



Construction and Analysis of Experimental Designs

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Declaration

I certify that except where due acknowledgement has been made, the work is that of the author alone; that work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and any editorial work, paid or unpaid, carried out by a third party is acknowledged.

Signed: ..
Talal Alanazi
Date

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Above all else acclaim is for 'Allah', who conceded me the quality and the capacity to continue my work effectively.

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Abstract

This thesis seeks to put into focus the analysis of experimental designs and their construction. It concentrates on the construction of fractional factorial designs (FFDs) using various aspects and applications. These different experimental designs and their applications, including how they are constructed with respect to the situation under consideration, are of interest in this study. While there is a wide range of experimental designs and numerous different constructions, this thesis focuses on FFDs and their applications.

Experimental design is a test or a series of tests in which purposeful changes are made to the input variables of a process or system so that we may observe and identify the reasons for changes that may be noted in the output response (Montgomery (2014)). Experimental designs are important because their design and analysis can influence the outcome and response of the intended action. In this research, analysing experimental designs and their construction intends to reveal how important they are in research experiments.

Chapter 1 introduces the concept of experimental designs and their principal and offers a general explanation for factorial experiment design and FFDs. Attention is then given to the general construction and analysis of FFDs, including one-half and one-quarter fractions, Hadamard matrices (H), Balanced Incomplete Block Design (BIBD), Plackett-Burman (PB) designs and regression modelling.

Chapter 2 presents an overview of the screening experiments and the literature review regarding the project.

Chapter 3 introduces the first part of the project, which is construction and analysis of edge designs from skew-symmetric supplementary difference sets (SDSs). Edge designs were introduced by Elster and Neumaier (1995) using conference matrices and were proved to be robust. One disadvantage is that the known edge designs in the literature can be constructed when a conference matrix exists. In this chapter, we introduce a new class of edge designs- these are constructed from skew-symmetric SDSs. These designs are particularly useful, since they can be applied in experiments with an even number of factors, and they may exist for orders where conference matrices do not exist. The same model robustness is archived, as with traditional edge designs. We give details of the methodology used and pro-

vide some illustrative examples of this new approach. We also show that the new designs have good D-efficiencies when applied to first-order models, then complete the experiment with interaction in the second stage. We also show the application of models for new constructions.

Chapter 4 presents the second part of the project, which is construction and analysis two-level supersaturated designs (SSDs) from Toeplitz matrices. The aim of the screening experiments was to identify the active factors from a large quantity of factors that may influence the response y . SSDs represent an important class of screening experiments, whereby many factors are investigated using only few experimental runs; this process costs less than classical factorial designs. In this chapter, we introduce new SSDs that are constructed from Toeplitz matrices. This construction uses Toeplitz and permutation matrices of order n to obtain $E(s^2)$ -optimal two-level SSDs. We also study the properties of the constructed designs and use certain established criteria to evaluate these designs. We then give some detailed examples regarding this approach, and consider the performance of these designs with respect to different data analysis methods.

Chapter 5 introduces the third part of the project, which is examples and comparison of the constructed design using real data in mathematics. Mathematics has strong application in different fields of human life. The Trends in International Mathematics and Science Study (TIMSS) is one of the worlds most effective global assessments of student achievement in both mathematics and science. The research in this thesis sought to determine the most effective factors that affect student achievement in mathematics. Four identified factors affect this problem. The first is student factors: age, health, number of students in a class, family circumstances, time of study, desire, behaviour, achievements, media (audio and visual), rewards, friends, parents' goals and gender. The second is classroom environment factors: suitable and attractive and equipped with educational tools. The third is curriculum factors: easy or difficult. The fourth is the teacher: well-qualified or not, and punishment. In this chapter, we detailed the methodology and present some examples, and comparisons of the constructed designs using real data in mathematics. The data comes from surveys contacted in schools in Saudi Arabia. The data are collected by the middle stage schools in the country and are available to Saudi Arabian citizen.

Two main methods to collect real data were used: 1/ the mathematics scores for students' final exams were collected from the schools; 2/ student questionnaires were conducted by disseminating 16-question questionnaires to students. The target population was 2,585 students in 22 schools. Data were subjected to regression analyses and the edge design method, with the finding that the main causes of low achievement were rewards, behaviour, class environment, educational tools and health.

Chapter 6 surveys the work of this thesis and recommends further avenues of research.

Contents

Declaration	iii
Acknowledgements	v
Abstract	vii
1 Introduction	1
1.1 Experimental design	1
1.2 Research aim and motivation	2
1.3 Research questions	3
1.4 Methodology	3
1.5 Background	4
1.5.1 Principle of experimental designs	4
1.5.2 Factorial experiment design	5
1.5.3 FFD	7
1.5.4 The general construction and analysis of FFD	8
1.5.5 H matrix	9
1.5.6 BIBD	9
1.5.7 PB Designs	9
1.5.8 Regression modeling	10
2 Background and literature review	13
2.1 Overview of screening experiments	13
2.2 Literature review	16
2.2.1 Literature review of the construction and analysis of edge designs	16
2.2.2 Literature review of the construction and analysis of two-level SSDs	17
2.2.3 Literature review of the construction and analysis of multi-level SSDs	18

2.2.4	Literature review of examples and comparison of the constructed design	20
3	Edge designs from skew-symmetric SDS	23
3.1	The description of the problem	24
3.2	Introduction and preliminaries	24
3.2.1	Edge designs	25
3.2.2	Conference matrix	26
3.2.3	SDSs	26
3.3	Edge designs from skew-symmetric SDSs	26
3.3.1	Algorithm of construction method	30
3.3.2	Constructional example of $n=14$	31
3.3.3	Constructional example of $n=26$	40
3.4	D-efficiency of the new edge designs	49
3.5	Applying the edge designs in screening experiments	51
3.5.1	Analysis example of $n = 14$	52
3.5.2	Analysis example of $n = 26$	55
3.5.3	Analysis for significant interaction effects with $n=14$	60
3.5.4	Analysis for significant interaction effects with $n=26$	60
3.6	Applying models for constructing Edge designs from skew-symmetric SDSs	61
3.6.1	Model 1:linear model	61
3.6.2	Model 2: Full second order	64
3.6.3	Model 3:Odd Polynomial	66
3.6.4	Model 4: Exponential model	67
3.7	Discussion	67
4	Two level SSD from Toeplitz matrices	69
4.1	Description of the problem	69
4.2	Introduction and preliminaries	70
4.2.1	SSDs	70
4.2.2	Toeplitz matrices	72
4.3	Two level SSDs from Toeplitz matrices	73
4.3.1	Algorithm of construction method	73
4.3.2	Constructional example of $n=6$	75
4.3.3	Constructional example of $n=8$	77
4.4	Results and comparison	81
4.5	Applying SSD in screening experiments	81
4.5.1	Analysis Example 7	82
4.5.2	Analysis Example 8	86
4.6	Discussion	87

5	Examples and comparison of design	93
5.1	Introduction and the description of the problem	93
5.2	Conceptual Framework	95
5.3	Real application analysis using real data.	95
5.3.1	Stating the main Concepts	95
5.3.2	Choosing the Place	96
5.3.3	Choosing the sample and making sure if it suits the different factors	98
5.3.4	Carrying out the experiment and collecting data	98
5.4	Data Analysis with experimental design approach	99
5.4.1	Alheverh School	101
5.4.2	The south Al-Shamly School	102
5.4.3	Almair School	103
5.4.4	Ibn Habban School	105
5.4.5	Sharf School	105
5.4.6	Amam Mohmmmed School	107
5.4.7	Sahama School	114
5.4.8	Alfadeelah School	116
5.5	Conclusion	117
6	Conclusion and Future work	119
6.1	Survey	119
6.2	Possible Future Research	120
	Chapter 4	133

List of Figures

1.1	General model of process or system.	2
1.2	Blocking test of drugs.	5
1.3	A two factor factorial with response (y).	6
1.4	The one-half fraction.	8
1.5	PB Designs.	10
1.6	Regression modeling in Minitab.	11
3.1	$X_1^T X_1$ in $n=14$ with one edge.	39
3.2	$X_2^T X_2$ in $n=14$ with two edge.	39
3.3	$X_3^T X_3$ in $n=14$ with one edge.	40
3.4	$X_1^T X_1$ in $n=26$ with one edge.	47
3.5	$X_2^T X_2$ in $n=26$ with two edge.	48
3.6	$X_3^T X_3$ in $n=26$ with one edge.	48
3.7	D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.3).	50
3.8	D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.7).	51
4.1	Analylsis stepwise of example 7 for Case 1 (SSD_1).	83
4.2	Analylsis regression of example 7 for Case 1 (SSD_1).	83
4.3	Analysis stepwise of example 7 for Case 2 (SSD_2).	84
4.4	Analylsis regression of example 7 for Case 2 (SSD_2).	85
5.1	TIMSS Report.	94

List of Tables

2.1	Effectiveness of screening experiment methods.	14
2.2	Relative case of screening experiment methods.	15
2.3	Efficiency and robustness of screening experiment methods.	16
3.1	Examples of skew-symmetric SDSs.	31
3.2	D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.3.)	50
3.3	D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.7).	50
3.4	Model-independent checks with the edge design X_1 ; The $z_{i,j}$ values for Example 1.	53
3.5	Model-independent checks with the edge design $-X_1$; The $z_{i,j}$ values for Example 1.	54
3.6	Step by step calculations for the analysis with the two edges of the design; The sequential approach for Example 1.	54
3.7	Mirror edges check $C - I$; $-C - I$; The $h_{i,j}$ values for Example 1.	54
3.8	Mirror edges check $C + I$; $-C + I$; The $h_{i,j}$ values for Example 1.	55
3.9	Model-independent checks with the edge design X_1 ; The $z_{i,j}$ values for Example 2.	56
3.10	Model-independent checks with the edge design $-X_1$; The $z_{i,j}$ values for Example 2.	57
3.11	Step by step calculations for the analysis with the two edges of the design; The sequential approach for Example 2.	58
3.12	Mirror edges check $C - I$; $-C - I$; The $h_{i,j}$ values for Example 2.	59
3.13	Mirror edges check $C + I$; $-C + I$; The $h_{i,j}$ values for Example 2	59
4.1	Example of defining sequences (w= +,h= -) for Toeplitz matrices.	88
4.2	The properties for SSD_2 ,Cyclic structure and Swarm Intelligence.	89
4.3	Sorted absolute contrasts values for example 7 in SSD_1 case.	90
4.4	Step by step calculations for the analysis example 7 with SSD_1 in the case 1.	90

4.5	Sorted absolute contrasts values for example 7 in SSD_2 case 2.	90
4.6	Step by step calculations for the analysis example 7 with SSD_2 in the case 2.	91
4.7	Sorted absolute contrasts values for Example 8 in SSD_1	91
4.8	Step by step calculations for the analysis Example 8 with SSD_1 in the case 1.	91
4.9	Sorted absolute contrasts values for Example 8 in SSD_2	92
4.10	Step by step calculations for the analysis of Example 8 with SSD_2 in the case 2.	92
5.1	Gender of participants.	96
5.2	Public Shcool in Al-Shamly.	97
5.3	Public Shcool in Ha'il.	97
5.4	Private Shcool in Ha'il.	98
5.5	Notes to be considered about data tables.	99
5.5	(cont.)	100
5.6	One replication of the runs of the design used for Alheverh School.	101
5.7	Model-independent checks with the edge design in Table 5.6.	101
5.8	Step by step calculations for the analysis of Example 9 with the edge of the design.	102
5.9	One replication of the runs of the design used for south Al-Shamly School.	102
5.10	Model-independent checks with the edge design in Table 5.9.	103
5.11	Step by step calculations for the analysis of Example 10 with the edge of the design.	103
5.12	One replication of the runs of the design used for Almair School.	103
5.13	Model-independent checks with the edge design in Table 5.12.	104
5.14	Step by step calculations for the analysis of Example 11 with the edge of the design.	104
5.15	One replication of the runs of the design used for Ibn Habban.	105
5.16	Model-independent checks with the edge design in Table 5.15.	105
5.17	Step by step calculations for the analysis of Example 12 with the edge of the design.	106
5.18	One replication of the runs of the design used Example 13 for Sharf School.	106
5.19	Model-independent checks with the edge design in Table 5.18.	106
5.20	Step by step calculations for the analysis of Example 13 with the edge of the design.	107
5.21	One replication of the runs of the design used Example 14 for Sharf School.	107
5.22	Model-independent checks with the edge design in Table 5.21.	107

5.23	Step by step calculations for the analysis of Example 14 with the edge of the design.	108
5.24	One replication of the runs of the design used for Amam Mohammed School.	108
5.25	Model-independent checks with the edge design in Table 5.24.	108
5.26	Step by step calculations for the analysis of Example 15 with the edge of the design.	109
5.27	One replication of the runs of the design used Example 16 for Ibrahim Althuwaini school.	109
5.28	Model-independent checks with the edge design in Tables 5.27.	109
5.29	Step by step calculations for the analysis of Example 16 with the edge of the design.	110
5.30	One replication of the runs of the design used Example 17 for Ibrahim Althuwaini school.	110
5.31	Model-independent checks with the edge design in Tables 5.30.	111
5.32	Step by step calculations for the analysis of Example 17 with the edge of the design.	111
5.33	One replication of the runs of the design used Example 18 for Ibrahim Althuwaini school.	112
5.34	Model-independent checks with the edge design in Tables 5.33.	112
5.35	Step by step calculations for the analysis of Example 18 with the edge of the design.	113
5.36	One replication of the runs of the design used Example 19 for Ibrahim Althuwaini school.	113
5.37	Model-independent checks with the edge design in Tables 5.36.	113
5.38	Step by step calculations for the analysis of Example 19 with the edge of the design.	114
5.39	One replication of the runs of the design used Example 20 for Sahama School.	114
5.40	Model-independent checks with the edge design in Table 5.39.	115
5.41	Step by step calculations for the analysis of Example 20 with the edge of the design.	115
5.42	One replication of the runs of the design used Example 21 for Sahama School.	115
5.43	Model-independent checks with the edge design in Table 5.42.	116
5.44	Step by step calculations for the analysis of Example 21 with the edge of the design.	116
5.45	One replication of the runs of the design used for Alfadeelah School.	116
5.46	Model-independent checks with the edge design in Table 5.45.	117

5.47	Step by step calculations for the analysis of Example 22 with the edge of the design.	117
6.1	Notes to be considered about the tables of fututr data.	121
6.1	(cont.)	122
2	The properties for new SSD_1 with n runs and m factors.	134
2	(cont.)	135
3	permutation generators for SSD_1 with n runs and m factors	136
3	Cont.	137
4	The properties for new SSD_2 with n runs and $m = k(n - 1)$ factors. 138	
4	(cont.)	139
4	(cont.)	140
5	Permutation generators for SSD_2 with n runs and m(k-1) factors. .	141
5	(cont.)	142
6	Generators permutation for new Es^2 - non optimal SSD_1 with n runs and m factors.	143
6	Cont.	144
6	Cont.	145
6	Cont.	146
6	Cont.	147
6	Cont.	148
6	Cont.	149
6	Cont.	150
6	Cont.	151
6	Cont.	152
6	Cont.	153
6	Cont.	154
6	Cont.	155
6	Cont.	156
6	Cont.	157
6	Cont.	158
6	Cont.	159
6	Cont.	160
6	Cont.	161
6	Cont.	162
7	Generators permutation for new Es^2 - optimal SSD_2 with n runs and m(k-1) factors.	163
7	Cont.	164
7	Cont.	165
7	Cont.	166

7	Cont.	167
7	Cont.	168
7	Cont.	169
7	Cont.	170
7	Cont.	171
7	Cont.	172
7	Cont.	173
7	Cont.	174
7	Cont.	175
7	Cont.	176

Chapter 1

Introduction

1.1 Experimental design

To learn a process, one should observe and understand how that process works. According to Montgomery (2014), "you can observe a lot just by watching". However, the problem is not watching, but knowing what happens when the factors under observation change. One can understand the causes and effects of the process or system by conducting an experiment. Designing an experiment involves testing the changed factors in the input variable to determine the reasons for the changed factors in the output variable. The design of an experiment is very important because it can influence the outcome and response of the intended action, such as technology, commercialisation and product realisation. Experimental design involves the following components:

1–Experimental unit:

This is the smallest department of experimental materials and is randomly assigned to a treatment.

2–Treatment:

This concept is selected by comparing different procedures or objects.

3–Sampling unit:

This alludes to the substance on which the units of measurement are made a review.

4–Factor:

This is a variable characterising an order that may be fixed or random. In the event that the specialist controls the level of factors, then the variable is fixed; but if the specialist randomly examines the level of factors, then the variable is random.

In general, well-designed experiment is led by empirical models that can be operated by engineers or any experts scientist. The aim is to examine the performance

of a system using the model shown in the Figure(1.1). The figure shows that the variables X_1, X_2, \dots, X_p are controllable for the purposes of a test, but the variables Z_1, Z_2, \dots, Z_q are uncontrollable Montgomery (2014). Experiments have

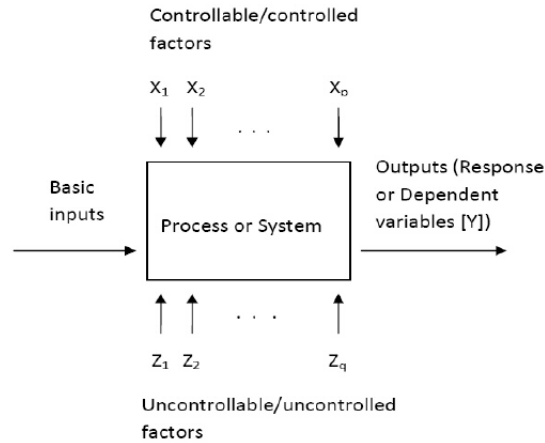


Figure 1.1: General model of process or system.

many factors, so the experimenter must identify those that influence the output using experimentation strategies, such as the best-guess approach and one factor at a time (OFAT). The OFAT strategy fails when there are interactions between the factors, so the best strategy is the fractional factorial experiment, which is discussed later.

1.2 Research aim and motivation

This research aims to be studied and developed the analysis of experimental designs and their construction. Specifically, new fractional factorial designs (FFDs) were constructed and new methods for their analysis explored. In FFDs, the experimenter can reasonably assume that certain high-order interactions are negligible, and information on the main effects and low-order interactions may be obtained by running only a few (selected) runs from the full factorial designs. FFDs can thus save time and money, as fewer experiments needed to be performed. The aim of this study is thus to extend the construction and analysis of experimental designs for FFDs. The construction of experimental designs is of great importance in research studies, and poses a notable challenge in terms of efficient experiment construction. The analysis depends on and benefits from the selected design. Fractional factorial experimental designs are significant in research, and their development

and improvement remains an issue. Therefore, this study seeks to attain new results and identify new construction methods for these designs and improve their statistical analysis.

1.3 Research questions

To ensure the research could achieve the intended objectives, measurable and achievable research questions were devised. The general research question was as follows:

RQ1: How can special types of FFDs be constructed and analysed?

The study also sought to answer following research questions:

RQ2: How does the construction and analysis of the design influence the outcome of a given experiment?

RQ3: How can special types of FFDs be constructed to improve their efficiency?

RQ4: Can the new FFDs perform better than those previously used and discussed in the literature?

RQ5: Can efficient algorithms be developed so that they construct new optimal experimental designs?

1.4 Methodology

The experimental method was adopted for this study, and allowed the study hypothesis to be tested. Variables were manipulated directly (the independent variable) the other potentially influential variables were controlled. Empirical observations from the experiments provided the strongest basis for inferring relationships among variables.

Random sampling of objects was performed and subjects placed into groups. Random sampling ensured the sample was a true representative of the population and not biased. Random assignment of subjects to treatment and comparison of groups ensured equivalency of treatment groups. After treatment, subjects between groups were compared. Factorial designs compared the effects of two or more independent variables; as a result, factors were crossed. The different combinations of independent variables formed cells, and those cells formed the units of analysis. Techniques were then developed and applied to generate new experiment designs. These techniques take advantage of new computer power and ideas, such as:

*merging and modifying different initial designs to generate improved designs

* implementing the new methods and ideas in algorithms and programs using Ex-

cel, Minitab, MATLAB and SPSS

*running simulation experiments to verify and further study findings

*studying the properties of theoretically constructed designs

*extending the method for models with main effects and interactions.

1.5 Background

This section presents a portion of the essential ideas that the peruser ought to be commonplace to continue with the assigned project.

1.5.1 Principle of experimental designs

The three basic principles of design are identified in Fisher et al. (1949): randomisation, replication and local control.

Randomisation

Randomisation refers to the allocation of experimental units; individual runs of the experiment are then performed randomly. Using the statistical method, it is assumed that the errors (observation) are independently random. The proper method of randomising the experiment involves following steps:

- * consider the impact factor in the treatment result
- * introduce the systemic basic to the treatment result
- * assign the representations sample from the population.

For example, in medical research, randomisation and controlled trials are used to test the viability or adequacy of social insurance administrations or health innovations, such as drugs, medicinal gadgets or surgery. If these steps are completed, the process is called complete randomisation.

Replication

Replication means that each factor combination is an independent repeat run to obtain a valid or estimated experimental error and the sample mean for one of the factor levels in the experiment. An example of this is $\sigma_y^2 = \sigma^2/n$, where σ_y^2 is the variance of the sample mean, σ^2 is the variance and n is the replicates.

local control

Local control is used with replication to reduce the experimental error, such as blocking that can be used to reduce the impact factors in the experimental re-

sponse. Blocks are homogeneous experimental units. For example, consider the simple experiment in the Figure 1.2 (see Fisher (1937)).

RANDOMIZED BLOCK DESIGN

TYPE OF HYPERTENSIVE DRUG	BLOCKS	BLOCKS	BLOCKS
	PATIRNT WITH HYPERTENSION (I)	DIABETIC PATIENT WITH HYPERTENSION (II)	PATIENT WITH HEART DISEASE AND HYPERTENSION (III)
A	A,1	A, I	A, III
B	B,1	B, I	B, II
C	C,1	C, I	C, III

Figure 1.2: Blocking test of drugs.

In next section, we present the factorial designs.

1.5.2 Factorial experiment design

Basic terminologies and principles

There are many factors in experiments to examine their process or system. This type of experiment is achieved by factorial designs, meaning that the experiment allows the specialist to examine the effect of each factor on the response variables, such as the effect of interactions between factors. The factorial design describes the aspect concepts in the following ways.

Main effect

The effect of an independent variable is defined as the change in dependent variables produced by a change in the level of an independent variable. For example, consider the simple experiment in the Figure(1.3). This is a two-factors factorial with two levels, high (+) and low (-). The main effect of factor T is the difference between the mean response at the low level of T and the mean response at the high level of T. Numerically, this is $T = (60 + 53)/2 - (69 + 64)/2 = -10$.

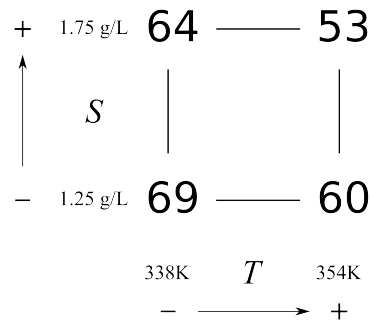


Figure 1.3: A two factor factorial with response (y).

Interaction

Two autonomous factors interact if the effect of one of the factors differs by relying on the level of the other variable. For example, consider the simple experiment in Figure (1.3). The low level of factor S is

$$T = 60 - 69 = -9$$

and the high level of factor S is

$$T = 53 - 64 = -11.$$

Therefore, there is an interaction between T and S, meaning that the interaction effect is the mean difference in the T effects, or $TS = [-9 - (-11)]/2 = 1$ (see Mee (2004)).

The advantage of factorials

The example in Montgomery (2014) help facilitate understanding of the advantage of factorials. Overall, these advantages are summarised as follows:

- * The factorial designs are better organised than OFAT.
- * When the interactions are presented, the factorial designs are very important for avoiding misleading conclusions.
- * The factorial designs allow the results to be estimated at various tiers of the different factors (see Addelman (1961)).

Two-level factorial designs

General 2^k design

The 2^k design refers to a full replicate with k factors at two levels, which is called a full factorial design. This includes the k main effect, the two factor interactions of $C(k,2)$ and the three factor interactions of $C(k,3)$. For example, consider a factorial design with three two-level factors. The set of all level combinations for $k = 3$ can be written independently:

$$X = \begin{pmatrix} A & B & C & AB & AC & BC & ABC \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

2^k designs are optimal designs

The 2^k designs optimal, as it achieves the following :

* All coefficients and effect estimates are least squares estimates that minimise the sum of the squares of the $\epsilon_i, i = 1, \dots, n$.

In general, to estimate the least squares, the following equation can be used:

$$\beta = [(X^T)X]^{-1}(X^T)y, \quad (1.1)$$

where X is the 2^k , T is the transpose, (-1) denotes the inverse and y is the response from the model with treatment combinations (see Mee (2009)).

For example, we can refer to the example in Montgomery (2014).

* All coefficients have the same variance, which means that the variance of coefficients must be minimised.

In general, to estimate the variance, we can use the following equation:

$$V(\beta) = (\sigma)^2/(n2^k) = (\sigma)^2/N, \quad (1.2)$$

where n is the replicated and N is the number of runs.

An example of this is provided in Montgomery (2014).

In the next section, we present FFDs.

1.5.3 FFD

Definition and how they can be used

FFDs are used when the experimenter can sensibly accept that specific high interactions are immaterial, and data on the fundamental effects and low request interactions might be acquired by running only a small portion of the total factorial test. The Fractional designs are expressed using the notation 2^{k-p} , where k is the number of factors used and p is the size fraction from full factorial designs. Fractional designs are successful when they meet three criteria:

- * The sparsity of the effect principle is used when there are many variables, so the system can drive via the main effect and lower interaction.
- * The projection property FFD can be estimated for stronger designs in the active factors.
- * Sequential experimentation is used when the fractional factorial is constructed using a large design to estimate the active factors and interactions(see Addelman (1961), Box and Hunter (1961), Montgomery (2014)).

1.5.4 The general construction and analysis of FFD

Construction and analysis of the one-half fraction

This is one-half of the number of runs, and is similar to the blocking procedure; the process is as follows:

- 1- choose a generator that divides the effects into two
- 2- base on the pluses and minuses of one factor
- 3- defining relation for FFD will be identity column I.

Consider the example in Figure (1.4).

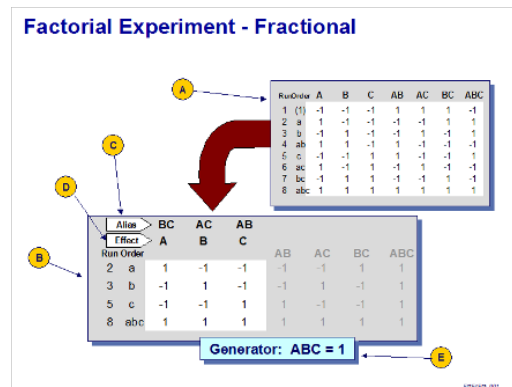


Figure 1.4: The one-half fraction.

Construction and analysis of the one-quarter fraction

This is one quarter the usual number of runs that contains 2^{k-2} . The procedure construction is as follows:

- 1- write down a full factorial in $k - 2$ factors
- 2- add two columns with chosen interactions
- 3- defining Relation: I = two generators, P and Q
- 4- determine the fraction based on the pluses and minuses of P and Q
- 5- all four fractions are the family.

Consider the example in Table 8.9 in Montgomery (2014).

Many computer programs can be used to analyse FFD, such as JMP and Minitab. The design can be analyzed based on the general principle using the following:

$$Effect_i = 2(contrast_i)/N = contrast_i/(N/2),$$

where $contrast_i$ is the plus and minus signs in column I and N is the total numbers of trials (see Box and Hunter (1961), Montgomery (2014)). In the next section, we present the Plackett-Burman (PB) designs.

1.5.5 H matrix

Hadamard matrix= (H) is $n \times n\{1,-1\}$ matrix satisfying $HH^T = nI_n$, all of whose entries are at most 1 in modulus. $\text{Det}(H)$ is equal to the volume of the n-dimensional parallelepiped spanned by the rows of H.

1.5.6 BIBD

A Balanced Incomplete Block Designs (BIBD) is an incomplete block design in which

- b blocks have the same number k of plots each and
- every treatment is replicated r times in the design.
- Each treatment occurs at most once in a block.
- Every pair of treatments occurs together in λ of the b blocks.

1.5.7 PB Designs

The most understood non-customary designs are the PB plans presented by Plackett and Burman (1946). These are known for their capacity to research a substantial number of components using a generally low number of exploratory runs. Because of this property, the PB outlines are frequently used for screening. The quantity of runs n in a PB designs is equivalent to four variables. PB outlines exist for $N = 12, [16], 20, 24, 28, [32], 36, 40, 44, 48$ and any number that is distinct by four. These plans are similar to Resolution III designs, and fundamental effects can be evaluated clear of other primary effects. Main effects are clear of one another, but are confounded with other higher interactions. PB designs are usually defined using a statistical program such as Minitab. Figure 1.5 shows an example.

The screenshot shows a software window titled "Create Factorial Design - Display Available Designs". It contains two main sections:

Available Factorial Designs (with Resolution)

Runs	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	Full	II												
8		Full	IV	II	II	II								
16			Full	V	IV	IV	IV	II	II	II	II	II	II	II
32				Full	VI	IV	IV	IV	IV	IV	IV	IV	IV	IV
64					Full	VII	V	IV	IV	IV	IV	IV	IV	IV
128						Full	VIII	VI	V	V	IV	IV	IV	IV

Available Resolution III Plackett-Burman Designs

Factors	Runs	Factors	Runs	Factors	Runs
2-7	12,20,24,28,...,48	20-23	24,28,32,36,...,48	36-39	40,44,48
8-11	12,20,24,28,...,48	24-27	28,32,36,40,44,48	40-43	44,48
12-15	20,24,28,36,...,48	28-31	32,36,40,44,48	44-47	48
16-19	20,24,28,32,...,48	32-35	36,40,44,48		

Buttons: Help, OK

Figure 1.5: PB Designs.

1.5.8 Regression modeling

Regression models describe the relationship between response variables (dependent variable) and predictor variables (independent variables). Regression helps investment and money-related directors to estimate resources and comprehend the connections between factors- for example, product costs and the loads of organisations managing in those wares. There are two kinds of regression: Simple linear regression and Multiple linear regression. Linear regression uses one independent variable to predict the outcome of response y , while multiple regression uses two or more independent variables to predict the outcome. The general form of each kind of regression is as follows:

Linear Regression:

$$y = \beta_0 + \beta_1 X + \epsilon,$$

Multiple Regression:

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \dots + \beta_n X_n + \epsilon,$$

where y is the dependent variable, X is an independent variables, β is the regression coefficient and ϵ is the residual error. Regression modeling fitting is almost always performed using a statistical program, such as Minitab (see Daniel (1976) and John (1998)). See Figure 1.6 for an example.

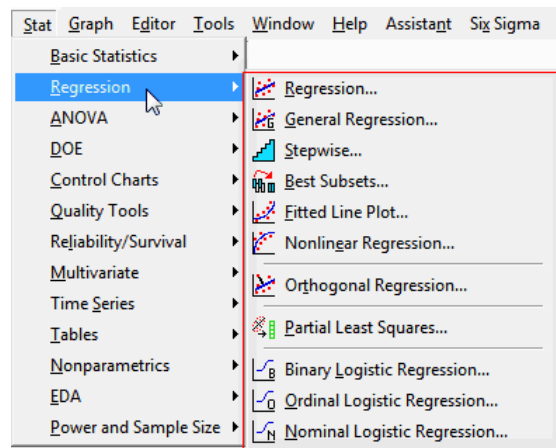


Figure 1.6: Regression modeling in Minitab.

Chapter 2

Background and literature review

This chapter lays the groundwork for the research underpinning this thesis. First, section 2.1 gives an overview of the screening experiments. Section 2.2 then reviews the literature concerning three new methods for constructing and analysing designs.

2.1 Overview of screening experiments

In industry processes, experimental design methods are commonly used to improve product realization and characterize the performance of a system. There are also many applications of experimental designs within the process, such as marketing and finance. Experimental design involves three steps: screening, response surface methodology, and model validation. This chapter focus on screening designs.

Screening designs refer to an experimental design; they are also the most popular designs in industry. The point of screening is to distinguish the dynamic components from countless others that may affects response y . If this stage is not completed during experimentation, all results may be erroneous. Myers et al. (2016) mentions that screening is Phase 0; following this, experimentation can move to Phase 1, which is constructing a first-order linear model. In general, the number of runs in screening is lower than in other designs. These runs do not cost much and can improve the process. Popular experimental designs used in screening experiments are full and fractional two-level factorial designs, PB and supersaturated designs (SSDs). Other types include two-stage group screening, Iterated fractional factorial designs (IFFD), Sequential bifurcation and the Trocine screening procedure (see Trocine and Malone (2001)).

To choose a screening method, a number of criteria must be considered. Efficiency is the first criterion-it is qualitative, and requires managing the number of runs depending on various factors. Effectiveness is the second criterion-this is a diffi-

cult one to determine, because the underlying coefficients are not known, although simulated cases can be used to resolve this. Robustness is the third criterion-it can be used when the condition of the problem is known. The last criterion is ease of use, which can be deployed as an easy experimentation method (such as a software program). Tables, 2.1, 2.3, and 2.2 indicate a comparison between different screening methods based on these criteria (see Trocine and Malone (2000)).

However, there are two noteworthy elements of screening strategies: design and

Table 2.1: Effectiveness of screening experiment methods.

Effectiveness in finding significant effects					
Design	Main effects	Interaction effects	Quadratic effects	No interaction without main effects	Robust to cancelled effects
Desired	Yes	Yes	Yes	Yes	Yes
One factor at time	Yes	No	No	No	Yes
Full factorial	Yes	Yes	1	Yes	Yes
Fractional factorial design	Yes	Yes	1	Yes	Yes
Edge designs	Yes	No	No	No	Yes
Two stage group screening	Yes	Some	No	Some	No
Sequential bifurcation	Yes	Some	No	No	No
Iterated fractional factorial designs	Yes	Some	Yes	No	Yes

analysis. Design has coded levels in each of the factors, numerous construction methods and criteria for selecting the design. Analysis method is very important when finding the active factor using data plots or computation (such as regression analysis). The construction and analysis of many designs are discussed later in this chapter. The analysis of SSDs is of great importance in terms of providing econometric estimates (Georgiou (2008a) and Georgiou (2014)). In the analysis of physical processes, computer simulation requires the application of Latin hypercube designs. Latin hypercube designs are evaluated using matrices. General methods for constructing Latin hypercube designs are detailed in the literature (for example, see Sun et al. (2010) and Steinberg and Lin (2006)). Combining orthogonal designs (ODs), together with generalised orthogonal designs and orthogonal hypercube designs, forms the basis for analysis in designs using computers. When designing an experiment, it is important to design it in such a way that the vari-

Table 2.2: Relative ease of screening experiment methods.

Design	Ease of use			
	Overall	Software Available	Design Phase	Analysis Phase
Desired	Easy	Yes	Easy	Easy
One factor at time	Easy	No	Easy	Easy
Full factorial	Easy	Yes	Easy	Modest
Fractional factorial design	Easy	Yes	Modest	Modest
Edge designs	Modest	No	Difficult	Modest
Two stage group screening	Modest	Yes	Difficult	Modest
Sequential bifurcation	Modest	No	Easy	Modest
Iterated fractional factorial designs	Modest	Yes	Easy	Difficult

ables will be orthogonal to one another (Georgiou (2009)). Recently, computer experiments have become progressively more mainstream surrogates for physical analyses. Circulant matrices with special properties are suggested for constructing designs for computer experiments. In addition, fold-over designs are used alongside the orthogonal matrices to produce desirable results.

However, because this thesis concentrates on analysing and constructing specific experimental designs (such as fractional factorial experimental designs), we define and describe a number of methods for how the designs are constructed and analysed in the following chapters. Here, we present the literature review concerning three new methods for constructing and analysing designs.

Table 2.3: Efficiency and robustness of screening experiment methods.

Screening experiment methods				
Design	Efficiency, Number of exper- iments needed	Robustness, Mono- tonicity required	Robustness, Number of variables	Issue, Sign of effect re- quired
Desired	Small	No	Large	No
One factor at a time	K	No	Small	No
Full factorial	2^k large	No	Small	No
Fractional factorial	2^{k-p} large	No	Small	No
Edge designs	$2k$	No	small	No
Two stage group screening	Varies	Yes	Medium	Yes
Sequential bifurcation	$O(k \log K)$	Yes	Large	No
Iterated fractional fac- torial designs	100 – 500	No	Large	No

2.2 Literature review

2.2.1 Literature review of the construction and analysis of edge designs

Seberry and Whiteman (1988) used Mathon's construction to generate new Hadamard matrices and conference matrices. They constructed a conference matrix of order 3646 and a Hadamard matrix of order 7292. Elster and Neumaier (1995) introduced a new class of screening designs that allowed a model-independent test for active variables. They used conference matrices to generate the desirable optimal edge designs. Skew-symmetric conference matrices of order $n \equiv 2 \pmod{4}$ cannot exist. Further work on constructing edge designs of $n \equiv 0 \pmod{4}$ can be found in other papers, such as Georgiou et al. (2004). They constructed some new skew Hadamard matrices of order 36, which can lead to the construction of conference designs of order $n \equiv 3 \pmod{4}$ and thus edge designs with $2n$ runs, n factors, and n edges. An alternative construction of edge designs from skew-symmetric weighing matrices was proposed in Koukouvinos and Stylianou (2006). They constructed new classes of the D-optimal edge design using weighing matrices of order n and weight k together with permutation matrices of order n . Stylianou (2010) focused on screening active factors to give a model-independent estimate. She generated the conference design with the fold-over design to obtain the significant factors.

2.2.2 Literature review of the construction and analysis of two-level SSDs

Satterthwaite (1959) used SSDs for screening experiments regarding the effect of a sparsity assumption. He constructed SSDs at random balanced designs. Booth and Cox (1962) introduced new SSDs via a computer search and algorithm. They generated effective designs regarding the $E(s^2)$ criterion. Lin (1993) introduced a new class of SSD with $m = t - 2$ factors and $n = t/2$ using the half fraction of the Hadamard matrix. They used a two-level saturated design with $t = 12$ and took a spreading segment that was identified with the positive components of the main section. Wu (1993) constructed SSDs with n runs and $n-1+(n-1)(n-2)/2$ factors using the two-level saturated designs with their two-factors interactions columns. He suggested removing the first column, which consisted of all ones, and minimizing the D and A criteria to find good designs. Lin (1995) suggested employing an algorithm to construct all possible SSD columns. He generated SSD columns in a random order and examined the candidate column using the maximum correlation, which is less than the various values. Nguyen (1996) introduced a Near Orthogonal Array (NOA) algorithm and incomplete block designs to construct SSD. He used the f criterion and the normalized Hadamard matrix, including in Lin (1993). Deng et al. (1996) introduced a new class of SSDs using marginally oversaturated designs. They constructed a two-level design H and incorporated two more balanced columns (v_1, v_2) to obtain the highest r-rank for $[H, v_1, v_2]$. Li and Jeff Wu (1997) constructed two-level SSDs using the k-exchange algorithm. They used M columns from k, in which deleting and replacing one column is essential to selecting the design. Cheng (1997) constructed an SSD via two-level orthogonal arrays and block designs using $E(s^2)$. Yamada and Un (1997) used an orthogonal basis to construct SSDs. Their technique combines two designs and then doubles their runs to obtain the effective designs. Liu and Zhang (2000) developed an algorithm to construct $E(s^2)$ -optimal SSDs via balanced incomplete design(BIBD). The researchers found, one by one, the blocks required for BIBD, and then performed a computer search and calculated the correlation. Butler et al. (2001) developed the lower bound (LB) for a large $E(s^2)$ via lesser known SSDs. Allen and Bernshhteyn (2003) introduced the SSDs class to maximise the probability by stepwise regression, which determines the active factor. Bulutoglu et al. (2004) built the designs regarding LB, and achieved this by the number of factors \times number of runs. Liu and Dean (2004) used k-circulant permutations and cyclic BIBD to construct $E(s^2)$ -optimal and near-optimal two-level SSDs, respectively. Butler (2005a) extended the Cheng method to obtain the mini max SSDs from 16 runs and up to 60 factors. Georgiou and Koukouvinos (2005) constructed infinite families of $E(s^2)$ -optimal two-level SSDs using Hadamard matrices of order $2n$ with $n \equiv 2(mod4)$ runs. Ryan and Bulutoglu (2007) established the NOA algorithm that was de-

fined by Nguyen (1996). They made a small change to the criterion used in the algorithm to achieve $E(s^2)$ -optimal and mini max-optimal SSDs for n runs and m factors. Liu et al. (2007) developed the k -circulant that was defined by Liu and Dean (2004) using interaction columns or deleting columns from it. Koukouvinos et al. (2007b) developed another class of $E(s^2)$ -ideal two-level k -circulant SSDs via methods for $k = 2, 3, 4, 5, 6, 7$. Nguyen and Cheng (2008) used BIBD and regular graph designs to construct $E(s^2)$ -optimal two-level SSDs. Georgiou (2008b) introduced generalized legendre pairs (GLP) to construct $E(s^2)$ -optimal two-level SSDs. Koukouvinos et al. (2008) used supplementary difference sets (SDS) to construct $E(s^2)$ -optimal two-level SSDs. Butler (2009) used Hadamard matrices, the Kronecker product, and ODs to construct $E(s^2)$ -optimal two-level SSDs with n runs and a maximum absolute correlation of $1/4$ between factors. Gupta et al. (2010a) extended the methods to construct $E(s^2)$ -optimal two-level SSDs via adding runs. Niki et al. (2011) constructed $E(s^2)$ -optimal two-level SSDs by selecting the columns according to non-orthogonality between columns. Jones and Majumdar (2014) introduced new criteria, called $UE(s^2)$, to construct SSDs, which were the same as $E(s^2)$ but without factor balance. Georgiou (2014) introduced an overview of two-level SSDs with a cyclic structure. He created the link between the k -circulant method and cyclic incomplete block designs to construct SSDs. Singh and Das (2015) constructed a new design of SSDs-optimal two-level called the Superior $UE(s^2)$. Phoa et al. (2016) introduced optimizing two-level SSDs using Swarm Intelligence Techniques. Sk and Ch (2016) proposed two new series of SSD constructions using mutual orthogonal Latin squares (OLS) and BIBD, and illustrating the methods with suitable examples. Gerorgiou and Stylianou (2016) introduced a new class of conference matrix with a Toeplitz matrix constructed by defining sequences with a symmetric and skew-symmetric conference matrix.

2.2.3 Literature review of the construction and analysis of multi-level SSDs

Yamada and Lin (1999) and Yamada et al. (1999) presented another class of three-level SSDs with an equivalent event property; they noted several criteria to assess the designs, such as the X^2 criterion. Fang et al. (2000) discovered the $E(f_{NOD})$ criterion, which compares multi-level SSDs and studies their properties, such as comparing U-type uniform designs with an orthogonal array. Fang et al. (2002) introduced multi-level SSDs using BIBD and saturated orthogonal arrays and then employed the $E(f_{NOD})$ and X^2 criteria for these designs. Georgiou et al. (2003) constructed three-level SSDs using weighing with an equal occurrence property. This method was used to add good runs to a weighing matrix, and selected the

permutations of rows. Chatterjee and Gupta (2003) introduced s -level SSDs for s^m experiments and provided two classes of designs to search one or two active effects where $s \geq 2$. Fang et al. (2004b) stated that the U-type designs are equal to SSDs by using the $E(f_{NOD})$ criterion. Aggarwal and Gupta (2004) used primitive roots and Galois field theory to construct $E(f_{NOD})$ -optimal multi-level SSDs. Koukouvinos and Stylianou (2004) constructed new optimal multi-level SSDs using a linear and quadratic function: design juxtaposition and permutation. Fang et al. (2004a) used the pressing technique to develop ideal multi-level SSDs. Xu et al. (2005) employed the generalized minimum aberration criterion, which studies and evaluates SSDs. They also provided a new LB and the Addelman-Kempthorne approach to construct optimal multi-level SSDs. Koukouvinos and Mantas (2005) used the juxtaposition method of orthogonal array to obtain multi-level SSDs. Butler (2005b) developed the SSDs Latin hypercube method by using trigonometric functions. Georgiou et al. (2006) used new k -circulant multi-level SSDs to generate the outcomes shown in Liu and Dean (2004). Yamada et al. (2006) deployed the Kronecker product method to determine the largest SSDs with good properties. Georgiou et al. (2006) used error-correcting codes and BIBD to obtain multi-level SSDs. Tang et al. (2007) presented another class of combinatorial outlines to construct $E(X^2)$ -optimal multi-level SSDs. Ai et al. (2007) introduced a new class of $E(X^2)$ -optimal multi-level SSDs using orthogonal arrays. Liu et al. (2007) constructed multi-level SSDs using the Kronecker product method and employed the X^2 and column correlation values in the initial designs. Koukouvinos et al. (2007a) constructed $E(f_{NOD})$ -optimal mixed-level SSDs using SDS and a new method to obtain OLS. Chen and Liu (2008) constructed optimal mixed-level SSDs by removing runs from familiar designs. Liu and Lin (2009) constructed a new method for (X^2) -optimal multi-level SSDs using the Kronecker sum and orthogonal arrays. Sarkar et al. (2009) used a genetic algorithm and a computer search to obtain good SSDs. Chai et al. (2009) used the $E_g(s^2)$ criterion to construct optimal multi-level SSDs and provide a LB. Gupta et al. (2010b) introduced the computer-aided construction of efficient multi-level SSDs using the $E(f_{NOD})$ and $E(s^2)$ criteria. Mandal et al. (2011) developed a construction algorithm of $E(f_{NOD})$ -optimal mixed-level SSDs for k -circulant to ensure that there are no aliased columns. Sun et al. (2011) proposed merging the equal distances of designs and different matrices to obtain optimal mixed-level SSDs using the $E(f_{NOD})$ and (X^2) criteria. Chatterjee et al. (2011) introduced a new LB for the multi-level SSDs and mixed-level SSDs, which was the A_2 -optimality measure. Liu and Liu (2011) suggested a new strategy for developing multi-level and blended level SSDs, which used the column and row juxtaposition of Liu and Lin (2009) and the level transformation of Yamada et al. (1999). Gupta et al. (2011) used BIBD only without a computer search to construct multi-level SSDs. Liu and Liu (2012) developed

the constructed mixed-level SSDs method of Liu and Lin (2009) to obtain new designs. Gupta and Morales (2012) extended their work from 2010 for s-level balanced SSDs. Gupta and Morales (2012) employed a cyclic generator to construct both $E(s^2)$ -optimal and k-circulant SSDs. Chatterjee et al. (2012) introduced the general construction of $E(f_{NOD})$ -optimal multi-level SSDs using SDS. Chen et al. (2013) introduced a new method that constructed $X^2(D)$ -optimal multi-level SSDs with a cyclic structure. Mandal et al. (2014) introduced a calculation to develop proficient adjusted multi-level k-circulant SSDs. Mandal and Koukouvinos (2014) constructed optimal multi-level SSDs through integer programming. Nguyen and Pham (2017) used cyclic generators to construct three-level saturated designs and SSDs.

2.2.4 Literature review of examples and comparison of the constructed design

Cross (2009) attempted to determine the reasons for the decline in primary students mathematics achievement levels in United States. The study provided many reasons for this decline, such as students lack of preparation, teachers failure to use interesting teaching methods, students negative experiences with mathematics and mathematics teachers and the difficulty of mathematical concepts (which were not explained well). Dunne and Gazeley (2008) aimed to understand the effect of the classroom environment on the decline of primary students achievement levels. The outcomes demonstrated that classroom cooperation positively affects students accomplishment levels. When a teacher succeeds in creating an effective classroom environment and building its social relations, students achievement levels increase. Gorard and Smith (2008) was conducted to discover the causes behind the decline of primary students mathematics achievement levels in the United Kingdom (UK). The study sample comprised 2,312 students (males and females) from different UK public schools. The study revealed that the level of accomplishment in mathematics was low, and that the main reasons for this were students negative attitudes towards mathematics and the use of traditional teaching methods (i.e., absence of modern teaching methods). Al-Ahmadi (2009) investigated the effect of using advisory programs to overcome problems of low self-esteem and lack of motivation to improve classrooms for female Grade 11 students from non-academic fields in Saudi Arabia. The study showed that using advisory programs had no effect on low self-esteem and lack of motivation. The researcher explained that the studys measurement time was too short to note any changes, and that there were other reasons for the results. Van Evra (2004) sought to discover the effects of watching television on the decline in achievement levels in Saudi Arabia, and showed an inverse correlation between watching television

and academic achievement: the more students watched television, the less they achieved. However, there was no evidence that the absence of television led to achieving higher grades. Jha et al. (2012) investigated the reasons for the decline of mathematics achievement levels for Grade 1 female students in secondary school. It also aimed to evaluate the percentage of decline objectively. The study sample comprised 367 female students from Mecca- Saudi Arabia, 45 female mathematics teachers and 12 supervisors. The study showed that there are many reasons for the decline in mathematics achievement levels, the important being poor facilities and teachers not understanding the nature of the age group they taught nor their students social, economic and cultural backgrounds (which would allow teachers to direct students more effectively and pinpoint the individual variance). Seeley (2004) sought to evaluate the effect of using computers in teaching (especially in grades 4 and 7) and the level at which students can achieve. The sample featured 68 students from Grade 4 and 64 students from Grade 7. The results showed that there were no significant contrasts identified in students attitudes and use of computers the group that used computers in teaching had high mathematics achievement levels. There were also significant differences in achievement levels according to gender. Mutai (2011) explored the relationship between attitudes towards mathematics and achievement levels for Grade 10 students in Tulkarm. The study sample consisted of 388 male and female students, and produced many results. The most important of these was the notable decline in mathematics achievement levels in general; there were also statistically significant differences between students attitudes towards mathematics and their mathematics achievements (depending on gender and place of living). There was also a direct relationship between attitudes towards mathematics and achievement levels. The study of the Palestinian National Center for Assessment Assaf (1997) sought to understand the achievement levels of Grade 6 Arabic and mathematics students in Palestine. The study sample was 4,114 male and female students from Palestinian schools. The study showed that the percentage of success in mathematics was 27.8 %, which is considered very low. Many researchers stated that students attitudes towards studying mathematics had the highest association with mathematics achievement (Sulayman and Yunan (2008)). Casanova et al. (2005) found that family factors (parental acceptance, control, success expectations) play a role in academic achievement. Hadi and Al-Omar (2013), while researching the factors associated with academic achievement, found that student factors (pre-dated achievement, self-concept) were considered more important than some school factors (gender, number of students, teacher satisfaction). Elekwa (2010) noted that students hold negative attitudes towards mathematics, even when they know that they need it to advance in their studies and in their life as a whole. Rosenthal and Jacobson (1968) explained that there are many determiners of a teachers expectations of

his or her students mental abilities, and Bamburg (1994) showed that a teachers expectations play a considerable role in determining how well and how much students learn. Some parents are eager to teach their children the basics of reading and mathematics at an early age (even before they enter school) (Epstein (1991)). In addition, governments have begun providing education to children, but most of these schools lack qualified staff and suitable equipment and maintenance (Ojaleye (2000)). Hanushek (1997) stated that the quality of a teacher affects almost all grade levels annual achievement growth. Ma (1999) also confirmed that providing the essential human resources and material assets goes far in terms of supporting and improving students execution. A 1991 study cited in Etsey (2005) showed that lack of inspiration and expert duty lead to poor student participation and amateurish traits towards students, which in turn influence student performance. Laz and Shafei (2014) evaluated the mathematics curriculum to determine the reasons behind low achievement levels in Saudi Arabia. The researchers found that there were many reasons for this among Grade 1 students undergoing secondary education-among these were the number of subjects studied, intensity, psychological changes and less memorisation; the most important reason was lack of parental follow-up. Moses et al. (2012) asked a teacher to spend a school day with their students and try to change their mode of teaching to suit the students attitudes (e.g., teaching sometimes, sending them to the playground and developing their art skills). Brown et al. (2008) showed that there are many reasons for the low participation in mathematics in the UK-perceived difficulties, lack of self-confidence, negative thoughts, boredom and mathematics lack of relevance are considered the main reasons for this problem.

Chapter 3

Construction and Analysis of edge designs from skew-symmetric SDSs

This chapter investigates the construction and analysis of edge designs from skew-symmetric SDSs. We explain the details of the methodology, outline the theoretical optimal designs and provide illustrative examples of this new approach. We also show that our proposals are the best realisable designs and possess good D-efficiencies when applied to first-order models.

We begin by introducing the description of the problem in section 3.1. In section 3.2, we delineate the class of edge designs, SDSs and some preliminaries. In section 3.3, we introduce a new class of edge designs constructed from skew-symmetric SDSs, and establish an algorithm for this new approach. Using this methodology, we present examples of new edge designs for 14 and 26 factors. In section 3.4, we examine the D-efficiency of the new edge designs and compare the D-efficiency of designs X_1 , X_2 and X_3 with the OFAT designs for the same number of variables. In section 3.5, we demonstrate the application of the new edge designs by using the analysis edge designs approach. In section 3.6, we show the application the of models for new construction. We conclude the chapter in section 3.7 with classical methodologies of edge designs that differ from those of the edge designs in the literature.

The work presented in this chapter was published as research article in Communications in Statistics - Theory and Methods (Alanazi et al. (2017)).

3.1 The description of the problem

Sometimes, linear models may not be effective for identifying main effects, and non-linear relationships within effects may destroy the identification power of simple regression methods. Therefore, a robust method is required to overcome these problems and screen out the active factors in the first stage of analysis. However, some robust methods are complicated and difficult to apply- a simple method is needed instead, and edge designs aim at to meet this need. They are both easy to apply and robust to the underlying model. Constructing new edge designs is challenging, but this problem is addressed in this thesis.

3.2 Introduction and preliminaries

Screening is usually performed in an initial experimentation stage. The purpose of screening experiments is to identify the dominant variables from a large number of variables that may affect the response y . Generally in screening experiments, the number of trials, N , needed to identify the active factors should be kept as small as possible. Since, in most cases, there is no clue regarding which of the variables under study will be active, a screening design should place its runs in such a way that the region of interest is well explored. More specifically, screening is intended to estimate a multivariate function y such that

$$y_i \simeq y(x_i), \quad x_i = (x_{i,1}, \dots, x_{i,n}) \in [a_1, b_1] \times \dots \times [a_n, b_n], i = 1, \dots, N$$

where y only depends on the p , active variables. Without loss of generality, we may assume that $x_{i,j}$ varies between -1 and 1.

In the following, I_n denotes the identity matrix of order n , $\mathbf{1} = (1, 1, \dots, 1)^T$ is a $n \times 1$ vector with all entries equal to 1, and $J = \mathbf{1}\mathbf{1}^T$. We denote Q the cube

$$\{x \in \mathfrak{R}^n : -1 \leq x_j \leq 1, j = 1, \dots, n\}$$

and by X the $N \times n$ design matrix with $x_{ij} = f_j(x_i)$. Here $x_i \in Q$ is the i th setting of the vector $x_i = (x_{i,1}, \dots, x_{i,n})^T$ and f is given by the assumed linear relation

$$y_i = a^T f(x_i) + \varepsilon_i = \sum_{j=1}^n a_j x_{i,j} + \varepsilon_i, \quad (3.1)$$

where y_i denotes the i th measurement, $a = (a_1, \dots, a_n)^T$ the coefficient column vector of the regression equation and ε_i the measurement error; the ε_i ($i = 1, \dots, N$) are assumed to be normally and identically distributed. Thus,

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \vdots & \vdots \\ x_{N,1} & \cdots & x_{N,n} \end{pmatrix}.$$

3.2.1 Edge designs

Elster and Neumaier (1995) introduced a new class of screening designs that allow a model-independent test for active variables. This is achieved by arranging the measurements into a set E of pairs; within these pairs, the coordinates differ in one component only. Such pairs are called 'edges' since, in the optimal case, they are located at the edges of the cube; designs consisting of a collection of edges are referred to as 'edges designs'. The mirror edge design is achieved by arranging the measurements into a set E of pairs; within these pairs, the coordinates differ in all but one component. Such pairs are called 'mirror edges' since, in the optimal case, they are located at the mirror edges of the cube; designs consisting of a collection of mirror edges are referred to as 'mirror edges designs'. Independent of any particular model, data collected with edge designs may be evaluated using the assumption that only a few, say p , of the n factors are active—that is, contribute to the variability in the observations. This so-called factor sparsity assumption, mentioned by Lenth (1989) and is very natural in screening experiments. It implies that almost all differences

$$z_{i,j} := y_i - y_j, \quad (i, j) \in E,$$

consist of noise only. If we assume that the noise in the data is additive, normally distributed with zero mean and variance σ^2 , the $n - p$ of the $z_{i,j}$ are normally distributed with zero mean and variance $2\sigma^2$. Because of the unknown number of outliers, the variance must be estimated in a robust way. For example, we can use the median estimate

$$\tilde{\sigma} = \frac{\text{median}\{|z_{i,j}| : (i, j) \in E\}}{2^{\frac{1}{2}} \times 0.675}. \quad (3.2)$$

Lenth (1989) show that the estimation given in Equation (3.2) is consistent when there are no active factors ($p = 0$) - thus, it is expected to give reasonable results when effect sparsity assumption holds (the number of active factors is small compared to the number of runs, i.e. $p \ll n$). Outliers then determine active factors. The procedure whereby more than one edge exists for each factor was described in Elster and Neumaier (1995) and is as follows. When each factor varies in the same number of r edges, one can improve on this by guessing the number of active factors p and discarding the rp largest absolute differences $|z_{i,j}|$ before calculating the median. That creates an estimate $\tilde{\sigma}(p)$ for each guess p . The active factors are determined by the outliers as $|z_{i,j}| > k2^{\frac{1}{2}}\tilde{\sigma}(p)$, using say $k = 3$. By sorting $|z_{i,j}|$ it is easy to calculate $\tilde{\sigma}(p)$ and $\omega(p)$ for all p . If q denotes the smallest possible

number with $\omega(q) < q$ then $p = \omega(q - 1)$ estimates the number of active factors and this procedure reveals the active factors themselves.

3.2.2 Conference matrix

A conference matrix is a square matrix C with 0 on the diagonal and ± 1 off the diagonal, such that $C^T C = C C^T$ is a multiple of the identity matrix I . Thus, if the matrix has order n , then $C^T C = (n - 1)I_n$. Symmetric ($C^T = C$) and skew-symmetric ($C^T = -C$) conference matrices are hard to construct, and may not exist in special cases. For example a conference matrix of order 22 does not exist and no skew-symmetric conference matrix of order $n \equiv 2 \pmod{4}$ can exist. Further work on the construction of edge designs of $n \equiv 0 \pmod{4}$ can be found in other papers including Georgiou et al. (2004), Koukouvinos and Stylianou (2006) and Stylianou (2010), and their references.

3.2.3 SDSs

Suppose we have a finite abelian group V and the subsets S_1, S_2, \dots, S_n of V . Write T_i for the totality of all possible difference between the elements of S_i and T for the totality of members of all the T_i . If T contains each non-zero element of V the same number of times, then the subsets S_i s are called SDSs.

In this chapter, we suggest constructing edge designs using SDSs. These edge designs are easy to construct using known SDSs. Note that SDSs with the required properties can exist for n even (including $n \equiv 2 \pmod{4}$) and have high D-efficiency. When the number of factors that need to be examined is even, then these new designs provide an interesting alternative to the existing ones. Using the suggested method, we obtain examples of new edge designs for examining 6, 14, 22, 26, 38, 42, 46, 58, and 62 factors. All the constructed designs are new, and this is the first time edge designs for studying 22 and 58 factors have been presented. This is because edge designs for these parameters cannot be constructed, as the conference matrices of the corresponding order cannot exist.

3.3 Construction of edge and mirror edge designs by skew-symmetric SDS

The notation and preliminaries for SDSs required for our approach are presented at the start of this section. A square matrix is called a circulant matrix if each of its row vectors is rotated one element to the right relative to the preceding row

vector.

If $n \equiv 2 \pmod{4}$ and A, B are $\frac{n}{2} \times \frac{n}{2}$ commuting matrices (that is matrices with elements ± 1), such that

$$A^T A + B^T B = (n - 2)I_{\frac{n}{2}} + 2J_{\frac{n}{2}} \quad (3.3)$$

where $J_{\frac{n}{2}}$ is an $\frac{n}{2} \times \frac{n}{2}$ matrix of 1's, then the $n \times n$ matrix

$$W = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix} \quad (3.4)$$

(known as the two circulant construction) has the maximum determinant (see Ehlich (1964)) among all $n \times n$ matrices with elements ± 1 . Such matrices are called D-optimal designs of order n . A special case of this construction is when A and B are circulant matrices- that is a matrix that every column vector is turned one component to the right in respect to the first line vector. It is straightforward that

$$W^T W = \begin{pmatrix} A^T A + B^T B & 0_{\frac{n}{2} \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times \frac{n}{2}} & A^T A + B^T B \end{pmatrix}. \quad (3.5)$$

If A, B are circulant matrices, then pre- and post-multiplying both sides of Equation (3.3) by 1^T and 1 respectively, we obtain:

$$\left(\frac{n}{2} - 2r\right)^2 + \left(\frac{n}{2} - 2s\right)^2 = 2n - 2 \quad (3.6)$$

where 1 is the $\frac{n}{2} \times 1$ vector of 1's and r, s is the number of -1's in every row of A, B respectively. If A, B satisfy (3.3) so do $(\pm A, \pm B)$, and $(\pm B, \pm A)$, which means we can always take $1 \leq r \leq s \leq (n - 2)/4$. Strictly a pair (A, B) is referred to as SDSs. Now form the two sets

$$P = \{p_1, p_2, \dots, p_r\}, \quad Q = \{q_1, q_2, \dots, q_s\}$$

where p_i, q_j denote the positions of -1's in the first row of A, B respectively. If the congruences

$$p_i - p_j \equiv a \pmod{\left(\frac{n}{2}\right)}, \quad q_i - q_j \equiv a \pmod{\left(\frac{n}{2}\right)}$$

have exactly $\lambda = r + s - (n - 2)/4$ solutions for any $a \not\equiv 0 \pmod{\left(\frac{n}{2}\right)}$, then P, Q called SDSs, denoted by $2 - \left\{\frac{n}{2}; r, s; \lambda\right\}$.

In general, for any 2-SDS, we have the following theorems:

Theorem 1 : [See Chadjipantelis and Kounias (1985)]

(i) If P, Q are SDS $2 - \left\{\frac{n}{2}; r, s; \lambda\right\}$ and A, B the corresponding incidence matrices, then

$$AA^T + BB^T = 4(r + s - \lambda)I_{\frac{n}{2}} + 2\left(\frac{n}{2} - 2(r + s - \lambda)\right)J_{\frac{n}{2}} \quad (3.7)$$

- (ii) Given two $\frac{n}{2} \times \frac{n}{2}$ circulant matrices A, B satisfying Equation (3.7), then the corresponding sets P, Q are SDS $2 - \{\frac{n}{2}; r, s; \lambda\}$, where r, s is the number of -1 's in each row of A, B respectively.

Therefore, the construction of the two circulant matrices A, B satisfying Equation (3.3) is equivalent to constructing the corresponding SDSs. Further, we assert that the SDS (P, Q) is skew-symmetric if, and only if, the matrix $C = W - I$, where W is generated by the two corresponding circulant matrices A and B in equation (3.4), is skew-symmetric.

Theorem 2 : Suppose that the $2 - \{\frac{n}{2}; r, s; \lambda\}$ SDS (P, Q) is skew-symmetric and W is the corresponding skew-symmetric matrix generated from Equation (3.4). Using matrix $C = W - I$, we define:

$$X_1 = \begin{pmatrix} C + I \\ C - I \end{pmatrix}, \quad X_2 = \begin{pmatrix} C + I \\ C - I \\ -C - I \\ -C + I \end{pmatrix}, \quad \text{and} \quad X_3 = \begin{pmatrix} C + I \\ C - I \\ -C - I \end{pmatrix}. \quad (3.8)$$

Then we have

- X_1 is an $2n \times n$ matrix and can be used as an edge design with one edge $(z_{i,j} = z_{i,n+i} = y_i - y_{n+i})$ for each of the n variables. $X_1^T X_1 = 2C^T C + 2I = 2W^T W$.
- X_2 is an $4n \times n$ matrix and can be used as an edge design with two edges $(z_{i,j} = z_{i,n+i} = y_i - y_{n+i}$ and $z_{i,j} = z_{2n+i,3n+i} = y_{2n+i} - y_{3n+i})$ for each of the n variables. Moreover, X_2 can evaluate two mirror edges $(h_{i,j} = h_{n+i,2n+i} = y_{n+i} + y_{2n+i}$ and $h_{i,j} = h_{i,3n+i} = y_i + y_{3n+i})$ for each of the n variables. $X_2^T X_2 = 4C^T C + 4I = 4W^T W$.
- X_3 is an $3n \times n$ matrix and can be used as an edge design with one edge $(z_{i,j} = z_{i,2n+i} = y_i - y_{2n+i})$ and one mirror edge $(h_{i,j} = h_{n+i,2n+i} = y_{n+i} + y_{2n+i})$ for each of the n variables. $X_3^T X_3 = 3C^T C + 3I = 3W^T W$.

Proof. The proof is straightforward by routine calculations. \square

Remark 1 : Design X_1 and X_3 is unbalanced while X_2 is balanced. The design balance has an equivalent number of perceptions for every single conceivable mix of component levels. Proper balance is not needed for the methodology and analysis of edge designs. Using the results of Table 3.1 and Theorem 2, we obtain new edge designs for 6, 14, 22, 26, 38, 42, 46, 58, and 62 factors. Of special interest are the designs for 22 and 58 factors because edge designs for these parameters cannot be constructed from conference matrices.

Remark 2 : Designs X_2 and X_3 provide a practical additional test for the linearity of our true model using $h_{i,j}$. In cases of linear true model of Equation (3.1) both sets of $z_{i,j}$ and $h_{i,j}$ will reveal the same active factors; otherwise, we may conclude that outliers in the set of $z_{i,j}$ s are the active factors and the underlying model is not linear. Examples 4 and 6 present examples and additional properties of the proposed designs.

Stylianou (2010) suggested constructing fold-over screening designs from conference matrices. The designs of that paper only exist when a conference matrix does, and they are unbalanced, able to study up to $n - 1$ variables. The designs have the same D-efficiency as those of X_3 in Theorem 2. The advantages of the designs constructed in Stylianou (2010) is that they are easily constructed but only when a conference matrix of the corresponding order exist. The designs presented here have the advantage to exist in cases where a conference matrix cannot exist. Moreover, the designs X_2 that is presented in Theorem 2 is also balance and thus mean orthogonal.

Proposition 1 : Under the assumption of linear model of Equation (3.1), the contrasts $z_{i,n+i} = y_i - y_{n+i}$, $h_{n+i,2n+i} = y_{n+i} + y_{2n+i}$ and $h_{i,3n+i} = y_i + y_{3n+i}$ will have normal distribution $N(\mu_i, 2\sigma^2)$ where $\mu = 2a_i x_{i,i}$.

Proof. For the linear model of Equation (3.1), we have that $y_i = \sum_{j=1}^n a_j x_{ij} + \varepsilon_i$, $j = 1, 2, \dots$ and $\varepsilon_i \sim N(0, \sigma^2)$. Then

$$\begin{aligned}
z_{i,n+i} &= y_i - y_{n+i} = \sum_{j=1}^{i-1} a_j x_{i,j} + \sum_{j=i+1}^n a_j x_{i,j} - \sum_{j=1}^{i-1} a_j x_{2n+i,j} \\
&\quad - \sum_{j=i+1}^n a_j x_{2n+i,j} + a_i x_{i,i} + a_i x_{2n+i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i} \\
&= 2a_i x_{i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i}. \\
h_{n+i,2n+i} &= y_{n+i} + y_{2n+i} = \sum_{j=1}^{i-1} a_j x_{i,j} + \sum_{j=i+1}^n a_j x_{i,j} + \sum_{j=1}^{i-1} a_j x_{2n+i,j} \\
&\quad + \sum_{j=i+1}^n a_j x_{2n+i,j} + a_i x_{i,i} + a_i x_{2n+i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i} \\
&= \sum_{j=1}^{i-1} a_j x_{i,j} + \sum_{j=i+1}^n a_j x_{i,j} - \sum_{j=1}^{i-1} a_j x_{i,j} - \sum_{j=i+1}^n a_j x_{i,j} + 2a_i x_{i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i}
\end{aligned}$$

$$= 2a_i x_{i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i}.$$

Similar calculations leads to the same result for $h_{i,3n+i}$.

So, $\mu_i = E(2a_i x_{i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i}) = 2a_i x_{i,i}$ and $Var(2a_i x_{i,i} + \varepsilon_{n+i} + \varepsilon_{2n+i}) = 2\sigma^2$.

We outline and illustrate the construction methodology for the suggested designs from skew SDSs using the detailed example explained in the next section.

3.3.1 Algorithm of construction method

We now briefly describe the simple procedures for constructing new edge designs using skew SDS:

Step 1: Find A and B from Skew SDS (A,B) in Table 3.1 and use circulant to complete the matrix.

Step 2: Find I the identity matrix and J regarding to the number of row or column(v).

Step 3: From all sequences satisfied in step 1, 2 and 3 , use these formulates in the above thermos to approve the sequences that are optimal or not by using equations 3.3 and 3.7.

Step 4: From all sequences that satisfy the conditions described in Step 3, find W , C and I that W is from Equation 3.4 , C matrix is the same W but the diagonal is zero , I is identify matrix (all diagonals are 1 and the other entries are zero).

Step 5: Then, find C+I , C-I, -C-I and -C+I and combine lists C+I and lists C-T by saving in new list or structure as defined in Equation 3.8.

We illustrate the construction methodology for the suggested designs from skew symmetric SDSs using following detailed example. In Table 3.1, we present some examples of skew-symmetric SDSs that will be useful for both methods in this chapter. A review on the existence of skew-symmetric SDSs can be found in Araya et al. (2016).

Remark 3 Note that the designs marked with a * in the Table 3.1 generate skew matrices W that satisfy Equation (3.3) and thus are D-optimal designs.

Table 3.1: Examples of skew-symmetric SDSs.

$2-(v; r, s; \lambda)$	P	Q
$(3, 1, 0, 0)^*$	$\{2\}$	\emptyset
$(7, 3, 1, 1)^*$	$\{3, 5, 6\}$	$\{0\}$
$(11, 5, 1, 2)$	$\{2, 6, 7, 8, 10\}$	$\{0\}$
$(13, 6, 3, 3)^*$	$\{4, 7, 8, 10, 11, 12\}$	$\{0, 2, 8\}$
$(19, 9, 1, 4)$	$\{2, 3, 8, 10, 12, 13, 14, 15, 18\}$	$\{0\}$
$(21, 10, 6, 6)^*$	$\{2, 3, 9, 11, 13, 14, 15, 16, 17, 20\}$	$\{0, 1, 7, 9, 12, 17\}$
$(23, 11, 1, 5)$	$\{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$	$\{0\}$
$(29, 14, 7, 8)$	$\{4, 5, 6, 8, 9, 10, 12, 13, 15, 18, 22, 26, 27, 28\}$	$\{0, 1, 11, 13, 15, 18, 21\}$
$(31, 15, 10, 10)^*$	$\{4, 6, 7, 12, 16, 17, 18, 20, 21, 22, 23, 26, 28, 29, 30\}$	$\{0, 1, 4, 5, 8, 11, 16, 18, 20, 29\}$

3.3.2 Constructional example of $n=14$

Example 1 : Let us consider the skew SDS $P = \{3, 5, 6\}$ and $Q = \{0\}$ with parameters $(7; 3, 1; 1)$ and use the following steps in order.

Step 1: To find A and B, we denote -1 if the position is SDS otherwise by 1. Then, we use the circulant matrix to complete all matrix. The result is the following :

$$A = \begin{pmatrix} + & + & + & - & + & - & - \\ - & + & + & + & - & + & - \\ - & - & + & + & + & - & + \\ + & - & - & + & + & + & - \\ - & + & - & - & + & + & + \\ + & - & + & - & - & + & + \\ + & + & - & + & - & - & + \end{pmatrix}$$

and for B, the result is the following :

$$B = \begin{pmatrix} - & + & + & + & + & + & + \\ + & - & + & + & + & + & + \\ + & + & - & + & + & + & + \\ + & + & + & - & + & + & + \\ + & + & + & + & - & + & + \\ + & + & + & + & + & - & + \\ + & + & + & + & + & + & - \end{pmatrix}.$$

Step 2: We find I and J. I is an identity matrix where the diagonal is 1; other entries are 0. In this example, we have $v = 7$, which means the matrix will be 7×7 , and we obtain the following:

$$I = \begin{pmatrix} + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}$$

and J is all entries are equal to 1. In this example, we have $v=7$ so the matrix will be 7×7 and we obtain the following:

$$J = \begin{pmatrix} + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \end{pmatrix}.$$

Step 3: We use Equation 3.3 to check the matrix it is optimal or not. We need to find A^T and B^T and we obtain the following:

$$A^T = \begin{pmatrix} + & - & - & + & - & + & + \\ + & + & - & - & + & - & + \\ + & + & + & - & - & + & - \\ - & + & + & + & - & - & + \\ + & - & + & + & + & - & + \\ - & + & - & + & + & + & - \\ - & - & + & - & + & + & + \end{pmatrix} \text{ and } B^T = \begin{pmatrix} - & + & + & + & + & + & + \\ + & - & + & + & + & + & + \\ + & + & - & + & + & + & + \\ + & + & + & - & + & + & + \\ + & + & + & + & - & + & + \\ + & + & + & + & + & - & + \\ + & + & + & + & + & + & - \end{pmatrix}.$$

Then we have two sides in Equation 3.3, denote that S_1 is the left side and calculate from above; we obtain the following:

$$S_1 = A^T A + B^T B = \begin{pmatrix} 14 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 14 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 14 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 14 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 14 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 14 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 14 \end{pmatrix}.$$

S_2 is the right side, and we calculate from above; we obtain the following:

$$S_2 = (n - 2)I_{\frac{n}{2}} + 2J_{\frac{n}{2}} = \begin{pmatrix} 14 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 14 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 14 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 14 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 14 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 14 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 14 \end{pmatrix}.$$

From S_1 and S_2 , it is obvious Equation 3.3 is satisfies and optimal so we complete the following steps of construction.

Step 4: We find W from Equation 3.4, C and I.

We obtain the following matrix :

$$W = \begin{pmatrix} + & + & + & - & + & - & - & - & - & + & + & + & + & + & + \\ - & + & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & - & + & + & + & - & + & - & - & + & + & + & + & + & + \\ + & - & - & + & + & + & - & - & - & + & + & + & + & + & + \\ - & + & - & - & + & + & + & - & - & + & + & + & + & + & + \\ + & - & + & - & - & + & + & + & - & + & + & + & + & + & - \\ + & + & - & + & - & - & + & + & + & + & + & + & + & + & - \\ + & - & - & - & - & - & - & - & - & + & - & - & + & + & + \\ - & + & - & - & - & - & - & - & - & + & + & - & - & + & + \\ - & - & + & - & - & - & - & - & - & + & + & - & - & + & - \\ - & - & - & + & - & - & - & - & - & - & + & + & - & - & + \\ - & - & - & - & + & - & - & - & - & + & - & + & + & - & - \\ - & - & - & - & - & + & - & - & - & - & + & + & + & - & - \\ - & - & - & - & - & - & + & - & - & - & - & + & + & + & 0 \end{pmatrix}.$$

From W, we change all diagonal to zero and we obtain the following :

$$C = \begin{pmatrix} 0 & + & + & - & + & - & - & - & - & + & + & + & + & + & + \\ - & 0 & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & - & 0 & + & + & - & + & - & - & + & + & - & + & + & + \\ + & - & - & 0 & + & + & - & - & - & + & + & + & - & + & + \\ - & + & - & - & 0 & + & + & - & - & + & + & + & - & + & + \\ + & - & + & - & - & 0 & + & - & - & + & + & + & + & - & + \\ + & + & - & + & - & - & 0 & - & - & + & + & + & + & + & - \\ + & - & - & - & - & - & - & 0 & - & - & + & - & + & + & + \\ - & + & - & - & - & - & - & + & 0 & - & - & + & - & + & + \\ - & - & + & - & - & - & - & + & + & 0 & - & - & + & - & - \\ - & - & - & + & - & - & - & - & + & + & 0 & - & - & + & + \\ - & - & - & - & + & - & - & + & - & + & + & 0 & - & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & 0 & - & - \\ - & - & - & - & - & - & + & - & - & + & - & + & + & 0 & 0 \end{pmatrix}.$$

Then, we find the I that is the identity matrix of order 14 and we obtain the

following:

$$I = \left(\begin{array}{cccccccc|cccccccc} + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 \end{array} \right).$$

Step 5: We use I and C , defined in the step 4, to find the $C+I$, $C-I$, $-C-I$ and $-C+I$ using Equation 3.8, we obtain the following:

$$C + I = \left(\begin{array}{cccccccc|cccccccc} + & + & + & - & + & - & - & - & - & + & + & + & + & + & + & + \\ - & + & + & + & - & + & - & - & + & - & + & + & + & + & + & + \\ - & - & + & + & + & - & + & - & + & + & - & + & + & + & + & + \\ + & - & - & + & + & + & - & - & + & + & + & - & + & + & + & + \\ - & + & - & - & + & + & + & - & + & + & + & + & - & + & + & + \\ + & - & + & - & - & + & + & - & + & + & + & + & + & - & + & + \\ + & + & - & + & - & - & + & - & + & + & + & + & + & + & + & - \\ \hline + & - & - & - & - & - & - & - & + & - & - & + & - & + & + & + \\ - & + & - & - & - & - & - & - & + & + & - & - & + & - & + & + \\ - & - & + & - & - & - & - & - & + & + & + & - & - & + & - & - \\ - & - & - & + & - & - & - & - & - & + & + & + & - & - & + & + \\ - & - & - & - & + & - & - & - & + & - & + & + & + & - & - & - \\ - & - & - & - & - & + & - & - & - & + & - & + & + & + & - & - \\ - & - & - & - & - & - & - & + & - & - & + & - & + & + & + & + \end{array} \right),$$

$$C - I = \left(\begin{array}{cccccc|cccc}
 - & + & + & - & + & - & - & - & + & + & + & + & + \\
 - & - & + & + & - & + & - & - & + & - & + & + & + \\
 - & - & - & + & + & - & + & - & + & + & - & + & + \\
 + & - & - & - & + & + & - & - & + & + & + & - & + \\
 - & + & - & - & - & + & + & - & + & + & + & - & + \\
 + & - & + & - & - & - & + & - & + & + & + & + & - \\
 + & + & - & + & - & - & - & - & + & + & + & + & - \\
 \hline
 + & - & - & - & - & - & - & - & - & - & - & + & + \\
 - & + & - & - & - & - & - & - & + & - & - & - & + \\
 - & - & + & - & - & - & - & - & + & + & - & - & - \\
 - & - & - & + & - & - & - & - & - & + & + & - & - \\
 - & - & - & - & + & - & - & - & + & - & + & - & - \\
 - & - & - & - & - & + & - & - & - & + & - & + & - \\
 - & - & - & - & - & - & - & + & - & - & + & + & - \\
 \hline
 - & - & - & - & - & - & - & + & - & - & + & + & - \\
 \end{array} \right),$$

$$-C - I = \left(\begin{array}{cccccc|cccc}
 - & - & - & + & - & + & + & + & + & - & - & - & - \\
 + & - & - & - & + & - & + & - & - & + & - & - & - \\
 + & + & - & - & - & + & - & - & - & - & + & - & - \\
 - & + & + & - & - & - & + & - & - & - & - & - & - \\
 + & - & + & + & - & - & - & - & - & - & - & + & - \\
 - & + & - & + & + & - & - & - & - & - & - & + & - \\
 - & - & + & - & + & + & - & - & - & - & - & - & + \\
 \hline
 - & + & + & + & + & + & + & + & - & + & + & - & - \\
 + & - & + & + & + & + & + & + & - & - & + & + & - \\
 + & + & - & + & + & + & + & + & - & - & - & + & + \\
 + & + & + & - & + & + & + & + & + & - & - & - & + \\
 + & + & + & + & - & + & + & + & - & + & - & - & + \\
 + & + & + & + & + & - & + & + & + & - & + & - & + \\
 + & + & + & + & + & + & - & + & + & + & - & - & - \\
 \hline
 + & + & + & + & + & + & - & + & + & - & + & - & - \\
 \end{array} \right),$$

$$-C + I = \left(\begin{array}{cccccc|cccccccc} + & - & - & + & - & + & + & + & - & - & - & - & - & - \\ + & + & - & - & + & - & + & - & - & + & - & - & - & - \\ + & + & + & - & - & + & - & - & - & - & + & - & - & - \\ - & + & + & + & - & - & + & - & - & - & - & + & - & - \\ + & - & + & + & + & - & - & - & - & - & - & + & - & - \\ - & + & - & + & + & + & - & - & - & - & - & - & + & - \\ - & - & + & - & + & + & + & - & - & - & - & - & - & + \\ \hline - & + & + & + & + & + & + & + & + & + & - & + & - & - \\ + & - & + & + & + & + & + & - & + & + & + & - & + & - \\ + & + & - & + & + & + & + & - & - & + & + & + & - & + \\ + & + & + & - & + & + & + & + & - & - & + & + & + & - \\ + & + & + & + & - & + & + & - & + & - & - & + & + & + \\ + & + & + & + & + & - & + & + & - & + & - & - & + & + \\ + & + & + & + & + & + & - & + & + & - & + & - & - & + \end{array} \right).$$

From Equation 3.8, we obtain the following matrix :

$$X_1 = \begin{pmatrix} C + I \\ C - I \end{pmatrix} = \left(\begin{array}{cccccccc|cccccccc} + & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & + & + & + & - & + & - & + & - & + & + & + & + & + \\ - & - & + & + & + & - & + & + & + & - & + & + & + & + \\ + & - & - & + & + & + & - & + & + & + & - & + & + & + \\ - & + & - & - & + & + & + & + & + & + & + & - & + & + \\ + & - & + & - & - & + & + & + & + & + & + & + & - & + \\ + & + & - & + & - & - & + & + & + & + & + & + & + & - \\ + & - & - & - & - & - & - & + & - & - & + & - & + & + \\ - & + & - & - & - & - & - & + & + & - & - & + & - & + \\ - & - & + & - & - & - & - & + & + & + & - & - & + & - \\ - & - & - & + & - & - & - & - & + & + & + & - & - & + \\ - & - & - & - & + & - & - & + & - & + & + & - & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & - & - \\ - & - & - & - & - & - & + & - & - & + & - & + & + & - \\ \hline - & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & - & + & + & - & + & - & + & - & + & + & + & + & + \\ - & - & - & + & + & - & + & + & + & - & + & + & + & + \\ + & - & - & - & + & + & - & + & + & + & - & + & + & + \\ - & + & - & - & - & + & + & + & + & + & + & - & + & + \\ + & - & + & - & - & - & + & + & + & + & + & + & - & + \\ + & + & - & + & - & - & - & + & + & + & + & + & + & - \\ + & - & - & - & - & - & - & - & - & - & + & - & + & + \\ - & + & - & - & - & - & - & + & - & - & - & + & - & + \\ - & - & + & - & - & - & - & + & + & - & - & - & + & - \\ - & - & - & + & - & - & - & - & + & + & - & - & - & + \\ - & - & - & - & + & - & - & + & - & + & + & - & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & - & - \\ - & - & - & - & - & - & + & - & - & + & - & + & + & - \end{array} \right).$$

X_1 is an 28×14 matrix and can be used as an edge design with one edge $z_{i,j} = z_{i,14+i} = y_i - y_{14+i}$ for each of the 14 variables. It also satisfies $X_1^T X_1 = 2C^T C + 2I = 2W^T W = \text{diag}(12I_7 + 2J_7, 12I_7 + 2J_7)$ (see Figure(3.1)).

$$X_2 = \begin{pmatrix} C+I \\ C-I \\ -C-I \\ -C+I \end{pmatrix} = \begin{pmatrix} + & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & + & + & + & - & + & - & + & - & + & + & + & + & + \\ - & - & + & + & + & - & + & + & + & - & + & + & + & + \\ + & - & - & + & + & + & - & + & + & + & - & + & + & + \\ + & - & + & - & - & + & + & + & + & + & + & + & - & + \\ + & + & - & + & - & - & + & + & + & + & + & + & + & - \\ + & - & - & - & - & - & - & + & - & - & + & - & + & + \\ - & + & - & - & - & - & - & + & + & - & - & + & - & + \\ - & - & + & - & - & - & - & + & + & + & - & + & - & + \\ - & - & - & + & - & - & - & + & - & + & + & + & - & - \\ - & - & - & - & + & - & - & + & - & + & + & + & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & + & - \\ - & - & - & - & - & + & - & - & + & - & + & + & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & - & - \\ - & - & - & - & - & + & - & - & + & - & + & + & - & - \\ + & - & - & + & - & + & + & + & - & - & - & - & - & - \\ + & + & - & - & - & + & - & - & + & - & - & - & - & - \\ - & + & + & - & - & - & + & - & - & - & + & - & - & - \\ + & - & + & + & - & - & - & - & - & - & - & + & - & - \\ - & - & + & + & + & + & - & - & - & - & - & - & + & + \\ + & - & + & + & + & + & - & + & + & - & + & - & + & - \\ + & + & - & + & + & + & + & - & - & - & + & + & - & + \\ + & + & + & - & + & + & + & + & - & - & + & + & + & + \\ + & + & + & + & + & - & + & + & - & + & - & - & + & + \\ + & + & + & + & + & + & - & + & + & - & + & - & - & + \\ + & - & - & + & - & + & + & + & - & - & - & - & - & - \\ + & + & + & - & + & - & - & - & + & - & - & - & - & - \\ - & + & + & + & - & - & + & - & - & - & + & - & - & - \\ + & - & + & + & + & - & - & - & - & - & - & + & - & - \\ - & + & - & + & + & + & - & - & - & - & - & - & + & - \\ - & - & + & + & + & + & + & + & - & + & - & + & - & - \\ + & - & + & + & + & + & + & + & - & + & + & - & + & - \\ + & + & - & + & + & + & + & - & - & + & + & + & - & + \\ + & + & + & - & + & + & + & + & - & - & + & + & + & - \\ + & + & + & + & + & - & + & + & - & - & + & + & + & + \\ + & + & + & + & + & + & - & + & + & - & + & - & + & + \\ + & + & + & + & + & + & - & + & + & - & + & - & + & + \end{pmatrix}.$$

X_2 is an 56×14 matrix and can be used as an edge design with two edges $z_{i,j} = z_{i,14+i} = y_i - y_{14+i}$ and $z_{i,j} = z_{28+i,42+i} = y_{28+i} - y_{42+i}$ for each of the 14 variables. Also, X_2 can evaluate two mirror edges $h_{i,j} = h_{14+i,28+i} = y_{14+i} + y_{28+i}$

and $h_{i,j} = h_{i,42+i} = y_i + y_{42+i}$) for each of the 14 variables. $X_2^T X_2 = 4C^T C + 4I = 4W^T W$ (see Figure(3.2)).

$$X_3 = \begin{pmatrix} C+I \\ C-I \\ -C-I \end{pmatrix} = \begin{pmatrix} + & + & + & - & + & - & - & - & + & + & + & + & + & + \\ - & + & + & + & - & + & - & + & - & + & + & + & + & + \\ - & - & + & + & + & - & + & + & + & - & + & + & + & + \\ + & - & - & + & + & + & - & + & + & + & - & + & + & + \\ - & + & - & - & + & + & + & + & + & + & + & - & + & + \\ + & + & - & + & - & - & + & + & + & + & + & + & + & - \\ + & - & - & - & - & - & - & + & - & - & + & - & + & + \\ - & + & - & - & - & - & - & + & + & - & - & + & - & + \\ - & - & + & - & - & - & - & + & + & + & - & - & + & - \\ - & - & - & + & - & - & - & + & + & + & - & - & + & - \\ - & - & - & - & + & - & - & + & + & + & - & - & - & + \\ - & - & - & - & - & + & - & - & + & - & + & + & + & - \\ - & - & - & - & - & - & + & - & - & + & - & + & + & + \\ - & - & - & - & - & - & + & - & - & + & - & + & + & - \\ - & - & - & - & - & + & + & + & - & - & - & - & - & - \\ + & - & - & - & + & - & + & - & + & - & - & - & - & - \\ + & + & - & - & - & + & - & - & - & + & - & - & - & - \\ - & + & + & - & - & - & + & - & - & - & + & - & - & - \\ + & - & + & + & - & - & - & - & - & - & - & + & - & - \\ - & + & - & + & + & - & - & - & - & - & - & - & + & - \\ - & - & + & - & + & + & - & - & - & - & - & - & - & + \\ - & + & + & + & + & + & - & - & + & + & - & + & - & - \\ + & - & + & + & + & + & - & - & + & + & - & + & - & - \\ + & + & - & + & + & + & - & - & - & + & + & - & + & - \\ + & + & + & - & + & + & + & + & - & - & - & + & + & - \\ + & + & + & + & - & + & + & - & + & - & - & - & + & + \\ + & + & + & + & + & - & + & + & - & + & - & - & - & + \\ + & + & + & + & + & + & - & + & + & - & + & - & - & + \end{pmatrix}.$$

X_3 is an 42×14 matrix and can be used as an edge design with one edge $z_{i,j} = z_{i,14+i} = y_i - y_{14+i}$ and one mirror edge $h_{i,j} = h_{14+i,28+i} = y_{14+i} + y_{28+i}$) for each of the 14 variables. $X_3^T X_3 = 3C^T C + 3I = 3W^T W$ (see Figure(3.3)).

In Figure (3.1), we observe that the diagonal of this matrix is the same, which is the number of runs (28), and has four parts, revealing that two parts of this matrix are orthogonal and have zero correlation. The other parts have a number around the diagonal (4), but they are not orthogonal and have high correlation. In Figure (3.2), we observe that the diagonal of this matrix is the same, which is the number of runs (56), and it has four parts, showing that two parts of this matrix are orthogonal and have zero correlation. The other parts have a number around the diagonal (8), but they are not orthogonal and have high correlation. In

28	4	4	4	4	4	4	4	0	0	0	0	0	0	0
4	28	4	4	4	4	4	4	0	0	0	0	0	0	0
4	4	28	4	4	4	4	4	0	0	0	0	0	0	0
4	4	4	28	4	4	4	4	0	0	0	0	0	0	0
4	4	4	4	28	4	4	4	0	0	0	0	0	0	0
4	4	4	4	4	28	4	4	0	0	0	0	0	0	0
4	4	4	4	4	4	28	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	28	4	4	4	4	4	4
0	0	0	0	0	0	0	0	4	28	4	4	4	4	4
0	0	0	0	0	0	0	0	4	4	28	4	4	4	4
0	0	0	0	0	0	0	0	4	4	4	28	4	4	4
0	0	0	0	0	0	0	0	4	4	4	4	28	4	4
0	0	0	0	0	0	0	0	4	4	4	4	4	28	4
0	0	0	0	0	0	0	0	4	4	4	4	4	4	28

Figure 3.1: $X_1^T X_1$ in $n=14$ with one edge.

56	8	8	8	8	8	8	8	0	0	0	0	0	0	0
8	56	8	8	8	8	8	8	0	0	0	0	0	0	0
8	8	56	8	8	8	8	8	0	0	0	0	0	0	0
8	8	8	56	8	8	8	8	0	0	0	0	0	0	0
8	8	8	8	56	8	8	8	0	0	0	0	0	0	0
8	8	8	8	8	56	8	8	0	0	0	0	0	0	0
8	8	8	8	8	8	56	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	56	8	8	8	8	8	8
0	0	0	0	0	0	0	0	8	56	8	8	8	8	8
0	0	0	0	0	0	0	0	8	8	56	8	8	8	8
0	0	0	0	0	0	0	0	8	8	8	56	8	8	8
0	0	0	0	0	0	0	0	8	8	8	8	56	8	8
0	0	0	0	0	0	0	0	8	8	8	8	8	56	8
0	0	0	0	0	0	0	0	8	8	8	8	8	8	56

Figure 3.2: $X_2^T X_2$ in $n=14$ with two edge.

Figure (3.3), we observe that the diagonal of this matrix is the same, which is the number of runs (42), and it has four parts, showing that two parts of this matrix are orthogonal and have zero correlation. The other parts have a number around the diagonal (6), but they are not orthogonal and have high correlation. In the above figures, we observe that $X_2^T X_2 = 2X_1^T X_1$ and $X_3^T X_3 = 3(X_1^T X_1/2)$.

42	6	6	6	6	6	6	6	0	0	0	0	0	0
6	42	6	6	6	6	6	6	0	0	0	0	0	0
6	6	42	6	6	6	6	6	0	0	0	0	0	0
6	6	6	42	6	6	6	6	0	0	0	0	0	0
6	6	6	6	42	6	6	6	0	0	0	0	0	0
6	6	6	6	6	42	6	6	0	0	0	0	0	0
6	6	6	6	6	6	42	6	0	0	0	0	0	0
0	0	0	0	0	0	0	0	42	6	6	6	6	6
0	0	0	0	0	0	0	0	6	42	6	6	6	6
0	0	0	0	0	0	0	0	6	6	42	6	6	6
0	0	0	0	0	0	0	0	6	6	6	42	6	6
0	0	0	0	0	0	0	0	6	6	6	6	42	6
0	0	0	0	0	0	0	0	6	6	6	6	6	42

Figure 3.3: $X_3^T X_3$ in $n=14$ with one edge.

3.3.3 Constructional example of $n=26$

Example 2 : Using the skew SDS $P = \{4, 7, 8, 10, 11, 12\}$ and $Q = \{0, 2, 8\}$ with parameters $(13; 6, 3; 3)$ and following these steps in order:

$$A = \begin{pmatrix} + & + & + & + & - & + & + & - & - & + & - & - & - \\ - & + & + & + & + & - & + & + & - & - & + & - & - \\ - & - & + & + & + & + & - & + & + & - & - & + & - \\ + & - & - & - & + & + & + & + & - & + & + & - & - \\ - & + & - & - & - & + & + & + & + & - & + & + & - \\ - & - & + & - & - & - & + & + & + & + & - & + & + \\ + & - & - & + & - & - & - & + & + & + & + & - & + \\ + & + & - & - & + & - & - & - & + & + & + & + & - \\ - & + & + & - & - & + & - & - & - & + & + & + & + \\ + & - & + & + & - & - & + & - & - & - & + & + & + \\ + & + & - & + & + & - & - & + & - & - & - & + & + \\ + & + & + & - & + & + & - & - & + & - & - & - & + \end{pmatrix} \text{ and}$$

From S_1 and S_2 , the condition is satisfied, meaning that it is optimal

$$W = \begin{pmatrix} \begin{array}{l} +++++-++--+--- \\ -++++-++--+--- \\ --++++-++--+- \\ ---++++-++--+- \\ +----++++-++-- \\ -+----++++-++- \\ --+----++++-++ \\ +-+----++++-++ \\ ++-+----++++-+ \\ +++-+----++++- \\ -++-+----+++++ \\ +-++-+----++++ \\ ++-++-+----++++ \\ +++-++-+----++ \\ +---++-+----++ \\ +++-++-+----++ \\ ---++-+----++ \\ ----++-+----++ \\ -----++-+---- \\ +-----++-+---- \\ -+-----++-+---- \\ --+-----++-+---- \\ ---+-----++-+ \\ -----+-----++ \\ +-----+-----++ \\ -+-----+-----++ \\ --+-----+-----++ \\ ---+-----+-----++ \\ -----+-----+-----++ \\ +-----+-----+-----++ \\ -+-----+-----+-----++ \\ --+-----+-----+-----++ \\ ---+-----+-----+-----++ \\ -----+-----+-----+-----++ \end{array} & \begin{array}{l} -+-++++-++++ \\ +-+----++++-++++ \\ ++-+----++++-++ \\ +++-+----++++-+ \\ ++++----++++-+ \\ -++++-+----++++ \\ +-++++-+----++++ \\ ++-++++-+----++ \\ +++-++++-+----++ \\ ++++----++++-+--- \\ +++++-+----++++ \\ -++++-+----++++ \\ +-++++-+----++++ \\ ++-++++-+----++ \\ +++-++++-+----++ \\ ++++----++++-+--- \\ +++++-+----++++ \\ -++++-+----++++ \\ +-++++-+----++++ \\ +---++++-+----++ \\ +---++++-+----++ \\ +++-++++-+----++ \\ -++++-+----++++ \\ +-++++-+----++++ \\ +---++++-+----++ \\ -+---++++-+----++ \\ ---+---++++-+----++ \\ ----+---++++-+----++ \end{array} \end{pmatrix} .$$

$$C + I = \left(\begin{array}{c|c}
++++-+-+---+--- & -+-+++++---++++ \\
-++++-+-+---+--- & +-+-+++++---++++ \\
---++++-+-+---+--- & +++-+-+++++---++++ \\
---++++-+-+---+--- & +++-+-+++++---++++ \\
+---+----+----+---- & +---+----+----+---- \\
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\end{array} \right).$$

By routine calculations you may verify that

- X_1 is an 52×26 matrix and can be used as an edge design with one edge $z_{i,j} = z_{i,26+i} = y_i - y_{26+i}$ for each of the 26 variables. It also satisfies $X_1^T X_1 = 2C^T C + 2I = 2W^T W$ (see Figure(3.4)).
- X_2 is an 104×26 matrix and can be used as an edge design with two edges $z_{i,j} = z_{i,26+i} = y_i - y_{26+i}$ and $z_{i,j} = z_{52+i,78+i} = y_{52+i} - y_{78+i}$ for each of the 26 variables. Also, X_2 can evaluate two mirror edges $h_{i,j} = h_{26+i,52+i} = y_{26+i} + y_{52+i}$ and $h_{i,j} = h_{i,78+i} = y_i + y_{78+i}$ for each of the 26 variables. $X_2^T X_2 = 4C^T C + 4I = 4W^T W$ (see Figure(3.5))
- X_3 is an 78×26 matrix and can be used as an edge design with one edge $z_{i,j} = z_{i,26+i} = y_i - y_{26+i}$ and one mirror edge $h_{i,j} = h_{26+i,52+i} = y_{26+i} + y_{52+i}$ for each of the 26 variables. $X_3^T X_3 = 3C^T C + 3I = 3W^T W$ (see Figure(3.6)).

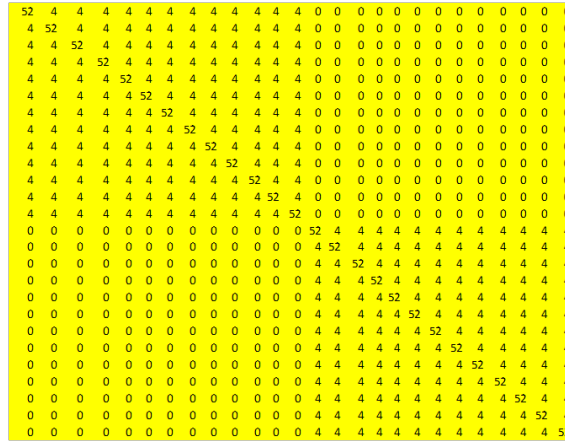


Figure 3.4: $X_1^T X_1$ in $n=26$ with one edge.

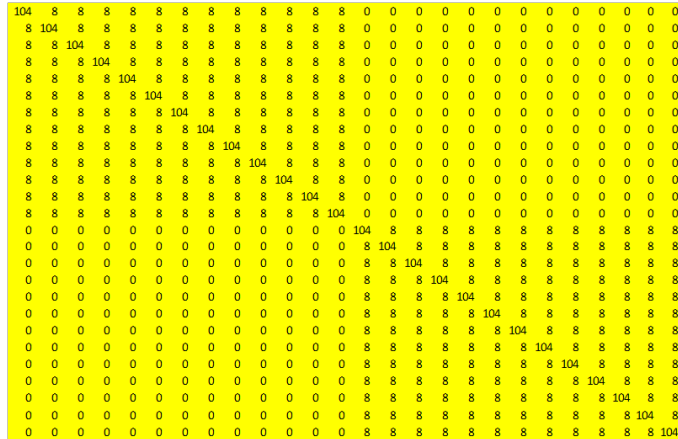


Figure 3.5: $X_2^T X_2$ in $n=26$ with two edge.

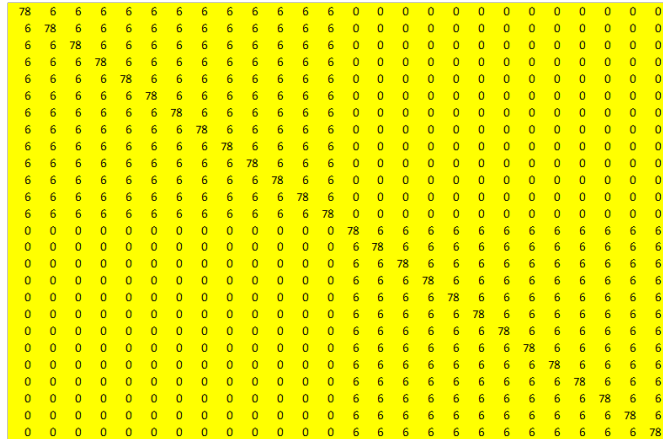


Figure 3.6: $X_3^T X_3$ in $n=26$ with one edge.

In the above figures, we observe that the diagonal of these matrices is the same, which is the number of runs-52, 104 and 78, respectively-and that they each have four parts, showing that two parts of these matrices are orthogonal and have zero correlation. The other parts have a number around the diagonal-4, 8 and 6, receptively-but they are not orthogonal and have high correlation. In addition, $X_2^T X_2 = 2X_1^T X_1$ and $X_3^T X_3 = 3(X_1^T X_1/2)$. In the next section, we study the D-efficiency of the generated designs. Although D-efficiency does not affect the performance of the edge analysis, it is very important when classical methods, such as regression analysis, are used. The constructed designs can be evaluated using both analysis methods: the robust edge analysis method and classical regression analysis.

3.4 D-efficiency of the new edge designs

Here, we present and compare the D-efficiency of new designs X_1 , X_2 and X_3 with the OFAT designs for the same number of variables. OFAT designs are systems for planning analyses directing, including testing factors or causes, particularly one at a time as opposed to testing various elements simultaneously. OFATs efficiency is defined as follows:

$$D_{eff} = 2^{2n/(n+1)}/(n+1) \quad (3.9)$$

However, it is known that the model matrix X of any two level design for the linear model (3.1) satisfies

$$\det X^T X \leq N^{n+1}, \text{ with equality if and only if } X^T X = NI. \quad (3.10)$$

The D_{eff} of a design is defined as follows:

$$D_{eff} = \left(\frac{D}{D^*} \right)^{1/(n+1)}, \quad (3.11)$$

where D is the determinant of $X^T X$, and $D^* = N^{n+1}$ is the upper bound given in Equation (3.10). A design X is called D-optimal if it achieves this bound and thus it has $D_{eff} = 1$ (see (Elster and Neumaier (1995))).

Now, we calculate $n=14$ and $n=26$ using equations (3.9) and (3.11).

For OFAT, the D_{eff} of a 14 is defined as:

$$D_{eff} = 2^{2*14/(14+1)}/(14+1) = 2^{28/15}/15 = 0.243. \text{ The } D_{eff} \text{ of a 26 is defined as:}$$

$$D_{eff} = 2^{2*26/(26+1)}/(26+1) = 2^{52/27}/27 = 0.141.$$

For the new edge designs, we have three previously defined cases, which means we calculate $n = 14$ and $n = 26$ for each case using Equation (3.11). For $n=14$, the D_{eff} of a x_1 is defined as: $D_{eff} = (1.06347/(28^{15}))^{(1/15)} = 0.020861993^{(1/15)} = 0.77260418$. The D_{eff} of a x_2 is defined as: $D_{eff} = (9.06045/(56^{15}))^{(1/15)} = 0.542411825^{(1/15)} = 0.960038412$. The D_{eff} of a x_3 is defined as:

$D_{eff} = (1.07626/(42^{15}))^{(1/15)} = 0.482143845^{(1/15)} = 0.952529515$. These steps can be used to find the D/efficiency of (26), so we generated the table (3.2) to show the values of these designs for different number of variables.

In Table 3.2 and Figure 3.7, we present the efficiencies of the designs generated from the corresponding skew D-optimal designs. Note that for this case, all three designs, X_1 , X_2 , and X_3 , have an increasing efficiency with the factor sizes. The designs presented in Table 3.3 and Figure 3.8 are the those constructed using non D-optimal skew SDSs. Note that these designs have also posses good efficiencies but there is no monotonic increase of efficiency with the factor size. Recall that the D-efficiency of the design matrix will influence a sequential step in the analysis,

Table 3.2: D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.3.)

Design	$n = 6$	$n = 14$	$n = 26$	$n = 42$	$n = 62$
X_1	0.661	0.773	0.846	0.889	0.918
X_2	0.918	0.960	0.978	0.986	0.990
X_3	0.902	0.952	0.973	0.983	0.988
OFAT	0.468	0.243	0.141	0.090	0.062

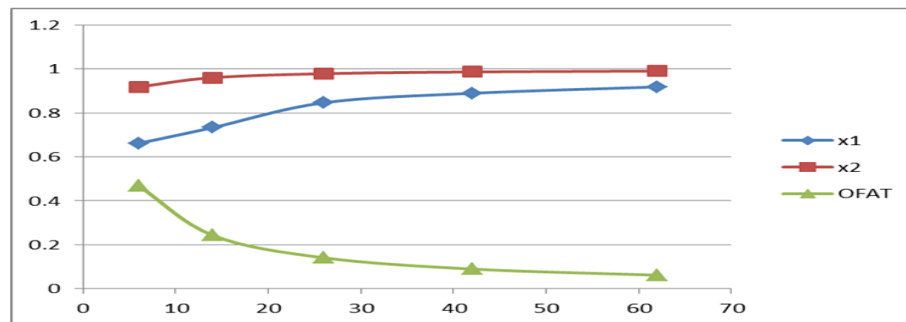


Figure 3.7: D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.3).

where the model matrix will be generated by projecting the full design matrix to all the active factors identified at the screening stage. The D-efficiencies of these designs are presented as indication only; a new calculation is needed when the active variables and the desirable fitted model are chosen in a sequential stage. Comparing the OFAT design with the same number of factors showed that the presented designs have much higher efficiency in all presented cases. It is also easy to see that the efficiency of the constructed edge designs from skew D-optimal designs (skew SDSs) will tend to one as the number of factors tends to infinity, while the corresponding efficiency of the OFAT design will tend to zero.

Table 3.3: D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.7).

Design	$n = 22$	$n = 38$	$n = 46$	$n = 58$
X_1	0.702	0.628	0.608	0.861
X_2	0.850	0.726	0.692	0.944
X_3	0.845	0.723	0.690	0.942
OFAT	0.146	0.099	0.083	0.066

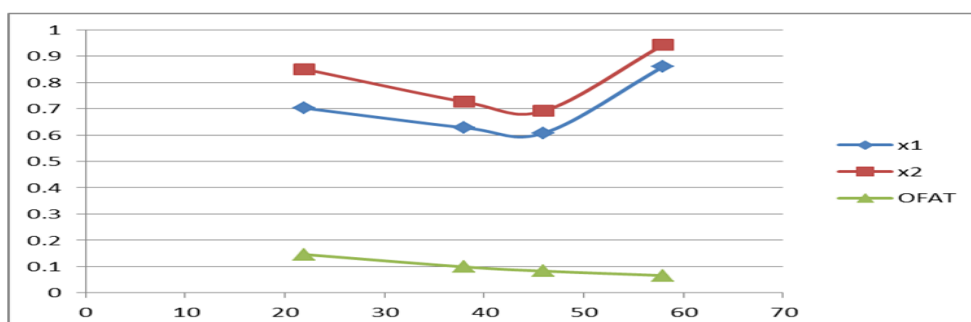


Figure 3.8: D-efficiencies of the Edge designs generated from skew SDSs satisfying (3.7).

3.5 Applying the edge designs in screening experiments

In this section, we illustrate, using several examples, the use and analysis of designs constructed from the skew SDSs. We use matrices X_1 , X_2 and X_3 , as these were generated in examples 1 and 2. The edge design X_1 is more economical and would be preferable for the first stage of the process. Then, using matrix X_2 and the same model, we apply both methods, edge design analysis and regression analysis, for comparison. We also illustrate the additional practical test that can be applied if the selected design matrix is X_3 .

Before introducing the examples, we denote the procedures analysis for edge designs. This procedure was applied by Elster and Neumaier (1995), who also constructed new designs for screening, called edge designs, using the following algorithm:

Step 1: Find the response y from the fitted model and find N and error.

Step 2: Find the difference

$$z_{i,j} := y_i - y_j, \quad (i, j) \in E,$$

and sort the results.

Step 3: Guess p which starts from zero until $w(p) < p$.

Step 4: These numbers multiply with the number of edges that $r = 1$ or $r = 2$ depending on the edge selected; discard the largest rp and then find the median estimate.

Step 5: Find sigma through Equation 3.2 and the best value of p can find by matching rp with the number of omega.

Step 6: Find $k2^{0.5}sigma$ using $k = 1, 2, 3, \dots$

Step 7: Find the number of omega through distinct factors varying in edges (i, j) .

Finally, in our design, we use $r=1$ and $k=1,2,3$ to find the number of active factors when $p=0$ and this number should be kept as small as possible.

3.5.1 Analysis example of $n=14$

In example 1, we first generate data using the design matrix X_2 and the simulated model $y = (x_1 - x_2)^2 + x_3x_4 + 0.5x_2x_5 + \epsilon$, where $\epsilon \sim N(0, 0.1)$. Although this may seem like an extreme case, in real experiments, similar non-linearities may be present. The simulated data we derived with this model produced the response vector $y^T = (-0.467, 4.344, 0.456, 2.536, 5.550, 3.432, -1.620, 5.430, 4.569, -0.503, -0.474, 0.542, 1.545, 1.656, 3.541, 1.411, -1.425, 4.441, 4.692, 3.437, -1.592, 5.500, 4.252, -0.256, -0.386, 0.448, 1.360, 1.465, -0.508, 4.331, 0.411, 2.497, 5.513, 3.424, -1.536, 5.394, 4.418, -0.608, -0.409, 0.440, 1.380, 1.398, 3.438, 1.517, -1.196, 4.398, 4.515, 3.571, -1.720, 5.581, 4.478, -0.420, -0.538, 0.542, 1.445, 1.478)$.

Note that $y^T = (y_1^T, y_2^T, y_3^T, y_4^T)$, where y_1, y_2, y_3, y_4 are $n \times 1$ response vectors that correspond to the runs generated from $C + I, C - I, -C - I, -C + I$ respectively. This means that the response vectors in this simulated experiment for X_1 is $(y_1^T, y_2^T)^T$, for X_2 is y and for X_3 is $(y_1^T, y_2^T, y_3^T)^T$.

Example 3 : We first use matrix X_1 and the corresponding response vector, as described above. An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_4, x_6 , and x_{14} because the p-values of (x_4, x_6 , and x_{14}) are less than 0.05 -they are 0.009, 0.028 and 0.005 respectively- and gives an estimated linear model $f(x) = 1.945 - 1.17x_4 + 0.947x_6 + 1.82x_{14} + \epsilon$, with $R-sg = 82.6\%$, ϵ of mean zero and standard deviation $\sigma = 1.41$. There is a Type II error equal to 80% (4/5) and a Type I error equal to 11.11% (1/9). Note that Type II errors are very important since any truly active variables not identified as active in this stage will not be considered in the follow up experiments for building more complicated models. Type I = the number of non active factors / (number of factors - main active factors) and Type II = the number of active factors / main active factors. In this case the screening process using regression analysis will miss 4 out of the 5 active variables. We now proceed with the edge analysis. First, we calculate all the 14 differences of the responses over the edges and the robust variance of the absolute values of these as defined by equation 3.2.

The analysis of the edges provided in Table 3.4 shows that five (bold values) of the 14 differences of response y along the 14 edges of the design are significant (outliers), with an absolute value of more than four times the robust estimation $\tilde{\sigma} = 0.221$ of σ defined in equation 3.2. This method leads to the identification of all the active factors leading to zero Type I and Type II errors. The edge analysis

Table 3.4: Model-independent checks with the edge design X_1 ; The $z_{i,j}$ values for Example 1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$z_{1,15}$	$z_{2,16}$	$z_{3,17}$	$z_{4,18}$	$z_{5,19}$	$z_{6,20}$	$z_{7,21}$
$y_1 - y_{15}$	$y_2 - y_{16}$	$y_3 - y_{17}$	$y_4 - y_{18}$	$y_5 - y_{19}$	$y_6 - y_{20}$	$y_7 - y_{21}$
-3.956	2.901	2.108	-2.060	0.958	-0.239	-0.0401
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$z_{8,22}$	$z_{9,23}$	$z_{10,24}$	$z_{11,25}$	$z_{12,26}$	$z_{13,27}$	$z_{14,28}$
$y_8 - y_{22}$	$y_9 - y_{23}$	$y_{10} - y_{24}$	$y_{11} - y_{25}$	$y_{12} - y_{26}$	$y_{13} - y_{27}$	$y_{14} - y_{28}$
-0.2283	-0.192	0.170	-0.102	-0.103	0.073	-0.065

with two or more edges is different and is illustrated in Example 4.

Example 4 : To check whether the results improves from changing the design matrix from X_1 to X_2 , we proceed as follows. We use matrix X_2 and the corresponding response vector, as described previously. An analysis of the data (using linear regression) with the software package Minitab for Windows, release 16, revealed no variable as active, and gave an estimated linear model $f(x) = 1.91 + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 2.69$. There is a Type II error equal to 100% and a Type I error equal to 0.00%. Even though we doubled the run by a fold over design, regression analysis becomes even worse, rejecting all variables as inactive. In this case the screening process using regression analysis will miss all five of the five active variables and will again be completely unsuitable for the screening process.

In analysing the edges for these cases, two edges, $z_{i,14+i}$ and $z_{28+i,42+i}$ are given in tables 3.4 and 3.5 for each variable i , so it is much stronger than before and again captures all five active variables.

The analysis with two edges per variable is as follows. By assuming the number p of active variables, we calculate the median of $|z_{i,j}|$ excluding the $2p$ larger values of $|z_{i,j}|$ s. Then we calculate $\tilde{\sigma}(p)$ and $\omega(p)$ as described previously. We do this starting from $p = 0$ and continuing up to $p = 7$ or until $\omega(p) < p$ for some guess p . The results are as follows: we have $\omega(5) = 5$ active factors (x_1, x_2, x_4, x_3, x_5 , i.e., the five distinct factors we excluded from calculating the median on Step 5). Thus, the analysis of the edges with $z_{i,j}$ provided in Tables 3.4 and 3.5, shows that 10 (bold values) of the 28 differences of response y along the 28 edges of the design are significant. These 10 differences correspond to the five distinct factors x_1, x_2, x_3, x_4, x_5 and thus this method, again, leads to the identification of all the active factors leading to zero Type I and Type II errors.

With no additional runs, this design can also be used to perform a linearity test

Table 3.5: Model-independent checks with the edge design $-X_1$; The $z_{i,j}$ values for Example 1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$z_{29,43}$	$z_{30,44}$	$z_{31,45}$	$z_{32,46}$	$z_{33,47}$	$z_{34,48}$	$z_{35,49}$
$y_{29} - y_{43}$	$y_{30} - y_{44}$	$y_{31} - y_{45}$	$y_{32} - y_{46}$	$y_{33} - y_{47}$	$y_{34} - y_{48}$	$y_{35} - y_{49}$
-3.946	2.813	1.607	-1.9008	0.997	-0.146	0.138
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$z_{36,50}$	$z_{37,51}$	$z_{38,52}$	$z_{39,53}$	$z_{40,54}$	$z_{41,55}$	$z_{42,56}$
$y_{36} - y_{50}$	$y_{37} - y_{51}$	$y_{38} - y_{52}$	$y_{39} - y_{53}$	$y_{40} - y_{54}$	$y_{41} - y_{55}$	$y_{42} - y_{56}$
-0.187	-0.059	-0.1875	-0.129	-0.101	-0.065	-0.0808

Table 3.6: Step by step calculations for the analysis with the two edges of the design; The sequential approach for Example 1.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$	Excluded $z_{i,j}$ at each stage p , new pairs (i, j) in this stage:	Total pairs
0	0.210	0.220	0.936	5	No	$\{\emptyset\}$	
1	0.192	0.202	0.857	5	No	(29,43), (1,15)	2
2	0.181	0.190	0.807	5	No	(30,44), (2,16)	4
3	0.170	0.178	0.757	5	No	(31,45), (4,18)	6
4	0.137	0.143	0.609	5	No	(3,17), (32,46)	8
5	0.103	0.108	0.461	5	No	(33,47), (5,19)	10
6	0.103	0.108	0.459	5	Yes	(41,55), (36,50)	12

for the true model. We need to calculate the values for the mirror edges $h_{i,j}$ of the designs. There are two mirror edges for each of the 14 variables of the design; the calculations are provided in tables 3.7 and 3.8.

Table 3.7: Mirror edges check $C - I$; $-C - I$; The $h_{i,j}$ values for Example 1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$h_{15,29}$	$h_{16,30}$	$h_{17,31}$	$h_{18,32}$	$h_{19,33}$	$h_{20,34}$	$h_{21,35}$
$y_{15} + y_{29}$	$y_{16} + y_{30}$	$y_{17} + y_{31}$	$y_{18} + y_{32}$	$y_{19} + y_{33}$	$y_{20} + y_{34}$	$y_{21} + y_{35}$
3.032	5.743	-1.01	6.939	10.20	6.862	-3.128
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$h_{22,36}$	$h_{23,37}$	$h_{24,38}$	$z_{25,39}$	$h_{26,40}$	$h_{27,41}$	$h_{28,42}$
$y_{22} + y_{26}$	$y_{23} + y_{37}$	$y_{24} + y_{38}$	$y_{25} + y_{39}$	$y_{26} + y_{40}$	$y_{27} + y_{41}$	$y_{28} + y_{42}$
10.894	8.671	-0.865	-0.796	0.888	2.741	2.863

Table 3.8: Mirror edges check $C + I$; $-C + I$; The $h_{i,j}$ values for Example 1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$h_{1,43}$	$h_{2,44}$	$h_{3,45}$	$h_{4,46}$	$h_{5,47}$	$h_{6,48}$	$h_{7,49}$
$y_1 + y_{43}$	$y_2 + y_{44}$	$y_3 + y_{45}$	$y_4 + y_{46}$	$y_5 + y_{47}$	$y_6 + y_{48}$	$y_7 + y_{49}$
2.971	5.861	-0.739	6.934	10.06	7.003	-3.340
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$h_{8,50}$	$h_{9,51}$	$h_{10,52}$	$z_{11,53}$	$h_{12,54}$	$h_{13,55}$	$h_{14,56}$
$y_8 + y_{50}$	$y_9 + y_{51}$	$y_{10} + y_{52}$	$y_{11} + y_{53}$	$y_{12} + y_{54}$	$y_{13} + y_{55}$	$y_{14} + y_{56}$
11.012	9.047	-0.924	-1.012	1.084	2.991	3.134

The standard deviation $\tilde{\sigma}$ for the two sets in tables 3.7 and 3.8 are calculated as in Equation 3.2 and are $\tilde{\sigma}_{3,7} = 2.94839216/0.954594155 = 3.088634207$ and $\tilde{\sigma}_{3,8} = 3.063381882/0.954594155 = 3.209093485$ respectively. None of the absolute values of the calculated $h_{i,j}$ s are larger than four times the corresponding standard deviation and thus no active factors (outliers) are detected. The analysis using the edges and the analysis using the mirror edges show a different set of active factors, prompting the conclusion that the true underlying model should be non-linear. We maintain only the active factors found by the edge analysis and, if we wish, we may proceed with a sequential experimentation and follow-up experiments using these factors only to fit a non-linear, more complicated model. The main benefits of this screening process are the robust identification of dominant effects, the conclusion that the underlying model is not linear and the reduced size of the necessary follow-up experimentation.

3.5.2 Analysis example of $n = 26$

In Example 2, we first generate data using the design matrix X_2 and the simulated model $y = (x_1 - x_2)^2 + x_3x_4 + 0.5x_2x_5 + \epsilon$, where $\epsilon \sim N(0, 0.1)$. Although this may seem like an extreme case, in real experiments, similar non-linearities may also be present. The simulated data we derived with this model produced the response vector $y^T = (0.574, 5.765, 0.664, -1.503, 4.402, 4.591, -0.514, 3.479, 1.502, 2.606, 5.507, -0.563, -0.258, 5.523, 4.173, 3.402, 2.461, -1.466, -0.493, 0.552, 1.540, 5.327, 4.418, -0.518, -0.463, 0.461, 4.363, 0.520, -1.500, 0.417, 5.387, 4.885, -0.465, 3.281, 1.536, 2.620, 5.511, -0.632, -0.600, 5.502, 4.411, 3.607, 2.282, -1.639, -0.522, 0.444, 1.442, 5.348, 4.558, -0.456, -0.343, 0.522, 0.403, 5.504, 0.352, -1.428, 4.475, 4.430, -0.545, 3.553, 1.350, 2.333, 5.632, -0.509, -0.464, 5.392, 4.492, 3.558, 2.527, -1.484, -0.582, 0.527, 1.471, 5.520, 4.883, -0.622, -0.493, 0.433, 4.417, 0.630, -1.484, 0.472, 5.484, 4.509, -0.600, 3.537, 1.243, 2.596, 5.423, -0.178, -0.388, 5.580, 4.622, 3.547, 2.490, -1.459, -0.394, 0.559,$

1.451, 5.359, 4.355, -0.467 , -0.453 , 0.467). Note that $y^T = (y_1^T, y_2^T, y_3^T, y_4^T)$, where y_1, y_2, y_3, y_4 are $n \times 1$ response vectors that correspond to the runs generated from $C + I, C - I, -C - I, -C + I$ respectively. This means that the response vectors in this simulated experiment for X_1 is $(y_1^T, y_2^T)^T$, for X_2 is y and for X_3 is $(y_1^T, y_2^T, y_3^T)^T$.

Example 5 : We first use matrix X_1 and the corresponding response vector, as described above. An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_1, x_5, x_7 and x_{23} because the p-values of (x_1, x_5, x_7, x_{23}) are less than 0.05- they are 0.002, 0.000, 0.001 and 0.038 respectively and gives an estimated linear model $f(x) = 2.34 + 0.938x_1 - 1.15x_5 + 1.00x_7 - 0.619x_{23} + \varepsilon$, with $R-sg = 87\%$, ε of mean zero and standard deviation $\sigma = 1.257$. There is a Type II error equal to 60% and a Type I error equal to 9.5%(2/21). Note that Type II errors are very important since any truly active variables not identified as active in this stage will not be considered in the followup experiments for building more complicated models. In this case the screening process using regression analysis will miss 3 out of the 5 active variables. We now proceed with the edge analysis. First, we calculate all the 26 differences of the responses over the edges and the robust variance of the absolute values of these as defined by equation 3.2. The analysis of the edges

Table 3.9: Model-independent checks with the edge design X_1 ; The $z_{i,j}$ values for Example 2.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$z_{1,27}$	$z_{2,28}$	$z_{3,29}$	$z_{4,30}$	$z_{5,31}$	$z_{6,32}$	$z_{7,33}$
$y_1 - y_{27}$	$y_2 - y_{28}$	$y_3 - y_{29}$	$y_4 - y_{30}$	$y_5 - y_{31}$	$y_6 - y_{32}$	$y_7 - y_{33}$
-4.172	4.897	1.978	-2.206	-0.725	-0.0635	-0.141
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$z_{8,34}$	$z_{9,35}$	$z_{10,36}$	$z_{11,37}$	$z_{12,38}$	$z_{13,39}$	$z_{14,40}$
$y_8 - y_{34}$	$y_9 - y_{35}$	$y_{10} - y_{36}$	$y_{11} - y_{37}$	$y_{12} - y_{38}$	$y_{13} - y_{39}$	$y_{14} - y_{40}$
-0.2109	-0.1637	0.1366	-0.017	0.2591	-0.1588	0.007
x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}
$z_{15,41}$	$z_{16,42}$	$z_{17,43}$	$z_{18,44}$	$z_{19,45}$	$z_{20,46}$	$z_{21,47}$
$y_{15} - y_{41}$	$y_{16} - y_{42}$	$y_{17} - y_{43}$	$y_{18} - y_{44}$	$y_{19} - y_{45}$	$y_{20} - y_{46}$	$y_{21} - y_{47}$
0.1301	0.1775	-0.0211	-0.3392	-0.1847	-0.01475	0.0777
x_{22}	x_{23}	x_{24}	x_{25}	x_{26}		
$z_{22,48}$	$z_{23,49}$	$z_{24,50}$	$z_{25,51}$	$z_{26,52}$		
$y_{22} - y_{48}$	$y_{23} - y_{49}$	$y_{24} - y_{50}$	$y_{25} - y_{51}$	$y_{26} - y_{52}$		
0.1485	0.2161	0.14975	0.1535	-0.1798		

provided in Table 3.9, shows that five (bold values) of the 26 differences of response y along the 26 edges of the design are significant (outliers), with an absolute value of more than four times the robust estimation $\tilde{\sigma} = 0.1689$ of σ defined in equation 3.2. This method leads to the identification of all the active factors leading to zero Type I and Type II errors. The edge analysis with two or more edges is different and is illustrated in Example 6.

Example 6 : To check whether the results improves from changing the design matrix from X_1 to X_2 , we proceed as follows. We use matrix X_2 and the corresponding response vector, as described previously. An analysis of the data (using linear regression) with the software package Minitab for Windows, release 17, revealed no variable as active, and gave an estimated linear model $f(x) = 1.96 + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 2.7$. There is a Type II error equal to 100% and a Type I error equal to 0.00%. Even though we doubled the run by a fold over design, regression analysis becomes even worse, rejecting all variables as inactive. In this case, the screening process using regression analysis will miss all five of the five active variables and again will be completely unsuitable for the screening process. In the analysis of the edges for these case, there are two edges $z_{i,26+i}$ and $z_{52+i,78+i}$ are given in tables 3.9 and 3.10 for each variable i , so it is much stronger than before and again captures all five active variables. The analysis with

Table 3.10: Model-independent checks with the edge design $-X_1$; The $z_{i,j}$ values for Example 2.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$z_{53,79}$	$z_{54,80}$	$z_{55,81}$	$z_{56,82}$	$z_{57,83}$	$z_{58,84}$	$z_{59,85}$
$y_{53} - y_{79}$	$y_{54} - y_{80}$	$y_{55} - y_{81}$	$y_{56} - y_{82}$	$y_{57} - y_{83}$	$y_{58} - y_{84}$	$y_{59} - y_{85}$
-4.014	4.873	1.836	-1.901	-1.009	-0.0786	0.0544
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$z_{60,86}$	$z_{61,87}$	$z_{62,88}$	$z_{63,89}$	$z_{64,90}$	$z_{65,91}$	$z_{66,92}$
$y_{60} - y_{86}$	$y_{61} - y_{87}$	$y_{62} - y_{88}$	$y_{63} - y_{89}$	$y_{64} - y_{90}$	$y_{65} - y_{91}$	$y_{66} - y_{92}$
0.0153	0.1067	-0.2633	0.2092	-0.3305	-0.07608	-0.1872
x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	
$z_{67,93}$	$z_{68,94}$	$z_{69,95}$	$z_{70,96}$	$z_{71,97}$	$z_{72,98}$	
$y_{67} - y_{93}$	$y_{68} - y_{94}$	$y_{69} - y_{95}$	$y_{70} - y_{96}$	$y_{71} - y_{97}$	$y_{72} - y_{98}$	
-0.1299	0.0113	0.0367	-0.0249	-0.1881	-0.0327	
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	
$z_{73,99}$	$z_{74,100}$	$z_{75,101}$	$z_{76,102}$	$z_{77,103}$	$z_{78,104}$	
$y_{73} - y_{99}$	$y_{74} - y_{100}$	$y_{75} - y_{101}$	$y_{76} - y_{102}$	$y_{77} - y_{103}$	$y_{78} - y_{104}$	
0.0196	0.1617	0.5275	-0.1543	-0.0404	-0.0338	

two edges per variable is as follows. By assuming the number p of active variables, we calculate the median of $|z_{i,j}|$ excluding the $2p$ larger values of $|z_{i,j}|$ s. Then we calculate $\tilde{\sigma}(p)$ and $\omega(p)$ as described previously. We do this starting from $p = 0$ and continuing up to $p = 7$ or until $\omega(p) < p$ for some guess p . The results are as follows: we have $\omega(5) = 5$ active factors $(x_1, x_2, x_4, x_3, x_5)$, i.e. the five distinct

Table 3.11: Step by step calculations for the analysis with the two edges of the design; The sequential approach for Example 2.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$	Excluded $z_{i,j}$ at each stage p , new pairs (i, j) in this stage:	
						$\{\emptyset\}$	Total pairs
0	0.161	0.168	0.716	5	No	$\{\emptyset\}$	2
1	0.158	0.166	0.706	5	No	(29,43), (1,27)	4
2	0.156	0.163	0.694	5	No	(30,44), (2,28)	6
3	0.153	0.160	0.682	5	No	(31,45), (4,18)	8
4	0.151	0.158	0.673	5	No	(3,17), (32,46)	10
5	0.149	0.156	0.665	5	No	(33,47), (5,19)	12
6	0.148	0.155	0.660	5	Yes	(41,55), (36,50)	

factors we excluded from calculating the median on step 5). Thus, the analysis of the edges with $z_{i,j}$ provided in Tables 3.9 and 3.10, shows that ten (bold values) of the 52 differences of response y along the 52 edges of the design are significant. These 10 differences correspond to the five distinct factors x_1, x_2, x_3, x_4, x_5 and thus this method, again, leads to the identification of all the active factors leading to zero Type I and Type II errors. With no additional runs, this design can also be used to perform a linearity test for the true model. We need to calculate the values for the mirror edges $h_{i,j}$ of the designs. There are two mirror edges for each of the 26 variables of the design; the calculations provided in tables 3.12 and 3.13.

The standard deviation $\tilde{\sigma}$ for the two sets in tables 3.12 and 3.13 are calculated as in equation 3.2 and they are $\tilde{\sigma}_{3,12} = 3.840676609/0.954594155 = 4.023360703$ and $\tilde{\sigma}_{3,13} = 3.972009167/0.954594155 = 4.160940173$ respectively. None of the absolute values of the calculated $h_{i,j}$ s are larger than four times the corresponding standard deviation and thus no active factors (outliers) are detected.

Remark 4 : We may also enjoy the benefits of X_2 using a more economical design such as X_3 . Note that this design has less size, even for the first stage of experimentation, during the screening process. This design maintains the robust identification of the dominant factors (screening process) using one edge for each factor, and also provides one mirror edge for each factor to perform the proposed additional test of linearity.

Table 3.12: Mirror edges check $C - I$; $-C - I$; The $h_{i,j}$ values for Example 2.

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$h_{27,53}$	$h_{28,54}$	$h_{29,55}$	$h_{30,56}$	$h_{31,57}$	$h_{32,58}$	$h_{33,59}$
$y_{27} + y_{53}y_{28} + y_{54}y_{29} + y_{55}y_{30} + y_{56}y_{31} + y_{57}y_{32} + y_{58}y_{33} + y_{59}$						
4.767	6.0242	-1.148	-1.011	9.862	9.315	-1.011
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$h_{34,60}$	$h_{35,61}$	$h_{36,62}$	$h_{37,63}$	$h_{38,64}$	$h_{39,65}$	$h_{40,66}$
$y_{34} + y_{60}y_{35} + y_{61}y_{36} + y_{62}y_{37} + y_{63}y_{38} + y_{64}y_{39} + y_{65}y_{40} + y_{66}$						
6.834	2.886	4.954	11.142	-1.142	-1.0655	10.895
x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}
$h_{41,67}$	$h_{42,68}$	$z_{43,69}$	$h_{44,70}$	$h_{45,71}$	$h_{46,72}$	$h_{47,73}$
$y_{41} + y_{67}y_{42} + y_{68}y_{43} + y_{69}y_{44} + y_{70}y_{45} + y_{71}y_{46} + y_{72}y_{47} + y_{73}$						
8.904	7.165	4.809	-3.124	-1.104	0.9718	2.9140
x_{22}	x_{23}	x_{24}	x_{25}	x_{26}		
$h_{48,74}$	$h_{49,75}$	$h_{50,76}$	$h_{51,77}$	$h_{52,78}$		
$y_{48} + y_{74}y_{49} + y_{75}y_{50} + y_{76}y_{51} + y_{77}y_{52} + y_{78}$						
10.866	9.441	-1.078	-0.8370	0.9558		

Table 3.13: Mirror edges check $C + I$; $-C + I$; The $h_{i,j}$ values for Example 2 .

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$h_{1,79}$	$h_{2,80}$	$h_{3,81}$	$h_{4,82}$	$h_{5,83}$	$h_{6,84}$	$h_{7,85}$
$y_1 + y_{79}$						
4.9920	6.396	-0.820	-1.031	9.887	9.1002	-1.114
x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
$h_{8,86}$	$h_{9,87}$	$h_{10,88}$	$h_{11,89}$	$h_{12,90}$	$h_{13,91}$	$h_{14,92}$
$y_8 + y_{86}$						
7.0176	2.7457	5.2037	10.9306	-0.741	-0.647	11.103
x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}
$h_{15,93}$	$h_{16,94}$	$z_{17,95}$	$h_{18,96}$	$h_{19,97}$	$h_{20,98}$	$h_{21,99}$
$y_{15} + y_{93}$						
8.795	6.9500	4.9519	-2.925	-0.887	1.1123	2.9920
x_{22}	x_{23}	x_{24}	x_{25}	x_{26}		
$h_{22,100}$	$h_{23,101}$	$h_{24,102}$	$h_{25,103}$	$h_{26,104}$		
$y_{22} + y_{100}y_{23} + y_{101}y_{24} + y_{102}y_{25} + y_{103}y_{26} + y_{104}$						
10.686	8.7744	-0.985	-0.916	0.928		

3.5.3 Analysis for significant interaction effects with n=14

Analysis design matrix X_1

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_1x_2 , x_2x_3 , x_2x_5 and x_3x_4 because the p-values of $(x_1x_2, x_2x_3, x_2x_5, x_3x_4)$ are less than 0.05 - they are 0.000, 0.037, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 2.004 - 2.02x_1x_2 - 0.0643x_2x_3 + 0.524x_2x_5 + 0.991x_3x_4 + \varepsilon$, with $R - sg = 99.96\%$, ε of mean zero and standard deviation $\sigma = 0.0670$. There is a Type II error equal to 0% and a Type I error equal to 0.083%.

Analysis design matrix X_2

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables $x_2, x_2 * x_5$ and $x_3 * x_4$ because the p-values of (x_1, x_2x_5, x_3x_4) are less than 0.05-they are 0.000, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 2 - 2x_2x_2 - 0.519x_2x_5 + 0.975x_3x_4 + \varepsilon$, with $R - sg = 99.83\%$, ε of mean zero and standard deviation $\sigma = 0.104$. There is a Type II error equal to 0.3333% and a Type I error equal to 0.083%.

Analysis design matrix X_3

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_1x_2, x_2x_5 and x_3x_4 because the p-values of (x_1x_2, x_2x_5, x_3x_4) are less than 0.05 -they are 0.000, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 2.028 - 2.02x_1x_2 - 0.500x_2x_5 + 0.979x_3x_4 + \varepsilon$, with $R - sg = 99.92\%$, ε of mean zero and standard deviation $\sigma = 0.0845$. There is a Type II error equal to 0% and a Type I error equal to 0%.

3.5.4 Analysis for significant interaction effects with n=26

Analysis design matrix X_1

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_1x_2 , x_2x_5 and x_3x_4 because the p-values of (x_2x_3, x_2x_5, x_3x_4) are less than 0.05-they are 0.000, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 1.968 - 2.005x_1x_2 + 0.486x_2x_5 + x_3x_4 + \varepsilon$, with $R - sg = 99.83\%$, ε of mean zero and standard deviation $\sigma = 0.120$. There is a Type II error equal to 0% and a Type I error equal to 0%.

Analysis design matrix X_2

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_1x_2 , x_2x_5 and x_3x_4 because the p-values of (x_1x_2, x_2x_5, x_3x_4) are less than 0.05 -they are 0.000, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 1.98 - 2.01x_1x_2 + 0.490x_2x_5 + 0.990x_3x_4 + \varepsilon$, with $R - sg = 99.87\%$, ε of mean zero and standard deviation $\sigma = 0.0960$. There is a Type II error equal to 0% and a Type I error equal to 0%.

Analysis design matrix X_3

An analysis of these data (using linear regression) with the software package Minitab for Windows, release 16, reveals as active variables x_3 , x_1x_2 , x_2x_5 and x_3x_4 because the p-values of $(x_3, x_1x_2, x_2x_5, x_3x_4)$ are less than 0.05 -they are 0.030, 0.000, 0.000 and 0.000 respectively and gives an estimated linear model $f(x) = 1.99 - 0.024x_3 - 1.99x_1x_2 + 0.500x_2x_5 + 0.989x_3x_4 + \varepsilon$, with $R - sg = 99.87\%$, ε of mean zero and standard deviation $\sigma = 0.0959$. There is a Type II error equal to 0% and a Type I error equal to 0%.

3.6 Applying models for constructing Edge designs from skew-symmetric SDSs

3.6.1 Model 1: linear model

Under the assumption of the linear model from Equation 3.1, it is easy to see that in two cases:

Case 1

$$X = \begin{pmatrix} C + I \\ C - I \end{pmatrix}.$$

$$X = \begin{pmatrix} 1 & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ c_{2,1} & 1 & c_{2,3} & \cdots & c_{2,n} \\ \vdots & & \vdots & & \vdots \\ c_{n,1} & & \cdots & & 1 \\ -1 & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ c_{2,1} & -1 & c_{2,3} & \cdots & c_{2,n} \\ \vdots & & \vdots & & \vdots \\ c_{n,1} & & \cdots & & -1 \end{pmatrix}.$$

We observe that everything is the same in this case, but the diagonals are different, so the linear model for this case is the following .

Let $\epsilon_i \sim N(0, \sigma^2)$

$$y_i = a_1 x_{i,1} + a_2 x_{i,2} + \dots + a_n x_{i,n} + \epsilon_i$$

$i = 1, 2, \dots, 2n$ where a_1, a_2, \dots, a_n are the unknown coefficient, x_i is variable, y_i is the i th measurement and ϵ_i is the error.

In general : $Z_i = y_i - y_{i+n}$, $i = 1, 2, \dots, n$.

$$Z_i = a_1 x_{i,1} + a_2 x_{i,2} + \dots + a_n x_{i,n} + \epsilon_i - [a_1 x_{i+n,1} + a_2 x_{i+n,2} + \dots + a_n x_{i+n,n} + \epsilon_{i+n}]$$

. For Example:

$$\begin{aligned} Z_1 &= a_1 x_{1,1} + a_2 x_{1,2} + \dots + a_n x_{1,n} + \epsilon_1 - [a_1 x_{1+n,1} + a_2 x_{1+n,2} + \dots + a_n x_{1+n,n} + \epsilon_{1+n}] \\ &= a_1 + a_2 c_{1,2} + \dots + a_n c_{1,n} + \epsilon_1 - [-a_1 + a_2 c_{1,2} + \dots + a_n c_{1,n} + \epsilon_{1+n}] \\ &= 2a_1 + \epsilon_1 - \epsilon_{1+n}. \end{aligned}$$

Similarity

$$Z_2 = 2a_2 + \epsilon_2 - \epsilon_{2+n}$$

.....

$$Z_i = 2a_i + \epsilon_i - \epsilon_{i+n} \text{ in general .}$$

Now the expected value (mean) in the following.

$$\begin{aligned} E(Z_i) &= E(2a_i + \epsilon_i - \epsilon_{i+n}), \text{ For } N(0, \sigma^2) \\ &= E(2a_i) + E(\epsilon_i) - E(\epsilon_{i+n}) \\ &= 2a_i + 0 + 0 \\ &= 2a_i. \end{aligned}$$

Now the variance in the following.

$$\begin{aligned} V(z_i) &= V(2a_i + \epsilon_i - \epsilon_{i+n}) \\ &= \text{Var}((1)\epsilon_i + (-1)\epsilon_{i+n}), \text{ that is from } \text{Var}(x+a) = \text{Var}(x) \\ &= 1^2 \text{var} \epsilon_i + (-1)^2 \text{var} \epsilon_{i+n} - 2(1)(-1) \text{cov}(\epsilon_i, \epsilon_{i+n}) \end{aligned}$$

3.6. APPLYING MODELS FOR CONSTRUCTING EDGE DESIGNS FROM SKEW-SYMMETRIC S

,that is from $var(ax - by) = a^2var(x) + b^2var(y) - 2abcov(x, y)$

$$= 1\sigma^2 + a\sigma^2 + 2(0)$$

$$= 2\sigma^2.$$

Case 2

$$X = \begin{pmatrix} C + I \\ -C + I \end{pmatrix}.$$

$$X = \begin{pmatrix} 1 & c_{1,2} & c_{1,3} \cdots & c_{1,n} \\ c_{2,1} & 1 & c_{2,3} \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \cdots & \cdots & 1 \\ 1 & -c_{1,2} & -c_{1,3} \cdots & -c_{1,n} \\ -c_{2,1} & 1 & -c_{2,3} \cdots & -c_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ -c_{n,1} & \cdots & \cdots & 1 \end{pmatrix}.$$

We observe that everything is different in this case, but the diagonals are the same, so the linear model is in the following.

Let $\epsilon_i \sim N(0, \sigma^2)$

$$y_i = a_1x_{i,1} + a_2x_{i,2} + \dots + a_nx_{i,n} + \epsilon_i, \quad i = 1, 2, \dots, n + 1, \dots, 2n$$

$$h_i = y_i - y + i + n, \quad i = 1, 2, \dots, n.$$

In general, $h_i = a_1x_{i,1} + a_2x_{i,2} + \dots + a_nx_{i,n} + \epsilon_i + [a_1x_{i+n,1} + a_2x_{i+n,2} + \dots + a_nx_{i+n,n} + \epsilon_{i+n}]$.

For Example:

$$h_1 = a_1x_{1,1} + a_2x_{1,2} + \dots + a_nx_{1,n} + \epsilon_1 + [a_1x_{1+n,1} + a_2x_{1+n,2} + \dots + a_nx_{1+n,n} + \epsilon_{1+n}]$$

$$= a_1 + a_2c_{1,2} + \dots + a_nc_{1,n} + \epsilon_1 + [a_1 + a_2(-c_{1,2}) + \dots + a_n(-c_{1,n}) + \epsilon_{1+n}]$$

$$= 2a_1 + \epsilon_1 + \epsilon_{1+n}.$$

Similarity

$$h_2 = 2a_2 + \epsilon_2 + \epsilon_{2+n}$$

.....

$$h_i = 2a_i + \epsilon_i + \epsilon_{i+n} \text{ in general .}$$

Now the expected value (mean) in the following.

$$\begin{aligned} E(h_i) &= E(2a_i + \epsilon_i + \epsilon_{i+n}), \text{ For } N(0, \sigma^2) \\ &= E(2a_i) + E(\epsilon_i) + E(\epsilon_{i+n}) \\ &= 2a_i + 0 + 0 \\ &= 2a_i. \end{aligned}$$

Now the variance in the following

$$\begin{aligned} V(h_i) &= V(2a_i + \epsilon_i + \epsilon_{i+n}) \\ &= \text{Var}((1)\epsilon_i + (1)\epsilon_{i+n}), \text{ that is from } \text{Var}(x+a)=\text{Var}(x) \\ &= 1^2 \text{var}\epsilon_i + (1^2) \text{var}\epsilon_{i+n} + 2(1)(1) \text{cov}(\epsilon_i, \epsilon_{i+n}) \end{aligned}$$

,that is from $\text{var}(ax + by) = a^2 \text{var}(x) + b^2 \text{var}(y) + 2ab * \text{cov}(x, y)$

$$\begin{aligned} &= 1\sigma^2 + a\sigma^2 + 2(0) \\ &= 2\sigma^2. \end{aligned}$$

3.6.2 Model 2: Full second order

$$\mathbf{y}_K = \sum_{i=1}^n a_i x_{k,i} + \sum_{i=1}^n \sum_{j>1}^n a_{i,j} x_{k,i} x_{k,j} + \sum_{i=1}^n a_{i,i} x_{k,i}^2.$$

We calculate each part that are $A = \sum_{i=1}^n a_i x_{k,i}$, $B = \sum_{i=1}^n \sum_{j>1}^n a_{i,j} x_{k,i} x_{k,j}$ and $C = \sum_{i=1}^n a_{i,i} x_{k,i}^2$.

A as defined in section 3.6.1.

Now , we calculate part B for two cases as follows.

Case 1 for B

$$\begin{aligned} Z_i &= y_i - y_{i+n} \\ &= a_{ij} x_i x_j + \dots + a_{in} x_i x_n + \dots + a_{n-i,n} x_{n-i} x_n + \epsilon_i - [a_{i+n,j} x_{n+i} x_j + \dots + \\ & a_{i+n,n} x_{i+n} x_n + \dots + a_{n+j-i,n} x_{n+j-i} x_n + \epsilon_{n+i}. \end{aligned}$$

For example:

$$i = 1$$

$$\begin{aligned} z_1 &= y_1 - y_n + 1 = a_{12} x_1 x_2 + a_{13} x_1 x_3 + \dots + a_{1n} x_1 x_n + \dots + a_{n-1,n} x_{n-1} x_n + \epsilon_1 - \\ & [a_{n+1,2} x_{n+1} x_2 + a_{n+1,3} x_{n+1} x_3 + \dots + a_{n+1,n} x_{n+1} x_n + \dots + a_{n+2-1,n} x_{n+2-1} x_n + \\ & \epsilon_{n+1}]. \end{aligned}$$

Then,

$$\begin{aligned} Z_1 &= a_{12} c_{12} + a_{13} c_{13} + \dots + a_{1n} c_{1n} + \dots + a_{2n} c_{12} c_{1n} + \dots + a_{n-1,n} c_{1,n-1} c_{1n} + \epsilon_1 - \\ & [(-1) a_{12} c_{12} + (-1) a_{13} c_{13} + \dots + (-1) a_{1n} c_{1n} + \dots + (-1) a_{n-1,n} c_{1,n-1} c_{1n} + \epsilon_{n+1}] \end{aligned}$$

3.6. APPLYING MODELS FOR CONSTRUCTING EDGE DESIGNS FROM SKEW-SYMMETRIC S

$$\begin{aligned}
 &= 2a_{12}c_{12} + 2a_{13}c_{13} + 2a_{1n}c_{1n} + \dots + \epsilon_1 - \epsilon_{n+1} \\
 &= 2(a_{12}c_{12} + a_{13}c_{13} + a_{1n}c_{1n} + \dots) + \epsilon_1 - \epsilon_{n+1}.
 \end{aligned}$$

Case 2 for B

$$\begin{aligned}
 h_i &= y_i + y_{i+n} \\
 &= a_{ij}x_i x_j + \dots + a_{in}x_i x_n + \dots + a_{n-i,n}x_{n-i}x_n + \epsilon_i + [a_{i+n,j}x_{i+n}x_j + \dots + a_{i+n,n}x_{i+n}x_n + \dots + a_{n+j-i,n}x_{n+j-i}x_n + \epsilon_{n+i}].
 \end{aligned}$$

For example:

$$i = 1$$

$$\begin{aligned}
 h_1 &= y_1 - y_n + 1 = a_{12}x_1x_2 + a_{13}x_1x_3 + \dots + a_{1n}x_1x_n + \dots + a_{n-1,n}x_{n-1}x_n + \epsilon_1 + \\
 &[a_{n+1,2}x_{n+1}x_2 + a_{n+1,3}x_{n+1}x_3 + \dots + a_{n+1,n}x_{n+1}x_n + \dots + a_{n+2-1,n}x_{n+2-1}x_n + \epsilon_{n+1}].
 \end{aligned}$$

Then,

$$\begin{aligned}
 h_1 &= a_{12}c_{12} + a_{13}c_{13} + \dots + a_{1n}c_{1n} + \dots + a_{n-1,n}c_{1,n-1}c_{1n} + \epsilon_1 + [(-1)a_{12}c_{12} + \\
 &(-1)a_{13}c_{13} + \dots + (-1)a_{1n}c_{1n} + \dots + (-1)(-1)a_{2n}c_{12}c_{1n} + \dots + (-1)a_{n-1,n}c_{1,n-1}c_{1n} + \epsilon_{n+1}]
 \end{aligned}$$

$$= 2a_{23}c_{12}c_{13} + 2a_{24}c_{12}c_{14} + \dots + 2a_{2n}c_{12}c_{1n} + \dots + \epsilon_1 + \epsilon_{n+1}.$$

$$= 2(a_{23}c_{12}c_{13} + a_{24}c_{12}c_{14} + \dots + a_{2n}c_{12}c_{1n} + \dots) + \epsilon_1 + \epsilon_{n+1}.$$

Now , we calculate part c for two cases in the following.

Case 1 for C

$$\begin{aligned}
 Z_i &= y_i - y_{n+i} \\
 &= a_{ii}X_i^2 + \dots + a_{nn}x_n^2 + \epsilon_i - [a_{ii}x_{i+n}^2 + \dots + a_{nn}x_{n+i}^2 + \epsilon_{n+i}].
 \end{aligned}$$

For example :

$$i = 1$$

$$z_1 = a_{11}X_1^2 + \dots + a_{nn}x_n^2 + \epsilon_1 - [a_{11}x_{1+n}^2 + \dots + a_{nn}x_{n+1}^2 + \epsilon_{n+1}].$$

Then,

$$= a_{11}1^2 + a_{22}c_{12}^2 + \dots + a_{nn}c_{1n}^2 + \epsilon_1 - [a_{11}1^2 + a_{22}c_{12}^2 + \dots + a_{nn}c_{1n}^2 + \epsilon_{n+1}].$$

$$= \epsilon_1 - \epsilon_{n+1}.$$

In general:

$$z_i \sim N(0, 2\sigma^2)$$

$$z_i = \epsilon_i - \epsilon_{n+i}.$$

Case 2 for C

$$\begin{aligned}
 h_i &= y_i + y_{n+i} \\
 &= a_{ii}X_i^2 + \dots + a_{nn}x_n^2 + \epsilon_i + [a_{ii}x_{i+n}^2 + \dots + a_{nn}x_{n+i}^2 + \epsilon_{n+i}].
 \end{aligned}$$

For example :

$$i = 1$$

$$h_1 = a_{11}X_1^2 + \dots + a_{nn}x_1^2n + \epsilon_i + [a_{11}x_{1+n}^2 + \dots + a_{nn}x_{n+1}^2 + \epsilon_{n+1}].$$

Then

$$= a_{11}1^2 + a_{22}c_{12}^2 + \dots + a_{nn}c_{1n}^2 + \epsilon_1 + [a_{11}(-1)^2 + a_{22}(-c)_{12}^2 + \dots + a_{nn}(-c)_{1n}^2 + \epsilon_{n+1}].$$

$$= 2a_{11}1^2 + 2a_{22}c_{12}^2 + \dots + 2a_{nn}c_{1n}^2 + \epsilon_1 + \epsilon_{n+1}$$

$$= 2(a_{11}1^2 + a_{22}c_{12}^2 + \dots + a_{nn}c_{1n}^2) + \epsilon_1 + \epsilon_{n+1}.$$

In general:

$$h_i \sim N(0, 2\sigma^2)$$

$$2(a_{11}1^2 + a_{ii}c_{12}^2 + \dots + a_{nn}c_{nn}^2) + \epsilon_1 + \epsilon_{n+1}.$$

3.6.3 Model 3:Odd Polynomial

$$\mathbf{y}_k = \sum_{i=1}^n a_i x_{ki} + \sum_{i=1}^n b_i x_{ki}^3 + \epsilon_k.$$

We calculate each part: $A = \sum_{i=1}^n a_i x_{ki}$ and

$$B = \sum_{i=1}^n b_i x_{ki}^3$$

A as defined in section 3.6.1.

Now, we calculate part B for two cases as follows:

Case 1

$$z_i = y_i - y_{n+i} = b_i x_{ki}^3 + \dots + b_n x_{kn}^3 + \epsilon_i - [b_i x_{k(n+i)}^3 + \dots + b_n x_{k(n+i)}^3 + \epsilon_{n+i}].$$

For example :

$$i=1$$

$$z_1 = b_1 x_{k1}^3 + \dots + b_n x_{1n}^3 + \epsilon_1 - [b_1 x_{k(n+1)}^3 + \dots + b_n x_{k(n+1)}^3 + \epsilon_{n+1}]$$

$$= b_1(1^3) + b_2 c_{12}^3 + b_3 c_{13}^3 + \dots + b_n c_{1n}^3 + \epsilon_1 - [b_1((-1)^3) + b_2 c_{12}^3 + b_3 c_{13}^3 + \dots + b_n c_{1n}^3 + \epsilon_{n+1}]$$

$$= 2b_1 + \epsilon_1 - \epsilon_{n+1}.$$

Case 2

$$h_i = y_i + y_{n+i} = b_i x_{ki}^3 + \dots + b_n x_{kn}^3 + \epsilon_i + [b_i x_{k(n+i)}^3 + \dots + b_n x_{k(n+i)}^3 + \epsilon_{n+i}].$$

For example :

$$i=1$$

$$h_1 = b_1 x_{k1}^3 + \dots + b_n x_{1n}^3 + \epsilon_1 + [b_1 x_{k(n+1)}^3 + \dots + b_n x_{k(n+1)}^3 + \epsilon_{n+1}]$$

$$= b_1(1^3) + b_2 c_{12}^3 + b_3 c_{13}^3 + \dots + b_n c_{1n}^3 + \epsilon_1 + [b_1(1)^3 + b_2(-c)_{12}^3 + b_3(-c)_{13}^3 + \dots +$$

$$\begin{aligned} & b_n(-c)_{1n}^3 + \epsilon_{n+1}] \\ & = 2b_1 + \epsilon_1 + \epsilon_{n+1}. \end{aligned}$$

Finally, if we have models with only odd Polynomial term and no interactions then z_i and h_i will give the same result.

3.6.4 Model 4: Exponential model

$$y_k = e^{\sum_{i=1}^n a_i x_{ki}} + \epsilon_k$$

Case 1

$$z_k = y_i - y_{n+i} = e^{\sum_{i=1}^n a_i x_{ki}} + \epsilon_i - [e^{\sum_{i=1}^n a_i x_{k(n+i)}} + \epsilon_{n+i}].$$

For example :

i=1

$$\begin{aligned} z_1 &= y_1 - y_{n+1} = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} + \epsilon_1 - [e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} + \epsilon_{n+1}] \\ &= e^{a_1 + a_2 c_{12} + \dots + a_n c_{1n}} + \epsilon_1 - [e^{-a_1 + a_2 c_{12} + \dots + a_n c_{1n}} + \epsilon_{n+1}]. \end{aligned}$$

let $A = a_2 c_{12} + \dots + a_n c_{1n}$

,then,

$$Z_1 = e^{a_1 + A} - e^{-a_1 + A} + \epsilon_1 - \epsilon_{n+1} = e^A (e^{a_1} - e^{-a_1}) + \epsilon_1 - \epsilon_{n+1}$$

Case 2

$$h_k = y_i + y_{n+i} = e^{\sum_{i=1}^n a_i x_{ki}} + \epsilon_i + [e^{\sum_{i=1}^n a_i x_{k(n+i)}} + \epsilon_{n+i}].$$

For example :

i=1

$$\begin{aligned} h_1 &= y_1 + y_{n+1} = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} + \epsilon_1 + [e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} + \epsilon_{n+1}] \\ &= e^{a_1 + a_2 c_{12} + \dots + a_n c_{1n}} + \epsilon_1 + [e^{a_1 + a_2(-c)_{12} + \dots + a_n(-c)_{1n}} + \epsilon_{n+1}] \end{aligned}$$

let $A = a_2 c_{12} + \dots + a_n c_{1n}$

,then,

$$h_1 = e^{a_1 + A} - e^{a_1 - A} + \epsilon_1 + \epsilon_{n+1} = e^{a_1} (e^A - e^{-A}) + \epsilon_1 + \epsilon_{n+1}.$$

3.7 Discussion

In this chapter, we introduced the construction of new edge designs using skew SDSs. These designs are particularly useful when the needed conference matrices of the same order do not exist. The methodology leads to the construction of a new family of edge designs that differ from those in the literature. The importance of this method is that it is also applicable in cases where the corresponding conference matrices cannot exist. Two such examples were also presented in this

chapter: edge designs with 14 and 26 factors. The robustness property is maintained and the new designs can provide a model-independent test for active factors. An additional mirror edge approach was provided here that gives an alternative test regarding whether the underlying model is linear. If the mirror edge approach yields different results, then we can conclude that the underlying model is not linear and sustain the revealed active factors for follow-up experiments and more complicated modelling. Using edge designs guarantees that irrelevant variables are never treated as relevant, in contrast to the above examples using classical methodologies, and there is only a very small chance that a relevant variable is not correctly recognised-this occurs when the two function values nearly agree on the corresponding edges. Thus, screening with edge designs is robust.

The number of runs and D-efficiency of the design is of great importance in the second stage of experimentation, where we select a design projected into the active factors and proceed to a supplementary second-stage statistical analysis to estimate and fit a more accurate prediction model. Projecting the suggested edge design to the active factors identified in the screening stage may be applied in a second analysis stage without the need for additional experimental runs.

Chapter 4

Construction and Analysis of Two level SSDs from Toeplitz matrices

This chapter investigates construction and analysis of Two level SSDs from Toeplitz matrices. We explain the details of the methodology, the theoretical optimal designs and provide the illustrative examples of this new approach. We also demonstrate that new designs are the most realisable and possess good D-efficiencies. We begin by describing the problem in section 4.1, and then delineating the class of SSDs and Toeplitz matrices in section 4.2. In section 4.3, we introduce a new class of two-level SSDs constructed from Toeplitz matrices and establish an algorithm for the new approach. Using this methodology, we present examples of two-level SSDs for 14 and 26 factors. In section 4.4, we examine the $E(s^2)$ and LB of the new designs (SSD_1, SSD_2), and compare their $E(s^2)$ and LB with the cyclic structure and Swarm Intelligence Techniques articles for the same number of variables. In section 4.5, we demonstrate the application of the new SSD designs by using the analysis contrast method, forward selection and regression approaches. Section 4.6 concludes the chapter with classical methodology for constructing new designs that differ from all SSDs discussed in the literature.

The work presented in this chapter was submitted as a research article in journal (Alanazi et al. (2018a)).

4.1 Description of the problem

Screening is necessary in cases where many factors must be studied. Traditional designs are very expensive, and sometimes not feasible to apply. Thus, there is a need for economical designs that can screen out active factors-SSDs can achieve this. However, constructing good SSDs are a significant research problem. We address this problem by developing a new method that generates optimal SSDs by

using the beautiful structure of Toeplitz matrices.

4.2 Introduction and preliminaries

Screening is very important in an initial experimentation stage when linear models are applied. The aim of screening experiments is to identify the active factors from a large number of factors that may influence the response y . Consider the following multivariate linear model:

$$y = X\beta + \epsilon \sim N_n(0_n, \sigma^2 I_n), \quad (4.1)$$

where y is the $n \times 1$ response vector, X is the $n \times k$ design model matrix $X = [x_1, x_2, \dots, x_k]$ and the experimental error is denoted by ϵ and assumed to be i.i.d. multivariate normal with dimension n , zero mean vector. This model is used to identify the significant main effects in the two-level designs, such as fractional factorial designs and Plackett-Burman designs. A screening design is said to be saturated if there are only enough degrees of freedom to estimate the parameters specified in the linear model, including the overall mean, and it is impossible to estimate the error variance without making additional assumptions, such as the effect of sparsity. For two-level saturated designs, the number of factors (m) equals the number of runs (n) minus one ($m = n - 1$).

In this chapter, we introduce a new class of SSDs constructed from Toeplitz matrices. In the next two sections, we provide basic notation and known results for SSDs and Toeplitz matrices.

4.2.1 SSDs

SSDs represent an important class of screening experiments, where many factors are investigated using only a few experimental runs; this costs less than classical factorial designs. There are two large classes for SSDs: two-level designs and mixed-level designs. Here, we study two-level SSDs. These designs have been studied for many years and feature attractive properties. In addition, we use certain criteria to evaluate the constructed SSDs. The first criterion is the classical $E(s^2)$ defined as follows:

$$E(s^2) = \sum s_{ij}^2 / (m/2), \quad (4.2)$$

where s_{ij} is the element of the i th row and j th column of $X^T X$. This criterion is used to measure the average of the aliased of the design factors. The second criterion to be used is the following :

$$\max_{i \neq j} |s_{ij}|. \quad (4.3)$$

This criteria is to show the maximum correlation between two effects. Booth and Cox (1962) suggested that these criteria are very important for evaluating the efficiency of designs. Wu (1993) commented that the average D-efficiency is maximised by $E(s^2)$ -optimal designs with just two main effects. Georgiou (2014) notes that when you combine these criteria, you can obtain good designs, and that these criteria are mostly used in the literature for the construction of two-level SSDs. SSDs is said to be Es^2 -optimal in the event that no other designs of a similar size have higher Es^2 -efficiency, characterized as Es^2/LB , where LB is Lower Bound. We now use the following theorem to obtain LB on $E(s^2)$.

Theorem 3 : [Das et al. (2008)] For an SSD with n runs and $m = p(n-1) \pm r$ factors (p positive, $0 \leq r \leq n/2$), the lower bound LB_D of $E(s^2)$ is

(1) $n \equiv 0 \pmod{4}$:

$$LB_D = \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)} \left(D(n,r) - \frac{r^2}{n-1} \right),$$

where

$$D(n,r) = \begin{cases} n+2r-3, & \text{for } |r| \equiv 1 \pmod{4}, \\ 2n-4, & \text{for } |r| \equiv 2 \pmod{4}, \\ n+2r+1, & \text{for } |r| \equiv 3 \pmod{4}, \\ 4r, & \text{for } |r| \equiv 0 \pmod{4}, \end{cases}$$

(2) $n \equiv 2 \pmod{4}$:

$$LB_D = \max \left\{ \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)} \left(D(n,r) - \frac{r^2}{n-1} \right), 4 \right\},$$

where

i. p is even:

$$D(n,r) = \begin{cases} n+2r-3+x/n, & \text{for } |r| \equiv 1 \pmod{4}, \\ 2n-4+8/n, & \text{for } |r| \equiv 2 \pmod{4}, \\ n+2r+1, & \text{for } |r| \equiv 3 \pmod{4}, \\ 4r, & \text{for } |r| \equiv 0 \pmod{4}, \end{cases}$$

ii. p is odd:

$$D(n,r) = \begin{cases} 2r-8r/n+n-16/n+9, & \text{for } |r| \equiv 1 \pmod{4}, \\ 4r-8r/n-8/n+8, & \text{for } |r| \equiv 2 \pmod{4}, \\ n+2r+8/n-3, & \text{for } |r| \equiv 3 \pmod{4}, \\ 2n-4+x/n, & \text{for } |r| \equiv 0 \pmod{4}, \end{cases}$$

and $x = 32$ if $\left\{ \frac{m-1-2i}{4} + \left\lceil \frac{m+(1+2i)(n-1)}{4(n-1)} \right\rceil \right\} \equiv (1-i \pmod{2})$ for $i = 0$ or 1 ; else $x = 0$.

4.2.2 Toeplitz matrices

A Toeplitz matrix is an $n \times n$ matrix $T_n = (t_{k,j})$, $k, j = 0, 1, \dots, n-1$ with the property that all elements on any fixed diagonal of the matrix are equal. Thus, $t_{k,j} = t_{k-j}$ for all $k, j = 0, 1, \dots, n-1$. For example $t_{5,3}$ will be the same as $t_{7,5}$ since $5-3 = 7-5$. A Toeplitz matrix is of the following form:

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2}, \dots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1}, \dots & t_{-(n-2)} \\ \vdots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3}, \dots & t_0 \end{pmatrix}. \quad (4.4)$$

The Toeplitz matrix of order n is constructed using two triangular matrices. The first is the upper triangular, defined by the sequence $t = (t_0, t_{-1}, \dots, t_{-(n-1)}, 0, \dots, 0)$. The upper triangular is of the following form:

$$U_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2}, \dots & t_{-(n-1)} \\ 0 & t_0 & t_{-1}, \dots & t_{-(n-2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \dots & t_0 \end{pmatrix},$$

where $t_i \in \{1, -1\}$, $t_i = (-1)^{n/2+i} t_{n-i}$, $i = 1, 2, \dots, n/2 - 1$.

The second is lower triangular, defined by the sequence $t = (t_0, 0, \dots, 0, t_1, \dots, t_{(n-1)})$. The lower triangular is of the following form:

$$L_n = \begin{pmatrix} t_0 & 0 & 0 \dots & 0 \\ t_1 & t_0 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} \dots & t_0 \end{pmatrix},$$

where $t_i \in \{1, -1\}$, $t_i = (-1)^{n/2+i} t_{n-i}$, $i = 1, 2, \dots, n/2 - 1$.

Gerorgiou and Stylianou (2016) introduced a new class of Toeplitz matrices constructed by defining sequences with symmetric and skew-symmetric structures. The construction method was described in Gerorgiou and Stylianou (2016) and is as following.

$$L_n P_n + (-1)^{n/2+1} P_n L_n^T, \quad (4.5)$$

where P_n is diagonal (p_1, p_2, \dots, p_n) , a signs-changer matrix with $p_i = (-1)^{i+1}$, and L_n is as defined above.

4.3 Two level SSDs from Toeplitz matrices

We present two construction methods for obtaining SSDs from Toeplitz matrices.

Case 1: The fold over of Toeplitz matrices can construct SSDs.

Theorem 4 : Suppose that L_n is the strictly-lower triangular defined in section (4.2.2), $P_n = \text{diag}(p_1, \dots, p_n)$ with $p_i = (-1)^{i+1}$. Define X matrix using Equation (4.5) and its fold-over as follow:

$$X = \begin{pmatrix} L_n P_n + (-1)^{n/2+1} P_n L_n^T + I_n \\ -(L_n P_n + (-1)^{n/2+1} P_n L_n^T) - I_n \end{pmatrix}, \quad (4.6)$$

where I_n is the identity matrix of order n . The desirable SSD is

$$SSD_1 = [X, X R_2, \dots, X R_k],$$

where R_k are any permutation matrices.

Proof: The proof is straightforward by routine calculations.

Remark 5 : Design X is fold-over with $2n$ runs, and thus X is balanced. Balance property is very important for the methodology and analysis of SSD.

Case 2: A column of $+1$ is generated by multiply the rows of $(L_n P_n + (-1)^{n/2+1} P_n L_n^T + I_n)$ that start with -1 by -1 . Then, we define the D matrix by removing the first column having all 1 's. The desirable SSD is generated using this matrix and any number of permutation. Then, we have

$$SSD_2 = [D, D R_1, D R_2, \dots, D R_k].$$

Remark 6 : Design D is a saturated design where the number of factors (m) equals the number of runs (n) minus one.

4.3.1 Algorithm of construction method

The steps for constructing two-level SSDs using Toeplitz matrices are provided in this section:

Step 1: Find P_n using $P_n = \text{diag}(p_1, \dots, p_n)$ with $p_i = (-1)^{i+1}$.

Step 2: Find L_n using the lower triangular form with the sequence $t = (t_0, t_{-1}, \dots, t_{-(n-1)}, 0, \dots, 0)$ form Table (4.1) :

$$L_n = \begin{pmatrix} t_0 & 0 & 0 \cdots & 0 \\ t_1 & t_0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} \cdots & t_0 \end{pmatrix}.$$

Step 3: Find $P_n \times L_n, P_n \times L_n^T$.

Step 4: Find $L_n P_n + (-1)^{n/2+1} P_n L_n^T$ and I_n the identity matrix of order n .

Step 5: Define X through Theorem 4 and D from case 2.

Step 6: Use MATLAB software to define the following:

$$SSD_1 = [X, XR_2, \dots, XR_k],$$

$$SSD_2 = [D, DR_1, DR_2, \dots, DR_k],$$

where R_k is selected permutation matrices. We illustrate the construction methodology for the suggested SSDs from Toeplitz matrices in the following detailed example. Table(4.1) presents some examples of defining sequences for Toeplitz matrices that are useful in this chapter.

4.3.2 Constructional example of n=6

Example 7 : Let us consider $n = 6$ and L_n by using the method and terminology given in Section 4.2.2 and Table 4.1. We follow these steps in this order:

Step 1: Find P_6 using $P_6 = \text{diag}(p_1, \dots, p_6)$ with $p_i = (-1)^{i+1}$. Thus, $p_1 = (-1)^{1+1} = 1, p_2 = (-1)^{2+1} = -1, p_3 = (-1)^{3+1} = 1, p_4 = (-1)^{4+1} = -1, p_5 = (-1)^{5+1} = 1, p_6 = (-1)^{6+1} = -1$. The result is the following :

$$P_6 = \begin{pmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & - \end{pmatrix}.$$

Step 2: Find L_6 . One defining sequence for the strictly lower triangular Toeplitz matrix L_6 is $t = (0, -1, -1, -1, 1, -1)$ (sequence hhh from Table 4.1, $h = -1$ and $w = 1$). Observe that this sequence satisfies $t_0 = 0, t_i \in [1, -1], t_i = (-1)^{3+i}t_{n-i}, i = 1, 2, \text{i.e. } t_1 = t_5 = 1, t_2 = t_4 = -1$. We have the following:

$$L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & - & - & 0 & 0 & 0 \\ + & - & - & - & 0 & 0 \\ - & + & - & - & - & 0 \end{pmatrix}.$$

Step 3: Find $P_6 \times L_6, P_6 \times L_6^T$

$$P_6 \times L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 \\ + & + & - & + & 0 & 0 \\ - & - & - & + & - & 0 \end{pmatrix} \text{ and } P_6 \times L_6^T = \begin{pmatrix} 0 & - & - & - & + & - \\ 0 & 0 & + & + & + & - \\ 0 & 0 & 0 & - & - & - \\ 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4: Find $L_n P_n + (-1)^{n/2+1} P_n L_n^T$ and I_n the identity matrix of order n. $L_n P_6 + (-1)^{n/2+1} P_n L_n^T = L_6 P_6 + (-1)^{6/2+1} P_6 L_6^T = L_6 P_6 + P_6 L_6^T$. We have the following:

$$= \begin{pmatrix} 0 & -1 & -1 & -1 & 1 & -1 \\ -0 & - & - & - & + & - \\ - & 0 & + & + & + & - \\ - & + & 0 & - & - & - \\ - & + & - & 0 & + & + \\ + & + & - & + & 0 & - \\ - & - & - & + & - & 0 \end{pmatrix} \text{ and } I_6 = \begin{pmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}.$$

Step 5: Define X through Theorem 4 and D from case 2.

From Theorem 4, we have two parts, and thus the results are the following:

$$L_6 P_6 + P_6 L_6^T + I_6 = \begin{pmatrix} + & - & - & - & + & - \\ - & + & + & + & + & - \\ - & + & + & - & - & - \\ - & + & - & + & + & + \\ + & + & - & + & + & - \\ - & - & - & + & - & + \end{pmatrix}$$

$$\text{and } -L_6 P_6 + P_6 L_6^T - I_6 = \begin{pmatrix} - & + & + & + & - & + \\ + & - & - & - & - & + \\ + & - & - & + & + & + \\ - & - & + & - & - & + \\ + & + & + & - & + & - \end{pmatrix}.$$

Then we have that $X =$

$$\begin{pmatrix} + & - & - & - & + & - \\ - & + & + & + & + & - \\ - & + & + & - & - & - \\ - & + & - & + & + & + \\ + & + & - & + & + & - \\ - & - & - & + & - & + \\ - & + & + & + & - & + \\ + & - & - & - & - & + \\ + & - & - & + & + & + \\ + & - & + & - & - & - \\ - & - & + & - & - & + \\ + & + & + & - & + & - \end{pmatrix}.$$

For case 2, we generated the column of +1 by multiply the minus row of $L_6 P_6 +$

$$P_6 L_6^T + I_6 \text{ by } -1, \text{ then, we have } = \begin{pmatrix} + & - & - & - & + & - \\ + & - & - & - & - & + \\ + & - & + & - & - & - \\ + & + & - & + & + & - \\ + & + & + & - & + & - \end{pmatrix}.$$

We then define D without the first positive column:

$$D = \begin{pmatrix} - & - & - & + & - \\ - & - & - & - & + \\ - & - & + & + & + \\ - & + & - & - & - \\ + & - & + & + & - \\ + & + & - & + & - \end{pmatrix}.$$

Step 6: Use MATLAB software to define

$$SSD_1 = [X, XR_2, \dots, XR_k],$$

$$SSD_2 = [D, DR_1, DR_2, \dots, DR_k],$$

where R_k is selected permutation matrices. We have that

$$SSD_1 = [X, XR_2] =$$

$$\begin{pmatrix} + & - & - & - & + & - & - & - & - & + & - & + & + & - & - & - & - & + \\ - & + & + & + & + & - & + & - & + & - & - & - & + & - & - & - & + & - \\ - & + & + & - & - & - & - & + & - & + & + & + & - & - & + & - & - & + \\ + & + & - & + & + & - & - & - & + & - & - & + & - & + & + & - & + & + \\ - & - & - & + & - & + & + & - & - & + & + & + & - & + & - & + & + & + \\ - & + & + & + & - & + & - & + & + & + & + & - & + & - & - & + & + & + \\ + & - & - & - & - & + & - & + & + & - & - & - & - & + & + & - & - & - \\ + & - & - & + & + & + & - & + & + & + & - & + & + & + & - & + & + & - \\ + & - & + & - & - & - & + & + & + & - & + & - & - & - & - & + & - & + \\ - & - & + & - & - & + & + & - & - & - & - & + & + & + & + & - & + & - \\ + & + & + & - & + & - & + & - & - & - & + & - & - & + & + & + & + & - \end{pmatrix}.$$

The above 12-runs designs of two permutation with $m = 18$ factors is not optimal designs.

$$SSD_2 = [D, DR_1] = \begin{pmatrix} - & - & - & + & - & - & - & + & + & + \\ - & - & - & - & + & + & + & - & + & - \\ - & - & + & + & + & - & - & - & - & + \\ - & + & - & - & - & - & - & - & + & - \\ + & - & + & + & - & + & - & + & + & - \\ + & + & - & + & - & - & + & - & - & - \end{pmatrix}.$$

The 6-run designs of one permutation with $m = 10$ factors are optimal designs, since $Es^2 = 4$ and $LB = 4$ implies that the design is 100% efficient.

4.3.3 Constructional example of $n=8$

Example 8 : Consider $n = 8$ and L_n by using the method and terminology given Section (4.2.2) and Table 4.1. One defining sequence for the strictly lower triangular Toeplitz matrix L_8 is $t = (0, 1, 1, 1, -1, 1, -1, 1)$ (sequence hhwh , $h = 1$ and

w = 1). We have the following:

$$L_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ - & + & - & - & 0 & 0 & 0 & 0 \\ - & - & + & - & - & 0 & 0 & 0 \\ - & - & - & + & - & - & 0 & 0 \\ + & - & - & - & + & - & - & 0 \end{pmatrix}.$$

Find P_8 using $P_8 = \text{diag}(p_1, \dots, p_8)$ with $p_i = (-1)^{i+1}$. We have the following:

$$P_8 = \begin{pmatrix} + & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \end{pmatrix}.$$

From Theorem 4, we obtain the following:

$$L_8 P_8 - P_8 L_8^T = \begin{pmatrix} 0 & + & + & - & + & + & + & - \\ - & 0 & - & - & + & - & - & - \\ - & + & 0 & + & + & - & + & + \\ + & + & - & 0 & - & - & + & - \\ - & - & - & + & 0 & + & + & - \\ - & + & + & + & - & 0 & - & - \\ - & + & - & - & - & + & 0 & + \\ + & + & - & + & + & + & - & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} + & + & + & - & + & + & + & - \\ - & + & - & - & + & - & - & - \\ - & + & + & + & + & - & + & + \\ + & + & - & + & - & - & + & - \\ - & - & - & + & + & + & + & - \\ - & + & + & + & - & + & - & - \\ - & + & - & - & - & + & + & + \\ + & + & - & + & + & + & - & + \\ - & - & - & + & - & - & - & + \\ + & - & + & + & - & + & + & + \\ + & - & - & - & - & + & - & - \\ - & - & + & - & + & + & - & + \\ + & + & + & - & - & - & - & + \\ + & - & - & - & + & - & + & + \\ + & - & + & + & + & - & - & - \\ - & - & + & - & - & - & + & - \end{pmatrix}.$$

Use MATLAB software to define SSD_1 , we obtain the following matrix SSD_1 whereby

$$SSD_1 = [X, XR_1, XR_2] = \begin{pmatrix} + & + & + & - & + & + & + & - & - & + & + & + & - & + & - & - & - & + & - & - & - & + & - & - & - & + & - \\ - & + & - & - & + & - & - & - & - & + & + & + & + & - & + & + & - & + & + & + & - & + & - & - & + & - & - \\ - & + & + & + & + & - & + & + & - & - & + & - & - & - & + & - & + & - & - & - & - & + & - & + & + & - & + \\ + & + & - & + & - & - & + & - & + & - & - & - & - & + & - & - & - & - & - & - & + & - & + & + & - & + \\ - & - & - & + & + & + & + & - & - & + & - & - & - & + & + & + & + & + & + & - & + & + & + & - & - & + & + \\ - & + & + & + & - & + & - & - & + & - & - & - & + & - & + & + & + & + & + & - & - & - & - & + & - & - & + \\ - & + & - & - & - & + & + & + & + & - & + & + & + & - & + & - & - & - & - & + & + & + & + & - & - & + & + \\ + & + & - & + & + & + & - & + & - & - & + & + & + & - & - & - & - & + & - & - & - & + & - & - & - & + & - \\ - & - & - & + & - & - & - & + & + & - & + & + & + & - & - & - & + & - & + & + & + & - & - & - & - & - & - \\ + & - & + & + & - & + & + & + & + & + & - & + & + & + & - & + & - & - & - & - & - & + & - & - & - & + & - \\ + & - & - & - & + & - & - & - & + & - & - & - & + & - & - & - & + & - & + & - & - & - & + & - & - & - & - \\ + & - & + & + & + & - & - & - & + & - & + & + & + & + & - & + & + & + & + & - & + & + & + & - & + & + & - & + \\ - & - & + & - & - & - & + & - & - & - & + & - & + & + & - & + & + & + & - & + & + & + & - & + & - & - & + & - \end{pmatrix}.$$

The above 16-run designs of two permutations with $m = 24$ factors are not optimal designs.

Now, we generate the column of + 1 by multiplying the minus row of $L_8P_8 + (-1)^{8/2+1}P_8L_8^T + I_8$ by -1; then, we have

$$= \begin{pmatrix} + & + & + & - & + & + & + & - \\ + & - & + & + & - & + & + & + \\ + & - & - & - & - & + & - & - \\ + & + & - & + & - & - & + & - \\ + & + & + & - & - & - & - & + \\ + & - & - & - & + & - & + & + \\ + & - & + & + & + & - & - & - \\ + & + & - & + & + & + & - & + \end{pmatrix}.$$

We then define D without the first positive column:

$$D = \begin{pmatrix} + & + & - & + & + & + & - \\ - & + & + & - & + & + & + \\ - & - & - & - & + & - & - \\ + & - & + & - & - & + & - \\ + & + & - & - & - & - & + \\ - & - & - & + & - & + & + \\ - & + & + & + & - & - & - \\ + & - & + & + & + & - & + \end{pmatrix}.$$

Use MATLAB software to define SSD_2 , we obtain the following matrix SSD_2 whereby

$$SSD_2 = [D, DR_1] = \begin{pmatrix} + & + & - & + & + & + & - & + & - & + & + & + & - & + \\ - & + & + & - & + & + & + & - & + & + & - & + & + & + \\ - & - & - & - & + & - & - & - & - & - & - & + & - & - \\ + & - & + & - & - & + & - & + & + & - & + & + & + & - \\ + & + & - & - & - & - & + & - & - & - & + & - & + & + \\ - & - & - & + & - & + & + & + & + & - & - & - & - & + \\ - & + & + & + & - & - & - & + & - & + & - & - & + & - \\ + & - & + & + & + & - & + & - & + & + & + & - & - & - \end{pmatrix}.$$

The 8-runs designs of one permutation with $m = 14$ factors are optimal designs, since $E(s^2) = 4.9231$ and $LB = 4.9231$ implies that the design is 100% efficient. In the next section we present the result for practitioners' use.

4.4 Results and comparison

Here, we present the $E(s^2)$ and LB of the new designs (SSD_1, SSD_2), and then compare these result for the same number of variables with the two-level SSDs with cyclic structure (Georgiou et al. (2009)) and Swarm Intelligence Techniques articles (Phoa et al. (2016)). Table 2 in Appendix 6.2 lists the SSD_1 with n runs and m factors that we have found via the MATLAB program for selected values n , m and Perm. From $n = 12$ to 160, the Table shows all these designs are not- $E(s^2)$ -optimal. The design is to be optimal if the Eff equals 1. Theses non $E(s^2)$ -optimal designs are generated in Table 3 in Appendix 6.2. Note that for this case, when the number of permutations increases, the efficiency values and s_{max} will also increase for each number of size. Table 4 in Appendix 6.2 lists the SSD_2 with n runs and m factors that we have found via the MATLAB program for selected values n , m and Perm. From $n=6$ to 80, the Table shows all these designs are $E(s^2)$ -optimal. Theses $E(s^2)$ -optimal designs are generated in Table 5 in Appendix 6.2. Now, we compare the properties for new $E(s^2)$ with cyclic structure and Swarm Intelligence designs with same number of runs and factors. In Table 4.2, we present the efficiencies of our designs and compare them with various designs. The results show that our designs are better than cyclic structure and Swarm Intelligence Techniques designs because they have much higher efficiency with 38, 57, 65, 76, 85, 91, 104, 114, 119, 170, 171, 190, 228 and 342 factors.

4.5 Applying SSD in screening experiments

In this section, we illustrate the analysis of SSD constructed from Toeplitz matrices using contrast method, forward selection and regression. We use matrices SSD_1 and SSD_2 as these were generated in example 7 and 8 and we then apply the aforementioned analysis methods, respectively.

Before introduce the examples, we delineate the procedures analysis for contrast method. This procedures were applied by Koukouvinos and Stylianou (2005), who introduced a method for analyzing SSDs that using a new contracts based method. Suppose there are p active out of m factors. The procedure as is described in Koukouvinos and Stylianou (2005) is as follows.

Step 1: Find all factor contrasts through

$$C = X^T y, \quad (4.7)$$

where y is the the response and X is the design model matrix. Then, calculate the absolute values and sort these absolute contracts.

Step 2: Set $i = 0$ and calculate the variance of the p largest absolute contracts; use $p = N/2$, where N is the run size.

Step 3: Calculate the upper and lower limits on the critical region through

$$ucl_i = |c_m| - t_{m-1, \alpha/2} \sigma_p \quad (4.8)$$

and

$$lcl_i = -|c_m| + t_{m-1, \alpha/2} \sigma_p, \quad (4.9)$$

where $t_{m-1, \alpha/2}$ is the percentage points of the t distribution.

Step 4: Remove the largest value $|c_{m-i}|$ and then set $i = i + 1$.

Step 5: Find σ_p for the p largest absolute contrasts using the remaining values only.

Step 6: From equations(4.8) and (4.9), if the variance in Step 5 is smaller than the variance that found before Step 3, go to Step 4 otherwise stop, and conclude the active factors from the contrasts falling outside the critical region.

4.5.1 Analysis Example 7

Analysis Case 1 (SSD_1)

Contrast method

Step 1: We generate data using the design matrix SSD_1 and the fitted simulation model $y = 5.8x_2 - 9x_4 + 10x_{17} - 0.6x_{18} + \varepsilon$, where the ε is $N(0,1)$. Then we find all factor contrasts through Equation 4.7, calculate the absolutes and sort these values. The result is shown in Table 4.3.

Step 2- Step 5: Set $i = 0$ and calculate the variance of the p largest absolute contracts- use $p = N/2$, where N is the run size. The result is shown in Table 4.4.

Step 6: We check if the variance in Step 5 is smaller than the variance found before (Step 3) , go to Step 4 otherwise stop, and conclude the active factors from the contrasts falling outside the critical region. We observe that σ_5^2 is more than σ_4^2 , and thus we stop and find the active factors. The final values for Stage 4 are $ucl_i = 33.62, lcl_i = -33.62$ and $\sigma_4^2 = 90.49$. As a result, there are actives factors outside the critical region: $x_{10}, x_{14}, x_1, x_6, x_3, x_{11}, x_{13}$ and x_{16} . This means there is a Type II error equals to 100% and Type I error equal to 57.14%. These eight active factors can be retained for future experimentation.

Forward selection

Analysis of these data (Forward selection) with software package SPSS, reveals as active variables x_{16} , x_4 and x_2 and gives an estimated linear model $y = -0.100 + 10066x_{16} - 9.083x_4 + 5.67x_2 + \varepsilon$ (see Figure 4.1). This means there is a Type II error equals to 50% and Type I error equal to 7% . These three active factors can be retained for future experimentation.

Coefficients^a

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.110	2.685		-.041	.968
	x16	7.038	2.685	.638	2.622	.026
2	(Constant)	-.110	1.797		-.061	.952
	x16	9.357	1.906	.848	4.908	.001
	x4	-6.955	1.906	-.631	-3.648	.005
3	(Constant)	-.110	.336		-.328	.752
	x16	10.066	.360	.913	27.988	.000
	x4	-9.083	.381	-.824	-23.811	.000
	x2	5.674	.360	.515	15.777	.000

a. Dependent Variable: Y

Figure 4.1: Analysis stepwise of example 7 for Case 1 (SSD_1).

Linear regression method

Analysis of these data (using regression) with software package SPSS, reveals as active variables x_2 and x_4 and gives an estimated linear model $y = 1.42 + 5.451x_2 - 8.861x_4 + \varepsilon$ (see Figure 4.2). This means there is a Type II error equals to 50% and Type I error equal to 0%. These two active factors can be retained for future experimentation.

Coefficients^a

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.142	.357		.398	.701
	x2	5.451	.382	.610	14.273	.000
	x4	-8.861	.405	-.991	-21.877	.000
	x16	-.444	.382	-.050	-1.164	.278

a. Dependent Variable: Y

Figure 4.2: Analysis regression of example 7 for Case 1 (SSD_1).

In this case, we prefer forward selection and linear regression because they give two of the four active variables with small Type I errors that can be retained for future experimentation.

Analysis Case 2 (SSD_2)

Contrast method

Step 1: We generate data using the design matrix SSD_2 and the fitted simulation model $y = 4.8x_2 + 6x_5 - 8.9x_9 + \varepsilon$, where the ε is $N(0,1)$, we then find all factor contrasts through Equation 4.7, calculate the absolutes and sort these values. The result is shown in 4.5.

Step 2- Step 5: Set $i = 0$ and calculate the variance of the p largest absolute contrasts; use $p = N/2$, where N is the run size. The result is shown in Table 4.6.

Step 6: We check if the variance in Step 5 is smaller than the variance found before Step 3, go to Step 4; otherwise, we stop, and conclude the active factors from the contrasts falling outside the critical region. We observe that σ_3^2 is more than σ_2^2 , and thus stop and find the active factors. The final values for Stage 2 are $ucl_i = 35.13$, $lcl_i = -35.13$ and $\sigma_4^2 = 2.800$. As a result, there are active factors outside the critical region: x_2, x_8, x_7, x_5 and x_9 . This means there is a Type II error equals to 33.33% and Type I error equal to 43% . These five active factors can be retained for future experimentation.

Forward selection

Analysis of these data (using stepwise) with software package SPSS, reveals as active variables x_9 and x_4 and gives an estimated linear model $y = -0.714 - 13.752x_9 - 5.648x_4 + \varepsilon$ (see Figure 4.3). This means there is a Type II error equals to 66.66% and Type I error equal to 0% . These two active factors can be retained for future experimentation.

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-3.537	2.498		-1.416	.230
	x9	-10.929	2.498	-.909	-4.374	.012
2	(Constant)	-.714	.682		-1.046	.372
	x9	-13.752	.682	-1.144	-20.163	.000
	x4	-5.648	.682	-.470	-8.280	.004

a. Dependent Variable: Y

Figure 4.3: Analysis stepwise of example 7 for Case 2 (SSD_2).

Linear regression method

Analysis of these data (using regression) with software package SPSS, reveals as active variable x_9 and gives an estimated linear model $y = -0.215 + 8.205x_9 + \varepsilon$ (see Figure 4.4). This means there is a Type II error equals to 33.33% and Type I error equal to 43% . This one active factor can be retained for future experimentation.

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.215	.651		-.331	.763
	x4	-.196	.651	-.023	-.301	.783
	x9	8.204	.651	.981	12.610	.001

a. Dependent Variable: Y

Figure 4.4: Analysis regression of example 7 for Case 2 (SSD_2) .

In this case, we prefer forward selection because it gives one of the three active variables with small Type I errors that can be retained for future experimentation.

4.5.2 Analysis Example 8

The analysis methods for SSD_1 are as follows. First, we apply the contrast method in Example 8. We generate data using the design matrix SSD_1 and the fitted simulation model $y = 5.6x_2 - 7x_{12} + 4x_{19} + 0.5x_{22} + \varepsilon$, where the ε is $N(0,1)$. The simulated data we derive with this model are $y^T = [2.220, 2.334, 7.104, 16.941, 4.227, 15.470, -5.984, -6.254, -10.432, -2.072, 4.647, -7.527, 9.246, -15.795, -17.184, -3.357]$. We then use Equation (4.7) to calculate the contrasts with absolute and sort values. These values are given in Table 4.7 .

Now, we set $i = 0$. We next explain the step-by-step analysis method as defined in Section 4.5. The result is shown in Table 4.8. We observe that σ_4^2 is more than σ_3^2 , so we stop the procedure to avoid any increase of type error 1. The final values for Stage 3 are $ucl_i = 30.10$, $lcl_i = -30.10$ and $\sigma_3^2 = 236.47$. As a result, there are actives factors that outside the critical region which are $x_{12}, x_2, x_{20}, x_{19}, x_8, x_5, x_{24}, x_{21}$ and x_{22} . This means there is a Type II error equal to 0% and a Type I error equal to 25%. These nine active factors can be retained for future experimentation.

Second, an analysis of these data (using stepwise) with the software package Minitab for windows, release 17, revealed as active variables $x_{12}, x_2, x_{19}, x_{17}, x_3, x_{23}, x_{10}, x_{21}, x_4, x_{24}, x_7, x_{22}$ and x_{13} and gives an estimated linear model $y = -0.4010 - 7.4047x_{12} + 5.1185x_2 + 3.9034x_{19} - 0.7954x_{17} + 0.3332x_3 - 0.5897x_{23} + 0.4036x_{10} + 0.1788x_{21} - 0.2794x_4 + 0.2569x_{24} + 0.1741x_7 - 0.0536x_{22} + 0.0265x_{13} + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 0.00392$. This means there is a Type II error equal to 0% and a Type I error equal to 45%. These 13 actives factors can be retained for further experimentation. An analysis of these data (using linear regression) with the software package Minitab for windows, release 17, revealed as active variables x_2, x_{12} and x_{19} and gives an estimated linear model $y = 0.0683 + 6.5305x_2 - 7.1117x_{12} + 3.8111x_{19} + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 1.65058$. This means there is a Type II error equal to 25% and a Type I error equal to 0%. These three actives factors can be retained for future experimentation. In this case, we prefer the contrast method because it gives all the active variables with small Type I errors.

The analysis methods for SSD_2 are detailed here. First, we apply the contrast method from Example 8. We generate data using the design matrix SSD_2 and the fitted model $y = 0.5x_3 - 5.7x_6 + 7x_{10} - 1.2x_{14} + \varepsilon$ where the ε is $N(0,1)$. The simulated data we derive with this model are $y^T = [11.630, 12.237, -10.889, 0.484, -12.934, -1.577, 3.496, 4.144]$. We then use Equation (4.7) to calculate the contrast with absolute and sort values. These values are given in Table 4.9.

We set $i = 0$. We now show the step-by-step analysis method, as defined in section 1. The result is shown in Table (4.10). We observe that σ_6^2 is more than σ_5^2 , so we stop the procedure to avoid any increase for type error 1. The final values for

Stage 5 are $ucl_i = 20.44$, $lcl_i = -20.44$ and $\sigma_5^2 = 2.35$. The active factors are those that fall outside the critical region : $x_8, x_2, x_9, x_5, x_4, x_3, x_6$ and x_{10} . This means there is a Type II error equal to 25% and a Type I error equal to 50%. These eight active factors can be retained for future experimentation.

An analysis of these data (using stepwise) with the software package Minitab for windows, release 17, revealed as active variables $x_{10}, x_6, x_{19}, x_{14}, x_7$ and x_8 , and gives an estimated linear model $y = 0.8240 + 7.0531x_{10} + 5.5400x_6 - 1.4834x_{14} + 0.4564x_7 + 0.1427x_8 + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 0.00842$. This means there is a Type II error equal to 25% and a Type I error equal to 20%. These five active factors can be retained for future experimentation. An analysis of these data (using linear regression) with the software package Minitab for windows, release 17, revealed as active variables x_6 and x_{10} and gives an estimated linear model $y = -0.1223 - 5.6773x_6 + 6.8014x_{10} + \varepsilon$, with ε of mean zero and standard deviation $\sigma = 1.19086$. This means there is a Type II error equal to 50% and a Type I error equal to 0%. These two active factors can be retained for future experimentation.

In this case, we prefer forward selection, because it gives three of the four active variables with small Type I errors that can be retained for follow-up experimentation and analysis.

4.6 Discussion

In this chapter, we have obtained and tabulated the optimal two-level SSDs constructed from Toeplitz matrices and permutation matrices from $n = 6$ runs to $n = 160$ runs. We have also proposed the methodology for constructing new designs that differ from all SSDs discussed in the literature. We have also presented the properties of $E(s^2)$ and LB for two cases (SSD_1, SSD_2), and compared the obtained results with those in the literature for the same number of runs and variables.

In this chapter, two such examples are also presented for SSDs with 14 and 24 factors. These examples are analysed using contracts variance, forward selection and regression analysis methods. The importance of these methods is that the active factors can be retained for follow-up experimentation and analysis when the Type I error is small.

Table 4.2: The properties for SSD_2 , Cyclic structure and Swarm Intelligence.

Design	n	m	Es^2	LB	Eff
New designs	14	65	12.3846	12.3385	99.63%
Cyclic structure	14	65	12.40	12.3385	99.50%
New design	14	91	13.11	13.1116	99.94%
Cyclic structure	14	91	13.14	13.1116	99.80%
New design	14	104	13.3264	13.3204	99.96%
Swarm Intelligence	14	104	13.3323	13.3204	99.91%
New design	18	85	15.6706	15.51	98.97%
Cyclic structure	18	85	15.73	15.51	98.60%
New design	18	119	16.5384	16.52	99.86%
Cyclic structure	18	119	16.55	16.52	99.80%
New design	18	170	17.27	17.2544	99.91%
Swarm Intelligence	18	170	17.2812	17.2544	99.85%
New design	20	38	10.8108	10.8108	100%
Swarm Intelligence	20	38	11.7440	10.8108	92.05%
New design	20	57	14.2857	14.2857	100%
Swarm Intelligence	20	57	14.6266	14.2857	97.67%
New design	20	76	16	16	100%
Swarm Intelligence	20	76	16.1853	16	98.86%
New design	20	114	17.6991	17.6991	100%
Swarm Intelligence	20	114	17.7885	17.6991	99.50%
New design	20	171	18.8235	18.8235	100%
Swarm Intelligence	20	171	18.8621	18.8235	99.80%
New design	20	190	19.0476	19.0476	100%
Swarm Intelligence	20	190	19.0744	19.0476	99.86%
New design	20	228	19.3833	19.3833	100%
Swarm Intelligence	20	228	19.4068	19.3833	99.88%
New design	20	342	19.9413	19.9413	100%
Swarm Intelligence	20	342	19.9575	19.9413	99.92%

Table 4.3: Sorted absolute contrasts values for example 7 in SSD_1 case.

j	1	2	3	4	5	6	7	8
x_n	x_8	x_9	x_7	x_{17}	x_5	x_{12}	x_{14}	x_{15}
$ c(j) $	1.60	2.69	6.17	9.57	12.73	21.74	22.45	23.08
j	9	10	11	12	13	14	15	16
x_n	x_2	x_{18}	x_{10}	x_{14}	x_1	x_6	x_3	x_{11}
$ c(j) $	31.75	33.54	35.76	46.03	50.52	54.17	60.06	65.42
j	17	18						
x_n	x_{13}	x_{16}						
$ c(j) $	74.80	84.45						

Table 4.4: Step by step calculations for the analysis example 7 with SSD_1 in the case 1.

i	σ_i^2	ucl_i	lcl_i	σ_i^2 more than σ_{i-1}^2
0	165.20	57.33	-57.33	
1	110.76	52.49	-52.49	No
2	110.24	43.04	-43.04	No
3	108.41	37.72	-37.72	No
4	90.49	33.62	-33.62	No
5	99.76	28.76	-28.76	Yes

Table 4.5: Sorted absolute contrasts values for example 7 in SSD_2 case 2.

j	1	2	3	4	5	6	7	8
x_n	x_4	x_6	x_1	x_3	x_{10}	x_2	x_8	x_7
$ c(j) $	7.80	13.81	14.12	19.73	21.14	35.74	37.37	39.09
j	9	10						
x_n	x_5	x_9						
$ c(j) $	44.70	72.64						

Table 4.6: Step by step calculations for the analysis example 7 with SSD_2 in the case 2.

i	σ_i^2	ucl_i	lcl_i	σ_i^2 more than σ_{i-1}^2
0	322.96	31.99	-31.99	
1	14.700	35.86	-35.86	No
2	2.800	35.13	-35.13	No
3	79.86	15.50	-15.50	Yes

Table 4.7: Sorted absolute contrasts values for Example 8 in SSD_1 .

j	1	2	3	4	5	6	7	8
x_n	x_{15}	x_{17}	x_9	x_{13}	x_1	x_7	x_3	x_{16}
$ c(j) $	1.388	3.22	3.32	4.88	10.08	12.98	14.21	15.67
j	9	10	11	12	13	14	15	16
x_n	x_6	x_{14}	x_{11}	x_{18}	x_{10}	x_4	x_{23}	x_{22}
$ c(j) $	15.86	16.51	17.86	18.75	20.59	22.01	27.57	30.72
j	17	18	19	20	21	22	23	24
x_n	x_{21}	x_{24}	x_5	x_8	x_{19}	x_{20}	x_2	x_{12}
$ c(j) $	50.88	51.30	55.33	57.01	62.17	81.10	88.57	110.83

Table 4.8: Step by step calculations for the analysis Example 8 with SSD_1 in the case 1.

i	σ_i^2	ucl_i	lcl_i	σ_i^2 more than σ_{i-1}^2
0	470.61	65.36	-65.36	
1	330.23	50.88	-50.88	No
2	291.11	45.61	-45.61	No
3	236.47	30.10	-30.10	No
4	244.08	24.31	-24.31	yes

Table 4.9: Sorted absolute contrasts values for Example 8 in SSD_2 .

j	1	2	3	4	5	6	7
x_n	x_{13}	x_{11}	x_1	x_7	x_{14}	x_{12}	x_8
$ c(j) $	0.0238	0.0576	0.0576	2.852	12.118	20.33	21.47
j	8	9	10	11	12	13	14
x_n	x_2	x_9	x_5	x_4	x_3	x_6	x_{10}
$ c(j) $	22.267	23.985	27.653	28.795	34.134	38.42	56.424

Table 4.10: Step by step calculations for the analysis of Example 8 with SSD_2 in the case 2.

i	σ_i^2	ucl_i	lcl_i	σ_i^2 more than σ_{i-1}^2
0	143.36	30.56	-30.56	
1	27.17	27.59	-27.59	No
2	17.61	24.89	-24.89	No
3	9.37	21.97	-21.97	No
4	7.54	21.44	-21.44	No
5	2.35	20.44	-20.44	No
6	21.9	11.18	-11.18	Yes

Chapter 5

Examples and comparison of the constructed designs using real data

In this chapter, we will present some examples and comparisons of the constructed designs using real data. The data comes from surveys conducted in schools in Saudi Arabia. The data are collected by the middle stage schools in the country and are available to Saudi Arabian citizen. These are to investigate the most influential factors on student achievement in mathematics in middle-stage schools in Ha'il, Saudi Arabia. We provide details of the comparison and provide some illustrative examples for the designs approach. We also show that the new designs have the best feasible design analysis than classical regression.

We begin by describing the problem in section 5.1. Section 5.2 presents the conceptual framework. Section 5.3 states the methodology of the constructed designs using real data. In sections 5.4, data analysis and results are presented, tabled and demonstrated using different examples. Finally, section 5.5 summarises the chapter and proposes the most influential factors on student achievement in mathematics.

The work presented in this chapter was submitted as a research article in journal (Alanazi et al. (2018b)).

5.1 Introduction and the description of the problem

Mathematics is thought to be the most essential school subject throughout the world. It is considered the basis of most scientific and technological knowledge

that benefits socio-economic development for any nation. For this reason, it is an obligatory subject at both primary- and middle-school levels in Saudi Arabia. In addition, it is used as a basic entry requisite for prestigious tertiary courses such as medicine, architecture and engineering. Although mathematics plays an important role in any society, performance in national mathematics examinations is still very poor (Aduda (2003)). In the labour market, workers with a strong mathematical and scientific background have a strong chance of finding more jobs than those with lower achievement, even if they do not have a university certificate (Darling-Hammond (1997)). Many studies have been conducted in this area, and researches have given various recommendations. However, the problem of low achievement in mathematics remains notable at all stages of school education. Unlikely, students at all levels of education perform poorly in mathematics in Saudi Arabia. The Trends in Mathematics and Science Study (TIMSS) Jones et al. (2015) reported that Saudi students have lower average achievement in mathematics (see Figure 5.1) . This achievement level is very low for both boys and girls in Saudi Arabia compared with other countries. We address this problem by choosing the sample one replication of the runs of design used by our new design, which we made based on real indicators and measurements in the field of mathematics and statistics.



Figure 5.1: TIMSS Report.

5.2 Conceptual Framework

As we mentioned above that there are many factors that affects this problem. Here are some of them:

Related to the Student

- age
- health
- number of students in class
- family circumstances
- time of study
- desire
- behavior
- achievements
- media (audible and visual)
- reward, Friends
- parents' goal
- gender

Related to the Classroom Environment

- Suitable and attractive
- equipped with educational tools.

Related to the Curriculum

- Easy or difficult

Related to The Teacher

- Well qualified one or no
- methods of punishment.

5.3 Real application analysis using real data.

5.3.1 Stating the main Concepts

The target sample: male and female students in the preparatory stage (2,585 students). Table (5.1) presents the gender split of participants.

The aim (the main variable): the level of student mathematical achievement.

The secondary variables:

- student gender
- student achievement in the primary stage
- student behaviour
- if teachers are effective and well qualified
- student health
- family circumstances

- number of the students in the classroom
- student attitudes towards mathematics
- time of study
- classroom environment
- if the curriculum is suitable and attractive
- media
- if classes are equipped with educational tools
- rewards
- advice from friends
- parents' goals
- punishment
- student age.

Table 5.1: Gender of participants.

Respondents	Number of male	Male %	Number of female	Female%
Students	1,690	65.37	895	34.62

5.3.2 Choosing the Place

The places of the data collect were preparatory schools for boys and girls in Al-Shamly governorate and in Ha'il city. Table (5.2) shows the number of participating students at Shmaly schools, Table (5.3) presents the number of participating students at Ha'il schools and Table (5.4) displays the number of participating students at private schools.

Table 5.2: Public Shcool in Al-Shamly.

School name	Students number	Gender	Code
Alheverh	126	M	1
Osboter	112	M	2
The South Al-Shamly	87	M	3
Almair	119	M	4
Dharahat	71	F	5
The first Alheverh	120	F	6
The first Almair	106	F	7
The first Al-Shamly	101	F	8
The first Osboter	129	F	9

Table 5.3: Public Shcool in Ha'il.

School name	Students number	Gender	Code
Ibn Habban	154	M	10
Almathnaa	114	M	11
Cordoba	166	M	12
Sharf	129	M	13
Thabet bin Qais	96	M	14
Amam Mohmmmed	200	M	15
Ibrahim Althuwaini	129	M	16
The Second Quran	89	F	17
Twenty one	113	F	18
Eighteen	90	F	19

Table 5.4: Private Shcool in Ha'il.

School name	Students number	Gender	Code
Sahama	126	M	20
Alfadeelah	132	M	21
Hail Agiel	76	F	22

5.3.3 Choosing the sample and making sure if it suits the different factors

We chose the sample using our new design, which we made based on real indicators and measurements in the field of mathematics and statistics.

5.3.4 Carrying out the experiment and collecting data

The data comes from surveys contacted in schools in Saudi Arabia. The data are collected by the middle stage schools in the country and are available to Saudi Arabian citizen. We could then calculate the achievement levels for the targeted student using the suitable sample. This is presented in Table (5.5), which explains the data collected.

Table 5.5: Notes to be considered about data tables.

Factor	signal	Description
student gender	+	Male
	-	Female
Achievement in primary	+	Excellent
	-	Poor
Behaviour	+	Excellent
	-	Moderate and bad
Teacher	+	good teaching method
	-	Moderate and bad
Health	+	Healthy
	-	Moderate and unhealthy
Family circumstances	+	Good
	-	Moderate and poor
Students number	+	20 and Fewer than 20
	-	More than 20
Desire	+	high
	-	Moderate and none
Study time	+	3 hours
	-	less or more than 3 hours

5.4 Data Analysis with experimental design approach

The collected data were analysed using edge design analysis (see [Elster and Neumaier (1995)]) and regression analysis, which was performed using SPSS computer software to determine the most influential factors on student achievement in mathematics in middle-stage school. We screen through the provided data and allocate one replication of the runs of the design used. The two analysis approaches are explained as the following.

We explain the two analysis methods used to determine the factors that cause low achievement in mathematics. The first is edge design, introduced by Elster and Neumaier (1995). The edge depends on a model-independent test that can be used for active variables. To ascertain the active variables, the measurements are arranged into a group of E pairs. With this approach, the measurements differ in only one component. It is very natural in screening experiments, and implies that almost all differences

$$z_{i,j} := y_i - y_j, \quad (i, j) \in E,$$

Table 5.5: (cont.)

Class environment	+	Attractive
	-	Unattractive
Curriculum	+	Moderate and easy
	-	Hard
Media	+	less than 3 hours
	-	More than 3 hours
Tools	+	Equipped
	-	Unequipped
Reward	+	Found
	-	Absent
Friend	+	Give advice
	-	Do not give advice
Parents' goals	+	Suitable
	-	Unsuitable
Punishment	+	no punishment
	-	Punishment-Moderate
Age	+	12 – 15
	-	More than 15 and Younger than 12
Y(Response)	Grades	First term Exams in 2016

consist of noise only. If we assume that the noise in the data is additive, normally distributed with zero mean and variance σ^2 , the $n - p$ of the $z_{i,j}$ are normally distributed with zero mean and variance $2\sigma^2$. Because of the unknown number of outliers, the variance must be estimated in a robust way. For example, we can use the median estimate

$$\tilde{\sigma} = \frac{\text{median}\{|z_{i,j}| : (i, j) \in E\}}{2^{\frac{1}{2}} \times 0.675}. \quad (5.1)$$

More details about this method can be found in Alanazi et al. (2017). The second method we use in this study is classical regression analysis. Using several examples, we now illustrate the use and analysis of these two methods of one replicate for the data collected from schools.

5.4.1 Alheverh School

Example 9 : Let $n = 4$ and, Reward x_1 , Friends x_2 , Parents goal x_3 and Punishment x_4 .

Table 5.6: One replication of the runs of the design used for Alheverh School.

Run	x_1	x_2	x_3	x_4	Y	Student number
1	1	1	1	1	50	57
2	-1	1	1	-1	17.88	94
3	-1	-1	1	1	17.5	24
4	-1	1	-1	1	31.5	6
5	-1	1	1	1	22	74
6	-1	-1	1	-1	25	12
7	-1	-1	-1	1	42	15
8	-1	1	-1	-1	22	7

Analysing the data in Table 5.6 using edge design is the following. First, we find all four differences of the response y over the edges and the absolute value, as provided in Table 5.7. Second, we calculate the median to assume the number p as active factors. Third, we find the sigma (σ) and calculate $w(p)$ and $k * 2^{0.5}\sigma$. Finally, if the $w(p)$ is greater than p for some guess p , we stop the procedure and find the active factor. Table 5.8 shows the results: we have $w(3) = 2$, which means there are active factors (rewards and parents' goals).

Table 5.7: Model-independent checks with the edge design in Table 5.6.

Reward x_1	Friends x_2	Parents goal x_3	Punishment x_4
28	-7.12	-24.5	9.5
28	7.12	24.5	9.5

Analysis of the data in Table 5.6 using linear regression and the software package SPSS reveals rewards as the active variable and gives an estimated linear model $y = 41.750 + 15.328x_1 + \epsilon$, with $R - sg = 86\%$ of mean 28.48 and standard deviation $\sigma = 11.85$. The result also shows that the residual is normal because the p-value (0.150) is more than 0.05.

From the above, we can observe that the robust edge design method shows two active factors, rewards and parents' goals, and the regression analysis method shows one active factor, reward. We thus conclude that the rewards is an active factor

Table 5.8: Step by step calculations for the analysis of Example 9 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	17	17.80	25.185	1	No
1	9.5	9.95	14.07	2	No
2	8.31	8.705	12.31	2	No
3	7.12	7.45	10.54	2	yes

with linear contributions, while the parents goals influence the response, but not in a linear manner.

5.4.2 The south Al-Shamly School

Example 10 : Let $n = 4$.

Table 5.9: One replication of the runs of the design used for south Al-Shamly School.

run	Friends x_1	Parent goal x_2	Punishment x_3	Age x_4	Y	Student number
1	1	1	1	1	27	50
2	-1	1	1	-1	47	86
3	-1	-1	1	1	34	33
4	-1	1	-1	1	43	19
5	-1	1	1	1	45	47
6	-1	-1	1	-1	18	40
7	-1	-1	-1	1	48	15
8	-1	1	-1	-1	17	21

Analysing the data in Table 5.9 as been described previously; the results are shown in tables 5.10, 5.11. We thus have $w(3) = 2$ active factors: parents' goals and age.

Analysing the data in Table 5.9 sing linear regression and the software package SPSS reveals no active variables and gives an estimated linear model $y = 23 + \epsilon$, with $R - sg = 50.7\% \epsilon$ of mean 34.87 and standard deviation $\sigma = 12.84$. The result also shows that the residual is normal, because the p-value (0.150) is more than 0.05.

From the above, we can observe that the robust edge design method shows two active factors, parents' goals, and age and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

Table 5.10: Model-independent checks with the edge design in Table 5.9.

Friends x_1	Parent goal x_2	Punishment x_3	age x_4
-18	29	-14	26
18	29	14	26

Table 5.11: Step by step calculations for the analysis of Example 10 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	22	23.04	32.59	0	No
1	18	18.85	26.66	1	No
2	16	16.76	23.70	2	No
3	14	14.66	20.74	2	Yes

5.4.3 Almail School

Example 11 : Let $n = 4$.

Table 5.12: One replication of the runs of the design used for Almail School.

run	Achievements x_1	Behavior x_2	Teacher x_3	Health x_4	Y	Student number
1	1	1	1	1	14	9
2	-1	1	1	-1	15	28
3	-1	-1	1	1	29	12
4	-1	1	-1	1	24	17
5	-1	1	1	1	20	7
6	-1	-1	1	-1	18	14
7	-1	-1	-1	1	25	57
8	-1	1	-1	-1	27	23

The edge analysis has been described previously; the results are shown in tables 5.13, 5.14. We thus have $w(2) = 1$ active factor: achievements.

Analysing the data in Table 5.12 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 19.5 + \epsilon$, with $R - sg = 64.4\%$ of mean 21.5 and standard deviation $\sigma = 5.58$. The result also shows that the residual is normal, because the p-value (0.150) is more than 0.05.

From the above, we observe that the robust edge design method shows one active factor, achievements, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

Table 5.13: Model-independent checks with the edge design in Table 5.12.

Achievements x_1	Behavior x_2	Teacher x_3	Health x_4
-6	-3	4	-3
6	3	4	3

Table 5.14: Step by step calculations for the analysis of Example 11 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	3.5	3.66	5.185185185	1	No
1	3	3.14	4.44	1	No
2	3	3.14	4.44	1	Yes

5.4.4 Ibn Habban School

Example 12 : Let $n = 4$.

Table 5.15: One replication of the runs of the design used for Ibn Habban.

run	Reward x_1	Friends x_2	Parents goal x_3	Punishment x_4	Y	Student number
1	1	1	1	1	33	43
2	-1	1	1	-1	20.5	73
3	-1	-1	1	1	28	21
4	-1	1	-1	1	21	71
5	-1	1	1	1	21	143
6	-1	-1	1	-1	24	14
7	-1	-1	-1	1	24	13
8	-1	1	-1	-1	36	6

The edge analysis has been described previously; the results are in tables 5.16, 5.17. We thus have $w(3) = 2$ active factors: rewards and punishment.

Analysing the data in Table 5.15 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 32.250 + \epsilon$, with $R - sg = 47.7\% \epsilon$ of mean 25.93 and standard deviation $\sigma = 5.78$. The result also shows that the residual is normal, because the p-value (0.186) is more than 0.05.

From the above, we observe that the robust edge design method shows two active factors, rewards and punishment, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

5.4.5 Sharf School

Example 13 : Let $n = 4$.

The edge analysis has been described previously; the results are shown in tables 5.19, 5.20. We thus have $w(2) = 1$ active factor: class environment.

Analysing the data in Table 5.18 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 41 + \epsilon$,

Table 5.16: Model-independent checks with the edge design in Table 5.15.

Reward x_1	Friends x_2	Parents goal x_3	Punishment x_4
12	-3.5	4	-15
12	3.5	4	15

Table 5.17: Step by step calculations for the analysis of Example 12 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	8	8.38	11.85	2	No
1	4	4.19	5.92	2	No
2	3.75	3.92	5.55	2	No
3	3.5	3.66	5.18	2	Yes

Table 5.18: One replication of the runs of the design used Example 13 for Sharf School.

run	class En x_1	Curriculum x_2	Media x_3	Tools Ed x_4	Y	Student number
1	1	1	1	1	20	32
2	-1	1	1	-1	50	39
3	-1	-1	1	1	21	8
4	-1	1	-1	1	30	54
5	-1	1	1	1	45	29
6	-1	-1	1	-1	39	10
7	-1	-1	-1	1	33	55
8	-1	1	-1	-1	47	79

with $R - sg = 79.4\%\epsilon$ of mean 38 and standard deviation $\sigma = 8.2$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05. From the above, we observe that the robust edge design method shows one active factor, class environment, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

Example 14 : Let $n = 4$.

The edge analysis has been described previously; the results are shown in tables 5.22, 5.23. We thus have $w(3) = 2$ active factors: tools and friends. Analysing the data in Table 5.21 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 29.5 + \epsilon$, with $R - sg = 74.3\%\epsilon$ of mean 28.1250 and standard deviation $\sigma = 6.95$. The result also shows that the residual is normal because the p-value (0.150) is more than

Table 5.19: Model-independent checks with the edge design in Table 5.18.

Class En x_1	Curriculum x_2	Media x_3	Tools Ed x_4
-25	11	-12	-17
25	11	12	17

Table 5.20: Step by step calculations for the analysis of Example 13 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	14.5	15.18	21.48	1	No
1	12	12.57	17.77	1	No
2	11.5	12.04	17.03	1	Yes

Table 5.21: One replication of the runs of the design used Example 14 for Sharf School.

run	Media x_1	Tools x_2	Reward x_3	Friends x_4	Y	Student number
1	1	1	1	1	21	8
2	-1	1	1	-1	27	24
3	-1	-1	1	1	26	61
4	-1	1	-1	1	25	11
5	-1	1	1	1	25	23
6	-1	-1	1	-1	39	4
7	-1	-1	-1	1	23	43
8	-1	1	-1	-1	39	36

0.05.

From the above, we observe that the robust edge design method shows two active factors, class tools and friends, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

5.4.6 Amam Mohmmed School

Example 15 : Let $n = 4$.

The edge analysis has been described previously; the results are shown in tables 5.25, 5.26. We thus have $w(3) = 2$ active factors: tools and friends.

Analysing the data in Table 5.24 using linear regression with the software package SPSS reveals no active variables and gives an estimated liner model $y = 39.125 + \epsilon$, with $R - sg = 44.8\% \epsilon$ of mean 32.40 and standard deviation $\sigma = 6.576$. The result

Table 5.22: Model-independent checks with the edge design in Table 5.21.

Media x_1	Tools x_2	Reward x_3	Friends x_4
-4	-12	3	-14
4	12	3	14

Table 5.23: Step by step calculations for the analysis of Example 14 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	8	8.38	11.85	1	No
1	4	4.19	5.92	2	No
2	3.5	3.66	5.18	2	No
3	3	3.14	4.44	2	Yes

Table 5.24: One replication of the runs of the design used for Amam Mohammed School.

Run	Tools Ed	Reward	Friends	Parents'goal	Y	Student number
1	1	1	1	1	36.5	53
2	-1	1	1	-1	38.5	7
3	-1	-1	1	1	25.5	41
4	-1	1	-1	1	32.25	63
5	-1	1	1	1	21.5	118
6	-1	-1	1	-1	36	18
7	-1	-1	-1	1	40	21
8	-1	1	-1	-1	29	13

also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows two active factors, tools and friends, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

Ibrahim Althuwaini School

Example 16 : let $n = 4$ and, Achievements x_1 , Behavior x_2 , Teacher x_3 and

Table 5.25: Model-independent checks with the edge design in Table 5.24.

Tools Ed	Reward	Friends	Parents' goal
15	2.5	-14.5	3.25
15	2.5	14.5	3.25

Table 5.26: Step by step calculations for the analysis of Example 15 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	8.87	9.29	13.14	2	No
1	3.25	3.40	4.81	2	No
2	2.87	3.01	4.25	2	No
3	2.5	2.61	3.70	2	Yes

Health x_4 .

Table 5.27: One replication of the runs of the design used Example 16 for Ibrahim Althuwaini school.

Run	achievements	Behaviour	Teacher	Health	Y	Student number
1	1	1	1	1	18.5	4
2	-1	1	1	-1	28	38
3	-1	-1	1	1	17.5	14
4	-1	1	-1	1	38	45
5	-1	1	1	1	39.5	3
6	-1	-1	1	-1	18.5	26
7	-1	-1	-1	1	15.5	22
8	-1	1	-1	-1	43	15

The edge analysis has been described previously; the results are shown in tables 5.28 and 5.29. We thus have $w(3) = 2$ active factors: achievements and behaviour.

Analysing the data in Table 5.27 using linear regression with the software package SPSS reveals one active variable, behaviour, and gives an estimated linear model $y = 18.62 + 9.781x_2 + \epsilon$, with $R-sg = 87.5\% \epsilon$ of mean 27.31 and standard deviation $\sigma = 11.34$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows two active

Table 5.28: Model-independent checks with the edge design in Tables 5.27.

Achievements x_1	Behaviour x_2	Teacher x_3	Health x_4
-21	9.5	2	-5
21	9.5	2	5

Table 5.29: Step by step calculations for the analysis of Example 16 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	7.25	7.59	10.74	1	No
1	5	5.23	7.407	2	No
2	3.5	3.66	5.18	2	No
3	5	5.23	7.40	2	Yes

factors, achievements and behaviour, and the regression analysis method shows one active factor, behaviour. We thus conclude that behaviour is an active factor with a linear contribution, while achievements influence the response, but not in a linear manner.

Example 17 : let $n = 4$ and, Study time (x_1), Class En (x_2), Curriculum (x_3) and Media (x_4).

Table 5.30: One replication of the runs of the design used Example 17 for Ibrahim Althuwaini school.

Run	Study time	Class En	Curriculum	Media	Y	Student number
1	1	1	1	1	21.5	18
2	-1	1	1	-1	36.5	19
3	-1	-1	1	1	28.5	4
4	-1	1	-1	1	31	13
5	-1	1	1	1	45.5	2
6	-1	-1	1	-1	18	29
7	-1	-1	-1	1	18	5
8	-1	1	-1	-1	39.5	7

The edge analysis has been described previously; the results are shown in tables 5.31 and 5.32. We thus have $w(3) = 2$ active factors: study time and class environment.

Analysing the data in Table 5.30 using linear regression with the software package SPSS reveals one active variable, class environment, and gives an estimate linear model $y = 18.25 + 9.125x_2 + \epsilon$, with $R - sg = 84.7\%$ of mean 29.81 and standard deviation $\sigma = 10.25$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows two active

Table 5.31: Model-independent checks with the edge design in Tables 5.30.

Study time	class En	Curriculum	Media
-24	18.5	10.5	-8.5
24	18.5	10.5	8.5

Table 5.32: Step by step calculations for the analysis of Example 17 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	14.5	15.18	21.48	1	No
1	10.5	10.99	15.55	2	No
2	9.5	9.95	14.07	2	No
3	8.5	8.90	12.59	2	Yes

factors, study time and class environment, and the regression analysis method shows one active factor, class environment. We thus conclude that class environment is an active factor with a linear contribution, while study time influences the response, but not in a linear manner.

Example 18 : let $n = 4$ and, Media(x_1), Tool ed(x_2), Reward (x_3) and Friends (x_4).

Table 5.33: One replication of the runs of the design used Example 18 for Ibrahim Althuwaini school.

Run	Media	Tools Ed	Reward	Friends	Y	Student number
1	1	1	1	1	28.5	4
2	-1	1	1	-1	14	33
3	-1	-1	1	1	31.5	49
4	-1	1	-1	1	15.5	27
5	-1	1	1	1	18	29
6	-1	-1	1	-1	36.5	14
7	-1	-1	-1	1	37.5	66
8	-1	1	-1	-1	25.5	128

The edge analysis has been described previously; the results are shown in tables 5.34 and 5.35. We thus have $w(2) = 1$ active factor: educational tools.

Analysing the data in Table 5.33 using linear regression with the software package SPSS reveals one active variable, educational tools, and gives an estimate liner model $y = 34.625 - 9.156x_2 + \epsilon$, with $R - sg = 91.5\%$ of mean 28.875 and standard deviation $\sigma = 9.237$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows one active factor, educational tools, and the regression analysis method shows one active factor, also educational tools. We thus conclude that educational tools is an active factor with a linear contribution.

Table 5.34: Model-independent checks with the edge design in Tables 5.33.

Media	Tools Ed	Reward	Friends
10.5	-22.5	-6	-10
10.5	22.5	6	10

Table 5.35: Step by step calculations for the analysis of Example 18 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	10.25	10.73	15.18	1	No
1	10	10.47	14.81	1	No
2	8	8.38	11.85	1	Yes

Example 19 : let $n = 4$ and Tool ed(x_1), Reward (x_2), Friends (x_3) and Parents goal(x_4).

Table 5.36: One replication of the runs of the design used Example 19 for Ibrahim Althuwaini school.

Run	Tools Ed	Reward	Friends	Parents goal	Y	Student number
1	1	1	1	1	36.5	19
2	-1	1	1	-1	27.5	74
3	-1	-1	1	1	40	65
4	-1	1	-1	1	36.5	41
5	-1	1	1	1	31.5	49
6	-1	-1	1	-1	41	50
7	-1	-1	-1	1	43.5	16
8	-1	1	-1	-1	39.5	6

The edge analysis has been described previously; the results are shown in tables 5.37 and 5.38. We thus have $w(3) = 2$ active factors: educational tools and reward. Analysing the data in Table 5.36 using linear regression with the software package SPSS reveals one active variable, rewards, and gives an estimated linear model $y = 41.375 - 4.469x_2 + \epsilon$, with $R - sg = 87.5\% \epsilon$ of mean 37 and standard deviation $\sigma = 5.27$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows two active

Table 5.37: Model-independent checks with the edge design in Tables 5.36.

Tools Ed	Reward	Friends	Parents goal
15	-13.5	-3.5	-3
15	13.5	3.5	3

Table 5.38: Step by step calculations for the analysis of Example 19 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	4.25	4.45	6.29	1	No
1	3.5	3.66	5.18	1	No
2	3.25	3.40	4.81	2	No
3	3	3.14	4.44	2	Yes

factors, educational tools and rewards, and the regression analysis method shows active factor, reward. We thus conclude that rewards is an active factor with a linear contribution.

5.4.7 Sahama School

Example 20 : Let $n = 4$ and x_1 Health, x_2 Family Circumstance, x_3 Student numbers class and x_4 Desire.

Table 5.39: One replication of the runs of the design used Example 20 for Sahama School.

Run	x_1	x_2	x_3	x_4	Y	Student number
1	1	1	1	1	23	14
2	-1	1	1	-1	36	63
3	-1	-1	1	1	48	45
4	-1	1	-1	1	39	66
5	-1	1	1	1	50	8
6	-1	-1	1	-1	26.5	25
7	-1	-1	-1	1	41	99
8	-1	1	-1	-1	33	30

The edge analysis has been described previously; the results are shown in tables 5.40 and 5.41. We thus have $w(2) = 1$ active factor: health.

Analysing the data in Table 5.39 using linear regression with the software package SPSS reveals two active variables, health and desire, and gives an estimated linear model $y = 424.125 - 13.219x_1 + 7.156x_4 + \epsilon$, with $R - sg = 89.6\%$ of mean 37.06 and standard deviation $\sigma = 9.511$. The result also shows the residual is normal because the p-value (0.150) is more than 0.05.

From the above, we observe that the robust edge design method shows one active factor, health, and the regression analysis method shows two active factors,

Table 5.40: Model-independent checks with the edge design in Table 5.39.

Health	Family Circumstance	Student numbers class	Desire
-27	9.5	7	6
27	9.5	7	6

Table 5.41: Step by step calculations for the analysis of Example 20 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	8.25	8.64	12.22	1	No
1	7	7.33	10.37	1	No
2	6.5	6.80	9.62	1	Yes

health and desire. We thus conclude that health is an active factor with a linear contribution, while desire influences the response, but not in a linear manner.

Example 21 : Let $n = 4$.

Table 5.42: One replication of the runs of the design used Example 21 for Sahama School.

run	class	En	Curriculum	Media	Tools Ed	Y	Student number
1	1	1	1	1	1	27	20
2	-1	1	1	1	-1	49	3
3	-1	-1	-1	1	1	24.5	21
4	-1	1	1	-1	1	44	101
5	-1	1	1	1	1	39	60
6	-1	-1	1	1	-1	28	19
7	-1	-1	-1	-1	1	40	36
8	-1	1	1	-1	-1	33	12

The edges analysis has been described previously; the results are shown in tables 5.43, 5.44. We thus have $w(2) = 1$ active factor: curriculum.

Analysing the data in Table 5.42 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 29 + \epsilon$, with $R - sg = 52\%$ of mean 35.56 and standard deviation $\sigma = 8.805$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows one active

Table 5.43: Model-independent checks with the edge design in Table 5.42.

class En	Curriculum	Media	Tools Ed
-12	21	-15.5	11
12	21	15.5	11

Table 5.44: Step by step calculations for the analysis of Example 21 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	13.75	14.40	20.37	1	No
1	12	12.57	17.77	1	No
2	11.5	12.04	17.037	1	Yes

factor curriculum, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

5.4.8 Alfadeelah School

Example 22 : Let $n = 4$.

Table 5.45: One replication of the runs of the design used for Alfadeelah School.

run	Media	Tools Ed	Reward	Friends	Y	Student number
1	1	1	1	1	35	3
2	-1	1	1	-1	44	7
3	-1	-1	1	1	32.5	43
4	-1	1	-1	1	21.5	54
5	-1	1	1	1	43.25	9
6	-1	-1	1	-1	36.25	6
7	-1	-1	-1	1	42	48
8	-1	1	-1	-1	43	16

The edge analysis has been described previously; the results are shown in tables 5.46, 5.47. We thus have $w(2) = 1$ active factor: friends.

Analysing the data in Table 5.45 using linear regression with the software package SPSS reveals no active variables and gives an estimated linear model $y = 37 + \epsilon$, with $R - sg = 20.3\% \epsilon$ of mean 37.18 and standard deviation $\sigma = 7.685$. The result also shows that the residual is normal because the p-value (0.200) is more than 0.05.

From the above, we observe that the robust edge design method shows one active

Table 5.46: Model-independent checks with the edge design in Table 5.45.

Media	Tools Ed	Reward	Friends
-8.25	7.75	-9.5	-21.5
8.25	7.75	9.5	21.5

Table 5.47: Step by step calculations for the analysis of Example 22 with the edge of the design.

p	Median	$\tilde{\sigma}(p)$	$k2^{\frac{1}{2}}\tilde{\sigma}(p)$	$\omega(p)$	$\omega(p) < p?$
0	8.875	9.29	13.14	1	No
1	8.25	8.64	12.22	1	No
2	8	8.38	11.85	1	Yes

factor, friends, and the regression analysis method shows no active factor. We thus conclude that there is no active factor and the model is not linear.

5.5 Conclusion

We can say that the analysis with the edge designs given results that are related and similar to the analysis that is performed with the whole data. Therefore, rewards, behaviour, class environment, educational tools and health were the factors that most affected students mathematical achievement.

Chapter 6

Conclusion and Future work

In this chapter, we survey the work detailed in this thesis and recommend further avenues of research. In section 6.1, we outline the exploration and result; in section 6.2, we examine different areas of research that are comparable and propose how these areas may be entwined in future investigations.

6.1 Survey

The research aim of this thesis has been to study and develop the analysis of experimental designs and their construction.

In Chapter 1, we presented some of the essential ideas underpinning this project. In Chapter 2, we gave an overview of the screening experiments and reviewed the literature relevant to the project.

In Chapter 3, we established the first sub-project: constructing and analysing edge designs from skew-symmetric SSDs. Edge designs were introduced by Elster and Neumaier (1995) using conference matrices and proved to be robust. We gave details of the methodology and provided illustrative examples of this new approach. We also showed that the new designs possess good D-efficiencies when applied to first-order models, and then completed the experiment with interactions in the second stage. We also demonstrated the application of models for new constructions.

In Chapter 4, we presented the second sub-project, constructing and analysing two-level supersaturated designs from Toeplitz matrices. We also studied the properties of the constructed designs and used established criteria to evaluate the constructed SSDs. We then gave detailed examples regarding this approach and considered the performance of these designs with respect to different data analysis methods.

In Chapter 5, we extended the work of Chapter 3 by presenting the examples and comparison of the constructed designs using real data. We detailed the

methodology and provided illustrative examples of this new approach. We also showed that the new designs have the best realisable design analysis than analytical normality.

6.2 Possible Future Research

In this section, we offer methods that could expand on the research conducted for this thesis.

In Chapter 3, we could have used new construction methods based on D-optimal designs in the case of k factors where $k = 2(\text{mod } 4)$. The edge analysis could be investigated and expanded to include more than two edges in a multi-level design. Other researchers could also include other SDSs and work with a more general structure, such as the difference in families. Symmetric designs could also be applied. The structure, properties and results would likely be different. Further investigation in this direction is needed.

Chapter 4 could be expanded by constructing and analysing four-level SSDs from Toeplitz matrices. We have two cases to describe how four-level SSDs can be obtained from Toeplitz matrices. The first is a fold-over of any two-level designs with $4n$ runs and kn factors-let X be an n -dimensional, two-level vector comprising equal numbers of -1 and 1 . We develop the constructed method for four-level SSDs from any two-level designs.

Chapter 5 also presents possibilities for future research. One is to give 16-question questionnaires to students and calculate the level of achievement for the targeted student from the suitable sample. All that is written in Table 6.1 in which we explain the data collected. Another possibility is to compare mathematics achievement between the middle stages of Bahrain and Saudi schools.

Table 6.1: Notes to be considered about the tables of fututr data.

Factor	signal	Description
Sex	+	Male
	-	Female
Achievement in primary	+	Excellent
	0	Moderate
	-	Poor
Behaviour	+	Excellent
	0	Moderate
	-	bad
Teacher	+	good teaching method
	0	Moderate
	-	bad
Health	+	Healthy
	0	Moderate
	-	unhealthy
Family circumstances	+	Good
	0	Moderate
	-	poor
Students number	+	20
	0	fewer than 20
	-	More than 20
Desire	+	high
	0	Moderate
	-	none
Study time	+	3 hours
	0	less than 3 hours
	-	more than 3 hours

Table 6.1: (cont.)

Class environment	+	Attractive
	0	Moderate
	–	Unattractive
Curriculum	+	easy
	0	Moderate
	–	Hard
Media	+	less than 3 hours
	0	3 hours
	–	More than 3 hours
Tools	+	Equipped
	0	Moderate
	–	Unequipped
Reward	+	Found
	0	Moderate
	–	Absent
Friend	+	Give advice
	0	Moderate
	–	Do not give advice
Parents goals	+	Suitable
	0	Moderate
	–	Unsuitable
Punishment	+	no punishment
	0	Moderate
	–	Punishment
Age	+	12 – 15
	–	More than 15 and Younger than 12
Y(Response)	Grades	First term Exams in 2016

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Chapter 4

Table 2: The properties for new SSD_1 with n runs and m factors.

n	m	Perm.	Es^2	LB	Eff	$smax$
12	18	2	9.9346	5.3904	0.5426	4
12	24	3	13.3333	7.3992	0.5549	8
12	30	4	12.9839	8.5768	0.6606	8
12	36	5	12.4444	9.3506	0.7514	8
12	42	6	12.2462	9.8980	0.8082	8
12	48	7	12.4539	10.3056	0.8275	8
12	66	10	12.7776	11.0769	0.8669	8
12	72	11	12.8075	11.2471	0.8782	8
12	78	12	12.7393	11.3908	0.8941	8
16	24	2	11.5942	6.6783	0.5760	8
16	32	3	12.3871	9.3591	0.7556	8
16	40	4	13.4564	10.9402	0.8130	8
16	48	5	13.9574	11.9830	0.8585	8
20	90	8	19.9231	16.7948	0.8430	12
20	100	9	20.3055	17.2249	0.8483	12
20	110	10	20.7293	17.5761	0.8479	12
24	84	6	21.2324	18.4054	0.8669	12
24	108	8	22.2305	19.8944	0.8949	12
28	126	8	27.3575	22.9973	0.8406	16
28	154	10	27.3240	24.1026	0.8821	16
28	182	12	27.6646	24.8660	0.8988	16
36	162	8	34.6464	29.2089	0.8431	20
36	198	10	34.9427	30.6379	0.8768	20
36	234	12	35.3918	31.6253	0.8936	20
40	180	8	36.3998	32.3163	0.8878	20
40	220	10	37.4263	33.9070	0.9060	20
40	260	12	38.0160	35.0064	0.9208	20
48	216	8	43.6589	38.5330	0.8826	24
48	264	10	44.6319	40.4472	0.9062	24
48	312	12	45.0913	41.7705	0.9264	24
52	234	8	48.9813	41.6420	0.8502	24
52	286	10	49.3568	43.7179	0.8858	24
56	252	8	50.6702	44.7513	0.8832	28
56	364	10	51.9704	46.9889	0.9041	28
56	308	12	52.8790	48.5361	0.9179	28
60	270	8	55.8381	47.8609	0.8571	28
60	330	10	56.9430	50.2602	0.8826	28
60	390	12	57.3711	51.9193	0.9050	28

Table 2: (cont.)

n	m	Perm.	Es^2	LB	Eff	$smax$
64	288	8	57.8304	50.9706	0.8814	28
64	352	10	59.2758	53.5316	0.9031	28
64	416	12	60.1209	55.3027	0.9199	28
76	342	8	70.1850	60.3008	0.8592	32
76	418	10	71.5720	63.3467	0.8851	32
76	494	12	72.6325	65.4535	0.9012	32
84	378	8	77.5744	66.5214	0.8575	36
84	462	10	79.2214	69.8906	0.8822	36
84	546	12	79.9034	72.2212	0.9039	36
88	396	8	79.2124	69.6318	0.8791	36
88	484	10	81.2257	73.1627	0.9007	36
88	572	12	82.1792	75.6052	0.9200	36
96	342	8	86.8677	75.8528	0.8732	36
96	528	10	88.3657	79.7069	0.9020	40
96	624	12	89.6348	82.3733	0.9190	40
100	450	8	91.6568	78.9634	0.8615	40
100	550	10	93.3670	82.9792	0.8887	40
100	650	12	94.5972	85.7574	0.9066	40
108	486	8	99.1737	85.1846	0.8589	40
108	594	10	100.6531	89.5237	0.8894	40
108	702	12	101.9681	92.5258	0.9074	40
120	540	8	107.6562	94.5168	0.8780	44
120	660	10	110.2497	99.3407	0.9011	44
120	780	12	111.6461	102.6785	0.9197	40
124	558	8	112.9581	97.6275	0.8643	44
124	682	10	114.8829	102.6131	0.8932	44
124	806	12	116.8868	106.0628	0.9074	44
136	612	8	121.9276	106.9600	0.8772	48
136	748	10	124.7287	112.4304	0.9014	48
136	884	12	126.5811	116.2158	0.9181	48
144	648	8	129.2859	113.1817	0.8754	48
144	792	10	132.2000	118.9754	0.9000	48
144	936	12	134.0367	122.9845	0.9175	52
148	666	8	134.4424	116.2925	0.8650	48
148	814	10	136.8497	122.2479	0.8933	52
148	962	12	139.0600	126.3689	0.9087	52
160	720	8	143.1123	125.6252	0.8778	52
160	880	10	146.8115	132.0655	0.8996	52
160	1040	12	148.8532	136.5222	0.9172	52

Table 3: permutation generators for SSD_1 with n runs and m factors

n	m	Perm.	Generators
12	18	2	R1=[6 10 4 5 11 9 2 3 7 12 8 1] R2=[8 1 11 10 7 4 9 3 5 6 12 2]
12	24	3	R1=[3 11 7 2 10 6 5 8 12 9 1 4]R2=[5 2 7 1 11 8 3 9 10 6 12 4]R3=[12 7 11 4 1 6 2 10 8 3 9 5]
12	30	4	R1=[9 10 6 7 11 12 5 1 3 8 4 2],R2=[6 3 2 1 10 11 12 9 4 5 8 7],R3=[10 4 3 6 1 11 8 5 9 12 7 2],R4=[1 2 6 9 12 7 8 10 11 5 3 4]
12	36	5	R1=[9 4 7 6 1 8 2 11 12 10 5 3],R2=[11 12 5 2 4 1 9 7 3 8 10 6],R3=[7 10 6 3 1 12 8 5 4 11 9 2],R4=[3 5 6 10 7 8 4 11 9 1 2 12],R5=[11 10 8 12 1 7 4 6 5 9 3 2]
12	42	6	R1=[3 1 12 2 10 4 8 9 5 6 7 11],R2=[7 10 11 12 9 6 4 3 8 2 5 1],R3=[10 8 9 5 4 2 12 1 11 6 7 3],R4=[1 10 3 9 12 2 11 6 4 5 8 7],R5=[7 2 8 3 6 1 9 4 5 11 10 12],R6=[11 4 1 8 7 9 10 2 12 6 3 5]
12	48	7	R1=[1 4 12 2 3 5 11 8 10 7 6 9],R2=[3 6 4 5 8 7 2 10 12 11 1 9],R3=[9 10 5 11 1 8 2 3 12 6 4 7],R4=[4 1 5 7 11 10 3 6 12 8 2 9],R5=[4 7 8 2 5 12 9 11 1 6 3 10],R6=[11 9 3 2 7 6 4 5 1 12 8 10],R7=[4 3 12 1 6 7 8 2 5 9 11 1]
12	66	10	R1=[8 1 11 9 4 3 7 12 10 6 5 2],R2=[12 2 3 7 6 4 1 8 9 5 10 11], R3=[5 11 3 12 1 4 2 8 7 9 10 6],R4=[7 6 12 10 5 2 3 9 8 11 1 4],R5=[1 3 8 12 9 4 7 2 10 5 11 6],R6=[3 2 12 10 6 7 1 5 9 4 11 8],R7=[5 10 1 12 11 2 4 8 6 7 9 3],R8=[9 3 12 1 2 11 7 5 6 10 8 4], R9=[6 2 1 12 11 10 5 4 3 8 9 7],R10=[10 12 9 7 4 2 6 11 5 8 3 1]
12	72	11	R1=[3 2 6 12 9 1 5 10 8 4 7 11],R2=[9 4 5 1 2 7 11 8 3 12 10 6],R3=[11 1 6 7 2 8 12 9 5 4 10 3],R4=[5 6 9 3 12 1 4 10 8 7 2 11],R5=[10 5 2 7 1 9 6 4 8 11 3 12], R6=[1 2 12 8 7 3 11 9 10 4 5 6],R7=[9 4 5 11 6 7 10 8 2 3 12 1], R8=[5 10 12 2 1 4 3 9 6 7 8 11],R9=[2 3 4 5 11 12 6 7 1 9 10 8],R10=[11 8 5 4 9 6 3 2 12 7 1 10],R11=[12 2 8 6 3 9 10 5 4 1 11 7]
12	78	12	R1=[7 4 8 11 1 10 5 2 3 6 12 9],R2=[5 7 2 9 6 8 10 12 4 1 3 11],R3=[9 3 7 8 11 4 5 12 2 1 6 10],R4=[12 7 3 9 5 2 8 1 10 6 4 11],R5=[3 1 4 2 5 7 10 9 12 6 11 8],R6=[6 1 9 11 12 10 8 4 2 5 7 3],R7=[1 9 10 11 8 6 2 12 3 4 5 7],R8=[1 8 10 7 11 2 3 4 5 9 6 12],R9=[2 4 6 10 3 8 12 11 1 5 9 7],R10=[4 5 10 12 8 7 3 11 9 1 2 6],R11=[8 7 6 11 4 12 2 10 5 3 9 1],R12=[10 11 9 4 8 5 1 6 2 12 3 7]

Table 3: Cont.

n	m	Perm.	Generators
16	24	2	R1=10 15 12 8 2 14 7 11 16 4 3 5 1 6 13 9],R2=[1 16 10 14 3 8 15 7 13 11 5 2 9 12 6 4]
16	32	3	R1=[12 11 15 1 16 6 8 4 14 9 3 13 7 5 2 10],R2=[5 15 4 2 13 14 9 7 1 8 10 16 3 12 11 6],R3=[11 9 1 10 5 3 2 6 7 14 4 13 15 16 8 12]
16	40	4	R1=[6 10 5 11 15 12 7 16 2 13 1 4 9 8 14 3], R2=[5 4 12 9 7 2 3 6 14 11 8 16 10 1 13 15], R3=[2 5 16 7 13 12 15 14 11 10 4 6 1 8 9 3],R4=[16 4 3 14 1 11 5 15 13 9 2 10 6 12 7 8]
16	48	5	R1=[5 6 9 16 10 12 15 11 3 1 2 14 8 13 4 7],R2=[13 1 8 6 14 16 5 3 9 10 7 11 2 12 4 15],R3=[11 16 3 15 14 7 9 4 8 13 2 12 10 6 1 5],R4=[3 11 8 1 4 2 15 7 12 10 9 14 16 5 6 13],R5=[1 5 13 7 6 10 4 14 2 12 11 9 8 3 16 15]
20	90	8	R1=[9 7 16 3 12 1 15 8 10 5 17 18 11 20 13 4 6 14 2 19], R2=[5 11 3 7 15 2 19 16 12 6 17 8 13 1 14 18 4 10 9 20], R3=[11 19 10 8 13 2 20 4 14 1 16 18 9 7 12 17 3 15 5 6],R4=[7 14 12 5 11 3 9 16 1 17 15 13 8 10 18 4 20 2 6 19],R5=[19 4 1 2 7 6 17 9 8 10 13 15 5 11 3 12 20 16 18 14],R6=[18 16 11 9 7 17 3 4 8 14 13 5 12 2 1 19 10 6 15 20],R7=[20 15 19 12 2 13 17 10 4 7 1 16 14 9 6 3 11 5 8 18],R8=[12 8 4 6 11 1 9 13 20 15 3 2 17 5 18 14 19 16 7 10]
20	100	9	R1=[18 7 12 2 15 5 19 13 3 9 14 11 1 8 20 4 6 17 10 16],R2=[8 19 7 16 3 13 5 1 10 11 20 18 6 15 14 9 12 2 4 17],R3=[12 8 10 7 6 4 19 3 14 9 16 18 2 15 17 13 5 11 20 1],R4=[18 11 17 7 2 9 16 15 4 6 13 5 3 12 8 20 10 19 14 1], R5=[9 10 8 5 3 16 7 14 13 12 18 1 4 20 15 17 19 6 2 11], R6=[15 7 12 17 8 14 10 13 20 1 19 11 6 2 9 16 3 18 5 4], R7=[15 8 20 11 5 6 4 9 19 1 13 7 3 17 10 16 2 12 18 14],R8=[2 18 9 5 11 7 14 8 4 15 12 1 6 17 19 3 16 10 13 20],R9=[9 5 12 8 14 10 3 16 15 6 18 13 17 4 7 11 2 20 19 1]

Table 4: The properties for new SSD_2 with n runs and $m = k(n - 1)$ factors.

n	m	Perm.	Es^2	LB	Eff	$smax$
6	10	1	4	4	1	2
8	14	1	4.9231	4.9231	1	4
10	27	2	7.9202	7.9202	1	6
12	22	1	6.8571	6.8571	1	8
12	33	2	9	9	1	8
12	44	3	10.0465	10.0465	1	8
12	55	4	10.6667	10.6667	1	8
12	66	5	11.0769	11.0769	1	8
12	77	6	11.3684	11.3684	1	8
12	88	7	11.5862	11.5862	1	8
12	99	8	11.7551	11.7551	1	8
12	110	9	11.8899	11.8899	1	8
14	26	1	9.12	7.8400	0.8596	6
14	39	2	10.9096	10.5641	0.9683	10
14	52	3	11.8673	11.5294	0.9715	10
14	65	4	12.3846	12.3385	0.9963	10
14	91	6	13.1194	13.1116	0.9994	10
14	104	7	13.3264	13.3204	0.9996	10
18	34	1	11.2442	9.8182	0.8732	10
18	51	2	14.0392	13.1859	0.9392	10
18	85	4	15.6706	15.51	0.9897	14
18	119	6	16.5384	16.52	0.9986	14
18	170	9	17.27	17.2544	0.9991	14
20	38	1	10.8108	10.8108	1	8
20	57	2	14.2857	14.2857	1	12
20	76	3	16	16	1	12
20	114	5	17.6991	17.6991	1	12
20	133	6	18.1818	18.1818	1	12
20	152	7	18.5430	18.5430	1	12
20	171	8	18.8235	18.8235	1	12
20	190	9	19.0476	19.0476	1	16
20	228	11	19.3833	19.3833	1	16
20	342	17	19.9413	19.9413	1	16
20	361	18	20	20	1	16
24	46	1	12.8000	12.8000	1	12
24	69	2	16.9412	16.9412	1	12
24	92	3	18.9890	18.9890	1	12

Table 4: (cont.)

n	m	Perm.	Es^2	LB	Eff	$smax$
26	50	1	15.6767	13.7959	0.8800	10
26	75	2	19.3023	18.4951	0.9582	14
26	100	3	21.2477	20.4848	0.9641	14
26	125	4	22.4279	21.8870	0.9759	14
26	150	5	23.0626	22.6846	0.9836	14
28	54	1	14.7925	14.7925	1	12
28	81	2	19.6000	19.6000	1	16
28	108	3	21.9813	21.9813	1	16
28	135	4	23.4030	23.4030	1	16
30	58	1	17.6286	15.7895	0.8957	14
30	87	2	22.0230	21.1590	0.9608	14
30	116	3	24.1883	23.4783	0.9706	18
32	62	1	16.7869	16.7869	1	12
32	93	2	22.2609	22.2609	1	16
32	124	3	24.9756	24.9756	1	16
32	155	4	26.5974	26.5974	1	16
38	74	1	21.8304	19.7808	0.9061	14
38	111	2	27.4195	26.4812	0.9658	18
38	148	3	30.3548	29.4694	0.9708	18
38	185	4	32.0273	31.4726	0.9827	18
42	82	1	23.6856	21.7778	0.9195	18
42	123	2	30.0845	29.1419	0.9687	18
42	164	3	33.4239	32.4663	0.9713	22
44	86	1	22.7765	22.7765	1	16
44	129	2	30.2500	30.2500	1	20
44	172	3	33.9649	33.9649	1	20
48	94	1	24.7742	24.7742	1	20
48	141	2	32.9143	32.9143	1	20
48	188	3	36.9626	36.9626	1	24
50	98	1	27.5169	25.7732	0.9366	18
50	147	2	35.4275	34.4702	0.9730	22
50	186	3	39.3574	38.4615	0.9772	22
50	245	4	41.7277	41.0639	0.9841	26
54	106	1	29.6230	27.7714	0.9375	22
54	159	2	38.0865	37.1362	0.9751	22
54	212	3	42.2910	41.4597	0.9803	26

Table 4: (cont.)

n	m	Perm.	Es^2	LB	Eff	$smax$
60	118	1	30.7692	30.7692	1	24
60	177	2	40.9091	40.9091	1	24
60	236	3	45.9574	45.9574	1	24
60	295	4	48.9796	48.9796	1	28
62	122	1	33.6762	31.7686	0.9434	22
62	183	2	43.4480	42.4661	0.9774	26
62	244	3	48.3106	47.4568	0.9823	26
68	134	1	34.7669	34.7669	1	24
68	201	2	46.2400	46.2400	1	28
68	268	3	51.9551	51.9551	1	28
72	142	1	36.7660	36.7660	1	24
72	213	2	48.9057	48.9057	1	28
72	284	3	54.9541	54.9541	1	28
74	146	1	39.7034	37.7655	0.9512	26
74	219	2	51.5006	50.4617	0.9798	30
74	292	3	57.3680	56.4536	0.9841	30
80	158	1	40.7643	40.7643	1	28
80	237	2	54.2373	54.2373	1	32
80	316	3	60.9524	60.9524	1	32

Table 5: Permutation generators for SSD_2 with n runs and $m(k-1)$ factors.

n	m	Perm.	Generators
6	10	1	R1=[3 6 2 1 5 4]
8	14	1	R1=[8 7 3 1 6 2 5 4]
10	27	2	R1=[7 1 8 2 3 5 4 10 9 6]R2=[8 4 9 6 1 2 5 10 7 3]
12	22	1	R1=[8 11 2 4 6 5 3 9 12 7 1 10]
12	33	2	R1=[10 7 11 9 2 4 12 8 1 5 6 3],R2=[3 8 11 6 9 12 5 2 1 7 10 4]
12	44	3	R1=[6 2 10 4 8 1 7 11 5 9 3 12],R2=[12 5 3 10 4 2 6 11 9 7 1 8],R3=[12 3 9 7 10 8 11 6 2 5 1 4]
12	55	4	R1=[1 4 10 8 5 11 12 6 9 2 7 3], R2=[5 11 1 4 7 6 8 12 2 3 9 10], R3=[2 4 9 3 7 6 1 10 11 5 8 12],R4=[5 12 4 11 3 2 8 9 7 10 1 6]
12	66	5	R1=[12 6 9 7 8 11 1 10 3 2 5 4],R2=[6 11 4 9 8 5 12 7 10 1 3 2],R3=[2 7 10 6 5 1 12 11 4 3 8 9],R4=[12 3 10 6 7 1 11 9 2 8 4 5],R5=[9 2 3 11 1 12 7 6 5 4 10 8]
12	77	6	R1=[5 6 1 2 3 12 7 4 11 10 8 9], R2=[4 10 9 8 2 7 5 6 11 3 12 1], R3=[7 8 11 10 12 1 2 6 3 9 5 4],R4=[6 5 1 9 12 3 2 10 11 7 8 4],R5=[9 5 4 10 6 1 3 2 8 7 11 12],R6=[5 11 10 3 4 8 6 2 1 9 7 12]
12	88	7	R1=[8 2 9 3 1 6 4 5 10 11 7 12],R2=[12 5 6 4 2 7 10 1 11 9 8 3],R3=[4 8 10 12 6 9 7 2 5 11 3 1],R4=[4 6 7 10 9 1 2 12 3 5 11 8],R5=[11 9 1 3 6 12 5 4 8 7 10 2],R6=[1 8 5 6 7 10 12 4 9 11 3 2],R7=[4 2 5 9 7 11 12 8 3 1 10 6]
12	99	8	R1=[11 3 6 4 8 7 9 5 2 12 10 1],R2=[2 11 9 3 1 4 7 6 8 5 12 10], R3=[9 12 4 5 8 6 3 10 7 1 11 2],R4=[8 12 3 5 1 2 11 6 7 4 10 9], R5=[3 8 11 10 5 1 9 7 4 2 12 6],R6=[12 4 6 2 3 5 9 8 10 7 11 1],R7=[3 10 8 7 1 5 4 11 9 12 6 2],R8=[11 7 10 4 9 1 12 5 3 6 8 2]
12	110	9	R1=[7 9 11 1 8 3 4 5 6 10 2 12],R2=[7 3 1 4 6 11 9 10 12 8 2 5], R3=[11 7 8 4 2 10 5 6 12 1 3 9], R4=[4 1 6 8 12 9 7 5 10 11 3 2],R5=[12 5 10 9 11 4 2 6 8 1 3 7],R6=[11 7 6 10 9 12 2 1 4 5 8 3],R7=[8 5 2 1 7 9 10 11 6 4 12 3],R8=[10 11 3 2 6 1 9 8 7 4 12 5],R9=[6 9 11 12 5 7 8 2 10 4 1 3]
14	26	1	R1=[5 8 3 2 1 9 7 12 6 10 13 11 14 4]
14	39	2	R1=[10 2 5 1 9 8 6 7 11 4 13 3 14 12], R2=[10 14 3 6 1 13 8 12 11 4 9 5 2 7]
14	52	3	R1=[5 7 4 9 8 2 3 14 6 11 10 12 1 13],R2=[2 5 10 12 14 1 6 4 7 8 9 13 11 3], R3=[2 10 4 8 9 3 12 6 14 13 5 1 7 11]

Table 5: (cont.)

14	65	4	R1=[2 7 1 13 5 12 4 11 9 10 8 3 6 14],R2=[6 12 14 5 4 3 8 2 1 7 13 11 10 9],R3=[3 9 14 7 13 8 11 4 2 1 6 10 5 12],R4=[8 1 6 5 14 11 10 3 13 9 12 4 2 7]
14	91	6	R1=[13 10 8 11 7 2 12 5 1 3 6 4 14 9],R2=[11 4 13 10 8 2 1 12 3 9 14 7 5 6],R3=[8 12 11 9 3 5 4 10 13 14 1 2 7 6],R4=[5 8 6 1 3 4 13 14 11 9 10 7 2 12],R5=[12 6 8 11 9 5 14 10 4 13 1 3 7 2],R6=[4 1 6 12 10 3 11 2 14 13 8 5 9 7]
14	104	7	R1=[4 6 10 12 9 4 11 5 3 2 1 7 13 8],R2=[11 5 1 10 7 8 13 3 2 6 14 4 12 9],R3=[1 10 12 3 8 14 11 7 9 4 2 5 6 13],R4=[10 11 9 14 2 5 8 6 4 3 13 12 1 7],R5=[12 1 3 2 11 5 4 14 9 8 13 6 10 7],R6=[2 6 9 14 12 3 10 5 13 4 11 8 1 7], R7=[4 3 5 9 10 12 1 13 11 14 2 6 7 8]
18	34	1	R1=[15 8 17 7 10 16 2 3 5 18 6 1 12 9 4 11 13 14]
18	51	2	R1=[6 17 8 4 9 13 1 18 12 5 3 14 10 2 15 16 7 11],R2=[16 5 9 15 17 8 7 13 1 14 18 2 11 12 4 10 6 3]
18	85	4	R1=[11 16 15 2 7 14 3 8 18 9 6 4 5 1 10 12 17 13],R2=[7 5 4 2 10 18 16 1 3 15 17 12 11 13 6 14 9 8],R3=[14 2 5 13 10 17 1 3 16 7 15 12 6 18 9 4 11 8], R4=[2 12 5 7 13 10 8 3 14 1 17 11 18 15 16 6 9 4]
18	119	6	R1=[17 9 12 18 3 6 16 13 11 4 10 8 14 7 2 1 5 15],R2=[11 1 7 9 6 14 16 12 2 10 18 5 3 17 13 4 15 8],R3=[13 15 9 2 10 7 3 14 11 6 8 1 17 12 4 18 16 5],R4=[7 16 15 12 11 14 17 2 4 18 3 6 1 13 9 10 8 5], R5=[6 12 8 7 18 17 9 4 3 13 5 14 11 2 16 15 10 1], R6=[9 14 15 17 12 1 11 7 18 5 6 13 16 10 4 2 3 8]
18	170	9	R1=[14 11 5 10 15 6 8 13 17 4 1 2 18 7 16 12 9 3],R2=[10 16 3 1 5 18 7 11 15 6 14 8 17 13 9 2 12 4],R3=[9 14 17 10 16 15 5 11 1 13 7 2 3 12 8 18 6 4],R4=[13 15 11 7 18 2 1 16 3 17 14 5 8 9 10 12 6 4],R5=[6 7 1 11 15 2 9 13 18 12 8 14 16 4 3 17 10 5],R6=[12 3 8 6 13 7 11 17 14 10 9 15 5 1 2 4 18 16],R7=[17 3 9 16 6 11 4 12 18 5 10 2 14 7 15 1 8 13],R8=[7 8 2 12 16 17 10 6 1 13 14 11 18 15 5 3 4 9],R9=[1 15 18 14 9 13 16 2 5 11 12 7 10 4 6 3 8 17]
20	38	1	R1=[12 16 17 19 3 4 2 1 10 13 14 18 15 8 11 5 9 7 20 6]
20	57	2	R1=[13 8 10 4 19 6 12 3 16 20 5 18 9 11 7 15 2 17 14 1],R2=[18 14 9 10 5 16 19 4 8 12 3 6 20 11 1 2 17 15 13 7]
20	76	3	R1=[13 9 19 20 2 5 15 16 10 1 11 12 14 7 8 18 6 17 4 3], R2=[5 19 20 2 1 11 16 18 8 7 6 12 10 14 15 9 13 3 4 17], R3=[18 5 6 8 1 16 13 10 2 19 17 7 14 9 11 4 15 12 3 20]
20	114	5	R1=[17 15 7 11 9 3 2 19 20 8 13 14 10 6 18 5 16 4 12 1], R2=[15 11 8 4 17 14 16 6 1 7 2 19 10 9 18 20 5 12 3 13],R3=[14 18 1 4 16 7 9 11 10 8 19 5 12 20 2 6 15 17 3 13],R4=[13 15 3 1 19 12 8 2 9 17 5 16 4 10 7 20 18 11 6 14],R5=[20 10 3 15 5 9 16 2 19 12 13 14 11 4 17 7 18 8 1 6]

Table 6: Generators permutation for new Es^2 - non optimal SSD_1 with n runs and m factors.

n	m	Perm.	Generators
12	18	2	R1=[6 10 4 5 11 9 2 3 7 12 8 1] R2=[8 1 11 10 7 4 9 3 5 6 12 2]
12	24	3	R1=[3 11 7 2 10 6 5 8 12 9 1 4]R2=[5 2 7 1 11 8 3 9 10 6 12 4]R3=[12 7 11 4 1 6 2 10 8 3 9 5]
12	30	4	R1=[9 10 6 7 11 12 5 1 3 8 4 2],R2=[6 3 2 1 10 11 12 9 4 5 8 7],R3=[10 4 3 6 1 11 8 5 9 12 7 2],R4=[1 2 6 9 12 7 8 10 11 5 3 4]
12	36	5	R1=[9 4 7 6 1 8 2 11 12 10 5 3],R2=[11 12 5 2 4 1 9 7 3 8 10 6],R3=[7 10 6 3 1 12 8 5 4 11 9 2],R4=[3 5 6 10 7 8 4 11 9 1 2 12],R5=[11 10 8 12 1 7 4 6 5 9 3 2]
12	42	6	R1=[3 1 12 2 10 4 8 9 5 6 7 11],R2=[7 10 11 12 9 6 4 3 8 2 5 1],R3=[10 8 9 5 4 2 12 1 11 6 7 3],R4=[1 10 3 9 12 2 11 6 4 5 8 7],R5=[7 2 8 3 6 1 9 4 5 11 10 12],R6=[11 4 1 8 7 9 10 2 12 6 3 5]
12	48	7	R1=[1 4 12 2 3 5 11 8 10 7 6 9],R2=[3 6 4 5 8 7 2 10 12 11 1 9],R3=[9 10 5 11 1 8 2 3 12 6 4 7],R4=[4 1 5 7 11 10 3 6 12 8 2 9],R5=[4 7 8 2 5 12 9 11 1 6 3 10],R6=[11 9 3 2 7 6 4 5 1 12 8 10],R7=[4 3 12 1 6 7 8 2 5 9 11 1
12	66	10	R1=[8 1 11 9 4 3 7 12 10 6 5 2],R2=[12 2 3 7 6 4 1 8 9 5 10 11], R3=[5 11 3 12 1 4 2 8 7 9 10 6],R4=[7 6 12 10 5 2 3 9 8 11 1 4],R5=[1 3 8 12 9 4 7 2 10 5 11 6],R6=[3 2 12 10 6 7 1 5 9 4 11 8],R7=[5 10 1 12 11 2 4 8 6 7 9 3],R8=[9 3 12 1 2 11 7 5 6 10 8 4], R9=[6 2 1 12 11 10 5 4 3 8 9 7],R10=[10 12 9 7 4 2 6 11 5 8 3 1]
12	72	11	R1=[3 2 6 12 9 1 5 10 8 4 7 11],R2=[9 4 5 1 2 7 11 8 3 12 10 6],R3=[11 1 6 7 2 8 12 9 5 4 10 3],R4=[5 6 9 3 12 1 4 10 8 7 2 11],R5=[10 5 2 7 1 9 6 4 8 11 3 12], R6=[1 2 12 8 7 3 11 9 10 4 5 6],R7=[9 4 5 11 6 7 10 8 2 3 12 1], R8=[5 10 12 2 1 4 3 9 6 7 8 11],R9=[2 3 4 5 11 12 6 7 1 9 10 8],R10=[11 8 5 4 9 6 3 2 12 7 1 10],R11=[12 2 8 6 3 9 10 5 4 1 11 7]
12	78	12	R1=[7 4 8 11 1 10 5 2 3 6 12 9],R2=[5 7 2 9 6 8 10 12 4 1 3 11],R3=[9 3 7 8 11 4 5 12 2 1 6 10],R4=[12 7 3 9 5 2 8 1 10 6 4 11],R5=[3 1 4 2 5 7 10 9 12 6 11 8],R6=[6 1 9 11 12 10 8 4 2 5 7 3],R7=[1 9 10 11 8 6 2 12 3 4 5 7],R8=[1 8 10 7 11 2 3 4 5 9 6 12],R9=[2 4 6 10 3 8 12 11 1 5 9 7],R10=[4 5 10 12 8 7 3 11 9 1 2 6],R11=[8 7 6 11 4 12 2 10 5 3 9 1],R12=[10 11 9 4 8 5 1 6 2 12 3 7]

Table 6: Cont.

n	m	Perm.	Generators
16	24	2	R1=10 15 12 8 2 14 7 11 16 4 3 5 1 6 13 9],R2=[1 16 10 14 3 8 15 7 13 11 5 2 9 12 6 4]
16	32	3	R1=[12 11 15 1 16 6 8 4 14 9 3 13 7 5 2 10],R2=[5 15 4 2 13 14 9 7 1 8 10 16 3 12 11 6],R3=[11 9 1 10 5 3 2 6 7 14 4 13 15 16 8 12]
16	40	4	R1=[6 10 5 11 15 12 7 16 2 13 1 4 9 8 14 3], R2=[5 4 12 9 7 2 3 6 14 11 8 16 10 1 13 15], R3=[2 5 16 7 13 12 15 14 11 10 4 6 1 8 9 3],R4=[16 4 3 14 1 11 5 15 13 9 2 10 6 12 7 8]
16	48	5	R1=[5 6 9 16 10 12 15 11 3 1 2 14 8 13 4 7],R2=[13 1 8 6 14 16 5 3 9 10 7 11 2 12 4 15],R3=[11 16 3 15 14 7 9 4 8 13 2 12 10 6 1 5],R4=[3 11 8 1 4 2 15 7 12 10 9 14 16 5 6 13],R5=[1 5 13 7 6 10 4 14 2 12 11 9 8 3 16 15]
20	90	8	R1=[9 7 16 3 12 1 15 8 10 5 17 18 11 20 13 4 6 14 2 19], R2=[5 11 3 7 15 2 19 16 12 6 17 8 13 1 14 18 4 10 9 20], R3=[11 19 10 8 13 2 20 4 14 1 16 18 9 7 12 17 3 15 5 6],R4=[7 14 12 5 11 3 9 16 1 17 15 13 8 10 18 4 20 2 6 19],R5=[19 4 1 2 7 6 17 9 8 10 13 15 5 11 3 12 20 16 18 14],R6=[18 16 11 9 7 17 3 4 8 14 13 5 12 2 1 19 10 6 15 20],R7=[20 15 19 12 2 13 17 10 4 7 1 16 14 9 6 3 11 5 8 18],R8=[12 8 4 6 11 1 9 13 20 15 3 2 17 5 18 14 19 16 7 10]
20	100	9	R1=[18 7 12 2 15 5 19 13 3 9 14 11 1 8 20 4 6 17 10 16],R2=[8 19 7 16 3 13 5 1 10 11 20 18 6 15 14 9 12 2 4 17],R3=[12 8 10 7 6 4 19 3 14 9 16 18 2 15 17 13 5 11 20 1],R4=[18 11 17 7 2 9 16 15 4 6 13 5 3 12 8 20 10 19 14 1], R5=[9 10 8 5 3 16 7 14 13 12 18 1 4 20 15 17 19 6 2 11], R6=[15 7 12 17 8 14 10 13 20 1 19 11 6 2 9 16 3 18 5 4], R7=[15 8 20 11 5 6 4 9 19 1 13 7 3 17 10 16 2 12 18 14],R8=[2 18 9 5 11 7 14 8 4 15 12 1 6 17 19 3 16 10 13 20],R9=[9 5 12 8 14 10 3 16 15 6 18 13 17 4 7 11 2 20 19 1]

Table 6: Cont.

n	m	Perm.	Generators
20	110	10	R1=[8 10 2 6 9 15 12 13 4 3 17 5 11 19 18 1 16 14 7 20],R2=[11 7 13 17 6 20 18 16 12 1 15 10 4 5 9 19 14 3 2 8],R3=[13 18 11 14 20 7 10 9 15 16 3 17 6 8 19 1 5 12 2 4],R4=[5 13 16 9 8 4 18 7 17 12 6 19 3 11 14 10 2 1 15 20],R5=[20 13 6 4 16 2 10 17 19 1 9 5 15 7 18 8 14 11 12 3], R6=[14 5 11 16 2 6 3 18 7 15 10 1 13 20 17 4 9 19 8 12], R7=[16 11 2 12 10 13 14 17 1 5 4 18 8 9 7 15 19 20 3 6], R8=[7 4 19 12 20 5 11 8 2 3 6 15 13 10 9 18 16 14 1 17],R9=[19 16 4 10 15 6 18 13 12 1 2 7 11 20 14 9 17 5 8 3], R10=[11 6 4 9 5 14 1 15 18 12 20 3 16 7 8 17 19 10 2 13]
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n	m	Perm.	Generators
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56	364	12	R1=[17 24 18 9 8 23 29 2 41 45 3 21 13 28 51 42 50 34 40 38 7 56 43 22 44 39 36 6 14 32 15 16 35 4 11 27 37 5 52 49 12 19 46 26 54 33 20 48 30 47 10 25 1 55 53 31],R2=[40 19 14 44 46 16 47 36 43 25 18 41 26 48 20 3 2 21 32 38 39 22 15 10 30 9 56 4 45 27 55 12 5 29 51 8 28 24 54 42 35 31 13 6 7 37 33 52 49 1 53 23 34 11 50 17], R3=[31 17 33 12 38 43 20 49 40 36 1 22 42 48 29 14 25 15 5 47 56 28 4 2 21 41 3 53 8 26 54 10 35 34 50 30 23 7 19 11 32 45 39 9 27 44 51 55 16 52 24 18 37 6 13 46], R4=[47 17 53 2 5 23 43 41 24 13 22 39 36 48 8 25 3 50 29 46 20 9 45 12 35 1 42 30 19 7 18 52 54 55 37 14 56 31 51 4 26 16 34 21 11 6 10 49 44 40 38 27 33 32 15 28],R5=[15 17 6 19 1 30 43 45 27 40 7 48 51 13 3 44 47 56 52 22 31 25 14 33 39 12 46 42 50 55 36 16 26 10 34 20 21 32 38 18 11 5 4 53 23 41 49 2 29 9 37 28 54 8 35 24],R6=[32 3 43 52 42 54 5 48 41 15 25 24 28 38 19 53 46 9 37 44 14 22 8 12 26 1 16 29 4 50 13 39 55 20 23 31 36 56 30 47 34 17 2 21 40 49 45 18 35 7 10 33 11 27 6 51],R7=[4 47 30 43 12 23 51 19 26 37 42 36 44 39 28 29 27 2 52 25 7 3 16 20 48 34 24 54 41 8 14 35 15 18 40 56 11 22 53 46 5 9 45 31 6 32 50 13 1 38 55 10 17 49 33 21],

Table 6: Cont.

n	m	Perm.	Generators
56	364	12	R8=[25 23 17 46 34 43 14 20 55 29 52 12 22 11 38 33 10 56 49 35 54 9 26 13 28 7 50 19 48 31 37 32 42 24 1 39 15 51 8 41 40 6 27 30 47 2 3 36 44 4 53 18 21 45 5 16], R9=[13 15 5 48 18 19 28 45 10 33 50 31 38 17 37 32 6 24 39 34 53 27 1 26 29 2 46 55 20 51 23 9 44 14 11 54 25 47 8 3 43 7 22 56 30 16 12 52 42 41 21 4 36 40 35 49], R10=[7 5 6 1 18 2 27 20 15 40 29 46 52 56 37 39 32 10 19 17 8 4 26 51 53 25 24 22 28 35 38 30 12 36 49 48 31 14 23 41 33 54 55 43 34 16 13 50 3 11 44 21 47 42 45 9], R11=[46 15 13 54 52 32 6 56 36 39 35 14 34 1 22 25 3 28 41 48 9 8 17 40 24 10 33 47 18 11 42 30 38 21 31 37 27 49 5 20 43 26 55 23 51 53 16 44 7 4 29 12 2 50 19 45], R12=[21 4 27 41 13 9 29 1 39 56 23 35 8 33 11 31 19 3 40 24 43 55 36 37 17 34 20 6 47 42 25 50 38 10 7 14 32 15 12 2 22 53 16 28 45 54 48 49 18 26 46 5 51 44 52 30]
60	270	8	R1=[2 12 54 22 58 15 43 51 33 23 42 11 29 26 32 52 48 31 7 41 39 60 59 17 38 18 40 1 16 5 10 3 13 37 21 28 9 20 14 57 49 8 6 19 47 56 45 30 34 50 25 36 4 27 55 35 53 46 24 44], R2=[19 4 22 34 6 7 27 36 17 55 33 50 10 5 35 1 52 25 56 45 60 8 39 15 14 49 51 12 11 30 46 20 59 16 2 32 3 21 53 26 43 23 31 29 41 44 18 40 58 57 54 28 42 47 37 24 9 38 13 48], R3=[37 59 8 45 20 43 40 3 26 1 44 6 14 19 53 10 34 41 55 57 25 17 4 5 46 54 21 11 30 47 33 7 60 39 23 52 38 2 18 50 15 22 48 16 13 49 35 32 51 36 42 29 12 31 28 24 9 58 27 56], R4=[4 53 58 54 42 43 20 14 49 29 41 8 1 59 5 37 32 15 31 60 26 6 23 36 19 55 50 22 12 56 18 45 11 57 10 2 25 35 44 33 39 52 34 17 21 47 27 3 40 9 13 38 48 51 16 46 28 30 24 7], R5=[54 3 11 50 2 27 34 31 58 26 4 37 28 13 16 48 15 22 45 23 47 40 41 56 33 44 18 1 14 19 51 25 46 29 20 6 36 9 8 17 42 12 49 52 53 55 10 43 7 24 35 59 39 57 21 30 60 38 5 32], R6=[10 15 48 17 52 18 13 2 8 39 38 11 45 35 56 43 23 12 19 3 41 40 33 7 46 34 16 25 31 24 28 36 14 21 20 55 1 51 27 5 54 32 60 57 47 50 29 22 58 4 59 49 30 26 9 44 37 6 42 53], R7=[16 60 55 25 36 7 4 20 34 22 56 57 37 51 38 44 52 2 49 11 59 33 19 8 6 1 54 15 27 47 14 40 46 39 21 48 53 43 45 26 10 28 42 41 13 12 31 50 32 17 35 58 30 3 18 23 24 29 9 5],

Table 6: Cont.

n	m	Perm.	Generators
60	270	8	R8=[60 29 51 31 20 16 14 19 50 47 42 37 7 9 25 53 17 41 38 5 10 11 18 45 52 12 59 27 48 30 46 36 24 35 49 3 34 4 22 55 23 26 43 8 33 28 2 6 13 32 39 58 15 44 40 1 21 57 56 54]
60	330	10	R1=[9 35 46 8 48 22 58 13 39 33 15 17 25 19 21 3 4 1 5 18 37 24 36 57 38 10 51 28 59 53 11 55 44 50 47 45 20 7 60 30 2 56 6 43 40 41 42 14 27 52 31 16 29 54 34 49 32 23 26 12], R2=[13 24 16 4 5 8 15 59 3 21 33 37 32 52 11 7 29 2 55 20 28 53 17 14 41 43 42 54 1 9 35 60 26 23 18 40 58 38 50 19 22 56 12 44 45 25 27 31 36 6 39 10 34 46 48 30 51 49 47 57], R3=[25 17 39 31 23 6 22 7 60 58 56 18 15 41 10 40 20 29 37 55 13 46 3 36 2 32 14 5 45 33 21 50 34 57 48 49 16 38 42 28 9 11 24 51 4 54 8 26 59 1 35 44 12 53 30 27 19 52 43 47],R4=[43 4 51 44 2 57 1 27 47 59 56 37 18 5 15 29 20 8 38 36 30 39 45 40 10 53 24 13 23 55 46 52 49 16 21 54 17 34 19 32 41 50 11 35 48 7 33 60 6 25 58 26 22 28 3 9 12 14 31 42],R5=[17 31 60 12 59 20 13 16 47 43 28 4 32 25 19 27 18 24 22 35 23 11 34 58 49 29 38 5 21 45 30 6 7 55 48 51 52 3 53 33 54 44 50 39 10 46 8 1 9 15 42 41 56 57 36 14 26 40 2 37],R6=[47 26 11 19 33 29 57 25 41 48 15 53 3 10 50 40 46 12 60 13 1 54 24 56 55 52 7 44 42 31 16 39 17 38 18 45 23 36 35 5 43 49 22 20 32 27 37 14 2 8 59 28 51 30 34 6 9 21 4 58],R7=[15 19 7 47 5 20 1 13 41 4 34 40 52 30 29 43 54 28 59 25 2 16 26 22 27 12 53 35 6 9 50 46 56 33 21 48 57 49 32 37 45 31 38 23 44 24 18 17 10 42 14 8 11 3 58 55 60 39 51 36],R8=[43 7 25 48 53 56 3 5 57 11 31 12 8 13 42 20 47 4 30 1 55 40 28 60 19 38 10 51 15 52 33 23 32 6 41 45 54 35 16 22 58 37 21 49 50 46 24 44 14 2 39 36 17 34 59 18 27 29 26 9],R9=[37 58 15 31 13 40 52 43 34 12 21 16 6 22 47 55 14 24 19 50 53 5 3 28 54 9 60 36 32 48 1 20 33 57 38 4 41 45 10 2 27 17 29 56 49 8 30 42 35 11 7 23 51 46 26 39 25 59 44 18], R10=[36 18 11 48 42 9 51 41 10 37 20 38 3 15 13 21 29 30 55 14 32 19 33 35 60 8 26 50 1 16 56 24 2 39 57 25 6 54 40 43 4 46 44 28 59 58 47 34 23 53 22 52 5 45 7 27 12 31 17 49]

Table 6: Cont.

n	m	Perm.	Generators
60	390	12	R1=[49 52 48 42 13 19 1 29 41 59 30 57 8 33 7 15 3 58 34 2 16 11 37 23 5 4 39 14 17 21 45 53 46 32 54 6 44 40 56 31 9 27 38 35 24 10 36 60 20 12 26 47 55 22 51 25 28 43 50 18], R2=[60 40 18 55 45 13 48 35 17 22 12 25 46 51 52 2 38 14 21 7 56 53 11 19 37 57 4 43 50 41 49 24 47 31 58 3 27 15 1 42 8 9 30 29 33 16 59 6 23 26 20 54 32 10 36 5 44 34 28 39], R3=[1 15 17 8 32 37 4 29 53 49 38 33 24 48 19 51 42 47 23 16 7 41 35 11 50 5 43 52 6 46 55 57 44 58 56 14 60 59 27 39 22 40 36 18 34 20 45 21 13 10 54 2 31 25 3 30 9 12 26 28],R4=[39 25 9 4 49 6 14 20 8 48 32 1 57 2 37 51 53 7 54 38 44 47 15 42 22 52 36 13 35 59 24 28 31 19 18 26 27 29 34 46 33 12 21 45 23 17 60 56 40 58 30 43 5 50 55 41 16 11 3 10],R5=[14 19 22 15 56 60 51 31 11 54 21 47 7 42 25 29 58 20 23 28 41 52 45 30 38 40 8 13 36 1 43 24 10 16 50 17 18 4 46 48 26 37 34 5 35 57 53 44 12 3 2 59 55 6 39 9 49 33 27 32],R6=[27 50 17 51 40 20 32 53 39 1 43 4 25 14 8 10 3 56 36 29 9 47 35 2 37 24 31 41 45 15 16 44 60 12 30 49 46 26 11 54 23 52 5 19 58 6 59 28 33 13 7 55 34 18 21 22 57 42 48 38],R7=[21 53 51 29 42 9 34 47 37 45 16 17 10 4 33 46 3132 35 20 28 59 55 40 7 41 13 1 18 23 30 24 5 19 38 6 43 56 60 12 8 26 14 2 49 39 57 48 25 27 58 3 54 52 22 15 11 50 36 44],R8=[53 27 26 21 30 38 24 8 25 42 43 6 40 10 56 35 59 15 3 32 7 2 50 19 49 36 17 23 9 45 51 11 44 39 12 29 48 37 4 58 18 41 52 5 1 28 22 55 33 57 31 20 16 13 60 14 46 54 47 34],R9=[60 37 50 33 25 19 10 6 11 17 44 7 27 47 12 52 5 8 38 42 21 56 49 43 13 30 28 23 32 40 59 4 15 22 26 9 48 35 34 39 41 54 14 18 29 46 57 3 45 2 51 24 36 58 55 16 53 31 1 20],R10=[32 8 56 2 24 16 43 21 10 25 58 23 26 35 55 46 7 29 41 51 27 28 49 6 13 60 45 44 14 52 4 34 36 5 38 59 20 17 19 50 18 30 1 9 33 31 12 37 47 15 42 48 53 22 54 57 11 40 3 39],R11=[39 56 19 16 35 59 14 7 5 8 32 58 44 52 37 31 23 33 18 15 24 51 34 4 12 17 55 47 48 60 43 45 13 9 41 38 50 3 6 49 36 1 54 46 27 11 20 28 10 30 40 2 22 42 21 57 29 25 53 26],R12=[26 2 50 33 37 12 51 43 15 17 45 16 59 42 29 34 27 54 20 48 56 44 38 5 35 1 28 41 21 6 25 4 19 32 8 11 49 24 52 18 39 40 13 30 47 55 22 58 3 53 57 9 46 31 60 23 36 10 7 14]

Table 6: Cont.

n	m	Perm.	Generators
64	288	8	R1=[42 53 45 8 62 47 58 55 29 12 2 38 14 33 5 59 35 54 56 36 3 61 32 51 6 40 13 30 4 9 27 50 7 21 1 28 26 20 10 57 64 18 48 23 60 43 34 52 41 16 49 24 11 46 15 17 63 37 19 44 31 25 22 39], R2=[4 23 42 61 52 5 29 47 8 46 17 57 35 55 26 33 30 3 24 16 10 20 6 15 11 63 60 39 13 45 49 37 43 59 2 50 32 31 19 51 36 48 34 18 28 25 7 9 53 27 21 14 40 58 1 44 64 54 12 38 56 62 22 41], R3=[41 50 26 18 53 29 35 12 22 9 15 13 34 48 2 6 28 64 1 33 10 25 17 16 59 39 23 21 45 14 7 62 3 27 61 55 19 58 32 57 4 54 5 47 49 56 51 46 36 37 52 11 8 40 31 60 63 30 20 44 42 24 38 43], R4=[51 20 59 48 5 33 11 26 52 30 39 37 27 63 28 54 24 49 40 1 56 36 64 62 21 3 45 35 57 23 61 2 19 17 34 47 42 60 15 38 8 6 16 31 7 41 22 9 13 32 4 44 25 43 18 10 50 46 29 12 14 58 55 53], R5=[3 57 7 15 38 23 48 24 43 41 42 59 46 8 22 18 14 40 56 20 63 60 30 16 47 49 26 9 64 27 21 10 51 6 17 35 45 32 61 12 11 39 13 29 44 31 2 58 1 33 52 25 5 37 53 50 4 62 34 28 19 54 36 55], R6=[63 43 4 18 60 39 5 8 40 45 38 33 7 37 13 44 14 2 17 11 15 29 55 32 52 9 41 1 59 62 46 28 27 19 35 34 3 42 54 36 21 61 10 57 49 58 23 64 22 47 16 26 12 48 25 20 51 30 56 53 50 24 6 31], R7=[39 16 52 2 60 54 3 35 43 5 53 44 59 51 37 8 22 41 6 10 61 58 25 48 36 7 13 49 18 26 40 11 30 33 34 31 14 21 23 32 1 62 12 28 19 55 29 57 27 20 47 45 42 64 24 63 56 17 4 46 38 15 50 9], R8=[15 22 47 32 33 41 39 57 60 28 52 34 25 50 30 17 46 49 18 21 4 16 20 43 19 64 6 3 54 29 38 5 42 13 11 63 31 51 35 55 48 58 10 36 14 44 59 23 8 9 56 7 61 45 2 1 27 26 37 53 24 12 62 40]
64	352	10	R1=[56 36 1 44 46 54 19 60 57 26 48 27 41 50 17 11 43 34 20 63 52 53 18 7 32 62 29 3 64 6 12 61 25 55 9 8 58 4 35 24 2 45 49 51 16 33 21 23 59 28 14 39 31 47 37 15 38 42 10 13 5 30 40 22], R2=[30 54 35 57 36 40 53 58 21 41 27 61 10 22 20 6 33 37 44 47 55 4 13 24 34 62 19 64 14 9 42 49 7 5 1 2 17 31 63 8 39 48 46 51 3 45 25 60 52 43 56 11 50 18 32 26 12 23 59 28 38 15 16 29], R3=[23 14 47 7 30 10 63 56 40 38 9 6 15 57 48 61 43 16 2 35 3 44 29 24 52 27 17 12 25 32 54 26 13 18 11 64 31 21 59 42 45 1 33 51 55 50 49 22 58 20 19 39 4 36 41 46 34 62 5 8 53 60 37 28],

Table 6: Cont.

n	m	Perm.	Generators
64	352	10	R4=[36 60 38 19 3 15 37 18 39 9 5 20 46 24 14 40 23 44 11 13 59 53 54 25 64 52 21 28 8 47 33 10 41 45 35 2 50 22 6 16 48 42 43 30 57 58 29 4 17 31 55 32 62 12 26 49 56 1 27 34 7 63 51 61],R5=[4 48 3 33 50 11 28 13 7 60 35 49 31 56 30 24 58 34 29 40 2 52 42 57 1 51 23 39 63 27 26 53 18 38 21 8 10 62 43 17 59 19 6 22 41 16 44 55 54 61 5 64 36 15 20 32 47 46 37 12 25 14 45 9],R6=[4 27 9 57 45 20 36 53 31 52 44 37 55 11 64 59 13 34 39 7 50 30 15 6 2 25 54 43 29 16 5 24 51 33 12 17 18 10 60 48 19 62 56 32 35 28 61 42 58 63 26 23 21 46 1 41 49 38 3 22 8 14 47 40],R7=[48 52 4 3 23 33 18 64 31 55 11 54 41 20 35 34 7 57 39 61 8 6 28 53 29 36 45 62 38 9 32 51 14 24 59 43 44 60 58 2 16 42 50 1 26 25 56 22 21 40 15 10 19 37 63 30 47 5 49 46 13 17 27 12],R8=[60 53 24 15 46 35 59 34 38 45 8 49 26 63 3 4 30 48 5 29 62 28 11 37 25 22 58 16 36 14 27 20 44 54 19 31 32 41 56 12 21 18 61 39 51 1 17 6 7 52 50 33 2 57 55 23 47 10 40 43 42 9 13 64],R9=[33 40 49 18 12 63 51 37 27 9 32 61 54 36 38 48 2 3 24 43 53 16 62 39 1 29 55 50 57 10 19 47 44 20 45 5 28 41 59 26 52 13 56 35 15 21 8 60 11 30 31 6 23 4 22 14 17 64 25 46 58 34 42 7],R10=[13 60 47 10 22 9 53 23 19 57 18 59 2 34 12 20 38 61 46 16 36 48 44 11 7 32 43 51 52 56 3 14 58 28 6 55 4 64 50 35 27 31 41 37 49 42 33 17 45 24 40 54 62 21 15 26 29 63 8 30 25 1 5 39]

Table 6: Cont.

n	m	Perm.	Generators
64	416	12	R1=[10 28 55 4 11 7 23 24 53 31 52 30 58 12 45 8 17 6 3 61 60 40 50 49 41 27 64 43 18 1 51 25 29 39 16 56 47 57 5 4415 62 63 13 38 42 54 33 48 36 19 46 20 9 35 34 32 14 2 21 26 22 59 37],R2=[14 12 47 39 38 30 40 6 37 64 55 57 4 53 20 7 43 42 56 3 34 15 18 52 59 1 62 5 54 45 33 24 32 21 16 31 50 58 19 60 8 49 2 63 25 61 41 23 9 26 36 22 51 13 11 46 35 17 29 44 48 27 10 28],R3=[60 2 11 50 10 57 64 4 15 6 7 43 41 53 20 22 13 47 56 27 46 16 8 38 34 36 5 17 1 31 62 12 23 40 32 14 26 48 52 25 30 33 49 18 19 45 3 63 35 58 29 59 28 51 9 44 61 42 24 21 55 39 54 37],R4=[17 8 14 49 51 44 32 12 42 10 21 53 9 50 3 29 43 61 62 147 34 16 23 60 54 55 37 35 36 56 48 26 4 38 45 2 39 15 30 6 28 18 40 20 58 22 33 11 64 19 7 57 59 5 24 13 52 27 63 31 41 46 25],R5=[36 25 50 63 41 20 12 4 11 17 44 59 13 26 33 35 27 19 1 55 58 61 43 39 6 60 56 31 62 15 42 53 64 49 54 46 9 52 32 54 5 8 37 18 29 16 30 40 10 28 34 23 2 57 48 22 3 21 14 47 7 38 24 51],R6=[3 17 53 52 48 19 40 26 9 49 64 36 11 13 25 14 21 24 41 63 5 58 18 6 12 2 27 39 54 55 50 4 59 23 57 46 29 8 47 7 37 51 42 20 33 34 1 44 43 10 45 35 31 30 61 28 22 56 16 62],R7=[7 5 28 1 57 12 20 14 56 15 42 53 58 31 8 40 29 21 54 247 17 4 62 30 60 34 9 37 64 23 19 3 52 25 27 61 22 24 11 38 55 45 46 39 26 6 50 59 33 43 36 16 49 10 48 18 44 51 63 32 13 35 41],R8=[44 58 37 25 10 42 48 62 14 1 20 57 18 11 19 43 12 53 28 622 47 41 64 55 35 2 21 38 13 4 32 34 49 40 33 5 52 26 3 29 16 39 56 8 27 7 54 59 63 30 9 61 17 45 24 31 46 60 23 36 50 51 15],R9=[10 60 44 4 27 57 12 55 23 30 56 58 37 62 61 5 26 53 29 45 51 36 64 2 14 3 22 20 24 39 19 7 6 11 43 41 38 8 49 15 9 42 35 31 21 48 25 17 59 13 47 50 16 40 33 63 1 32 52 18 34 28 54 46],R10=[4 16 21 51 36 62 61 60 27 46 53 15 54 29 58 37 10 13 33 34 38 7 11 55 17 32 57 14 52 1 59 3 64 6 18 25 39 49 35 2 19 22 50 42 24 9 43 48 40 44 63 41 5 26 8 12 23 45 28 47 56 30 31 20],

Table 6: Cont.

n	m	Perm.	Generators
			R11=[54 42 45 23 19 13 7 3 32 25 5 31 58 53 57 46 6 4 21 61 8 52 56 55 30 59 29 27 50 24 14 60 49 16 26 9 22 41 15 39 36 33 64 48 28 40 38 43 63 35 51 44 18 34 62 12 11 1 47 17 2 10 37 20],R12=[64 50 35 1 31 18 30 13 10 62 27 48 11 54 4 22 59 25 45 37 39 26 21 52 56 41 23 28 16 60 44 34 7 57 19 55 36 53 38 24 29 8 12 2 20 9 58 49 47 46 61 43 15 32 17 63 6 51 5 40 14 33 42 3]

Table 7: Generators permutation for new Es^2 - optimal SSD_2 with n runs and $m(k-1)$ factors.

n	m	Perm.	Generators
6	10	1	R1=[3 6 2 1 5 4]
8	14	1	R1=[8 7 3 1 6 2 5 4]
10	27	2	R1=[7 1 8 2 3 5 4 10 9 6]R2=[8 4 9 6 1 2 5 10 7 3]
12	22	1	R1=[8 11 2 4 6 5 3 9 12 7 1 10]
12	33	2	R1=[10 7 11 9 2 4 12 8 1 5 6 3],R2=[3 8 11 6 9 12 5 2 1 7 10 4]
12	44	3	R1=[6 2 10 4 8 1 7 11 5 9 3 12],R2=[12 5 3 10 4 2 6 11 9 7 1 8],R3=[12 3 9 7 10 8 11 6 2 5 1 4]
12	55	4	R1=[1 4 10 8 5 11 12 6 9 2 7 3], R2=[5 11 1 4 7 6 8 12 2 3 9 10], R3=[2 4 9 3 7 6 1 10 11 5 8 12],R4=[5 12 4 11 3 2 8 9 7 10 1 6]
12	66	5	R1=[12 6 9 7 8 11 1 10 3 2 5 4],R2=[6 11 4 9 8 5 12 7 10 1 3 2],R3=[2 7 10 6 5 1 12 11 4 3 8 9],R4=[12 3 10 6 7 1 11 9 2 8 4 5],R5=[9 2 3 11 1 12 7 6 5 4 10 8]
12	77	6	R1=[5 6 1 2 3 12 7 4 11 10 8 9], R2=[4 10 9 8 2 7 5 6 11 3 12 1], R3=[7 8 11 10 12 1 2 6 3 9 5 4],R4=[6 5 1 9 12 3 2 10 11 7 8 4],R5=[9 5 4 10 6 1 3 2 8 7 11 12],R6=[5 11 10 3 4 8 6 2 1 9 7 12]
12	88	7	R1=[8 2 9 3 1 6 4 5 10 11 7 12],R2=[12 5 6 4 2 7 10 1 11 9 8 3],R3=[4 8 10 12 6 9 7 2 5 11 3 1],R4=[4 6 7 10 9 1 2 12 3 5 11 8],R5=[11 9 1 3 6 12 5 4 8 7 10 2],R6=[1 8 5 6 7 10 12 4 9 11 3 2],R7=[4 2 5 9 7 11 12 8 3 1 10 6]
12	99	8	R1=[11 3 6 4 8 7 9 5 2 12 10 1],R2=[2 11 9 3 1 4 7 6 8 5 12 10], R3=[9 12 4 5 8 6 3 10 7 1 11 2],R4=[8 12 3 5 1 2 11 6 7 4 10 9], R5=[3 8 11 10 5 1 9 7 4 2 12 6],R6=[12 4 6 2 3 5 9 8 10 7 11 1],R7=[3 10 8 7 1 5 4 11 9 12 6 2],R8=[11 7 10 4 9 1 12 5 3 6 8 2]
12	110	9	R1=[7 9 11 1 8 3 4 5 6 10 2 12],R2=[7 3 1 4 6 11 9 10 12 8 2 5], R3=[11 7 8 4 2 10 5 6 12 1 3 9], R4=[4 1 6 8 12 9 7 5 10 11 3 2],R5=[12 5 10 9 11 4 2 6 8 1 3 7],R6=[11 7 6 10 9 12 2 1 4 5 8 3],R7=[8 5 2 1 7 9 10 11 6 4 12 3],R8=[10 11 3 2 6 1 9 8 7 4 12 5],R9=[6 9 11 12 5 7 8 2 10 4 1 3]
14	26	1	R1=[5 8 3 2 1 9 7 12 6 10 13 11 14 4]
14	39	2	R1=[10 2 5 1 9 8 6 7 11 4 13 3 14 12], R2=[10 14 3 6 1 13 8 12 11 4 9 5 2 7]
14	52	3	R1=[5 7 4 9 8 2 3 14 6 11 10 12 1 13],R2=[2 5 10 12 14 1 6 4 7 8 9 13 11 3], R3=[2 10 4 8 9 3 12 6 14 13 5 1 7 11]

Table 7: Cont.

n	m	Perm.	Generators
14	65	4	R1=[2 7 1 13 5 12 4 11 9 10 8 3 6 14],R2=[6 12 14 5 4 3 8 2 1 7 13 11 10 9],R3=[3 9 14 7 13 8 11 4 2 1 6 10 5 12],R4=[8 1 6 5 14 11 10 3 13 9 12 4 2 7]
14	91	6	R1=[13 10 8 11 7 2 12 5 1 3 6 4 14 9],R2=[11 4 13 10 8 2 1 12 3 9 14 7 5 6],R3=[8 12 11 9 3 5 4 10 13 14 1 2 7 6],R4=[5 8 6 1 3 4 13 14 11 9 10 7 2 12],R5=[12 6 8 11 9 5 14 10 4 13 1 3 7 2],R6=[4 1 6 12 10 3 11 2 14 13 8 5 9 7]
14	104	7	R1=[4 6 10 12 9 4 11 5 3 2 1 7 13 8],R2=[11 5 1 10 7 8 13 3 2 6 14 4 12 9],R3=[1 10 12 3 8 14 11 7 9 4 2 5 6 13],R4=[10 11 9 14 2 5 8 6 4 3 13 12 1 7],R5=[12 1 3 2 11 5 4 14 9 8 13 6 10 7],R6=[2 6 9 14 12 3 10 5 13 4 11 8 1 7], R7=[4 3 5 9 10 12 1 13 11 14 2 6 7 8]
18	34	1	R1=[15 8 17 7 10 16 2 3 5 18 6 1 12 9 4 11 13 14]
18	51	2	R1=[6 17 8 4 9 13 1 18 12 5 3 14 10 2 15 16 7 11],R2=[16 5 9 15 17 8 7 13 1 14 18 2 11 12 4 10 6 3]
18	85	4	R1=[11 16 15 2 7 14 3 8 18 9 6 4 5 1 10 12 17 13],R2=[7 5 4 2 10 18 16 1 3 15 17 12 11 13 6 14 9 8],R3=[14 2 5 13 10 17 1 3 16 7 15 12 6 18 9 4 11 8], R4=[2 12 5 7 13 10 8 3 14 1 17 11 18 15 16 6 9 4]
18	119	6	R1=[17 9 12 18 3 6 16 13 11 4 10 8 14 7 2 1 5 15],R2=[11 1 7 9 6 14 16 12 2 10 18 5 3 17 13 4 15 8],R3=[13 15 9 2 10 7 3 14 11 6 8 1 17 12 4 18 16 5],R4=[7 16 15 12 11 14 17 2 4 18 3 6 1 13 9 10 8 5], R5=[6 12 8 7 18 17 9 4 3 13 5 14 11 2 16 15 10 1], R6=[9 14 15 17 12 1 11 7 18 5 6 13 16 10 4 2 3 8]
18	170	9	R1=[14 11 5 10 15 6 8 13 17 4 1 2 18 7 16 12 9 3],R2=[10 16 3 1 5 18 7 11 15 6 14 8 17 13 9 2 12 4],R3=[9 14 17 10 16 15 5 11 1 13 7 2 3 12 8 18 6 4],R4=[13 15 11 7 18 2 1 16 3 17 14 5 8 9 10 12 6 4],R5=[6 7 1 11 15 2 9 13 18 12 8 14 16 4 3 17 10 5],R6=[12 3 8 6 13 7 11 17 14 10 9 15 5 1 2 4 18 16],R7=[17 3 9 16 6 11 4 12 18 5 10 2 14 7 15 1 8 13],R8=[7 8 2 12 16 17 10 6 1 13 14 11 18 15 5 3 4 9],R9=[1 15 18 14 9 13 16 2 5 11 12 7 10 4 6 3 8 17]
20	38	1	R1=[12 16 17 19 3 4 2 1 10 13 14 18 15 8 11 5 9 7 20 6]
20	57	2	R1=[13 8 10 4 19 6 12 3 16 20 5 18 9 11 7 15 2 17 14 1],R2=[18 14 9 10 5 16 19 4 8 12 3 6 20 11 1 2 17 15 13 7]

Table 7: Cont.

n	m	Perm.	Generators
20	76	3	R1=[13 9 19 20 2 5 15 16 10 1 11 12 14 7 8 18 6 17 4 3] ,R2=[5 19 20 2 1 11 16 18 8 7 6 12 10 14 15 9 13 3 4 17], R3=[18 5 6 8 1 16 13 10 2 19 17 7 14 9 11 4 15 12 3 20]
20	114	5	R1=[17 15 7 11 9 3 2 19 20 8 13 14 10 6 18 5 16 4 12 1], R2=[15 11 8 4 17 14 16 6 1 7 2 19 10 9 18 20 5 12 3 13],R3=[14 18 1 4 16 7 9 11 10 8 19 5 12 20 2 6 15 17 3 13],R4=[13 15 3 1 19 12 8 2 9 17 5 16 4 10 7 20 18 11 6 14],R5=[20 10 3 15 5 9 16 2 19 12 13 14 11 4 17 7 18 8 1 6]
20	133	6	R1=[13 10 19 7 18 20 12 6 1 2 15 5 8 17 14 11 9 16 4 3],R2=[8 2 7 19 14 10 11 5 6 9 13 1 17 16 12 15 3 20 4 18],R3=[12 4 3 13 5 2 1 16 9 7 10 20 19 18 8 14 11 17 15 6],R4=[12 3 13 9 17 4 15 6 1 19 10 2 11 14 8 7 18 20 5 16],R5=[9 18 5 11 3 15 12 2 8 17 7 16 20 13 4 1 6 14 10 19],R6=[20 12 5 17 7 18 8 1 4 13 10 6 16 3 14 11 19 9 2 15]
20	152	7	R1=[9 10 13 1 7 6 5 11 3 12 14 18 4 17 15 16 2 19 20 8], R2=[18 14 10 12 13 7 15 3 17 2 6 9 16 11 1 19 8 5 4 20],R3=[10 20 6 11 9 13 1 5 17 18 2 7 3 8 19 16 14 12 4 15],R4=[8 10 4 15 1 14 6 2 5 12 3 17 18 16 9 19 11 7 20 13],R5=[12 11 5 3 17 4 13 14 20 1 16 18 19 10 7 15 2 6 9 8],R6=[11 9 13 4 1 15 8 6 14 17 12 16 7 20 2 18 19 3 5 10],R7=[1 4 7 16 20 19 18 17 8 15 6 9 12 3 5 11 10 2 13 14]
20	171	8	R1=[15 16 17 12 7 8 9 2 1 14 4 20 11 6 3 13 10 18 19 5],R2=[1 17 13 8 20 16 5 2 14 19 15 6 18 7 12 11 3 9 10 4],R3=[4 11 3 14 5 7 9 6 2 17 10 19 20 16 12 8 15 18 13 1],R4=[9 5 3 10 12 7 17 18 15 8 19 20 13 1 2 11 4 16 6 14], R5=[11 9 15 2 19 5 1 8 7 16 12 18 4 6 17 10 20 14 13 3],R6=[9 7 5 13 10 4 3 16 1 8 14 2 12 18 6 11 15 17 19 20],R7=[16 11 17 15 6 12 2 5 19 10 13 7 8 14 9 3 18 4 20 1],R8=[16 7 18 20 13 15 3 19 9 4 8 2 12 17 14 6 1 10 11 5]
20	190	9	R1=[16 9 13 8 1 20 14 18 11 5 12 7 3 19 4 2 17 15 10 6],R2=[18 17 13 3 11 5 20 1 10 16 2 8 19 6 12 4 7 9 14 15],R3=[18 3 14 7 5 11 17 15 19 1 4 2 13 12 10 20 8 9 16 6],R4=[4 7 20 15 6 12 16 1 18 3 5 8 17 11 13 2 9 19 14 10],R5=[12 1 11 5 4 8 17 16 14 9 18 13 15 20 6 19 7 10 2 3],

Table 7: Cont.

n	m	Perm.	Generators
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20	361	18	R1=[10 11 16 8 2 1 5 7 19 6 17 14 15 20 12 4 9 3 18 13],R2=[7 1 13 10 5 14 16 4 12 20 2 15 18 6 3 17 19 9 11 8],R3=[16 3 4 18 6 9 17 13 5 8 7 20 14 12 10 15 1 11 2 19],R4=[13 9 20 2 3 19 8 6 15 18 17 14 4 5 1 11 10 12 16 7],R5=[7 11 10 9 13 18 1 8 16 15 17 19 6 12 20 5 14 3 4 2],

Table 7: Cont.

n	m	Perm.	Generators
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24	46	1	R1=[14 21 19 10 6 13 2 9 12 17 16 1 5 4 7 11 3 23 24 8 15 22 18 20]
14	69	2	R1=[22 16 7 18 11 21 13 12 9 3 10 8 1 15 19 2 24 5 4 6 17 14 20 23],R2=[13 11 21 20 17 24 15 9 10 22 19 1 8 7 18 3 6 16 23 12 2 14 5 4]
24	92	3	R1=[17 3 16 2 6 9 12 5 7 20 22 24 15 19 11 1 23 18 13 8 21 4 10 14],R2=[6 24 2 17 21 15 20 19 14 7 10 8 16 12 3 13 1 9 23 11 5 18 22 4],R3=[24 22 5 16 11 15 18 10 7 14 9 2 21 17 4 8 19 12 6 23 20 3 1 13]
26	50	1	R1=[6 2 24 11 18 21 13 25 9 22 12 5 16 23 19 20 8 1 14 26 3 10 17 7 4 15]
26	75	2	R1=[16 20 14 22 12 21 1 3 17 19 25 18 15 13 26 11 4 7 2 9 24 6 5 8 23 10],R2=[18 11 7 15 21 14 17 12 1 6 19 8 22 26 2 24 3 4 10 23 9 13 25 20 16 5]

Table 7: Cont.

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26	125	4	R1=[2 20 8 18 25 17 12 1 22 7 24 21 19 23 26 9 3 5 10 14 4 6 11 13 16 15],R2=[15 21 25 16 10 13 12 1 17 7 5 4 11 19 23 18 14 2 20 9 26 3 24 22 6 8],R3=[22 19 11 8 20 4 2 17 24 16 9 6 15 13 3 10 14 21 5 18 1 26 7 25 23 12], R4=[13 17 8 21 18 5 23 1 12 3 14 25 26 22 9 10 11 2 16 7 15 19 4 20 24 6]
26	150	5	R1=[19 16 15 14 26 4 8 6 11 12 1 10 5 3 23 22 2 24 18 25 20 7 9 21 13 17],R2=[15 12 11 20 9 8 2 6 24 22 1 7 5 21 14 4 3 16 13 26 25 17 23 19 18 10],R3=[21 5 8 13 12 19 6 9 22 26 14 16 7 11 25 15 17 4 3 23 10 2 20 24 18 1] , R4=[7 6 11 17 15 21 10 2 26 14 13 12 4 1 9 16 5 25 18 8 24 23 20 19 22 3],R5=[10 9 7 18 6 22 24 8 26 2 17 1 23 12 13 11 19 14 15 21 3 20 4 16 5 25]
28	54	1	R1=[23 12 19 25 8 22 20 15 9 17 27 1 26 6 16 28 3 13 21 2 4 14 7 24 10 5 11 18]
28	81	2	R1=[25 16 27 23 4 7 11 8 20 18 13 10 14 9 2 26 6 15 21 28 17 1 24 5 3 12 22 19] , R2=[23 8 26 16 22 18 14 11 12 10 9 4 27 5 15 19 1 20 17 24 25 2 13 6 7 3 21 28]
28	108	3	R1=[4 19 5 18 26 24 8 20 3 15 9 23 10 22 27 14 7 11 21 2 13 1 17 6 16 28 25 12],R2=[15 10 26 9 3 22 8 28 23 21 14 6 20 13 11 16 18 25 5 4 12 19 2 24 17 7 27 1],R3=[15 1 14 16 12 21 2 20 7 4 11 9 27 22 5 28 17 18 23 3 6 13 26 19 10 24 8 25]
28	135	4	R1=[18 6 15 2 20 21 28 14 10 1 11 22 25 23 7 16 27 4 26 12 13 19 5 17 24 3 8 9],R2=[5 19 16 6 12 7 2 20 18 10 13 17 28 23 14 21 1 27 3 4 26 11 9 25 15 8 22 24],R3=[2 11 22 25 26 4 17 16 15 12 6 7 8 3 20 24 9 21 13 14 27 23 5 10 18 1 28 19],R4=[2 17 8 19 24 14 3 6 9 28 11 12 16 15 5 4 23 21 22 25 7 26 20 13 1 10 18 27]

Table 7: Cont.

n	m	Perm.	Generators
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30	87	2	R1=[24 29 13 2 4 17 28 22 10 23 27 12 19 3 26 8 14 1 15 11 6 30 21 16 7 5 20 18 25 9], R2=[27 29 9 28 5 7 23 17 18 25 8 20 19 21 1 12 22 6 4 2 13 14 16 11 24 15 26 30 10 3]
30	116	3	R1=[9 19 27 6 14 15 5 3 25 18 22 30 12 8 17 10 4 1 23 7 13 21 29 11 2 26 24 16 28 20],R2=[18 9 24 1 17 13 29 11 21 28 26 2 19 3 8 5 20 4 10 16 12 27 6 22 15 14 7 25 30 23],R3=[30 19 21 24 4 11 18 22 10 27 3 1 25 6 7 15 17 5 13 2 12 20 28 9 29 26 8 23 14 16]
32	62	1	R1=[13 4 9 17 14 26 6 16 23 1 27 18 11 12 15 2 29 10 8 30 20 28 7 31 5 21 32 24 3 19 25 22]
32	93	2	R1=[5 9 23 20 21 6 26 18 24 22 30 7 19 14 27 17 25 1 29 12 4 28 3 8 11 2 16 10 15 31 13 32],R2=[10 32 18 9 14 4 21 3 19 29 2 25 5 22 11 8 13 24 23 31 6 20 28 7 12 1 15 16 27 30 26 17]
32	124	3	R1=[12 10 6 1 15 13 9 28 21 11 5 25 19 18 14 26 16 8 7 17 22 2 31 23 3 30 4 24 27 29 32 20],R2=[21 14 3 10 9 26 11 13 18 27 17 6 22 28 19 2 23 7 16 15 8 30 32 12 31 25 20 4 5 24 1 29],R3=[14 17 23 22 29 16 6 12 9 32 25 7 15 20 11 24 27 31 5 19 13 8 2 30 1 26 10 21 18 28 3 4]
32	155	4	R1=[3 23 28 18 21 30 12 8 10 15 16 31 29 11 5 24 6 17 2 22 19 20 14 25 27 7 26 9 4 32 3 1],R2=[13 25 19 29 16 17 7 27 6 22 18 21 3 15 1 10 14 11 5 23 26 20 24 32 30 8 9 4 28 12 2 31],R3=[9 26 12 1 18 13 24 5 27 31 8 2 25 21 29 19 15 32 3 6 4 22 16 28 7 20 14 17 30 11 10 23],R4=[20 9 18 5 27 11 25 16 15 21 3 17 26 28 7 32 24 31 4 14 23 13 22 19 6 30 10 12 8 2 1 29]
38	74	1	R1=[2 36 8 5 7 26 22 9 32 13 14 3 30 33 19 38 25 35 1 20 31 34 17 24 11 12 15 27 4 28 6 18 23 29 10 21 37 16]
38	111	2	R1=[32 29 18 34 1 2 19 5 15 6 13 20 17 16 31 12 7 38 37 24 36 21 26 25 11 27 9 10 22 28 35 14 8 33 23 4 30 3],R2=[22 7 23 5 21 14 6 3 15 27 16 29 9 32 4 37 11 34 18 35 1 33 26 19 13 20 31 2 28 38 10 30 8 17 12 36 24 25]

Table 7: Cont.

n	m	Perm.	Generators
38	148	3	R1=[24 26 31 38 20 13 37 17 2 10 8 4 28 32 16 1 9 36 6 19 34 3 18 25 27 22 30 23 12 11 35 21 29 5 15 33 14 7],R2=[25 26 13 17 4 7 34 16 24 28 3 18 29 15 6 38 32 19 11 8 12 30 37 23 33 27 35 5 22 36 2 9 31 21 1 10 20 14],R3=[1 37 22 8 23 13 33 38 14 30 16 34 11 32 27 2 12 18 25 10 31 24 28 19 20 17 9 36 15 21 26 35 5 3 7 29 4 6]
38	185	4	R1=[22 26 18 3 28 37 6 27 13 10 9 7 24 36 29 23 30 33 25 17 4 5 15 32 2 31 21 34 35 38 14 20 16 1 19 12 11 8],R2=[34 28 15 4 18 14 27 37 23 2 33 26 7 32 20 21 6 5 10 11 22 35 8 9 31 19 3 38 24 30 1 36 25 29 13 17 16 12],R3=[23 26 9 14 19 38 20 15 4 32 29 31 16 22 28 17 35 25 2 36 37 1 30 18 34 12 13 24 27 3 5 21 8 11 6 10 7 33],R4=[38 18 36 34 4 37 27 17 21 24 6 15 29 14 33 28 3 32 7 31 10 13 30 5 19 16 23 2 11 8 20 35 25 9 26 22 1 12]
42	82	1	R1=[10 39 32 21 29 33 25 37 11 31 2 20 41 22 15 34 18 12 40 38 6 36 35 7 14 30 24 27 13 4 16 3 9 8 5 42 1 28 19 26 23 17]
42	123	2	R1=[7 26 27 3 33 14 4 24 5 38 18 19 40 13 15 6 41 35 34 22 2 28 23 16 39 12 21 25 10 31 7 8 32 36 20 29 1 37 11 30 9 42],R2=[8 2 31 10 16 37 42 41 22 6 18 17 21 13 11 5 4 33 14 35 24 7 25 26 15 29 36 27 39 9 19 40 1 3 32 23 34 38 12 28 30 20]
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44	86	1	R1=[1 16 43 18 37 14 36 25 5 38 40 10 35 22 21 26 20 29 13 24 8 12 31 33 7 41 3 39 15 19 30 2 4 23 44 32 11 9 42 34 6 17 28 27]
44	129	2	R1=[20 36 23 18 37 6 39 41 12 31 35 21 10 13 28 44 22 42 3 11 33 29 2 5 19 27 43 17 8 30 38 34 25 26 14 16 32 1 24 15 40 4 9 7], R2=[4 20 31 12 3 43 10 25 27 1 33 30 38 39 14 18 24 36 28 40 29 42 44 19 15 37 16 41 21 8 17 32 13 7 5 26 6 22 23 35 2 9 34 11]

Table 7: Cont.

n	m	Perm.	Generators
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48	94	1	R1=[27 23 36 34 25 7 46 13 26 11 41 33 31 18 38 43 35 45 15 30 37 3 48 20 12 17 9 5 28 29 47 19 22 14 21 4 2 44 6 16 1 8 40 32 42 10 24 39]
48	141	2	R1=[20 18 3 13 5 1 25 4 19 9 33 47 29 37 8 24 10 31 39 15 27 28 43 17 14 35 45 46 12 23 26 6 16 22 21 44 40 48 7 36 11 41 2 30 42 32 34 38],R2=[8 44 2 46 23 38 45 20 41 40 34 19 42 4 27 47 10 32 16 11 15 7 21 3 1 14 29 31 25 26 37 22 33 18 24 30 43 48 36 12 17 28 5 35 6 39 13 9]
48	188	3	R1=[32 13 8 6 39 44 45 26 24 30 7 47 40 2 20 15 5 33 27 38 17 19 25 43 28 41 11 4 1 42 36 48 18 29 46 35 37 3 23 14 9 10 16 31 22 12 21 34],R2=[47 18 20 15 7 41 40 12 2 36 44 43 24 46 16 37 22 13 29 25 6 21 30 28 27 38 26 10 34 35 19 32 9 39 5 11 33 8 48 42 45 1 23 17 4 14 3 31],R3=[46 28 31 29 16 40 18 27 26 6 5 13 39 44 37 4 8 36 33 19 41 11 22 2 10 15 25 42 1 9 3 45 24 35 34 47 48 21 23 14 32 20 17 38 43 7 12 30]
50	98	1	R1=[5 42 33 39 32 8 14 45 44 37 27 38 25 34 30 48 50 29 46 35 19 16 18 7 43 22 1 5 17 10 23 24 26 12 41 40 3 36 4 9 20 49 47 2 28 11 21 31 6 13]
50	147	2	R1=[3 6 38 13 36 11 35 10 28 19 14 22 21 8 20 15 47 18 41 25 27 33 45 9 39 2 40 5 32 46 43 34 24 29 17 26 50 42 30 31 16 12 7 23 44 37 1 48 49 4],R2=[10 50 43 29 26 22 36 14 8 6 42 40 2 24 1 3 28 21 4 25 49 15 39 48 34 7 13 38 12 9 17 11 33 16 35 44 41 46 23 27 45 20 32 30 47 37 31 19 5 18]
50	186	3	R1=[24 19 47 6 40 21 35 44 43 28 17 22 30 33 4 38 15 42 41 16 49 20 46 32 31 11 8 50 12 39 10 9 5 34 2 7 25 13 26 3 1 23 14 27 36 37 45 48 18 29],R2=[1 25 2 15 8 19 31 48 46 45 11 18 16 5 41 14 24 44 33 35 4 9 29 38 40 7 37 28 43 34 30 49 36 21 39 23 6 32 12 47 13 22 20 50 27 26 10 17 3 42],

Table 7: Cont.

n	m	Perm.	Generators
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54	106	1	R1=[16 49 54 27 43 6 18 2 21 19 7 23 42 8 37 22 10 4 44 14 5 29 51 26 12 13 9 17 35 33 1 52 15 50 28 48 11 53 31 24 34 32 40 25 3 46 30 38 20 41 45 36 47 39]
54	159	2	R1=[24 18 16 22 53 10 19 23 38 27 54 6 28 29 25 17 8 1 41 34 36 42 40 9 37 51 26 35 46 11 44 14 2 12 50 20 3 33 47 48 39 13 49 52 4 31 7 15 21 45 5 30 43 32],R2=[51 28 3 19 17 50 6 8 42 16 48 44 47 2 33 10 35 18 45 25 27 54 23 40 31 53 21 41 39 52 11 20 30 4 14 22 37 13 46 15 36 38 7 49 43 24 29 5 1 32 34 26 12 9]
54	212	3	R1=[10 29 23 3 27 31 19 8 9 43 2 26 18 52 15 54 7 39 41 17 34 13 4 53 33 46 6 47 38 16 44 35 25 48 28 5 21 50 42 1 32 20 12 24 51 40 36 14 11 45 22 49 37 30],R2=[47 38 41 25 16 42 1 29 44 34 11 54 28 3 21 50 6 31 45 12 8 2 37 20 30 39 19 53 5 24 43 32 27 46 15 7 48 22 23 36 26 52 13 51 18 33 49 35 14 17 4 40 9 10],R3=[17 30 44 16 47 10 13 52 41 53 38 9 35 6 19 32 1 42 33 8 4 27 5 28 37 50 54 21 24 3 7 20 25 14 22 23 46 26 18 39 29 12 34 49 31 45 51 40 15 43 2 11 36 48]

Table 7: Cont.

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60	236	3	R1=[39 30 53 4 14 26 6 48 2 16 18 51 58 54 10 38 35 56 9 7 34 44 5 32 36 3 37 1 60 41 13 50 55 52 33 43 17 12 45 23 25 47 19 31 28 46 27 59 49 42 15 24 57 29 21 20 40 8 11 22],R2=[58 11 26 3 54 4 42 6 5 40 32 24 14 15 23 47 743 55 1 34 52 51 41 28 21 22 31 16 53 60 46 8 3638 39 13 48 18 12 35 10 44 20 49 29 56 30 37 17 59 33 25 50 2 57 45 27 9 19],R3=[2 4 51 3 35 8 29 22 58 32 56 45 43 60 42 37 38 48 27 23 24 53 59 44 1 5 49 28 41 21 57 15 9 30 7 16 12 13 25 20 34 19 46 26 14 36 11 55 39 10 17 31 40 33 54 47 18 52 50 6]
60	295	4	R1=[56 10 52 5 36 19 58 30 9 35 1 28 50 43 8 51 54 24 11 33 23 37 20 32 22 7 16 14 6 18 2 45 25 59 31 34 4 15 3 40 17 12 49 57 48 42 53 41 29 39 26 46 44 13 38 27 55 60 21 47],R2=[22 1 42 17 50 55 26 33 36 60 58 49 47 23 31 13 44 28 45 53 15 29 52 11 39 40 30 46 48 34 37 16 51 56 27 10 7 59 9 12 43 3 25 8 6 54 14 57 38 20 4 2 24 5 32 18 19 35 41 21],R3=[20 14 21 53 46 36 41 59 4 54 52 30 33 13 34 38 28 26 45 19 50 40 16 29 27 55 22 1 37 57 24 15 42 23 11 43 18 32 31 2 17 6 58 3 47 44 56 25 35 7 49 8 12 60 48 10 5 39 51 9],R4=[40 35 23 14 54 37 56 20 32 59 33 7 51 43 10 30 3 41 42 34 50 25 8 15 29 31 58 2 11 27 44 17 12 55 24 38 18 19 5 52 60 45 9 46 16 28 4 1 49 57 39 48 26 47 6 13]
62	122	1	R1=[61 2 29 22 36 54 42 24 1 15 30 35 39 18 44 20 28 49 5 38 45 50 47 25 7 56 6 11 53 57 14 23 16 4 21 26 19 10 33 37 8 48 51 34 46 55 60 9 12 52 32 58 13 59 17 31 41 62 27 43 40 3]

Table 7: Cont.

n	m	Perm.	Generators
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62	244	3	R1=[44 48 53 5 56 38 12 16 34 7 1 3 41 57 35 22 25 13 60 29 32 55 17 52 54 15 8 26 21 58 24 62 43 31 36 33 40 61 45 23 10 2 46 47 30 14 59 18 20 4 19 11 39 9 49 6 37 42 28 27 50 51],R2=[34 28 3 53 33 46 24 51 9 56 42 35 41 14 55 2 50 47 45 37 48 18 25 57 58 32 21 6 29 1 8 19 44 22 31 62 27 4 40 10 59 39 12 61 49 13 38 7 15 5 54 26 23 36 30 60 17 11 16 43 52 20],R3=[22 61 12 51 11 25 38 30 36 60 3 20 18 7 2 35 46 31 9 32 4 59 49 41 45 52 15 14 21 54 53 44 34 17 24 37 39 8 56 47 29 10 19 55 43 13 62 27 1 23 33 50 40 5 48 6 28 26 16 57 42 58]
68	134	1	R1=[32 18 4 59 16 44 66 60 35 40 9 42 23 24 47 28 41 68 65 45 10 53 27 54 34 17 3 67 20 37 62 46 58 30 36 33 2 14 51 25 43 19 26 49 50 1 31 29 57 15 52 39 38 63 7 61 22 21 56 55 8 5 48 6 11 13 64 12]
68	201	2	R1=[57 31 61 40 5 62 36 49 66 6 55 37 56 12 47 46 22 39 35 1 25 60 59 15 64 54 67 27 2 7 30 18 28 941 32 34 8 4 53 14 44 24 50 26 63 11 43 16 3 52 29 51 23 48 19 45 13 10 58 20 17 65 33 42 68 38 21],R2=[67 23 50 48 20 66 15 37 32 62 56 46 58 1 9 21 40 53 22 31 54 27 42 36 26 7 13 33 44 35 14 12 30 10 34 47 59 41 60 68 2 57 8 45 16 18 25 49 17 38 63 55 6 51 39 29 43 3 19 61 64 24 52 11 5 65 4 28]
68	268	3	R1=[22 35 57 30 40 27 16 9 20 8 10 61 24 54 50 32 44 3 19 60 41 31 45 64 29 28 2 39 68 46 33 47 6 55 59 12 13 11 25 56 34 26 4 5 1 38 65 51 63 7 42 53 58 23 21 49 18 66 67 43 15 37 17 62 48 14 52 36],R2=[28 57 63 39 10 64 46 60 30 32 6 43 29 37 35 24 42 58 38 21 56 55 41 27 22 19 3 45 2 49 61 16 18 11 5 34 14 65 4 26 47 66 48 20 7 25 1 62 12 51 53 9 31 40 33 67 54 13 50 52 23 8 36 44 17 15 68 59],R3=[61 25 11 47 17 29 63 15 58 36 20 16 24 32 2 8 64 10 60 55 7 26 40 3 6 57 31 5 28 49 65 38 50 14 34 12 44 67 21 53 37 18 39 62 48 68 66 35 1 30 59 54 42 9 51 27 46 22 45 23 19 13 52 33 4 41 56 43]

Table 7: Cont.

n	m	Perm.	Generators
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72	213	2	R1=[47 42 23 1 67 26 28 38 56 6 17 30 63 29 8 61 43 32 2 10 66 18 50 69 36 11 62 53 48 57 70 9 46 60 45 49 44 37 19 34 25 52 14 33 72 20 59 58 16 35 21 15 51 7 41 54 65 39 64 3 12 31 71 24 68 40 4 22 55 13 27],R2=[63 15 11 52 68 35 24 64 17 5 51 28 41 53 13 14 37 46 2 61 50 66 9 7 72 48 45 39 54 71 25 26 32 34 21 16 44 58 36 12 47 43 59 22 38 55 23 19 62 33 65 42 60 49 56 40 8 18 6 67 20 57 3 27 4 69 70 30 10 1 31 29]
72	284	3	R1=[69 40 17 57 11 28 5 56 31 34 46 32 25 29 63 70 14 18 42 54 19 6 51 48 7 38 53 45 41 3 8 1 37 26 59 12 61 39 10 52 24 66 47 49 44 9 21 72 15 68 4 27 55 20 36 60 23 58 13 50 35 71 33 65 16 30 64 2 22 43 62 67],R2=[19 26 7 23 24 17 33 11 50 57 61 42 60 29 38 6 71 62 35 56 72 13 27 65 21 45 63 36 68 37 64 40 53 41 70 69 31 4 49 16 25 28 44 59 3 67 22 30 51 48 39 15 5 9 32 18 58 2 52 8 1 10 47 46 14 43 34 12 20 54 55 66],R3=[64 42 70 32 45 20 33 58 72 54 62 66 1 7 21 35 44 65 12 61 39 15 60 46 2 19 6 23 43 52 63 16 13 9 49 27 17 51 68 47 26 34 71 29 55 25 36 41 22 30 48 38 59 57 53 18 37 67 10 14 69 40 24 11 56 3 28 31 4 50 5 8]
74	146	1	R1=[73 31 74 65 66 19 70 40 43 67 1 53 59 13 12 55 25 64 36 58 72 21 60 51 9 11 62 50 61 2 69 71 33 63 52 45 7 20 8 42 16 14 23 44 15 46 30 6 32 34 18 28 39 48 35 29 56 57 26 4 41 27 5 3 10 17 38 68 24 22 54 49 37 47]
74	219	2	R1=[59 15 70 26 34 57 52 47 1 22 23 19 24 45 9 33 18 32 56 61 48 71 55 28 39 30 53 38 14 37 62 25 63 51 40 49 29 7 12 21 5 66 68 3 6 35 54 58 50 2 44 73 8 36 65 10 64 42 20 31 46 4 72 43 27 13 11 17 67 16 60 74 41 69],R2=[29 69 36 20 49 24 3 71 14 66 19 18 67 48 43 53 1 45 26 54 72 5 15 12 40 21 28 7 10 42 65 37 64 56 34 35 9 58 6 51 13 38 47 32 59 61 27 22 11 31 55 33 52 30 63 60 4 46 57 62 41 44 39 2 25 16 70 8 17 50 73 74 68 23]

Table 7: Cont.

n	m	Perm.	Generators
74	292	3	R1=[58 57 70 54 10 46 31 21 74 27 40 20 13 8 63 39 67 12 34 61 36 53 18 19 24 22 41 7 73 28 69 11 62 35 60 71 38 43 49 50 47 65 15 37 26 55 6 56 42 17 14 25 52 23 64 16 30 59 48 32 33 9 2 68 44 51 45 1 66 4 3 5 29 72],R2=[62 12 13 29 11 41 73 52 57 40 16 55 1 66 54 60 45 46 34 58 26 67 44 31 51 9 61 18 10 71 53 4 70 59 33 19 56 39 23 47 72 6 30 28 38 20 42 15 2 36 3 21 25 68 32 35 64 65 50 48 74 14 69 8 24 17 63 22 49 43 7 37 5 27],R3=[64 4 23 14 55 15 11 63 18 69 28 44 2 54 21 40 50 17 47 51 42 57 26 22 38 49 68 72 70 56 10 59 34 65 71 25 12 74 30 16 37 13 29 1 66 58 35 60 46 62 52 31 67 33 9 24 48 7 43 32 20 39 8 53 19 6 3 36 61 73 45 5 41 27]
80	158	1	R1=[37 74 59 76 1 48 6 52 14 79 77 2 69 24 34 49 9 73 72 41 60 71 39 61 26 8 23 68 29 20 64 63 75 19 67 43 58 22 78 42 30 21 12 40 18 4 25 16 50 35 65 62 45 31 55 11 3 15 70 56 44 38 32 33 54 57 47 13 53 5 80 28 51 66 10 27 7 36 46]
80	237	2	R1=[25 3 49 27 13 46 58 34 40 24 59 15 68 31 70 14 11 35 78 28 9 30 63 69 73 61 7 47 6 50 43 20 18 22 74 67 57 62 2 16 48 65 17 64 76 32 71 26 12 66 52 75 72 45 80 4 44 10 33 36 8 1 21 37 56 79 60 23 39 19 29 53 51 42 55 38 77 54 5 41], R2=[61 55 29 48 43 5 77 71 47 32 78 34 23 53 69 46 66 20 67 25 28 68 33 17 1 76 30 26 54 36 10 6 42 14 16 64 22 74 4 39 63 70 2 45 52 59 3 13 80 50 24 65 27 56 35 18 73 11 79 60 31 40 12 38 37 15 44 7 8 72 58 19 62 51 9 41 75 21 49 57]
80	316	3	R1=[45 12 76 35 44 55 2 62 13 71 15 16 24 72 22 25 80 51 75 77 4 31 8 46 48 40 47 59 73 10 1 27 60 78 50 52 64 61 49 9 32 65 54 58 66 70 37 68 20 36 14 17 63 56 69 18 41 11 79 30 67 26 3 5 42 6 38 19 34 28 23 7 33 57 43 53 39 74 29 21],R2=[29 76 46 8 75 54 15 13 5 57 39 22 40 7 73 32 20 63 42 43 25 59 27 77 55 69 11 51 31 33 65 58 9 50 80 14 68 37 2 16 45 56 28 62 35 60 6 23 21 1 24 52 38 67 4 41 18 78 66 44 48 12 74 49 61 70 19 30 3 64 10 17 79 47 53 26 34 36 72 71],R3= [15 47 63 29 68 9 74 34 59 54 44 71 17 1 16 41 37 60 64 31 80 40 78 66 56 69 39 23 77 14 10 21 43 25 75 8 18 62 13 33 3 67 27 57 22 42 36 79 26 20 70 49 11 19 7 4 61 76 53 52 65 5 72 73 38 2 12 30 55 58 28 24 48 32 45 46 50 35 51 6]