

Height-volume characteristics of a fuel tank

Problem presented by

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Problem statement

Airbus UK are concerned with finding a function which links the height h of fuel measured in a tank to its volume V for a variety of pitch and roll angles. A large amount of data of such relations was provided to the Study Group, and the challenge was to find a suitable function which fitted this data well.

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1 Introduction

A key feature of any aircraft is its fuel system. Fuel is stored in large tanks (typically within the wing of the aircraft) which are approximately rectangular, but have detailed internal structure. A typical tank is illustrated in Figure 1, in which we can see the internal stringers and other small details.

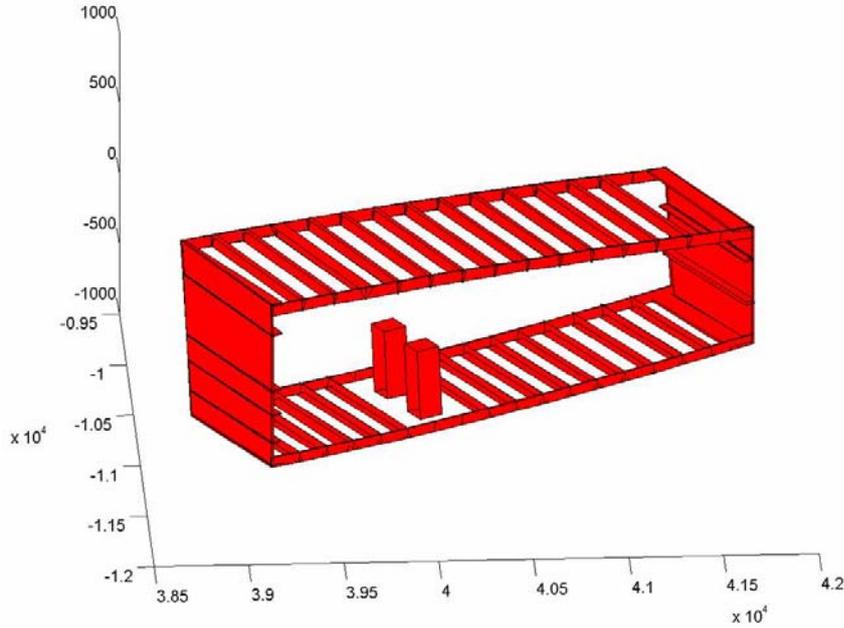


Figure 1: An example of a typical fuel tank showing the internal structures including corners and stringers.

A key aspect of aircraft safety is an accurate measurement of the amount of fuel in the tank. To determine this, the height h of the fuel at certain points within the tank is measured and the fuel volume V is determined from this. However the relationship between height and volume depends crucially on the pitch angle, α , and roll angle, β , of the tank. (This is clear to all of us when we drive a car and see that the indicated fuel level in the tank depends upon whether we are going up or down hill at the time.) At present Airbus uses lookup tables to link height to volume for a set of pitch and roll angles. As the fuel tanks become more and more intricate, the huge amount of data required becomes an issue. The problem posed to the Study Group was whether the lookup tables can be replaced by suitable functions. (Note that we only considered the static problem, which assumed that the fuel was in equilibrium. The much harder problem of determining the volume of a *sloshing* fuel was not considered, but might make a very interesting problem for a future Study Group.)

To solve this problem we require a combination of numerical analysis and geometry. Numerical methods are needed to determine suitable approximations to the h - V function. However, generating such approximations turns out to be very subtle, because of the

way that the fuel interacts with the tank geometry. This leads to h - V curves which are nonsmooth and have to be approximated very carefully. An understanding of the way that this nonsmoothness arises and how the resulting curves can be approximated forms the basis of this Study Group report.

2 The nature of the h - V curve

Imagine that we have a tank at a fixed orientation (α, β) and that we steadily fill it up with fuel. If we measure the height of the fuel, h , at a fixed point, then there is a continuous, monotone increasing curve linking h and the fuel volume V . The form of this curve changes as we change the pitch α and the roll β .

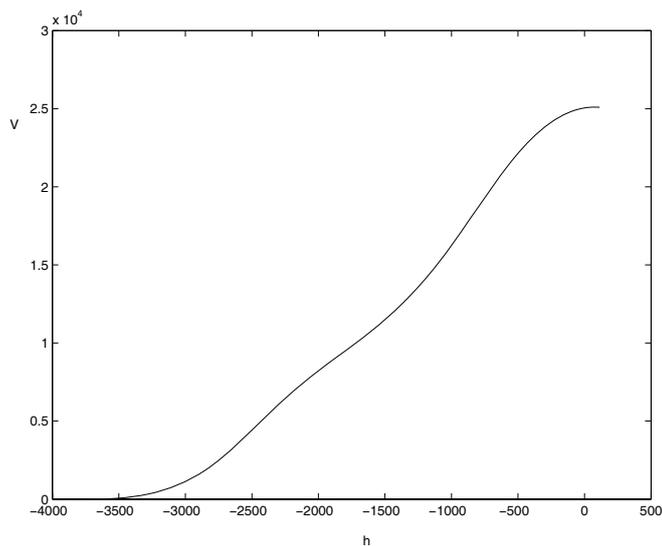


Figure 2: An example of a typical h - V curve for fixed values $\alpha = 200^\circ$ and $\beta = 90^\circ$. Although it looks smooth to the eye, it is far from smooth in practice.

This curve is always monotonic¹ and it looks smooth to the eye. In fact it is continuous and also has a continuous first derivative. However, it is far from smooth in practice, and indeed it has an irregular second derivative. This lack of smoothness arises when, as the tank is filled up, the fuel comes into contact with a corner of the tank and/or some internal structure. To see this, we present graphs of both the first and second derivatives of the above graph, obtained by numerically differentiating the h - V graph. The graph of the second derivative is very revealing and exhibits two distinctive features. Firstly, there are distinct points where the second derivative jumps abruptly. We will show that these arise when the fuel encounters a corner in the fuel tank. The second feature are the oscillations seen in the second derivative. We think that these are probably due to

¹The fuel does not spill over a weir or fill a sump in such a way that volume increases without a corresponding increase in fuel height.

the internal fine structure of the tank, in particular the internal stringers. The period of these oscillations is certainly consistent with the known dimensions of the stringers (around 80mm).

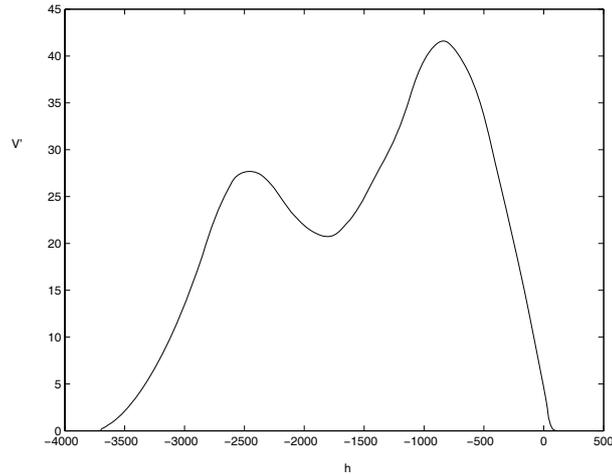


Figure 3: The (numerical) first derivative of the h - V curve.

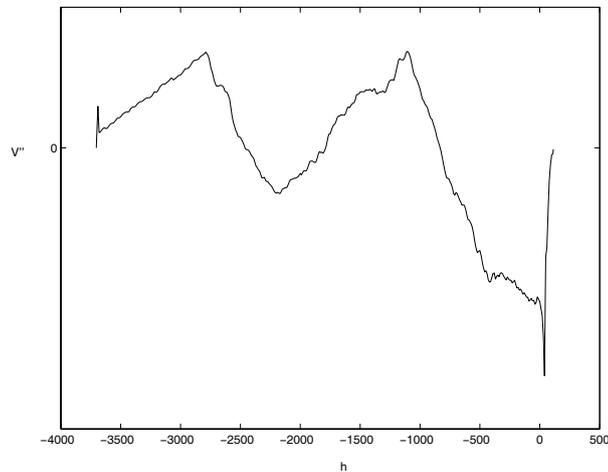


Figure 4: The (numerical) second derivative of the h - V curve for fixed values of $\alpha = 200^\circ$ and $\beta = 90^\circ$. In this figure we can see the large jumps in the second derivative which arise when the fuel encounters a corner in the tank and also the high oscillation associated with the internal structure.

We conclude from this preliminary analysis that it is far from straightforward to generate a simple function which will approximate the h - V curve. Any such function needs to account both for second derivative discontinuities and the presence of the internal structure. We now consider two approaches to this problem. The first involves an

analysis of the form of the h - V curve, and the second a discussion of the use of an adaptive numerical method to perform the approximation.

3 The analytic form of the h - V curve

To see why there are jumps in the data, we start by considering a rectangular tank which is being slowly filled up with fuel in the manner shown in Figure 5.

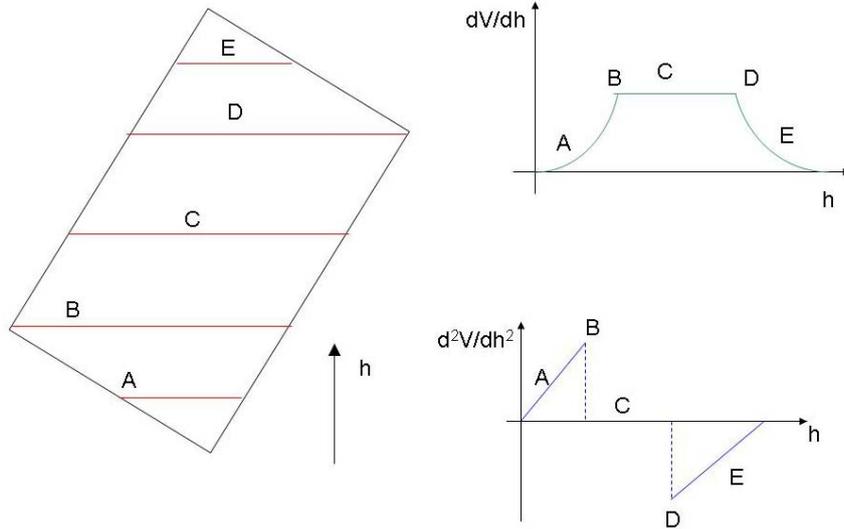


Figure 5: An example of a cuboid fuel tank at an angle showing the analytic form of the curves dV/dh and d^2V/dh^2 .

We show in this figure schematic plots of dV/dh and d^2V/dh^2 . If we start filling up the fuel from a corner, then for small values of h the volume V increases proportionally to h^3 . The surface area of the fuel is given by dV/dh and this grows quadratically from zero (as h^2) away from the lowest corner. When the fuel reaches the corner at B its surface area remains constant until it reaches the point D when it decreases back to zero again. There is thus a jump discontinuity in the second derivative of V at the points B and D . More generally we can think of a general fuel tank as being made up of a set of tetrahedral simplices. As each of these fill up with fuel, we see the same behaviour as described above. Note that this is a simplified view, since the sides of the tank can in general be smooth but curved.

To consider this in more detail, we now consider a perfectly rectangular tank (with 8 corners) with side lengths 2, 4 and 1, and at pitch and roll angles α and β , respectively. The heights of these 8 corners are critical values of h , at which the h - V curve loses smoothness. Indeed, the h - V characteristics of this particular tank are largely determined by these heights, which each depend upon α and β . The form of the h - V curve thus

depends critically upon the overall *rotation* of the tank. This rotation is given by the matrix O , where

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix},$$

so that

$$O = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ -\sin(\alpha) \sin(\beta) & \cos(\beta) & -\cos(\alpha) \sin(\beta) \\ \sin(\alpha) \cos(\beta) & \sin(\beta) & \cos(\alpha) \cos(\beta) \end{bmatrix}.$$

If the rotation O is then applied to the box of sides 2, 4 and 1, with one corner held at the origin, then the heights of the 8 corners C_i ($i = 1 \dots 8$) are given by

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \sin(\alpha) \cos(\beta) \\ 2 \sin(\alpha) \cos(\beta) + 4 \sin(\beta) \\ 4 \sin(\beta) \\ \cos(\alpha) \cos(\beta) \\ 2 \sin(\alpha) \cos(\beta) + \cos(\alpha) \cos(\beta) \\ \cos(\alpha) \cos(\beta) + 2 \sin(\alpha) \cos(\beta) + 4 \sin(\beta) \\ \cos(\alpha) \cos(\beta) + 4 \sin(\beta) \end{bmatrix}.$$

A plot of the height of the corners as a function of α with β fixed at 45° is given in Figure 6

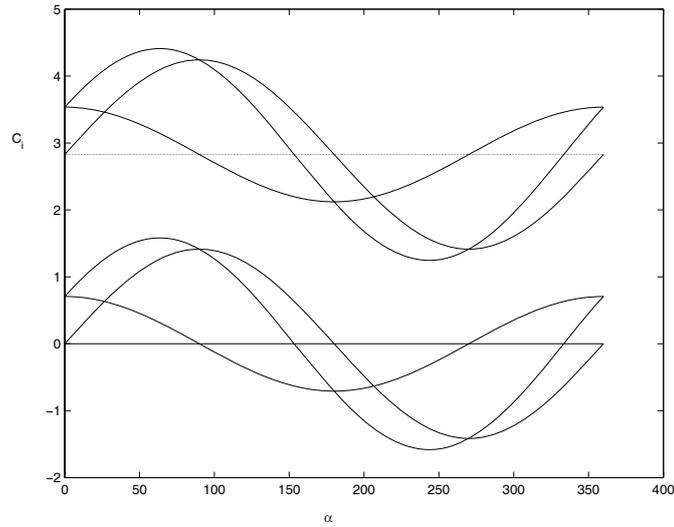


Figure 6: The height of the 8 corners as a function of α when $\beta = 45^\circ$.

From the perspective of computing the h - V curve accurately over a range of values of h , α and β for even the simple shape of the rectangular box, this graph demonstrates a number of problems. Firstly, it is difficult even for this shape to find a simple expression for the corner locations. Secondly, there are a number of values of α where the heights of two corners coincide. More generally there are curves (branch lines) of α and β values where the heights coincide. Any approximation method is likely to have problems close to such lines.

However, the rotation procedure described above can be applied easily to any shape of fuel tank with a known internal structure, and of course the corner heights are smooth functions of α and β . Thus, we could in principle determine the discontinuity lines directly from this calculation, although it would require significant investment of time (and would have to include the fine structure such as the stringers which are not included above).

4 Numerical approximation of the data

4.1 Outline

We now consider the problem of simply starting from the data and finding a numerical approximation to it, ideally over all ranges of h , α and β . The theory of the approximation of smooth functions is well developed, but much less is known about the approximation of nonsmooth functions. The lack of a second derivative will cause any method which assumes smoothness to have large errors (typically oscillations close to the discontinuity). Essentially what is needed in the context of this problem is a way of finding the surfaces across which the second derivative is discontinuous and using a fine mesh close to these surfaces. This is a difficult problem in general and two approaches can in principle be used. The first is to use *a priori* information about the location of the discontinuities, as indicated in the last section, and the second is to use an *a posteriori* estimate of the error of an approximation and to then adapt the approximation where this error is high. We now consider the second of these two approaches.²

4.2 Approximation of the data on a uniform mesh

We assume that the $V(h, \alpha, \beta)$ data is given on a regular grid in h , α and β , and that this data is accurate (and periodic in α and β) but with discontinuities in the second derivatives. At the Study Group the data was given as a series of sets of h - V data for fixed values of α and of β , so that for each such pair we had a table of values

$$(h_k, V_k), \quad k = 1, \dots, K$$

²It would be interesting to investigate further whether this approach does in fact refine the mesh near to discontinuities in the second derivative, *i.e.* whether the discontinuities correspond to regions where the approximation error is high.

with $K \approx 400$. The tables were given for pairs of α and β over a uniform mesh of pitch and roll angles with 1° separation. If α and β are fixed then the approximation problem of the h - V curve consists of finding an approximation $\tilde{V} = f(h)$ so that for a given tolerance which depends on V , *e.g.*

$$\epsilon(V) = \epsilon_a + \epsilon_r |V|,$$

we want

$$|f(h_k) - V_k| \leq \epsilon(V_k), \quad \forall k$$

with a small number of parameters describing \tilde{V} .

A useful observation from the previous section is that the h - V curve is a sequence of piecewise *cubic* functions, and thus it makes sense to approximate it by cubic B-splines or similar (*e.g.* Chebyshev) functions [1]. Various approximation approaches can be applied to this problem, including

- (1) Data fitting using piecewise cubic polynomials (typically cubic B-splines) on a very fine mesh.
- (2) Data fitting using piecewise cubic polynomials with H-adaptivity, in which we increase the density of the mesh at points where there are jumps in the second derivative.
- (3) Interpolating splines with a fixed number of free knots which are concentrated at the discontinuity points.
- (4) Wavelets using the best n -term approximation (giving compression where needed).

Of these four methods, the third requires the *em a priori* information discussed in the previous section. We did not consider this method further in the Study Group. The fourth method is potentially a powerful one as wavelets have the property that they can capture local structure very well. Although we did not have time to consider this further during the Study Group, we consider this a procedure worth future investigation.

Initially, during the Study Group we considered Method (1) above. The basic idea of this method, with α and β fixed, is to take the original data set with K data points, which we assume is too large. (Of course the data would be considerably larger if all values of α and β are included.) Now, for a value $n \ll K$ we take a subset of size $n + 2$ of these data points, spaced at equal intervals and including h_1 and h_K . We can easily construct a basis set of cubic B-splines on this subset of points. (Details of the form of the cubic B-splines are given in the monograph by Powell [2].) Using this basis set, we can then construct an approximation $\tilde{V} = f(h)$ to V given by

$$f(h) = \sum_{i=1}^{n+2} \gamma_i B_i(h),$$

where the $B_i(h)$ are the cubic-splines and the γ_i are appropriate calculated coefficients.

In the Study Group we considered two approaches to finding the coefficients γ_i . The first is to find the values which minimise a weighted least squares L^2 error of the form

$$E_2 = \sum_{k=1}^K w_k^2 (f(h_k) - V_k)^2, \quad \text{where} \quad w_k = 2 - \frac{V_k}{V_K}.$$

This approximation emphasises errors made when V_k is small, which are likely to be more significant. The procedure for implementing this method was to choose an initial value of n and then to increase n until the point errors were sufficiently small so that

$$|V_k - f(h_k)| \leq 0.005 \frac{V_K}{w_k}, \quad k = 1, \dots, K.$$

In general this method worked and converged well. (The coefficients were determined by using the inbuilt Matlab optimisation routines.) It was found that, over a range of values of α and β , between 6 and 12 coefficients γ_i were needed for each such approximation. This gave a data reduction factor of about 60 over the original data set.

We also considered using an L^∞ minimisation in which the coefficients γ_i were chosen to minimise

$$E_\infty = \max_k w_k |f(h_k) - V_k|.$$

This is a much harder approximation problem and did not always work.

To see how well these methods worked we show in Figure 7 a plot of the error (darkest values) of this method over a range of values of α and β . The lines of largest errors arising from the discontinuities can be seen clearly in these figures. The error could be reduced by adapting the mesh close to these lines and we consider this in the next section.

4.3 An adaptive approach

4.3.1 Outline

In practice a fine mesh might not be practical for computations, and an adaptive approach could well be needed. The adaptive approach that we considered in the Study Group was Method (2) described above, in which we used H -*adaptivity* with piecewise

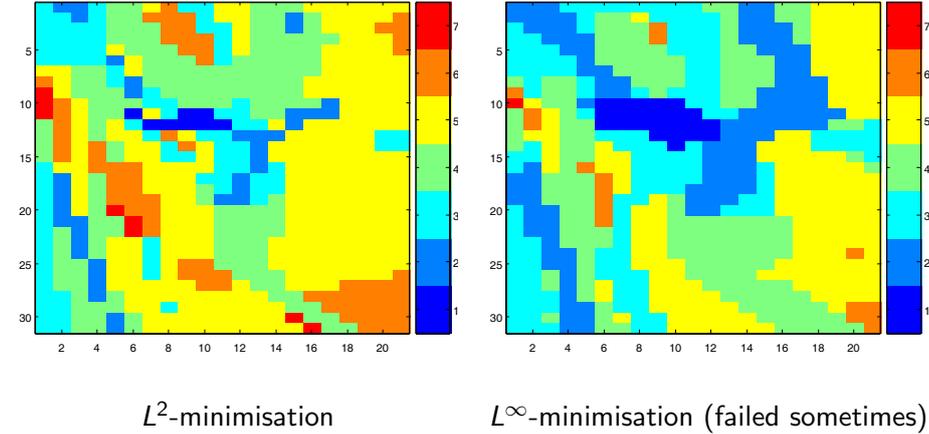


Figure 7: The maximum error obtained from the minimisation problems described above over a range of values of α and β .

cubic polynomials. To implement this for the problem of interest to Airbus it would be necessary to impose a mesh over the full (h, α, β) space. However, this was not possible in the time available, and instead we considered in detail only the case of keeping α and β fixed and approximating the curve over some interval $h \in [a, b]$ (although the basic idea can be in principle extended to any dimension). The approach behind the adaptive method is to start the approximation with a uniform mesh (of mesh size H) and initially to approximate the h - V curve on this mesh using smooth functions as described above. We then used the following algorithm to improve the approximation.

- A best least squares approximation to the h - V curve on $[a, b]$ is generated using piecewise cubic polynomials on the current mesh.
- The actual weighted error is calculated for each subinterval.
- If the prescribed tolerance is not reached on a particular subinterval then a new mesh is generated by recursion. In this process the subintervals of $[a, b]$ where the error is too high are subdivided into two and the approximation process is repeated.
- The resulting data structure takes the form of a binary tree, with the midpoints in nodes, and the coefficients in the leaves.

The advantages of this approach are that it is easy to implement, can be applied to any dimension and any basis functions and can use errors obtained from standard approximation theory. It is described in more detail in the monograph by DeVore and Lorentz [3].

However, it is not as powerful as using free knots (in that too many mesh points may be needed to obtain the approximation) and it is, in general, hard to impose continuity.

4.3.2 Results of the 1D calculation

To implement this approach we considered a mesh with 229 points, and used local Chebyshev polynomials of degree 3. The relative and absolute tolerances used were $\epsilon_r = 2.5 \cdot 10^{-3}$, $\epsilon_a = 10^{-5}$, which were tighter than the earlier tolerances. The procedure outlined above converged and the resulting approximation had $11 \times 4 = 44$ coefficients, with 10 interval midpoints. Looking at the resulting curve it is clear that the adaptive procedure has clustered most of the points at the start of the curve where V is close to zero. It is possible that this clustering is unnecessary and is due to the way that the errors are calculated. It could be possibly avoided if more care is used in incorporating an analytical description of this part of the curve into the numerical method.

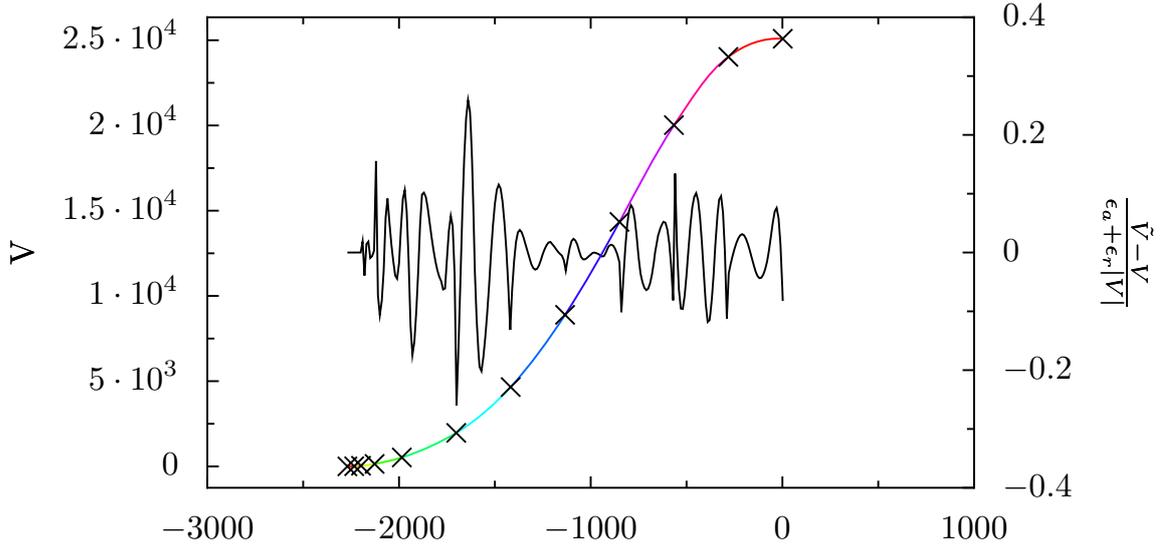


Figure 8: A plot of the results of the adaptive calculation showing both the computed curve and the error. Note the clustering of the points at the origin.

5 Conclusions and further work

The results of the one-dimensional calculation are encouraging and show that approximation of the h - V curve for fixed α and β is possible using both a fixed mesh and an adaptive approach. We see a reduction in data by a factor of approximately 60. Approximation in the pitch and roll directions is likely to be much harder due to the complex nature of the discontinuity surfaces in this case.

For future work we recommend that the adaptive approach be considered further using as much a priori information as possible. We would also suggest that the use of wavelets might give good results [4, 5].

References

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