Robustly Efficient Equilibria in Non-Convex Economies

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Abstract

An economy has a robustly efficient marginal cost pricing equilibrium (mcpe) if it has an mcpe that is Pareto efficient and if this property is preserved under small variations in preferences endowments and technologies.

We consider economies in which there is a finite number of equilibria, each of which varies continuously with preferences and endowments. We prove that there exist no robustly efficient marginal cost pricing equilibria.

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1. Introduction.

This paper is about the existence of Pareto efficient marginal cost pricing equilibria in non-convex economies with smooth preferences and a finite number of equilibria. We prove that such economies which have Pareto efficient marginal cost pricing equilibria (mcpe) are not robust, i.e., do not survive small variations in the data of the economy such as preferences, endowments and technologies. In other words, the property of having a Pareto efficient mcpe is satisfied on a nowhere dense set.

It was pointed out in Guesnerie (1975) and Brown-Heal (1979) that economies with non-convex technologies have welfare properties very different from those with convex technologies. The most widely noticed difference is the difficulty in obtaining Pareto efficient equilibria. The first welfare theorem is true in any economy, convex or not, so that a competitive equilibrium of a non-convex economy is Pareto efficient but in general competitive equilibria fail to exist. We study instead the existence of marginal cost pricing equilibria because they satisfy the necessary (first order) conditions for Pareto efficiency. But a mcpe may not be Pareto efficient. Indeed, all the mcpe of a non-convex economy may be Pareto inefficient. This point was noted by Guesnerie (1975) and developed by Brown and Heal (1979). Beato and Mas-Colell (1985) then observed that in fact one can have a non-convex economy where all of the mcpe are not only Pareto inefficient but also productively inefficient. Recently Chichilnisky (1990) showed that it is possible to construct technologies for which all of the mcpe are inefficient for a generic set of preference profiles. So the problem of obtaining efficient equilibria in non-convex economies goes deep (for a review, see Vohra (1990)).

We introduce the concept of "robustly efficient" marginal cost pricing equilibria: an economy has a robustly efficient mcpe if it has a Pareto efficient mcpe and if there exists a neighborhood of that economy in which every economy has a Pareto efficient mcpe. This formalizes the idea that we are interested in Pareto efficient equilibria whose efficiency properties are in some sense robust or stable under small perturbations in the specification of the economy. We cannot measure the data that specify an economy exactly, so we are not interested in properties that are not robust to small errors in the description of the economy. We prove in Theorem 2 that there are no robustly efficient mcpe in the class of non-convex economies having a finite number of equilibria which vary continuously with preferences and endowments. Having a Pareto efficient mcpe is not a stable property in a non-convex economy.

A natural and related question is: Are there necessary and sufficient conditions for the existence of at least one Pareto efficient mcpe in a non-convex economy? One can approach this question in three ways. One is to seek conditions on preferences that ensure the existence of a Pareto efficient mcpe. A second is to seek such conditions on production sets. A third is to look for such conditions on the relationships between the production sets and preferences. In general the efficiency of equilibria depends on this latter relationship. However there is no convenient framework for stating conditions on the relationships between preferences and technologies, and conditions of this sort may be hard to verify. Conditions stated on preferences (or production sets) alone have the merit of being verifiable by inspection of preferences (or production sets) alone, as is true of convexity. A condition that is verifiable by inspecting each element of the economy separately is easier to check that one that has to be verified on all elements simultaneously.

We present below in Theorem 1 a condition on preferences that is necessary and sufficient for the existence of at least one Pareto efficient mcpe whatever the technology of the economy. This condition is that the aggregate demand of the economy be independent of the distribution of endowments in the economy. This result is of independent interest in its own right: it is also a crucial building block in the proof of our main proposition, the non-existence of robustly efficient mcpe (Theorem 2).

We state in Corollary 1 a condition sufficient to ensure that all mcpe are Pareto efficient, so that in the sense of the welfare properties of its equilibria, the non-convex economy behaves like a convex economy.

Finally, we establish in Corollary 2 a connection between the efficiency of the mcpe of a non-convex economy with production, and the uniqueness of the competitive equilibria of an underlying exchange economy. Uniqueness of the competitive equilibrium of the underlying exchange economy is necessary for the efficiency of at least one mcpe.

2. The Model.

The commodity space is R^n. We consider an economy with a single increasing returns firm whose production possibility set is Y_I and K convex firms whose production possibility sets are Y_j. Firms are indexed by j which runs over the set $\{1,2,...K,I\}$. A typical production plan of firm j is $y_j \in Y_j$. We assume:

(A1) For all j, Y_j is closed, contains 0 and $Y_j - R^n_+ \subseteq Y_j$. For all $j \neq I$, Y_j is convex. Y_I is __not__ assumed to be convex.

(A2) Let $Y = \sum_{j \neq I} Y_j$, $\tilde{Y} = Y_I + Y$. Then $\tilde{Y}$ is closed and $A(-\tilde{Y}) \cap A(\tilde{Y}) = \{0\}$ where $A(\tilde{Y})$ denotes the asymptotic cone of $\tilde{Y}$.

There are m consumers indexed by i. Each has preferences $\succ_i$, endowments ω_i and shares in the firms of θ_{ij} where $\theta_{ij} \geq 0$ and $\sum_i \theta_{ij} = 1$. Let $\sum_i \omega_i = \omega$, the total endowment of goods and services. i's consumption set is X_i. A typical consumption plan for consumer i is $x_i \in X_i$. Let $X = \sum_i X_i$ be the aggregate consumption set. We assume:

(A3) X_i is the positive cone for all consumers.

(A4) Preferences are convex, smooth, regular and monotone. Formally, $\succ_i$ can be represented by a quasi concave utility function U_i which is twice continuously differentiable (C^2) and has bounded derivatives on R^n_+ the positive cone of R^n. The first derivative of U_i never vanishes on R^n_+ and is positive. The gradient $DU_i(x)$ of U_i is normalized so that its norm is equal to one.

As in Chichilnisky (1990) the space of preferences denoted P is given the topology induced by the C^1 sup norm on utilities. The C^1 sup norm of U is denoted $\|U\|_1$ and defined by $\|U\|_1 = \sup_{x\in P}\|U(x), DU(x)\|$.

A price vector will be denoted p and is assumed to be in the unit simplex Δ in R^n.

Let $X = \sum_i X_i$, the sum of the individual consumption sets. Let $A = X \cap (Y + \omega)$, where $Y = \sum_i Y_i$ is the sum of the economy's production sets and $\omega = \sum_i w_i$ is the sum of the individual endowments. A is the set of aggregate attainable states of the economy. This is assumed to be representable in polar coordinates by a function $a(\theta)$ where $\theta \in R_+^{n-1}$ is an angular direction and $a(\theta)$ is the distance from the origin at which the boundary of A intersects a ray in the direction θ. Under our assumptions $a(\theta)$ is a continuous but not necessarily smooth function. Let $\upsilon = (U_1, \ldots, U_m)$ be a profile of preferences. An economy E is represented by $\left\{a(\theta), \upsilon, \{w_i\}_{i=1,\ldots,m}\right\}$ The space of all possible economies is topologized by the C^1 sup norm product topology on preference profiles and the C^0 topology on functions $a(\theta)$. This latter choice is natural as the function $a(\theta)$ need not be smooth.

We shall often work with the exchange economy created by removing the production possibilities of a productive economy, and call this the underlying exchange economy. In this context we shall need the following definitions. For an exchange economy, let $(\tilde{x}_1, \ldots \tilde{x}_m)$ be a Pareto efficient allocation and $\tilde{p}$ the supporting price (unique by the smoothness assumption (A4)). Let $\psi(\tilde{x}_1, \ldots \tilde{x}_m)$ be the set of all initial endowments from which individuals facing prices $\tilde{p}$ will choose to consume $(\tilde{x}_1, \ldots \tilde{x}_m)$, i.e., the set of endowments for which $(\tilde{x}_1, \ldots \tilde{x}_m)$ and $\tilde{p}$ form a competitive equilibrium. Call this the preimage of the efficient allocation $(\tilde{x}_1, \ldots \tilde{x}_m)$. Formally,

(D1) $\psi(\tilde{x}_1, \ldots \tilde{x}_m) = \left\{(w_1, \ldots w_m) \in \Re^{nm}:\ \forall i, \underset{p.x_i \le \tilde{p}.w_i}{ArgMax} U_i(x_i) = \tilde{x}_i\right\}$

Next we define $\phi(\tilde{x}_1, \ldots \tilde{x}_m)$: this is the same concept subject to the additional restriction that the initial endowments should sum to w, the aggregate endowment of the initial exchange economy. That is, $\phi(\tilde{x}_1, \ldots \tilde{x}_m)$ is the set initial endowments from which individuals facing prices $\tilde{p}$ will consume $(\tilde{x}_1, \ldots \tilde{x}_m)$ *and* that also sum to ω. Formally,

(D2) $\phi(\tilde{x}_1, \ldots \tilde{x}_m) = \left\{(w_1, \ldots w_m) \in \Re^{nm}:\ \sum_i w_i = w \text{ and } \forall i, \underset{p.x_i \le \tilde{p}.w_i}{ArgMax} U_i(x_i) = \tilde{x}_i\right\}$

We shall use maps θ which assigns to each aggregate endowment vector a list of individual consumption vectors which sum to that aggregate endowment. They represent rules for distributing a given total of goods and services between individuals. Formally,

$$\theta: \Re^n \to \Re^{mn},\ \theta(x) = (x_1, \ldots. x_m),\ \sum_i x_i = x$$

We next introduce the concept of a robustly efficient mcpe. Intuitively, this is an Pareto efficient mcpe in a non-convex economy E such that small variations in the parameters (preferences, endowments and productions sets) defining the economy lead to another economy E^1 also having a Pareto efficient mcpe.

(D3) An economy E has a robustly efficient mcpe if it has a Pareto efficient mcpe and if there exists a neighborhood N(E) of E within which every economy also has a Pareto efficient mcpe.

3. Community Preferences.

The concept of community preference has lengthy history, dating back to Scitovsky (1941), Stolper (1950), Gorman (1953), Samuelson (1956) and Chipman (1974). The reader is referred to Chichilnisky and Heal (1983) or to those sources for the original motivation.

For each $x \in P$, let $N_i(x)$ be the vector normal to the indifference surface of i's preference $\succ_i$ at x, i.e., $N_i(x)$ is the gradient of $\succ_i$ at x. $N_i(x)$ is normalized so that $\|N_i(x)\| = 1$. Let $G_i(\hat{x}) = \{x: U_i(x) \geq U_i(\hat{x})\}$. Now define

(D4) $G(x) = \{y: y \in \sum_i G_i(x_i),\ \sum_i x_i = x \text{ and } \exists N: \forall i, N_i(x_i) = N\}$

$G(x)$ is the set of points preferred to x according to the community preference. It is obtained by summing a set of individual preferred-or-indifferent sets $G_i(x_i)$ for points x_i that sum to x at which all consumers have the same marginal rates of substitution between commodities, given by the normal vectors $N_i(x)$.

For each point x there are many different consumption allocations x_i that sum to x and at which all consumers have equal marginal rates of substitution. In general the various consumption allocations with a common marginal rate of substitution that sum to x will have different common marginal rates of substitution. They will therefore give rise to different sets $G(x)$. So corresponding to each x there may be $G^1(x), G^2(x), G^3(x), \ldots.$ constructed as above with $G^1(x) \neq G^2(x) \neq G^3(x) \ldots$ In Chichilnisky and Heal (1983) we give necessary and sufficient conditions for $G^1(x) = G^2(x) = G^3(x) \ldots.$, i.e., for all sets $G(x)$ constructed as above to be identical. Under these conditions, one and only one set $G(x)$ is assigned to each point x, and these sets define the preferred-or-indifferent sets of a complete transitive preference constructed from the profile $\{\succ_1 \succ_2 \succ_3 \ldots \succ_m\}$ of individual preferences. This preference is the *social preference* or *community preference*. If the conditions in Chichilnisky-Heal (1983) are not satisfied, the operation defined in (D4) can of course be carried out for every x, but it will in general lead us to assign several distinct sets $G(x)$ to this point, depending on the choice of the x_i that sum to x. In this latter case the boundary of each such set defines a different indifference curve through x, so that the community preference so constructed is not transitive. The following definition introduces a condition that is necessary and sufficient for the definition (D1) to lead to a unique set $G(x)$ for each x, and so to a transitive social or community preference:

(D5) A family of preferences represented by a set of C^2 utility functions $\{U_1, U_2, \ldots U_m\}$ is said to be affinely homothetic (AH) if there exists a C^2 linear homogeneous utility function U_0 and a set of constant vectors $Z_i \in R^n$ such that for all $x \in X_i$,

$$U_i(x) = U_0(x + Z_i) \quad i = 1,2,\ldots\ldots m.$$

(D5) requires that all preferences be translates of one homothetic preference. It does not require that they be either homothetic or identical. The *tangency path* of a preference $\succ_i$ is the set of points where its indifference surfaces have a given support, i.e., it is a set of the form $\{x \in R^n : N_i(x) = N^*\}$. Note that this is not the same as an expansion path: because of non-negativity constraints an expansion path may contain boundary segments, which can never be true of a tengency path. Chichilnisky and Heal (1983, pages 43-5) elaborate on this point in the context of the difference between conditions for aggregation of preferences and of demands. We can now state

Theorem (Chichilnisky and Heal (1983)). If preferences are such that their tangency paths have a single point of intersection[1], a necessary & sufficient condition for (D1) to lead to a unique set $G(x)$ for each x, and so to a complete and transitive social or community preference, is that individual preferences $\{\succ_1 \succ_2 \succ_3 \ldots\ldots \succ_m\}$ are affinely homothetic.

4. Marginal Cost Pricing Equilibria.

(D6) The Clarke tangent cone (hereafter tangent cone) to $Y \subset R^n$ at the point $y \in Y$, denoted $T_c(Y,y)$, is the set

$\{x \in R^n$: For any sequence (t^k, y^k) in $R_+ \times Y$ with $t^k > 0$ and tending to $(0, y)$, there exists another sequence $(x^k) \in R^n$ tending to x such that $(y^k + t^k x^k) \in$ Y for large enough k$\}$

[1]The condition that preferences are such that their tangency paths have a single point of intersection, is discussed in Chichilnisky and Heal (1983) and is a mild restriction on preferences. From now on we assume preferences to satisfy this condition, i.e.,

(A5) The tangency paths of any preference have a unique point of intersection.

A more complex condition than being affinely homothetic is given in Chichilnisky and Heal (1983) for preferences not meeting this single intersection condition. Note that a profile of preferences can be AH without any of them actually being homothetic, and without any two of them being identical. Chichilnisky and Heal (1983) establishes that condition (D5) is equivalent to the following statement: "..differences between preferences, and their departures from homotheticity, can be removed by suitable changes in initial endowments" (p. 51). Note also that (D5) is equivalent to the fact that when all individual consumption vectors are strictly positive, then aggregate demand is independent of the distribution of endowments.

(D7) The normal cone to $Y \subset R^n$ at the point $y \in Y$, denoted $N_c(Y,y)$, is the set

$$N_c(Y,y)=\{x \in R^n:(x,y) \leq 0 \forall y \in T_c(Y,y)\}$$

Let $(\hat{x})=(x_1,x_2,..x_i..x_m)$, *and* $(\hat{y})=(y_1,y_2,.y_j..y_K,y_l)$.

(D8) A *<u>marginal cost pricing equilibrium</u>* *is a triple* $\{(\hat{x}),(\hat{y}),p\}$ *where the price p is in the unit simplex in* R^n, *satisfying*

$$x_i = \underset{p.x\leq p.\omega_i+\sum_j\theta_{ij}p.y_j}{ArgMax} U_i(x_i) \text{ for all } i \text{ (utility maximization by consumers).}$$

$y_j \in \partial(Y_j)$ *where* ∂ *denotes the set-theoretic boundary, and p is in the normal cone* $N_c(Y_j,y_j)$ *to* Y_j *at* y_j *for all j (first order conditions for profit maximization by firms).*

$$\sum_i x_i \leq \sum_j y_j + \omega \text{ (for all commodities demand does not exceed supply).}$$

In words, an mcpe is a price vector and a set of consumption and production plans such that consumers maximize utility subject to budget constraints defined by the value of their endowments and their profit income at the equilibrium prices and production levels, and all production plans meet the first order conditions for profit maximization at the equilibrium prices.

5. Results.

Our main result, the non-existence of robustly efficient mcpe, is in Theorem 2. In Theorem 1 we give condition on preferences $\succ_i$ and endowments of goods and services ω_i that are necessary and sufficient to ensure that whatever the technologies Y_j at least one mcpe will be efficient. These conditions on preferences and on endowments of goods and services guarantee an efficient mcpe, independently of the structure of the production side of the economy and the allocation of its profits (or losses). They are an essential input in the proof of Theorem 2.

Note that we do not give here conditions sufficient to ensure the existence of a mcpe. Such conditions are discussed in Beato and Mas-Colell (1985), Brown Heal Khan and Vohra (1986) and Dierker Guesnerie and Neuefeind (1985), and the reader may verify that the sufficient conditions in for example Brown Heal Khan and Vohra (1986) are satisfied by our assumptions. We concentrate here on the efficiency of mcpe.

Theorem 1. *Consider an economy as defined in section 2 and satisfying (A1) to (A5). A necessary and sufficient condition on preferences* $\succ_i$ *for the existence of at least one efficient mcpe whatever the endowments* $\{w_i\}_{i=1,..,m}$ *and technologies* Y_j *is that the profile of individual preferences* $\{\succ_1\succ_2\succ_3.....\succ_m\}$ *is AH as defined in (D5).*

Proof. *Sufficiency*. Suppose that the preference profile $\{\succ_1,\succ_2,\succ_3,......,\succ_m\}$ is affinely homothetic as defined in (D5). Then there exists a family of convex sets $G(x)$ for each x as defined in (D4) which define a transitive preference. Represent this preference by the continuous utility function $U(x)$ (a continuous representation may be chosen because the preferred-or-indifferent sets $G(x)$ are closed, being the sums of closed sets whose asymptotic cones are positively semi-independent (Debreu, (1959), p 23.)). This function $U(x)$ is defined on $X = \sum_i X_i$, the sum of the individual consumption sets.

Let $A = X \cap (Y + \omega)$, where $Y = \sum_i Y_i$ is the sum of the economy's production sets and $\omega = \sum_i w_i$ is the sum of the individual endowments. A is the set of aggregate attainable states of the economy. An application of Lemma 3 of Brown and Heal (1982) establishes that by (A1) and (A2) A is compact. Define $x^* = \underset{x \in A}{ArgMax}\, U(x)$, which exists by continuity of $U(x)$ and compactness of A.

We first show that x^* represents a Pareto efficient allocation in the initial economy. By construction of $G(x)$ any allocation which is Pareto superior to x^* is in the interior of $G(x^*)$. But no point in the interior of $G(x^*)$ is feasible because x^* maximizes $U(x)$ over A and hence there can be no point in the interior of $G(x^*)$ that is also in A. Hence x^* is Pareto efficient.

It remains to show that there exist prices such that at these prices x^* can be disaggregated as an mcpe. This is an application of Theorem 1 of Khan and Vohra (1987), noting that aggregate demand is independent of the distribution of endowments.

Necessity. The proof of necessity proceeds via two lemmas. The first, Lemma 1, (stated and proved in the Appendix) shows that "almost no" preference profiles are AH.

Lemma 2. *Let E_1 and E_2 be exchange economies with total endowment w^1 and $w^2 \neq w^1$ respectively. They have the same preferences, which are not affinely homothetic (violate (D5)). Let $\tilde{x}^1$ be a Pareto efficient allocation in E_1 with supporting prices $\tilde{p}^1$, and let $\tilde{x}^2$ be a Pareto efficient allocation in E_2 with supporting prices $\tilde{p}^2$, $\tilde{p}^1 \neq \tilde{p}^2$. Then the set of initial endowments at which $\tilde{x}^1$ is a competitive equilibrium intersects a translate of the set of initial endowments at which $\tilde{x}^2$ is a competitive equilibrium. Formally:*

$$\phi(\tilde{x}^2) \cap \left\{ \phi(\tilde{x}^1) + \theta(w^2 - w^1) \right\} \neq \varnothing$$

where $\phi(.)$ is defined in (D2).

Proof. We have to show that for any $y^2 \in \phi(\tilde{x}^2)$ there exists $y^1 \in \phi(\tilde{x}_1)$ such that $y^2 = y^1 + \theta(w^2 - w^1)$, where $\theta(w^2 - w^1)$ is a distribution over all of the agents of the difference $(w^2 - w^1)$ between the aggregate endowments of the two exchange economies i.e., $\phi(\tilde{x}^2) \cap \left\{ \phi(\tilde{x}^1) + \theta(w^2 - w^1) \right\} \neq \varnothing$. For this we need some notation.

Define a generalization of the Edgeworth box denoted B as:

$$B^k = \left\{ (x_1, \dots x_m) \in \Re^{mn} : \textstyle\sum_i x_i = w^k \right\},\ k=1,2.$$

We shall express B, $\phi(\tilde{x})$ and $\psi(\tilde{x})$ as solutions of systems of simultaneous linear equations. To start, note that B can be expressed as $B^k = \{y \in \Re^{mn} : y.A = w^k \in \Re^n\}$ where

$$A = \begin{bmatrix} I_n \\ . \\ . \\ . \\ I_n \end{bmatrix}$$ is an nm by n matrix made up of the nxn identity matrix $[I_n]$ stacked vertically m times.

We can also express $\psi(\tilde{x})$ as the solution of a system of equations: $\psi(\tilde{x}^k) = \{y \in \Re^{mn} : y.E^k = v^k\}$ where $v^k \in \Re^m$, $v_i^k = \tilde{p}^k.\tilde{x}_i^k$, $v_i^k \in \Re$, $\tilde{x}_i^k$ is a consumption vector in $\Re^n$, $\tilde{p}^k$ is a supporting price vector and

$$E^k = \begin{bmatrix} \tilde{p}^k & 0 & . & 0 & 0 \\ 0 & \tilde{p}^k & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & \tilde{p}^k & 0 \\ 0 & 0 & . & 0 & \tilde{p}^k \end{bmatrix}$$ where $$\tilde{p}^k = \begin{bmatrix} \tilde{p}_1^k \\ \tilde{p}_2^k \\ . \\ . \\ . \\ \tilde{p}_n^k \end{bmatrix}$$, so that $\tilde{p}^k$ is an nx1 matrix and E^k is an nm x m matrix. It follows immediately from the definitions that $\phi(\tilde{x}^k) = \psi(\tilde{x}^k) \cap B^k$

Clearly this intersection is not empty as it contains $\tilde{x}^k$. Note that $\phi(\tilde{x})$ can be expressed as the set of points satisfying both of the above equation systems. Let A and E^k be as defined above and $[A, E^k]$ be the mn by (m+n) matrix whose first n columns are those of A and whose remaining m columns are those of E^k. Similarly, (w^k, v^k) is the row vector whose first n components are those of w^k and whose last m components are those of v^k. Then $\phi(\tilde{x}^k) = \{y \in \Re^{mn} : y.[A, E^k] = (w^k, v^k)\}$.

To see that $\phi(\tilde{x}^2) \cap \{\phi(\tilde{x}^1) + \theta(w^2 - w^1)\} \neq \emptyset$ let $y^1.[A, E^1] = (w^1, v^1)$ and $y^2.[A, E^2] = (w^2, v^2)$. Then set $y^2 = y^1 + \Delta$ for some $\Delta \in \Re^{mn}$, which we can always do. Now $(y^1 + \Delta).[A, E^2] = (w^2, v^2)$, and $(y^1 + \Delta).A = w^2$, $y^1.A = w^1$, so that $\Delta.A = (w^2 - w^1)$. Hence $\Delta = \theta(w^2 - w^1)$, proving the Lemma. •

To prove necessity, assume that preferences are not affinely homothetic (i.e., (D5) is violated). We shall use a modification and extension of the argument of Chichilnisky (1990 Theorem 3.2) which establishes the existence of an aggregate endowment, a production possibility set and a set of shares in the outputs of firms such that all mcpe are Pareto inefficient. Chichilnisky's Theorem 3.1 (used in her proof of Theorem 3.2) does not apply here and is replaced by Lemma 1 of the Appendix.

Given that preferences are not affinely homothetic, from Lemma 1 of the Appendix and

Chichilnisky (1990 Theorem 3.2) it is routine but lengthy to establish the existence of:

* a technology Y,
* points β and ξ in the aggregate attainable set A with the aggregate endowment of the economy being β and $\xi - \beta$ being the net aggregate output,
* distributions of the totals β and ξ between individuals such that β and ξ are the only productively efficient points and neither is Pareto efficient.

The next step is to prove that the distributions of the total net outputs β and ξ between individuals that ensure that both aggregate points are inefficient, are consistent in that both can arise within a standard specification of ownership patterns in a general equilibrium model. This is a consequence of Lemma 2, as follows. We apply Lemma 2 and set $\beta = w^1$ and $\xi = w^2$. Clearly $\tilde{x}^1 = (\tilde{x}_1^1, \ldots \tilde{x}_m^1)$ is an efficient distribution of the endowment $\beta = w^1$ arising from a no-production equilibrium. This distribution gives rise to the community preferred or indifferent set $G^4(\beta)$. Similarly $\tilde{x}^2 = (\tilde{x}_1^2, \ldots \tilde{x}_m^2)$ is an efficient distribution of the supply vector $\xi = w^2$ at the mcpe with production which in turn gives rise to the community preferred or indifferent set $G^1(\xi)$ whose boundary intersects that of $G^4(\beta)$. By construction $\tilde{x}^2 = (\tilde{x}_1^2, \ldots \tilde{x}_m^2)$ is in the interior of $G^4(\beta)$ and therefore Pareto superior to $\tilde{x}^1 = (\tilde{x}_1^1, \ldots \tilde{x}_m^1)$. At the same time, $\tilde{x}^1 = (\tilde{x}_1^1, \ldots \tilde{x}_m^1)$ is in the interior of $G^1(\xi)$ and therefore Pareto superior to $\tilde{x}^2 = (\tilde{x}_1^2, \ldots \tilde{x}_m^2)$, so that neither allocation can be Pareto efficient in the production economy.

Lemma 2 shows that there are initial endowments $(w^*_1\, w^*_2, \ldots, w^*_m)$ from which the equilibrium consumption vector is $\tilde{x}^1 = (\tilde{x}_1^1, \ldots \tilde{x}_m^1)$ and that when we change this endowment vector $(w^*_1\, w^*_2, \ldots, w^*_m)$ by an amount $\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_m^*)$ which sums to the net production vector $w^2 - w^1 = \xi - \beta$ then we arrive at a set of endowments in the economy with production for which the production equilibrium is $\tilde{x}^2 = (\tilde{x}_1^2, \ldots \tilde{x}_m^2)$. Clearly this change in endowments is equivalent to giving individual i a fraction θ_i^* of the profit from operating the technology at the point $\xi = w^2$. Hence both equilibria are consistent with a single normally-specified distribution of endowments of goods and services and of profits (or losses) in the production economy. Note furthermore that all individuals have non-negative incomes at the production equilibrium, as consumption vectors and prices are non-negative[2]. This completes the proof of Theorem 1.•

In Theorem 2 we address the issue of the robustness of Pareto efficient mcpe. We assume the economy to have a finite number of equilibria, each of which depends continuously on preferences and endowments. Smale (1974, theorem, page 120) establishes that this is a generic property for exchange economies using the Whitney topology, and in his Main Theorem (page 126) extends this to

[2] Note that at the production equilibrium, some individuals may of course be worse off than at the no production equilibrium, both because prices are different and because they are shareholders in firms that are making losses at this equilibrium.

production economies. Genericity in the Whitney topology implies genericity in the C^1 sup norm topology: however, Smale imposes a smoothness condition on production functions which we do not adopt, so that his results can not be used to justify our assumption.

Theorem 2. *In the class of non-convex economies with a finite number of locally unique equilibria depending continuously on preferences and endowments, there are no robustly efficient mcpe.*

Proof. There are two stages to the proof. **Stage 1**: we consider any Pareto efficient mcpe, and demonstrate that arbitrarily small changes in the economy suffice to ensure that this equilibrium is no longer an efficient mcpe. We repeat this process for all of the finite number of equilibria of the economy. **Stage 2**: we show that these changes do not introduce a new efficient mcpe.

The argument in stage 1 is summarized in figures 1a and 1b. Figure 1a shows an economy with a Pareto efficient mcpe. Figure 1b shows another economy obtained by modifying the preferences and attainable set of the first very slightly. Preferences have been modified so that in 1b there is an intersection of two community indifference curves corresponding to two different income distributions at the point in the attainable set A corresponding to the equilibrium in 1a. The attainable set has been modified by altering technologies so that its boundary follows the segments abcde in a neighborhood of the equilibrium in 1a. That preferences can be so modified by an arbitrarily small perturbation, follows from the next lemma.

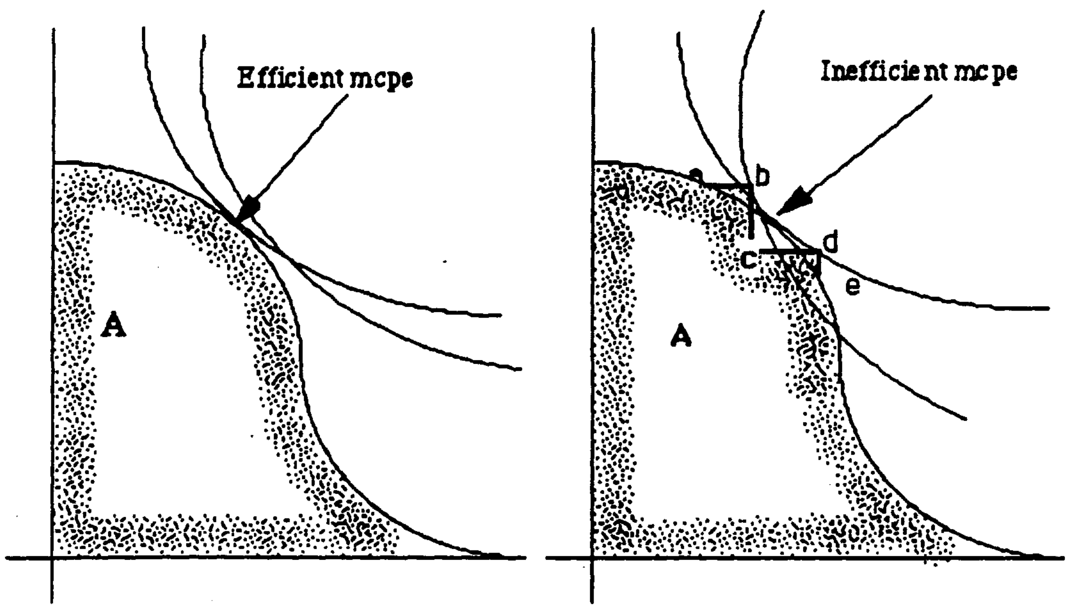

Fig 1a *Fig 1b*

Figure 1: small perturbations in preferences and the attainable set remove a Pareto efficient mcpe

Lemma 2. *For any $\omega \in R^n$, the set of preference profiles $p \in P^m$ giving an intransitive community preference at $\omega \in R^n$, denoted $p_{in}(\omega)$, is open and dense in P^m in the C^1 sup norm topology.*

Proof. For the social preference to be transitive at $\omega \in R^n$, it is necessary that at all sets of consumption vectors $\{x_1^1, x_2^1, x_3^1, ...\}, \{x_1^2, x_2^2, x_3^2, ...\}, \{x_1^3, x_2^3, x_3^3, ...\}$,......... satisfying

(a) $\sum_i x_i^j = \omega$ and

(b) for all consumers $N_i(x_i^j) = N_j$,

we have $N_j = N \; \forall j$. Consider now an allocation and a preference profile that together satisfy (a) and (b). Alter every preference slightly in a neighborhood of the distribution $\{x_1^1, x_2^1, x_3^1, ...\}$ of ω. Ensure that $\forall i, N_i(x_i^1) = N_1$, but choose $N_1 \neq N$. Such a preference profile can be arbitrarily close to the original but is not transitive at ω. This establishes that the complement of the set of preference profiles that are transitive at ω is a dense set. This complement is open because the set of profiles transitive at ω is closed, by arguments similar to those used in Lemma 1 in establishing that (a) and (b) are closed conditions. •

The two attainable sets in figure 1 can be made as close as we wish in the C^0 topology on functions $a(\theta)$ as discussed in section 2.

It remains to be shown that the social indifference curve through b in figure 1b corresponds to the distribution of welfare when b is an equilibrium, and the social indifference curve through d corresponds to the distribution of welfare when d is an equilibrium. This follows from the arguments used in the sufficiency part of the proof of Theorem 1 above. This observation completes stage 1 of the proof of Theorem 2.

Stage 2 of the proof of Theorem 2 uses the assumption that the number of equilibria is finite and that equilibria vary continuously with preferences and endowments. It follows that small changes in preferences, as discussed in stage 1, will lead to small changes in equilibria. Hence the modification of preferences described in stage 1 will not create new equilibria. The only way in which new efficient equilibria could arise, is if existing inefficient mcpe become Pareto efficient as a result of the perturbations of preferences described in stage 1. But a mcpe is Pareto inefficient if it is not a maximal element of the set of attainable utility values under the vector ordering. This is clearly an open property with respect to changes in preferences in the C^1 sup norm topology. Hence a small perturbation of preferences cannot result in a previously inefficient mcpe becoming Pareto efficient. This completes the argument of stage 2 of the proof and hence proves the theorem. •

In Corollary 1 we consider an economy where preferences meet the condition of Theorem 1 so that there is always at least one Pareto efficient mcpe. We give an additional condition under which *all* mcpe are Pareto efficient.

Corollary 1. *Let consumer preferences be affinely homothetic (as in (D5)) and let the community preference be represented by the utility function* $U(x)$. *Assume that the attainable set* $A = X \cap (Y + \omega)$ *can be represented by the equation* $F(x) \geq 0$ *and that both* $F(x)$ *and* $U(x)$ *are continuously differentiable. Also let there be an* x *such that* $F(x) > 0$. *Then a sufficient condition for all mcpe to be Pareto efficient is that there exists a regular* C^1 *diffeomorphism* T *from* R^n *to* R^n *such that the "transformed" programming problem maximize* $U \circ T(x)$ *subject to* $F \circ T(x) \geq 0$ *is concave.*

Proof. This follows from Theorem 1 (i) of Heal (1985) and the arguments in the sufficiency proof of Theorem 1 above. •

Corollary 1 is isolating a class of optimization problems where in spite of the feasible set being non-convex, any critical point is a global maximum. Intuitively this means that preferences are "more curved than" the non-convexities in the feasible set (see figure 2). Economies that are log-linear but non-convex can satisfy this condition. This property is related to the concept of "generalized convexity" discussed in the optimization literature (see Heal (1985) for details). Intuitively, there is a clear explanation for Corollary 1: the structure of critical points of a real-valued function on a manifold is invariant under diffeomorphisms of the commodity space, so that we are characterizing economies that in terms of this structure of critical points are like convex economies. However, the economic interpretation of these conditions is less clear than that of Theorems 1: this is why we have focussed in Theorem 1 on conditions that can be verified simply and by inspection of preferences alone. (For a related condition see Quinzii (1988)).

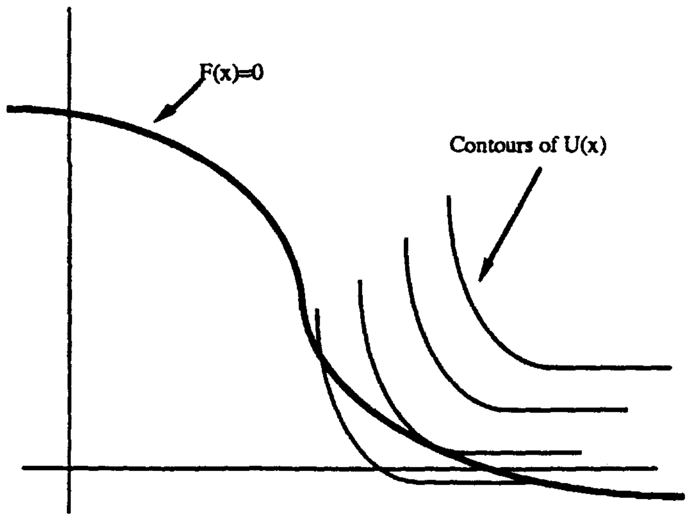

Figure 2: a non-convex economy where all mcpe are Pareto efficient.

The next Corollary establishes a connection between the efficiency of mcpe in an economy with production and the uniqueness of competitive equilibrium in an associated exchange economy. It says that if preferences are such that we can find a set of individual endowments that together with these preferences form an exchange economy whose competitive equilibrium is non-unique, then we can also find a technology that together with the preferences forms an economy whose mcpe are inefficient. Thus the possibility of non-uniqueness of competitive equilibrium implies the possible inefficiency of all mcpe.

Corollary 2. *Let individual preferences $\succ_i$ be such that for some total endowment w^0 there exists a distribution w_i^0, $i = 1, \ldots, m$, of w^0 amongst individuals with $w_i^0 > 0$ $\forall i$, at which the exchange economy $\{(\succ_i) i = 1, \ldots m, (w_i^0) i = 1, \ldots m\}$ has more than one competitive equilibrium. Then there exists a production economy satisfying (A1) to (A4) having $\succ_i$ as individual preferences in which all mcpe are Pareto inefficient.*

Proof. Let w_i^0, $i = 1, \ldots, m$, be the distribution of the total endowment leading to two different competitive equilibria of the exchange economy $CE^1 = (\tilde{x}_1^1, \ldots \tilde{x}_m^1, p^1)$ and $CE^2 = (\tilde{x}_1^2, \ldots \tilde{x}_m^2, p^2)$. Then

(*) w_i^0, $i = 1, \ldots, m \in \phi(\tilde{x}^1) \cap \phi(\tilde{x}^2)$ where $\tilde{x}^k = (\tilde{x}_1^k, \ldots \tilde{x}_m^k)$ $k = 1, 2$.

Let $G^1(w^0) = \sum_i G_i(\tilde{x}_i^1)$ and $G^1(w^0) = \sum_i G_i(\tilde{x}_i^1)$ which have supports p^1 and p^2 respectively at w^0. Pick $\xi \in \partial G^1(w^0)$ and $\beta \in \partial G^2(w^0)$ in such a way that $\xi \in \operatorname{int} G^2(w^0)$ and $\beta \in \operatorname{int} G^1(w^0)$. This can be done because of the arguments of Chichilnisky (1990) Theorem 3.2 Step 1. Clearly β is Pareto superior to ξ and ξ is Pareto superior to β (this is illustrated in Figure 3). Now we note that the existence of two (or more) competitive equilibria from a single initial endowment vector implies the intersection of social indifference curves in which case preferences are not affinely homothetic and Corollary 2 follows from Theorem 1.

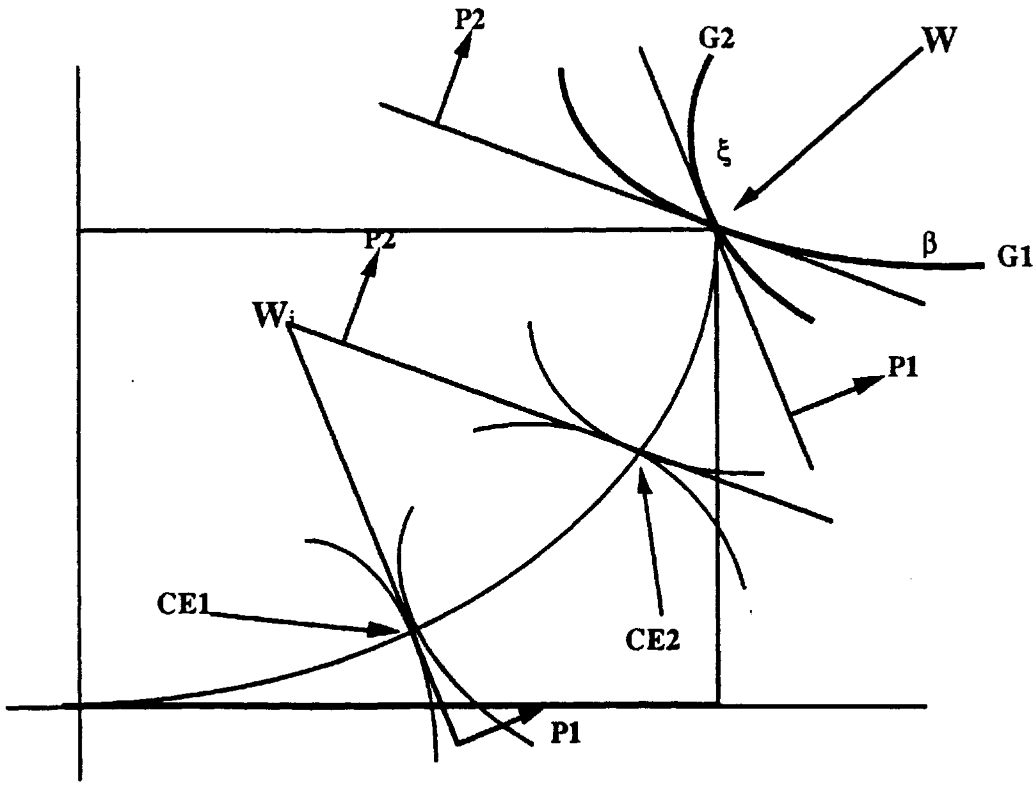

Figure 3: an exchange economy with non-unique competitive equilibria CE1 and CE2 supported by prices P1 and P2. The associated social indifference curves through W, G1 and G2, intersect.

Appendix.

Lemma 1. *Let P be the space of preference satisfying (A4), with the topology given by the C^1 sup norm $\|f\|_1 = \sup_{x \in P} \|f(x), Df(x)\|$. Then the set of preference profiles in P^m that are AH has a complement that is open and dense.*

Proof. First we show that being affinely homothetic (AH) is a closed property in the C^1 topology. Hence the complement of the set of AH profiles is open. Consider a set of vectors $Z_1, Z_2, Z_3, \ldots \rightarrow Z^* \in R^n$. Let U_1, U_2, U_3,... be a sequence of preferences generated by shifting the origin of a fixed homothetic preference by the vectors $Z_1, Z_2, Z_3, \ldots$, i.e., $U_i(x) = U(x + Z_i)$ where U is a homothetic preference. Clearly every element of this sequence of preferences is AH and the limit is also an AH preference.

A homothetic preference satisfying (A4) can be represented by a C^2 function from S^{n-1}, the (n-1) dimensional sphere in R^n, to S^{n-1}, f_i: $S^{n-1} \rightarrow S^{n-1}$. This is because on each ray in the commodity space the preference has a single normal vector. Such a function defines a normal of unit norm to the tangent to the indifference surface along each ray in the commodity space. This is unique on a given ray by homotheticity. Both the space of normals of unit norm and the space of rays are equivalent to

S^{n-1}. An affine translation of a homothetic preference can be represented by such a functions plus a point z_i, the vector by which the origin of a homothetic preference is translated. Let $f_1, f_2, f_3, \ldots$ be a sequence of such functions having f^* as a limit. Because the space of smooth functions between manifolds is a complete space in the C^1 topology, f^* is also an affine translation of a homothetic preference[3]. Hence the space of AH profiles is closed under limits of origin shifts and also under limits of homothetic preferences, and is therefore closed. Hence the complement of the set of AH profiles is open.

Next we have to show that the complement of the set of AH profiles is dense. We show this by demonstrating that an arbitrarily small deformation of any AH profile takes it out of the space of AH profiles. Take the first preference in an AH profile and pick a point x in $\Re^n$. Modify this preference only in a small neighborhood of x , continuing to respect the assumptions (A4). Then the modified preference is no longer a homothetic preference with its origin translated, because there is a ray in the commodity space on which the normal to the indifference curves is not constant. The result is a profile that is not AH, but can be made arbitrarily close to the initial AH profile. This establishes that the complement of the set of AH profiles is dense, and completes the proof of Lemma 1. •

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[3]We do not spell out the argument for the convergence of the vectors z_i, which is obvious.

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