14	179	15	171	16	172	17	195	1	188	191	253	be 177.
5	233	234	235	236	254	255	242	30	1	20	19	18	13	14	247	

Fig.4-14

In Fig. 4-14, he made a mistake in 14 degree, it is the underlined 176, it should be 177. He made mistakes in odd degree magic squares, too, they are,

eg. 15 degree: 299→199, 298→198

In that age, the "stratiform pair method" was popular among Japanese mathematicians. But Shimada Sadatsugu could not find a general method to arrange pairs, his method was by trial and error. Thus larger magic squares are more difficult. Therefore, the theme in that age was making larger magic squares. Japanese mathematicians probably knew about "compound magic squares". So the theme was the magic squares which could not be made using the "compound magic squares" method, i.e., the magic squares of prime number degree. The typical theme was to make 19 degree magic squares.

This question was posed in the *Sampō Ketsugi-shō* 算法闕疑抄 (Solving Mathematical Questions) in 1659. And the first answer was given in the *Sanso* 算組 (Mathematical Cutting Board, 1663) by Muramatsu Shigekiyo 村松茂清 (d.1695), the magic square is as follows [76];

359	2	5	7	8	10	11	13	14	326	328	331	332	334	335	339	341	343	1
358	323	296	41	42	43	45	47	48	49	294	299	300	301	302	304	306	37	4
356	52	291	70	72	75	76	79	80	266	267	269	272	273	276	280	69	310	6
353	53	94	263	98	100	102	103	105	242	244	246	249	250	254	97	268	309	9
350	54	92	261	222	231	230	229	228	227	129	127	125	121	122	101	270	308	12
347	55	91	258	123	171	176	169	216	221	214	153	158	151	239	104	271	307	15
346	57	88	256	124	170	172	174	215	217	219	151	154	156	238	106	274	305	16
345	59	87	255	126	175	168	173	220	213	218	157	150	155	236	107	275	303	17
35	64	85	119	128	162	167	160	180	185	178	198	203	196	234	243	277	298	367
33	65	84	117	130	161	163	165	179	181	183	197	199	201	232	245	278	297	329
32	67	83	115	223	166	159	164	184	177	182	202	195	200	139	247	279	295	330
29	311	289	114	224	207	212	205	144	149	142	189	194	187	138	248	73	51	333
26	312	288	111	225	206	208	210	143	145	147	188	190	192	137	251	74	50	336
25	316	285	110	226	211	204	209	148	141	146	193	186	191	136	252	77	46	337
24	318	284	109	240	131	132	133	134	135	233	235	237	241	140	253	78	44	338
22	322	281	265	264	262	260	259	257	120	118	116	113	112	108	99	81	40	340
20	324	293	292	290	287	286	283	282	96	95	93	90	89	86	82	71	38	342
18	325	66	321	320	319	317	315	314	313	68	63	62	61	60	58	56	39	344
361	360	357	355	354	352	351	349	348	36	34	31	30	28	27	23	21	19	3

Fig. 4-15

It is certain that he used the "stratiform pair method" to make this magic square because the core magic square is the "Luo Shu" with 176 added.

180	185	178		4	9	2
179	181	183		3	5	7
184	177	182		8	1	6

Fig. 4-16

After Muramatsu Shigekiyo, Japanese mathematicians continued to study magic squares. The largest one during that period was a 30 degree magic square by Andō Yūeki 安藤有益 (1624-1708) in the *Ki Gū Hō Sū* 奇偶方數 (Odd and Even Squares, 1694). However, they had no general method to arrange pairs, except by trial and error. The era when Japanese mathematicians competed by intuition was soon over, and they tried to find the general method.

(5) Works of Seki Kōwa

(a) Terms of magic squares

In the early Edo period, magic squares were usually called "Narabemono" 並べ物. And the name "Luo Shu" was introduced into Japan. But Seki Kōwa called magic squares a new name, "Hōjin" 方陳 (陣) (lit. square formation). I think that the "Luo Shu" in the *Yang Hui Suan Fa* (Yang Hui's Method of Computation) was mistaken, thus he did not want to use the name "Luo Shu", and must use other names for magic squares.

In China, this term, "Fang Chen" 方陳 (陣), has never used. But there is a similar term to it in the *Shu Shu Jiu Zhang*, it is "Fang Bian Rui Chen (Zhen)" 方變鋭陳 (陣) (Changing Formation from Square to Sharp Triangle) which is the title of question 2 of chapter 15 [77]. It does not mean magic squares, thus it is difficult to conclude that Seki Kōwa used *Shu Shu Jiu Zhang*'s term. But is it an accident that Seki Kōwa's unique term, which has never been used in both China and Japan, is very similar to Qin Jiushao's one?

Seki Kōwa's methods are also "stratiform pair methods". He used many technical terms and explained the structure of magic squares numerically. For example, "Zō Sū" 増數 (lit. increase numbers) is the number of items in outmost sturatum.

I wonder why Seki Kōwa did not refer only to Japanese mathematicians' works but also *Yang Hui Suan Fa*, when he considered these numbers. Yang Hui described the question of square's strata, which is question 12 of chapter 1 of *Tian Mu Bi Lei Cheng Chu Jie Fa* (Practical Rules of Arithmetic for Surveying);

> A bundle of arrows with a square cross-section has 40 arrows on the boundary of the square. Find the total numbers of arrows.
>
> Answer: 121 arrows.

i) The original method: Add 8 to the number of arrows on the boundary and then multiply this by the number on the boundary. Divide by 16, and add the arrow in the centre.

ii) By the method of square ("Fang Tian Fa" 方田法): Halve the number of arrows on the boundary twice, add 1 to give the length of one side of square ("Fang Mian" 方面). Square this result.

iii) Alternatively use the method of the trapezium ("Ti Tian Fa" 梯田法): Add the number of the innermost and outermost layers. Halve the sum and multiply it by the number of layers ("Ceng Shu" 層數). Finally add the arrow in the centre to obtain the resolt[78]. To obtain the number of layers, divide the number of arrows on the boundary by 8. [79]

Yang Hui indicated the connection between total number and the numbers on the boundary clearly, (let the length of one side of the square be n, the number on the boundary c), these are:

i) $n^2 = c(c+8)/16 + 1$

ii) $n^2 = (c/4 + 1)^2 \quad (\therefore n = c/4 + 1)$

iii) $n^2 = ((c+8)/2) \cdot (c/8) + 1$

These formulae are not the same as Seki Kôwa's, but this question was a good exercise that Seki Kôwa studied concerning the relations between the number of total items and the number of items on the boundary.

Seki Kôwa's terms and the formulae described in the *Hōjin no Hō* 方陣之法 (Method for Magic Squares) are:

"Sôshisû" 総子數 (number of items) ; Put "Hôsû" 方數 (degree), square it, obtain "Sôshisû".

"Kyôsekisû" 共積數 (sum of item numbers); Put two "Sôshisû" (number of items) on two rows (counting board rows, not magic square rows), add one to the lower, multiply it by the upper, divide it by two, obtain "Kyôsekisû" (sum of item numbers).

"Jyûô Shakaku Heisekisû" 縦横斜角併積數 (sum of one side); Put "Kyôsekisû" (sum of item numbers), divide it by "Hôsû" (degree), obtain "Jyûô Shakaku Heisekisû" (sum of one side).

"Zôsû" 增數 (increase of item); Add new "Hôsû" (degree) with the previous "Hôsû" (degree), to obtain "Zôsû" (increase of item). Then add it to each number of previous magic square, to obtain the numbers of now inner magic square.

"Sôtaisû" 相對數 (sum of each pair); Put "Sôshisû" (number of items), add one to it, to obtain "Sôtaisû" (sum of each pair).

"Hyô(sû),Risû" 表裏數; Put "Hôsû" (degree), take away one from it, multiply it by two. "Hyôsû" 表數 (front numbers) are numbers from 1 to this number. "Risû" 裏數 (back numbers) are numbers from "Sôshisû" (number of items) going downwards[80]

That is, his formulae are (in algebraic notation):

"Hôsû" $= n$

"Sôshisû" $= n^2$

"Kyôsekisû" $(\sum_{k=1}^{n} k^2) = n^2(n^2+1)/2$

"Jyûô Shakaku Heisekisû" $= n^2(n^2+1)/2n$

"Zôsû" $= 2n-2$

"Sôtaisû" $= n^2+1$

"Hyôsû" $: 1, 2, \cdots, 2(n-1)$

"Risû" $: n^2, n^2-1, \cdots, n^2-2(n-1)+1$

"Zôsû" (increase of item) is half the numbers of items in the outermost boundary, and is the number to add to the core magic square. Seki Kôwa

described these numbers clearly, then he explained the general method to arrange pairs. He classified all magic squares into three categories, "Ki Hôjin" 奇方陣 (odd number degree magic squares), "Tan-gu Hôjin"單偶方陣 (oddly-even number （4n－2） degree magic squares) and "Sô-gu Hôjin" 雙偶方陣 (doubly-even number （4n） degree magic squares). The details will be described from section Ⅱ-4-b to Ⅱ-4-d.

(b) Odd number degree magic squares

Seki Kôwa computed how many items needed to be added to make the next large magic square, then he arranged "Hyôsû" (front numbers) and "Risû" (back numbers) around the outer circumference of the new magic square symmetrically. The method of arrangement for odd degree magic squares is as follows.

The degree to be is $n=2m+1$: We will demonstrate with an eleven degree magic square, therefore the value m, (his term was "Kô Dan Sû" 甲段數 (number of A, see Fig.5-1)), is 5. The direction of row is from right to left normally.

First, arrange of "Hyô sû" (Front Numbers) (see Fig. 5-1);

> Put 1 next to top-right corner, arrange m numbers, from 1 to m from here. The direction is clockwise, arriving at the corner to the below.
>
> Arrange $(m-1)$ numbers, his term was "Otsu Dan Sû" 乙段数 (number of B, from $(m+1)$ to $2m-1$), from the third position on the top row to the left.
>
> Arrange $(m+1)$ numbers, his term was "Hei Dan Sû" 丙段數 (number of C, from $2m$ to $3m$), from the next position of A downwards.
>
> Arrange m numbers, his term was "Tei Dan Sû" 丁段數 (number of D, from $(3m+1)$ to $4m$), from the next position from B to the left. [81]

Second, arrange "Ri Sû" (back numbers) symmetrically with the front numbers; the corner numbers are arranged diagonally.

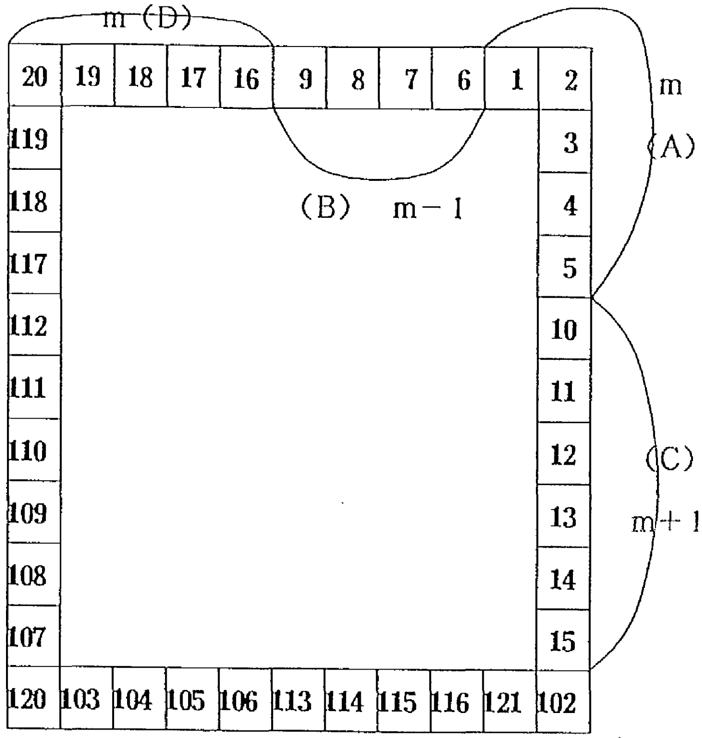

m (D)
20
19
18
17
16
9
8
7
6
1
2
m
(A)
119
3
118
4
117
5
112
10
111
11
110
12
109
13
108
14
107
15
(B) m－1
(C)
m＋1
120
103
104
105
106
113
114
115
116
121
102

Fig. 5-1

Third, interchange some items, the method is;

Interchange m columns from 1 symmetrically.

Interchange m rows similar with it. [82]

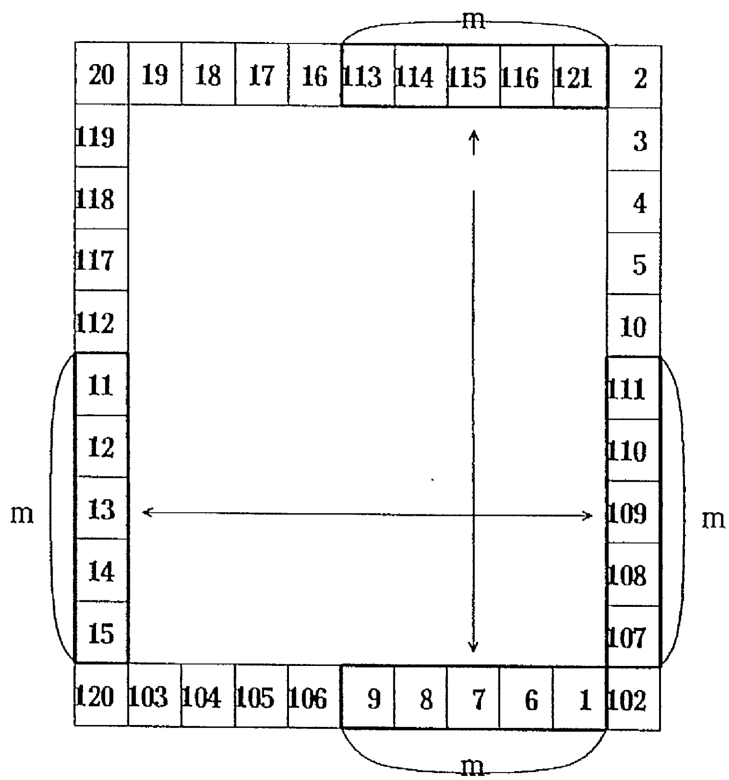


Fig. 5-2

We cannot know how he found this interchanging method, but Fig. 5-2 is certainly a perimeter magic square.

(c) Doubly-even degree magic squares

"Sô Gû Hô" (doubly-even degree magic squares) were also obtained by a similar method, as follows. We will explain the case of a twelve degree magic square, $n = 4m$, so m is 3 (see Fig. 5-3).

Arrange $(4m-2)$ numbers, his term was "Kô Dan Sû" (A, from 1 to $(4m-2)$), from the third position of the top row to the left.

Arrange two numbers, his term was "Otsu Dan Sû" (B, $(4m-1)$ and $4m$, from the top-right corner to the left.

Arrange $(4m-2)$ numbers, his term was "Hei Dan Sû" (C, from $(4m+1)$ to $(8m-2)$), from just below the top-right corner downwards. (83)

Interchange within the columns;

Interchange two numbers, the fourth and fifth position from top-right corner. Do not interchange next two numbers. Interchange next two numbers, do not interchange next two numbers. Continue interchanging in the same way. (84)

Interchange within the rows;

Interchange two numbers from the top-right corner. Do not interchange next two numbers, interchange next two numbers. Do not interchange next two numbers, interchange next two numbers. Interchange in the same way until the third position from bottom-right corner is reached. (85)

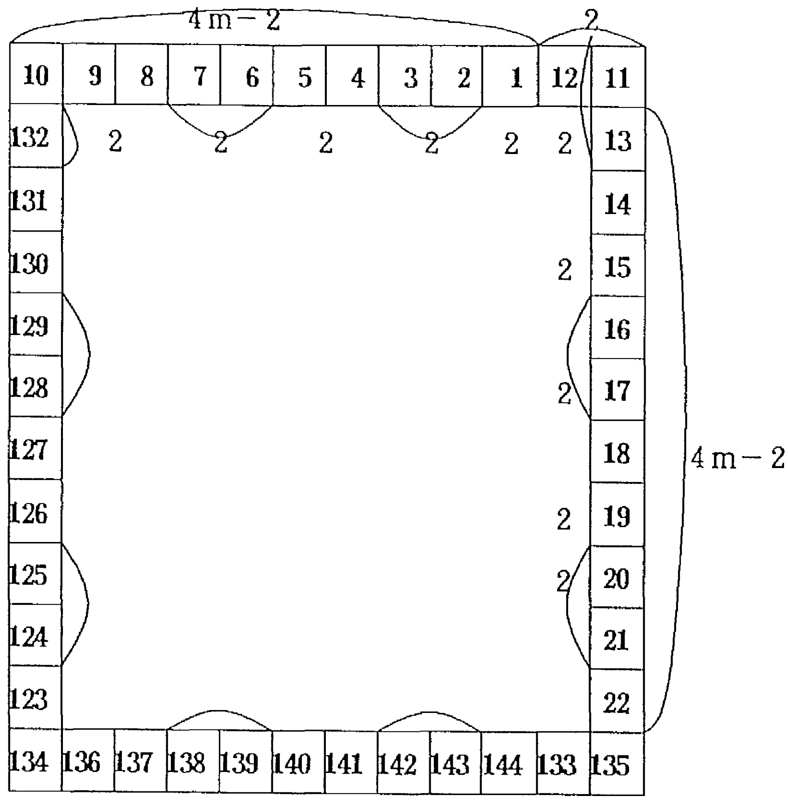

4m−2
2
10
9
8
7
6
5
4
3
2
1
12
11
132
131
130
129
128
127
126
125
124
123
13
14
15
16
17
18
19
20
21
22
4m−2
134
136
137
138
139
140
141
142
143
144
133
135

Fig. 5-3

11	9	8	138	139	5	4	142	143	1	12	10
13											132
131											14
130											15
16											129
17											128
127											18
126											19
20											125
21											124
123											22
134	136	137	7	6	140	141	3	2	144	133	135

Fig. 5-4

(d) <u>Oddly-even degree magic squares</u>

"Tan Gû Hô" (oddly-even degree magic squares) were also obtained by a similar method, as follows. We will explain the case of a fourteen degree magic square, $n=4m+2$, so m is 3. (see Fig. 5-5)

Arrange $4m$ numbers, his term was "Kô Dan Sû" (A, from 1 to $4m$), from the third position on the top row to the right.

Arrange $4m$ numbers, his term was "Otsu Dan Sû" (B, from $(4m+1)$ to $8m$), from the top-right corner downwards until the third position from the bottom-right corner.

Then arrange one number, his term was "Hei Dan Sû" (C, the number $(8m+1)$), left of the top-right corner, arrange one number, his term was "Tei Dan Sû" (D, $(8m+2)$) above the right-bottom corner. (86)

Interchanging by columns;

Interchange three numbers from the top-right corner. Do not interchange next two numbers, interchange next two numbers. Do not interchange next two numbers, interchange next two numbers. Continue until the third position from top-left corner is reached. (87)

Interchanging by rows;

Interchange in one row below top-right corner (2nd row). Do not interchange next two numbers, interchange next two numbers. Do not interchange next two numbers, interchange next two numbers. Continue interchanging until second last row. (88)

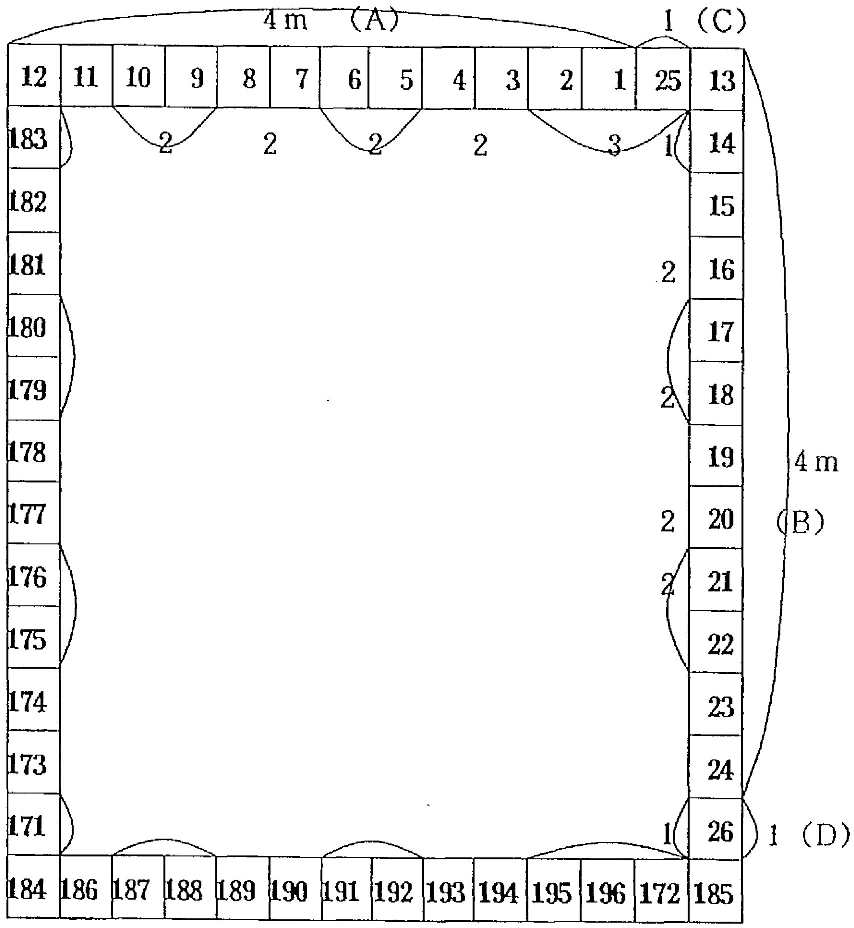
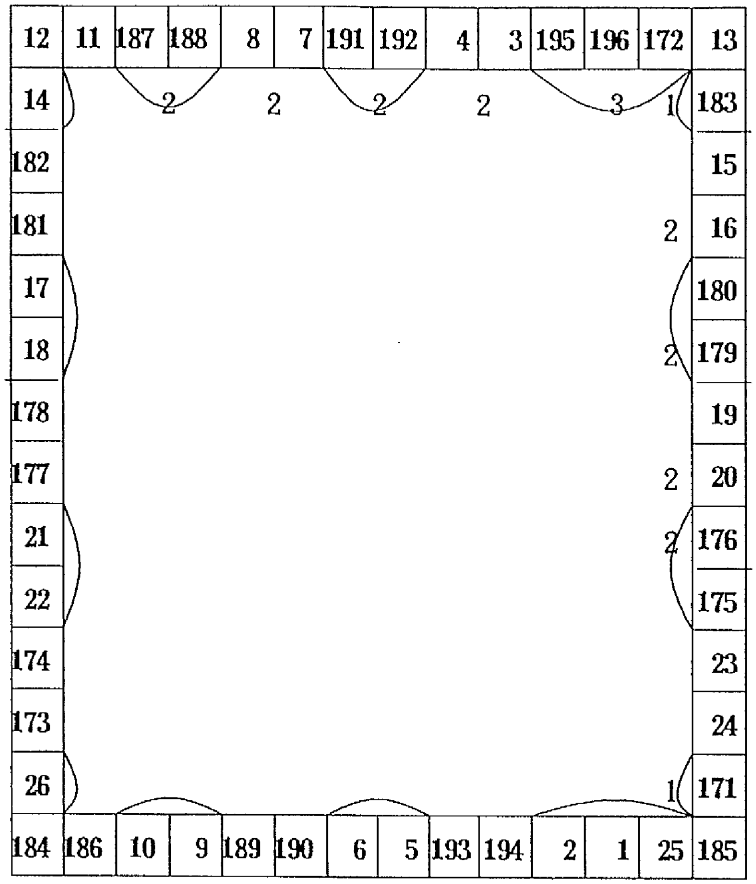

4 m (A)
1 (C)
12 11 10 9 8 7 6 5 4 3 2 1 25 13
2 2 2 2 3 1
14 15 16 17 18 19 20 21 22 23 24 26
183 182 181 180 179 178 177 176 175 174 173 171
2 2 2 2
4 m
(B)
1 1 (D)
184 186 187 188 189 190 191 192 193 194 195 196 172 185

Fig. 5-5



12
11
187
188
8
7
191
192
4
3
195
196
172
13
14
2
2
2
2
3
1
183
182
15
181
2
16
17
180
18
2
179
178
19
177
2
20
21
2
176
22
175
174
23
173
24
26
1
171
184
186
10
9
189
190
6
5
193
194
2
1
25
185

Fig. 5-6

(6) Conclusion

(a) Studies of magic squares in the Edo period, Japan

Seko Kōwa's method was based on calculation. The magic square is a geometrical matter at first glance, but he solved the problems by calculation. He probably used a particular calculating tool such as the abacus. Seki Kōwa could make any degree magic squares by those methods. His method was the application of the "stratiform pair method". He was influenced by the *Yang Hui Suan Fa* and some Japanese mathematicians' methods, so his method was not totally original. However, he was first to find the method of arrangement [89]. He advanced step by step, and this was a most important advance. The study of magic squares as "normal science" had been transfered to another culture, i.e., from China to Japan. During the Edo period in Japan, mathematicians continued to study magic squares.

Now, we list the studies of magic squares by Japanese mathematicians [90].

Author	Title	year	Subject
Isomura Yoshinori	*Sampō Ketsugi Shō*	1659	the question of making 19
礒村吉徳 (d.1710)	算法闕疑抄		degree magic square
Muramatsu Shigekiyo	*Sanso*	1663	the 19 degree magic square
村松茂清 (d.1695)	算組		
Satō Masaoki	*Sampō Kongen-ki*	1669	the 19 degree magic square
佐藤正興 (fl.1669)	算法根源記		
Hoshino Sanenobu	*Ko Kō Gen Shō*	1672	the 20 degree magic square
星野實宣 (1628-1699)	股句弦抄		
Isomura Yoshinori	*Kashiragaki*	1684	the 19 degree magic square
	Sampō Kongen-ki		(It is the same as *Sanso*'s
礒村吉徳 (d.1710)	頭書算法闕疑抄		magic square.)
Seki Kōwa	*Hōjin no Hō*	1683	any degree magic squares

關 孝和 (1642-1708)	方陣之法		
Tanaka Yoshizane	*Rakusho Kigan*	1683	any degree magic squares
田中吉眞 (1651-1719)	洛書龜鑑		another method of Seki's one
Andō Yūeki	*Ki Gū Hō Sū*	1694	up to 30 degree magic
安藤有益 (1624-1708)	奇偶方數		squares
Suzuki Shigetsugu	*Sampō Chōhō-ki*	1694	Yang Hui's method for 5
鈴木重次 (fl.1694)	算法重寶記		degree magic squares
Takebe Katahiro	*Hōjin Shin-jutsu*	(1760)	"diagonal methods"
建部賢弘 (1664-1739)	方陣新術		
Matsunaga Yoshisuke	*Hōjin Shin-jutsu*	?	Yang Hui's method for odd
松永良弼 (d.1744)	方陣新術		degree magic squares
Kurushima Yoshita	*Ku-shi Ikō*	1755	"totter method"
久留島義太 (d.1755)	久氏遺稿		
Kurushima Yoshita	*Ritsu Hōjin*	1755	a 4 degree magic cube
久留島義太 (d.1755)	立方陣		
Nakane Hikodate?	*Kanja Otogizōshi*	1743	magic squares using the
中根彦楯 (1701-1761)	勘者御伽雙子		same value items
Murai Chūzen	*Sampō Dōshi Mon*	1784	new diagonal method
村井中漸 (1708-1797)	算法童子問		
Matsuoka Yoshikazu	*Hōjin Enjin Kai*	?	using matrix symbols
松岡能一 (1737-1803)	方陣圓陣解		
?	*Hōjin Genritsu*	1790's?	5 degree magic squares
	方陣元率		
Nakata Takahiro	*Hōjin Genkai*	?	a new stratiform method
中田高寛 (1739-1802)	方陣諺解		
?	*Shi Hōjin Ren-jutsu*	?	4 degree magic squares
	四方陣廉術		
?	*Shi Hōjin Tan-jutsu*	?	4 degree magic squares
	四方陣探術		
Yamaji Shuju	*Sekiryū Go Hōjin Hensū*		a new stratiform method

	Jutsuro narabini Sūkai	1771	
山路主路 (1704-1772)	關流五方陣變數術路並數解		
?	*Go Hōjin Hensū-jutsu*	?	a new stratiform method
	五方陣變術		
Aida Yasuaki	*Hō-Enjin no Hō*	?	transforming magic squares
會田安明 (1747-1817)	圓陣之法		
Aida Yasuaki	*Hōjin Henkan no Jutsu*	?	computing how many varia-
會田安明	方陣變換之術		tions of the same M.S.
Ishiguro Nobuyoshi	*Shi Hōjin Hensū*	?	four degree magic squares
石黒信由 (1760-1836)	四方陣變數		
?	*Hōjin Henkan-jutsu*	?	valiations of four degree
	方陣變換術		magic squares
Uchida Kyūmei	*Hōjin no Hō*	1825	a new interchange method
内田久命 (d.1868)	方陣之法		
Ichikawa Yukihide	*Gōrei Sampō*	1836	a new interchange method
市川行英	合類算法		
Okayu Yasumoto	*Sampō Semmon Shō*	1840	diagonal methods (the same
御粥安本	算法淺問抄		as Takebe's method)
Komatsu Donsai	*Hōjin Furetsu-hō*	?	a new interchange method
小松鈍齋 (1800-1868)	方陣布列法		(similar to Matsunaga's)
Satō Mototatsu	*Sampō Hōjin Enjin-jutsu*	?	a new diagonal method
佐藤元龍 (1819-1896)	算法方陣圓陣術		
Hagiwara Teisuke	*Hachi Hōjin*	?	a perfect 8 degree magic
萩原禎助 (1828-1909)	八方陣		square

Seki Kōwa had already created the method for composing all magic squares in 18th century, therefore later mathematicians could continue to study more advanced methods of making magic squares. Some mathematicians, Tanaka, Matsunaga and Uchida, studied magic squares in a similar way to that of Seki Kōwa, i.e., classifying magic squares into three categories, then considering separate methods of composition.

Another important method is Takebe Katahiro's 建部賢弘 (1664-1739). He is a student of Seki Kōwa. His method, for example in the case of a seven degree magic square, is:

i) Arrange a natural square (Fig. 6-1).

ii) Thus four lines or rows are already the same as the lines in a magic square. These are the middle column, the middle row and the two diagonal lines. The diagonal line from top-right to bottom-left is called "Usha" 右斜 (right diagonal) and the other is called "Sasha" 左斜 (left diagonal).

iii) Arrange the original middle row as the new "Usha", the original "Usha" as the new middle column, the original middle column as the new "Sasha", the original "Sasha" as the new middle row (Fig. 6-2).

iv) Reverse the middle column and row, interchange the "Usha" and the "Sasha" (Fig. 6-3).

v) There are some irregularities (Fig. 6-4), but these are able to be corrected by interchanging one item in each column and row. For example, interchange 15 and 29, or 21 and 35 for correcting third and fifth row.

43	36	29	22	15	8	1
44	37	30	23	16	9	2
45	38	31	24	17	10	3
46	39	22	25	18	11	4
47	40	33	26	19	12	5
48	41	34	27	20	13	6
49	42	35	28	21	14	7

Fig. 6-1

22			1			4
	23		9		11	
		24	17	18		
43	37	31	25	19	13	7
		22	33	26		
	39		41		27	
46			49			28

Fig. 6-2

4			49			28
	11		41		27	
		18	33	26		
7	13	19	25	31	37	43
		24	17	32		
	23		9		39	
22			1			46

Fig. 6-3

	+42	+28	+14		-14	-28	-42
-6	4	<36>	(29)	49	(15)	<8>	28
-4	<44>	11	(30)	41	(16)	27	<2>
-2	<45>	(38)	18	33	26	(10)	<3>
	7	13	19	25	31	37	43
+2	<47>	(40)	24	17	32	(12)	<5>
+4	<48>	23	(34)	9	(20)	39	<6>
+8	22	<42>	(35)	1	(21)	<14>	46

Fig. 6-4

Takebe Katahiro's method was also applied by later Japanese mathematicians such as Murai Chûzen 村井中漸 (1708-1797), Mikayu Yasumoto 御粥安本 (fl.1840) and Satō Mototatsu 佐藤元龍 (1819-1896).

As given above, Japanese mathematicians regarded the works of Japanese mathematicians in the 18th century as immortal, and the studies of magic squares were advanced in Japan. In any case, it is certain that Japanese mathematicians', at least Seki Kôwa's, most important reference book was the *Yang Hui Suan Fa* whether directly or indirectly referred to.

(b) Studies of magic squares in the Qing dynasty, China

On other hand, Chinese mathematics in the Qing dynasty did not admit that the magic square was a part of mathematics. Mei Juecheng 梅瑴成 (1681-1763) suggested;

> The original book *(Suan Fa Tong Zong)* drew "He Tu" and "Luo Shu" in the initial volumes. They seemed to be mathematics at first glance because they used numbers. However they were used for future-telling, and all books about divination commented on them. Since they were not useful for arithmetic, they were omitted in this book [91].

Most works in *Suan Fa Tong Zong* were imitations of *Yang Hui Suan Fa*'s magic squares, however an important explanation was given in *Suan Fa Tong Zong* (see section Ⅱ-3-d).

But, in the Qing dynasty, magic squares were no more considered mathematics. Magic squares of *Shu Du Yan* 數度衍 (Generalisations on Numbers) by Fang Zhongtong 方中通 (17c) in 1661 was an imitation of *Suan Fa Tong Zong* [92], although he corrected Cheng Dawei's mistake. (concerning a five degree magic square.) Let us make a table of magic squares in *Yang Hui Suan Fa*, *Suan Fa Tong Zong* and *Shu Du Yan*. In the table, "=" is the same as *Yang Hui Suan Fa*'s magic square, "≒" is similar to *Yang Hui Suan Fa*'s magic square, and "no" is no figure.

degree	*Yang Hui Suan Fa* (1275)	*Suan Fa Tong Zong* (1592)	*Shu Du Yan* (1661)
3	"Luo Shu"	=	=
4	"Hua Shi-Liu Tu"	no	no
	"Hua Shi-Liu Yin Tu"	=	=
5	"Wu Wu Tu"	≒ (error)	≒ original
	"Wu Wu Yin Tu"	no	no
6	"Liu Liu Tu"	no	no
	"Liu Liu Yin Tu"	≒	≒ *SFTZ*
7	"Yan Shu Tu"	=	=
	"Yan Shu Yin Tu"	no	no
8	"Yi Shu Tu"	=	=
	"Yi Shu Yin Tu"	no	no
9	"Jiu Jiu Tu"	=	=
10	"Bai Zi Tu"	=	=

TABLE 4 MAGIC SQUARES IN THE QING DYNASTY

It was the age of introduction of modern western science. It cannot be said that the studies of magic squares were influential although it was very influential in Japan. I suspect it was no accident that Mao Jin's manuscript of *Yang Hui Suan Fa* omitted the part concerning magic squares, which is the first chapter of the *Xu Gu Zhai Qi Suan Fa*.

One reason for this was that there were no systematic schools for mathematics in China [93]. In contrast, there were many schools called Terakoya 寺子屋 and they were competitors among themselves in Japan. Most Japanese mathematicians were teachers of mathematics, and magic squares were a good advertisement for their schools.

Magic squares were an embodiment of philosophy in China. But it was also a kind of hobby for Japanese mathematicians. It was often said that Japanese mathematics was "Muyô no Yô"無用の用 (the practical use for impractical use).

The magic square is a typical example of this.

Notes

(*1): Chapter 1 of *Yi Tu Ming Bian* 易圖明辯 (Clarification of the Diagrams in the Book of Changes). *SKQS*, vol. 44: 652.

(*2): Andrews, 1908.

(*3): Li Yan, 1933; 1954, vol. 1: 175-229. It was printed in *Xueyi Zazhi* 學藝雜誌 vol. 8 (1927), no. 9: 1-40.

(*4): Cammann, 1963.

(*5): Lam Lay-Yong, 1977.

(*6): Graham, 1989.

(*7): Ho Peng-Yoke, 1991.

(*8): Mikami Yoshio, 1917.

(*9): Katô Heuzaemon, 1956, Zatsuron vol. 3.

(10): Hirayama Akira and Abe Gakuhô, 1983. The study of *Yang Hui Suan Fa* 's magic squares by; Hirayama Akira, Abe Gakuhô and Toya Seiichi, 1984.

(11): Nihon Gakushiin, 1956. 5 vols.

(12): *SKQS*, vol. 968: 12.

(13): See Major, 1976: 163. Graham, 1989: 348. and Ho Peng-Yoke, 1991.

(14): The idea of "Wu Xing" (Five Phases) was created from Chinese astronomy, and it became philosophy (Nihon Gakushiin, 1960: 42).

(15): Zhu Xi was a Confucianist in the Southern Song dynasty. He completed the studies of "Xing Li" 性理 (principle of human and nature). His other names were Zhuzi 朱子 and Zhu Wengong 朱文公.

(16): Chapter 1 of the *Yi Tu Ming Bian* 易圖明辯 (Clarification of the Diagrams in the Book of Changes), *SKQS*, vol. 44: 671. And chapter 1 of the *Yi Xue Qi Meng* 易學啓蒙 (Elementary of Changes Studies, quarted on *Yi Xue Qi Meng Tong Shi* 易學啓蒙通釋 (Commentary of *Yi Xue Qi Meng*), *SKQS*, vol. 20: 663 has the similar section.

(17): See Li Yan, 1933; 1954, vol. 1: 176. and Sun Guozhong, 1989: 445-71.

(18): Chapter 47, Kongzi Shi-jia 孔子世家 (Biography of Confucius) on the *Shi*

Ji 史記 (Historical Record), (ZHSJ, vol. 6: 1937).

(19): Ouyang Xiu was a scholar and a statesman, later known as one of the Eight Great Poets in the Tang and Song dynasties. He was born at Luling 廬陵 (now Jishui, Jiangxi province 江西省吉水縣) in the Song dynasty. His other names were Yongshu 永叔, Cuiweng 醉翁, Liuyijushi 六一居士 and Wenzhong 文忠.

(20): *Yi Tong Zi Wen* 易童子問 (Pupils Question about Changes) written by Ouyang Xiu 歐陽脩 (chapter 4 of the *Jing Yi Kao* 經義考 (Studies of Chinese Classics) written by Zhu Yicun 朱彝尊 in 1679, *SKQS*, vol. 677: 31-2).

(21): Wilhelm (trs.), 1950; 1989: 320.

(22): Kong Anguo was a scholar in the Western Han 西漢 dynasty. His other name was Ziguo 子國. 12th generation descendant of Kongzi. He found the *Gu Wen Shang Shu* 古文尚書 (Old Edition of Historical Classic), but this version was destroyed and *Wei Gu Wen Shang Shu* 僞古文尚書 (Pseudo-Historical Classic) written by Mei Ze 梅賾 in the Jin 晉 dynasty was made known to the public at large.

(23): Chapter 11 of *Yi Jing*. *SKQS*, vol. 54: 239.

(24): Since chapter 2 of the *Zhou Yi Qian Zhuan Du* 周易乾鑿度 (Prophecy Book of "Qian", *SKQS*, vol. 53: 875) commented on by Zheng Xuan 鄭玄 (127-200) in the Eastern Han 東漢 dynasty, the three degree magic square had been called "Jiu Gong" 九宮 (Li Gaishi, 1975; 1982: 208).

(25): Yan Dunjie, 1978. and Yin Difei, 1978.

(26): In some edition, this section is section 67 (Editor's Note of the *Si Ke Quan Shu* Wenyange 文淵閣 ed.).

(27): Written by Dai De 戴德 about 50 A.D. in the Eastern Han dynasty, text-book of "Da Dai Xue" 大戴學 (School of Dai De). *SKQS*, vol. 128: 488.

(28): Section of "Jiangren" 匠人 (Carpenter), Chapter of *Dong-Guan Kao-Gong-Ji Xia* 冬官考工記下 (Office of Winter, Record of Artificers, Chapter 2) of *Zhou Li* 周禮 (Record of the Rites of the Zhou Dynasty). *SKQS*, vol. 90: 770.

According to this evidence, "Ming Tang" 明堂 (Hall of Light) was called

"Shi Shi" 世室 (Royal Family Room) in the Xia 夏 dynasty and called "Chong Wu" 重屋 (stacked building) in the Shang 商 dynasty.

(29): Chapter 68, Yuwen Kai Zhuan 宇文愷 (Bibliography of Yuwen Kai) of *Sui Shu* 隋書 (History of the Sui Dynasty). China Press edition, vol. 6: 1588.

(30): *Shu Shu Ji Yi* 數術記遺 (Memoir on some Traditions of Mathematical Art) is said to be written by Xu Yue 徐岳 in the 3c, however it was in fact written by Zhen Luan 甄鸞 in the 6c (Qian Baocong, 1964: 92).

(31): Qian Baocong (ed.), 1963, vol. 2: 544.

(32): Liu Mu was a scholar of the *Yi Jing* in the Song dynasty.

(33): *SKQS*, vol. 8: 154-5.

(34): His name was Guan Lang 關郎. Guan Ziming was a Confucianist in the Northern Wei 北魏 dynasty of Southern-Northern period. He wrote the *Guan-Shi Yi Zhuan* 關氏易傳 (Mr. Guan's Commentary of *Yi Jing*).

(35): Quoted in *Yi Xue Qi Meng Tong Shi* 易學啓蒙通釋 (Commentary of *Yi Xue Qi Meng*) written by Hu Fangping 胡方平 in 1289 (*SKQS*, vol. 20: 663).

(36): Zhu Xi's opinion was influenced by Cai Yuanding 蔡元定 (1145-1198), his student.

(37): Nihon Gakushiin, 1960: 43.

(38): Chapter of *Xi Ci Zhuan Shang* of the *Yi Jing*. *SKQS*, vol. 7: 536. Wilhelm (trs), 1950; 1989: 310.

(39): *SKQS*, vol. 806: 343.

(40): Abe Gakuhō, 1976.

(41): Kodama Akio, 1966: 91. Translated by; Lam Lay-Yong, 1977: 112.

(42): *SKQS*, vol. 7: 98, translated by; Whilhelm (trs.), 1950; 1989: 313. Liu Hui 劉徽 (fl. 263) also quoted this paragraph in the preface of *Jiu Zhang Suan Fa* 九章算術 (Nine Chapters on the Mathematical Arts) (Bai Shangshu, 1983: 1).

(43): Quoted in *Yi Xue Qi Meng Tong Shi* 易學啓蒙通釋 (Commentary of *Yi Xue Qi Meng*), *SKQS*, vol. 20: 671. Moreover Ding Yidong 丁易東 also used the term "Zong Heng" 縱横 (vertical and horizontal) in chapter 2 of *Da Yan Suo Yin*

大衍索隱 (Studies about Da Yan) at end of 13c. (*SKQS*, vol. 806: 353)

(44): Kodama Akio, 1966: 71. See Lam Lay-Long, 1977: 146.

(45): see Kanô Toshi, 1980: 20.

(46): Some mathematicians used this method generally. Suzuki Shigetsugu 鈴木重次 used the "arrangement method" for all odd degree magic squares in the *Sampō Chōhō-ki* 算法重寶記 (Treasure of Mathematics) written in 1692. Then Matsunaga Ryôsuke 松永良弼 completed this method in *Hōjin Shin-jutsu* 方陣新術 (New method for Magic Squares) (Katô Hezaemon, 1956 zatsuron vol. 3: 271-6).

(47): Li Yan wrote "Yi Huan Shu" 易換術 in Li Yan, 1933; 1954, vol. 1: 179, but we must abide by the Korean ed, which is "Huan Yi Shu" 換易術. The term in *Suan Fa Tong Zong* was also "Huan Yi Shu" although it is another method (see section II-3-d).

(48): Lam Lay-Yong, 1977: 146-7.

(49): Kanô Toshi, 1980: 33.

(50): Lam Lay-Yong, 1977: 297.

(51): Abe Gakuhô, 1976.

(52): The magic square whose sum of each diagonal is also the same is called a "middle level magic square", and if all diagonal lines are the same, then it is called a "perfect magic square".

(53): Xiong Jisheng, 1955.

(54): Li Yan, 1933; 1954, vol. 1: 187-8.

(55): P.10B of chapter 12 of *Suan Fa Tong Zong* (Systematic Treatise on Arithmetic) (Katô Heizaemon, 1956, zatsuron vol. 3: 226-7).

(56): Of course, exchanging 1 and 5, 21 and 25, it would be the "Wu Wu Tu" (Five by Five Figure, Fig. 2-17).

(57): It was named "bordered magic square" by Andrews, 1908.

(58): Hirayaam Akira, Abe Gakuhô and Toya Seiichi, 1984: 139.

(59): Hirayama Akira, Abe Gakuhô and Toya Seiichi: 1984: 139.

(60): Li Yan, 1933; 1954, vol. 1: 188-9.

(61): It was named "compound magic square" by Andrews, 1908: 44.

(62): This is Ding Yidong's interpretation of "The number of "Da Yan" is fifty, and so use forty-nine" (*Xi Ci Zhuan Shang* (Interpretation, chapter 1) of *Yi Jing* (the Book of Changes)),(see Ho Peng-Yoke, 1991) in chapter 2 of *Da Yan Suo Yin* (Studies about Dayan) (*SKQS*, vol. 806: 341-3).

(63): Lam, Lay-Yong, 1977: 303-4.

(64): Li Yan, 1933; 1954, vol. 1: 189.

(65): Li Yan, 1933; 1954. vol. 1: 190-1.

(66): Ding Yidong, chapter 2 of *Da Yan Suo Yin* (Studies about Da Yan), *SKQS*, vol. 806: 353.

(67): Li Yan, 1933; 1954. vol. 1: 192-3.

(68): Abe Gakuhô found that this material is a magic square. (Noguchi Taisuke, 1991).

(69): The manuscript kept by Noguchi Taisuke is that "Kaku" is written "Yô" 甬 (pipe), but *ZGR* edition of *Kuchizusami* (Humming) is "Kaku". Thus I changed "Yô" to "Kaku" (see Noguchi Taisuke, 1991: 13).

(70): The manuscript which was written in 1263, and republished in 1924 (Noguchi Taisuke, 1991: 13).

(71): The manuscript which is written in 1739 (Noguchi Taisuke, 1991: 15).

(72): *Nichūreki* is the dictionary which has the *Shō Chū Reki* 掌中歴 (Hand Almanac) and *Kai Chū Reki* 懐中歴 (Pocket Almanac), 13 chapters. It was said that Miyoshi Tameyasu 三善爲康 who was a "San no Hakase" 算博士 (Doctor of mathematics) in the Heian 平安 period wrote it, however it was written between 1444 and 1448.

(73): Sawada Goichi, 1927: 82.

(74): *ZGR*, vol. 30A: 94.

(75): It was said that this book was destoryed by the conflagration in 1657 (Andô's *Kigū Hōsū* , preface. see Mikami Yoshio, 1917: 34. Toya Seiichi, 1987: 19-20.) However a manuscript of "*Kigū Hōsū* " was kept at Waseda 早稲田 Univ. Lib. Even the author is unknown, however the contents are not the same as Andô's *Kigū Hōsū*. According to the preface of Andô's *Kigū Hōsū* , Shimada also wrote "*Kigū Hōsū* ", thus this manuscript is probably Shimada's *Kigū Hōsū* .

(76): Mikami Yoshio, 1917: 16.

(77): ZHSJ, vol. 2: 376-8.

(78): There is no passage of "Finally add the arrow in the centre to obtain the result" 加増心箭 in Korean edition. However, there is the same kind of question in question 18 of chapter 1 of *Tian Mu Bi Lei Cheng Chu Jie Fa,* which has this passage.

(79): Lam Lay-Yong, 1977: 92-3.

(80): Hirayama Akira et al (eds.), 1974: 199.

(81): Hirayama Akira et al (eds.), 1974: 199-200.

(82): Hirayama Akira et al (eds.), 1974: 200.

(83): Hirayama Akira et al (eds.), 1974: 200.

(84): Hirayama Akira et al (eds.), 1974: 200.

(85): Hirayama Akira et al (eds.), 1974: 200.

(86): Hirayama Akira et al (eds.), 1974: 200.

(87): Hirayama Akira et al (eds.), 1974: 200.

(88): Hirayama Akira et al (eds.), 1974: 200.

(89): In 1683, Tanaka Yoshizane 田中吉真 found another method of arrangement, which makes the magic square from the outer circumference to the inner circumference in *Rakusho Kigan* 洛書亀鑑 (Mirror of "Luo Shu" on turtle's Shell)(Katô Hêzaemon, 1956 zatsuron vol. 3: 266-71). Most of his terms were influenced by the study of *Yi Jing.*

(90): Mikami Yoshio, 1917.

(91): The explanatory note of the *Zeng Shan Suan Fa Tong Zong* 増刪算法統宗 (Systematic Treatise on Arithmetic with Mei Juecheng's Comments) commented by Mei Juecheng 梅瑴成 (1681-1763) in 1757.

(92): Li Yan, 1933; 1954, vol. 1: 201.

(93): Hua Yinchun, 1987: 85-6.

Ⅲ: THE ANALYSIS FOR SOLVING INDETERMINATE EQUATIONS

(1) Study history

It is said that indeterminate equations are classified into two categories (*1), "the Sunzi Theorem" 孫子定理 (lit. the Master Sun's Theorem, the Chinese remainder theorem) and development, and "Bai Ji Shu" 百鶏術 (One Hundred Fowls problem) (*2). But these two problems are not the same in their origin, their solution or their developments, such as the way Yang Hui (fl.1274) classified them in traditional Chinese mathematical thought (*3).

In this chapter, we will only consider "the Sunzi Theorem" and its development, because we wish to compare with Seki Kôwa's work.

These fields of indeterminate equations have been studied by many historians of mathematics (*4). Since Wylie (fl.1852) (*5) introduced the subject in 1852, all books about the history of Chinese mathematics have devoted several pages to it. The work of Mikami Yoshio 三上義夫 (1875-1950) (*6) is one of the oldest analyses using modern mathematics, while the work of Li Yan 李儼 (1892-1963) (*7) was one of the most complete studies about "Da-Yan Qiu Yi Shu" 大衍求一術 (The Technique of Acquiring "One" in Dayan), development of "the Sunzi Theorem" (the Chinese remainder theorem). Qian Baocong 錢寶琮 (1892-1974) (*8) discussed the relation of indeterminate equations with astronomy, and the method of "Jiu Ding Shu" 求定數 (the method of finding mutually prime numbers).

In the Occident, Needham's (b.1900) works (*9) provide one of the most complete discussions. Libbrecht (fl.1973) (10) compared "Da-Yan Zong Shu Shu" 大衍総數術 (The General Solution of Dayan Problems) with Indian and Western mathematics; moreover he solved most of the problems concerning relations between "Da-Yan Qiu Yi Shu" and Indian work, and posed the problem of making mutually prime numbers. Lu Zifang 呂子方 (1895-1964) (11) and Li Jimin 李繼閔 (b.1940) (12) analyzed the problems for astronomy, and specifically Li Jiming analyzed, using modern theorems, the remaining mathematical problems, that is, ascertaining what methods Qin Jiushao used, to greater or lesser effect, to make the divisors mutually prime.

On the other hand, Japanese mathematicians from the Meiji period onwards, in spite of being trained in Western mathematics, also were based on the traditional pre-Meiji mathematics and therefore were able to analyse the old mathematics using Western mathematics. Hayashi Tsuruichi 林鶴一 (1873-1946) [13] did his work in this age. Japanese mathematics was analysed completely by Fujiwara Shôzaburô 藤原松三郎 (1881-1946) [14], including all problems of indeterminate equations. In 1964, Katô Heizaemon 加藤平左エ門 (fl.1956) [15] concluded historical studies of number theory in China and Japan. These three scholars understood that "Da-Yan Jiu Yi Shu" (or "Senkan-jutsu" 翦管術 which was Seki Kôwa's term) involved the application of "Geng Xian Jian Sun" 更相減損 (Mutual Subtraction Algorithm, "Chinese Euclid's Algorithm") [16]. They concluded that the traditional method is not the same as a series of numerators of continuing fractions [17].

We will refer to the works of these scholars and compare Chinese and Japanese mathematics. Then we will consider the historical influence of Chinese mathematics on Japanese mathematics. Next we will consider how Japanese mathematicians solved the problems remaining at the beginning of the early Edo period, and whether they made good use of Chinese works in the 13th century.

(2) "The Sunzi Theorem" (Chinese Remainder Theorem)

(a) In China

The origin of indeterminate equations is probably to be found in the construction of the calendar [18]. When a new calendar was to be made, the first thing that was done was to compute "Shang Yuan Ji Nian" 上元積年 (the accumulated years from the epoch) from certain conditions. In Chinese philosophy, "Shang Yuan" 上元 (the epoch) must be the starting point of the astronomical period. It was commonly thought that that year had to be the first year of sexagenary cycle, "Jia Zi" 甲子. This year, in 1993, is the year of "Gui You" 癸酉, the 10th year of the sixty year cycle, let the "Shang Yuan Ji Nian" be x, it is the same value as the congruence expression;

$$
\begin{aligned}
x &\equiv (10-1) \pmod{60} \\
&\equiv 9 \pmod{60}
\end{aligned}
$$

Some specific conditions were assumed in the construction of each calendar. The first day of the epoch year, which is the winter solstice, had to be a new moon, which is the first day of a lunar month. For example, if it were 5th day of 11th lunar month at the winter solstice [19], the total number of days from the first day of the epoch year, be expressed as y, and one tropical year be given as 365 1/4 days. Thus it is the same value as the congruence expression;

$$
\begin{aligned}
y &\equiv (5-1) \pmod{365\ 1/4} \\
&\equiv 4 \pmod{1461/4}
\end{aligned}
$$

Every four years, 1461 days, the fraction is eliminated, thus;

$$
y \equiv 4 \pmod{1461}.
$$

Because the calendar was one of the most important things in an agricultural state like China, the method of solving indeterminate equations, which was the method of computing the length of "Shang Yuan Ji Nian", was also important in all ages in China.

In the field of mathematics, indeterminate equations also appeared early. It was recorded in question 26, chapter 3 of the *Sun-zi Suan Jing* 孫子算經 (Master Sun's Mathematical Manual) , which is known as the question of "Wu Bu Zhi Qi Shu" 物不知其數 (The Question of Unknown Number of Articles):

> QUESTION: Now there are an unknown number of things. If we count by threes, there is a remainder 2; if we count by fives, there is a remainder 3; if we count by sevens, there is a remainder 2. Find the number of things.
>
> ANSWER: 23
>
> METHOD: If we count by threes and there is a remainder 2 put down 140. If we count by fives and there is a remainder 3 put down 6 3. If we count by sevens and there is a remainder 2 put down 30. Add them to obtain 233 and subtract 210 to get the answer. If we count by threes and there is a remainder 1 put down 70. If we count by fives and there is a remainder 1 put down 21. If we count by sevens and there is a remainder 1 put down 15. When a number exceeds 106, the result is obtained by subtracting 105 to get the answer. [20]

The "question" of "Wu Bu Zhi Qi Shu" is equivalent to the following set of indeterminate equations:

$$
\begin{aligned}
x &\equiv 2 \pmod{3} \\
&\equiv 3 \pmod{5} \\
&\equiv 2 \pmod{7}
\end{aligned}
$$

The "method", it is called "The Sunzi Theorem", gave the answer;

$$x \equiv 2\times 70 + 3\times 21 + 2\times 15 \pmod{3\times 5\times 7}$$

$$\equiv 233 \pmod{105}$$

$$\therefore \quad x_{min} = 23$$

This answer is the correct one, but Sunzi 孫子 (fl. 4c) did not comment on the process. He suggested three numbers, 70, 21 and 15. We can understand that these numbers are the products of the other two divisors. 21 is 3 times 7, and 15 is 3 times 5.

Sunzi, however, did not give any comments about the meaning of these numbers. Moreover, the exception to this pattern occurs in the case of the first equation. For its treatment, the instruction is to "put 70 which when divided by 3 gives a remainder of 1" yet the product of the divisors of the second and third equations, 5 and 7, is 35, not 70. There is an integral factor of 2 that seems to come into play that is not accounted for in the description of the general method.

In the 13th century, Qin Jiushao suggested the meaning of these numbers, so Sunzi's method was very probably the same as Qin Jiushao's method. His method, however, described only the "junction" (the procedure of computation), but it is difficult to understand his method. Then historians of mathematics after Gauss (1777-1850) (21) explained Qin Jiushao's method using Gauss's mathematical symbols (22) . Let us express the problem in modern notation, Chinese terms are Qin Jiushao's terms.

The Chinese remainder theorem is the method for solving simultaneous indeterminate equations, thus given equations are expressed generally;

$$x \equiv R_i \pmod{a_i} \quad (i=1,2,3 \cdots\cdots n) \quad \cdots\cdots \quad (1.1)$$

and with $m = \prod a_i$ ("Yan Mu" 衍母 (lit. extension mother)), $m_i = m / a_i$ ("Yan Shu" 衍數 (lit. extension number)), that is to say, m_i are the products of other divisors. And this m_i has a strange character, it is $\exists$ constants of k_i, which satisfies modulus equations, if m_i and a_i are mutually prime, i.e., $(m_i, a_i) = 1$;

$$k_i m_i \equiv 1 \pmod{a_i} \qquad \cdots\cdots\cdots\cdots(1.2)$$

Then multiply two sides of modulus equations (1.2) by R_i;

Because if;

$$a \equiv b \pmod{d},$$
$$ac \equiv bc \pmod{d}.$$

$$\therefore \qquad R_i k_i m_i \equiv R_i \pmod{a_i}$$

Thus each $R_i k_i m_i$ satisfies one of modulus equations (1.1).
On the other hand, from definition of m_i, we can see that

$$m_i \mid a_j \qquad (i \neq j)$$

thus $R_i k_i m_i \mid a_j$, that is to say,

$$R_i k_i m_i \equiv 0 \pmod{a_j}$$

And if;

$$a \equiv b \pmod{d},$$
$$a + c \equiv b + c \pmod{d}.$$

Therefore:

$$R_1 k_1 m_1 + R_2 k_2 m_2 + \cdots + R_n k_n m_n \equiv R_1 + 0 + \cdots + 0 \pmod{a_1}$$

$$\equiv 0 + R_2 + \cdots + 0 \pmod{a_2}$$

$$\vdots$$

$$\equiv 0 + 0 + \cdots + R_n \pmod{a_n}$$

Thus $\sum_{i=1}^{n} R_i k_i m_i$ fulfils the condition of all modulus equations (1.1).

And there are $\prod_{i=1}^{n} a_i$ cases in the permutation of the remainders, so x_{min} is less than $\prod_{i=1}^{n} a_i$,

$$x \equiv \sum_{i=1}^{n} R_i k_i m_i \pmod{m}$$

$$x_{min} = \sum_{i=1}^{n} R_i k_i m_i - pm \quad (p \in I,\ p \geqq 0)$$

So in the case of "Wu Bu Zhi Qi Shu", $a_i = 3,5,7$ (23); $R_i = 2,3,2$;

$\therefore$

$$k_1 = 2$$

$$k_2 = 1$$

$$k_3 = 1$$

Thus;

$$k_1 m_1 = 70$$

$$k_2 m_2 = 21$$

$$k_3 m_3 = 15$$

These are just the numbers which Sunzi suggested, i.e., Sunzi described the value of $k_i m_i$ ("Fan Yong" 泛用 (lit. extensive use) which is Qin Jiushao's term) in the "method".

$$\therefore\ R_1 (2) \times k_1 (2) \times m_1 (35) \equiv 2 \pmod 3 \equiv 0 \pmod 5 \equiv 0 \pmod 7$$

$$R_2 (3) \times k_2 (1) \times m_2 (21) \equiv 0 \pmod 3 \equiv 3 \pmod 5 \equiv 0 \pmod 7$$

$$R_3 (2) \times k_3 (1) \times m_3 (15) \equiv 0 \pmod 3 \equiv 0 \pmod 5 \equiv 2 \pmod 7$$

$$\therefore \quad \sum_{i=1}^{n} R_i \; k_i \; m_i \; (233) \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 2 \pmod{7}$$

This process gives a correct answer as above. But two important points were not explained in *Sun-zi Suan Jing*.

One is: there are no comments on how k_i is found. If Qin Jiushao had not explained this method, it would have been very difficult to understand it (this method will be described at section Ⅲ-3-c).

The other is: the condition computing k_i (see modulus equation (1.2) is;

$$(m_i \, , \; a_i) = 1 .$$

$$\therefore \quad (a_i \, , \; a_j) = 1$$

Thus all divisors must be mutually prime numbers. The question of "Wu Bu Zhi Qi Shu" is that divisors are already mutually prime, but when Chinese astronomers computed "Shan Yuan Ji Nian", the divisors were sometimes not mutually prime numbers. Therefore, if divisors were not mutually prime, the problem divisors must be changed to be mutually prime numbers. But Sunzi did not approach this subject. These two subjects remained for future ages.

(b) In Japan

The *Sun-zi Suan Jing* was one of the most important mathematical texts used in the "Daigaku-ryô" 大學寮 (university) [24] during the Ritsuryo 律令 age (710-1192), so "The Sunzi Theorem" was also studied there.

Then when the "Shôgun age" 將軍 (1192-1867) [25] started. "Daigaku-ryô" fell into inactivity and decay but "The Sunzi Theorem" was not lost. We cannot find historical references in the Kamakura 鎌倉 (1192-1333) period but there are some from the Muromachi 室町 period (1336-1573): in the question of "Shichi Go San" 七五三 (Seven-Five-Three) in the *Kemmon Zakki* 見聞雑記 (Things Seen and Heard);

> First, put 15 which when divided by 7 gives a remaining stone of 1. Secondly, count 21 which when divided by 5 gives a remaining stone of 1. Thirdly, count 70 which when divided by 3 gives a remaining stone of 1. Then add these together, take away 105, and the remainder is the required number. [26]

This is describing $k_i m_i$, the describing contents was actually the same as that found in the *Sun-zi Suan Jing* . Even divisors were not changed.

In the early Edo 江戸 period, *Jinkō-ki* 塵劫記 (Permanent Mathematics), which is one of the most famous mathematical books of that age also mentions indeterminate equations; question 42 of chapter 5 [27], "Hyaku-go Gen to Iukoto" 百五減と云事 (Subtracting 105):

A problem is stated which is equivalent to:

$$x \equiv 2 \pmod{7} \equiv 1 \pmod{5} \equiv 2 \pmod{3}$$

The method was;

First, put 15 which when divided by 7 gives a remainder of 1, so put 30. And put 21 which when divided by 5 gives a remainder of 1. And put 70 which when divided by 3 gives a remainder of 1, so put 140.

Then add these three numbers together, take away 105 [28], a remainder, 86, is the required number.

It is about one thousand years ago that "The Sunzi Theorem" was received in Japan. But it was only imitated, not applied. From the above evidence we can conclude that Japanese mathematics could not create a method to compute k_i and could not even change a_i from the values originally used in the *Sun-zi Suan Jing* .

(3) The General Solution of Dayan Problems

(a) The name of "Da-Yan Shu" 大衍術 (Dayan Technique)

"Da-Yan Zong Shu Shu" (The General Solution of Dayan Problems) is a method used in Qin Jiushao's main work, the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections). The term "Da-Yan" also occured in the *Da Yan Li* 大衍曆 (Dayan Calendar) (29). But there are not any similarities between "Da-Yan Zong Shu Shu" and the method of the *Da Yan Li* (30).

This mysterious term, "Da-Yan Zong Shu Shu", is from one of the most important terms "Da-Yan" 大衍 (lit. great extension) in chapter *Xi Ci Zhuan Shang* 繫辭傳上 (Commentary on the Appended Judgment, part Ⅰ) of the *Yi Jing* 易經 (the Book of Changes);

> The number of total ("Da-Yan" 大衍) is fifty. Of these, forty-nine are used. (31)

We cannot understand what the "Da-Yan" is, but anyhow the "Da-Yan" is a symbol of the *Yi Jing*.

The similarity between the "Da-Yan Zong Shu Shu" and the *Yi Jing* is suggested by Li Jimin. "'Da-Yan Shu' essentially means the solution of one degree indeterminate equations, and its primary meaning is based upon the concept of 'residue'. The *Yi Jing* is based on the concept of 'Yin Yang Qi Ou' 陰陽奇偶 (even and odd numbers), the even and odd numbers composing the most simple and primitive system of residues" (32). Therefore Qin Jiushao chose the name of "Da-Yan".

(b) The content of "Da-Yan Zong Shu Shu" (The General Solution of Dayan Problems)

The key to solving indeterminate equations involves two points, both found in the modulus equation (1.2) in section Ⅲ-2-a. One is "Da-Yan Qiu Yi Shu" 大衍求一術 (The Technique of Acquiring "One" in Dayan), which means the method for computing the k_i, for which Qin Jiushao's term[33] is "Cheng Lü" 乘率 (Multiplying Ratio). The other is to make mutually prime numbers, "Ding Shu" 定數 (Definite Numbers, called a_i hereafter) from "Wen Shu" 問數 (Problem Numbers, called A_i hereafter), the original divisors in the questions.

So the "Da-Yan Zong Shu Shu" consists of three steps, as follows:

[1] Changing the divisors of the question into mutually prime numbers.

$(A_1, A_2, \cdots A_n) \rightarrow (a_1, a_2, \cdots a_n)$

[2] Computing the k_i.

"Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan)

[3] "Sunzi Theorem" (Chinese Remainder Theorem).

Let us consider each step in the following sections, Ⅲ-2-c, d and e.

(c) "Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan)

Among these steps of "Da-Yan Zong Shu Shu" (The General Solution of Dayan Problems), step [2] is very important. Qin Jiushao creates a mechanical method for computing k_i using the algorithm of "Geng Xiang Jian Sun" 更相減損 ("Chinese Euclid's Algorithm"). Libbrecht [34] translated it into English; let us follow his version and analyse the algorithm:

① You multiply the "Ding Shu" 定數 (definite numbers, a_i) with each other, and you get the "Yan Mu" 衍母 (extension mother, called m hereafter).

② You divide the "Yan Mu" (extension mother) by all the "Ding Shu" and you obtain the "Yan Shu" 衍數 (extension numbers, called m_i hereafter).

③ Or you set up all the "Ding Shu" as "Mu" [factors] in the right column, and before all these, you set up the "Tian Yuan" 天元 (celestial element) 1 as "Zi" [factors] in the left column. By the "Mu" you mutually multiply the "Zi" and you get the "Yan Shu", too.

④ From all the "Yan Shu" you subtract all the [corresponding] "Ding Mu" as many times as possible. The part that does not suffice any more [literally, the incomplete part], is called the "Qi" [remainder].

⑤ On the "Qi" 奇 (remainder of step 0, called r_0 hereafter) and "Ding Shu" (definite numbers) one applies the "Da-Yan Qiu Yi Shu". With this method one will find the "Cheng Lü" (multiplying ratios). [Those of which one gets the remainder 1 are the "Cheng Lü"].

⑥ The "Da-Yan Qiu Yi Shu" method says: Set up the "Qi" at the right hand above, the "Ding Shu" at the right hand below. Set up "Tian Yuan" 1 at the left hand above.

⑦ First divide the "right below", and the quotient obtained, multiply it by the "left below". Set it up the left hand below [in the second disposition].

⑧ After this, on the "upper" and "lower" of the right column, divide the larger number by the smaller one. Transit [the numbers to the following diagram] and divide them by each other. Next bring over the quotient obtained and cross-multiply with each other. Add the "upper" and the "lower" of the left column.

⑨ One has to go on until the last remainder ["Qi"] of the "upper right" is 1 and then one can stop. Then you examine the result on the "upper left"; take it as the "Cheng Lü".

⑩ Sometimes the "Qi" [remainder] is already 1; this is then the "Cheng Lü". (35)

A short explanation is in order concerning Qin Jiushao's use of the terms "left", "right", "below" and "above" in his instructions for following "Geng Xiang Jian Cun" ("Chinese Euclid's Algorithm").

Chinese mathematicians computed numerical values using counting rods on the counting board. They divided the counting board into four parts for this computation.

left above	right above
left below	right below

In my explanation and analysis, I will respect this "matrix organization" by representing the result obtained at each step of the algorithm in the same graphical form (see Table 2-1, 2-2).

Qin Jiushao also used the very suggestive terms, "Tian Yuan" (celestial element), "Qi" (remainder) that can be more transparently represented using algebraic symbols. Li Yan's work (36) is the best in this regard, so I will use algebraic symbols which he defined to explain this algorithm.

The modern representation of the original problem is

$$x \equiv R_i \pmod{a_i}.$$

Qin Jiushao set his terms up in a way that suggests the following correspondence:

algebraic symbols	Qin Jiushao's original text
$m = \prod_{i=1}^{n} a_i$	: ① "Yan Mu" 衍母 (extension mother)
$m_i = m / a_i$	: ② "Yan Shu" 衍數 (extension numbers)
$r_0 = m_i - q_0 \times a_i$	: ④ "Qi Shu" 奇數 (remainder of step 0)

The "Geng Xiang Jian Cun" computes the k_i, "Cheng Lü" which satisfy the indeterminate equations;

$$k_i r_0 \equiv 1 \pmod{a_i}.$$

The method is a sort of "Euclid's Algorithm", i.e., divide the former divisor by the remainder until the last remainder becomes 1 (because the last remainder becomes the Greatest Common Divisor (called G.C.D. hereafter), thus it becomes 1).

$$m_i = q_0 \times a_i + r_0$$ [37]

$$a_i = q_1 \times r_0 + r_1$$
$$r_0 = q_2 \times r_1 + r_2$$
$$r_1 = q_3 \times r_2 + r_3$$
$$\vdots$$
$$r_{(n-2)} = q_n \, r_{(n-1)} + r_n \quad (r_n = 1)$$

In order to transform these expressions into expressions using only r_0 and a_i, Li Yan set up the following equations (some formulae were already explained by Qin Jiushao, as indicated by the circled indices in the translated text, e.g., ⑦, ⑧, etc.).

algebraic symbols		Qin Jiushao's original text
$\rho_n = q_n \, \rho_{(n-1)} + \rho_{(n-2)}$	:	none
$\alpha_n = q_n \, \alpha_{(n-1)} + \alpha_{(n-2)}$	:	⑦⑧
$\rho_0 = 0$	:	none
$\rho_1 = 1$	:	none
$\alpha_{-1} = 0$	:	(zero)
$\alpha_0 = 1$	:	⑥ "Tian Yuan" (celestial element) [38]

Now we can clearly see how the remainders at each step (r_n) can be expressed in terms of r_0 and a_i

$r_1 = \rho_1 a_i - \alpha_1 r_0 \quad : \rho_1 = 1 \qquad , \alpha_1 = q_1 \alpha_0 + \alpha_{-1} = q_1$

$r_2 = -(\rho_2 a_i - \alpha_2 r_0) : \rho_2 = q_2 \rho_1 + \rho_0 , \alpha_2 = q_2 \alpha_1 + \alpha_0$

$r_3 = \rho_3 a_i - \alpha_3 r_0 \quad : \rho_3 = q_3 \rho_2 + \rho_1 , \alpha_3 = q_3 \alpha_2 + \alpha_1$

$\vdots$

$$r_n = (-1)^{n+1} (\rho_n a_i - \alpha_n r_0) : \left.\begin{array}{l} \rho_n = q_n \rho_{(n-1)} + \rho_{(n-2)} \\ \alpha_n = q_n \alpha_{(n-1)} + \alpha_{(n-2)} \end{array}\right\} \cdots\cdots (2.1)$$

Because $r_n = 1$,

$$\alpha_n r_0 \equiv 1 \pmod{a_i}$$

$$\alpha_n m_i \equiv 1 \pmod{a_i}$$

$$\therefore \quad k_i = \alpha_n .$$

Qin Jiushao computed this algorithm on the counting board mechanically. According to Qin Jiushao's text, the α_{2k} were put at "left above" and the $\alpha_{(2k-1)}$ were put at "left below" (see ⑦- ⑨) as in table 2-1. But we must pay attention to the paragraph which states that "One has to go on until the last remainder of the 'upper right' is 1 and then one can stop." That is to say, this operation is done an even number of times.

$$\begin{array}{|ll|} \hline \alpha_0 = 1 & r_0 \\ \alpha_{-1} = 0 & a_i \\ \hline \end{array} \xrightarrow{(q_1)} \begin{array}{|ll|} \hline 1 & r_0 \\ \alpha_1 & r_1 \\ \hline \end{array} \xrightarrow{(q_2)} \begin{array}{|ll|} \hline \alpha_2 & r_2 \\ \alpha_1 & r_1 \\ \hline \end{array} \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\xrightarrow{(q_{2m})} \begin{array}{|ll|} \hline \alpha_{2m} & r_{2m} = 1 \\ \alpha_{(2m-1)} & r_{(2m-1)} \\ \hline \end{array} \quad (2m=n)$$

TABLE 2-1 OPERATION OF "DA-YAN QIU YI SHU"

Sometimes $r_{(2k-1)}$ becomes 1 before reaching the computation of r_{2k}, in an odd number of steps (39). In this case, the answer is $\alpha_{(2k-1)}$ at "left below". However the answer is a negative (see expression (2.1)), so Qin Jiushao regards q'_{2k} as $q_{2k}-1$ (or $r_{2k}-1$ (40)) and continues computing one more time to change the answer from negative to positive. Thus he is always able to obtain the answer α'_{2k} at "left above".

$(q_{(2k-1)})$	$\alpha_{(2k-2)}$	$r_{(2k-2)}$	$(q_{2k}-1)$	$\alpha_{2k}-\alpha_{(2k-1)}$	1	true
........→	$\alpha_{(2k-1)}$	$r_{(2k-1)}=1$	→	$\alpha_{(2k-1)}$	1	

(q_{2k})	$\alpha_{2k}=a_i$	0	false
→	$\alpha_{(2k-1)}$	1	

TABLE 2-2 SPECIAL OPERATION OF "DA-YAN QIU YI SHU"

Let us consider that α'_{2k} is the positive answer using algebraic symbols.

Because $r_{2k}=0$ in formula (2.1)

$$\therefore \qquad \alpha_{2k} r_0 \equiv 0 \pmod{a_i}$$

but $(a_i, r_0)=1$, therefore

$$a_i \mid \alpha_{2k} \text{ [41]} \quad \cdots\cdots\cdots\cdots (2.2)$$

and Chinese (and Japanese) mathematicians regarded q'_{2k} as $q_{2k}-1$,

$$\alpha'_{2k}=(q_{2k}-1)\,\alpha_{(2k-1)}+\alpha_{(2k-2)} \cdot \quad \cdots\cdots\cdots\cdots (2.3)$$

On the other hand

$$\alpha_{2k}=q_{2k}\,\alpha_{(2k-1)}+\alpha_{(2k-2)}$$

$$\therefore \qquad \alpha'_{2k}=\alpha_{2k}-\alpha_{(2k-1)}$$

after formula (2.2)

$$\alpha'_{2k}\equiv\alpha_{2k}-\alpha_{(2k-1)} \pmod{a_i}$$

$$\equiv -\alpha_{(2k-1)} \pmod{a_i}$$

and $\alpha_{(2k-1)} < 0$

$\therefore \qquad \alpha'_{2k} > 0$

on the other hand

$$(q_{2k} - 1) \geqq 1, \quad \alpha_{(2k-2)} \geqq 1$$

and from formula (2.3)

$\therefore \qquad |\alpha'_{2k}| > |\alpha_{(2k-1)}|$

Qin Jiushao, however, did not explain the case of table 2-2. This method was the same as the last step of computing "Deng Shu" (G.C.D) [42], so he regarded this case as normal. In the *Shu Shu Jiu Zhang,* he used fourteen cases [43] of table 2-2.

(d) Changing the divisors of the question into mutually prime numbers

In "Da Yan Qiu Yi Shu" (The Method of Acquiring "One" in Dayan) computations, the r_n (or r_{2k}) must be 1, otherwise the k_i can not be obtained. r_n (or r_{2k}) is the G.C.D. between m_i and a_i, and $m_i = (\prod a_m)/a_i$. Thus "Da-Yan Qiu Yi Shu" requires that all of a_i be mutually prime. In order to treat problems in which the A_i "Wen Shu" (problem numbers) are not mutually prime, Qin Jiushao created the method of acquiring mutually prime a_i ("Yuan Shu") from the original divisors A_i.

This procedure seeks to construct a set of moduli whose numbers meet the following three conditions (44) :

(1) $A_i \mid a_i \qquad (i=1,2,\cdots,n)$

(2) $(a_i, a_j) = 1 \qquad (\forall i, j; i \neq j)$.

(3) $\prod_{m=1}^{n} a_m = [A_1, A_2, \cdots, A_n]$

According to these three conditions, transform

$$\begin{cases} A_i = Ka'_i \\ A_j = Ka'_j \\ A_k = a'_k \end{cases} \qquad \begin{matrix} (a'_m, a'_n) = 1 \\ (k=1, 2, \cdots n; k \neq i, j) \end{matrix}$$

into

$$\begin{cases} a_i = a'_i \\ a_j = Ka'_j \\ a_k = a'_k \end{cases}$$

$A_n \mid a_n$, i.e., condition (1), thus R_n is not changed;

$$b \equiv R_i \pmod{A_i} \Rightarrow b \equiv R_i \pmod{a_i}$$
$$b \equiv R_j \pmod{A_j} \Leftrightarrow b \equiv R_j \pmod{a_j}$$
$$b \equiv R_k \pmod{A_k} \Leftrightarrow b \equiv R_k \pmod{a_k}.$$

According to the "Chinese Remainder Theorem", the answer X is as follows when divisors are a_n,

$$X = R_i k_i m_i + R_j k_j m_j + R_k k_k m_k - cm$$

$$(m = \Pi a_n, \ m_m = \Pi a_n / a_m, \ c \in N)$$

Letting $X' = R_i k_i m_i + R_j k_j m_j$,

we have

$$X = X' + R_k k_k m_k - cm.$$

Let us check whether this answer meets the original conditions:

$$X \equiv R_p \pmod{A_p}, \qquad (p = 1, 2, \cdots n)$$

All A_p, except A_i, are a_p, therefore it is enough to consider the remainders of the divisor A_i.

Moreover, the remainders of $R_k k_k m_k$ and cm, when divided by A_p are:

$$R_k k_k m_k \equiv 0 \pmod{A_p}$$

$$cm \equiv 0 \pmod{A_p}$$

because $m_k \mid K$, $m_k \mid a_p$ and $m \mid K$, $m \mid a_p$, i.e., condition (3).

Therefore, we will only consider the remainder of X' by the divisor A_i, or Ka_i. But the remainder of $R_i k_i m_i$ divided by a_i is not the same as the remainder of $R_i k_i m_i$ when divided by Ka_i, and the remainder of $R_j k_j m_j$ divided by a_i is also not the same as the remainder of $R_j k_j m_j$ divided by Ka_i. Because

$$R_i k_i m_i \equiv R_i \pmod{a_i} \qquad \cdots\cdots\cdots (2.4)$$

and $$R_j k_j m_j \equiv 0 \pmod{a_i}, \qquad \cdots\cdots\cdots (2.5)$$

then

$$R_i k_i m_i \equiv R_i + p a_i \pmod{K a_i}$$

and $$R_j k_j m_j \equiv 0 + q a_i \pmod{K a_i}.$$

$$(0 \leqq p < K, \ 0 \leqq q < K)$$

Thus we cannot consider the remainder of $R_i k_i m_i$ and $R_j k_j m_j$ separately, so they must be considered together.

Let us consider the remainder of X' divided by K.

Since $m_i \mid K$,

$$R_i k_i m_i \equiv 0 \pmod{K} \qquad \cdots\cdots\cdots (2.6)$$

On the other hand,

$$R_j k_j m_j \equiv R_j \pmod{a_j}$$

or $$R_j k_j m_j \equiv R_j \pmod{K a'_j}$$

$\therefore$ $$R_j k_j m_j \equiv R_j \pmod{K} \qquad \cdots\cdots\cdots (2.7)$$

From expressions (2.6) and (2.7), it follows that

$$(X' - R_j) \mid K$$

From expressions (2.4) and (2.5),

$$(X' - R_i) \mid a_i$$

$\therefore$ $$(X' - R_i)(X' - R_j) \mid K a_i$$

$\therefore$ $$(X' - R_i)(X' - R_j) \equiv 0 \pmod{A_i}$$

$$X'^2 - (R_i + R_j) X' - R_i R_j \equiv 0 \pmod{A_i}$$

$\therefore$ $$X' \equiv R_i \pmod{A_i}$$

or $$X' \equiv R_j \pmod{A_i}$$

but $(X' - R_i) \mid a_i$, therefore

$$X' \equiv R_i \pmod{A_i}$$

and

$$X \equiv R_i \pmod{A_i}$$

Therefore, by meeting the three conditions (1)(2) and (3), it is possible to transform the "Wen Shu" numbers to "Ding Shu" numbers.

Then he considered how to change "Yuan Shu" numbers into mutually prime numbers. The general method, which he called "Yue Qi Fu Yue Ou" 約奇弗約偶 (divide odd, do not divide even), was "using the algorithm of 'Geng Xiang Jian Sun' ('Chinese Euclid Algorithm'), compute the 'Deng Shu' (G.C.D.). Divide the 'odd numbers', do not divide 'the even ones'" (see historical material following).

Qin Jiushao used the terms "odd numbers" and "even numbers", but these are not actual odd and even numbers. In fact sometimes both are odd numbers or both are even numbers. These terms must simply refer to "one side" and "the other".

Remaining one side and dividing the others, so the L.C.M. of processed numbers is the same as the L.C.M of original numbers.

We only considered the case when "Wen Shu" numbers are integral numbers above, however Qin Jiushao also considered the cases of decimal fractions and general fractions. He classified "Wen Shu" (problem numbers) into four categories, "Yuan Shu"元數 (lit. original numbers, integral numbers), "Shou Shu" 收數 (decimal fractions), "Tong Shu" 通數 (fractions) and "Fu Shu" 復數 (multiples of ten). Qin Jiushao developed these categories in order to treat astronomical constants, which are either decimal fractions or general fractions. For example, in question 2 of chapter 1 in *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections), Qin Jiushao used 365 1/4, i.e., 365.25, to number the days in one tropical year, and 29 499/940 to number the days in one lunar month. Thus these fractions had to be changed to "Yuan Shu" numbers, then to "Ding Shu" (definite numbers).

His method was:

"Yuan Shu" (integral numbers); first of all, using the algorithm of "Geng Xiang Jian Sun" ("Chinese Euclid Algorithm"), compute the "Deng (Shu)" 等 (數) (lit. equal number, G.C.D.). Divide the odd numbers, do not divide the even ones[45]. [Sometimes one divides and gets 5, and the other number is 10. In this case, one has to divide the even numbers and not the odd one.] Sometimes all the numbers are even. After all the numbers have been divided, we may keep only one even number.

Sometimes after dividing all the numbers there still remain numbers with "Deng (Shu)" (G.C.D.). Provisionally set them up until you can divide them with the others. Finally, find "Deng (Shu)" (G.C.D.) of ones you have provisionally set up and divide them. Or (if) all numbers cannot be changed (to "Ding Shu" (definite numbers)), manage them as "Fu Shu" (multiply number).

"Shou Shu" (decimal fractions) are the numbers whose last part is "Fen" 分 (1/10) or "Li" 釐 (1/100). Move "Wen Shu" (problem numbers) some columns until they become integral numbers. Then manage them as "Yuan Shu" (integral numbers). Or choose any number as the denominator, change ("Wen Shu" (problem numbers)) into fractions (using chosen denominator). Then manage them as "Tong Shu" (fractions).

"Tong Shu" (fractions): Set up "Wen Shu" (problem numbers), then transform the improper fraction into a proper fraction[46]. These are named "Tong Shu" (fractions).

Compute the "Zong Deng (Shu)" 總等 (數) (G.C.D. of all numbers), (choose one number and) do not divide it, divide the others by "Zong Deng (Shu)" (G.C.D.). You obtain "Yuan Fa Shu" 元法數 (lit. original method numbers). Manage them as "Yuan Shu" (integral number).

If the denominators are not mutually prime numbers, i.e., these are able to be reduced to a common denominator, do not use the original denominators. Compute "Deng (Shu)" (G.C.D.) of all denominators, do not divide one number, divide the others by "Deng (Shu)" (G.C.D.). You also obtain "Yuan Fa Shu" (original method numbers). Manage them as "Yuan Shu" (integral numbers).

"Fu Shu" 復數 (lit. multiplied number) is the "Wen Shu" (problem number) whose last part is ten or more. Compute "Zong Deng (Shu)" (G.C.D. of all numbers) from all numbers, do not divide one number, divide the others by it. You obtain "Yuan Shu" (integral numbers).

Using the algorithm of "Geng Xiang Jian Sun" ("Chinese Euclid Algorithm"), compute "Deng (Shu)" (G.C.D.), divide the odd numbers, do not divide the even ones. (Compute next "Deng (Shu)", divide the even numbers,) multiply the odd numbers by it ("Fu Cheng" 復乘 (multiply again). Or divide the even numbers, do not[47] divide the even ones. (Compute new "Deng (Shu)" (G.C.D.), divide the odd numbers,) multiply the even numbers by it. Or if numbers still have "Deng (Shu)" (G.C.D.), compute "Xu Deng (Shu)" 續等 (數) " (next G.C.D.) and divide the one by next G.C.D. and multiply another by next G.C.D.. You obtain "Ding Shu" (Definite Numbers). In three categories of "Yuan Shu" (integral number), "Shou Shu" (decimal fraction) and "Tong Shu" (fraction), you must sometimes do "Fu Cheng" (multiply again). In this case, do this procedure. [48]

He explained these four categories, and considered the cases in which the divisors were fractions, decimal fractions, multiplies of ten and integral numbers. First, he transformed them into integral numbers.

In the case of "Shou Shu" (decimal fraction); "move 'Wen Shu' some columns until it becomes an integral number". That is to say,

365.25 →36525.

There is no example of "Shou Shu" numbers in the *Shu Shu Jiu Zhang*. And "Shou Shu" numbers are sometimes transformed into "Tong Shu" (fractions). It is very probable that "Shou Shu" numbers were more difficult than "Tong Shu" numbers for Chinese mathematicians in that age.

365.25 →365 1/4

The general method of "Tong Shu" numbers is; "transform the improper fraction into a proper fraction", and use only the numerator.

365 1/4=1461/4 →1461

That is, Qin Jiushao regarded "1/4" as the unit.

The question of "Gu Li Kuai Ji" 古暦會積 (L.C.M. of the ancient calendar), question 2 of chapter 1 in the *Shu Shu Jiu Zhang* treats the cases in which "Wen Shu" numbers are 365 1/4, 29 499/940 and 60.

365 1/4	→ 1461/4	→ 1461 ×940 ×1	→1373340
29 499/940	→27759/940	→27759 × 4 ×1	→ 111036
60	→ 60/1	→ 60 ×940 ×4	→ 225600

Then "Compute the 'Deng Shu' (G.C.D.), do not divide one by it, divide the others by the 'Deng Shu' (G.C.D.). You obtain 'Yuan Fa Shu' numbers."

The G.C.D. is 12;

(1373340, 111036, 225600)= 12

Thus "Yuan Fa Shu" numbers are;

$$1373340 \rightarrow 1373340/12 = 114445$$

$$111036 \rightarrow 111036/12 = 9253$$

$$225600 \rightarrow 225600 = 225600.$$

Generally, a "Yuan Fa Shu" number is one of "Yuan Shu" numbers, thus he used the general method;

$$(225600,\ 9253) = 1,$$

$$\begin{cases} 225600 \rightarrow 225600 \\ 9253 \rightarrow 9253 \end{cases}$$

then $$(114445,\ 225600) = 235,$$

∴ $$\begin{cases} 114445 \rightarrow 114445/235 = 487 \\ 225600 \rightarrow 225600 = 225600 \end{cases}$$

then $$(487,\ 9253) = 487,$$

∴ $$\begin{cases} 487 \rightarrow 487 = 487 \\ 9253 \rightarrow 9253/487 = 19 \end{cases}$$

Thus

$$\begin{cases} A_1 = 114445 \\ A_2 = 9253 \\ A_3 = 225600 \end{cases} \rightarrow \begin{cases} a_1 = 487 \\ a_2 = 19 \\ a_3 = 225600. \end{cases}$$

The special case is when the numerators are not mutually prime numbers, i.e., these are able to be reduced to a common denominator. This question, question 2 of chapter 1, is this special case, but Qin Jiushao did not use this special procedure —"Do not use the original denominators. Compute "Deng (Shu)" (G.C.D.) of all denominators, do not divide one number, divide the others by 'Deng Shu' (G.C.D.). You also obtain "Yuan Fa Shu" (original method numbers)".

Thus

$$(4,\ 940) = 4$$

$$
\begin{array}{llllllll}
365\ 1/4 & \rightarrow & 1461/4 & \rightarrow & 1461 \times 940 \times 1/4 & \rightarrow & 343335 \\
29\ 499/940 & \rightarrow & 27759/940 & \rightarrow & 27759 \times 4 \times 1/4 & \rightarrow & 27759 \\
60 & \rightarrow & 60/1 & \rightarrow & 60 \times 940 \times 4 & \rightarrow & 225699
\end{array}
$$

These "Yuan Fa Shu" numbers are not the same as the above, but by the general method of "Yuan Shu" numbers, we can obtain the same "Ding Shu" numbers as the above.

$$(225600,\ 27759) = 3,$$

$$\therefore \quad \begin{cases} 27759 \rightarrow 27759/3 = 9253 \\ 225600 \rightarrow 225600 = 225600 \end{cases}$$

then

$$(343335,\ 225600) = 705,$$

$$\therefore \quad \begin{cases} 343335 \rightarrow 343335/705 = 487 \\ 225600 \rightarrow 225600 = 225600 \end{cases}$$

then

$$(487,\ 9253) = 487,$$

$$\therefore \quad \begin{cases} 487 \rightarrow 487 = 487 \\ 9253 \rightarrow 9253/487 = 19. \end{cases}$$

Thus

$$\begin{cases} A_1{}' = 343335 \\ A_2{}' = 27759 \\ A_3{}' = 225699 \end{cases} \rightarrow \begin{cases} a_1 = 487 \\ a_2 = 19 \\ a_3 = 225600. \end{cases}$$

In this example, this method is a similar process to the above, thus Qin Jiushao used the general method of "Tong Shu" numbers.

In the case of "Fu Shu" numbers, the method is "Compute 'Deng Shu' (G.C.D.)

from all numbers, do not divide one number by it, divide the others by the G.C.D. You obtain 'Yuan Shu'."

The question of "Cheng Xing Ji Di" 程行計地 (measuring distance), question 2 of chapter 2 in the *Shu Shu Jiu Zhang*, is a typical example of "Fu Shu" numbers, Qin Jiushao made from "Wen Shu" numbers;

$$\begin{cases} A_1 = 300 \ (= 2^2 \times 3 \ \times 5^2) \\ A_2 = 240 \ (= 2^4 \times 3 \ \times 5 \ ) \\ A_3 = 180 \ (= 2^2 \times 3^2 \times 5 \ ), \end{cases}$$

G.C.D. among them is 60, do not divide A_1, but divide A_2 and A_3 by the G.C.D., into "Yuan Shu" numbers;

$$\begin{cases} A_1' = 300 \\ A_2' = \ \ 4 \\ A_3' = \ \ 3. \end{cases}$$

These "Yuan Shu" numbers are not mutually prime, thus we must continue to reduce. But "'Yuan Shu'" numbers transformed from "Fu Shu" numbers are special characters; the L.C.M. of "'Yuan Shu'" numbers tranaformed from "Fu Shu" numbers is equal to the G.C.D. of "Wen Shu" numbers, i.e.,

$$[A_1', \ A_2', \ A_3'] = (A_1, A_2, A_3),$$

thus they must be transformed by the "Fu Cheng" operation —"Compute next 'Deng Shu', divide the even numbers, and multiply the odd numbers by it".

These numbers are transformed into

$$\begin{cases} a_1 = 25 \\ a_2 = 16 \\ a_3 = \ \ 9, \end{cases}$$

by the "Fu Cheng" operation twice;

$$(300, 4) = 4$$

$$\therefore \quad \begin{cases} 300 \rightarrow 300/4 = 75 \\ 4 \rightarrow 4 \times 4 = 16, \end{cases}$$

then

$$(75, 3) = 3$$

$$\therefore \quad \begin{cases} 75 \rightarrow 75/3 = 25 \\ 3 \rightarrow 3 \times 3 = 9. \end{cases}$$

And these "Ding Shu" numbers meet three conditions.

Concluding the above, the relationship of the four categories of "Wen Shu" numbers and "Ding Shu" numbers is as in table 2-3.

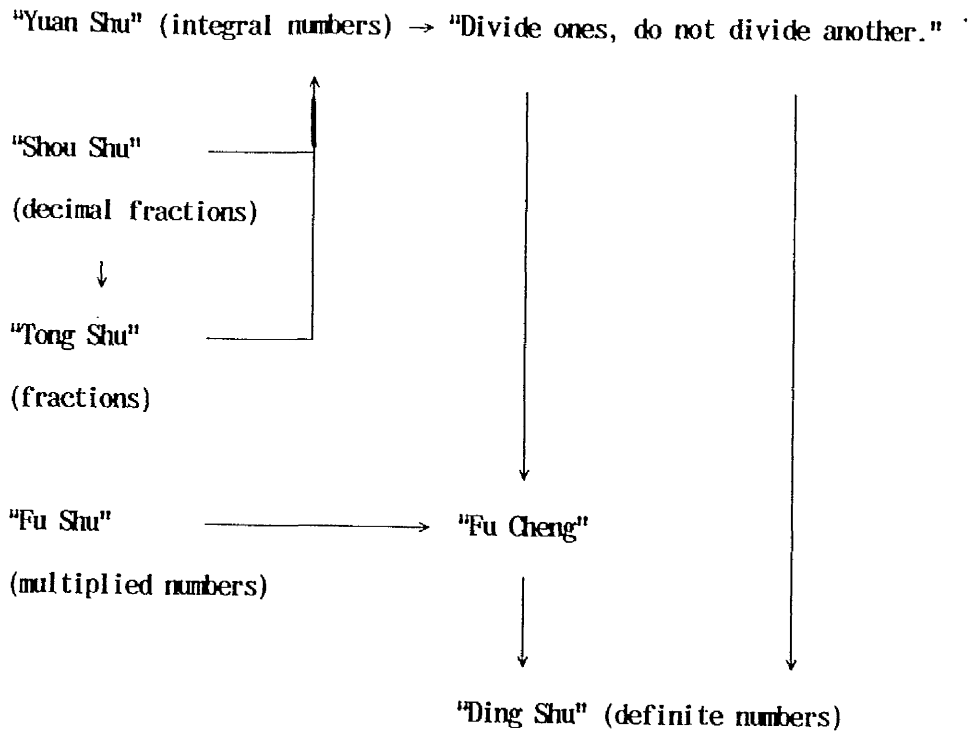


TABLE 2-3 The relationship of the four categories and "Ding Shu"

(e) "Fu Cheng" (multiply again)

The "Fu Cheng" (multiply again) operation is the special case of "Yuan Shu" numbers, while the general case of "Fu Shu" numbers.

The "Fu Cheng" operation is carried out when there is a remaining (next) G.C.D. after dividing one side by the first G.C.D.. In this case, operators must continue dividing one side by the next G.C.D., and, to assume the L.C.M. remains unchanged, multiply the other by the next G.C.D.. Qin Jiushao expressed it thus: "using the algorithm of 'Geng Xiang Jian Sun' ('Chinese Euclid Algorithm'), compute 'Deng Shu' (G.C.D.), divide the odd numbers, do not divide the even ones. (Compute next 'Deng Shu', divide the even numbers,) multiply the odd numbers by it".

This explanation is given in a part of "Fu Shu" numbers, that is, he regarded the "Fu Cheng" operation as the general method of "Fu Shu" numbers. But the "Fu Cheng" operation is neither a necessary condition nor a sufficient condition for transforming "Fu Shu" numbers into "Ding Shu" numbers. Of course, there is the first G.C.D., which is a multiple of ten, among "Fu Shu" numbers. Thus next operation is "Fu Cheng" operation after the first transformation, if there is a next G.C.D.. But there is sometimes no next G.C.D. among "Fu Shu" numbers after the first transformation. On the other hand, in some cases of "Yuan Shu" numbers, the "Fu Cheng" operation must be done (I will consider the case in which we must perform the "Fu Cheng" operation in the next paragraph). Therefore the "Fu Cheng" operation must be independent from his four categories as in table 2-3.

Why did Qin Jiushao describe the "Fu Cheng" operation in the part concerning "Fu Shu"? Are not there any relations between the "Fu Cheng" operation and "Fu Shu" numbers? Let us consider this problem. First, we will consider the general formula of "Wen Shu" in the case of "Fu Cheng" operation, then consider the relation between the "Fu Cheng" operation and "Fu Shu" numbers.

First of all, resolve two "Wen Shu" into factors. And let us consider the case in which there is one kind of common prime factor, i.e.,

$$A_l = \alpha^{n+k}\ \phi \qquad\qquad A_l\ ,\ A_m \in I$$

$$A_m = \alpha^{n}\ \ \omega. \qquad\qquad \alpha\text{: prime numbers}$$

ϕ, ω: possibly composite factors

k, $n \geqq 1$

The G.C.D. between A_l and A_m is α^n, so divide one of them by it, do not divide the other. If A_l were divided and A_m was not divided, they would become

$$A_l{}' = \alpha^k\ \phi$$

$$A_m\ \ = \alpha^n\ \omega.$$

There is new G.C.D. between $A_l{}'$ and A_m, $\alpha^{\min(k,n)}$. Thus we must do the "Fu Cheng" operation, divide A_m by the new G.C.D., and multiply $A_l{}'$ by new G.C.D..

$$A_l{}'' = \alpha^{n+k}\ \phi$$

$$A_m{}' = \alpha^0\ \ \omega$$

But if A_m were divided first, the answer would be the same as $A_l{}''$ and $A_m{}'$ directly. Thus it is not necessary to do the "Fu Cheng" operation.

Considering the above, in the case of one kind of common prime factor, we can avoid doing the "Fu Cheng" operation if we choose to divide first by the A_m with larger power of α.

There are, however, some cases in which we must do the "Fu Cheng" operation in the cases of multiple common factors between "Wen Shu". This is the case in which A_i and A_j can be written in the following forms:

$$A_i = \alpha^{n+k} \beta^m \phi$$
$$A_j = \alpha^n \beta^{m+l} \omega \qquad \cdots\cdots\cdots\cdots (2.8)$$

A_i, $A_j \in I$

α, β: prime numbers.

ϕ, ω: possibly composite factors

$k, l, m, n \geqq 1$

The first G.C.D. is $\alpha^n \beta^m$; divide A_i by it and do not divide A_j, or divide A_j by it and do not divide A_i, but there is still new G.C.D. in both cases. Thus we must do the "Fu Cheng" operation.

$$\begin{cases} A_i \\ A_j \end{cases} \longrightarrow [\alpha^n \beta^m] \quad \begin{array}{l} \nearrow \begin{cases} \alpha^k \quad \phi \quad (\text{"Fu Cheng"}) \\ \alpha^n \beta^{m+l} \omega \rightarrow [\alpha^{\min(k,n)}] \end{cases} \\ \searrow \begin{cases} \alpha^{n+k} \beta^m \phi \quad (\text{"Fu Cheng"}) \\ \beta^l \omega \rightarrow [\beta^{\min(l,m)}] \end{cases} \end{array} \longrightarrow \begin{cases} a_i = \alpha^{n+k} \phi \\ a_j = \beta^{m+l} \omega \end{cases}$$

If there is still G.C.D. after the first "Fu Cheng" operation, we continue to do the "Fu Cheng" operation, so we can obtain the "Ding Shu" numbers [49]. This method was the same as Seki Kōwa's (see section Ⅲ-3-h).

In any event, the final result is the same, and this meets three conditions;

(1) $a_i \mid A_i$, $a_j \mid A_j$

(2) $(a_i, a_j) = 1$

(3) $a_i \times a_j = [A_i, A_j] \quad (= \alpha^{n+k} \beta^{m+l} \phi\omega)$

Next, we must consider "Fu Shu" numbers. These are the numbers of multiple of ten. And the number ten is the product of two prime numbers, two and five, i.e., it is the case of $\alpha = 2$ and $\beta = 5$ in formula 2-8. Of course, only the number ten itself does not meet the condition of formula 2-8. For example, the case of $A_i = 30$, $A_j = 70$, it is not necessary to do the "Fu Cheng" operation;

$$(30, 70) -[10]\rightarrow(3, 70)$$
$$-[10]\rightarrow(30, 7)$$

The numbers 2 and 5, however, are two of the simplest prime numbers. Sometimes A_i, rather than $A_i/10$, is a multiple of 2 and A_j, rather than $A_j/10$, is a multiple of 5. In this case, we must do the "Fu Cheng" operation. And these cases are not rare case. For example, if $A_i = 20$ and $A_j = 50$,

$$(20, 50) -[10]\rightarrow(2, 50) -[2]\rightarrow(4, 25)$$
$$-[10]\rightarrow(20, 5) -[5]\rightarrow(4, 25)$$

If the other factors of "Fu Shu" numbers met the condition of formula 2-7, of course, it would be the case of "Fu Cheng" operation. Therefore the probability of "Fu Cheng" operation in "Fu Shu" numbers is stronger than in "Yuan Shu" numbers. If the cases in which the one side met the condition of formula 2-7 were "Fu Cheng" operation, they would be stronger than the "general" operation. I think very probably that Qin Jiushao found many examples of "Fu Cheng" operations among "Fu Shu" numbers, and that he therefore regarded the "Fu Cheng" operation as the general operation in "Fu Shu" numbers.

(f) "Jian Guan Shu" 翦管術 (the Technique of Cutting Lengths of Tube) in the *Yang Hui Suan Fa* (Yang Hui's Method of Computation)

The indeterminate equations in the *Yang Hui Suan Fa* (Yang Hui's Method of Computation) are very simple though the *Yang Hui Suan Fa* was published after *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections). And even the terms of the *Yang Hui Suan Fa* also differ from the *Shu Shu Jiu Zhang*. Two key issues for solving indeterminate equations: "Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan) and for making a_i mutually prime, are not described in the *Yang Hui Suan Fa*.

The problems of the *Yang Hui Suan Fa* are:

$$x \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 2 \pmod{7}$$

$$x \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 0 \pmod{7}$$

$$x \equiv 1 \pmod{7} \equiv 2 \pmod{8} \equiv 3 \pmod{9}$$

$$x \equiv 3 \pmod{11} \equiv 2 \pmod{12} \equiv 1 \pmod{13}$$

$$x \equiv 1 \pmod{2} \equiv 2 \pmod{5} \equiv 3 \pmod{7} \equiv 4 \pmod{9}$$

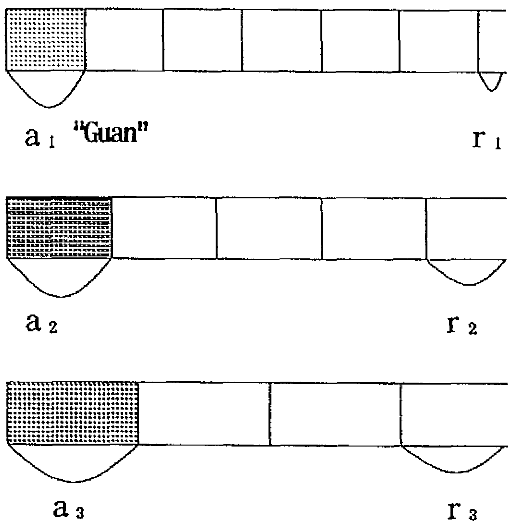
These divisors are not the same as in *Sun-zi Suan Jing* (Master Sun's Mathematical Manual) — Yang Hui had advanced a little beyond the 4th century treatise. But the divisors in the problems are already mutually prime as given. The methods put forth in the *Yang Hui Suan Fa* to solve these problems are on the level of those given in the *Sun-zi Suan Jing*, and do not attain the level of the solutions found in the *Shu Shu Jiu Zhang*. As indicated in chapter 1, above, there is good evidence that Seki Kōwa had access to the *Yang Hui Suan Fa*, but did not study the *Shu Shu Jiu Zhang*. As for the comparison of his methods and the methods presented in the *Shu Shu Jiu Zhang*, I shall return to this point in my conclusion to this chapter, after a thorough analysis of the methods of the *Yang Hui Suan Fa*.

The *Yang Hui Suan Fa* usually uses the term "Jian Guan Shu" 翦管術 (the

Technique of Cutting Lengths of Tubes), but also introduces other names; "Qin Wang An Dian Bing" 秦王暗點兵 (Prince Qin's[50] secret method of counting soldiers) and "Fu She" 覆射 (guess the thing covered by the cup).

The two former terms refer to fairly obvious situations producing indeterminate equations.

"Guan" (tube) of "Jian Guan Shu" must be lengths of bamboos' cells, and it is equivalent to with the divisors, "Ding Shu" (fixed numbers) or "Wen Shu" (problem numbers). And "Jian" (cut) suggests dividing. That is to say, there are some bamboos whose length are the same. Also we know the lengths of the bamboo's cells and the remaining length at the end. The question is to compute the whole length.



"Qin Wang An Dian Bing" (the Prince Qin's secret method of counting soldiers) refers to the situation where soldiers are assembled in rank formations, and from the remainder of each line the total number of soldiers may be found.

"Fu She" (guess the thing covered by the cup) was explained by Li Jimin [51], "Fu She" means the same as "She Fu". "She Fu" was a game linked with the *Yi Jing* method of divination. The oldest material on "Fu She" can be found in the chapter entitled *Dong-Fang Shuo Zhuan* 東方朔傳 (biography of Dongfang Shuo),

which is chapter 65 of the *Han Shu* 漢書 (History of the Western Han Dynasty).

> The emperor let "Shu Jia" 數家 ("Reckoner") guess the matter covered by some cups.
>
> Yan Shigu's 顏師古 (581-645) comment: "Shu Jia" means "Shu Shu Zhi Jia" 術數之家 (numerological diviners). Something is hidden by the cup which is laid face down, and guess the object was to them, so it was named "She Fu". [52]

Yang Hui probably collected questions of indeterminate equations from books, since there is no original work by Yang Hui in the *Yang Hui Suan Fa*.

(g) Before Seki Kōwa in Japan

There is no direct evidence that the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections) was ever brought to Japan. The evidence suggests that indeterminate equations were probably introduced into Japan by the *Yang Hui Suan Fa* (Yang Hui's Method of Computation) and the *Suan Fa Tong Zong* (Systematic Treatise on Arithmetic): the same term, "Jian Guan Shu" 翦管術 (or Japanese style pronunciation "Sen Kan Jutsu" 翦管術) to describe the method of cutting tubes was used. The achievement of these books, however, is not more advanced than that of the *Shu Shu Jiu Zhang*. They use many sets of divisors; a_i, but these a_i are mutually prime. And there are no rationales for the most important part; "Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan).

Some applications from this age can be found. "Metsuke-Ji" 目付字 (Find the Chinese Character in the Tables) was one of the most popular mathematical games in Japan. The most typical one is described in the *Jinkō-ki* (Permanent Mathematics), where some Chinese characters are set up in matrices, as in tables 3 and 4.

b_{11}	b_{21}	b_{31}
b_{12}	b_{22}	b_{32}
b_{13}	b_{23}	b_{33}

TABLE 3

b_{11}	b_{12}	b_{13}
b_{21}	b_{22}	b_{23}
b_{31}	b_{32}	b_{33}

TABLE 4

An optional character is selected from table 1. Let it be known in which file it occurs in, and let this number be k. Then let it be known in which column it is shown in table 4, and let this number be m. The character is found at a_{km} in table 4.

Isomura Yoshinori 礒村吉徳 (1640?-1710) applied indeterminate equations to "Metsuke-ji" (53) in the chapter entitled, "Iroha Metsuke-ji" いろは目付字 (Find the Chinese Character in the Tables by Japanese Character's Order, written in 1659 [54],

He used 47 Japanese Hiragana-characters 平假名, which are numbered 1 to 47, and selected any character in it. Then it is counted by three types of measure. These are by 7, "I Ro Ha Ni Ho He To" いろはにほへと which are the first seven Hiraganas of the old order, by 5, "Ye Hi Mo Se Zu" ゑひもせず which are the last five Hiraganas, and by 3, Sho Chû Go 初中後 (First, Middle, Last), and if the selected character were pointed up at "Ni" に, "Ye" ゑ and "Chû" 中, it means;

$$x \equiv 4 \pmod 7 \equiv 1 \pmod 5 \equiv 2 \pmod 3$$

Then he solves this indeterminate equations and gets the answer of 11 which is "Ru" る.

He stated that this method is the same as the computation of "Hyaku-go Gen" 百五減 (Take Away 105) of the *Jinkō-ki* 塵劫記 (Permanent Mathematics). In "Hyaku-go Gen" computation, 105 answers are possible, from 0 to 104. But in "Iroha Metsuke-ji" must be 47 answers, the number of Japanese characters, not 105. Thus many cases of combination of R_i are impossible, for example

$$6 \pmod 7 \equiv 3 \pmod 5 \equiv 0 \pmod 3).$$

He probably did not consider the relationship between the product of divisors ($=m$, "Yan Mu") and how many answers are possible. The product of divisors is the cycle, i.e., the numbers of remainder term, for example, 1 and 106 are the same remainders, R_i , in "Hyaku-go Gen". Anyhow, it is the first case of original application in Japan.

Hoshino Sanenobu 星野實宣 (1628-1699) also tried to apply indeterminate equations to "Metsuke-ji" game in "Metsuke 60-ji" 目付字六十字 (Find the Sixty Chinese Characters in the Tables) on the *Ko Kō Gen Shō* 股句弦鈔 (Manuscript of Triangle's Three Side) written in 1672;

He used 60 Chinese characters, which are numbered 1 to 60 now, and made these three matrices having 3, 4 and 5 rows, respectively (see table 5);

	rows (R_i)		$R_i\,k_i\,m_i$ (mod m)
GROUP 1	1	1 4 7 10 ··········58	40
	2	2 5 8 11 ··········59	20
	3	3 6 9 12 ··········60	0
GROUP 2	1	1 5 9 13 ·······57	45
	2	2 6 10 14 ·······58	30
	3	3 7 11 15 ·······59	15
	4	4 8 12 16 ·······60	0
GROUP 3	1	1 6 11 16·····56	36
	2	2 7 12 17·····57	12
	3	3 8 13 18·····58	48
	4	4 9 14 19 ····59	24
	5	5 10 15 20 ····60	0

TABLE 5 Matrix of "Metsuke 60-ji"

Select any character, and let it be known to which row it belongs in each of the groups. Determining the identity of the character, given knowledge of only this information, is equal to solving the indeterminate equations

$$x \equiv R_1 \pmod{3} \equiv R_2 \pmod{4} \equiv R_3 \pmod{5}.$$

There are two new points to this. One is that he understood the relationship between the products of divisors and the cycle. In this computation, 60 answers are possible, and the number is the same as the product of divisors. And Hoshino Sanenobu use the value of $R_i\,k_i\,m_i$ which takes away m, is not original. Because $R_i k_i m_i$ easily becomes a huge value, so he computed $R_i k_i m_i \pmod{m}$ before computing $\Sigma R_i k_i m_i$. The other point is that we are no longer limited to the case of $a_i = 3,5,7$, which is the case of the *Sun-zi Suan Jing*. This was the first case in Japan.

Next, Hoshino Sanenobu computed the case that a_i are not mutually prime.

$$x \equiv 5 \pmod{6} \equiv 7 \pmod{8} \equiv 5 \pmod{10} \qquad \cdots\cdots(2.1)$$

The answer, given without comment, is

$$x \equiv 5 \times 40 + 7 \times 45 + 5 \times 36 \pmod{120}$$
$$= 95$$

This answer is correct, but it entails the following equations:

$$k_1 m_1 = 40 \qquad \cdots\cdots(2.2)$$
$$k_2 m_2 = 45 \qquad \cdots\cdots(2.3)$$
$$k_3 m_3 = 36 \qquad \cdots\cdots(2.4)$$

If we compute according to the method of "The General Solution of Dayan Problems";

$$A_1 = 6 = 2 \times 3 \quad \rightarrow \quad a_1' = 3$$
$$A_2 = 8 = 2^3 \quad \rightarrow \quad a_2' = 8$$
$$A_3 = 10 = 2 \times 5 \quad \rightarrow \quad a_3' = 5$$
$$x \equiv 5 \pmod{3} \equiv 7 \pmod{8} \equiv 5 \pmod{5} \qquad \cdots\cdots(2.5)$$

Therefore, the values in formula (2.1) and (2.5) are the same. Solving formula (2.5),

$k_1' m_1' = 40$ ····················(2.6)

$k_2' m_2' = 105$ ····················(2.7)

$k_3' m_3' = 96$ ····················(2.8)

The values of formula (2.7) and (2.8) are not the same, and

$$k_1 m_1 = k_1' m_1'$$
$$k_2 m_2 = k_2' m_2' - \tfrac{1}{2}m$$
$$k_3 m_3 = k_3' m_3' - \tfrac{1}{2}m$$

The values in formula (2.3) and (2.4) are acquired by taking away $\frac{1}{2}m$ from the value computed by "The General Solution of Dayan Problems". This is done with the aim of simplifying the computing of the sum total. However, the value taken away is not m, but $\frac{1}{2}m$, so this calculation does not hold good generally.

However, $|a_2 - a_3| = 2$

So $R_2 + R_3 \mid 2$

$\therefore$ $R_2 (k_2 m_2 - \tfrac{1}{2}m) + R_3 (k_3 m_3 - \tfrac{1}{2}m)$

$= R_2 k_2 m_2 + R_3 k_3 m_3 - \tfrac{1}{2}m (R_2 + R_2)$

$\equiv R_2 k_2 m_2 + R_3 k_3 m_3 \pmod{m}$ ················(2.9)

This case is a special one. We do not know how well Hoshino Sanenobu knew indeterminate equations and how he arrived at formula (2.9), but he was at least more advanced than the *Sun-zi Suan Jing* and the *Yang Hui Suan Fa*.

He did not only use indeterminate equations for mathematical games, but could also compute indeterminate equations whose a_i are not mutually prime.

(h) Works of Seki Kôwa

The first work which surmounted the level of the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections) is that of Seki Kôwa. His method of solving indeterminate equation is called "Senkan-Jutsu" (the Technique of Cutting Lengths of Tubes), and he probably received help from the *Yang Hui Suan Fa* (Yang Hui's Method of Computation) because the term is similar to one used in the latter work. However, "Senkan-Jutsu" is only the third step of "Da-Yan Zong Shu Shu" (general method of Dayan rule), which consisted of three steps. Seki Kôwa's method included the other two steps.

This method was described at chapter 2 of the *Katsuyō Sampō* 括要算法 (Essential Points of Mathematics) published in 1712. He described many questions in this chapter, but these are not only explanations for solving indeterminate equations. We can pick up three subjects;

step	Seki Kôwa's work	the function of Qin Jiushao's one
[1]	Goyaku 互約 (reduce each other)	"Fu Cheng" (multiply again)
[2]	Jôichi-jutsu 剰一術 (method of one remainder)	"Da-Yan Qiu Yi Shu"
[3]	Senkan-jutsu 翦管術 (cutting tube method)	"Sunzi Theorem" (Chinese Remainder Theorm)

TABLE 6 CONSTRUCTION OF SEKI KOWA'S METHOD

Seki Kôwa's method also consists of three steps, the basic stracture is the same as "Da-Yan Zong Shu Shu". However each step shows some advance. Let us consider each steps of Seki Kôwa's method.

First step is the making mutually prime numbers. Qin Jiushao touched the case of fraction or decimal fraction. But Seki Kôwa studied only the case of integral numbers.

He wrote at question 1 of "Goyaku" 互約 (reduce each other) in chapter 2 of the *Katsuyō Sampō*.

There are two numbers, six and eight. Divide each other, how many the numbers are obtained?

Answer: six becomes 3, eight is not divided.

Method: G.C.D. is obtained using the Chinese Euclid Algorithm, it is two. Then reduce six by it, six becomes three. [G.C.D. of three and eight becomes one. That is, if G.C.D. became one, stop reducing. Do the same method as the later problems.]

The other method: G.C.D. is obtained using the Chinese Euclid Algorithm, it is two. Then reduce eight by it, eight becomes four. G.C.D. of four and six is two, so multiply four by it, four becomes eight. Reduce six by it, six becomes three [55].

This example is for making a pair of numbers, (6,8) mutually prime numbers. The G.C.D. of (6,8) is 2, so reduce one of the numbers by 2;

$$(6,8) \to (3,8)$$

or

$$(6,8) \to (6,4)$$

But in the latter case, there still remains the G.C.D., 2, so reduce 6 by 2, and multiply 4 by 2;

$$(6,8) \to (6,4) \to (3,8)$$

This solution is the same as "Fu Cheng" (multiply again) of the *Shu Shu Jiu Zhang*. Seki Kōwa commented clearly, "if G.C.D. became one, stop reducing." i.e., he pointed out the need to continue to "Fu Cheng" computation. He supplements Qin Jiushao's brief explanation.

Let us consider step 3 firstly. Although Seki Kōwa used the same term as that of Yang Hui, but the content is not only the imitation, but it is also the solution of general indeterminate equation of first degree. He solved the following indeterminate equations;

$$b_i x \equiv R_i \pmod{a_i} \qquad \cdots\cdots (2.10)$$

His new work was written at chapter 2 of the *Katsuyō Sampō*. This time, he used "Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan), which he called "Jōichi-jutsu" 剰一術 (Technique of One Remainder) twice. He sets up the equivalence

$$l_i b_i \equiv 1 \pmod{a_i} \qquad \cdots\cdots (2.11)$$

and he computes l_i. Then he continues as follows, computing k_i:

$$k_i m_i \equiv 1 \pmod{a_i} \qquad \cdots\cdots (2.12)$$

He multiplies the left side of expression (2.11) by the left side of expression (2.12),

$$b_i k_i l_i m_i \equiv 1 \pmod{a_i}$$

Then he also multiplies by R_i

$$b_i R_i k_i l_i m_i \equiv R_i \pmod{a_i} \qquad \cdots\cdots (2.13)$$

and posits $i \neq j$, because $m_i \mid a_j$, so

$$b_i R_i k_i l_i m_i \equiv 0 \pmod{a_j}$$

$$\therefore \quad \Sigma b_i R_i k_i l_i m_i \equiv R_i \pmod{a_i}$$

$$\therefore \quad x = \Sigma R_i k_i l_i m_i$$

$$x_{min} = \Sigma R_i k_i l_i m_i - pm$$

This procedure is not "Sunzi Theorem" (Chinese Remainder Theorem) itself, but the way of thinking is the same in both. Seki Kôwa only improved "Sunzi Theorem" a little.

And step 2 is; in computing l_i, k_i, he used "Jōichi-jutsu" (method of one remainder): This method is presented in question 2, chapter 2 of the *Katsuyō Sampō* (Essential Points of Mathematics), his question is the solving the equation as follows;

$$179k - 74p = 1$$

Or $$179k \equiv 1 \pmod{74}$$

And he comments:

A number, which is a multiple of 179, is on the left, from which a multiple of 74 is subtracted, giving a remainder of 1. What is the "Sôsû" (general number, $= 179k$) on the left?

ANSWER: 7697.

METHOD: Reduce 179, on the left, by 74, on the right [If the left is smaller than the right, do not reduce, or the remainder is (just) the first number of left]. Reduce 74, on the right, by 31 (which is "Qi Shu" of "Da Yan Qiu Yi Shu"), on the left, and the quotient is 2. The remainder, which is named "kô" 甲, is 12.

Reduce 31, on the left, by 12, "kô", and the quotient is 2. The remainder, which is named "Otsu" 乙, is 7.

Reduce 12, "kô" by 7 of "Otsu" and the quotient is 1. The remainder, which is named "Hei" 丙, is 5.

Reduce 7, "Otsu" by 5, "Hei", and the quotient is 1. The remainder, which is named "Tei" 丁, is 2.

Reduce 5, "Hei" by 2, "Tei", and the quotient is 2. The remainder, which is named "Bo" 戊, is 1.

Take away 1, "Bo" from 1 of "Tei", the quotient is regard as 1.

The remainder, which is named "Ki" 己, is 1. (Because the remainder on the left became 1, stop computing) .

Multiply the quotient of "Kô" and the quotient of "Otsu", then add 1, the answer, which is named "Shi" 子, is 5.

Multiply "Shi" and the quotient of "Hei", then add the quotient of "ko" and the answer, which is named "Chû"丑, is 7.

Multiply "Chû" and the quotient of "Tei", then add "Shi", the answer, which is named "In" 寅, is 12.

Multiply "In" and the quotient of "Bo", then add "Chû"; the answer, which is named "U" 卯, is 31.

Multiply "U" and the quotient of "Ki", then add "In", the answer is 43 (that is, the number on the left) .

Then multiply 179, on the left, by it. The answer, the "Sôsû" (general number) on the left, is 7697, and that is right answer [56].

$179=2\times 74+31$ (the left)

$74=2\times 31+12$ ("Kô")

$31=2\times 12+7$ ("Otsu")

$12=1\times 7+5$ ("Hei")

$7=1\times 5+2$ ("Tei")

$5=2\times 2+1$ ("Bo")

$2=1\times 1+1$ ("Ki")

This method is similar to the one used in "Da-Yan Qiu Yi Shu", but has been improved by the use of algebraic symbols, like "kō" and "Otsu". This calculation is limited, so this advantage of Seki Kōwa's method is not clear. However, when calculating unlimited values, e.g., the approximate value of the root by fraction, his method exhibits its power [57].

We must examine the computation of "Bo" and "Ki". "Bo" had already been 1, but Seki Kōwa continued to compute, and obtained "Ki" of 1. This method is exactly the same as the special case of "Da-Yan Qiu Yi Shu".

Moreover, it is interesting that Seki Kôwa uses the terms "left" and "right". Doesn't that remind us of the position in the method of Qin Jiushao? (see Table 2 in section Ⅲ-3-c).

(i) Advanced works in China and Japan in the 19th and 20th centuries

After Seki Kôwa, some Chinese and Japanese mathematicians improved the solutions of indeterminate equations. They improved each of the three steps, but the construction was the same as "Da-Yan Qiu Yi Shu" (The Technique of Acquiring "One" in Dayan).

The method of making mutually prime numbers from problem numbers, Qin Jiushao's method, gives the correct answer, but it is necessary to compute G.C.D. several times. And it is quite difficult to choose which number to divide first. So there is scope for improving it, as seem for example in Huang Zongxuan's 黃宗憲 (fl. 19c) method [58] —the divisors are resolved into factors, (his term was "Fan Mu" 泛母, lit. extention mother), then the highest degree of each prime numbers is retained and the others are erased, for example,

$$6 = \underline{2} \times 3 \quad \rightarrow 3$$
$$8 = 2^3 \times \underline{3^0} \quad \rightarrow 8.$$

The other method of "Da-Yan Qiu Yi Shu" was discovered in 1929. This formula was as follows:

Let $$k_i m_i \equiv 1 \pmod{a_i}$$

$$k_i = r_0 \left\{ -\frac{1}{r_0 r_1} + \frac{1}{r_1 r_2} + \cdots + (-1)^{n-1} \frac{1}{r_{n-1} r_n} \right\}$$

"Da-Yan Qiu Yi Shu" utilised used the sequence of quotients q_n, but these quotients can be rewritten in the following form (see section Ⅲ-3-c);

$$r_n = r_{n-2} - q_n r_{n-1}$$

We see by this transformation the equivalence of "Da Yan Qiu Yi Shu" and the more modern formulation (i.e., complex fractions), that is, between the q_n and r_n. But this computation, involving complex fractions, is not in the Chinese tradition [59]. This method was found by Euler before 1734, and Tanigawa Hideyuki 谷川榮幸 (19-20c) and Hayashi Tsuruichi (1873-1935) rearranged it to illustrate "Da-Yan Qiu Yi Shu" [60].

Saitô Naonaka 齋藤尚中 (1773-1844), who was not in Seki Kôwa's School but in the competing School —Saijô-ryû 最上流 (lit. Best School) [61], endeavored to apply Seki Kôwa's works. In his manuscript, *Saitō Naonaka Sōkō* 齋藤尚中草稿 (Saito Naonaka's Manuscript), which is preserved at Nihon Gakushiin 日本學士院 (The Japanese Academy) [62], he described the improved "Sunzi Theorem" which is the third step of "Da-Yan Zong Shu Shu" (The General Solution of Dayan Rule) (see Table 1).

When we solved indeterminate equations of the form

$$x \equiv R_i \pmod{a_i},$$

we had to compute "Da-Yan Qiu Yi Shu" i times; if there were four congruence expressions, we would have to compute four "Cheng Lu" (Multiplying Ratio, k_i). But the method of "Da-Yan Qiu Yi Shu" is very complex. It is easy to make a mistake. Therefore, Saitô Naonaka improved the method so that it used "Da-Yan Qiu Yi Shu" only one time.

He solved as follows: let

$$x \equiv R_1 \pmod{a_1} \quad \text{------------------(2.14)}$$

$$\equiv R_2 \pmod{a_2} \quad \text{------------------(2.15)}$$

He designated $(a_1 \times R_2 + a_2 \times R_1)$ as "Ko" 甲 (A), $(a_1 + a_2)$ as "Hidari" 左 (Left), $(a_1 \times a_2)$ as "Migi" 右 (Right). Firstly, he computed x' as it appears in the following expression:

$$(a_1 + a_2)\, x' \equiv 1 \pmod{a_1 a_2} \quad \cdots\cdots\cdots\cdots(2.16)$$

left right

Then he computed the answer y, which depends on x', as follows:

$$y \equiv x' (a_1 R_2 + a_2 R_1) \pmod{a_1 a_2} \quad \cdots(2.17)$$

ko

This formula gives the correct answer, because y of formula (2.17) is equivalent to x of formula (2.14) and (2.15).

Let us show it.

Multiplying formula (2.14) by a_2, we have;

$$a_2 x \equiv a_2 R_1 \pmod{a_1 a_2} \quad \cdots\cdots\cdots(2.18)$$

and multiplying formula (2.15) by a_1, yields

$$a_1 x \equiv a_1 R_2 \pmod{a_1 a_2} \quad \cdots\cdots\cdots(2.19),$$

add (2.18) and (2.19),

$$(a_1 + a_2)\, x \equiv a_1 R_2 + a_2 R_1 \pmod{a_1 a_2}$$

and divide by $(a_1 R_2 + a_2 R_1)$, to obtain

$$\frac{a_1 + a_2}{a_1 R_2 + a_2 R_1}\, x \equiv 1 \pmod{a_1 a_2} \quad \cdots\cdots(2.20)$$

On the other hand, from formula (2.17),

$$y = (a_1 R_2 + a_2 R_1)\, x' + p\, (a_1 a_2)$$

when $p = 0$,

$$x' = \frac{y}{a_1 R_2 + a_2 R_1}$$

substitute it into formula (2.16)

$$\frac{a_1 + a_2}{a_1 R_2 + a_2 R_1} y \equiv 1 \pmod{a_1 a_2} \quad \cdots\cdots (2.21)$$

Comperativing formula (2.20) and (2.21), we can conclude

$$y = x.$$

Saitō Naonaka gives details of his method for application of "Senkan Jutsu" (cutting tube method) increasing up to three simultaneous congruences, but we can obtain the general formula;

$$\sum_{i=1}^{n} a_i \, x' \equiv 1 \left(\bmod \prod_{i=1}^{n} a_i\right)$$
$$x \equiv x' \left(\sum_{k=1}^{n} \left(\prod_{i=1}^{n} a_i / a_k \right) R_k \right) \left(\bmod \prod_{i=1}^{n} a_i\right)$$

Because he also used algebra method it is easy to obtain this formula.

His originality is finding the multiplication method in the indeterminate equations;

$$x \equiv R \pmod{a} \quad \Leftrightarrow \quad kx \equiv kR \pmod{ka}$$

He probably obtained a hint from the works of Seki Kōwa; because Seki Kōwa developed the method of solving indeterminate equations whose coefficients of first degree are not one, such as;

$$b_i x \equiv R_i \pmod{a_i}$$

Because it looks unlikely that he considered only formula for computing "Shang Yuan Ji Nian" (the accumulated years from the epoch);

$$x \equiv R_i \pmod{a_i}$$

Moreover his terms are similar to those of Seki Kôwa, "Migi" (right) "Hidari" (left) "Kô" (A) "Jô Ichi Jutsu" (Technique of One Remainder) and so on. So I conclude that Saitô Naonaka's work was based on Seki Kôwa's.

(4) Conclusion

Before Seki Kôwa, the influence of the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections) in Japan was very small. Japanese mathematicians studied indeterminate equations from the *Sun-zi Suan Jing* (Master Sun's Mathematical Manual), *Suan Fa Tong Zong* (Systematic Treatise on Arithmetic) and perhaps *Yang Hui Suan Fa* (Yang Hui's Method of Computation). Their work was not on a very high level.

Seki Kôwa also studied these books including the *Yang Hui Suan Fa,* but these books also would not provide sufficient basis for his work.

We do not have proof that *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections) was brought to Japan, but I strongly suggest that Seki Kôwa had studied it. There are many points of similarity between the "Da-Yan Zong Shu Shu" and "Sen Kan Jutsu". That is, there was the same value method with "Da-Yan Zong Shu Shu". In other words, Qin Jiushao's method was one stage of developing the Chinese Remainder Theorem, not the final method. But Seki Kôwa's method was "Da-Yan Zong Shu Shu". However Seki Kôwa's method was similar to Qin Jiushao's method. Can it be a coincidence?

	Da-Yan Zong Shu Shu		another method
[1]	Make mutually prime numbers	⇔	Huang Zongxuan's method
[2]	"Da-Yan Qiu Yi Shu"	⇔	Hayashi's method
[3]	"Sunzi Theorem" (C.R.T.)	⇔	Saitô's method

TABLE 7 Another Methods of "Da-Yan Zong Shu Shu"

Seki Kôwa's work did not constitute a theoretical advance on the process of three steps method of "Da-Yan Zong Shu Shu". However his method was a great improvement from the part of view of facility of application.

Firstly he abandoned the calculating rods with their systematic layout and used the symbols of algebra. Secondly he computed as follows;

$$b_i x \equiv R_i \pmod{a_i}$$

His purpose was not to compute "Shang Yuan Ji Nian" (the accumulated years from the epoch).

Of course, the counterargument that he derived the principles involved from Chinese astronomical method is possible. Because he researched the *Shou Shi Li* 授時暦 (Season-granting Calendar) [63] deeply. But this calendar is the first one which had not used the system of "Shang Yuan Ji Nian" (the accumulated years from the epoch), I think it is very unlikely that Seki Kôwa derived "Senkan Jutsu" by induction.

At any rate, the works of indeterminate equations during this age in Japan also applied the works of the Song dynasty in China. Japanese mathematicians following Qin Jiushao's method. While Japanese mathematicians did not create something entirely new, neither did they merely imitate.

(*1): Needham, vol.3: 119-22. Li Jimin added them to the method of approximative value of the fraction (Li Jimin, 1987c: 246). Shen Kangshen classified the Chinese Euclid Algorithm into five categories, the greatest common divisor (G.C. D.), the least common multiple (L.C.M.), the method of approximative value of the fraction, "Da-Yan Qiu Yi Shu" and "Bai Ji Shu" (Shen Kangshen, 1982: 210-21.).

(*2): Question 38, chapter 3, on the *Zhang Qiu-Jian Suan Jing* 張丘建算經 (Zhang Qiujian's Mathematical Manual)(Qian Baocong (ed.), 1963, vol.2: 402-5) is:

> QUESTION: There are cocks whose cost is five coins, hens whose cost is three coins and chickens whose cost is one coins for three. I have one hundred coins and want to buy one hundred birds. How many cocks, hens and chickens can I buy?
>
> ANSWER: Four cocks whose cost is twenty coins, eighteen hens whose cost is fifty-four coins, seventy-eight chickens whose cost is twenty-six coins.
>
> Eight cocks whose cost is forty coins, eleven hens whose cost is thirty-three coins, eighty-one chickens whose cost is twenty-seven coins.
>
> Twelve cocks whose cost is sixty coins, four hens whose cost is twelve coins, eighty-four chickens whose cost is twenty-eight coins.
>
> METHOD: Every increasing of four cocks, implies a decreasing of seven hens and increasing of three chickens, so you can compute.

This means;

$$\begin{cases} 5x+3y+1/3\ z=100 \\ x+\ y+\ z=100 \end{cases} \qquad \begin{cases} x=0+4t \\ y=25-7t \\ z=75+3t \quad (t=1,2,3) \end{cases}$$

(*3): In Occidental categories, "the Sunzi Theorem" (the Chinese Remainder Theorem) and "Bai Ji Shu" 百鷄術 (One Hundred Fowls problem) are the same.

However, Yang Hui thought "Bai Ji Shu" was the application of the question of "Zhi Tu Tong Long" 雉兎同籠 (Pheasants and Rabbits in the Same Basket, Simultaneous Linear Equations Involving Two Unknown) in question 31, chapter 3 of the *Sun-zi Suan Jing* (Master Sun's Mathematical Manual)(Qian Baocong (ed.), 1963, vol.2: 320). Therefore Yang Hui described "Bai Ji Shu" in chapter 2 of the *Xu Gu Zhai Qi Suan Fa* 續古摘奇算法 (Continuation of Ancient Mathematical Method for Elucidating the Strange), however "the Sunzi Theorem" was described at another part which is chapter 1 of the *Xu Gu Zhai Qi Suan Fa* .

Additionally, we think the question of "Zhi Tu Tong Long" is derived from the method of "Qi Lu Shu" 其率術 (The Ratio Method) at questions 38-43, chapter 2, *Jiu Zhang Suan Shu* (Nine Chapters on the Mathematical Arts) (personal communication of Bai Shangshu 白尚恕).

(*4): See Li Di , "Guan Yu Qin Jiushao Yu *Shu Shu Jiu Zhang* De Yan-Jiu Shi" 關於秦九韶與數書九章的研究史 (On the History of Studies of Qin Jiushao's "Mathematical Treatise in Nine Sections")(Conference Paper of ISCMTNC, Beijing, 1987).

(*5): Wylie, 1897: 175-81.

(*6): Mikami Yoshio, 1912: 65-9.

(*7): Li Yan, 1937: 118-20.

(*8): Qian Baocong, 1964b, 66-77.

(*9): Needham, 1959, vol.3: 119-22.

(10): Libbrecht, 1973.

(11): Lu Zifang, 1982, vol.1: 20-96.

(12): Li Jimin, 1987b.

(13): Hayashi Tsuruichi, 1937, vol.1.

(14): Nihon Gakushiin, 1954, 5 vols.

(15): Katô Heizaemon, 1956-1964. Seisûron is published in 1964.

(16): "Gen Xian Jian Sun" (Chinese Euclid's Algorithm) was described in question 5 and 6, chapter 1, *Jiu Zhang Suan Shu* (Nine Chapters on the Mathematical Arts)(Bai Shangshu, 1983: 15-7). At first, this method was used to compute "Deng Shu" (G.C.D.), then used to apply computing the approximate value of the fraction (Nihon Gakushiin, 1954, vol.2: 394-5).

(17): Nihon Gakushiin, 1954, vol.2: 395

(18): Mikami Yoshio, 1912: 65.

(19): This condition is question 2 of chapter 1 in *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections).

(20): Qian Baocong (ed.), 1963, vol.2: 318. Translated into English by Lam Lay-Yong and Ang Tian Se, 1992: 178-9.

(21): See Gauss, 1801.

(22):One of the oldest studies is Mikami Yoshio, 1912: 65-9.

(23):The longest calendar cycle, which is a primitive "Shang Yuan Ji Nian" (the accumulated years from the epoch) in the *Zhou Bi Suan Jing* 周髀算經 (The Arithmetical Classic of the Gnomon and the Circular Path of Heaven), is 31,920 years (1 "Ji"極)(Qian Baocong (ed.), 1963, vol.1: 75-7).

And it is resolved into factors as follows;

$$31{,}920 = 19 \times 7 \times 5 \times 3 \times 2^4$$

The combination of (7,5,3), therefore, is one of the most important significance for computing "Shang Yuan Ji Nian". Especially we must attend to the number of 7. In the *Zhou Bi Suan Jing* , each number had the astronomical meanning, for example, 19 is the cycle of Meton. However, only 7 did not have any meaning. I suspect that 7 was derived from the problem of indeterminate equations. Because 7 is the biggest prime number of one decimal place, therefore it can be "Ding Shu" (Fixed Number) directly not "Wen Shu" (Problem Number).

(24): The *Sun-zi Suan Jing* (Master Sun's Mathematical Manual) was listed as the first of mathematical textbooks in the section entitled Oyoso Sankyô no Jô 凡算經條 (mathematical texts) of the *Gaku Ryō* 學令 (educational law);

Students of Mathematics Department must study *Son-shi* 孫子, *Go Sō* 五曹, *Kyū Shō* 九章, *Kai Tō* 海島, *Tetsu Jutsu* 綴術, *Sankai Jūsa* 三開重差, *Shū Hi* 周髀 *Kuji* 九司, each book becomes 1 unit.

So the *Son-shi*, which means *Sun-zi Suan Jing*, was the first mathematical text to be studied. Probably it became the most popular text in this age. (See Jôchi Shigeru, 1987).

(25): These ages were usually called the "Middle Ages". However, historians of culture have opined that there was no "Middle Age" in Japan, but that the ages should be named 8c-13c "Ancient Age", 13c-19c "pre Modern Age" (Ishida Ichirô, 1989: 363-5). Therefore I call these age individually in this paper, Kamakura period 鎌倉 (1192-1333), Muromachi period 室町 (1333-1573), Azuchi-Momoyama period 安土桃山 (1573-1603) and Edo period 江戸 (1603-1867).

(26): Oya Shin'ichi, 1980: 137. The original book is kept at Kunaichô Shôryôbu 宮内庁書陵部, see *ZGR*, vol.30-A, Chapter 872: 93-4.

(27): The question of indeterminate equations, "Hyaku-go Gen to Iukoto", was not described at 1st edition of the *Jinkō-ki* 塵劫記 (Permanent Mathematics), which was a four volume book. It was described in and after 2nd edition, which was a five volumes book (Yamazaki Yoemon, 1966: 4-5).

(28): The original book has "100", but it should be emended to "105".

(29): The *Da Yan Li* (Da Yan Calendar) was made by Yi Xing 一行 (683-727) in 727 and used from 729 to 757.

(30): Needham, 1959, vol.3: 37. Qian Baocong, 1964a: 208.

(31): Wilhelm (trs.), 1950; 1989: 310.

(32): Li Jimin, 1987a.

(33): Qin Jiushao's terms were very complex, therefore there were many translations, as follows (Libbrecht, 1973: 328-30)

Wen Shu	問數	A_i	Problem Numbers (p.328)
Yuan Shu	元數	—	Original Numbers (p.328)
Ding Shu	定數	a_i	Definite Numbers, Fixed Numbers (p.329)
Yan Mu	衍母	m	Extension Mother (p.329), Multiple Denominator (Needham, p.120)
Yan Shu	衍數	m_i	Extension Numbers (Wylie), Operation Numbers) Erweitrungszahlen (Biernatzki,translated from Wylie) Multiple Numbers (Needham) Extension Mother (p.330),
Cheng Lu	乘率	k_i	Multiplying Factors (p.330)
Qi Shu	奇數	r_0	Remainder (p.330)

(34): Libbrecht, 1973: 329-31. from *SKQS*, vol. 797: 329-30.

(35): Another text is; Qian Baocong (ed.), 1963, vol.1: 2-3.

(36): Li Yan, 1933; 54, vol.1: 123-74.

(37): If m_i were larger then r_0, perform this procedure. If m_i were less then r_0, let m_i become r_0.

(38): This "Tian Yuan" (Element of Heaven) was thought to be a reference to "Tian Yuan Shu"(Chinese Algebra)(Wylie, 1897: 93-94), however this function is not "Tian Yuan Shu" (Chinese Algebra)(Katô Heizaemon, 1956: 54). But it means numerical value of unknown quanity, so Qin Jiushao used this term (Mo Shaokui, 1987: 185). In my opinion "Tian Yuan" means the value of one (see section 2-7 chapter 3, Ding Yidong's material).

(39): This case was studied by Libbrecht (Libbrecht, 1973, 344 and 357-8), he ex plained

$$-\alpha_{(2k-1)} m_i \equiv 1 \pmod{a_i}$$
$$a_i \quad m_i \equiv 0 \pmod{a_i}$$

$$(a_i - \alpha_{(2k-1)}) m_i \equiv 1 \pmod{a_i}$$

Exactly $\alpha_{2k} = a_i$, so his explanation becomes the same as my explanation. But Seki Kôwa and Huang Zongxian (fl. 19c) did not use the value of a_i directly, thus I will explain this case according to their methods.

Takebe Kataaki 建部賢明 (1661-1716), who was a student of Seki Kôwa, suggested $\alpha_{2k} = a_i$ using the same concept as Libbrecht at section 5 of chapter 6 in *Taise Sankyo* 大成算經 (Complete Mathematical Manual) (Nihon Gakushiin, 1954, vol.2: 394-5).

(40):

$$q_{2k'} = q_{2k} - 1$$
$$= r_{(2k-2)} / r_{(2k-1)} - 1$$

and $r_{(2k-1)} = 1$,

thus $q_{2k'} = r_{(2k-2)} - 1$

(41): Actually, α_{2k} is a_i.

(42): The computation of "Deng Shu" 等數 (G.C.D.) was used "Geng Xiang Jian Sun" (Chinese Euclid Algorithm), however the quotient of the last step was $q_n - 1$, not q_n (see Bai Shangshu, 1983: 15-7):

(present method)	(Chinese method)
$a = q_1 b + r_1$	$a = q_1 b + r_1$
$b = q_2 r_1 + r_2$	$b = q_2 r_1 + r_2$
$r_1 = q_3 r_2 + r_3$	$r_1 = q_3 r_2 + r_3$
⋮	⋮
$r_{n-2} = q_n r_{n-1}$	$r_{n-2} = (q_n - 1) r_{n-1} + r_{n-1}$

(43): Qin Jiushao computed 38 examples of "Da-Yan Qiu Yi Shu" (The Method of Acquiring "One" in Dayan) in the *Shu Shu Jiu Zhang*.

$r_0 = 0$	($k = 0$, need not to compute)	7 times
$r_0 = 1$	($k = 1$, need not to compute)	7 times
$r_{2m} = 1$	(normal)	10 times
$r_{21} = 0$	(special)	14 times

(44): Li Jimin, 1987b.

(45): These terms, "Qi" (odd numbers) and "Ou" (even numbers), are not literal.

Libbrecht thought that they meant "belonging to the same class" (Libbrecht, 1973: 333). Then Li Jimin concluded that they meant the positions on calculating board (Li Jimin, 1987b: 222).

(46): The original term is "Tong Fen Na Zi" 通分内子. It means "transforming into a complex fraction".

(47): The original text is "or". But the similar paragraph of "Yuan Shu" (integral number) is "divide the even numbers and do not the odd one". Thus this "or" must be corrected to "do not".

(48): SWYSG, vol.1: 1-2. or *SKQS*, vol.797: 327-9.

(49): Li Jimin, 1987b.

(50): The general who counted soldiers was changed to Han Xin 韓信 (?-B.C.196) in the *Suan Fa Tong Zong* (p.21B of Chapter 5).

(51): Li Jimin, 1987a.

(52): ZHSJ, vol. 9: 2843.

(53): Seki Kōwa also studied "Metsuke-ji" and his term was "Kempu" 験符 (Check the Sign) in the *Sandatsu no Hō, Kempu no Hō* 算脱之法験符之法 (Methods of Solving Josephus problems, Methods of the Check of Sign) in 1683.

(54): *Sampō Ketsugi Shō* 算法闕疑抄 (Solving Mathematical Questions) was said to be published in 1660, but Shimodaira Kazuo 下平和夫 found the 1659 edition (private communication).

(55): Hirayama Akira et al (eds.), 1974: 297.

(56): Hirayama Akira et al (eds.), 1974: 301-2.

(57): See Jōchi Shigeru. 1991b.

(58): Chapter 1 of the *Qiu Yi Shu Tong Jie* 求一術通解 (Comments of the Technique of Acquiring One).

His method was influenced by western mathematics. There was no concept of prime number in Chinese mathematics. Euclid's *Elements*, the first book that described prime numbers, contains a part on prime numbers in vol.7, which had been translated to Chinese by Wylie, Alexander (1818-1887) and Li Shanlan 李善蘭 (1811-1882) in 1858. Moreover Li Shanlan had published *Kao*

Shu Gen Si Fa 考數根四法 (Four Methods of Studying Prime Number) in 1872, which was one of the oldest works on prime numbers in China. Then Huang Zongxuan's work was planned (see Li Di, 1984: 350-88).

(59): Chinese mathematicans avoided calculation of fraction if possible, the preface of *Zhang Qiu-Jian Suan Jian* 張丘建算經 (Zhang Qiujian's Mathematical Manual) said, "Generally, multiplication and division are not difficult in mathematics, reducing fractions to a common denominator is difficult."(Qian Baocong (ed.), 1963, vol. 2: 329)

(60): Hayashi Tsuruichi, 1937, vol.1: 720.

(61): This School was founded by Aida Yasuaki 會田安明 (1747-1817) in 1785. He was born at Yamagata 山形 prefecture, and the most famous river is the river Mogami 最上川 so he was named in reference to the river name of his home town, but in the pronunciation "Onyomi" 音讀 (ancient Chinese pronunciation) its name was Saijô 最上. Saijô means "the best"; he wished his school to become the best, especially better than Seki Ryû 關流, which was Seki Kôwa's School.

There were disputes between the Seki Ryû and the Saijô Ryû, which grew ever sharper as time went by. The weak point of Aida Yasuaki had been the problem of indeterminate equations, but Saitô Naonaka, who was the third genaration of Saijô Ryû, created new methods.

(62): Nihon Gakushiin, 1954, vol.5: 279-282.

(63): *Shou Shi Li* 授時暦 (Season-granting Calendar) was made by Wang Xun 王恂 and Guo Shoujing 郭守敬 in 1280 and used from 1281 to 1368, and it was also used from 1368 to 1384 under the name *Da Tong Li* 大統暦 (Large Unification Calendar).

Seki Kôwa studied it, then wrote the *Juji Hatsumei* 授時發明 (Comments of the Works and Days Calendar) in 1680, *Juji Reki Kyō Rissei no Hō* 授時暦經立成之法 (Methods of Manual Tables of the Works and Days Calendar) in 1681.

CONCLUSION

In China, the magic square was not only a mathematical game, but was also bound up with Chinese philosophy. Chinese philosophers believed that the smallest magic square, which was called "Luo Shu" (a writing from Luo river) a magic square of order three, had mysterious power. That is to say, Chinese magic squares not only had the property that the sums of each column and row were the same numerical value, but also that each item was also arranged in counter clockwise order in accordance with the notion of Five Elements. "Counter clockwise" is the direction of Heaven moving. Chinese philosophers thought that this mysterious magic square embodied a micro-cosmos.

Chinese mathematicians had to consider the philosophical aspects of magic squares. Thus they were limited within the framework of philosophical practicality when they tried to make larger magic squares.

Japanese mathematicians, however, only considered the mathematical interest of magic squares. Thus it was easier for Japanese mathematicians to make larger magic squares than it was for Chinese mathematicians. In other words, Seki Kōwa took only the mathematical aspects of magic squares from *Yang Hui Suan Fa* (Yang Hui's Method of Computation). He used Chinese mathematical books, but it was not only imitation, he also applied the essential points of Chinese mathematical books for his own original researches. The metaphysical aspect of magic squares do not seem to have interested him.

If Seki Kōwa used Chinese mathematical books like this, what was his approach to the problem of solving indeterminate equation?

In China, indeterminate equations were studied for computing "Shang Yuan Ji Nian" (accumulated years from the initial epoch). For this purpose, it was enough to solve the simultaneous modular equations of;

$$
\begin{aligned}
x &\equiv r_1 \pmod{a_1} \\
&\equiv r_2 \pmod{a_2} \\
&\vdots \\
&\equiv r_n \pmod{a_n}.
\end{aligned}
$$

That is, the coefficient of the unknown number is always one. It was not necessary to solve the general indeterminate equation of first degree.

As with magic squares, "Shang Yuan Ji Nian" also had a metaphysical significance for Chinese mathematicians. Qin Jiushao, who was limited by this notion, nevertheless succeeded in developing a method for solving the case when the divisors were fractions (including decimal fractions). The divisors were astronomical constants, such as one tropical year, i. e., 365.25, and thus he was confronted with this case.

On the basis of his use of the *Yang Hui Suan Fa* , our expectation would be that if Seki Kōwa had studied the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections), he would ignore mataphysics and concentrate on the purely mathematical aspect of solving indeterminate equations, i.e., solving the general indeterminate equation of first degree. This is exactly what we find in the works of Seki Kōwa.

We do not know whether Seki Kōwa studied the *Shu Shu Jiu Zhang* or not. If he did, he evidently ignored significant parts of its content, just as he had in the case of *Yang Hui Suan Fa.* He works contain no reference to the philosophical significance of "Shang Yuan Ji Nian" problem.

Nevertheless his method of computing k_i , which fills the modular equation;

$$k_i\, m_i \equiv 1 \pmod{a_i},$$

was very similar to the method of *Shu Shu Jiu Zhang.* It was not only the same process, but the case when k_i was negative was also processed by similar methods.

Seki Kôwa's contribution was limited to developing methods already basically contained within the *Shu Shu Jiu Zhang*. Although his achievements in his respect were substantive, his work did not constitute a fundamentally new theoretical departure.

CHRONOLOGICAL TABLES OF EASTERN ASIA

CHINA			KOREA	JAPAN
Xia	夏	(BC 21c-BC 16c)		
Shang	商	(BC 16c-BC 1066?)		
Western Zhou	西周	(BC 1066-BC 771)		
Eastern Zhou	東周	(BC 770-BC 256)		
Spring and Autumn	春秋	(BC 770-BC 476)		
Warring States	戰國	(BC 475-BC 221)		Jōmon 縄文 (?-BC 1c)
Qin	秦	(BC 221-BC 206)		
Western Han	西漢	(BC 206-AD 23)		
Xin	新	(AD 9- 25)	Three Kingdoms 三國 (BC 1c-668)	Yayoi 弥生 (BC 1c-AD 3c)
Eastern Han	東漢	(25- 220)		
Centuries of Disunity	三國	(220- 265)		Large Tombs 古墳 (AD 3c-6c)
Western Jin	西晉	(265- 316)		
North and South Dynasties	南北朝	(317- 581)		
Sui	隋	(581- 618)		Asuka 飛鳥 (538- 645)
Tang	唐	(618- 907)	Silla 新羅 (668- 936)	Nara 奈良 (645- 794)
Five Dynasties and Ten Kingdoms	五代	(907- 960)		Heian 平安 (794-1192)
Northern Song	北宋	(960-1127)	Koryo 高麗 (918-1392)	
Southern Song	南宋	(1127-1279)		Kamakura 鎌倉 (1192-1333)
Jin	金	(1115-1234)		
Yuan	元	(1279-1368)		Muromachi 室町 (1336-1573)
Ming	明	(1368-1644)	Yi 李 (1392-1910)	
Qing	清	(1644-1911)		Edo 江戸 (1603-1867)
Republic	民國	(1912-1949)	←	Empire (1867-1945)
People's Republic		(1949-	Pepublics(1945-	present (1945-

ABBREVIATIONS

CSJC *Cong-Shu Ji-Cheng* 叢書集成. SWYSG.

ESWS *Edo Shoki Wasan Sensho* 江戸初期和算選書. see Shimodaira Kazuo et al (eds.), 1990-.

GXJBCS *Guo Xue Ji-Ben Cong-Shu* 國學基本叢書. SWYSG.

KXCBS Kexue Chubanshe 科學出版社

SKQS *Si Ku Quan Shu* 四庫全書

SSJZQJ *Shu Shu Jiu Zhang Yu Qin Jiushao* 數書九章與秦九韶. see Wu Wenxun (ed.), 1987.

SWYSG Shangwu Yinshuguan 商務印書館 (Commercial Press)

SKZ *Seki Kōwa Zenshū* 關孝和全集. see Hirayama Akira, Shimodaira Kazuo and Hirose Hideo (eds), 1974.

SJSS *Suan Jing Shi Shu* 算經十書. see Qian Baocong (ed.), 1963.

MZNSS *Meiji-Zen Nihon Sūgaku-shi* 明治前日本數學史. see Nihon Gakushiin (ed.) 1954.

WHM *Wasan no Hōjin Mondai* 和算の方陣問題. see Mikami Yoshio, 1917.

ZHSJ Zhonghua Shuju 中華書局 (China Press)

ZGR *Zoku Gunsho Ruijū* 續群書類從. see *Zoku Gunsho Ruijū*

ZGSXSJB *Zhongguo Shuxue-shi Jian Bian* 中國數學史簡編. see Li Di, 1984.

ZSSRC *Zhong Suan-shi Lun Cong* 中算史論叢. see Li Yan, 1933; 1954.

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律令期の数学教育

城地　　茂

奈良時代は，律令体制が整備され，全国規模の行政が行われていた．班田収受を行い，税を徴収し，橋梁を架け，寺院を造り，都を造営するためには，かなり高度な数的処理が必要であったはずである．従って，数学も発達していたであろうし，史料が少なく，その実態はあまり知られていない．これは，算道官人の政治的地位の低さのためである．

しかし，算道の教育については比較的史料も残っており，律令期の数学の一端を知るために，教育制度を探ってみたい．

(1)算生の大学寮所管

律令制度の下には，それを運営する為に多くの方伎職があるが，それらは，算道を除き，すべて所管の官庁で自家養成されている．

ところが，実務職である算師は民部省主計寮・主税寮に置かれながら，算生は式部省大学寮に所属している．これは，一つには算道を重視された証左であろう．又，汎用性のある技能であったとも言える（註1）．大学寮に在る為，将来国家の中枢で活躍するであろう貴族の子弟と同窓になる訳で，これは，有形・無形で有益であったことだろう（註2）．

(2)大学（算道）入学資格

大学入学資格は，『学令』大学生条に依れば，１３歳から１６歳で聰明な，

(a)五位以上の子・孫（外五位の場合は嫡子のみ・釈云）

(b)東西史部の子（倭川内文忌寸．他の史姓は除外・穴云．分家は可・古記云）

(c)許可された八位以上の子（畿外，外六位以下は不可能・一云．）

この他に，特例として，

(d)国学を卒業し，考試に合格した者

である．

問題は小学とされた算道もこれと同じかどうかである．橋本義彦氏の見解に依れば，算道入学資格は，明法道と同様，雑任及び庶人の聰慧の者を選抜す

ると言うものである（註３）．同氏は，

(イ)　『職員令』大学寮条には「学生」と「算生」が別に記されていること．

(ロ)　『学令集解』所引の天平三年式部解の「諸国貢挙算生」が大学算生の多数を占めていたであろうこと．

(ハ)　唐の算学生が文武八品以下及び庶人の子であること．

(ニ)　唐では算生と同等な明法生が，日本では雑任及び庶人の聰慧の者を選抜したこと．（従って，算生も明法生と同様になるはずである．）

(ホ)　算道出身者が卑姓出身の下級官吏であること．

の以上５点からこの見解に達している．

しかし，この説には従い難い．

(イ)については，「学生」と「算生」等を統括するものとして，「大学生」という表現がある．大学の生という意味である．従って，「算生」も「学生」も同じ「大学生」ということになる．『学令』大学生条は，この「大学生」の入学規定を定めたものであるから，算生もこの規定になるはずである．

(ロ)(ホ)については，(d)で説明が可能である．国学には，庶人の入学も可能だからである．

(ハ)については，日本がどこまで唐の教育制度を模倣しかのか疑問が残る．これについては(ニ)の検討のところで詳しく述べる予定である．

問題は(ニ)である．『職員令集解』大学寮条釈云は，天平二年三月辛亥（２７日）の太政官奏として，

「直講四人．〔一人文章博士．〕律学博士二人．　已上同助教．明法生十人．文章生二十人．簡取雑任及白丁聰慧．不須限年多少也．得業生十人．明経生四人．文章生二人．明法生二人．算生二人．並取生内性識聰慧芸業優長者．夏人別絁一疋．布一端．冬人別絁二疋．綿四屯．布二端．食料米日二升．堅魚海藻雑魚各二両．塩二夕．」（史料１）

が発せられたことを述べている（註４）．

これを見る限り，明法生は橋本氏の指摘するように，「簡取雑任及白丁聰慧．」である．しかし，そうすると，文章生も同じにならなければならない．橋本氏の論法は，唐では算学生（日本では算生）と律学生（明法生）の規定

が同じであるということが前提になっている．ところが，文章生は唐では三品以上の公卿が入学する規定である国子学で教育されている（註5）．この文章生が律学生と同じ規定になってしまうのは唐の制度では考えられないことである．そうなると，日本の教育制度がどこまで唐の制度と近いのか疑問になってくる．そうなれば，明法生の規定がどうあろうと，算生が史料１の規定になるとは限らない．

更に，明法生の規定にも問題が残る．これは，天平二年の格が出ていない（あるいは別の日に出ていた）可能性があるということである．

『類聚三代格』神亀五年七月二十七日の格に，

「勅　大学寮　律学博士二人，直講三人，文章学（ママ）士一人，（生二十人）以前．一事已上同助教．」（史料２）

がある．（註６）

天平２年３月２７日に新設されたはずの明法博士他が神亀５年７月２７日に設置されていたとすれば，史料１を信頼することは出来ない．そうなると，史料１の入学規定が明法生のことを述べたものではない可能性があるのである．

史料１の再検討をしてみよう．釈は，史料１と『職員令集解』陰陽寮条釈云，同典薬寮条を見る限り，天平２年３月２７日に，

（ｉ）直講４人〔内，文章博士１人〕，律学博士２人

（ii）明法生１０人，文章生２０人

（iii）大学得業生１０人

（iv）陰陽得業生３人，暦得業生２人，医得業生３人

の４種類の官員が新設されたと記録している．

しかし，史料２によれば，（ｉ）（ii）は，神亀５年に設置されているのだから，この日は，『続日本紀』天平二年三月辛亥（２７日）条に記録されているように（iii）と（iv）だけになる（註７）．

ここで仮に，これが正しいとしよう．医生，陰陽生，暦生は，「先取薬部及世習．次取庶人年十三以上．十六以下．聰令者為之．」（註８）という規定である．そうすると，史料１に見られる「簡取雑任及白丁聰慧．」は，『学令』の規定を無視した（ii）の規定と考えるより，（iv）の規定とした方が自然

である．

つまり，天平二年には，（iii）と（iv）が設置されたが，その内の（iv）の入学規定と（ii）の入学規定が混乱してしまって記録された可能性が無いとは言えないのである．

史料1と史料2の何れの史料が正しいのか今となっては分からない．従って，明法生の入学規定については断言することは出来ない．しかし，明法生が「簡取雑任及白丁聰慧．」の規定だとしても，それを算生にも適用するには，先に述べたように無理がある．従って，算生の入学規定は，『学令』大学生条の規定通りと考えるべきである．

(3) 使用した教科書

『学令』算経条には，学習すべき教科書が列挙されている．

「凡算経．孫子．五曹．九章．海島．六章．綴術．三開重差．周髀．九司．各為一経．学生分経習業．」（史料3）

と9部の数学書を学習することになっていた．しかし，全員が全てを学習するのではなく，卒業試験の規定である『学令』書学生条に，

「（前略）其算学生．弁明術理．然後為通．試九章三条．海島．周髀．五曹．九司．孫子．三開重差．各一条．試九．全通為甲．通六為乙．若落九章者．雖通六．猶為不第．其試綴術．六章者．准前綴術六条．六章三条．試九．全通為甲．通六為乙．若落経者．雖通六．猶為不第．其得第者叙法．一准明法之例．」（史料4）

と，あるように，『九章算術』他6部を学習する班と『六章』，『綴術』を学習する班に分かれていた．多分，30名が算生の定員であるから，15名ずつであろう．

このなかで，『孫子算経』『五曹算経』『九章算術』『海島算経』および『周髀算経』は，唐の教科書『算経十書』のなかに収められ，現在に伝わっている．

残り4部は現存していないが，現存する史料を使って内容を復元してみよう．

先ず，『六章』は，唐には無く，新羅と日本で使われた教科書である．史

料2で『九章算術』と同等の地位を与えられていることから，『九章算術』の9章のうちの6章であろうとされている(註9)．しかし，どの章が削除してあるか分かっていない．

『綴術』は，『隋書』律暦志備数条から，祖沖之による球体の研究書であることが分かる．円周率の計算がなされ，3.1415926＜π＜3.1415927 の値が求められている．しかし，「学官莫能究其深奥，是故廃而不理.」という難しい数学書であったために日本では勿論，唐でもよく理解できていなかったようである．しかし，新羅でも教科書になっていた．

『三開重差』は，唐にはなく，新羅と日本で使われていた．確証はないが，書名から『海島算経』に類する数学書であった可能性がある．すなわち，『海島算経』で記述してあるような三種の重差術の開法を指すものと考えることができよう．

最後の『九司』は，日本でのみ使われた数学書である．藤原佐世『日本国見在書目録』には，1巻のものと5巻のものであることが記録されていることから，『九章算術』のように9章編成になっておらず，『学令集解』算経条古記の「事雑計也.」というような様々な雑題の寄せ集めであったようである．

次に，これらの教科書の履修期間を調べてみよう．

『延喜式』大学寮講書条には，

「凡応講説書籍者．礼記(大)．左伝(大)各限七百七十日．周礼(中)．儀礼(中)．詩(中)．律各四百八十日．周易(小)三百一十日．尚書(小)．論語．令各二百日．孝経六十日．三史．文選各准大経．公羊．穀梁．孫子．五曹．九章．六章．綴術．各准小経．三開重差．周髀共准小経．海島．九司亦共准小経.」(史料5)

と履修期間が列挙されている．

括弧内は，『学令』礼記左伝各為大経条に規定されている大経・中経・小経の区別である．算経各書は，「准小経」か「共准小経(2部で准小経)」であるから，小経の規定を求めれば算経の履修期間が分かる訳である．

『学令』礼記左伝各為大経条の規定で，「小経」となっているのは，明経

道の教科書である『周易』と『尚書』だけであるから，これに準じることになるのだが，この履修期間はそれぞれ310日と200日となっており，「准小経」が何れになるのか確定出来ない．そこで，明経道の規定について詳しく検討してみよう．

明経道で二経を学習する場合は，「大経内通一経．小経内通一経．」若しくは，「中経即併通両経．」（註10）となっており，大経と小経各一部を学習する場合と中経二部を学習すると同等になっている．明経道では何経学習したかに依って与えられる位階が異なってくるので，同じ二経であれば履修期間も同じか，かなり近いはずである．この事から，310日なのか200日なのかを計算することが可能である．

大経である『礼記』と『左伝』は770日である．中経である『周礼』，『儀礼』と『毛詩』は480日になっている（註11）．従って，

大経＋小経 ≒ 中経×2

小経 ＝ 中経×2 －大経

＝ 480日×2 － 770日 ＝ 190日 ≒ 200日

となり，小経は200日であるか，寧ろ，それ以下の方が適当である．

また，算道について考えてみても，『九章算術』班の場合は，『九章算術』・『海島算経』・『周髀算経』・『五曹算経』・『九司』・『孫子算経』と『三開重差』の7部を学習するが，このうち，「三開重差．周髀共准小経．海島．九司共准小経．」なので，小経5経分の日数になる．従って，小経を200日と仮定すると1000日になる．これは，明経道における中経2経の場合の960日，大経と小経である『周易』の1080日，大経と小経である『尚書』の970日と極めて近い数字である．

これらの事を考えあわせると，「准小経」は200日と考える方が適当であろう．

1年は，360日で計算されている（註12）．大学寮では，旬仮が10日毎に1日（註13），田仮が5月の農繁期に15日，9月には授衣仮の休みがある（註14）．更に，旬仮の前には考試があり，この日も講義は出来ないであろう．その他，年終之試，釈奠，掃除（註15）などで講義の出来ない日が考えられる．そうすると，実際に講義をする日は，360日から2割を減じ，更に30

日内外を引いた250日～260日程度になる．これは，大学の出席日数が260日以上の規定（註16）と矛盾しない．従って，大学寮で200日の講義の規定の算経各書は，約9か月，最大10か月を越えない程度で教えていたと考えられる．

従って，『九章算術』班は1000日（約4年），『綴術』班は400日（約1年半）で教育していたことになる．

(4) 卒業後の叙位

前項で述べた教科書を使って教育が終わると，試験を受けて位階を得た．その試験の出題規定は，史料4にあるように，『九章算術』班は，『九章算術』3題と他の算経6書から各1題の合計9題，『綴術』班は，『綴術』6題と『6章』3題の合計9題が出題された．

合格は全問正解の甲題と6題正解の乙題があり，甲題は大初位上，乙題は大初位下を与えられた（註17）．しかし，『九章算術』班では『九章算術』が，『綴術』班では『六章』が必修で，3題共出来なかった場合は，他の6題が出来ても不合格になっていた．

この大初位上と大初位下は30階ある位階のうち27位と28位であり，あまり高い位階とは言えない．しかし，実務職である主計算師と主税算師の位階が26位の従八位下，大宰算師の正八位上，算道の最高職の算博士の従七位上（註18）ということを考えれば，適当なものと言える．

まとめ

以上，簡単に算生の入学から卒業までを見てきたが，古代の数学を研究する上で教育制度という構造を捉える事は，様々な利点があるはずである．

例えば，計算方法は，使用された教科書から算盤ではなく算木であったことが分かる．これは，日本では『算経十書』のうち，算盤を扱った『数術記遺』（註19）が採用されず，算木の教科書とも言える『孫子算経』が教科書の筆頭に挙げられていることから容易に推測できるのである．

『孫子算経』が和算に大きな影響を及ぼしていることは周知の事実であり，この意味からも奈良時代の数学研究の必要があるだろう．

（註）

1. 『軍防令』内六位条には，内六位以下八位以上の嫡子の無任の者のうち，「儀容端正．工於書算．」であれば，「為上等．」として，大舎人に成れたことが記述されている．五位以上も同五位子孫条の規定で「性識聰敏．」であれば，内舎人になれたが，この時にも書算が重視された可能性もある．例えば，恵美押勝は，幼少の頃大納言阿部少麻呂に算術を学び，それに優れていたので，内舎人を経て大学少允になっている（『続日本紀』天平宝字八年九月壬子（18日）条）．但し彼の場合は三位以上の子として，無試験で内舎人になれるので，どの程度算術の知識が役立ったかは分からない．
2. しかし，学問的な交流は余り活発ではなかったらしい．例えば，『学令集解』算経条の『五曹算経』の注釈には，釈云，古記云とも「即人名也．」とある．『五曹算経』は，5種類の曹という意味であり，全く的外れである．明法学者は，『五曹算経』を紐解くことすらしていなかったようである．
3. 橋本義彦，「官務家小槻氏の成立とその性格」（『平安貴族社会の研究』，吉川弘文館，1976）
4. 弘仁十二年二月十七日の格で，文章博士の位階を正七位下から正五位上に引き上げた時に，「去天平二年三月二十七日格．置件官員（文章博士）．」とあることから正しい記録と考えられる．また，天平二年三月二十七日の格が，十七日の可能性もある．これは，東山文庫本の傍書，及び，旧輯国史大系所引鈴鹿氏所蔵本所引イ本には十七日になっているからである（新訂増補国史大系25，『類聚三代格』巻五定官員併官位事，弘仁十二年二月十七日太政官符頭注.）．尚，『続日本紀』には17日，27日の何れの記事もない．
5. 『旧唐書』巻44職官志国子監条，『新唐書』巻48百官志国子監条，『大唐六典』巻21国子監条
6. 『類聚三代格』貞観十三年十二月二十七日の応加増算博士位階事の所引に，「去神亀五年初置律学博士，為正七位下．」とあり，これに依って裏付けられる．
7. 『続日本紀』天平二年三月辛亥（27日）条には，「選性識聰恵（慧）芸業優長者十人以下五人以上専精学問．以加善誘．仍賜夏冬・服并・食料．」及び，「吉田連宜．大津連首．御立連清道．難波連吉成．山口忌寸田主．

私部首石村．志斐連三田次等七人．各取弟子将令習業．其時服食析亦准大学生．其生徒陰陽医術各三人．曜暦各二人．」と，記載されている．前者は，大学四道の得業生10人，後者は陰陽得業生3人，暦得業生2人，及び，医得業生3人の事と釈は考えている（『職員令集解』陰陽寮条釈云．同典薬寮条．）天平二年には，（iii）大学得業生10人（iv）陰陽得業生3人，暦得業生2人，医得業生3人が新設されたことは確実である．

8. 復元『医疾令』医生等取薬部及世習条．これは，『職員令集解』典薬寮条私・同国博士医師条釈云・『政事要略』巻95至要雑事（学校）により復元可能である（井上光貞他校注，日本思想大系『律令』，岩波書店，1976初版，1985，p.421）．

9. 金容雲・金容局，『韓国数学史』，槇書店，1978，p.85〜87

10 『学令』礼記左伝各為大経条

11 『弘仁式』では，中経は460日であったことが紅葉山本『学令義解』礼記左伝各為大経条から分かっている．しかし，460日だとしても小経の期間が短くなりこそすれ，310日にはならない．

12 『選叙令集解』以理解条釈云に，「以三百六十日為一年」とある．

13 『学令』先読経文条

14 『学令』放田仮条，『仮寧令』給休仮条

15 平安時代には，左京職が「凡並年二月．八月前上丁三日．各雇夫掃除大学．〔二月五十人．八月一百人．〕（後略）」（『延喜式』左京職大学掃除条）を行っている．奈良時代でも行っていた可能性は高く，そうなれば，この日は講義ができないとみるべきだろう．又，大掃除があるのだから，この日を境にして，大学が二期制になっていた事が考えられる．

16 『学令』不得作楽条．又，『考課令』内外初位条で，長上官は，240日以上勤務しなければ，考課の対象にならなかった事が分かる．

17 『選叙令』秀才出身条に，算道の準じる明法道の合格者の位階の記述がある．

18 『類聚三代格』に依れば，貞観十三年十二月二十七日に正七位下になった．

19 『数術記遺』一計算条には，珠算の記述がある．

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数学史研究

(通 巻 117 号)

1988年4月～6月

目　　　次

日 本 数 学 史 学 会

資　料

中国湖北省江陵県張家山遺跡出土『算数書』について

城地　茂

1983年12月から翌年1月にかけて，湖北省江陵県で，前漢初期の墳墓の発掘調査が行われた．数々の副葬品が出土したが，その中に『算数書』という竹簡も含まれていた．これは，最古の数学書とされる『九章算術』を遡ること200年，東洋数学の起源を書き替えるものとして，数学史界の注目を引いた．

未だ竹簡の全てが公開されていない状態であるが，報告者が江陵県へ行く機会に恵まれたので，現在までの中国考古学界の研究状況を報告してみたい．

(1) 出土場所

江陵県は，湖北省の省都武漢市の西方約270km，長江（揚子江）北岸の町である．1979年の統計によれば，人口は約7万2900人，面積2421,9km²である．古くは，春秋戦国時代の楚の都として栄え，三国時代には，三か所の争点として三国志に名を止めている．蜀の武将関羽の戦死した荆州と言った方が通りがいいかもしれない．その荆州城が現在の江陵県城である．また，現在でも，「地区」行政府（註1）の所在地でもある．

『算数書』の出土したのは，張家山*M*247西漢（前漢）墓という墳墓である．張家山遺跡は，江陵県城の西1.5km，煉瓦工場の敷地内にある．煉瓦の材料となるのは粘土である．幸いにも，この粘土層に棺があったために，竹簡が2000年を経ても腐敗することなく保存されていたのである．

(2) 年　代

同時に出土した文字資料から，被埋葬者は，楚国人で，秦国統治下の楚の古都紀南城付近に生まれ，前漢王朝の下級文官として9年間勤務している．そして，亡くなったのは，呂后2年（B.C.186年）もしくは，そのやや後である（註2）．したがって，『算数書』は，それ以前に成立していたことになり，これは，『九章算術』が成立したとされる紀元後25年（註3）より，200年以上古いということになる．

また，これは，『九章算術』劉徽序の

「往者暴秦焚書、経術散壊．自時厥后、漢北平侯張蒼、大司農中丞耿寿昌皆以善命世．」

とある張蒼と同じ年代になるが，『算数書』が張蒼の手によるもの（註4）かどうか断

定するに到っていない.

(3) 「算数書」の内容

『算数書』竹簡は，総数約200支，このうち180余は，完全なものであったが，残る10余は，断片であった．竹簡には，三か所で綴られていた形跡が残されている．

これらの竹簡にある問題数は約60題，総字数は約7000字である．問題は，大きく二つに分類できる．一つは，算術部分である．具体的な問題ではなく，

「一乘十、十也、十乘万、十万也.」

というような計算を示したものである．このように整数の乗法（「乗」）から始まっている．整数の四則演算は，『九章算術』には無いもので，『算数書』が，『九章算術』以上に算術を重視していたことが分かる．このほかに，「合分（分数加法）」「増減分（分数加減法）」「分乘（分数乘法）（註5）」「径分（分数除法）」「約分」「増乘」「相乘」「合乘」などがある．内容的には『九章算術』の範囲内であり，数量的にも少なく60題あるうちの10題程である．

もう一つは，『九章算術』に類似した応用問題の部分である．しかし，『九章算術』のように，類似した問題を章立して整理してはおらず，個別に術の名が記されている（註6）．

『算数書』「少広」には，

「広一歩半歩、以一為二、半為一、同之三、以為法．即直二百四十歩、亦以一為二、除如法得従歩．為従百六十歩.」

という問題がある．これは，『九章算術』巻4「少広」第1題の

「今有田広一歩半．求田一畝、問従幾何．

答曰．一百六十歩．

術曰．下有半、是二分之一．以一為二、半為一、併之得三、為法．

置田二百四十歩、亦以一為二乘之、為実．実如法得従歩.」

と数値まで同じである（註7）．

また，『算数書』の「息銭」には，

「貸銭百、月息三．今貸六十銭、月未盈、十六日帰、請息幾何．

得曰．二十五分銭之二十四．

術曰．計百銭一月積銭数為法、置貸銭、以一月百銭息乘之、有（又）

以日数乘之為実、如〔法〕得息一銭.」

とあるが，これも，『九章算術』巻3衰分第20題

「今有貸人千銭、月息三十．今有貸人七百五十銭、九日帰之、問息幾何．

答曰．六銭四分銭之三．

術曰．以月三十日乗千銭為法．以息三十乗今所貸銭数、又以九日乗之、

為実．実如法得一銭．」

と基本的に同じものである（註８）．

さらに，『算数書』には，

「粺米四分升之一、為粟五十四分升之二十五、二十七母、五十子．」

という問題があるが，これも，換算率が『九章算術』と同じであり，『九章算術』巻２「粟米」第２題には，以下のような類似の問題がある（註９）．

「今有粟二斗一升、欲為粺米．問得幾何．

答曰．為粺米一斗一升五十分升之十七．

術曰．以粟求粺米、二十七之、五十而一．」

以上のことから，『算数書』は，『九章算術』の母体であるといって大過ないだろう．『算数書』が整理，系統化され『九章算術』の源流の一つになったと考えられる．『算数書』には，「方程」「句股」が無いがこれが，埋葬時点から無かったのか，腐敗散逸してしまったのかは分からない．今後の考古学の発見が期待される．

（註）

1．中国の行政組織は，大体，省→地区→県→郷となっている．江陵県城に荊州地区人民政府がある．

2．陳耀鈞・閻頻，「江陵張家山漢墓年代及相関問題」（『考古』1985年12期，科学出版社，中国北京）

3．白尚恕他編，『中国数学簡史』，山東教育出版社，中国済南，1986，p，16）

4．陳躍鈞・陳燕萍，「『算数書』与『九章算術』」（『湖北省考古学会論文選集(1)』武漢大学学報編輯部，中国武漢，1987年）

5．「少半（1/3）乗少半，九分之一也．」

「四分乗五分，二十分之一．」

「半（1/2）乗一，半也．」といった問題がある．

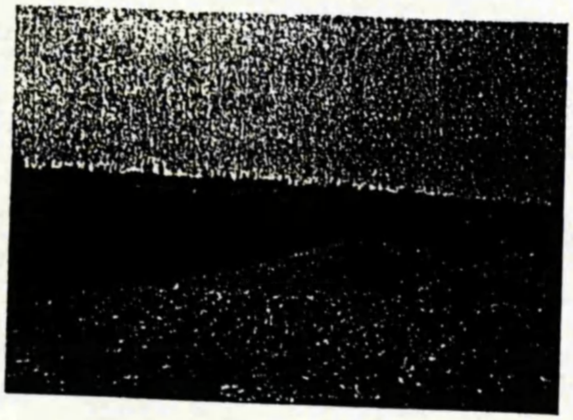
6．「里田」「税田」「金賈」「程禾」「出金」「銅耗」「方田」「買塩」「息銭」「負炭」「少広」「石衡」といった術名がある．これらは，７つに大別でき，「方田」「粟米」「衰分」「少広」「商功」「均輸」「盈不足」に相当する．

7．張家山漢墓竹簡整理小組編，「江陵張家山漢簡概述」（『文物』，1985年１期，文物出版社，中国北京）

8．前出，「『算術書』与『九章算術』」

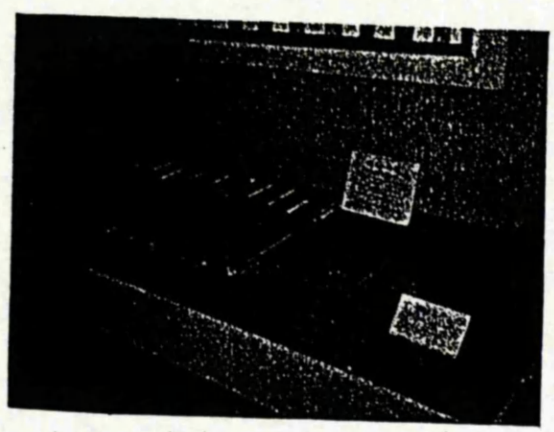
9．前出，「『算数書』与『九章算術』」

（昭和62年12月５日受理）

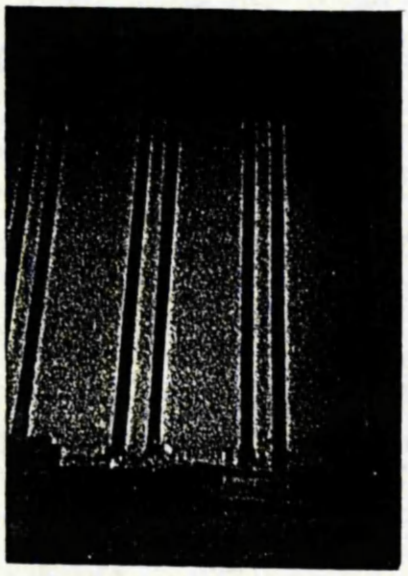


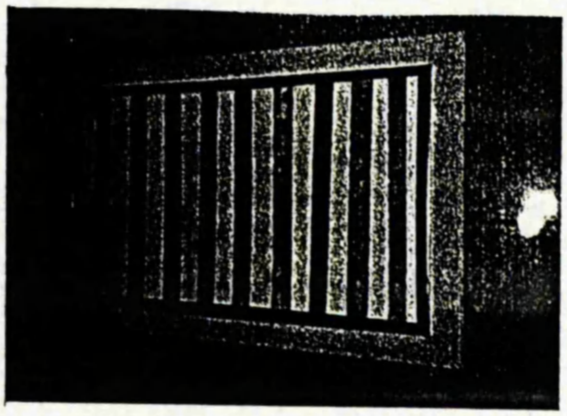
◀ 張家山遺跡

張家山遺跡 ▶



◀ 「漢律」 竹簡





「漢律」　竹簡

▲
「算数書」竹簡（復製品）
右より3，4番目

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科学史研究

祖冲之的《大明历》与圆周率计算

城地 茂

(数 学 系)

摘 要

《大明历》计算了回归年长度，其主要方法是应用了"率".《大明历》的数学部分是从以前的数学、特别是刘徽的数学思想延伸出来的.如果祖冲之计算圆周率方法与刘徽一样，应该计算到圆内接正两万四千五百七十六边形的面积.但不管增加多少有效数位，用定点运算，都不能计算出这一结果.有可能祖冲之利用了刘徽的"率".

关键词 回归年，率，定点运算.

祖冲之是中国古代的大数学家.他的主要著作《缀术》已失传，我们无法直接了解他的工作.但是从一些资料之中还可以看到有关于祖冲之的记载.比如唐代著名数学家李淳风评价祖冲之说："古之九数，圆周率三，圆径率一，其术疏舛.自刘歆、张衡、刘徽、王蕃、皮延宗之徒，各设新率，未臻折衷.宋末，南徐州从事史祖冲之，更开密法，以圆径一亿为一丈，圆周盈数三丈一尺四寸一分五厘九毫二秒七忽.朒数三丈一尺四寸一分五厘九毫二秒六忽，正数在盈朒二限之间.密律(率)，圆径一百一十三，圆周三百五十五.约率，圆径七，周二十二.又设开差幂，开差立，兼以正圆(负)参之.指要精密，算氏之最者也.所著之书，名为《缀术》，学官莫能究其深奥，是故废而不理."(《隋书·律历志》)

据此可知，祖冲之得出了$3.1415926<\pi<3.1415927$.

本文目的是探讨他的圆周率计算方法.

1 《大明历》中回归年的计算方法

《大明历》是划时代的历法.其特点有二：一是第一次使用"岁差"；二是改变"闰周期".

"岁差"是1回归年与1恒星年的差数.由于月球、太阳及行星的引力影响地球自转轴的改变方向，以至春分点每年西迁50多秒.所以使用"岁差"，是为了得出1回归年的正确时间.

"闰周期"的决定也是为了得出1回归年的时间.古代中国使用太阴太阳历，以12朔望月为"1年".所以1太阴年与1回归年的误差大约是$365.25-12\times29.5=11.25$(日).即每年少了11日

本文1989年3月15日收到.本文于1988年8月在美国召开的中国科学史第6届国际会议上宣读，在此发表时略作修改.

多.所以设立"闰月"以修正误差.在《大明历》以前是19年7闰月，祖冲之改为391年144闰月.

《大明历》测定了1回归年的正确时间，也可以说，测定了正确的冬至点时刻.

古代中国用"表"测定日影的长度，以确定季节.日影最长时是冬至，最短时是夏至.但是冬至点未必上中天，所以只可以决定冬至日，不能得出正确的冬至点时刻.

祖冲之计算冬至点时刻的方法在《宋书·律历志》中有记载，即在冬至日前后3次用"表"测定日影，借以计算冬至点："据大明五年十月十日，影一丈七寸七分半.十一月二十五日，一丈八寸一分太.二十六日，一丈七寸五分强.折取其中，则中天冬至，应在十一月三日，求其蚤晚，令后二日影相减，则一日差率也.倍之为法；前二日减，以百刻乘之为实，以法除实，得冬至加时在夜半后三十一刻."如表1.

表1 时间与影长

时间X	影长(尺)y
大明五年十月十日(x_1)	10.775(y_1)
十一月二十五日(x_2)	10.817 5(y_2)
十一月二十六日(x_3)	10.758 3(y_3)

祖冲之得出"一日差率"即1天的日影变化率.$\Delta y=y_2-y_1=0.066\ 67$.

然后他假定从十月十日到十一月二十六日每天日影按这比例变化，而且这个比例对于冬至点前后是对称的.所以"其蚤晚"即x_1与x_2的中间：十一月三日上午0点与冬至点的差数等于$\frac{1}{2}\cdot\frac{y_2-y_1}{\Delta y}\approx 0.318\ 73$. 1天是100刻，舍去小数而得31刻.

祖冲之可能认为时间与日影是一次函数.但实际上是下面函数式(假定地球公转轨道是正圆)：

$$y=h\cdot \mathrm{tg}\{\varphi-\sin^{-1}(\sin\varepsilon\cdot\sin x')\},$$

此处$x'=(360/365.242\ 199)\cdot x(°)$为春分点0点；$h=8$尺为"表"高；$\varphi\approx 32.07(°)$为南京纬度；$\varepsilon\approx 23.64(°)$为461年黄赤交角.

此函数图象是直线，所以"蚤晚"的计算只是近似值.不少中国数学史的研究者指出："率"是汉代以来中国传统数学的基础.而祖冲之的冬至点计算方法也是应用"率"来表示复杂函数近似值的方法.

2 计算圆周率

《大明历》中计算回归年长度的手法是传统的，从数学角度来说并不是划时代的.所以计算圆周率的方法也应该考虑传统方法，就是刘徽的方法.我们先来研究一下刘徽的方法.

2.1 刘徽的第1方法

刘徽首先计算圆内接正多边形的面积，并根据刘徽不等式

$$S_{2n}<S<2\cdot S_{2n}-S_n,$$

从正六边形开始计算到正一百九十二边形面积

$$314\frac{64}{625}<S<314\frac{169}{625}.$$

刘徽认为314是确定的，所以他以3.14作为圆周率.

在理论上这方法可以继续计算下去，但是刘徽在开平方过程中把小数舍去，所以发生误差，不收敛确切值.特别是刘徽时代没有浮点运算，只有定点运算.如果继续计算的话，可得七

百六十八边形的面积，则有

$$314\frac{94}{625}<S<314\frac{100}{625}=\frac{3927}{1250}\times 100,$$

较接近π，以后却误差变大.

虽然《九章算术》方田章第22题注文中有"当求一千五百三十六觚之一面，得三千七十二觚之幂，而裁其微分，数亦宜然，重其验耳"，但是设圆半径1尺的条件下；不能继续计算三千零七十二边形的面积，有可能只计算到一百九十二边形的面积.如表2.

表2　用第1方法计算的面积

边数n	正n边形面积S_n	边数n	正n边形面积S_n
6		192*	$314\frac{64}{625}$
12	300	384	$314\frac{88}{625}$
24	$310\frac{364}{625}$		
48	$313\frac{164}{625}$	768	$314\frac{94}{625}$
96	$313\frac{584}{625}$	1536	$314\frac{94}{625}$

* 刘徽计算到192边形，384边形以下是按刘徽方法计算的.

表3　面积差的比

边数n	正n边形与正$2n$边形面积差ΔS_n	$\Delta S_n/\Delta S_{2n}$
12		
	$\Delta S_{12}=S_{24}-S_{12}=\frac{6614}{625}$	
24		$\frac{\Delta S_{12}}{\Delta S_{24}}=\frac{6614}{1625}=3\frac{1589}{1675}$
	$\Delta S_{24}=S_{48}-S_{24}=\frac{1675}{625}$	
48		$\frac{\Delta S_{24}}{\Delta S_{48}}=\frac{1675}{420}=3\frac{415}{420}$
	$\Delta S_{48}=S_{96}-S_{48}=\frac{420}{625}$	
96		$\frac{\Delta S_{48}}{\Delta S_{96}}=\frac{420}{105}=4$
	$\Delta S_{96}=S_{192}-S_{96}=\frac{105}{625}$	
192		$\frac{\Delta S_{96}}{\Delta S_{192}}=\frac{105}{24}=4\frac{9}{24}$
	$\Delta S_{192}=S_{384}-S_{192}=\frac{24}{625}$	
384		

2.2　刘徽的第2方法

"此术微少，而差幂六百二十五分寸之一百五.以十二觚(如果三上义夫的解释正确的话，应改为"一百九十二觚"——笔者注)之幂为率消息，当取此分寸之三十六，以增于一百九十二觚之幂以为圆幂."

三上义夫认为刘徽发现多边形的面积差很近于公比1/4的等比数列.所以

$$S=S_{192}+(S_{192}-S_{96})\times\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{S_{192}-S_{96}}{S_{96}-S_{48}}\right)^k$$

$$=314\frac{64}{625}+\frac{105}{625}\times\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{4}\right)^k$$

$$=314\frac{64}{625}+\frac{105}{625}\times\frac{1}{3}\approx 314\frac{4}{25}=\frac{3\,927}{1\,250}\times 100.$$

2.3　祖冲之的圆周率计算

祖冲之计算到小数点以下第7位.以圆直径为1丈计算到忽位，可以计算到8位.但是由于定点运算，误差越来越大.若用刘徽的第1方法，只可以得出"朒数"，不能得出"盈数".

如果用刘徽的第2方法，那么只有当n适大时，可以得到$S<31\,415\,927$，就是"盈数".

2.4　刘徽的继承者：祖冲之

从以上考察可知，祖冲之计算到一万二千二百八十八边形面积，然后用刘徽的第2方法公式，得出"朒数"和"盈数".

在前面曾考察过回归年的计算方法是传统性的手法.因此我们推测圆周率的计算方法也是传统性的.如果祖冲之和刘徽的方法相同的话，若就刘徽的第2方法公式而论，率为1/3最为合适.

祖冲之与刘徽是一脉相承的，那么"学官莫能究其深奥"的李淳风的评价如何？虽然祖冲之的计算工作量很大，比刘徽大几倍、几十倍，但是计算"朒数"与"盈数"之中，没有发现"深奥"的方法.

许多现代数学史家认为得出"约率"与"密率"的方法是崭新的.这种理论大约有3种不同看法，就是：1)调日法，2)连分数法，3)求一术.特别值得注意的是2)和3).这些方法应用了"辗转相除"(或"更相减损术")，笔者认为这个方法有可能由计算最大公约数("等数")的过程中发现的.在(a,b)的计算中.

$$\begin{aligned}a&=q_1b+r_1\\ b&=q_2r_1+r_2\\ r_1&=q_3r_2+r_3\\ &\cdots\ \cdots\\ r_k&=q_{k+2}r_{k+1}+r_{k+2}\\ &\cdots\ \cdots\\ r_{n-2}&=q_nr_{n-1}+r_n\\ r_{n-1}&=q_{n+1}r_n+r_{n+1}\end{aligned}$$

表4

	a	b	q_1
q_2	r_1	r_2	q_3
q_4	r_3	r_4	q_5
⋮	⋮	⋮	⋮
q_k	r_{k-1}	r_k	q_{k-1}
⋮	⋮	⋮	⋮
q_{n-2}	r_{n-2}	r_{n-1}	q_n
q_{n+1}	r_n	r_{n+1}	

此处$r_{n+1}=0$时$(a,b)=r_n$.

设a/b的近似值P_k.P_k可以表示为$(a>b)$.

$$P_k=\{q_k,\ q_{k-1},\ q_{k-2},\ \cdots,\ q_1\}=q_k+\cfrac{1}{q_{k-1}+\cfrac{1}{q_{k-2}+\cdots}},\quad (k\leqslant n+1).$$

或者　$P_k=P_{k-1}/P_{k-2}$.

此处　$P_m=P_{m-1}\cdot q_{m+1}+P_{m-2}$，$P_{-1}=1$，$P_0=q_1$.　○印小字

这是假设$r_k=0$时的近似值，这就是所谓连分数法.

"求一术"的方法是假设$r_k=1$的近似值，所以与连分数法的结果差不多.也可以说，2)和3)同源.

问题是以什么原数得出的呢？梅荣照认为"朒数"和"盈数"得出，就是$a=314\,159\,265$，$b=100\,000\,000$，辗转相除得：

$$\pi\approx\{3,7,15,1,288,1,2,1,3,1,7,4\}$$

所以，　　约率$=P_2=22/7$，密率$=P_4=355/113$.

另外看法是由刘徽的第2方法的结果.

$$\pi\approx 3927/1250\approx\{3,7,16,11\},（即a=3\,927,b=1\,250）.$$

所以，　　约率$=P_2'=22/7$，密率$=P_3'=355/113$.

所得结果都一样.

虽然连分数法汉代已经产生，但可以映证李淳风的评价.

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ZU CHONGZHI'S DA MING ALMANAC AND COMPUTE π

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(Department of Mathematics)

Abstract

Da Ming Almanac's new device is the method of computing a tropical year, which uses the concept of 'lü' (linear ratio), a Chinese traditional one.

So the present writer set up by a hypothesis that the method of Zu Chongzhi's computing π and Liu Hui's one was the same. If Zu Chongzhi used same method and computed down to seven decimal places, he had to compute 24 576 polygonal area. But even if the figures were increased, this method cannot compute down here because it used fixed point operation, not floating point operation. So he perhaps used the formula of Liu Hui, and had the approximate value of π.

Key words　computing a tropical year, the concept of 'lü' (linear ratio), fixed point operation.

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論　　説

日 本 数 学 史 学 会

論　　説

日中の方程論再考—『楊輝算法』と『古今算法記』

城地　茂

従来，和算を含めた東洋数学で，方程式の解を最初に２つ求めたのは，沢口一之の『古今算法記』(1671年）とされている．正確に述べれば，沢口一之は，正の解が２つ求まる方程式は出題の誤り－「翻狂」として，定数項を改めている．

しかし，中国では，南宋末に楊輝が，『楊輝算法』(1275年）の中で１つの２次方程式から２つの解を求めているのである．『楊輝算法』を構成する３種の数学書の１つ，『田畝比類乗除捷法』巻下第12，13題の問題がそれである．『楊輝算法』はあまり研究されていなかったので，この事実は報告されていなかったが，明らかに正の解が２つ求められているのである．従って，沢口一之に与えられていた評価は，楊輝に帰すことになるが，日中の方程論は，従来考えられていた水準を遙かに越えており，『楊輝算法』と『古今算法記』の方程式の解法をもう一度検討し直す必要がありそうである．

しかし，本稿は，一番乗りは誰か，という素朴な問いに答えるものではない．なぜなら，そのような比較は，「比較の対象が唯一の数学である」という命題を先験的に真として議論を進めているからである．単純に比較するのは極めて危険である．また，ここで言う方程論とは，西洋数学のそれではなく，解の求め方の思惟形式という意味である．本稿では，徒に数式に頼ることなく，方程式解法を再考し，西洋数学と異なった発展をした東洋数学の一端を探ってみたい．また，中国数学と和算との関係も考えてみたいと思う．

(1) 解法の変遷　幾何から代数へ

中国では，所謂ホーナー法（Horner Method）による高次方程式の解法が発展した．もちろん，計算機具として籌（算木）を使っており，西洋のものとは趣を異にするが，原理は驚くほど似ている．これは，『九章算術』(註１）巻４の「開平方術」が説明するとおり，幾何学的発想に基づいて考案された．

この方法は，現在でも使われている珠算による方法に似ているが，少し異なっている。

先ず，開く数値に相当する面積の正方形を考える．(これを「実」とする.）そして，これを越えない最大の平方数を探し，それを面積から取り去る．このとき，その平方数を探

し易いように，その指数部だけを別に表示しておくという方法が考えられている．「借算」と呼ばれる1本の算木を「実」の末位から2桁毎に最上位まで進めるのである．『算法統宗』（註2）では，「実」を2桁毎のブロックに分けているが，これと機能は同じである．しかし，『算法統宗』では「1」の表示を省略しているので，2次の係数が1のもの以外は開き難いが，『九章算術』の方法では，「借算」に任意の有理数を置くことが可能なので，一般の2次方程式を解く可能性を残している．

こうして，「商」の最上位 x_1 が決まった訳であるが，2桁目以降の値 x' と既知の「商」x_1 とは次の関係にある．

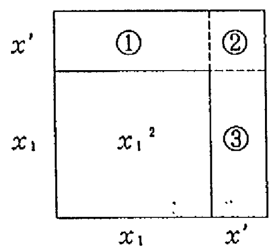

図 1

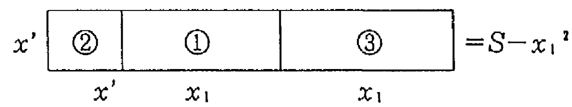

図 2

この関係を利用して2桁目 x_2 を求める訳である．余った「実」の面積は，図1で矩形で表されている。これは，①～③の3つの部分で分解することができる。長方形①と③の長さについては x_1 であることが分かっている．縦は x' であり，この長方形が2つある。また，正方形② x'^2 もある．しかし，この面積は，①③に比べて遙かに小さいものである．そこで，既知の x_1 を2倍して（「法」とする)，これで残った「実」を割れば，大体，x' の見当がつく．$S-x_1^2$ を越えない範囲で0に最も近くなるようにすれば x' が求められる。このとき，「借算」を2桁目の指数を表しているように，1桁目のときより2桁退けておくことを忘れてはならない．

3桁目以降は2桁目と同様に求めればよい．

つまり，2桁目以降は既に求めた数値を x_k として，次以降の値 x' は，

$$S=(x_k+x')^2$$

$$\underset{\text{「実」}}{\underline{S-x_k^2}}=\underset{\text{「法」}}{\underline{2x_k}}\,x'+\underset{\text{「借算」}}{\underline{1}}\,x'^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

で表されることを利用して次々に数値を決めていくのである．

次に，一般の2次方程式，$y^2+ay=S$ について考えてみよう。『九章算術』巻9第20題の術には，

「北門を出ずる歩数を以て西行の歩数を乗し，之を倍し（$=S$)，実と為す．南

門を出ずる歩数を併せ（＝a），従法と為し，開方して之を除く」
とある。これが，「開帯従平方」である．

これも幾何的に考えることが出来る．これは，図３のような長方形と考えて，その広さyを求めることと同じである．

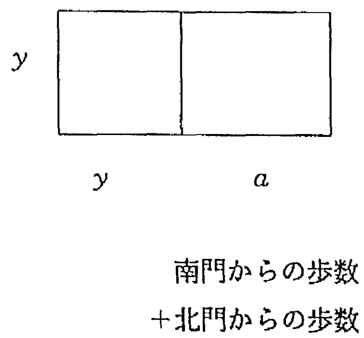

図　2

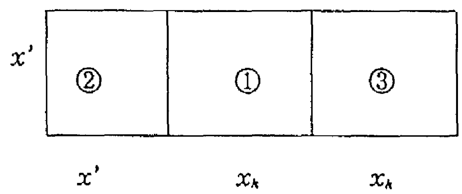

図　3

ここで，図３をみると，図２と同じであることが分かる．開平方の２桁目以降は，つぎつぎに(1)式のような２次方程式に変換して，これを解いてみいることになる。$a=2x_k$となっている訳で，予め１次の項aを「従法」として「法」の位置（３列目）に置いて計算をすれば良いのである．

$$\underbrace{S-(y_k^2+ay_k)}_{\text{「実」}}=\underbrace{(2y_k+a)}_{\text{「法」}}y'^2+\underbrace{y'^2}_{\text{「借算」}}$$

として，次の桁を決めていけばよい。

３次方程式も立体模型を使うことによって解くことができる．係数は複雑になるが，既知のx_kでの各次の係数，指数を表せばよい訳で，特に２次方程式と異なるものはない．『九章算術』では「開立方術」までで，一般の３次方程式は明記されていないが，唐代になると『緝古算経』（註３）が著され，３次方程式が解かれている。

この幾何を応用した方法だと，実際に模型で作れる３次方程式までは可能であるが，４次以上は模型が出来ないので解くことは出来なかった．幾何的に思考する方法は，非常に有効であったが，それは同時に３次方程式までしか解けないという桎梏となってしまったのである．

この限界を突破したのは，宋代の劉益・賈憲である．

彼らは，高次方程式を解く鍵は，既知のx_kと次に求めようとするx'との関係であると看破し，その係数を表す面白い図を考えた。数字を俵積にして，その最も外側を１にする。そして，内部はその上部の２数を足したものとするのである．

0次							1						
1次						1		1					
2次					1		2		1				
3次				1		3		3		1			
4次			1		4		6		4		1		
5次		1		5		10		10		5		1	
6次	1		6		15		20		15		6		1

図　4

これが,「開方作法本源図（註４）」で，各項の係数を表している.「パスカルの三角形」に相当するものを発見したのである．また，これを機械的に求める「増乗開方法」（第３節で実例を示す）も述べている．劉益も賈憲も特殊な例を解いただけであるが，こうして代数的に解くことが考案され，秦九韶の『数書九章』（註５）で,「正負開方術」として完成した。

このとき,「隅」(最高次数の係数,『九章算術』では「借算」）は，任意の有理数が可能である.「隅」は「実」を区切るという機能より,「商」の冪乗と係数を表示する機能を果している.

(2)『楊輝算法』

このような発展を遂げた中国数学は,『楊輝算法』に至って，２次方程式の解が２つ求められることを示した。

方程式 $(x-\alpha)(x-\beta)=S$　　　$\alpha>\beta>0$

を解く場合，従来のように面積を削っていく方法では，小さい方の解βだけしか求めることが出来ない．大きい方の解αを「商」に立てると,「実」が一時的に負になってしまうのである．面積という考え方からすれば，負の面積というものは存在しないから，これはどうしても解くことができない．しかし，代数的に考えれば,「実」の符号を一時的に負にしても何ら不都合はないのである.

『田畝比類乗除捷法』（註６）巻下第13題の問題は,

$$-x^2+60x=860$$

を解くものである。その解き方は,

「草に曰く．積を置きて実と為し，六十歩を以て従方と為し，一算を置き負隅と為す(1).」

実の上に商，長さ三十歩を置き，負隅と命じ，従三十を減ず(2).

上商を以て余る従に命じ，合ず．積九百を除く．而れども積及ばず．乃ち翻法

と命じ，商数の下，積数の上に置く．合わせて積九百より反りて元積八百六十四を減じ，余り正積（註７）三十六とす(3).

上商を以て負隅と命じ，従三十を減ぜば尽きる．負隅を二退す(4).

又，上商長さ六歩を負隅に命じ，六を負方（註８）に置く(5).

以下，複た上商と命じ実を除かば尽きる．長さ三十六歩を得，問に合ふ(6)」

というものである．

「商」		3	3	36	36	36
「実」	864	864	−36	−36	−36	0
「方法」	−60	30	30	0	−6	−12
「隅（借算)」	−1(−1)	−1	−1	−1	−1	−1
図５	(1)	(2)	(3)	(4)	(5)	(6)

(1) 題意のように数値を並べ，「隅」を２桁，「法」を１桁進ませる．

(2) 十位の「商」を３として，「法」から引く．

(3) 残った「法」と「商」を掛けて900となり，「実」から引く．このとき，「実」の符号が変わっている．

(4) 「法」からもう一度「商」×「隅」を引き０になる。「隅」を２桁（「法」を１桁）退ける．

(5) 個位の「商」を６として(2)と同様にする．

この方法を楊輝は「翻積法」と言っている．面積を翻すという意味である．楊輝以前にも「実」の符号が変わる例が知られていたが，解が複数求まることを明示したのは楊輝が最初である．こうして，先に大きい方の解でも小さい方の解でも任意に求めることができるようになったのである．

ここでは，分かり易いように，「実」を『九章算術』のように正として計算を示したが，宋代になると，「実」が面積であるという考え方は希薄になっている．他の項と同様の扱いで，０次の項と捉えている．したがって，楊輝も「実」を負で始めて，途中で正に翻している．ここにも，幾何的発想でなく，代数的発想を見ることができる．

(3)『古今算法記』

日本でも戦国時代の戦乱が収まり，商工業が盛んになると，珠算が普及した．『算法統宗』が伝来し，珠算による「開平方」も伝わっている。

しかし，日本で普及した方法は，『算法統宗』の方法ではなく，算盤を使ってはいるが，

算木と同じ事を行っている訳である.『塵劫記』では，これを「商実法」と言った.

この方法では，算盤の軸間の規格が同じものを何台も容易しなければならず，当時の工業技術では困難が予想される．実用的には3次以下の計算しか出来なかったのではないだろうか.

珠算より算木による計算方法に近かったことは，高次方程式を解くには有利だった.

1671年に天元術を使った数学書,『古今算法記』が沢口一之の手によって刊行された.『古今算法記』は，単に日本最初の天元術の数学書だけではなく，方程論の大きな進歩があった．それは，高次方程式の解が一つとは限らないという事を発見したことである．このことは，前節で述べたように『楊輝算法』でも指摘されていたが，沢口一之は「翻積」とならないものも発見したのである.

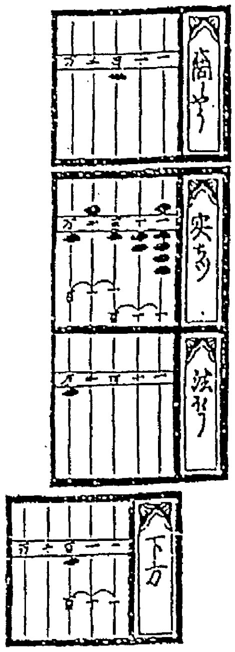

図6　「商実法」による高次方程式の解法　『塵劫記』(註9)より

方程式　$(x-\alpha)(x-\beta)=0$

において，解α，βは,

$$\begin{cases}\alpha>\beta>0\\ \alpha=n\times10^{m}\end{cases}\quad(9\geqq n\geqq1\text{ の自然数，}m\text{は整数})\cdots\cdots\cdots\cdots(2)$$

という条件を満たすとする．つまり，大きい方の解の有効桁数が1桁であるというものである．このような方程式を解くのに,「商」αを立てると，その時点で「実」が丁度0になってしまうので,「実」の符号が変わる訳ではないから「翻積」とは言えない。つまり,「翻積」の特例であるが，計算過程では全く普通の状態で解が2つ出てしまうのである.

『古今算法記』では,『算法根源記』(註10) の遺題第16問を解いている．この問題は，4次方程式であるが，原理は上記のものと同じである.

$$\begin{cases}\frac{1}{4}\pi y^2-x^2=A=47.6255\ (\pi=3.142)\\ y-\sqrt{x}=B=7\end{cases}$$

という問題で，これを解くと,

$$\begin{cases}x=4\\ y=9\end{cases}\qquad\begin{cases}x=0.67932764\cdots\\ y=7.8242133\cdots\end{cases}$$

となってしまう．これは，先にxについて解いてもyについて解いても，(2)式の条件を満たしていることが分かる.

先に y について解くと，

$$-y^4+28y^3-293.2145y^2+1372y=2448.6255$$

「商」		9	
「実」	2448.6255	0	
「方」	1372	272.0695	「方」×「商」
「一廉」	−293.2145	−122.2145	「一廉」×「商」
「二廉」	28	19	「二廉」×「商」
「隅」	−1	−1	「隅」×「商」

図　7－1

となり，yの大きい方の解を先に出しても，「実」は0までにしかならず，符号が変わることはない．「実」が0にならず継続して行うときは，「方」以下の係数を「増乗開方法」で求めておく．

「商」	9	9	9
「実」	0	0	0
「方」	−17.861	−17.861	−17.861
「一廉」	−32.2145	−23.2145	−23.2145
「二廉」	10	1	−8
「隅」	−1	−1	−1

図　7－2

尚，yの小さな方の解の個位（1桁目）7を立てても「実」は正のままであり，そのまま計算を続けることになる．

「翻積法」を使わなくとも2つの解が出てしまい，これを「翻狂」として，出題の失敗とした．そこで，A，Bの数値を

$$A=12.278 \qquad B=4$$

と変えて，

$$x=4 \qquad y=9$$

と解が一意に決まるようにしたのである．

この「翻狂」という名称は，『楊輝算法』から考えるべきだろう．大きい方の解を求める「翻積法」のうち「実」が「狂って」負にならないことがあることを表した名称と言える．したがって，偶然，解が2つになることを発見したのではなく，「翻積法」の特例と意識しているのである．これは，明らかに『楊輝算法』を発展させたものである．

その後，関孝和は方程論を更に進歩させ，「和算」と呼ぶに値する日本独特の数学を完成させる．しかし，沢口一之の業績は，既に南中国文化の模倣の段階を越えて，応用，発

展させていることが分かる.

沢口一之が『楊輝算法』を入手したかどうかについては, 記録が残っていない. しかし, 関孝和が 1661 年に『楊輝算法』を写本しているという事実がある. 沢口一之が誤って関孝和の弟子とされている記録もある（註11）ぐらいなので, 両者の交流は確実で, 当然, 関孝和所蔵の『楊輝算法』を目にする機会があったはずである. 寧ろ, 両者の師弟関係から考えて, 関孝和の写本も種本が沢口一之の蔵書である可能性も否定できないと思う.

(6) まとめ

『楊輝算法』は, 李氏朝鮮で官吏養成の為の教科書として採用されたことが示すように, 初等数学の集大成であり, 従来, 数学史家の注意を余り引かなかった. しかし, 2次方程式の解を2つ求めただけでなく, それと「翻積法」との関係まで把握していたのである.

そして, 現代数学史家が見逃していたこの史実(ママ)を沢口一之は理解していただけではなく, 応用し,「翻狂」という概念にまで達していたのである. これは, 仮定であるが,「翻積法」を使うことによって解が2つ以上出たのであれば, 沢口一之は認めていた, つまり, 出題の誤りとはしていなかったのではないだろうか.

このように考えると, 和算（註12）が中国数学の正統な後継者のように思われてならない. 西洋数学の影響を強く受け, 変質した清代の数学より, 寧ろ, 鎖国により, 西洋文明の摂取を制限された和算が, 中国伝統数学の延長線にあったように思えてならないのである. そうだとすれば, 冒頭で述べた, 誰が最初に2つ目の解を求めたかという問いも, 意味あるものになる. 同じ文明圏の「数学」の中で機能を比較するのならば, それは客観性を保証できるからである. そして, その答えは楊輝である.

（註）

1　撰者不祥,『九章算術』, 9巻. A.D.1C 頃.

2　程大位撰,『算法統宗』, 17巻, 1592 年.

3　王孝通撰,『緝古算経』, 1巻, 620 年頃.

4　賈憲撰,『黄帝九章算法細草』, 1050 年頃. これは散逸してしまい,『永楽大典』巻 16344 に転載されたものが現存している.

5　秦九韶撰,『数書九章』, 18巻, 1247 年.

6　楊輝撰,『田畝比類乗除捷法』, 1275 年.『楊輝算法』を構成する3部の1つである.

7　後に詳解するが,「実」を負として計算を始めている.

8　各本「負積」となっているが, 今,「負方」に改める. 尚,『九章算術』では1次の項を「法」と呼んでいたが, 宋代辺りから「方法」「方」という名称に変化している. 本

稿では，以下の和算も術語を統一せず，原典に従った．

9　吉田光由，『塵劫記』，3巻，1627年序．巻3，第19，開平方を商実法にて除之事．図は大矢真一校注，岩波文庫，1978年版による．

10　佐藤正興撰，『算法根源記』，5巻，1666年序．

11　松永良弼撰，『荒木先生茶談』，18世紀前半．

12　通常，和算とは，関孝和の『発微算法』（1674年）以降を指すが，関孝和の業績の多くはそれ以前の日本数学の系譜を引くものであり，明確に時代区分するのはむずかしい．むしろ，『塵劫記』（1627年）から『発微算法』（1674年）までを過渡期と考えるべきだろう．

（平成2年10月24日受理）

論文の訂正

国外に居りましたので，訂正が遅れましたが，拙論を以下のように訂正させていただきます．

「中国湖北省江陵県張家山遺跡出土『算数書』について」(『数学史研究』　117号)

場所	誤	正
p.21　ℓ.13	三か所	三か国
p.23　ℓ.11	『算数署』	『算数書』

「中国の「圭表」の考案－清朝十尺の「圭表」についての仮説」(『数学史研究』　124号)

場所	誤	正
p.12　ℓ.19	$y=\frac{gx}{b-x}$	$y=\frac{bg}{b-x}$
p.12　ℓ.31	普通傾斜角	黄通傾斜角
	$\varepsilon=24.03°$	$\varepsilon=23.47°$
p.13　ℓ.10	$\phi=35.55°$	$\phi=39.90°$
p.13　ℓ.27	$\phi=35.55°$	$\phi=39.90°$
	$\varepsilon=24.03°$	$\varepsilon=23.47°$

(城地　茂)

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目　　　次

日本数学史学会

資　料

英国王立協会図書館蔵『算法童蒙須知』について

“Sampo Domo Suchi”kept at The Royal Society Library

城　地　　茂

知識公開制度の確立は，近代科学成立のための必要条件と言えるだろう．科学が知識の連続的な積み重ねか，或いは，不連続な革命的なものかは，意見の分かれる所であるが，いずれにせよ，それを公開し，討論しなければならない．しかし，知識を公開した人物・機関が，経済的・名誉的に保護されなければ，知識は秘匿され，大いなる進歩は望めなくなってしまう．著作権という制度は，知識の保護と公開を両立させるものと言える．

英国王立協会（The Royal Society）は，世界で最初にこの知識公開制度を確立させた，換言すれば，近代科学誕生の地の一つになった機関である．その英国王立協会図書館に秘密主義（*1）であるはずの和算書が保管されていたというのも歴史の意外性を感じさせるものである．しかも，これは，日本では散逸してしまった写本であった．勿論，所持していた和算家が公開しても差し支えのないと考えた和算書であるから，数学的水準は高いものではないが，英国王立協会へ到った経路など興味深いものがあるので，報告してみたいと思う．

(1)王立協会と『算法童蒙須知』の寄贈者

ロンドンの中心，ピカデリー・サーカスに近い一等地に位置する王立協会は，1660年の創立であり，ギルド的な大学の枠を越えて研究を進めるという趣旨の元に，新進研究者が集まった．「見えざる大学」(Invisial College）という俗称がそのことを物語っている．1662年には，国王チャールズ二世（Charles II）の勅許状も得られて，英国最高の学会として発展してゆく．1665年以来，機関紙『哲学紀要』(Philosophical Transactions）の発行を続け，知識の保護に努めている（*2）。この方法が最高のものであったかどうかは分からないが，ニュートン卿（Isaac Newton）の『プリンキピア』(Principia，1687年初版，図2右）に代表される近代科学を生み出し，この学会の名声を不動のものとした（*3）。

英国の研究者にとって会員（Fellow of the Royal Society）になることは最高の栄誉である．現会長は，位相幾何学の「k理論」（*4）のアティア卿（Michael Atiyah）で，ニュートンの母校ケンブリッジ大学トリニティ学院院長でもある．毎年，40名の新入会員と6名

の外国人会員を迎え続け，会員は，約850名，中国科学史のニーダム博士（Joseph Needham）もその一員である．外国人会員は約40名，昭和天皇も名誉会員であった．

『算法童蒙須知』は，会員のジェームス・レニィ教授（James Rennie，1787－1867）が1868年１月に寄贈したものである．同教授はロンドン大学キングス校で自然科学史の講座を担当した後，1867年，オーストラリアのアデレードへ移住しているが，日本に立ち寄った形跡はない[*5]．『算法童蒙須知』が中国の歴史書『資治通鑑』（司馬光，1084年成立）と王立協会図書館内部の同じ場所に保管されていることから考えて，香港或いは別な英国の植民地でこれを入手した可能性が高いのではないだろうか．入手したのは，1867年のことであろう．

(2)『算法童蒙須知』の目録と成立時期

題箋には，『最上流，算法童蒙須知』とあり，内題は『算法童蒙須知』である（以下，協会本と記す）．最上流三伝，安永惟正の写本である．大きさは，18.7cm×12.4cm，毎半葉６行×16字である．３編22巻682章[*6]のうち，初編は全失，中編５～10巻，後編１～５巻の11巻が現存していた．

安永惟正は，最上流の四天王と言われた市瀬惟長の門弟で，『二一天作五』（1811年）『算法約術知津』（1816年）を著し，次いで，甲斐の石和に遊歴し，『甲陽算鑑童蒙知津（本朝算鑑）』（1816年序，1820年刊）を著している．最晩期の著作は，『最上流算法中伝目録』（1831年）である[*7]．しかし，協会本は，巻頭と巻末が散逸してしまっており，著された年代は不祥である．他に残された史料から推定してみよう．

下平和夫氏は市瀬惟長の門弟，久松彝之が著した『算法童蒙須知』（以下，下平本と記す）を所持している．下平本は，中編の６，７，10巻の３冊だけであるが，協会本と比較すると，酷似していることが分かった．

	協会本	下平本
大きさ	写本　18.7cm×12.4cm	写本　18.1cm×12.0cm
著　者	最上流三伝　東都本石町十軒店　処士 檜山堂[*8]安永伝語橘篤愛　別名格斎	最上流三伝　東都日本橋稲荷新街 呉橋堂　久松梅司管彝之　観斎
筆記者	門人　鈴木伊三郎[*9]	門人　関金太郎
目　録	中編　巻５　米穀諸術 内外眷減 御蔵米相場	欠

	杉形算		
	諸国引合		
中編　巻6	年利割	年利割	
	月利割	月利割	
	何両一分	何両一分	
	諸利足定法	諸利足定法	
	利平均	利割平均	
中編　巻7	材木売買	材木売買	
	尺〆定法	尺〆定法	
	円法	円法	
	周法	周法	
	本挽定法	本挽定法	
中編　巻8	砂糖類	欠	
	芋種		
	中ケ間引合		
	外店引合		
	上方引合		
中編　巻9	地代勘定	欠	
	沽券割合		
	線香紙鰹節類		
	的中矢数之事		
	平均相場		
中編　巻10	金物類	金物類	
	給金日割	給金日割	
	引合ケ算	引合ケ算	
	賃銀割合	賃銀割合	(「金」相違)
	飛脚再会	飛脚再会	
	運賃割合	運賃割合	

	残物之法		残物之法
	平均利割		平均利割
	奇偶算		奇偶算
	干支用法		干支用法

下編　巻1	位附法	40章	（図3下）欠
2	平歩諧	41	（図3上）
3	〃	30	
4	立坪諧	33	
5	皮積（附析法）	28	
欠（6	土方普請	32）	
欠（7	幾裏諸術	28）	
欠（8	差分（附分術）	18）	
欠（9	盈朒方程	17）	
欠（10	九章算術	26）	

このように，協会本と下平本は，殆ど同じ物と考えられる．安永惟正と久松彝之が市瀬惟長の同門で，共に「最上流三伝」を名乗り，初等教科書として同じ物を使っていたのである．しかも，その著作権を二人の人物が主張しているのである．このような混乱は，師の市瀬惟長の生存中は考えられなかったであろうから，いずれが著者だとしても，『算法童蒙須知』の成立は，市瀬惟長の没後と考えられる．

著者の問題に関しては，次のように考えたい．著者と考えられるのは，二人の門弟[10]以外にも，市瀬惟長（遺稿）であった可能性もある．しかし，『算法童蒙須知・附編・地方算法』も安永惟正が著しているので，著者は，安永惟正であろう．

市瀬惟長と安永惟正の住所（多分，塾の住所）は同じである．したがって，市瀬惟長の没後，最上流三伝として，安永惟長の塾を引き継いだと考えるのが自然だろう．市瀬惟長の年記が残る最後の著作は，『宅間流系譜』(1819年)[11]であり，市瀬惟長は健在である．市瀬惟長が没したのは，市瀬惟長の遺稿[12]が纏められた間である．すなわち，1819年から1824年の間である．

しかし，1820年まで安永惟正は甲斐に逗留しており，江戸を留守にしている[13]ので，師と同じ住所を記することは考え難い．したがって，協会本の成立は，少なくとも1820年以降である．また，遺稿の整理で，1824年までは，忙殺されていたであろうから，それ以降に著された可能性が高いだろう．また，安永惟正の総括とも言える『最上流算法中伝目

録』が著わされたのが，1831年である．したがって，1824年から1831年までの間に成立したと考えられる．

(3)『算法童蒙須知』の内容

それでは，『算法童蒙須知』は，どの程度の数学的水準であったのだろうか．題名から見て，教科書と考えられるが，その対象となった生徒の水準を考えてみたい．

上編は全く散逸してしまっているので，想像するしかないわけであるが，『甲陽算鑑童蒙知津（本朝算鑑）』(1820年刊）には，「八算見一」の説明がなく，この部分を補うような，算盤の基本操作のようなものではないだろうか．

中編になると，巻7の「円法」では，日本初等数学の伝統的円周率，$\pi \fallingdotseq 3.16$を使っている．つまり，「円理」までは教えていないのである．

また，中編，巻10の「引合ケ算」では，

今，小銭（一文銭）ト四文銭[14]ト交テ，四十二銭アル．此ノ銭ヲ以テ，一ツニ付三[15]文ツツノ桃三十九ケ也．小銭，四文銭ノ数ヲ問．

答曰．小銭十七文，四文銭二十五文．

術曰．桃三十九ケニ三文ヲ乗ジ，甲トシ，又，四十二文ニ四文ヲ乗ジ，内甲ヲ引余リ実トシ，別ニ四文ノ内一文引余ル三文ヲ以テ実ヲ除キ，小銭ヲ知ル．

という，所謂「鶴亀算[16]」の応用問題を出している．

代数的に表記すれば，一文銭の数をx，四文銭の数をyとして，

$$\begin{cases} x + y = 42 \\ x + 4y = 117 \ (= 3 \times 39) \end{cases}$$

という連立2元1次方程式を解く訳であるが，このような代数的方法は，欠巻になっている下編，巻9「盈朒方程」で教授していたのであろう．ここでは，一風変わった，大小の差が3である「鶴亀算」で解いている．

42枚の硬貨が全部四文銭であると仮定すれば，168文あることになる．ところが，題意では，1個3文の桃が39個であるから117文である．仮定との差が51文であるから，小銭（一文銭）が四文銭に代わってゆけば，1枚につき四文銭との差3文が修正される．したがって，51÷3＝17枚が小銭の数である．

この問題は，桃1個の価格3文と，小銭（一文銭）と四文銭の差3文が同じになっており，工夫を凝らしたものと言える．最初の仮定のように，168文あるとすれば，桃は56個になり，一文銭が1枚増える毎に桃1個（3文）が減少する．ここから直ちに，56－39＝17として，一文銭の数が計算できる．

このように，普通の「鶴亀算」で「足」の部分が，「桃」で表されているために，仮定的思考に習熟していなければ，混乱してしまう．したがって，この問題を解決できる生徒は，一般的な寺子屋で，普通の「鶴亀算」を既に学習した水準の生徒と考えられる．

また，中編，巻10の「奇偶算」も一見，『孫子算経』巻下第26題「物不知其数」問題の剰余方程式（翦管術，不定方程式）を思わせる問題であるが，実は級数を使う問題である．

今，十露盤ニ物数アリ．其数ヲ知ラス．只云，奇数ヲ以テ〔一，三，五，七，九，十一，十三，逐如此〕是ヲ累減ノ余リ三個．又云，偶数ヲ以テ〔二，四，六，八，十，十二，十四，逐如此〕是ヲ累減ノ余リ八個．

答曰．物数二十八箇．

術曰．偶ノ余ル内，奇ノ余リヲ引キ，五個トナルヲ自乗シテ，奇ノ余リヲ加へ，物数ヲ知ル．

という問題で，

求める数を x，偶数の余りを r_e，奇数の余りを r_o とすると，

$$x = \sum_{k=1}^{n-1} 2K + r_e = n^2 - n + r_e \quad \cdots\cdots (1)$$

$$x = \sum_{k=1}^{n} (2K - 1) + r_o = n^2 + r_o \quad \cdots\cdots (2)$$

(1)(2)式を整理して，n^2 を消去すると，

$$n = r_o - r_e \quad \cdots\cdots (3)$$

求める x は，(2)式に(3)式を代入して，

$$x = n^2 + r_o = (r_o - r_e)^2 + r_o$$

となり，術文のようになる．

このような問題は，翦管術への導入[17]や級数の初心者用の問題としてよく出来た，最上流らしい問題と言える．また，問題に有るように算木を使わず，算盤で解いていたことが分かる．

このように，中編は，中級の生徒を対象にした教科書と考えられる．

下編では，平面幾何や立体幾何（位附法，平歩詰，立坪詰，皮積（附枡法））[18]であり，下編巻5までを見るかぎり四則演算の範囲を超えるものではない．開平方・開立方はなく，「算術」の水準である．これ以後は，『附編・地方算術』や『甲陽算鑑童蒙知津（本朝算鑑）』を学習させたのではないだろうか．

(4)まとめ

以上の考察から，『算法童蒙須知』は，初心者から中級者までの教科書と考えられる．比較的よくできた教科書で，そのため，少なくとも，安永惟正と久松彝之の二つの塾で使われていた．しかし，協会本が海外へと流失したのは，単なる偶然だったのだろうか．

和算が，西洋科学とは異なった結果となったのは，冒頭で述べたように，その流派毎の奥義を最後まで秘匿し続けた事とと無関係ではあるまい．勿論，和算にも公開する制度はあり，それは，2種類に大別できる．

一つは出版（含写本）である．遺題継承の時代[19]には，短期間に中国数学の吸収を成し得ていた．しかし，著作権，出版権は確立しておらず，『塵劫記』（1627年初版）の例に見るまでもなく，他者が自由に出版し，著者吉田光由は偽者に苦慮[20]していた．

もう一つは日本独自の制度で，「算額奉掲」と呼ばれるものである．関流と最上流の論争の発端となったのが，会田安明が愛宕神社に奉納した算額を藤田貞資が訂正を示唆したことから始まっている[21]．このことからも，この制度が論壇の一部を担っていたことが分かる．しかし，算額は印刷されず1枚だけのものである．別な地方の和算家が情報を得ようとしても，近世の交通事情から考えれば，限界は否めない．また，算額は1枚の絵馬なので，その中に盛り込まれる情報量も制限されよう．

このような知識公開制度が完成しなかった和算界にあって、輸出できる和算書は限られてくる。秘伝を公開する訳にはいかない。しかし、一方では、圧倒的な西洋文明との邂逅は、民族主義を芽生えさせた。和算という民族科学を誇示しようとしたに違いない。できるだけ完成度の高い和算書を、と考えただろう。この矛盾する状況のなかで、和算家や書店にとって、協会本の選択は、最善の選択ではなかったのだろうか。

末筆ながら、資料を提供して下さった、王立協会図書館司書、デビット・フォスター氏（David Foster）、ロンドン大学アジア・アフリカ学院。クリストファー・カレン博士（Christopher Cullen）、日本数学史学会、下平和夫博士、佐藤健一氏及び早稲田大学図書館、日本学士院図書館、東北大学図書館に対し、御礼申し上げます。

注釈

1　日本学士院編（藤原松三郎編）、『明治前日本数学史』第4巻、岩波書店、1959年、pp.179－180

2　他に『会報』（Proceedings of the Royal Society,1800～）、『記録』（Note and Records of the Royal Society,1938～）も発行している。

3　ニュートン卿は、反射望遠鏡（図1中央）を寄贈した功績により、1672年に会員と

なり、1703年～27年まで会長を務めるとともに、近代物理学・数学を創立した。図2左は、デス・マスク。

4　Atitah,F.Michael,K-Theory,W.A.Benjamin Inc.,New York,1967.

5　Dictionaray of National Biography（up to 1900,POCOCK to ROBINS）,Royal Society,London,1909,pp.904－905.

6　安永惟正、『算法童蒙須知・附・地方算術』（早稲田大学図書館小倉文庫蔵、写本5巻のうち残2巻）凡例による。本編も附編と同様に、1巻が1冊になっており、1章が1問題である。大きさは本編と殆ど同じく、18.7cm×12.7cmである。これでは、安永惟正は、「最上流再伝」となっている。市瀬惟長は「最上流直伝」となっている（註12参照）事が多いので、このように自称していたようである。

7　前出、『明治前日本数学史』第5巻、1960年、p.275。

8　忠怒（恕カ）堂とも号す。また、安永伊織時正之供ともある（遠藤利貞、『増修日本数学史』、1896年初版、1981年、p.470、林鶴一頭注）。

9　安永惟正、『甲陽算鑑童蒙知津（本朝算鑑）』（1816年序、東北大学図書館林文庫蔵）の跋には、安永惟正の甲斐での門弟の名が列挙されているが、その中には見られない名前である。したがって、江戸へ戻ってからの門弟であろう。

10　確認できる市瀬惟長の門弟は、このほかに、森川徳次郎尺明がいる（前出、『明治前日本数学史』第5巻、p.296）。

11　前出、『明治前日本数学史』第5巻、pp.296－297。

12　市瀬惟長遺稿、『最上流珠盤術自三乗至六乗』（1824年序、早稲田大学小倉文庫蔵、筆者未見）は、安永惟正序、久松彝之編となっている。また、その巻末には、安永惟正、『天正法起源』が付されている（前出、『明治前日本数学史』第5巻、pp.273－274）。

13　1819年の会田安明の三回忌が浅草観音で行われ、算子塚が築かれたが、その時の石碑に安永惟正の名前は上がっていない。（前出、『増修日本数学史』、p.503）ので、江戸不在だったようである。したがって、江戸に戻ったのは『甲陽算鑑童蒙知津（本朝算鑑）』の跋文に記された1820年以降と考えられる。

14　1768年に鋳造された真鍮銭で、江戸時代後期にかけて、よく流通した。なお、1863年の文久銭（銅銭）も有名である。

15　下平本は、五となっているが、術文により三に改める。

16　「鶴亀算」は、『孫子算経』（400年頃）巻下第31題「雉兎同籠」以来、大小の差は2である。2足の鳥と4足獣の合計が何頭、足の合計が何本で、それぞれの数を問うものであった。鶴と亀という目出たい動物になったのは、坂部広胖、『算法点竄指南

録』(1810年刊、序)からである(下平和夫、『和算の歴史』(上)、富士短期大学出版部、1965年初版、1970年、p.45)。

村井中漸、『算法童子問』(1781年序、1784年刊)で大小の差が2でない場合も計算している(佐藤健一、『数学の文明開化』、時事通信社、1989年、p.195)。

尚、『九章算術』(A.D.1c)巻2第38～43題の「其率術」は大小の差が1の「鶴亀算」とする説がある(北京師範大学白尚恕教授、未発表)。

17 斎藤尚中、『斎藤尚中草稿』(1829年、日本学士院蔵)(前出、『明治前日本数学史』第5巻、pp.276－283)など、最上流では剰余方程式の造詣が深かった。

18 安永惟正、『歩詰坪詰解』(成立不祥、東北大学図書館林文庫蔵)には、点竄に類する記号があるが、これと『算法童蒙須知』とは別なものである。

19 吉田光由、『新編塵劫記』(1641年刊)から沢口一之、『古今算方記』(1671年刊)の間が遺稿によって数学が発展した時期とされている(前出、『和算の歴史』(上)、p.167)。

20 前出、『明治前日本数学史』第1巻、1954年、p.41。

21 1781年のことである(前出、『和算の歴史』(下)、1970年、pp.130－134)。以後、藤田貞資、『精要算法』(1781年刊)に反駁する会田安明、『改精算法』(1782年稿、1785刊)から会田安明、『算法非撥乱』(1801年稿)、神谷定令、『福成算法』(1802年稿)の間にかけての論争は関流と最上流の相互に好結果をもたらした(前出、『明治前日本数学史』第4巻、pp.490－504)。

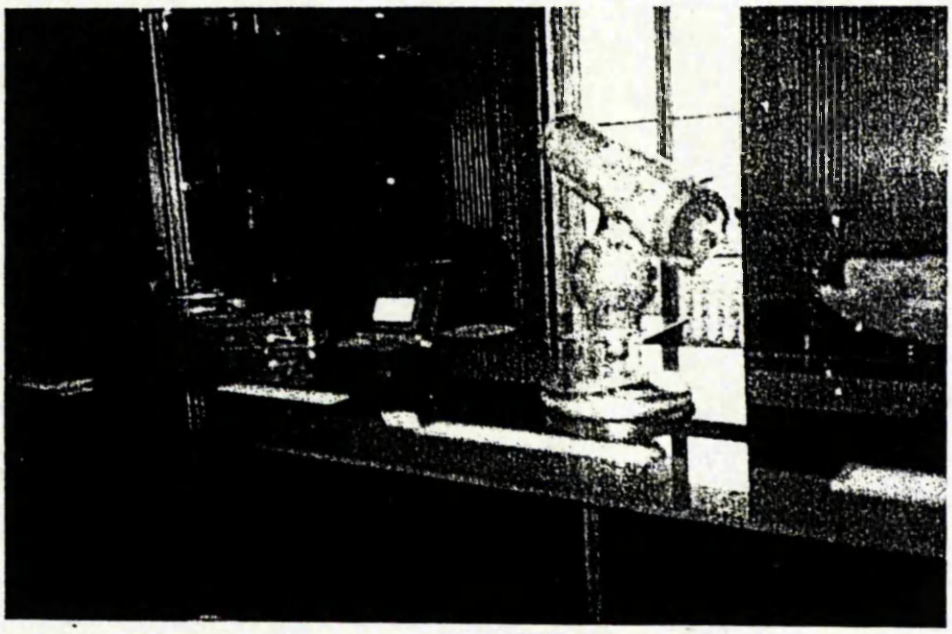

図1 ニュートンの反射望遠鏡

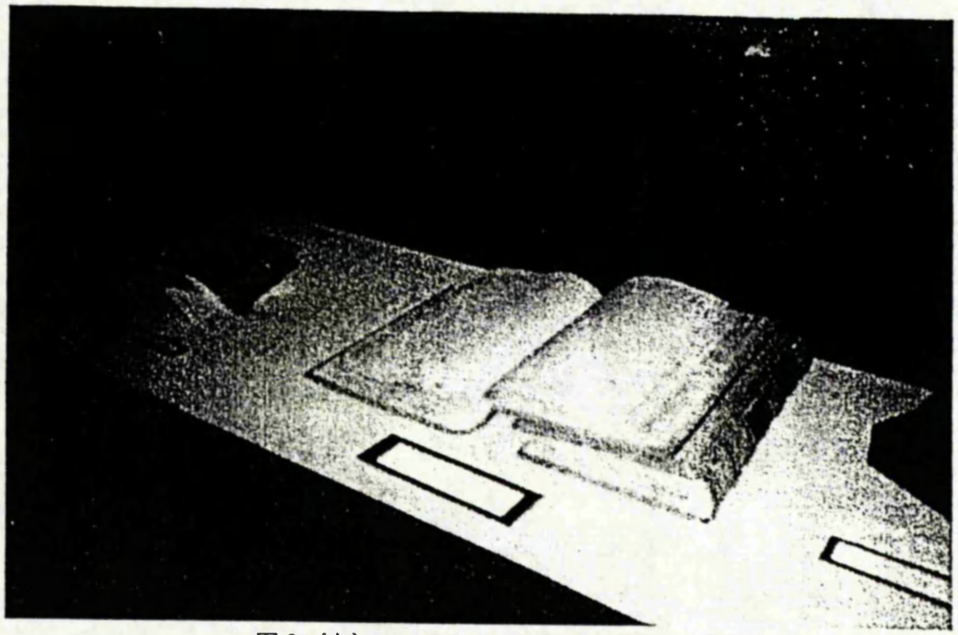

図2（左）ニュートンのデス・マスク
（右）『プリンキピア』初版本

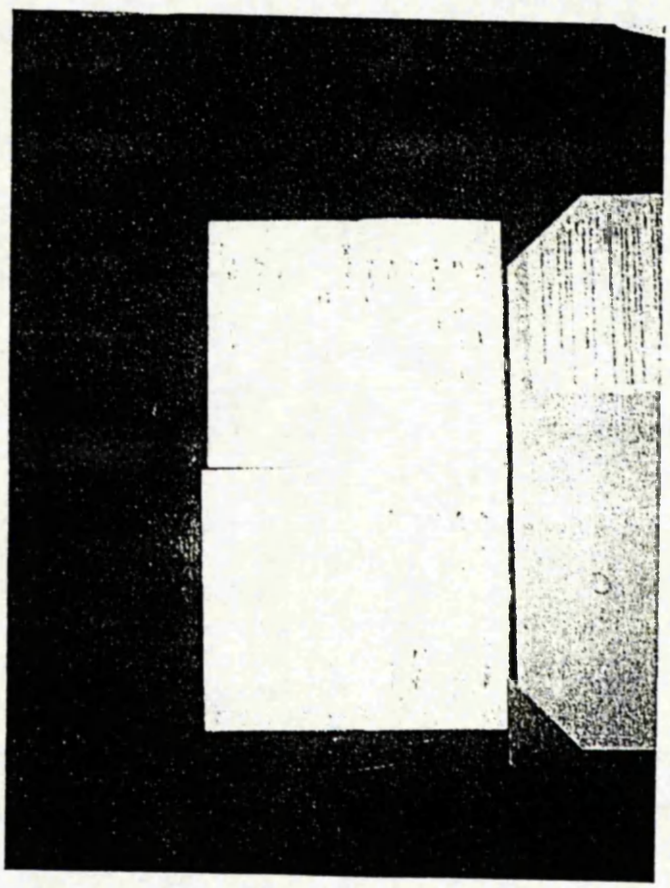

図3『算法童蒙須知』影印
（上）下編　巻2
（下）下編　巻1