Indian J. Phys. **81(11)**, 1115-1175 (2007)

Review

IJP

An introduction to nonlinear oscillators : a pedagogical review

J K Bhattacharjee*

Department of Theoretical Physics, Indian Association for the Cultivation of Science, 2A & 2B Raja S C Mullick Road, Kolkata-700 032, India

A K Mallik

Department of Mechanical Engineering, Indian Institute of Technology, Kanpur-208 016, India

and

Sagar Chakraborty

S.N. Bose National Centre for Basic Sciences, Salt Lake, Kolkata-700 098, India

E-mail : tpjkb@iacs.res.in

Abstract : Oscillators are omnipresent; most of them are inherently non-linear. Though a non-linear equation mostly does not yield an exact analytical solution for itself, plethora of elementary yet practical techniques exist for extracting important information about the solution of equation. The purpose of this review is to introduce the readers to such techniques which are carefully illustrated using mainly the examples of anharmonic oscillator, Van-der-Pol oscillator and Duffing's oscillator.

Keywords : Lindstedt Poincare method, self consistent method, Pade approximant, Duffing oscillator, Van der Pol oscillator, Mathien equation

PACS No : 05.45.-a

Plan of the Article

*Corresponding Author

1. Introduction

We begin with a discussion of the simple harmonic oscillator which has equation of motion

$$m\ddot{x} = -m\omega^2 x \tag{1}$$

where the potential corresponding to the restoring force is written as $V(x) = \frac{1}{2}m\omega^2 x^2$. The '$\omega$' appearing in the potential is the frequency of the periodic motion as can be seen from the integral of equation (1) which is

$$x(t) = A\cos\omega t + B\sin\omega t \tag{2}$$

where A and B are constants which can be determined from the initial values of $x(t)$ and $\dot{x}(t)$. The energy E of the particle is given by the sum of the kinetic (K) and the potential energies and is found to be

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\omega^2\left(A^2 + B^2\right). \tag{3}$$

The frequency of motion is a constant ω determined by the strength of the potential and is independent of the initial condition. The energy which determines the size of the orbit in the phase space (the $x - \dot{x}$ space) is, however, dependent on the initial condition. We can calculate the average kinetic and the potential energies, $\langle K \rangle$ and $\langle V \rangle$ respectively, over a cycle as

$$\langle K \rangle \equiv \frac{1}{T}\int_0^T \frac{1}{2}m\dot{x}^2 dt = \frac{1}{2}m\omega^2\left(\frac{A^2}{2} + \frac{B^2}{2}\right) \tag{4a}$$

and $\langle V\rangle \equiv \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 x^2 dt = \frac{1}{2}m\omega^2\left(\frac{A^2}{2}+\frac{B^2}{2}\right).$ (4b)

Clearly, $\langle K\rangle = \langle V\rangle$ for simple harmonic oscillator.

If we consider the more general potential $V = m\frac{\Lambda}{2\mu}x^{2\mu}$ (with $\mu > 0$), then equation of motion is

$$\ddot{x} = -\lambda x^{2\mu-1} \tag{5}$$

which for all $\mu \neq 1$ is a non-linear equation. The first integral is the energy expression

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2\mu}m\lambda x^{2\mu}. \tag{6}$$

If 'a' is the amplitude of the motion, then the total energy is given by :

$$E = \frac{1}{2\mu}m\lambda a^{2\mu} \tag{7}$$

since at the maximum displacement $\dot{x} = 0$. Combining equations (6) and (7) we get :

$$\dot{x}^2 = \frac{\lambda}{\mu}\left(a^{2\mu} - x^{2\mu}\right) \tag{8}$$

$$\Rightarrow \mathrm{d}t = \sqrt{\frac{\lambda}{\mu}}\frac{dx}{\sqrt{\left(a^{2\mu} - x^{2\mu}\right)}}. \tag{9}$$

The time period T is the time required to go from x = 0 to x = a and thence to x = -a and return to the origin. Since things are symmetric about x=0, we have

$$T = 4\sqrt{\frac{\lambda}{\mu}}\int_0^a \frac{dx}{\sqrt{\left(a^{2\mu} - x^{2\mu}\right)}} = 4\sqrt{\frac{\lambda}{\mu}}a^{1-\mu}\int_0^{\frac{\pi}{2}}\frac{\cos\theta d\theta}{\sqrt{\left(1-\sin^{2\mu}\theta\right)}}. \tag{10}$$

Setting $\sin^\mu\theta = \sin\phi$, with $\mu\sin^{\mu-1}\cos\theta d\theta = \cos\phi d\phi$ and using the relation (7)

$$T = 4\sqrt{\frac{\lambda}{\mu\lambda}}a^{1-\mu}\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{\mu}-1}\phi d\phi \tag{11}$$

$$T = 2\sqrt{\frac{\lambda}{\mu\lambda}}a^{1-\mu}\beta\left(\frac{1}{2\mu},\frac{1}{2}\right) = 2\sqrt{\frac{1}{\mu\lambda}}\left(\frac{2\mu E}{m\lambda}\right)^{\frac{1-\mu}{2\mu}}\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2\mu}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2\mu}\right)}\right]. \tag{12}$$

For $\mu=1$, we have $T = \frac{2\pi}{\sqrt{\lambda}} = \frac{2\pi}{\omega}$, if we write $\lambda = \omega^2$ to make the potential agree with that taken in equation (1). This is as expected from our previous analysis. For $\mu \neq 1$, the period depends on the energy and hence on the initial conditions. Interestignly enough, T increases with increasing energy if $\mu < 1$ and decreases with increasing energy E for $\mu > 1$. The latter could be understood if we associate with larger E, a faster motion and hence a shorter period. This does not explain the situation for $\mu < 1$. To get a different perspective, we calculated the average kinetic and potential energies

$$\langle K \rangle = \frac{1}{T}\int_0^T \frac{1}{2} m\dot{x}^2 dt = \frac{m}{2T}\int_0^T \dot{x}^2 \frac{dt}{dx} dx = \frac{m}{2T}\oint \dot{x} dx$$

$$\Rightarrow \langle K \rangle = \frac{2m}{T}\int_0^a \sqrt{(a^2\mu - x^2\mu)} = \frac{2m}{T}\sqrt{\frac{\lambda}{\mu}} a^{\mu+1}\int_0^1 \sqrt{(1-y^{2\mu})}dy \tag{13}$$

while,

$$\langle V \rangle = \frac{1}{T}\int_0^T \frac{m\lambda}{2\mu} x^{2\mu} dt = \frac{m\lambda}{2\mu T}\oint \frac{x^{2\mu}}{\dot{x}} dx = \frac{m}{2T}\sqrt{\frac{\lambda}{\mu}}\oint \frac{x^{2\mu}}{\sqrt{(a^{2\mu} - x^{2\mu})}} dx$$

$$\Rightarrow \quad \langle V \rangle = \frac{2m}{T}\sqrt{\frac{\lambda}{\mu}}\int_0^a \frac{x^{2\mu}}{\sqrt{(a^{2\mu} - x^{2\mu})}} dx = \frac{2m}{T}\sqrt{\frac{\lambda}{\mu}} a^{1+\mu}\int_0^1 \frac{y^{2\mu} dy}{\sqrt{(1-y^{2\mu})}}. \tag{14}$$

The ratio of the average kinetic to the average potential energy turns out to be

$$\frac{\langle K \rangle}{\langle V \rangle} = \frac{\int_0^1 \sqrt{(1-y^{2\mu})}dy}{\int_0^1 \frac{y^{2\mu} dy}{1-y^{2\mu}}} = \frac{\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{\mu}-1}\theta \cos^2\theta d\theta}{\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{\mu}+1}\theta d\theta}$$

$$\Rightarrow \quad \frac{\langle K \rangle}{\langle V \rangle} = \frac{\beta\left(\frac{1}{2\mu}, \frac{3}{2}\right)}{\beta\left(\frac{1}{2\mu}+1, \frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2\mu}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2\mu}+1\right)\Gamma\left(\frac{1}{2}\right)} = \mu. \tag{15}$$

For $\mu = 1, \langle K \rangle = \langle V \rangle$ and the time period is energy independent. For $\mu > 1$, the kinetic energy dominates, hence the particle moves faster relative to the simple harmonic oscillator and the time period decreases. For $\mu < 1$ on the other hand, the kinetic energy diminishes and the period increases with increasing energy.

We would like to end this section by pointing out the existence of a non-simple harmonic potential for which the time period is independent of the amplitude. This is the potential (see Figure 1)

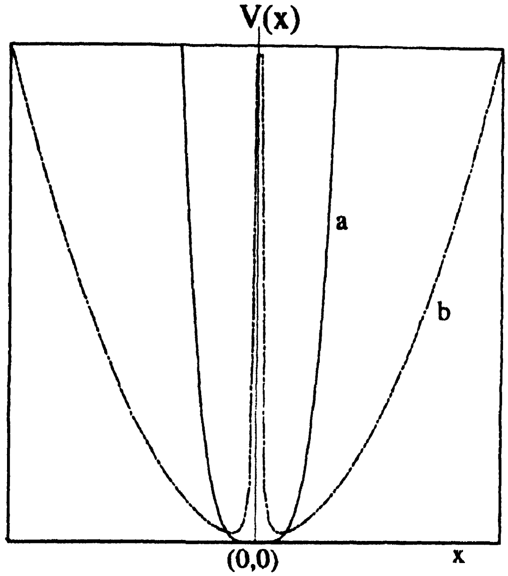

Figure 1. In the sketch, the curve 'a' represents $V(x) \sim x^{2\mu}$, where μ is an integer. Compare it with the shape of the curve 'b' which is the plot for the potential

$$V(x) = \left(x^2 + \frac{1}{x^2}\right); \quad \lambda > 0 . \tag{16}$$

Suppose the total energy of the oscillator E is entirely due to potential energy at the positions $x = a$ and $x = b$ (at these points the oscillator is at rest), *i.e.*,

$$E = \lambda\left(a^2 + \frac{1}{a^2}\right) = \lambda\left(b^2 + \frac{1}{b^2}\right). \tag{17}$$

Hence, the time-period T is given by

$$T = 2\left|\int_a^b \frac{dx}{\sqrt{2\left[E - \lambda\left(x^2 + \frac{1}{x^2}\right)\right]}}\right| . \tag{18}$$

Using the expression (17) to find a and b in terms of λ and E, we can calculate T from relation (18)

$$T = \frac{\pi}{\sqrt{2\lambda}} \tag{19}$$

which evidently is independent of the amplitude!

2. Frequency of Undamped Anharmonic Oscillator : Direct calculation

The typical anharmonic oscillator has a potential energy which can be written as :

$$V(x)=\frac{1}{2}m\omega^2x^2+\frac{\lambda m}{2\mu}x^{2\mu} \tag{20}$$

with equation of motion

$$\ddot{x}=-\omega^2x-\lambda x^{2\mu-1}. \tag{21}$$

The motion is bounded and periodic with the first integral expressing the energy conservation as :

$$E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+\frac{\lambda m}{2\mu}x^{2\mu}. \tag{22}$$

The period of motion T is given by the integral

$$T=\frac{4}{\omega}\int_0^a\frac{dx}{\sqrt{a^2-x^2+\frac{\lambda}{\mu\omega^2}\left(a^{2\mu}-x^{2\mu}\right)}}. \tag{24}$$

For any value of λ and the total energy E (this determines the amplitude 'a' from equation (23)), the time period can be found from equation (24) by a numerical integration for all positive values of μ. In general, the integral in equation (24) cannot be evaluated analytically and so we can resort to some approximations to get a feeling for what the time period looks like.

The first approximation that we can try is a small value of non-linear term in equation of motion. In this case, we can write the expression (24) as

$$T=\frac{4}{\omega}\int_0^1\frac{dy}{\sqrt{a^2-y^2+\frac{\lambda a^{2\mu-2}}{\mu\omega^2}\left(1-y^{2\mu}\right)}} \tag{25}$$

$$\Rightarrow\quad T=\frac{4}{\omega}\int_0^1\frac{dy}{\sqrt{1-y^2}}\left\{1-\frac{\lambda a^{2\mu-2}}{2\mu\omega^2}\frac{\left(1-y^{2\mu}\right)}{1-y^2}\right\} \tag{26}$$

(the expansion parameter is $\lambda a^{2\mu}/\left(\omega^2a^2\right)$, the ratio of the potential energy for the anharmonic term to that for the simple harmonic term). The period to $O(\lambda)$ accuracy

$$T=\frac{2\pi}{\omega}-\frac{2\lambda a^{2\mu-2}}{\mu\omega^3}\int_0^1\frac{\left(1-y^{2\mu}\right)}{\left(1-y^2\right)^{\frac{3}{2}}}dy+\ldots$$

$$=\frac{2\pi}{\omega}-\frac{2\lambda a^{2\mu-2}}{\mu\omega^3}\int_0^{\frac{\pi}{2}}\frac{1-\sin^{2\mu}\theta}{1-\sin^2\theta}d\theta+\ldots$$

$$\frac{2\pi}{\omega}\quad\frac{2\lambda a^{2\mu-2}}{\mu\omega^{\smile}}\left[\sum_{n=0}^{\mu-1}\int_0^{\frac{\pi}{2}}\sin^{2n}\theta d\theta\right]+\ldots$$

$$:\frac{2\pi}{\omega}-\frac{2\lambda a^{2\mu-2}}{\mu\omega^3}\sum_{n=0}^{\mu-1}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}+\ldots .$$

Since we need the correction only upto $O(\lambda)$, we can substitute for 'a' from equation (23) as $a=2E/\left(m\omega^2\right)$, so that the above relation becomes:

$$T=\frac{2\pi}{\omega}-\frac{2\lambda a}{\mu\omega^3}\left(\frac{2E}{m\omega^2}\right)^{\mu-1}I_{\mu} \tag{27}$$

where $I_{\mu}=\sum_{n=0}^{\mu-1}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}$. For the most studied case of the potential $V(x)=\frac{1}{2}m\omega^2x^2+\frac{\lambda}{4}x^4$ *i.e.*, $\mu=2$, we have to $O(\lambda)$,

$$T=\frac{2\pi}{\omega}-\frac{3}{2}\frac{\pi\lambda E}{2m\omega^3}\quad\frac{2\pi}{\omega}\left(1-\frac{3}{4}\frac{\lambda E}{m\omega^4}\right) \tag{28}$$

From equation (27), we note that the dimensionless parameter for the perturbation series is $\lambda E^{\mu-1}/\left(m^{\mu-1}\omega^{2\mu}\right)$ *i.e.* for $\mu>1$, the expansion will break down as easily at high λ as at high E.

If $\lambda E^{\mu-1}/\left(m^{\mu-1}\omega^{2\mu}\right)\gg 1$, then the second term in the denominator of R.H.S. of the relation (24) dominates and the leading term in the time period is found to be given by equation (12). Specialising for the case of μ = 2, we and for $\lambda E/\left(m\omega^4\right)\gg 1$

$$T=\sqrt{2\pi}\left(\frac{m}{4E\lambda}\right)^{\frac{1}{4}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} . \tag{29}$$

The next correction in equation (29) can be obtained by reverting to equation (24), setting μ=2 and expanding as

$$T=\frac{4}{\omega}\int_0^a\frac{dx}{\left[\frac{\lambda}{2\omega^2}\left(a^4-x^4\right)\right]^{\frac{1}{2}}}\left\{1-\frac{\omega^2}{\lambda}\left(\frac{a^2-x^2}{a^4-x^4}\right)+\ldots\right\}$$

$$=\frac{4}{\omega}\sqrt{\frac{2\omega^2}{\lambda a^4}}a\left[\int_0^1\frac{dy}{\sqrt{1-y^4}}-\frac{\omega^2}{\lambda a^2}\int_0^1\frac{dy}{\left(1+y^2\right)\sqrt{1-y^4}}+\ldots\right]$$

$$=2\sqrt{2}\sqrt{\frac{4}{\lambda a^4}}a\int_0^1\frac{dy}{\sqrt{1-y^4}}-2\sqrt{2}\sqrt{\frac{4}{\lambda a^4}}a\frac{\omega^2}{\lambda a^2}\int_0^1\frac{dy}{\sqrt{(1+y^2)^3(1-y^2)}} \tag{30}$$

At this point, we note that the energy expression has to be expanded as

$$E=\frac{1}{4}m\lambda a^4+\frac{1}{2}m\omega^2a^2 \tag{31}$$

$$\Rightarrow a^4=\frac{4E}{m\lambda}-\frac{2m\omega^2a^2}{\lambda}\approx\frac{4E}{m\lambda}-\frac{2m\omega^2}{\lambda}\left(\frac{4E}{m\lambda}\right)^{\frac{1}{2}} \tag{32}$$

$$\Rightarrow a=\left(\frac{4E}{m\lambda}\right)^{\frac{1}{4}}\left\{1-\frac{m\omega^2}{8E}\left(\frac{4E}{m\lambda}\right)^{\frac{1}{2}}+\ldots\right\}. \tag{33}$$

Inserting the expression in equation (30)

$$T=\frac{1}{\sqrt{2}}\sqrt{\frac{m}{E}}\;1+\frac{m\omega^2}{4E}\left(\frac{4E}{m\lambda}\right)^{\frac{1}{2}}\left|\left(\frac{4E}{m\lambda}\right)^{\frac{1}{4}}\;1-\frac{m\omega^2}{8E}\left(\frac{4E}{m\lambda}\right)^{\frac{1}{2}}\;\frac{\frac{1}{2})\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\;2\sqrt{2}\sqrt{m}\,\frac{\omega^2}{4}\frac{m}{E}\left(\frac{4E}{m\lambda}\right)^{\frac{3}{4}}\right.$$

$$=\sqrt{\pi}\left(\frac{m}{\lambda E}\right)^{\frac{1}{4}}\frac{\Gamma\Big|\frac{1}{4}}{\Gamma\Big|\frac{3}{4}}\;1+\frac{m\omega^2}{4E}\left(\frac{E}{m\lambda}\right)^{\frac{1}{2}}\;-2\omega^2\frac{m^4}{E^{\frac{3}{4}}\lambda^{\frac{3}{4}}}$$

$$=\sqrt{\pi}\left(\frac{m}{\lambda E}\right)^{\frac{1}{4}}\frac{\Gamma\Big|\frac{1}{4}}{\Gamma\left(\frac{3}{4}\right)}\;1+\frac{m\omega^4}{4\sqrt{mE\lambda}}-\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}\sqrt{\left(\frac{E}{m\lambda}\right)}$$

$$=\sqrt{\pi}\left(\frac{m}{\lambda E}\right)^{\frac{1}{4}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\;1+\sqrt{\left(\frac{m\omega^4}{E\lambda}\right)}\;\frac{}{4}-\frac{\Gamma\left(\frac{3}{4}\right.}{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right.} \tag{34}$$

where, $I=\int_0^1\frac{dy}{\sqrt{(1+y^2)^3(1-y^2)}}$ Once again the expansion parameter is the dimensionless quantity $\lambda E/(m\omega^4)$, which we hereafter denote by β. The perturbation series of equations (28) and (34) can now be expressed as

$$T=\frac{2\pi}{\omega}\left(1-\frac{3}{4}\beta+\ldots\right)\qquad\beta\ll 1 \tag{35a}$$

$$T=\frac{\sqrt{\pi}}{\omega}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\beta^{\frac{1}{4}}}\left(1-\frac{1}{\sqrt{\beta}}+\ldots\right. \qquad \beta\gg 1\ . \tag{35b}$$

The series for $\beta \gg 1$ is what known as an asymptotic series.

The $\beta \ll 1$ and $\beta \gg 1$ forms of equations (35a) and (35b) allow us to introduce a common technique for connecting the low β and the high β ends. We note that what is required is a function $f(\beta)$ such that

$$\frac{\omega T}{2\pi}=f(\beta) \tag{36}$$

with

$$f(\beta)=1-\frac{3}{4}\beta+\ldots, \qquad \beta\ll 1 \tag{37a}$$

$$f(\beta)=\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\beta^{\frac{1}{4}}}\ 1-\frac{\cdot}{\sqrt{\beta}}+\ldots \qquad \beta\gg 1\ . \tag{37b}$$

A Padé approximant constructs $f(\beta)$ as $N(\beta)/D(\beta)$, where $N(\beta)$ and $D(\beta)$ are polynomials of the appropriate power of β. From equation (37b), we notice that $\beta^{\frac{1}{4}}=\alpha$ is the appropriate choice of the variable for a polynomial and we have

$$f(\beta)=g(\alpha)=\begin{cases}1-\frac{3}{4}\alpha^4+\ldots, & \alpha\ll 1\\ \frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\alpha}\left(1-\frac{1}{\alpha^2}+\ldots\right), & \alpha\gg 1\end{cases} \tag{38}$$

Clearly, $N(\alpha)$ has to be a polynomial of order three, while $D(\alpha)$ has to be a polynomial of order four to match the known results of equation (38). We try

$$g(\alpha)=\frac{1+a_1\alpha+a_2\alpha^2+a_3\alpha^3}{1+a_1\alpha+a_2\alpha^2+a_3\alpha^3+b_1\alpha^4} \tag{39}$$

the coefficients a_1's are the same in the numerator and denominator so that for $\alpha \ll 1$,the coefficients of α, α^2 and α^3 may cancel out and

$$g(\alpha)=1-b_1\alpha^4 \tag{40}$$

This fixes

$$b_1=\frac{3}{4}\ . \tag{41}$$

The large α expansion for $g(\alpha)$, reads

$$g(\alpha)=\frac{a_3}{b_1\alpha}\left\{1+\left(\frac{a_2}{a_3}-\frac{a_3}{b_1}\right)\frac{1}{\alpha}+\left(\frac{a_1}{a_3}-\frac{a_2}{b_1}\right)\frac{1}{\alpha^2}+\ldots\right\}. \tag{42}$$

So that comparing with the expression (38) we can set

$$\frac{a_3}{b_1}=\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{43}$$

$$\Rightarrow \quad a_3=\frac{1}{8\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{44}$$

where we have used the relation (41). And,

$$\frac{a_2}{a_3}-\frac{a_3}{b_1}|=0 \tag{45}$$

$$\Rightarrow a_2=\frac{a_3^2}{b_1}=\frac{3}{16\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{46}$$

Finally,

$$\frac{a_3}{b_1}\left(\frac{a_1}{b_3}-\frac{2a_2}{b_1}\right)=-\frac{1}{2\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{47}$$

leading to

$$a_1=\frac{\Gamma\left(\frac{1}{4}\right)}{8\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}\left[\frac{1}{2\pi}\left\{\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right\}^2-1\right] \tag{48}$$

Thus,

$$f(\beta)=\frac{1+\left(\frac{3}{8\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left[\frac{1}{2\pi}\left\{\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right\}^2-1\right]\right)\beta^{\frac{1}{4}}+\left(\frac{3}{16\pi}\left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]^2\right)\beta^{\frac{1}{2}}+\left(\frac{3}{8\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)\beta^{\frac{3}{4}}}{1+\left(\frac{3}{8\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left[\frac{1}{2\pi}\left\{\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right\}^2-1\right]\right)\beta^{\frac{1}{4}}+\left(\frac{3}{16\pi}\left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]^2\right)\beta^{\frac{1}{2}}+\left(\frac{3}{8\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)\beta^{\frac{3}{4}}+\left(\frac{3}{4}\right)\beta}$$

is the required Padé approximant.

3. Lindstedt poincare perturbation theory

In this section, we demonstrate how a perturbation theory can be developed for the anharmonic oscillator of equation (21) written as

$$\ddot{x}+\omega^2 x=-\lambda x^{2\mu-1}. \tag{50}$$

We first begin with the naive expansion

$$x=x_0+\lambda x_1+\lambda^2 x_2+\ldots. \tag{51}$$

and inserting in equation (50) equate the same power of λ on either side. This leads to, at the zeroth order,

$$\ddot{x}_0+\omega^2 x_0=0 \tag{52}$$

with the solutions

$$x_0=A\cos\omega t \tag{53}$$

with the initial conditions: $x_0=A$ and $\dot{x}_0=0$. Turning to $O(\lambda)$,

$$\ddot{x}_1+\omega^2 x_1=-x_0^{2\mu-1}=A^{2\mu-1}\cos^{2\mu-1}\omega t. \tag{54}$$

We note that $\cos^{2\mu-1}\omega t$ can be expanded as a Fourier series. The coefficient of $\cos^{2\mu-1}\omega t$ is $\frac{-}{\pi}\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right)$, so

$$\cos^{2\mu-1}\omega t=\frac{2}{\pi}\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right)\cos\omega t+\text{other harmonics }\omega t. \tag{55}$$

Consequently, equation (54) becomes

$$\ddot{x}_1+\omega^2 x_1=-\frac{2}{\pi}A^{2\mu-1}\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right)\cos\omega t+\text{other harmonics of }\omega t.$$

The cos ωt term on the R.H.S. of equation (56) causes a resonance which is spurious since the solution of equation (50) is quite well defined as we have seen in the earlier section. Terms causing spurious resonances are known as secular terms and they have arisen because we have not paid attention to the fact that the frequency of motion is no longer ω, but has to be changed to some new value Ω which can be expanded in power of λ as

$$\Omega^2 = \omega^2 + \lambda\omega_1^2 + \lambda^2\omega_2^2 \ . \tag{57}$$

This is what we have seen in the earlier sections and this is what we need to do to make the perturbation theory work.
We rewrite equation (50) as

$$\ddot{x} + \Omega^2 x = -\lambda x^{2\mu-1} + \lambda\omega_1^2 + \lambda^2\omega_2^2 \, . \tag{58}$$

This is an identity since equations (57) and (58) simply reproduce equation (50). We now carry out the expansion of equation (51) and equating the same power of λ^n on the either side of equation (58), obtain at the $O(1)$

$$\ddot{x}_0 + \Omega^2 x_0 = 0 \tag{59}$$

with

$$x_0 = A\cos\Omega t \tag{60}$$

for the same initial condition as before. At $O(\lambda)$, we now get :

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^{2\mu-1} + \omega_1^2 x_0 \tag{61}$$

$$\Rightarrow \quad \ddot{x}_1 + \omega^2 x_1 = -\frac{2}{\pi}A^{2\mu-1}\beta\left(\mu + \frac{1}{2}, \frac{1}{2}\right)\cos\Omega t + \omega_1^2 A\cos\Omega t + \text{other harmonics of } \Omega t \, . \tag{62}$$

The correction to the frequency is an unknown and we choose it so that the resonance inducing $\cos\Omega t$ term is removed from the R.H.S. of equation (62). This leads to :

$$\frac{2}{\pi}A^{2\mu-2}\beta\left(\mu + \frac{1}{2}, \frac{1}{2}\right) \, . \tag{63}$$

Since the energy E is given by $E = \frac{1}{2}m\omega^2 A^2 + \frac{\lambda}{2\mu}A^{2\mu}$, as have

$$A = \sqrt{\frac{2E}{m\omega^2}} + O(\lambda) \tag{64}$$

and hence

$$\omega_1^2 = \frac{2}{\mu}\left(\frac{2E}{m\omega^2}\right)^{\mu-1}\beta\left(\mu + \frac{1}{2}, \frac{1}{2}\right) . \tag{65}$$

Correct to $O(\lambda)$, we get

$$\Omega^2 = \omega^2 + \lambda\frac{2}{\pi}\left(\frac{2E}{m\omega^2}\right)^{\mu-1}\beta\left(\mu + \frac{1}{2}, \frac{1}{2}\right) \tag{66}$$

$$\Rightarrow \quad \Omega = \omega + \frac{\lambda}{\pi\omega}\left(\frac{2E}{m\omega^2}\right)^{\mu-1} \left(\mu+\frac{1}{2},\frac{1}{2}\right) + O(\lambda)^2 . \tag{67}$$

For the special case of $\mu = 2$

$$\Omega = \omega + \frac{3}{4}\left(\frac{\lambda E}{m\omega^3}\right) + O(\lambda)^2 \tag{68}$$

in exact agreement with the result obtained in the previous equation (28). This method of doing perturbation theory is known as the Lindstedt Poincare technique.

4. Multiple time scale perturbation theory

We can imagine to do the perturbation theory in yet another fashion. The final result, where the frequency gets shifted means that if we consider the displacement of the oscillator to occur with the unperturbed frequency, then the phase of the oscillator, instead of being constant gets shifted slightly in time *i.e.*, it picks up a slow time variation. We picture the displacement x being a function of different time scales t_0, t_1, t_2, ... *etc.*, where $t_0 = \lambda^n t$, n=0,1,2,3... Thus, $x = x(t_0, t_1, t_2, \ldots)$ and

$$\frac{dx}{dt} = \sum_n \frac{\partial x}{\partial t_n}\frac{dt_n}{dt} = \sum_n \lambda^n \frac{\partial x}{\partial t_n} . \tag{69}$$

Similarly, for the acceleration

$$\frac{d^2x}{dt^2} = \sum_n \frac{\partial}{\partial t_n}\left(\frac{dx}{dt}\right)\frac{dt_n}{dt} = \frac{\partial^2 x}{\partial t_0^2} + 2\lambda\frac{\partial^2 x}{\partial t_0 \partial t_1} + \ldots . \tag{70}$$

Returning to equation of motion (50) and specialising to the case of $\mu = 2$,

$$\frac{\partial^2 x}{\partial t_0^2} + 2\lambda\frac{\partial^2 x}{\partial t_0 \partial t_1} + \ldots + \omega^2 x = -\lambda x^3 . \tag{71}$$

We also need to expand $x = x_0(t_0, t_1, t_2, \ldots) + \lambda x(t_0, t_1, t_2, \ldots) + \ldots$. At the zeroth order,

$$\frac{\partial^2 x_0}{\partial t_1^2} + +\omega^2 x_0 = 0 \tag{72}$$

leading to $x_0 = A\cos(\omega t_0 + \theta)$, where A and θ can be functions of t_0, t_1, t_2, ... *etc.* At $O(\lambda)$, equation (71) yields

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 + 2\frac{\partial}{\partial t_1}\frac{\partial x_0}{\partial t_1} = -x_0^3 = -A^3\cos^3(\omega t_0 + \theta) \tag{73}$$

Now, $\frac{\partial x_0}{\partial t_0} = -A\omega\sin(\omega t_0 + \theta)$ and hence $\frac{\partial}{\partial t_1}\left(\frac{\partial x_0}{\partial t_0}\right) = -\frac{\partial A}{\partial t_1}\omega\sin(\omega t_0 + \theta) - A\omega\cos(\omega t_0 + \theta)\frac{\partial \theta}{\partial t_1}$, which leads to

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 + 2\left[-\frac{\partial A}{\partial t_1}\omega \sin(\omega t_0 + \theta) - A\omega\cos(\omega t_0 + \theta)\frac{\partial \theta}{\partial t_1}\right] = -\frac{A^3}{4}\left[3\cos(\omega t_0 + \theta) + \cos 3(\omega t_0 + \theta)\right]. \tag{74}$$

Equating the coefficients of the different time harmonics, we have

$$\frac{\partial A}{\partial t_1} = 0 \tag{75a}$$

$$-2\omega A\frac{\partial \theta}{\partial t_1} = -\frac{3}{4}A^3 \tag{75b}$$

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = -\frac{A^3}{4}\cos 3(\omega t_0 + \theta)\,. \tag{75c}$$

We note that A remains a constants at this order while $\theta = \frac{3A^2}{8\omega}t_1$. The solution for x becomes

$$x = A\cos\left| \omega t_0 + \frac{3A^2}{8\omega}t_1 \right| + \lambda x_1 + \ldots$$

$$= A\cos\left(\omega t + \lambda\frac{3A^2}{8\omega}t\right) + \lambda x_1 + \ldots$$

$$= A\cos\Omega t + \lambda x_1 + \ldots \tag{76}$$

where,

$$\Omega = \omega + \lambda\frac{3A^2}{8\omega} + \ldots = \omega + \frac{3\lambda E}{4m\omega^3} + O(\lambda^2) \tag{77}$$

in exact agreement with the results of the previous sections.

The perturbation theory, delineated in this section, is the multiple scales perturbation theory and is designed to work when we have a periodic and an almost periodic solution which can be captured by a slowly varying amplitude and/or phase over an unperturbed periodic state.

5. Coordinate perturbation

In this section we again fall back on equation (50); setting $\lambda = -\varepsilon$ and $\omega = 1$, for convenience, we write

$$\ddot{x} + x = \varepsilon x^3. \tag{78}$$

Suppose we formulate it into an initial value problem by supplementing this equation (78) with initial conditions :

$$x(\varepsilon, 0) = 1$$

$$\dot{x}(\varepsilon, 0) = 0. \tag{79}$$

We assume $|\epsilon| \ll 1$ and so write the following perturbative form of the solution to equation (78)

$$x(\varepsilon,t) = x_0(\varepsilon) + \varepsilon x_1(t) + \varepsilon x_2(t) + ... \tag{80}$$

and set out to find an approximate solution. We shall encounter a subtleness involved in finding the solution and in the process touch upon the possible existence of a class of problems known as singular perturbation problems and a method to deal with them. Substituting the expression (80) in equations (78) and (79) and subsequently equating the coefficients of the power of ε, we get following set of infinite set of coupled differential equations and corresponding initial conditions :

$$\begin{aligned} &\ddot{x}_0 + x_0 = 0 && x_0(0) = 1,\ \dot{x}_0(0) = 0 \\ &\ddot{x}_1 + x_1 = -x_0^3 && x_1(0) = 0,\ \dot{x}_1(0) = 0\,. \\ &\quad \textit{etc.} && \quad \textit{etc.} \end{aligned} \tag{81}$$

Solution to the leading equation of the set (81) is cos *t* putting which in the next equation of the set (81) we arrive at :

$$x_1(\varepsilon,t) = \frac{1}{32}\cos t + \frac{3}{8} t \sin t - \frac{1}{32}\cos 3t\,. \tag{82}$$

Hence, the solution (80) approximates to :

$$x(\varepsilon,t) = \cos t + \varepsilon\left(\frac{1}{32}\cos t + \frac{3}{8} t \sin t - \frac{1}{32}\cos t\right) + O(\varepsilon^2)\ . \tag{83}$$

Obviously, this approximation is non-uniform in the range $0 \le t < \infty$ due to the the presence *t* sin *t* term. This point about non-uniformity is clarified later in a section. For the time-being it suffices to note that for large t (*i.e.*,$\gg O(1/\varepsilon)$),terms of $O(\varepsilon^2 t^2)$ dominator and the series (80) is no longer convergent. So, what is the way out? Well, setting

$$t = T(\varepsilon,\tau) = \tau + \varepsilon T_1(\tau) + \varepsilon T_2(\tau) + ... \tag{84}$$

we, expand the independent variable *t* in the powers of ε. Here τ is called the strained or perturbed coordinate. Putting the expression (84) in the relation (83), we get :

$$x(\varepsilon,t) = X(\varepsilon,\tau) = \cos\tau + \varepsilon\left(-T_1(\tau)\sin\tau + \frac{1}{32}\cos\tau + \frac{3}{8}\tau\sin\tau - \frac{1}{32}\cos 3\tau\right) + O(\varepsilon^2)\ . \tag{85}$$

Now is the time to force $T_1(\tau)$ to take such a value so that the term (*t* sin *t*) which causes non-uniformity of the approximation (83) vanishes. This method is called coordinate perturbation method. This means that from the expression (85), we choose :

$$T_1(\tau) = \frac{3}{8}\tau\,. \tag{86}$$

Hence, within the error of $O(\varepsilon^2)$ we have the solution to equation (78) as :

$$x(\varepsilon,t) = X(\varepsilon,\tau) = \cos\tau + \frac{\varepsilon}{32}(\cos\tau - \cos 3\tau) \tag{87a}$$

where,

$$t = \tau + \frac{[\text{illegible}]}{8}\varepsilon\tau + O(\varepsilon^2)\,. \tag{87b}$$

This process can be carried out *ad infinitum* reducing the error at each step by finding successive values of T_2, T_3, *etc.* so that culprit terms are forced out of the solution.

6. Non-perturbative approximation ("Hartee Approximation" or equivalent linearisation)

Returning once again to equation (50), we want to explore how we can think of it as a linear system. This would require replacing $x^{2\mu-1}$ by a linear term. Keeping the dimensionality of the term unaltered, we can imagine making the replacement

$$x^{2\mu-1} = \alpha\left\langle x^2\right\rangle^{\mu-1} x \tag{88}$$

where α is a dimensionless number. Since the motion is periodic, we can think of $\langle x^2\rangle$ as an average over a cycle. We do not *a priori* know what $\langle x^2\rangle$ is therein lies the strength of the method. The determination of $\langle x^2\rangle$ in self consistent manner leads to a non-perturbative determination of the frequency of the oscillator. This is similar to the mean field theory of the ferromagnetic transition and the Hartee approximation in the many body context. The equation of motion (50) is, under the approximation of relation (88),

$$x + \omega^2\left(1 + \frac{\lambda a}{\omega^2}\left\langle x^2\right\rangle^{\mu-1}\right)x = \ddot{x} + \Omega^2 x = 0 \tag{89}$$

where,

$$\Omega^2 = \omega^2\left(1 + \frac{\lambda\alpha}{\omega^2}\left\langle x^2\right\rangle^{\mu-1}\right) \tag{90}$$

from which we find the time period of the oscillation. The energy of the oscillator of equation (89) is $E = \frac{1}{2}m\Omega^2 A^2$ and the amplitude A is given by $2\langle x^2\rangle$ for the simple harmonic motion. Consequently,

$$\left\langle x^2\right\rangle : \quad \frac{\quad}{m\Omega^2} \quad \frac{\quad}{m\omega^2\left(1 + \frac{\lambda\alpha}{\omega^2}\left\langle x^2\right\rangle^{\mu-1}\right)} \tag{91}$$

$$\Rightarrow \quad \frac{\alpha\lambda}{\omega^2}\left\langle x^2\right\rangle^{\mu} + \left\langle x^2\right\rangle - \frac{E}{m\omega^2} = 0 \tag{92}$$

This is the self consistent equation which determines $\langle x^2\rangle$ and thence determines Ω from equation (90). It is instructive to examine the limiting cases. If $\frac{\lambda\langle x^2\rangle^{\mu-1}}{\omega^2}$ is small, then from equation (92), we can write $\langle x^2\rangle \approx \frac{E}{m\omega^2}$ and equation (90) becomes

$$\Omega^2 = \omega^2 + \alpha\lambda\left(\frac{E}{m\omega^2}\right)^{\mu-1}. \tag{93}$$

Comparing with the expression (67), we can see that the correct structure has been obtained and the exact agreement can be achieved if we can choose

$$\alpha = \frac{2^\mu}{\pi}\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right). \tag{94}$$

Now, we switch to the other extreme, $\lambda\langle x^2\rangle^\mu / \omega^2 \gg 1$, a limit that is totally inaccessible in the perturbation theory. In this case, the relation (92) yields

$$\langle x^2\rangle^\mu \approx \frac{E}{\alpha m\lambda} \tag{95}$$

and from equation (90), we find

$$\Omega^2 \approx \alpha\lambda\langle x^2\rangle^{\mu-1} = \alpha\lambda\left(\frac{E}{\alpha m\lambda}\right)^{1-\frac{1}{\mu}} = 2\lambda^{\frac{1}{\mu}}\left(\frac{E}{m}\right)^{1-\frac{1}{\mu}}\left[\frac{1}{\pi}\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right)\right]^{\frac{1}{\mu}}. \tag{96}$$

The time period in this limit is

$$T = \frac{2\mu}{\Omega} = 2\pi\frac{\lambda^{-\frac{1}{2\mu}}}{\sqrt{2}}\left(\frac{m}{E}\right)^{\frac{1}{2}-\frac{1}{2\mu}}\left[\frac{\pi}{\beta\left(\mu+\frac{1}{2},\frac{1}{2}\right)}\right]^{\frac{1}{2\mu}}. \tag{97}$$

The exact answer in this limit, according to equation (12) should have been

$$T_{exact} = 2^{1+\frac{1}{2\mu}}\mu^{\frac{1}{2\mu}-1}\lambda^{-\frac{1}{2\mu}}\left(\frac{m}{E}\right)^{\frac{1}{2}-\frac{1}{2\mu}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2\mu}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2\mu}\right)}. \tag{98}$$

The λ, m and E dependences tally exactly; the perfecter, however, is different. For the usual case of m =2, the ratio of the two prefactors is $\sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{8}{3}\right)^{\frac{1}{4}}$, which is quite close to unity and is an indicator of the accuracy of our method.

For μ=2, we can solve equation (92) exactly and get:

$$\langle x^2\rangle = \frac{-1 \pm \sqrt{1+\dfrac{4\alpha E\lambda}{m\omega^4}}}{\dfrac{2\alpha\lambda}{\omega^2}}. \tag{99}$$

From eq. (94), α = 3/2 and keeping only the positive square root

$$\langle x^2\rangle = \frac{\sqrt{1+\dfrac{6\lambda E}{m\omega^4}}-1}{\dfrac{3\lambda}{\omega^3}} \tag{100}$$

and from eq. (90)

$$\Omega^2 = \omega^2 + \frac{\omega^2}{2}\left(\sqrt{1+\frac{6\lambda E}{m\omega^4}}-1\right) \tag{101}$$

This formula gives an extremely good rendering of the frequency of the oscillator with the potential $V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{1}{4}m\lambda x^4$ for all values of λ. By noting that Ω is dependent on the dimensionless parameter $6\lambda E / m\omega^4$ which is nothing but the β used in expression (49), this result (101) may be compared with the Padé approximant.

To end this section, we show that the perturbation theory of Lindstedt and Pioncare can be vastly improved by using some appropriate scalings coming from the non-perturbative results of equations (100) and (101). Starting with

$$\ddot{x} + \omega^2 x = -\lambda x^3 \tag{102}$$

we write it as

$$\ddot{x} + \left(\omega^2 + \frac{3}{2}\langle x^2\rangle x\right)x = -\lambda\left(x^3 - \frac{3}{2}\langle x^2\rangle x\right) \tag{103}$$

$$\Rightarrow \quad \ddot{x} + \Omega^2 x = -\lambda\left(x^3 - \frac{3}{2}\langle x^2\rangle x\right) \tag{104}$$

with Ω given by equation (101) and $\langle x^2\rangle$ by equation (100). We define $\tau = \Omega t$ and $y = \dfrac{x}{\langle x^2\rangle^{\frac{1}{2}}}$, to write

$$\frac{d^2y}{d\tau^2} + y = \frac{\lambda\langle x^2\rangle}{\Omega^2}\left(y^3 - \frac{3}{2}y\right). \tag{105}$$

We note that

$$\frac{\Omega^2}{\langle x^2\rangle^2}=\frac{3\lambda\left\{1+\frac{1}{2}\left(\sqrt{1+\frac{6\lambda E}{m\omega^4}}-1\right)\right\}}{\left(\sqrt{1+\frac{6\lambda E}{m\omega^4}}-1\right)}. \tag{106}$$

This quantity goes to a constant *viz.* $m\omega^4/E$ for $\lambda\to 0$ and goes to $3\lambda/2$ for $\lambda\to\infty$. Ergo, the scaled coupling

$$\bar{\lambda}=\lambda\frac{\langle x\rangle^2}{\Omega^2} \tag{107}$$

is of the $O(\lambda)$ for $\lambda\to 0$ and is 2/3 for $\lambda\to\infty$. This $\bar{\lambda}$ never becomes too large and is ideally suited for carrying out a perturbation calculation for the dynamics of equation (105) which can be rewritten as :

$$\frac{d^2y}{d\tau^2}+y=-\bar{\lambda}\left(y^3-\frac{3}{2}y\right) \tag{108}$$

we can now carry out the standard Lindstedt Pioncare perturbation theory.

7. Renormalisation and the non-linear oscillator

The nonlinear oscillator can be used to illustrate another very significant development in the theoretical physics-the idea of renormalisation. The necessity for renormalisation arose when one tried to calculate physical quantities like cross-section in the quantum theory of the electromagnetic field-quantum electrodynamics. Perturbation calculations lead to divergence which had to be removed by invoking the concept of dressed masses and charges. In what follows, we will see how that approach can be invoked in our case. We have already seen, that a blind perturbation theory will lead to resonances. Let us take that approach at its face value and write down the divergent solution upto $O(\lambda)$. We write the zeroth order solution as (see sec. 3)

$$x_0=A_0\cos(\omega t+\theta_0) \tag{109}$$

Specialising to the case m = 2 the $O(\lambda)$ correction, x_1, is

$$\ddot{x}_1+\omega^2x_1=-x_0^3=-A_0\cos^3(\omega t+\theta_0) \tag{110}$$

$$\Rightarrow\quad \ddot{x}_1+\omega^2x_1=-\frac{A_0}{4}\left[3\cos(\omega t+\theta_0)-\cos 3(\omega t+\theta_0)\right]. \tag{111}$$

The solution can be written down as

$$x_1(t)=R\cos(\omega t+\theta_0)-\frac{3A_0^3}{8\omega}(t-t_0)\sin(\omega t+\theta_0)-\frac{A_0^3}{32\omega^2}\cos 3(\omega t+\theta_0) \tag{112}$$

If we maintain the boundary condition that $x = x_0 + \lambda x_1 + ... = A_0$ at $t = -\theta_0 / \omega$, then R is forced to be $A_0^3/(32\omega^2)$ and the solution for $x(t)$ to $O(\lambda)$ is

$$x(t) = A_0 \cos(\omega t + \theta_0) + \lambda\left[\frac{A_0^3}{32\omega^2}\{\cos(\omega t + \theta_0) - \cos 3(\omega t + \theta_0)\} - \frac{3A_0^3}{8\omega}(t - t_0)\sin(\omega t + \theta_0)\right] + ... \tag{113}$$

We note that the initial condition can be considered arbitrary and the solution can be written starting from any point in true. This freedom allows us to set up a renomalisation group flow.

This is the divergent solution which is obtained at $O(\lambda)$. Our aim is to define the renormalisation constants Z_1 and Z_2 for the amplitude and the phase to absorb the infinities that occur. Accordingly,

$$A(t_0) = A_0 = Z_1(t_0, \tau)A(\tau) \tag{114a}$$

$$\theta(t_0) = \theta_0 = \theta(\tau) + Z_2(t_0, \tau). \tag{114b}$$

In the above τ is an arbitrary time scale which can always be introduced since $t - t_0 = t - \tau + \tau - t_0$. The $Z_{1,2}$ are to be perturbatively calculated as

$$Z_1 = 1 + \sum_{n=1}^{\infty} a_n \lambda^n \tag{115a}$$

$$Z_2 = \sum_{n=1}^{\infty} b_n \lambda^n. \tag{115b}$$

Working to $O(\lambda)$, we can write

$$x(t) = A(\tau)(1 + a_1\lambda)\cos(\omega t + \theta(\tau) + b_1\lambda)$$
$$+\lambda\left[\frac{A_0^3}{32\omega^2}\{\cos(\omega t + \theta_0) - \cos 3(\omega t + \theta_0)\} - \frac{3A_0^3}{8\omega}(t - t_0)\sin(\omega t + \theta_0)\right] + O(\lambda^2) \tag{116}$$

$$\Rightarrow \quad x(t) = A(\tau)(1 + a_1\lambda)\left(\cos(\omega t + \theta(\tau)) - b_1\lambda \sin(\omega t + \theta(\tau))\right)$$
$$+\lambda\left[\frac{A_0^3}{32\omega^2}\{\cos(\omega t + \theta_0) - \cos 3(\omega t + \theta_0)\} - \frac{3A_0^3}{8\omega}(t - t_0)\sin(\omega t + \theta_0)\right] + O(\lambda^2) \tag{117}$$

$$\Rightarrow \quad x(t) = A(\tau)\cos(\omega t + \theta(\tau)) + \lambda\left[a_1 A(\tau)\cos(\omega t + \theta(\tau)) - b_1 A(\tau)\sin(\omega t + \theta(\tau))\right)$$
$$+\frac{A^3(\tau)}{32\omega^2}\{\cos(\omega t + \theta(\tau)) - \cos 3(\omega t + \theta)\} - \frac{3A^3(\tau)}{8\omega}(t - t_0)\sin(\omega t + \theta(\tau)) + O(\lambda^2). \tag{118}$$

Now, a_1 and b_1 have to be chosen such that they connect t_0 and τ. This is possible for b_1 since $t-t_0$ in the coefficient of $\sin(\omega t+\theta)$ in equation (118) can be split as $t-\tau+\tau-t_0$. However the coefficient of $\cos(\omega t+\theta)$ does not allow for this facility. Ergo, we choose

$$a_1 = 0 \tag{119a}$$

$$b_1 = -\frac{3A^2}{8\omega}(\tau - t_0)\,. \tag{119b}$$

With this choice, the solution of equation (118) becomes upto $O(\lambda)$:

$$x(t,\tau) = A\cos(\omega t+\theta) + \lambda\left[\frac{A_0^3}{32\omega^2}\{\cos(\omega t+\theta) - \cos 3(\omega t+\theta)\} - \frac{3A^3}{8\omega}(t-\tau)\sin(\omega t+\theta)\right]. \tag{120}$$

But τ is an arbitrary time on which the solution cannot depend. Hence, we demand

$$\frac{dx}{d\tau} = 0\,. \tag{121}$$

Differentiating the relation (120) with respect to τ we arrive at :

$$\frac{dx}{d\tau} = \frac{dA}{d\tau}\cos(\omega t+\theta) - A\frac{d\theta}{d\tau}\sin(\omega t+\theta) + \lambda\frac{3A^3}{8\omega}\sin(\omega t+\theta) + o(\lambda)\text{terms in } \frac{dA}{d\tau}, \frac{d\theta}{d\tau}\,. \tag{122}$$

Which in accordance with the condition of equaton (121) yield :

$$\frac{dA}{d\tau} = 0 \tag{123a}$$

$$\frac{d\theta}{d\tau} = \lambda\frac{3A^2}{8\omega} \tag{123b}$$

leading to $\theta = \frac{3A^2}{8\omega}\tau$. Inserting this solution for θ in the expression (120), we get :

$$x(t,\tau) = A\cos\left(\omega t + \frac{3\lambda A^2}{8\omega}\tau\right) + \lambda\left[\frac{A^3}{32\omega^2}\{\cos(\omega t+\theta) - \cos 3(\omega t+\theta)\} - \frac{3A^3}{8\omega}(t-\tau)\sin(\omega t+\theta)\right] \tag{124}$$

we now remove the divergence in equation (124) by choosing $\tau = t$, whence

$$x(t,\tau) = A\cos\left(\omega + \frac{3\lambda A^2}{8\omega}\tau\right)t + \lambda\left[\frac{A^3}{32\omega^2}\{\cos(\omega t+\theta) - \cos 3(\omega t+\theta)\}\right] \tag{125}$$

and we recognise the new frequency as

$$\Omega = \omega + \frac{3\lambda A^2}{8\omega} \tag{126}$$

which is in accordance with the previous results in relations (28) and (68).

8. Resonances in forced systems

The forced system

$$\ddot{x} + \omega^2 x = f(t) \tag{127}$$

has a particularly straightforward solution when *f(t)* is oscillatory *i.e. f(t) = Fcos ωt*. The solution

$$x(t) = A\cos(\omega t + \theta) + \frac{}{\omega^2 + \Omega^2}\cos \Omega t \tag{128}$$

where A and θ are to be determined from the initial condition. Resonance occurs when $\Omega = \omega$. The solution can no longer be written in the form given in equation (128). Instead it becomes

$$x(t) = A\cos(\omega t + \theta) + \frac{Ft}{2\omega^2}\sin \Omega t \tag{129}$$

which clearly shows that even if we have $x = \dot{x} = 0$ at $t = 0$, the solution $x(t)$ exists and actualy increase with time. The solution changes if we are dealing with a non-linear oscillator

$$\ddot{x} + \omega^2 x + \lambda x^3 = F\cos \Omega t \cdot \tag{130}$$

For small λ, we can replace the non-linear oscillator

$$x + \left(\omega^2 + \frac{3}{}\lambda A^2\right)x = F\cos \Omega t \tag{131}$$

with the solution

$$x(t) = A_1 \cos\tilde{\omega} + A_2 \sin\tilde{\omega}t + \frac{F\cos \Omega t}{\omega^2 - \Omega^2 + \frac{3}{}\lambda A^2} \tag{132}$$

If $\omega \approx \Omega$, and λ is small, the last term in the R.H.S. of eqation (132) will dominate (also in any realistic system, the damping will eventually diminish the first two terms) and we have (with $\omega \approx \Omega(1+\delta)$)

$$x(t) \approx \frac{F\cos \Omega t}{\left(2\delta\Omega^2 + \frac{3}{4}\lambda A^2\right.} = \pm A\cos \Omega t \tag{133}$$

depending on the sign and magnitude of δ and this implies

$$A^2 = \frac{F^2}{2\delta\Omega^2 + \frac{3}{4}\lambda A^2} \tag{134}$$

This determines the amplitude 'A' self-consistently via the cubic equation in A^2. We pause rather abruptly here to jump onto the topic of parameteric resonance in the next section. Discussion on the issue discontinued here will be taken up later in this review after extending the scope to damped oscillators.

9. Parametric resonance

Unlike in the previous section, where resonance was produced because of a coordinate independent periodic external force, a parametric resonance occurs when a parameter in equation of motion becomes a periodic function of time. In the context of our simple harmonic oscillator $\ddot{x} + \omega^2 x = 0$, this means that the frequency ω becomes a function of time, *i.e.*, we now have

$$\ddot{x} + \omega^2(t)x = 0 \tag{135}$$

with

$$\omega(t) = \omega(t+T) \tag{136}$$

making ω a periodic function of time. We imaging that $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of equation (135). Then any other solution can be written as a linear combination of $x_1(t)$ and $x_2(t)$. In particular, we note that if $x(t)$ is a solution of equation (135), so is $x(t+T)$ due to the relation (136). Then both $x_1(t+T)$ and $x_2(t+T)$ are solutions of equation (135) and we must have

$$x_1(t+T) = c_{11}x_1(t) + c_{12}x_2(t) \tag{137a}$$

$$x_2(t+T) = c_{21}x_1(t) + c_{22}x_2(t) \tag{137b}$$

The matrix of the coefficient c_{ij} can be diagonalised with the eigenvalues $\lambda_{1,2}$ and in that basis the solutions $X_1(t)$ and $X_2(t)$ have the property

$$X_{1,2}(t+T) = \lambda_{1,2}X_{1,2}(t). \tag{138}$$

Returning to equation (135), we note that :

$$\ddot{x} + \omega^2(t)x_1 = 0 \tag{139a}$$

$$\ddot{x} + \omega^2(t)x_2 = 0. \tag{139b}$$

Multiplying equation (139a) by x_2, the second on *i.e.*, equation (139b) by x_1 and subtracting one from the other, we get

$$\frac{d}{dt}(\dot{x}_1x_2 - \dot{x}_2x_1) = 0 \tag{140}$$

$$\Rightarrow \quad \dot{x}_1x_2 - \dot{x}_2x_1 = \text{constant} \tag{141}$$

Thus,

$$\dot{x}_1(t)x_2(t)-\dot{x}_2(t)x_1(t)=\dot{x}_1(t+T)x_2(t+T)-\dot{x}_2(t+T)x_1(t+T) \tag{142}$$

for any pair of solution $x_1(t)$ and $x_2(t)$. When applied to our pair of $X_1(t)$ and $X_2(t)$ (see equation (138)), equation (142) yields the vital relation

$$\lambda_1\lambda_2=1. \tag{143}$$

From equation (138)

$$X_{1,2}(t+nT)=\lambda_{1,2}^{n}X_{1,2}(t) \tag{144}$$

for any integer *n*. We now note that if the λ's are real then, we have two possibilities :

1. $\lambda_1>1\ \lambda_2<1$: In this case, one of the solutions $X_{1,2}$ grows with time and the motion is unbounded leading to instability.
2. $\lambda_1=\lambda_2=\pm 1$: If we have the positive sign, then the solution is periodic with period *T*. If the negative sign holds, then the solution has period 2*T*- the phenomenon of period doubling.

If the $\lambda_{1,2}$ are complex, then they need to be complex conjugate and if we write $\lambda_{1,2}$ in the form $re^{i\theta}$, equation (143) yields $r=1$ making $\lambda_{1,2}=e^{\pm i\theta}$. The solution *X(t)* has the general structure

$$X(t)=\lambda^{\frac{t}{T}}\Pi(t) \tag{145}$$

where $\Pi(t)$ is a periodic function of *t* with period *T*. This ensures $X(t+T)=\lambda X(t)$. The motion $X(t)$ can be have the following form :

1. If $\lambda>1$, $X(t)$ is unbounded.
2. If $\lambda=\pm 1$, $X(t)$ is periodic with period *T* and for $\lambda=-1$, $X(t)$ is periodic with period 2*T*.
3. If $\lambda=e^{\pm i\theta}$, $X(t)$ is bounded and quasi-periodic in general since $e^{\pm i\theta\frac{t}{T}}$ is periodic with frequency $\omega=\dfrac{\theta}{2\pi T}$.

The periodic orbits separate regions of bounded and unbounded motion. The eigenvalues λ are usually known as Floquet multipliers.

The most studied equation of this class is the Mathieu equation, where

$$\ddot{x}+\left(\omega_0^2+\varepsilon\cos\Omega t\right)x=0. \tag{146}$$

The situations which can be tackled analytically is the one where $\varepsilon\ll 1$ and perturbation theory can be used. We note that for $\varepsilon=0$, the solution is $x_0=A\cos\omega_0 t+B\sin\omega_0 t$. When ε is switched on, we find the term $\varepsilon\cos(\Omega t)x$ will contribute a response at frequency ω_0 with

$x(t)$ approximated as x_0 provided $\Omega = 2\omega_0$. This implies that $\Omega = 2\omega_0$ will provide a response at $O(\varepsilon)$. Any other Ω will respond at a higher order. We accordingly let $\Omega = 2\omega_0 + \delta$ and seek a response of frequency $\omega_0 + \delta/2$. The trial solution for $x(t)$ is written as

$$x(t) = A(t)\cos\left(\omega_0 + \frac{\delta}{2}\right)t + B(t)\sin\left(\omega_0 + \frac{\delta}{2}\right)t. \tag{147}$$

We have deliberately made A and B weakly time dependent since the response is not expected to be exactly periodic from our previous discussion. We will be able to explore the region near the resonance by the form that we have assumed in equation (147). Weak time dependence of A and B implies that $\dot{A} \gg \ddot{A}$, $\dot{B} \gg \ddot{B}$. With this in mind, we try equation (147) as a solution to equation (146). So we have :

$$\begin{aligned}\ddot{x} = &-A\left(\omega_0 + \frac{\delta}{2}\right)^2 \cos\left(\omega_0 + \frac{\delta}{2}\right)t - 2\dot{A}\left(\omega_0 + \frac{\delta}{2}\right)\sin\left(\omega_0 + \frac{\delta}{2}\right)t \\ &-B\left(\omega_0 + \frac{\delta}{2}\right)^2 \sin\left(\omega_0 + \frac{\delta}{2}\right)t + 2\dot{B}\left(\omega_0 + \frac{\delta}{2}\right)\cos\left(\omega_0 + \frac{\delta}{2}\right)t\end{aligned} \tag{148a}$$

$$\omega_0^2 x = \omega_0^2 A\cos\left(\omega_0 + \frac{\delta}{2}\right)t + \omega_0^2 B\sin\left(\omega_0 + \frac{\delta}{2}\right)t \tag{148b}$$

$$\begin{aligned}\varepsilon x\cos(2\omega_0 + \delta t)t = \frac{\varepsilon}{2}\Bigg[&A\cos\left(3\omega_0 + \frac{3\delta}{2}\right)t + A\cos\left(\omega_0 + \frac{\delta}{2}\right)t \\ &+B\sin\left(3\omega_0 + \frac{3\delta}{2}\right)t - B\sin\left(\omega_0 + \frac{\delta}{2}\right)t\Bigg].\end{aligned} \tag{148c}$$

The trial solution *x(t)* gives (on adding equations (148a) to (148c))

$$\begin{aligned}\ddot{x} + \left(\omega_0^2 + \varepsilon\cos(2\omega_0 + \delta)t\right)x = &-2\dot{A}\left(\omega_0 + \frac{\delta}{2}\right)\sin\left(\omega_0 + \frac{\delta}{2}\right) + 2\dot{B}\left(\omega_0 + \frac{\delta}{2}\right)\cos\left(\omega_0 + \frac{\delta}{2}\right)t \\ &-A\left[\left(\omega_o + \frac{\delta}{2}\right)^2 - \omega_0^2\right]\cos\left(\omega_o + \frac{\delta}{2}\right)t - B\left[\left(\omega_o + \frac{\delta}{2}\right)^2 - \omega_0^2\right]\sin\left(\omega_o + \frac{\delta}{2}\right)t \\ &+\frac{\varepsilon}{2}A\cos\left(\omega_o + \frac{\delta}{2}\right)t - \frac{\varepsilon}{2}B\sin\left(\omega_o + \frac{\delta}{2}\right)t \\ &+\frac{\varepsilon}{2}A\cos\left(3\omega_o + \frac{3\delta}{2}\right)t - \frac{\varepsilon}{2}B\sin\left(\omega_o + \frac{3\delta}{2}\right)t\end{aligned} \tag{149}$$

For $x(t)$ to be a solution, the R.H.S. has to be zero. The different harmonics and the sines and cosines have to separately vanish. We have to ignore the harmonic $3(\omega_0 + \delta/2)$ since the trial solution did not have such term to begin with. It is easy to check that if equation (147) had contained terms $A_1 \cos 3(\omega_0 + \delta/2)t$ and $B_1 \sin 3(\omega_0 + \delta/2)t$, then the corresponding equation (149) would have immediately shown that $A_1 \sim O(\varepsilon A_0)$. Consequently, for the lowest order calculation in ε, A_1 and hence the higher harmonics can be dropped. The sine terms in equation (149) need to satisfy:

$$2\dot{A}\left(\omega_0\delta + \frac{\delta}{2}\right) + B\left(\omega_0\delta + \frac{\delta^2}{2}\right) + \frac{\varepsilon}{2}B = 0 \tag{150}$$

while the cosine terms yield

$$2\dot{B}\left(\omega_0 + \frac{\delta}{2}\right) - A\left(\omega_0\delta + \frac{\delta^2}{4}\right) + \frac{\varepsilon}{2}A = 0 \tag{151}$$

Keeping only the leading order terms in δ,

$$\dot{A} + B\left(\frac{\delta}{2} + \frac{\varepsilon}{4\omega_0}\right) = 0 \tag{152a}$$

$$\dot{B} + A\left(-\frac{\delta}{2} + \frac{\varepsilon}{4\omega_0}\right) = 0 \tag{152b}$$

The solutions for A and B are of the form $e^{\mu t}$, with

$$\begin{vmatrix} \mu & \dfrac{\delta}{2} + \dfrac{\varepsilon}{4\omega_0} \\ -\dfrac{\delta}{2} + \dfrac{\varepsilon}{4\omega_0} & \mu \end{vmatrix} = 0 \tag{153}$$

yielding

$$\mu = \pm\frac{1}{2}\sqrt{\left(-\delta^2 + \frac{\varepsilon^2}{4\omega_0^2}\right)} \tag{154}$$

If $\delta < |\varepsilon / 2\omega_0|$, μ has a positive root and the solution is unbounded. If $-\varepsilon/2\omega_0 > \delta > +\varepsilon/2\omega_0$, then μ is purely imaginary and the solution $x(t)$ of equation (147) does not have a definite periodicity. Periodic solutions are obtained on the dividing line: $\delta = \pm\varepsilon/2\omega_0$. The periodicity of the solution is twice the periodicity of the forcing term corresponding to the eigen value $\lambda = 1$ in equation (138). Thus, for $\omega_0/\Omega \approx 1/2$, we have in the $\varepsilon - \delta$ plane two branches starting out from $\omega_0/\Omega \approx 1/2$, along which period doubled solutions exist separating the regions of unbounded and bounded trajectories. The higher order corrections to the curve $\delta = \pm\varepsilon/2\omega_0$ is obtained by keeping the higher harmonics in a systematic manner.

Shifting to the region $\omega_0/\Omega \approx 0$, we have the differential equation :

$$\ddot{x} + \left(\epsilon\,\delta_1 + \epsilon^2\,\delta_2 + \ldots + \varepsilon\cos\Omega t\right)x = 0 \tag{155}$$

where we have used $\omega_0^2 = 0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + .$ Using the perturbative expansion of $x(t)$ in the powers of ε in equation (155), we get

$$\begin{aligned} &\ddot{x}_0 = 0 \\ &\ddot{x}_1 = -x_0 \cos \Omega t - \delta_1 x_0 \\ &\ddot{x}_2 = -x_1 \cos \Omega t - \delta_2 x_0 - \delta_1 x_1 \\ &etc... \end{aligned} \tag{156}$$

From these equations we have

$$x_0 = A_0 \tag{157}$$

as the periodic solution. Where A_0 is a constant. Putting relation (157) in the second equation of the system (156), we obtain

$$x_1 = \frac{A_0}{\Omega^2}\cos \Omega t + A_1 - \delta_1 A_0\left(\frac{t^2}{2} + C_1 t\right). \tag{158}$$

Where A_1 and C_1 are constant. Periodic solution sequences $\delta_1 = 0$. Again, using (157) and *(158) in the second equation of (156), we arrive at*

$$x_2 = -\left(\delta^2 A_0 + \frac{A_0}{2\Omega^2}\right)\frac{t^2}{2} + \frac{A_0 \cos 2\Omega t}{8\Omega^2} + \frac{A_1 \cos \Omega t}{\Omega^2} + A_3 t + A_4 \tag{159}$$

and demanding periodicity of $2\pi / \Omega$ implies :

$$\delta_2 A_0 + \frac{A_0}{2\Omega^2} = 0; \qquad A_3 = 0. \tag{160}$$

Therefore, periodic solutions are obtained for

$$\delta_2 = -\frac{1}{2\Omega^2} \tag{161}$$

$$\Rightarrow \quad \omega_0^2 = -\varepsilon^2 \frac{1}{2\Omega^2} . \tag{162}$$

As a final example, we explore the region where $\Omega \approx \omega_0$ *i.e.*, we write equation (146) in the form

$$\ddot{x} + \omega_0^2 x + \varepsilon x \cos(\omega_0 + \delta)t = 0 \tag{163}$$

We seek a solution which will have the same period as the forcing *i.e.*,

$$x(t) = A(t)\cos(\omega_0 + \delta)t + B(t)\sin(\omega_0 + \delta)t \tag{164}$$

clearly, this solution is inadequate and we need to introduce

$$X(t) = A(t)\cos(\omega_0 + \delta)t + B(t)\sin(\omega_0 + \delta)t + \varepsilon x_1(t) \tag{165}$$

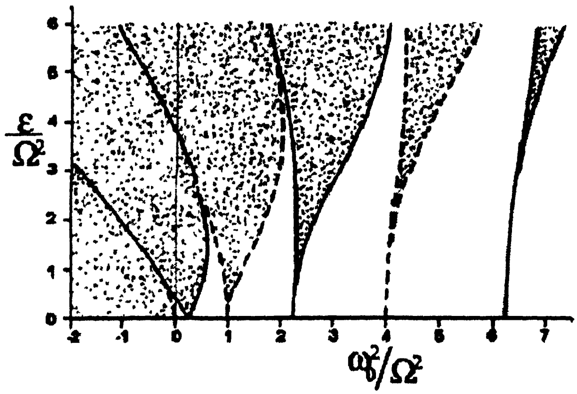

Figure 2. This is a rather exact plot obtained numerically in contrast to the perturbative analysis done in the review. The shaded region is unstable while the unshaded one have all bounded solutions and hence is unstable. The dashed lines correspond to the periodic solutions with angular frequency Ω whereas the solid curves represent solutions of angular frequency $\Omega/2$.One may check that the perturbation technique adopted gives correct hints about the solutions.

with

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{A}{2}\left(1 + \cos 2(\omega_0 + \delta)t\right) - \frac{B}{2}\cos 2(\omega_0 + \delta)t \tag{166}$$

leading to

$$x_1 = \frac{A}{2\omega_0^2} + \frac{A}{2}\frac{\cos 2(\omega_0 + \delta)t}{\left[2(\omega_0 + \delta)\right]^2 - \omega_0^2} + \frac{B}{2}\frac{\sin 2(\omega_0 + \delta)t}{\left[2(\omega_0 + \delta)\right]^2 - \omega_0^2} \ . \tag{167}$$

Trying equation (165) in equation (163), we find

$$\begin{aligned}
&-2\dot{A}(\omega_0 + \delta)\sin(\omega_0 + \delta)t + \left[\omega_0^2 - (\omega_0 + \delta)^2\right]A\cos(\omega_0 + \delta)t \\
&+2\dot{B}(\omega_0 + \delta)\cos(\omega_0 + \delta)t + \left[\omega_0^2 - (\omega_0 + \delta)^2\right]B\cos(\omega_0 + \delta)t \\
&-\left(\frac{\varepsilon^2 A}{2\omega_0^2}\right)\cos(\omega_0 + \delta)t + -\left(\frac{\varepsilon^2 A}{4}\right)\frac{\cos(\omega_0 + \delta)t}{3\omega_0^2} + \left(\frac{\varepsilon^2 B}{4}\right)\frac{\sin(\omega_0 + \delta)t}{3\omega_0^2} = 0
\end{aligned} \tag{168}$$

where we have kept only the leading order terms in ε and δ and the lowest harmonics. Equating the coefficients of $\cos(\omega_0 + \delta)t$ and $\sin(\omega_0 + \delta)t$ separately to zero.

$$\dot{A} = \left(-\delta + \frac{\quad}{12\omega_0^3} \right) B \tag{169a}$$

$$\dot{B} = \left(\delta + \frac{5\varepsilon^2}{12\omega_0^3}\right)A \ . \tag{169b}$$

Trying $A,B \sim \exp(\mu t)$, we have

$$\mu^2 = -\left| \delta - \frac{}{12\omega_0^3} \quad \delta + \frac{5\varepsilon^2}{12\omega_0^3} \right. \tag{170}$$

Clearly, μ^2 is positive for $\delta < \frac{\varepsilon}{12\omega_0^3}$ and $\delta > -\frac{5\varepsilon^2}{12\omega_0^3}$ where there is unbounded motion. The peri odic motion occurs on the curves $\delta = \frac{\varepsilon^2}{12\omega_0^3}$ and $\delta = -\frac{5\varepsilon^2}{12\omega_0^3}$ and outside these curves the motion is bounded but in general aperiodic. Starting from the points $\omega_0 / \Omega = 0, 1/2, 3/2, \dots$, we have curves which limit the unbounded motion. As the amplitude of the modulation increases, the width of the region of the unbounded motion increases and at a critical ε, the regions starting from different parts on the ω_0 / Ω axis merge and the bounded motion disappears altogether. Refer to Figure 2.

It is instructive to consider an exactly solvable version of this problem. Instead of the sinusoidal modulation of equation (135), we consider a situation, where the modulation is such that $\omega^2(t) = \omega_0^2 + \varepsilon$ for $0 < t < T/2$ and $\omega_0^2 - \varepsilon$ for $T/2 < t < T$. We now have

$$\ddot{x} + \left(\omega_0^2 + \varepsilon\right)x = 0 \quad 0 < t < T/2$$

$$\ddot{x} + \left(\omega_0^2 - \varepsilon\right)x = 0 \quad T/2 < t < T \tag{171}$$

yielding respectively :

$$x(t) = A\cos\sqrt{\left(\omega_0^2 + \varepsilon\right)}t + B\sin\sqrt{\left(\omega_0^2 + \varepsilon\right)}t, \qquad 0 < t < T/2 \tag{172}$$

and $$x(t) = C\cos\sqrt{\left(\omega_0^2 - \varepsilon\right)}t + D\sin\sqrt{\left(\omega_0^2 - \varepsilon\right)}t, \qquad T/2 < t < T. \tag{173}$$

Matching the position and the velocity at $t = T/2$, one gets respectively (defining $\Omega\pm \equiv \sqrt{\omega_0^2 \pm \varepsilon}$

$$A\cos\Omega + \frac{T}{2} + B\sin\Omega + \frac{T}{2} = C\cos\Omega - \frac{T}{2} + D\sin\Omega - \frac{T}{2} \tag{174}$$

$$-A\Omega + \sin\Omega + \frac{T}{2} + B\Omega + \cos\Omega + \frac{T}{2} = -C\Omega - \sin\Omega - \frac{T}{2} + D\Omega - \cos\Omega - \frac{T}{2}. \tag{175}$$

Again, matching the values of the position and the velocity at the boundaries of the periodic domain $[0, T]$ we obtain respectively :

$$\pm A = C\cos\Omega\_T + D\sin\Omega\_T \tag{176}$$

$$\pm B\Omega_+ = -Cos\Omega\_T + D\Omega_- \sin\Omega\_T \tag{177}$$

$\pm$ sign is to include two different directions of the possible periodic motion — clockwise and anti-clockwise respectively. For a non-trivial solution for A;B;C;D, the determinant of the matrix formed from the coefficients of the A,B,C,D in equations (174) to (177) should be zero. This means

$$\begin{vmatrix} \cos\Omega_+\frac{T}{2} & \sin\Omega_+\frac{T}{2} & -\cos\Omega_-\frac{T}{2} & -\sin\Omega_+\frac{T}{2} \\ -\Omega_+\sin\Omega_+\frac{T}{2} & -\Omega_+\sin\Omega_+\frac{T}{2} & \Omega_-\sin\Omega_-\frac{T}{2} & -\Omega_+\sin\Omega_+\frac{T}{2} \\ \pm 1 & 0 & -\cos\Omega_- T & -\sin\Omega_- T \\ 0 & \pm\Omega_+ & \Omega_-\sin\Omega_-\frac{T}{2} & -\Omega_+\cos\Omega_- T \end{vmatrix} = 0 \quad (178)$$

Expanding which one gets :

$$2\left(\pm 1-\cos\Omega_+\frac{T}{2}\cos_-\frac{T}{2}\right)+\left(\frac{\Omega_+}{\Omega_-}+\frac{\Omega_-}{\Omega_+}\right)\sin\Omega_+\frac{T}{2}\sin\Omega_-\frac{T}{2}=0. \quad (179)$$

Thus, if $\Omega = 2\pi / T$ then in the limit $\varepsilon \to 0$, periodic response are available for $\omega_0 / \Omega = n\pi$ and anti-periodic responses are available for $\omega_0 / \Omega = (n+1/2)\pi$ (where, $n \in \{0,1,2,...\}$).

10. Rapidly oscillating external force

We consider a particle subjected to a potential $V(x)$ and an osicillating force $f(x,t)$ that can be expressed as

$$f(x,t) = f_1(x,t)\cos\omega t + f_2(x,t)\sin\omega t. \quad (180)$$

Equation of motion is

$$m\ddot{x} = -\frac{dV}{dx} + f(x,t). \quad (181)$$

The frequency ω is high compared to the frequency associated with the motion under V. The magnitude of f is not small but because of its high frequency, we assume that the oscillations produced by f have a small amplitude.

The motion is assumed to be a regular motion with small oscillations about this smooth part. Accordingly we split

$$x(t) = X(t) + \xi(t) \quad (182)$$

where $\xi(t)$ is the effect of the oscillations. Over a period $2\pi / \omega, \xi(t)$ averages to zero and

$$\bar{x}(t) = X(t). \quad (183)$$

Trying out the solution of equation (182) in equation (181)

$$m\left(\ddot{X}+\ddot{\xi}\right) = -\frac{dV}{dx} - \xi\frac{d^2V}{dX} + f(X,t) + \xi\frac{\partial f}{\partial X} \quad (184)$$

This equation has both "oscillatory" and "smooth" terms, which must separately be equated. For the oscillatory part,

$$m\ddot{\xi} = f(X,t) + \xi \frac{\partial f}{\partial X}. \tag{185}$$

We drop the second term since it involves the small terms ξ. It should be noted that $\ddot{\xi}$ can not be dropped because of the high frequency. Thus,

$$\xi = -\frac{f}{m\omega^2} \tag{186}$$

Taking the time average of equation (184), we find

$$m\ddot{X} = -\frac{dV}{dX} + \overline{\xi \frac{\partial f}{\partial X}}$$

$$= -\frac{dV}{dX} - \frac{\partial}{\partial X}\left(\frac{\overline{f^2}}{2m\omega^2}\right)$$

$$= -\frac{dV}{dX} - \frac{1}{4m\omega^2}\frac{\partial}{\partial X}\left(f_1^2 + f_2^2\right)$$

$$= -\frac{dV}{dX}V_{eff} \tag{187}$$

where

$$V_{eff} = V + \frac{1}{4m\omega^2}\left(f_1^2 + f_2^2\right). \tag{188}$$

The regular motion occurs in an effective potential field which is the regular potential augmented by the mean kinetic energy of the oscillation.

11. Unforced damped linear oscillator

A. Unforced Damped Linear Oscillator

We commence our study on damped systems by considering linear oscillators :

$$\ddot{x} + k\dot{x} + \omega^2 x = 0, \qquad k > 0. \tag{189}$$

The oscillator has been assumed to have unit mas, $\omega^2 x^2 / 2$ is the potential corresponding to the restoring fore and multiplying both the sides of equation (189) by $\dot{x}$ and rearranging, we get :

$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2}\right) = -k\dot{x} \tag{190}$$

$$\Rightarrow \frac{dE}{dt} = -k\dot{x}^2 < 0 \tag{191}$$

where E is energy (the sum the kinetic energy and the potential energy at every instant) of the particle. It is seen to decrease with time; hence $k\dot{x}$ in equation (189) is acting as a damping term.

Trying solutions of the form $x = Ae^{mt}$, where A and m are constants, we get from equation (189) the characteristic equation :

$$m^2 + km + \omega^2 = 0 \tag{192}$$

which has two roots viz. :

$$m = m_{1,2} = \frac{1}{2}\left(-k \pm \sqrt{D}\right) \tag{193}$$

when the discriminant D is $k^2 - 4\omega^2$. According as the discriminant D is equal to zero or not, we have respectively the following solutions to equation (189) :

$$x(t) = (A_1 + A_2 t)e^{-\frac{k}{2}t} \tag{194a}$$

$$x(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} \tag{195}$$

. we focus on the trajectories of the system (196) in the 2-D phase space (x,y). Equation (196) has only one fixed point at the origin (0,0). To get an idea of stability of the point, we formally note that the matrix

$$M \vdots \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} \quad \begin{matrix} 0 & 1 \\ -\omega^2 & -k \end{matrix} \tag{196}$$

has the eigenvalue equation (m being the eigenvalue)

$$\begin{vmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{vmatrix} = 0$$

which yields nothing but the relation (192). So, (i) for the case of critical damping $(D = 0, m_1 = m_2 = -k/2 < 0)$ origin is degenerate stable node, (ii) for weak damping $\left(D < 0, m = m_{1,2} = \frac{1}{2}\left(-k \pm i\sqrt{|D|}\right)\right)$, the fixed point is stable spiral, and (iii) for the strong damping $\left(D > 0, m = m_{1,2} = \frac{1}{2}\left(-k \pm \sqrt{|D|}\right)\right)$, one has stable node at the point (0,0).

12. Unforced damped non-linear oscillator

Having recalled linear oscillator, let us ponder over the non-linear damping. Let the 2-D autonomous system be

$$\dot{x} = y$$
$$\dot{y} = -h(x, y) - g(x). \quad (199)$$

Unifying them, one may write

$$\ddot{x} + g(x) = -h(x, \dot{x}). \quad (200)$$

If it is assumed that there's only one isolated fixed point and that too at the origin implying $h(0; 0) + g(0) = 0$. The condition $h(x, 0)+g(x) = 0 \Rightarrow x = 0$, so that we have $h(0, 0) = 0$. These simplifying assumption lead us to interpret (200) as a model for a particle on a spring where free motion is a conservative system $\ddot{x} + g(x) = 0$ and is acted upon by an external force $-h(\dot{x}, x$ which can inject or take out energy of the system.

However one must resist from wrongly thinking that the mere presence of $\dot{x}$ -dependent terms in an equation means the existence of damping. Taking for example the following equation :

$$x^2\ddot{x} + x\dot{x}^2 + ax - 1 = 0, \quad a = \text{constant} \quad (201)$$

where the second term in the L.H.S. depends on $\dot{x}$, we simply rewrite equation (201) in terms of $\tilde{x} \equiv x^2/2$,

$$\ddot{\tilde{x}} + \frac{a\sqrt{2\tilde{x}}}{\sqrt{2\tilde{x}}} = 0 \quad (202)$$

which is conservative. Hence, the presence of $\dot{x}$ in a system is not enough for the system to be dissipative.

Historically, the research on non-linear oscillators in large scale was initiated with the development of vacuum tube technology wherein it was observed that many oscillating circuits follow Liénard's equation which basically is a second order differential equation of the type:

$$\ddot{x} + h(x)\dot{x} + g(x) = 0 \quad (203)$$

where we have written equation (200) with a special form of $h(x, \dot{x})$. In other words,

$$\dot{x} = y$$
$$\dot{y} = -g(x) - h(x)y. \quad (204)$$

Liénard's theorem : Without proof, we state an extremely interesting and useful result concerning Liénard's system. If in equation (203) $g(x)$ and $h(x)$ are such that :

1. $g(x)$ and $h(x)$ are continuously differentiable.

2. $g(-x) = -g(x)$ $\forall$x.

3. $g(x) > 0$ $\forall$x > 0

4. The integral $I(x) \equiv \int_0^x h(x')dx'$ has the following properties :

(a) $I(-x) = -I(x) \quad \forall x$

(b) It has exactly one positive zero at $x = x_0$.

(c) $I(x) < 0 \quad \forall x \in (0, x_0)$

(d) $I(x) > 0 \quad \forall x > x_0$

(e) $\frac{dI}{dx} \geq 0 \quad \forall x > x_0$

(f) $I(x) \to \infty$ as $x \to \infty$.

then equation (203) has a unique, stable limit cycle surrounding the origin in the 2-D phase space.

By the way, limit cycle is essentially a non-linear phenomenon, for, linear systems cannot have an isolated closed orbit, thanks to the fact that if $\vec{x}$ is a solution to $\dot{\vec{x}} = A\vec{x}$ then $c\vec{x}$ will also invariably be a solution for any constant c. (Here, $\vec{x}$ is a n-dimensional vector and $A = n \times n$ matrix Limitcydes basically model systems that exhibit self-sustained oscillations *i.e.*, the system which oscillates even in absence of external forcing. Obviously, conditions needed to satisfy Liénard theorem indicate that $g(x)$, being odd, behaves like a restoring force to reduce any displacement and $f(x)$ is such that large oscillations are damped down whereas smaller ones are energised, thereby helping the systèm to settle into a limit cycle for some intermediate amplitude.

13. Van-der-Pol oscillator : weak non-linear limit

$$\ddot{x} + \varepsilon(x^2 - \mu)\dot{x} + \omega^2 x = 0, \qquad \varepsilon, \mu, \omega = \text{constant} \tag{205}$$

is a Liénard equation which satisfies the condition written earlier while stating the Liénard theorem. For the sake of convenience, we use $\tau \equiv \omega t$ and let prime denote differentiation w.r.t. τ to rewrote (205) as :

$$x'' + \varepsilon(x^2 - \mu)x' + x = 0 \tag{206}$$

where ε has been redefined accordingly. In what follows in this section, we shall always assume that $\mu > 0$ and the non-linearity is small *i.e.*, $|\varepsilon| \ll 1$.

As we have already mentioned that equation (206) has a limit cycle in accordance with the Liénard theorem. What we wish to illustrate in this section are the various techniques useful in determining the properties of that eriodic solution without making the difficult attempt of solving the equation altogether. Let us begin with the use of power of polar coordinates to investigate equation (206) that can be split as:

$$\dot{x} = y$$
$$\dot{y} = -\varepsilon\left(x^2 - \mu\right)y - x. \tag{207}$$

We remind ourselves that any point $\left(x(t), y(t)\right)$ on a trajectory in 2-D Cartesian coordinate when expressed using the polar coordinates $\left(r(t), \theta(t)\right)$ follows the transformation rules according to the relations :

$$r^2 = x^2 + y^2 \tag{208a}$$

$$\tan\theta = \frac{y}{x} \tag{208b}$$

Relations (208a) and (208b) imply, upon differentiation w.r.t. time :

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} \tag{209a}$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}. \tag{209b}$$

Using equations (207) in the (209a) and (209b), we get the polar form for the Van-der-Pol equation :

$$\dot{r} = -\varepsilon r \sin^2\theta\left(r^2\cos^2\theta - \mu\right)$$
$$\dot{\theta} = -\varepsilon \sin\theta\cos\theta\left(r^2\cos^2\theta - \mu\right) - 1 \tag{210}$$

which on mutual division yields :

$$\frac{dr}{d\theta} = \frac{\varepsilon r \sin^2\theta\left(r^2\cos^2\theta - \mu\right)}{\varepsilon \sin\theta\cos\theta\left(r^2\cos^2\theta - \mu\right) + 1} \tag{211}$$

which is the equation for the trajectories. Now that we are for the features of the limit cycle *i.e.*, a periodic solution with a period T(say) — $r(t+T) = r(t)\ \ \forall t$ — we shall denote its amplitude by $a(t)$. We choose $a = a_0,\ \theta = 2\pi$ at $t = 0$ we have $a = a_0,\ \theta = 0$ at $t = T$ so that as t increases, θ decreases just to be consistent with the clockwise evolution of the orbits in accordance with the first of equations (207). Equation (211) was quite general in the sense that there was no restriction on the values on ε which we now put as $|\varepsilon| \ll 1$. Therefore from expression (211), we have

$$\frac{da}{d\theta} = \varepsilon a \sin^2\theta\left(a^2\cos^2\theta - \mu\right) + O\left(\varepsilon^2\right) \tag{212}$$

$$\int_{a(2\pi)}^{a(\theta)} da = \int_{2\pi}^{\theta} \varepsilon a \sin^2\theta\left(a^2\cos^2\theta - \mu\right)d\theta + O\left(\varepsilon^2\right) \tag{213}$$

$$a(\theta) = a_0 + \int_{2\pi}^{\theta} \varepsilon a \sin^2\theta\left(a^2\cos^2\theta - \mu\right)d\theta + O\left(\varepsilon^2\right) \tag{214}$$

$$\Rightarrow \quad a(\theta) = a_0 + \varepsilon\int_{2\pi}^{\theta} a_0 \sin^2\theta\left(a^2\cos^2\theta - \mu\right)d\theta + O\left(\varepsilon^2\right) \tag{215}$$

where in the Ist step, due to very mild non-linearity, we have put $a = a_0 + O\left(\varepsilon^2\right)$ in the second term on the R.H.S. for $\theta = 0$, the preceding equation (215) yield on equating the coefficient of ε on both the sides :

$$\int_{2\pi}^{\theta} a_0 \sin^2\theta\left(a_0^2\cos^2\theta - \mu\right)d\theta = 0 \tag{216}$$

$$\Rightarrow \quad a_0 = 2\sqrt{\mu} \tag{217}$$

which is the approximate estimate of the amplitude of the limit cycle. The polar coordinates may even be used to find the estimate for the time-period 'T' in the following manner:

$$T = \int_0^T dt = \int_{2\pi}^{0} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\sin\theta\cos\theta\left(a^2\cos^2\theta - \mu\right)} \tag{218}$$

where we have used second of equations (210). Again substituting $a = a_0 + O(\varepsilon)$ and keeping terms upto $O(\varepsilon)$ after expansion, we arrive at :

$$T = \int_0^{2\pi}\left[1 + \varepsilon\sin\theta\cos\theta\left(a_0^2\cos^2\theta - \mu\right)\right]d\theta + O(\varepsilon) \tag{219}$$

$$\Rightarrow \quad T = 2\pi + o\left(\epsilon^2\right). \tag{220}$$

Hence, the angular frequency of the limit cycle is $2\pi / T$ *i.e.*, $1 + O\left(\epsilon^2\right)$

For this Van-der-Pol oscillator, alongwith the limit cycle being talked about, origin is a fixed point and the phase diagram for the system in a bounded region wherein the origin lies consists of trajectories which can be determined approximately. By looking at the structure of Eq. (212). One can use a Fourier series valid for all θ to expand the R.H.S. as:

$$\frac{da}{d\theta} = \varepsilon A_0(a) + \varepsilon\sum_{n=1}^{\infty} A_n(a)\cos n\theta + \varepsilon\sum_{n=1}^{\infty} B_n(a)\sin n\theta + + O\left(\varepsilon^2\right) \tag{221}$$

For $0 \le \theta \le 2\pi$ this represents of one loop of the spiral. Therefore, using $a(\theta) = a(0)$ on the loop to the lowest order of accuracy

$$\frac{da}{d\theta} = \varepsilon A_0(a(0)) + \varepsilon\sum_{n=1}^{\infty} A_n(a(0))\cos n\theta + \varepsilon\sum_{n=1}^{\infty} B_n(a(0))\sin n\theta + O\left(\varepsilon^2\right) \tag{222}$$

When averaged over the range $0 \le \theta \le 2\pi$, there can be no contribution from the higher harmonic terms in the order $O(\varepsilon)$ and using the relations (212) and (222) $A_0(a(0))$ may be obtained as :

$$\varepsilon A_0(a(0)) = \frac{1}{2\pi}\int_0^{2\pi}\left[\varepsilon a(\theta)\sin^2\phi\left(a(\theta)^2\cos^2\phi - \mu\right)\right]d\phi = \frac{\varepsilon a}{2}\left(\frac{1}{4}a^2 - \mu\right) \tag{223}$$

that basically is the average value of $\varepsilon a(\theta)\sin^2\theta\left(a(\theta)^2\cos^2\theta - \mu\right)$ over a loop passing through the point having $(a(\theta), \theta)$ as the polar representation. So the separation between successive loops of the trajectory is generated by

$$\frac{da}{d\theta} = \frac{\varepsilon a}{2}\left(\frac{1}{4}a^2 - \mu\right) \tag{224}$$

is correct upto $O(\varepsilon)$ as no contribution come from the higher harmonics.

This procedure inherently assumes $|\varepsilon| \ll 1$ so that the curves generated by equation (224) are nearly circular. Using second of the relations (210) and chain rule, we get :

$$\frac{da}{dt} = \dot{\theta}\frac{da}{d\theta} = \frac{da}{d\theta} + O(\varepsilon^2). \tag{225}$$

From equations (224) and (225) one has :

$$\frac{da}{dt} = \frac{\varepsilon a}{2}\left(\frac{1}{4}a^2 - \mu\right) \tag{226}$$

The value for $\dot{a} = 0$ *i.e.*, $a = 2\sqrt{\mu}$ corresponds to the limit cycle as discussed earlier. Solution of equation (226) is obtained as :

$$\int_{a(0)}^{a(t)} \frac{da}{a\left(a - 2\sqrt{\mu}\right)\left(a + 2\sqrt{\mu}\right)} = \int_0^t \frac{1}{8}\varepsilon dt \tag{227}$$

$$\Rightarrow \frac{1}{8}\int_{a(0)}^{a(t)} da\left(\frac{-2}{a} + \frac{1}{a - 2\sqrt{\mu}} + \frac{1}{a + 2\sqrt{\mu}}\right) = \frac{\varepsilon}{8}\int_0^t dt \tag{228}$$

$$\Rightarrow ln\left(\left|\frac{a^2 - 4\mu}{a^2(0) - 4\mu}\right|\frac{a^2(0)}{a^2}\right) = -\varepsilon\mu t \tag{229}$$

$$\Rightarrow a(t) = \frac{2\sqrt{\mu}}{\sqrt{\left[1 - \left(1 - \frac{4\mu}{a^2(0)}\right)e^{-\varepsilon\mu t}\right]}} \tag{230}$$

As $t \to \infty, a(t) \to 2\sqrt{\mu}$ *i.e.*, towards the limit cycle.

To find $x(t)$ we need to know $\theta(t)$. One could start ab initio to perform similar averaging procedure to get a relevant expression for $\theta(t)$. But one can guess the result having understood the essence of the calculation done to arrive at (226). Basically, all hat has been done is that one has replaced the R.H.S. of the first of the relations (210) with the average value of it over the loop containing the point $(a(\theta,\theta))$ in the limit $|\varepsilon| \ll 1$ so that inaccuracies of only order higher than $O(\varepsilon)$ creeps in; which, thus, are negligible. In the same manner, one hopes to replace the R.H.S. of the second of the relations (210) by the similar value averaged over looped trajectory. So, it yields :

$$\frac{d\theta}{dt} = -1 - \frac{\varepsilon}{2\pi}\int_0^{2\pi} \sin\phi\cos\phi\left[a^2(\theta)\cos^2\phi - 1\right]d\phi + O(\varepsilon^2) = -1 + +O(\varepsilon^2) \quad (231)$$

$$\Rightarrow \quad \theta(t) = -t + (\varepsilon^2) \quad (232)$$

where we have put $\theta(0) = 0$. Using equations (230) and (232), we note that upto the , $O(\varepsilon)$ the approximate time solutions to the Van-der-Pol equation are given by :

$$x(t) = a(t)\cos\theta(t) = \frac{2\sqrt{\mu}\cos t}{\sqrt{\left[1 - \left(1 - \frac{4\mu}{a^2(0)}\right)e^{-\varepsilon\mu t}\right]}}. \quad (233)$$

The approximate amplitude and frequency, and the approximate solution to the Van-der-Pol equation can be found in yet another fashion, *viz.*, harmonic balance which is a rather more formal way of arriving at the results already obtained. Let us illustrate the procedure: We first of all rewrite the Van-der-Pol equation as:

$$\ddot{x} + x = -\varepsilon(x^2 - \mu)\dot{x} \quad (234)$$

Assume,

$$x = a\cos\omega t \quad (235)$$

and put it in the relation (234), to get :

$$(-a\omega^2 + a)\cos\omega t = \varepsilon(a^2\cos^2\omega t - \mu)a\omega\sin\omega t \quad (236)$$

$$\Rightarrow \quad (1 - \omega^2)a\cos\omega t = \varepsilon\omega(a^2 - \mu)a\sin\omega t - \varepsilon a^3\omega\sin^3\omega t \quad (237)$$

$$\Rightarrow \quad (1 - \omega^2)a\cos\omega t = \varepsilon\omega\left(\frac{a^2}{4} - \mu\right)\sin\omega t + \frac{\varepsilon a^3\omega}{4}\sin 3\omega t \quad (238)$$

Ignoring the higher harmonic -sin $3\omega t$ and matching the terms cos ωt and sin ωt on both the sides of the expression (238), we arrive at :

$$1 - \omega^2 = 0 \Rightarrow \omega = 1 \quad (239a)$$

$$\frac{a^2}{4} - \mu = 0 \Rightarrow a = 2\sqrt{\mu} \tag{239b}$$

respectively. The results (239a) and (239b) matches what was earlier in the results (220) and (217).

To obtain approximate solutions to the Van-der-Pol equation, harmonic balance may be used to pseudo-linearise the non-linear equation. By this we actually mean that a linear substitute would be found out as follows. As discussed in he equations (234) to (238), we have

$$-\varepsilon\left(x^2 - \mu\right)\dot{x} = \varepsilon a\omega\left(\frac{a^2}{4} - \mu\right)\sin\omega t + \frac{\varepsilon a^3\omega}{4}\sin 3\omega t \tag{240}$$

Again, using the assumption (235), have

$$\dot{x} = -a\omega\sin\omega t \tag{241}$$

Substituting the relation (241) in the expression (240) and ignoring the higher harmonics, we get :

$$-\varepsilon\left(x^2 - \mu\right)\dot{x} = -\varepsilon\omega\left(\frac{a^2}{4} - \mu\right)\dot{x} \tag{242}$$

which can be used to rewrite equation (234) as :

$$\ddot{x} + \varepsilon\omega\left(\frac{a^2}{4} - \mu\right)\dot{x} + x = 0. \tag{243}$$

It may be noted that a = 2 corresponds to limit cycle in which case the damping term vanishes. As far as the non-periodic solution are concerned, we use the initial conditions $x(0)$ = a and $\dot{x}(0) = 0$ that is equivalent to the initial conditions used to arrive at the result (233) which is what we wish to verify. Obviously, the solution of equation (243), in consistence with the initial conditions, is

$$x(t) = a\ \exp^{\left[-\frac{\varepsilon}{2}\left(\frac{a^2}{4} - \mu\right)t\right]}\cos\left(\sqrt{\left[1 - \frac{\varepsilon^2}{4}\left(\frac{a^2}{4} - \mu\right)^2\right]}\,t\right) \tag{244}$$

To verify that the solutions (233) and (244) are same, one may expand them to see that in the limit $\varepsilon t \ll 1$ both are same.

An even more important fact that may be extracted from here is that the spiral inside a = 2 grows and the spiral outside $a = 2$ gets damped onto the limit cycle described by the loop $a = 2$; thereby, hinting at the fact that the limit cycle is stable. This stability test is better checked using energy balance method discussed below; which by the way, as we shall see, prove to be another way of getting a hint of amplitude of the limit cycle.

We focus on the form (234) of the Van-der-Pol oscillator. If we set $\varepsilon = 0$, solution to it would be

$$x(t) = a\cos(t+\phi) \qquad a,\phi = \text{integration constants} \tag{245a}$$

Obviously,

$$y(t) = \dot{x}(t) = -a\sin\ (t+\phi). \tag{245b}$$

For convenience we settle at $\phi = 0$ and A > 0. The relations (245a) and (245b) describe a circular orbit in the phase plane; the circle has a period $T = 2\pi$. Now as $\varepsilon \neq 0$ but $\varepsilon \to 0$, one may still assume to approximate:

$$x(t) \approx a\cos t,\ y(t) \approx -a\ \sin\ t, \quad \text{and } T \approx 2\pi. \tag{246}$$

As the term in the R.H.S. of equation (234) gives estimate for the change in the energy E over the period $0 \le t \le T$ (treating $-\varepsilon(x^2-\mu)\dot{x}$ as an external force) is :

$$\Delta E = E(T) - E(0) = \int_0^T \left[-\varepsilon(x^2-\mu)\dot{x}\right]\dot{x}dt \tag{247}$$

One may think of the integrand of the expression (247) as the input power. For a limit cycle ΔE should be zero; hence using the expressions (246) in the relation (247) we arrive at, for the limit cycle:

$$\Delta E = -\varepsilon a^2 \int_0^{2\pi} \left(a^2\cos^2 t - \mu\right)\sin^2 t dt = 0 \tag{248}$$

$$\Rightarrow \quad -\varepsilon a^2 \pi\left(\frac{a^2}{4} - \mu\right) = 0 \tag{249}$$

$$\Rightarrow \quad a = 2\sqrt{\mu}\ . \tag{250}$$

Thus, we again arrive at the approximation to the limit cycle for mild non-linearity. Again, if $\Delta E < 0$ for any spiral trajectory in the region outside $a = 2$ and $\Delta E < 0$ for any spiral trajectory in the region inside $a = 2$, both the trajectories respectively, loose and gain energies over a loop to fall onto the limit cycle which hence could be termed stable. More formally, stability would mean

$$\left[\frac{d}{da}\Delta E(a)\right]_{\text{Limit Cycle}} < 0. \tag{251}$$

For the Van-der-Pol oscillator this cirterion would mean :

$$\left[\frac{d}{da} - \varepsilon a^2 \pi\left(\frac{a^2}{4} - \mu\right)\right]_{2\sqrt{\mu}} < 0 \tag{252}$$

$$\Rightarrow \quad \left[-\varepsilon a\pi\left(a^2 - 2\mu\right)\right]_{2\sqrt{\mu}} < 0 \tag{253}$$

$$-2\varepsilon\mu\pi a < 0 \tag{254}$$

which is true and hence the limit cycle is stable.

The stability of the limit cycle may be analysed in yet another fashion by invoking the "method of slowly varying amplitude and phase", also known as KBM (Krylov-Mitropusky) method. Once more we rearrange the Van-der-Pon oscillator's equation to look like.

$$\ddot{x} + x = -\varepsilon\left(x^2 - \mu\right)\dot{x} \tag{255}$$

Had the R.H.S. of equation (255) been zero, the solution would have been sinusoidal in time, hence, inspiring one to write

$$x(t) = a(t)\cos\left(t + \phi(t)\right) \tag{256b}$$

There is no approximation involved. Differentiating the expression (256) with respect to '*t*' one gets :

$$\dot{x} = \dot{a}\cos(t + \phi) - a\left(1 + \dot{\phi}\right)\sin(t + \phi)\,. \tag{257}$$

Again differentiating with respect to '*t*'

$$\ddot{a}\cos(t + \phi) - 2\dot{a}\left(1 + \dot{\phi}\right)\sin(t + \phi) - a\ddot{\phi}\sin(t + \phi) - a\left(1 + \dot{\phi}\right)^2\cos(t + \phi)\,. \tag{258}$$

Inserting the relations (256), (257) and (258) in equation (255), we arrive at :

$$\ddot{a}\cos(t + \phi) - 2\dot{a}\left(1 + \dot{\phi}\right)\sin(t + \phi) - a\ddot{\phi}\sin(t + \phi) - a\left(2\dot{\phi} + \dot{\phi}^2\right)\cos(t + \phi)$$

$$+ \in\left[\dot{a}\cos(t + \phi) - a\left(1 + \dot{\phi}\right)\sin(t + \phi)\right]\left[a^2\cos^2(t + \phi) - \mu\right] = 0 \tag{259}$$

At this point we introduce the idea of KBM method. We assume, in the light of the small non-linearity *i.e.*, $|\varepsilon| \ll 1$, the amplitude and the phase vary slowly over the time scale of the oscillation period for the trajectory near the limit cycle. In other words, for such trajectories we may treat $\dot{a}$ and $\dot{\phi}$ as constants over one full period of oscillation. Further $\dot{a}/a$ and $\dot{\phi}/\phi \ll 1$, so that Eq. (259) reduces to

$$-2\dot{a}\sin(t + \phi) - 2a\phi\cos(t + \phi) + \in\left[\dot{a}\cos(t + \phi) - a\left(1 + \dot{\phi}\right)\sin(t + \phi)\right]$$

$$\left[a^2\cos^2(t + \phi) - \mu\right] = 0\,. \tag{260}$$

Using the principle of harmonic balance at this point, we see that to $O(\varepsilon)$

$$\dot{a} = \frac{\in}{2}a\left(\mu - \frac{a^2}{4}\right. \tag{261}$$

$$\dot{\phi} = o\left(\in^2\right) \tag{262}$$

which brings us back to eq. (226).

Now, for doing the limit cycle's stability analysis we note that $\dot{a} = 0$ gives the limit cycle's amplitude $a = 2\sqrt{\mu} = a^{*}$ (say). Let d be the infinitesimal perturbation on the limit cycle *i.e.*,

$$d = a - a^{*} \tag{263}$$

So, putting the expression (263) in equation (262a), we get

$$\dot{d} = f(d + a^{*}), \qquad f(a) = -\frac{\varepsilon a}{2}\left(\frac{a^2}{4} - \mu\right. \tag{264}$$

$$\Rightarrow \dot{d} = f(a^{*}) + d \frac{}{da}\Big|_{a^{*}} + O(d^2) \tag{265}$$

$$\Rightarrow \dot{d} = -\varepsilon\mu d \tag{266}$$

$$\Rightarrow \ d \sim e^{-\varepsilon\mu t}. \tag{267}$$

Thus, as the perturbation decreases with time, the limit cycle is stable in conformity with what has been arrived at earlier.

14. Van-der-Pol oscillator : strong non-linear limit

As has been already mentioned, the Liénard equation $\ddot{x} + h(x)\dot{x} + g(x) = 0$, when it satisfies certain conditions, does possess a closed path *i.e.*, a periodic solution. There is a particular choice of variable (x; y) say, in terms of which a particular phase plane — Liénard plane — may be defined; in this plane the Liénard equation is given by :

$$\begin{aligned} \dot{x} &= y - I(x) \\ \dot{y} &= -g(x) \end{aligned} \tag{268}$$

where $I(x) = \int_0^x h(x')dx'$. If one defines $f(x) = y^2/2 + \int_0^x g(x')dx'$, then by calculating $\dot{f}$ on closed path "C" in the plane one can arrive at the Liénard criterion :

$$\oint I(x)dy = 0. \tag{269}$$

To be specific, the variables for the Liénard plane in the case of Van-der-Pol equation $\ddot{x} + \varepsilon(x^2 - \mu)\dot{x} + x = 0$ may be constructed as follows :

$$\ddot{x} + \varepsilon(x^2 - \mu)\dot{x} + x = 0 \tag{270}$$

$$\Rightarrow \frac{d}{dt}\left[\dot{x} + \varepsilon\left(\frac{x^3}{3} - \mu x\right)\right] = -x \tag{271}$$

$$\Rightarrow \dot{y} = -x \tag{272}$$

where $y = \dot{x} + \varepsilon\left(\frac{x^3}{3} - \mu x\right)$; thus by setting $l(x) = \varepsilon\left(\frac{x^3}{3} - \mu x\right)$, we arrive at :

$$\begin{aligned} \dot{x} &= y - l(x) \\ \dot{y} &= -x \ . \end{aligned} \tag{273}$$

It can be used to find the approximate time period of the limit cycle of the Van-der-Pol oscillator in the strongly non-linear limit viz., $|\varepsilon| >> 1$ without going into the mathematical rigour which could be seen to be too involved as would be discussed sketchily later. To do that let us further change/define the variables

$$v \equiv \frac{y}{\varepsilon}, \ u \equiv x, \quad F(u) \equiv \frac{l(x)}{\varepsilon}. \tag{274}$$

So that we have

$$\begin{aligned} \dot{u} &= \varepsilon[v - F(u)] \\ \dot{v} &= -\frac{u}{\varepsilon} \ . \end{aligned} \tag{275}$$

Now consider the nullcline $v = F(u)$ in theLiénard plane $(u;v)$ (see Figure 3). The nullcline is a cubic and has maximum at $u = -\sqrt{\mu}$ and minimum at $u = \sqrt{\mu}$ which let be denoted by the points A and C respectively. If we have an initial condition above the nullcline, $v - F(u) > 0 \Rightarrow \dot{u} > 0$, then the trajectory moves sideways toward the nullcline and that too with the horizontal velocity much greater than the vertical velocity. This is so because if $v - F(u) \sim O(1)$, then $|\dot{u}| / |\dot{v}| \sim O(\varepsilon) / O(\varepsilon^{-1}) \sim O(\varepsilon^2) >> 1$. As soon as the trajectory comes so close to $v = F(u)$ such that $v = F(u) \sim O(\varepsilon^{-2})$, then $|\dot{u}| / |\dot{v}| \sim O(\varepsilon^{-1}) / O(\varepsilon^{-1}) \sim 1$, the nullcline is crossed vertically by the trajectory which then in effect crawls slowly along the segment . $\mathrm{B} \rightarrow \mathrm{C}$. Again as the point C is reached, the picture discussed repeats for the path $\mathrm{C} \rightarrow \mathrm{D} \rightarrow \mathrm{A}$ symmetrically w.r.t. the path $\mathrm{A} \rightarrow \mathrm{B} \rightarrow \mathrm{C}$. Thus, we claim that all the trajectories, including the limit cycle behaves the way discussed in the Liénard plane. Obviously, justification of this claim can come only from the rigourous calculations which we are not quite going into.

Now, what we discussed above implies that total time period 'T' of the limit cycle is the total time spent on the slower branches *viz.*, $\mathrm{B} \rightarrow \mathrm{C}$ and $\mathrm{D} \rightarrow \mathrm{A}$. As on the slower branches $v \approx F(u)$, so

$$\dot{v} = \frac{dF}{dt}\frac{du}{dt} = (u^2 - \mu)\dot{u} \tag{276}$$

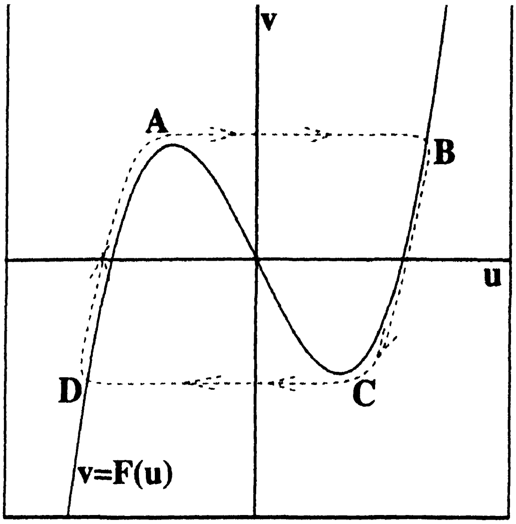

Figure 3. Typical trajectory of strongly non-linear van-der-Pol Oscillator in Liénard plane.

Using the second of equations (275), we get

$$dt = -\frac{\varepsilon\left(u^2 - \mu\right)}{u} d. \tag{277}$$

Therefore,

$$T = \int_B^C dt + \int_D^A dt = 2\int_B^C \, dt = \int_{2\sqrt{\mu}}^{\sqrt{\mu}} -\frac{\varepsilon\left(u^2 - \mu\right)}{u} du \tag{278}$$

$$\Rightarrow \quad T = \varepsilon\mu(3 - 2 \ln 2) \,. \tag{279}$$

By the way, if one carefully considers the time needed to traverse the knees as A and C, one gets correction terms of $O\left(\varepsilon^{-1/3}\right)$. But this requires quite a lot of mathematical calculations. Actually just because the regular perturbation theory fails in the case of strongly non-linear Van-der-Pol oscillator, it becomes tough to analyse it; by the regular perturbation we simply mean the procedure we have been carrying out extensively by using the perturbative series solution of the form $x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$ and expecting it to converge. Let us illustrate with a simple example how in general the regular perturbation theory fails. Consider the differential equation with initial conditions:

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \; |\varepsilon| \ll 1; \quad \dot{x}(0) = 1 \,. \tag{280}$$

It can be easily be solved to get :

$$x(t,\varepsilon)=\frac{2}{\sqrt{4-\varepsilon^2}}\exp\left(-\frac{\varepsilon t}{2}\right)\sin\left[\frac{\sqrt{4-\varepsilon^2}}{2}t\right]. \tag{281}$$

And, the regular perturbation method would yield :

$$x(t,\varepsilon)=\sin t-\frac{\varepsilon t}{2}\sin t+O(\varepsilon^2). \tag{282}$$

If the expression (281) is expanded in the power series in ε, the first two terms are given by the expression (282). For fixed t, the relation (282) is a good approximation as long as ε is small enough so that $\varepsilon t \ll 1$. But since in practice, ε is kept fixed and t grows, we can expect the relation (282) to hold good only for $t \ll O(1/\varepsilon)$ because only then the error $O(\varepsilon^2 t^2)$) are small enough to be neglected (otherwise error dominates). Thus, regular perturbation may be said to have failed for larger t and we may say that the expression (282) does not provide an approximation that is valid uniformly in the range $[0, \infty 1)$ of t.

In fact it is of very common occurrence that regular perturbation theory fails to provide a uniform approximation to the solution of the corresponding differential equation; such cases of failure are generally termed as singular perturbation problems, already introduced earlier in the section on coordinate perturbation method. Such problems are also showcased by certain boundary value problems in which a small parameter multiples the highest derivative terms. These cases may be treated by so-called boundary-layer method in which different parts of the solution curve are approximated differently and then matched appropriately. It is this very technique that we shall now briefly introduce and in the process learn the concepts of the stretched coordinates and the method of dominant balance to apply them to the Van-der-Pol oscillator in the large non-linearity limit.

For this we resort to some illustrative examples as follows. Let us use the perturbative expansion :

$$x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\ldots \tag{283}$$

to solve the following algebric equations :

$$x^2+\varepsilon x=1 \tag{284a}$$

$$\varepsilon x^2+x=1. \tag{284b}$$

Substituting the expression (283) in equation (284a) and equating the coefficient of the like powers of ε step by step we recover the following two series :

$$x=1-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{8}+\ldots \tag{285a}$$

$$x=-1-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{8}+\ldots \tag{285b}$$

which corresponds to the expansion of the exact roots $\left(-\varepsilon/2 \pm \sqrt{1+\varepsilon^2/4}\right)$ of equation (284a). The solutions (285a) and (285b) are the examples of regular perturbation series. One may note that since $|\varepsilon| << 1$, solutions (285a) and (285b) to equation (284a) differ very slightly from the result when $\varepsilon = 0$ $(x = \pm 1, \text{ when } \varepsilon = 0)$. Again, putting the trial series (283) in equation (284b), on similar considerations we manage to get only one solution, *viz*,

$$x = 1 - \varepsilon + 2\varepsilon^2 + \ldots \;. \tag{286}$$

The exact solution of equation (284b) is

$$x = \frac{1}{2\varepsilon}\left(-1 \pm \sqrt{1+4\varepsilon}\right) \tag{287}$$

the solution (286) corresponds to the expression (287) with positive sign chosen from $\pm$ -sign; the solution that could not be recovered is the one with the negative sign *i.e.*,

$$x = \frac{1}{2\varepsilon}\left(-1 - \sqrt{1+4\varepsilon}\right) = -\frac{1}{\varepsilon} - 1 + \varepsilon + \ldots \;. \tag{288}$$

This is an example of singular perturbation expansion *i.e.*, result for small ε differs greatly from the result for $\varepsilon = 0$. The root (288) no way could have been obtained using the series (283) due to the presence of the $1/\varepsilon$ term in the solution (288) and absence of such a term in the series (283). So the perturbation expansion fails when a small perturbation multiplies the highest power of x in the algebric equation (284b). It is in such scenario that the method of dominant balance comes in as a handy tool in which one assumes a trial expansion of the form :

$$x = x_0\varepsilon^{\lambda_0} + x_1\varepsilon^{\lambda_1} + x_2\varepsilon^{\lambda_2} + \ldots, \qquad \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{289}$$

and investigates all the possible leading order balances, checking each for self-consistency. So, substituting (289) in equation (284) we get :

$$x_0^2\varepsilon^{2\lambda_0+1} + x_0\varepsilon^{\lambda_0} = 1 \tag{290}$$

where we have kept only the largest contributions to each term. Now, the possible leading order balances in equation (290) are :

where we have kept only the largest contributions to each terms. Now, the possible leading order balances in equation (290) are :

1. $x_0^2\varepsilon^{2\lambda_0+1}$ balances $x_0\varepsilon^{\lambda_0}$: This implies $\lambda_0 = -1$ and $x_0 = 0, -1$. Thus, the non-trival solution is $x = -1/\varepsilon$ (upto the given order) which obviously corresponds to the root (288).

2. $x_0\varepsilon^{\lambda_0}$ balances 1 : This implies $\lambda_0 = 0$ and $x_0 = 1$ which obviously indicates the root (286).

3. $x_0^2\varepsilon^{2\lambda_0+1}$ balances 1 : This implies $\lambda_0 = -1/2$ and $x_0 = 1$. But this is a contradiction of the of the philosophy of balancing, for, then $x_0\varepsilon^{\lambda_0}$ becomes the most dominating term contrary to what has been assumed in this case.

Having illustrated the method of dominant balance to treat the singular expansions, let us see how it applies in the similar vein to the boundary value problem for ordinary differential equations. Consider a rather simple example:

$$\varepsilon\ddot{x} + x = 1 \quad x(0) = 0, x(1) = a, \quad |\varepsilon << 1| \tag{291}$$

where, *a* is a positive numerical constant. One can readily check that the trial solution :

$$x(\varepsilon,t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ... \tag{292}$$

fails to yield solution for equation (291) because substituting the expression (292), we get following infinite set of differential equations :

$$x_0 = 1; \qquad x_0(0) = 0, x_0(1) = a \tag{293a}$$

$$\ddot{x}_0 + x_1 = 0\,; \qquad x_1(0) = 0, x_1(1) = 0 \tag{293b}$$

etc. *etc.*,

and the solution to equation (293a) is $x_0(t) = t + A$, where unfortunately, A — the integration constant — cannot be uniquely determined so as to let the solution satisfy both the boundary conditions simultaneously. This anomaly arises from the possibility that on $0 < t < 1$, $x(t)$ can change rapidly enough so as to make $\varepsilon\ddot{x}$ too large to be ignored even for $\varepsilon \to 0$. This situation is tackled by considering the existence of a boundary layer(s) near $t = 0$ or(and) $t = 1$ where x changes quite rapidly with t. Let's assume that a boundary layer exists near $t = 0$. Guessing the position of the boundary layer is an art which comes to one with experience. One may check that in this problem, there exists no boundary layer near $t = 1$. Now, we introduce the very important idea of stretched coordinates. To get an approximation near $t = 0$, it is of no use to work with t =fixed which is covered (as we shall see) by an approximation (292). Hence, we consider $t \to 0$ with ε as follows:

$$t(\varepsilon) = \varepsilon^{\lambda}\tau \tag{294}$$

in which τ can take any value. This τ is called stretched coordinate; it has helped us to zoom the boundary layer to the width $O(1)$. Now, in the boundary layer, we assume :

$$x(t) = X_0(\tau) + \varepsilon X_1(\tau) + ... \tag{295}$$

Using the relations (294) and (295) to rewrite (291), we get upto the leading order :

$$\varepsilon^{1-2\lambda}\frac{d^2X_0}{d\tau^2}+\varepsilon^{-\lambda}\frac{dX_0}{d\tau}=1\ . \tag{296}$$

Invoking the method of dominating balance, the three possible candidates for balancing in the expression (296a) are :

1. $\varepsilon^{1-2\lambda}\frac{d^2X_0}{d\tau^2}$ blances $\varepsilon^{-\lambda}\frac{dX_0}{d\tau}$: This implies $\lambda=1$ and $x(t)=B+Ce^{-t/\varepsilon}$. B and C are constants.
2. $\varepsilon^{1-2\lambda}\frac{d^2X_0}{d\tau^2}$ balances 1 : This suggests $\lambda=0$ and $x(t)=t+A$ which is nothing but the solution to equation (293a).
3. $\varepsilon^{1-2\lambda}\frac{d^2X_0}{d\tau^2}$ balances 1 : This means $\lambda=1/2$ which obviously is a contradiction as the term $\varepsilon^{-\lambda}\frac{dX_0}{d\tau}$ in equation (296) turns out to be the most dominating.

Thus, to the leading order we have the solution to equation (292) as :

$$x(t)=t+A=x_{out}(say)\qquad 0<t<1$$

$$x(t)=B+Ce^{-t/\varepsilon}=x_{bl(say)}\quad \text{near } t=0. \tag{297}$$

Obviously, in accordance with the boundary condition : $A=a-1$ and $B=-C$. Again, matching the solutions at the edge of the boundary layer *i.e.*, imposing

$$\lim_{t\to 0}x_{out}=\lim_{t\to\infty}x_{bl} \tag{298}$$

we get, $A=B$, *i.e.*, $B=a-1$. Hence, we have ultimately found an approximate solution to (291) which has uniform validity in the range of $t\in[0,1]$; the solution is formally written as :

$$x(t)=x_{out}+x_{bl}-\text{common terms} \tag{299}$$

$$\Rightarrow x(t)=(t+a-1)+(a-1)\left(1-e^{-t/\varepsilon}\right)-(a-1) \tag{300}$$

$$\Rightarrow x(t)=a-1+t-(a-1)e^{-t/\varepsilon}. \tag{301}$$

Now let us see how the concept of stretched coordinates may be used to get the limit cycle in the usual phase plane (x, y) (and not in the Liénard plane has we have done earlier) for the van-der-Pol oscillator for strong non-linearity $(|\varepsilon|>>1)$.

Putting $\varepsilon=\delta^{-1}$ that $|\delta|<<1$, we rewrite the Van-der-Pol equation as:

$$\delta\ddot{x}+\left(x^2-\mu\right)\dot{x}+\delta x=0 \tag{302}$$

which may be broken into following two differential equations of first order :

$$\dot{x}=y$$

$$\dot{y}=-\frac{\left(x^2-\mu\right)y}{\epsilon}-x. \tag{303}$$

Dividing these two equations (303), we get the differential equation of the trajectories in the phase space (x,y) as :

$$\frac{dy}{dx}=\frac{x^2-\mu}{\delta}-\frac{x}{y}. \tag{304}$$

The thing to note in equation (304) is that as $x\to -x$ and $y\to -y$, equation remains invariant. Thus, if the part of limit cycle is found for y>0, then the lower part may be constructed using this symmetry. As in the boundary layer problems, we go on to solve equation (302) by putting

$$t=\delta^{\lambda}\tau. \tag{305}$$

Therefore, we get (using prime to denote the differentiation w.r.t. τ) :

$$\delta^{1-2\lambda}x''+\delta^{-\lambda}\left(x^2-\mu\right)x'+\delta x=0. \tag{306}$$

The first term in the L.H.S. of equation (306) cannot balance the last term as it would led to a contradiction. However, the first term can balance the second term to lead to $\lambda=1$ *i.e.*,

$$t=\delta\tau \tag{307}$$

and,

$$x''+\left(x^2-\mu\right)x'=0 \tag{308}$$

If at $y=0$, $x=a$ then the solution to equation (308) may be written as :

$$x'=\frac{1}{3}(x-a)\left(3\mu-a^2-ax-x^2\right) \tag{309}$$

$$x'=-\frac{1}{3}(x-a)\left[x-\left(-\frac{a}{2}+\frac{\sqrt{12\mu-3a^2}}{2}\right)\right]\left[x-\left(-\frac{a}{2}-\frac{\sqrt{12\mu-3a^2}}{2}\right)\right] \tag{310}$$

This means that in case $a>+\sqrt{\mu}$, then $x'>0$ (*i.e.*, $y>0$) in the range

$$\frac{1}{2}\left(-a+\sqrt{12\mu-3a^2}\right)<x<a. \tag{311}$$

Again the second and the third terms in the L.H.S. of the relation (306) can balance each other to yield :

$$\lambda = -1 \Rightarrow t = \frac{\iota}{\delta} \tag{312}$$

and $$\left(x^2 - \mu\right)x' + x = 0 \tag{313}$$

$$\Rightarrow x' = -\frac{x}{x^2 - \mu}\,. \tag{314}$$

This equation (314), owing to the singularity present at $-\sqrt{\mu}$ is valid in the region $y > 0$ only for $-a < x < -\sqrt{\mu}$. The region in the neighborhood of $x = -\sqrt{\mu}$ is problematic; the solutions (310) and (314) may be joined together therein by another equation. Before we seek that equation, just note that, to the required order, the trajectory given by the expression (310) should touch the x-axis at $x = -\sqrt{\mu}$; which means that by stretching the solution to $x = -\sqrt{\mu}$ and using the relation (311) we get $a = 2\sqrt{\mu}$ on equating $\left(-a + \sqrt{12\mu - 3a^2}\right)/2 = -\sqrt{\mu}$,. Now, to get the approximation about x = p, we zoom the region in the phase plane by substituting

$$x - \left(-\sqrt{\mu}\right) = x + \sqrt{\mu} = \delta^{\lambda}\xi \tag{315}$$

where ξ is the stretched coordinate. Substituting the expressions (305) and (315) in equation (302) we reach at

$$\delta^{1-2\lambda+\bar{\lambda}}\xi'' + \delta^{-\lambda+3\bar{\lambda}}\xi^2\xi' - 2\sqrt{\mu}\delta^{-\lambda+2\bar{\lambda}}\xi\xi' + \delta^{1+\bar{\lambda}+}\xi - \delta\sqrt{\mu} = 0 \tag{316}$$

where first, third and fifth terms in the L.H.S. can balance each other with values for λ and $\bar{\lambda}$ and 2/3 respectively, *i.e.*,

$$\xi'' - 2\sqrt{\mu}\xi\xi' - \sqrt{\mu} = 0 \tag{317}$$

is the approximate differential equation giving solution in the region about $x = -\sqrt{\mu}$. Similarly, we zoom the neighbourhood around $x = -2\sqrt{\mu}$ by defining a stretched coordinate η such that

$$x + 2\sqrt{\mu} - \delta^{\bar{\lambda}}\eta\,. \tag{318}$$

Inserting the expressions (305) and (318) in equation (302) we arrive at :

$$\delta^{1-2\lambda+\bar{\lambda}}\eta'' + \delta^{-\lambda+3\bar{\lambda}}\eta^2\eta' - 4\sqrt{\mu}\delta^{-\lambda+2\bar{\lambda}}\eta\eta' + 3\mu\delta^{-\lambda+\bar{\lambda}}\eta' + \delta^{1+\bar{\lambda}}\eta - 2\delta\sqrt{\mu} = 0\,. \tag{319}$$

A possible balance is obtained n equation (319) among the first, the fourth and the sixth terms, *i.e.*, with $\lambda = 1$ and $\bar{\lambda} = 2$, which gives :

$$\eta'' + 3\mu\eta' - 2\sqrt{\mu} = 0\,. \tag{320}$$

Therefore, without solving every differential equation (in fact, equation (317) does not have an elementary solution) we merely state that the upper half of the limit cycle is given by the solutions of the differential equations (308), (318), (314) and (320) valid in the range $2\sqrt{\mu} \geq x > -\sqrt{\mu} + O(\delta^{2/3})$, $-\sqrt{\mu} + O(\delta^{2/3}) > x > \sqrt{\mu} - O(\delta^{2/3})$, $-\sqrt{\mu} + O(\delta^{2/3}) > x > -2\sqrt{\mu} + O(\delta^2)$ and $-2\sqrt{\mu} + O(\delta^2) > x \geq -2\sqrt{\mu}$ respectively and the lower half is obtained by invoking the symmetry earlier just after writing equation (304).

15. Forced damped oscillators

A. Linear forced damped oscillator

To start with, we sketchily consider a damped oscillator (the damping force being proportional to the instantaneous velocity) with unit mass and being acted upon by an external sinusoidal force. Mathematically, we thus have a non-autonomous system:

$$\ddot{x} + k\dot{x} + \omega^2 x = F\cos\Omega t;\quad \omega, \Omega, F, k \text{ are constants and } \omega, \Omega > 0, k \in (0, 2\omega). \quad (321)$$

The solution is given by

$$x(t) = A\exp\left(-\frac{k}{2}t\right)\cos\left[\left(\omega^2 - \frac{k^2}{4}\right) - \theta\right] + \frac{F\cos\left[\Omega t - \tan^{-1}\dfrac{k\Omega}{\left(\omega^2 - \Omega^2\right)^2}\right]}{\sqrt{\left(\omega^2 - \Omega^2\right)^2 + k^2\Omega^2}} \quad (322)$$

which is, respectively, combination of free oscillation and forced oscillation. A and θ are the arbitrary constants of the complementary function that basically is a transient term vanishing at $t \to \infty$. Initial conditions decide the values for A and θ. In the steady state, thus, $x(t)\left(= x_s(t) \text{ say}\right)$ settles into the particular solution

$$x_s(t) = \frac{F\cos\left[\Omega t - \tan^{-1}\dfrac{k\Omega}{\left(\omega^2 - \Omega^2\right)^2}\right]}{\sqrt{\left(\omega^2 - \Omega^2\right)^2 + k^2\Omega^2}} \quad (323)$$

whose amplitude is

$$a = \frac{|F|}{\sqrt{\left(\omega^2 - \Omega^2\right)^2 + k^2\Omega^2}}. \quad (324)$$

Treating F, k, ω as fixed, one may verify that $da/d\Omega^2 = 0 \Rightarrow \Omega^2 = \omega^2 - k^2/2$ at which $d^2a/d\left(\Omega^2\right)^2 < 0$. Hence the maximum value of a is a_{max} given by

$$a_{max} = \frac{|F|}{k\sqrt{\left(\omega^2 - \frac{k^2}{4}\right)^2}} \tag{325}$$

As $\Omega \to \omega$ (damping is very samll), a_{max} ›. This state of the system is well-known as the state of resonance.

16. Duffing's equation

A non-autonomous equation of form :

$$\ddot{x} + a\dot{x} + bx + cx^3 = F(t), \text{ a,b,c = constants} \tag{326}$$

is known as Duffing's equation with a forcing term. Actually such an equation emanates from the standard form of equation of a forced pendulum :

$$\ddot{\theta} + k\dot{\theta} + \omega^2 \sin\theta = F(t) \tag{327}$$

θ is the angular displacement from the vertical. For small, θ, sin $\theta \approx \theta$ to give us back the linear damped oscillator which we had discussed earlier. However, approximating $\sin\theta \approx \theta - \theta^3/6$, one gets a Duffing's equation. Now, let us take periodic forcing F(t) = $F(t) = F\cos\Omega t$, put $\theta = x$ and define $\tau = \Omega t, \omega_r^2 = (\omega/\Omega)^2$, $\varepsilon_r = \omega_r^2/6$, $k_r = k/\Omega$ and $f = F/\Omega$ to get

$$\frac{d^2x}{d\tau^2} + kr\frac{dx}{d\tau} + \omega_r^2 x - \varepsilon_r x^3 = f\cos\tau \ . \tag{328}$$

Note that ε_r is a constant, a are k_r,ω_r and f. In order to treat ε_r as a parameter we replace it by ε which is a continuous variable occupying an interval that has $\varepsilon = 0$ and also $\varepsilon = \varepsilon_r$ as elements. So, letting prime to denote differentiation w.r.t. τ, we shall consider the following equation :

$$x'' + k_r x' + \omega_r^2 x - \varepsilon x^3 = f\cos\tau. \tag{329}$$

We again assume $|\varepsilon| << 1$ in what follows.

1. Resonances

This phenomenon, as we have seen earlier, is most vivid when the damping is small; so let $k_r = 0$ for the time being and represent the solution of the resulting equation as:

$$x(\varepsilon,\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots \ . \tag{330}$$

Inserting which in the relation (329) (remembering k_r=0) and equating the coefficients of the like powers of ε, we arrive at following set of infinite differential equations

$$\begin{aligned} &x_0'' + \omega_r^2 x_0 = f\cos\tau \\ &x_1'' + \omega_r^2 x_1 = x_0^3 \\ &\ldots \qquad \ldots \\ &etc. \qquad etc. \end{aligned} \tag{331}$$

The dominating equation of this set, *i.e.*, the first of equations (331), has the same periodic solution as the linearised undamped version of equation (329) has. Hence, the perturbation series helps us in finding only those solutions that bifurcate (due to the presence of mild non-linearity) from the solution of the linearised equation. And, thus by invoking this procedure we are basically restricting ourselves into the investigation of the periodic solutions having period 2π of the forcing term, *i.e.*,

$$x(\varepsilon,\tau+2\pi) = x(\varepsilon,\tau) \quad \forall \varepsilon,\tau \tag{332}$$

$$\Rightarrow \quad x_i(\tau+2\pi) = x_i(\tau) \quad i = 0,1,2,3, \tag{333}$$

which follows from the expression (330). Now, the solution to the first of equations (331) subject to the condition (333) is :

$$x_0(\tau) = \frac{f\cos\tau}{\omega_r^2 - 1} \tag{334}$$

putting which in the second of equations (331) and subsequently solving the resulting equation subject to the condition (333), we have

$$x_1(\tau) = \frac{3}{4}\frac{f^3\cos\tau}{(\omega_r^2-1)^4} + \frac{1}{4}\frac{f^3\cos 3\tau}{(\omega_r^2-1)^3(\omega_r^2-3^2)} \,. \tag{335}$$

Note that implicitly we have assumed in what has been calculated above $\omega_r \neq 1,3,5,\ldots$ so that the series (330) converges. These values of ω_r correspond to near-resonance condition. Evidently, $\omega_r \approx 1$ corresponds to linear resonance while the other values can be associated with non-linear resonances as they can be attributed to the higher harmonics being fed back by the non-linear term x^3. Using the solutions (334) and (335), the solution

$$x_1(\varepsilon,\tau) = \frac{f\cos\tau}{\omega_r^2-1} + \varepsilon\left(\frac{3}{4}\frac{f^3\cos\tau}{(\omega_r^2-1)^4} + \frac{1}{4}\frac{f^3\cos 3\tau}{(\omega_r^2-1)^3(\omega_r^2-3^2)}\right) \cdot O(\varepsilon^2) \tag{336}$$

thus, corresponds to the forced osicllations far from resonance.

2. Periodic solutions near resonance with weak excitation

"Near resonance" would obviously mean that

$$\omega_r^2 = 1 + \varepsilon_r \omega \tag{337}$$

and that the excitation is weak would allow us to write

$$f = \varepsilon_r \phi \tag{338}$$

Again, consider the damping to be weak for simplicity, *i.e.*,

$$k_r = \varepsilon_r \chi \tag{339}$$

Thus, putting the relations (337), (337) and (339) in equation (329), we get on rearranging :

$$x'' + x = \varepsilon\left(\phi\cos\tau - \chi x' - \omega x + x^3\right), \quad \phi, \chi > 0\,; \;\; \phi, \chi, \omega = \text{constant} \tag{340}$$

where we use ε for ε_r the sake of generality. Again using the series (330) as solution of equation (340), we get following set of infinite number of differential equations

$$\begin{aligned} &x_0'' + x_0 = 0 \\ &x_1'' + x_1 = \phi\cos\tau - \chi x_0' - \omega x_0 + x_0^3 \\ &\textit{etc.} \qquad\qquad \textit{etc.} \end{aligned} \tag{341}$$

We, being in search of periodic solution of period 2π of the forcing term, still assume condition (333) to hold; therefore, solution to the first equation of the set (341) is

$$x_0(\tau) = A_0\cos\tau + B_0\sin\tau \tag{342}$$

A_0 and B_0 being the integration constants that can be determined from the initial conditions. Using the solution (342) in the second equation of the set (341), we get:

$$x_1'' + x_1 = \left[\phi - \chi B_0 + A_0\left(-\omega + \frac{3}{4}a_0^2\right)\right]\cos\tau + \left[\chi A_0 + B_0\left(-\omega + \frac{3}{4}a_0^2\right)\right]\sin\tau + \text{higher harmonics} \tag{343}$$

where $a_0 = \sqrt{A_0^2 + B_0^2}$ = amplitude f the "generating" periodic solution (342). Condition (333) implies that the coefficient of cos τ and sin τ must be zero *i.e.*,

$$\chi B_0 - A_0\left(-\omega + \frac{3}{4}a_0^2\right) = \phi \tag{344a}$$

$$\chi A_0 - B_0\left(-\omega + \frac{3}{4}a_0^2\right) = 0. \tag{344b}$$

Squaring and adding the relations (344a) and (344b), we get

$$a_0^2\left[\chi^2+\left(\frac{3}{4}a_0^2-\omega\right)^2\right]=\phi^2 \tag{345}$$

$$\Rightarrow\ \frac{9}{16}\left(a_0^2\right)^3-\frac{3}{2}\omega\left(a_0^2\right)^2+\left(\chi^2-\omega^2\right)\left(a_0^2\right)-\phi^2=0. \tag{346}$$

Invoking Descartes rule of sign, we notice that this equation for a_0^2 can have at most three positive roots and hence a_0, being positive, also can have three values. Of course, this will depend on the value of ω, χ and ϕ. Whatever it may be, the point is that there can very well be three distinct solutions of the periodically weakly forced During equation (329) near resonance. Each of the solutions basically bifurcate from a distinct generating solution given by the expression (342).

Amplitude-phase perturbation technique

There is yet another technique of getting the relation (345) starting from equation (340). It is worthwhile to ponder over that technique known as amplitude-phase perturbation technique because it gives a way of obtaining higher approximations to the amplitude obtained earlier. The trick is to assume following form of solution for equation (340):

$$x(\varepsilon,\tau)=a\cos(\tau+\alpha)+\text{higher harmonics} \tag{347}$$

where, $a=a_0+\varepsilon a_1+\varepsilon^2 a_2+\ldots$

and, $\alpha=\alpha_0+\varepsilon\alpha_1+\varepsilon^2\alpha_2+\ldots$

As might be guessed that since both the amplitude and the phase are perturbed simultaneously we call the technique amplitude-phase perturbation technique. The state of art suggests to make following relabellings:

$$\bar{\tau}\equiv\tau+\alpha,\quad y(\varepsilon,\bar{\tau})\equiv x(\varepsilon,\tau),\ \ \bar{y}\equiv\frac{dy}{d\bar{\tau}}$$

which helps us to rewrite equation (340) as :

$$\bar{\bar{y}}+y=\varepsilon\left[\phi\cos(\bar{\tau}-\alpha)-\chi\bar{y}-\omega y+y^3\right] \tag{348}$$

One may carefully note that all we have managed to achieve by the innocent looking changes is that α has crept into equation (348) explicitly; this is the key step and what follows is just the repetition of what we have been doing all the time in the earlier sections. We put two conditions:

$$\bar{y}(\varepsilon,0) = 0 \tag{349a}$$

$$y(\varepsilon,\bar{\tau}+2\pi) = y(\varepsilon,\bar{\tau}) \tag{349b}$$

which respectively are to adjust the origin of time (and hence the phase) and to probe for only the periodic solution of period 2π of the forcing term. The conditions, in view of the following assumed solution of equation (348)

$$y(\varepsilon,\bar{\tau}) = y_0(\bar{\tau}) + \varepsilon y_1(\bar{\tau}) + \ldots \tag{350}$$

reassert themselves as (for $i = 1, 2, 3, \ldots$)

$$\bar{y}_i(\varepsilon,0) = 0 \tag{351a}$$

$$y_i(\varepsilon,\bar{\tau}+2\pi) = y_i(\varepsilon,\bar{\tau}). \tag{351b}$$

Putting the series (350) in equation (348) and collecting the coefficients of the like powers of ε, we obtain following set of differential equations:

$$\bar{\bar{y}}_0 + y_0 = 0$$

$$\bar{\bar{y}}_1 + y_1 = \phi\cos(\bar{\tau}-\alpha_0) - \chi\bar{y}_0 - \omega y_0 + y_0^3 \tag{352}$$

etc. etc.

The term $\cos(\bar{\tau}-\alpha_0)$ in the R.H.S. of the second equation of the set (352) comes due to the use of Taylor series $\cos(\bar{\tau}-\alpha_0) = \cos(\bar{\tau}-\alpha_0) + \varepsilon\alpha_1\sin(\bar{\tau}-\alpha_0)$. Solutions of the first equation of the set (352) obeying the conditions (351a) and (351b) invariably are

$$y_0(\bar{\tau}) = a_0\cos\bar{\tau},\ \ a_0 > 0\,. \tag{353}$$

Substituting which in the second equation of the set (352) we are left with

$$\bar{\bar{y}}_1 + y_1 = \left(\phi\cos\alpha_0 - \omega a_0 + \frac{3}{4}a_0^3\right)\cos\bar{\tau} + (\chi a_0 + \phi\sin\alpha_0)\sin\bar{\tau} + \text{higher harmonics}\,. \tag{354}$$

As argued earlier, imposing the condition (351b) we have from preceding equation (354) the vanishing of the coefficient of $\cos\bar{\tau}$ and $\sin\bar{\tau}$

$$\phi\cos\alpha_0 = \omega a_0 - \frac{3}{4}a_0^3 \tag{355a}$$

$$\phi\sin\alpha_0 = -\chi a_0\,. \tag{355b}$$

Dividing the expression (355b) by the expression (355a), we get :

$$\alpha_0 = \tan^{-1}\left(\frac{-\chi}{\omega - \frac{3}{4}a_0^2}\right) \tag{356}$$

where a_0 is the solution of

$$a_0^2\left[\chi^2 + \left(\omega - \frac{3}{4}a_0^2\right)^2\right] = \phi^2 . \tag{357}$$

This relation has been obtained by squaring and adding the relations (355a) and (355b). One can note that the relation (345) has been rediscovered. By the way, if one wishes to find the improvement $\varepsilon\alpha_1$ over a_0, then one merely has to solve equation (354) with higher harmonics on the R.H.S. and use the found solution that obeys the conditions (351a) and (351b) alongwith the solution of the first equation of the set (352) in the differential equation for y_2; impose the condition (333) to set the coe ficients of cos τ and sin τ to zero and hence proceed to find a_1 and α_1. Further improvement are possible to find in a similar manner.

Stability of solutions and jump phenomenon

We now ask the question that out of the three possible amplitudes (see equation (357)) which one is settled on by the oscillator. Actually, it is the set of initial conditions which decides which state of oscillation the system is ultimately going to adopt. Here in this section we shall sketchily deal with this issue and also on the stability of the solutions. For convenience, lets deal with the following form of the forced Duffing's equation :

$$\ddot{x} + k\dot{x} + x + \lambda x^3 = F\cos\Omega t \cdot \tag{358}$$

Assuming the truncated Fourier series

$$x(t) = A_0\cos\Omega t + B_0\sin\Omega t \tag{359}$$

as an approximate solution to equation (358). Here, A_0 and B_0 are constants. One may compare it with relation (342). Substituting (359) in (358) and matching the coefficients of $\cos\Omega t$ and $\sin\Omega t$, we get

$$B_0\left(\Omega^2 - 1 - \frac{3}{4}\lambda a^2\right) + k\Omega A_0 = 0 \tag{360a}$$

$$A_0\left(\Omega^2 - 1 - \frac{3}{4}\lambda a^2\right) - k\Omega B_0 = -F \tag{360b}$$

where $a = \sqrt{A_0^2 + B_0^2}$ =amplitude. We have neglected the higher harmonics. One can combine equations (360a) and (360b) to yield:

$$a^2\left[k^2\Omega^2 + \left(\Omega^2 - 1 - \frac{3}{4}\lambda a^2\right)^2\right] = F^2 . \qquad (361)$$

So, we have at most three possible values for a. If one wishes to study the stability of these oscillations, one should better look at the transient states by letting A_0 and B_0 depend very slowly on time and investigate if the transient states converge towards or diverges away from the corresponding periodic state. Mathematically, we start by assuming:

$$x(t) = A_0(t)\cos\Omega t + B_0(t)\sin\Omega t ; \quad \ddot{A}_0 << \dot{A}_0,\ \ddot{B}_0 << \dot{B}_0 . \qquad (362)$$

Putting this in equation (358) and neglecting the higher harmonics as before, we arrive at the following autonomous system of equations on matching the coefficients of cost and sint.

$$\dot{A}_0 = -\frac{B_0}{2\Omega}\left(\Omega^2 - 1 - \frac{3}{4}\lambda a^2\right) - \frac{kA_0}{2}$$

$$= -\frac{A_0}{2\Omega}\left(\Omega^2 - 1 - \frac{3}{4}\lambda a^2\right. - \frac{kB_0}{2} + \frac{F}{2\Omega} \qquad (363)$$

Note that equation (361) correspond to the equilibrium point of this system of equations. Hence the possible states of oscillations are the fixed points in the van-der-Pol plane (*i.e.*, the phase plane of $A_0;B_0$). Now, the usual linear stable analysis about these fixed points can be carried out to find (a) if there exists only one response then that response is stable and (b) if all the three responses are present then only the responses with minimum and maximum amplitude are stable.

Again if one carefully notes that initial conditions for equations (363) can be given in terms of the initial conditions of equation (358) as follows

$$A_0(0) = x(0)$$

$$B_0(0) = \frac{\dot{x}(0)}{\Omega} \qquad (364)$$

(where we have taken $\dot{A}_0(0)$, $\dot{B}_0(0)$ to be negligible) then it should be immediately evident that it is the initial condition of the original forced Duffing equation that decides which state of oscillation the system is going to adopt. More clearly speaking, for the initial conditions residing in the basin of attraction of a particular amplitude (that is the stable fixed point) in van-der-Pol plane, the system would eventually oscillate with that particular amplitude.

The stability of the forced oscillation of the Duffing equation can be studied by the general method of so-called solution perturbation technique set up below. Let the solution (359) of equation (358) be denoted by x^* and let $\dot{x}^* = \dot{y}^*$ Then, the solution $(\dot{x}^* = \dot{y}^*)$ for equation (358) rewritten in the following form

$$\dot{x} = y$$
$$\dot{y} = -ky - x - \lambda x^3 - F\cos\Omega t \tag{365}$$

may be checked for stability (linear stability) by considering a perturbation $(\xi(t), \eta(t))$. So putting

$$\xi = x - x^*$$
$$\eta = y - y^* \tag{366}$$

in equations (365) and churning out linearised equations :

$$\dot{\xi} = \eta$$
$$\dot{\eta} = -k\eta - \xi - 3\lambda(x^*)^2\xi . \tag{367}$$

Using expression (359) and the two equations (367) together, we arrive at :

$$\xi'' + K\xi' + (\nu + \varepsilon\cos\tau)\xi = 0 \tag{368}$$

where we have put : $A_0\cos\Omega t + B_0\sin\Omega t = -a\cos(\Omega t + \phi)$, $\tau = 2\Omega t + 2\phi$, $K = k/2\Omega$, $\nu = (2 + 3\lambda a^2)/8\Omega^2$ and $\varepsilon = 3\lambda a^2/8\Omega^2$ and defined $\xi' \equiv d\xi/d\tau$. Further substituting $\xi(\tau) = \exp(-K\tau/2)\rho(\tau)$ yields :

$$\rho'' + \left(\nu - \frac{K^2}{4} + \varepsilon\cos\tau\right)\rho = 0 \tag{369}$$

which is nothing but the Matheiu's equation already discussed extensively in the section on parametric resonance. One can, thus, now extend the arguments developed therein to get at the stability of the solutions here. By the way, whenever $\nu - K^2/4$ and ε take values such that equation (369) has only bounded solutions, equation (368) must have only bounded solutions owing to the relationship deffined between ρ and ξ.

We close this section with a brief discussion of the so-called jump phenomenon. We rewrite equation (361) as:

$$\alpha\left[k^2\Omega^2 + \left(\Omega^2 - 1 + \frac{3}{4}\alpha\right)\right] = f^2 \tag{370}$$

where we have assumed k; $k, \Omega, F > 0$ and $\lambda < 0$ and defined $\alpha \equiv -\lambda a^2$ and $f = F\sqrt{-\lambda}$ In experimental situation if one keeps k and f fixed at a favourable value and increases Ω from zero then the amplitude of the oscillation suddenly jumps at a critical frequency. This is called jump phenomenon.

Let us see how one explains it within the domain of the theory developed here. It may be noted that equation (370) is a cubic polynomial $\left(\phi(\alpha)=0, \text{ say}\right)$ in α and has at least one positive root; we, by the way, are interested only in $\alpha-\Omega$. For certain parameter values ($\alpha \geq 0$ (α, k, Ω, f), $\phi(\alpha)=0$ will have three real roots if $\frac{\partial \phi}{\partial \alpha}=0$ has two distinct real roots; on exploiting the positivity of α, this condition yields:

$$\Omega < \frac{1}{2}\left(\sqrt{3k^2+4}-k\sqrt{3}\right). \tag{371}$$

Thus keeping k, Ω fix in accordance with inequality (7), one has the generic curve in $f-\alpha$ plane as in Figure 4. This figure shows the existence of two stable oscillations as discussed earlier in this section. The jump phenomenon is best illustrated in the $\alpha-\Omega$ plane. The generic figure for the phenomenon is given in the Figure 4. Obviously, the dotted path will not be followed and the amplitude will jump from α_1 to α_2 at $=c$ as the frequency in increased from zero.

Reconsider equation (329), rewritten here for the sake of convenience

$$x'' + k_r x' + \omega_r^2 x - \varepsilon x^3 = f\cos\tau. \tag{372}$$

In the preceding few pages we have confirmed ourselves in investigating the periodic solution of period 2π of the forcing term. Can we have other kinds of periodic solutions? The answer to this question, as we shall see in what follows, is in affirmative. Suppose $x(\tau)$ is a periodic solution with period T. Then the Fourier series corresponding to it is :

$$x(\tau) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{2\pi n}{T}\tau + \sum_{n=1}^{\infty} B_n \sin\frac{2\pi}{T}\tau. \tag{373}$$

Substituting this series in equation (372), we obtain a relation of the form

$$\tilde{A}_0 + \sum_{n=1} \tilde{A}_n \cos\frac{2\pi n}{T}\tau + \sum_{n=1}^{\infty} \tilde{B}_n \sin\frac{2\pi}{T}\tau = f\cos\tau \tag{374}$$

where each $\tilde{A}_n$ and each $\tilde{B}_n$ are functions of A_n's ans B_n's. Now ponder over the case when $T = 2\pi n$ which when applied in the relation (374) yields

$$\tilde{A}_m = f,\ \tilde{A}_m = 0 \quad \forall m \neq n,\ \tilde{B}_m = 0 \quad \forall m \in N.$$

Thus, if such periodic solutions exist then we have the oscillator responding with the angular frequencies Ω/n that are aptly known as subharmonics of order $1/n$ ($n = 2; 3; 4; \ldots$). Obviously, all of subharmonics will, in general, not exist; stability check must be done on a subharmonic solution to see if at all the response is possible. In fact, for the forced Duffing oscillator with small damping and non-linearity, only the subharmonic of order 1/3 is stable and hence existent. The possible existence of subharmonics indicates that the discussion in the preceding section is not complete because the van-der-Pol plane there does not identifies any region of initial conditions which would lead to the subharmonic responses. Detailed analysis would show that the subharmonics show up for a relatively narrower ranges of initial conditions. The mode of such an analysis, by the way, would involve only the techniques discussed already in the context of periodic solutions taken up earlier; hence re-discussion of the techniques is dispensed with herein.

References

[1] L D Landau and E M Lifshitz "*Course of Theoretical Physics*" **Vol. 1** (Oxford : Mechanics Pergamon Press) (1958)

[2] D W Jordan and P Smith "*Nonlinear Ordinary Differential Equations*" (Oxford : Oxford University Press) (1977)

[3] A H. Nayfeh and D T Mook "*Nonlinear Oscillations*" (New York : J. Wiley and Sons) (1995)

[4] J Guckenheimer and P Holmes "*Nonlinear Oscillations, Dynamical Systems and Bifurcation for Vector Fields*" (Springer)

[5] N Minorsky "*Nonlinear oscillations*" (Melbourne : F L Krieger Publishing) (1974)

[6] S H Strogatz "*Nonlinear Dynamics and Chaos*" (New York : Perseus Books) (2001)

[7] C Hayashi, "*Nonlinear Oscillations in physical Systems*" Princeton : Princeton University Press (1986)

[8] A B Pippard "*Response and Stability : An introduction to physical theory*" (Cambridge : Cambridge University Press) (1985)

[9] J P Den Hartog "*Mechanical Vibrations*" (New York : Dover Publication) (1985)

[10] A K Mallik and J K Bhattacharjee "*Stability Problems in Applied Mechanics*" (Narosa) (2005)

[11] J K Bhattacharjee and A K Mallik edited "*Complex Systems*" (Narosa) (1995)