<table>
<thead>
<tr>
<th>Title</th>
<th>Tropical Ideals, genera of tropicalization of curves and the minimum finishing time of projective networks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ito, Takaaki</td>
</tr>
<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 代数幾何学シンポジウム記録 代数幾何学シンポジウム記録</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2017</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/229106">http://hdl.handle.net/2433/229106</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University provided by
View metadata, citation and similar papers at core.ac.uk
Tropical Ideals, genera of tropicalization of curves and the minimum finishing time of projective networks

Takaaki Ito

Graduate School of Science and Technology, Tokyo Metropolitan University

~Tropical Ideals~

● Background
  - For a tropical polynomial \( f \in \mathbb{T}[x_1, ..., x_n] \), the tropical variety \( V(f) \subset \mathbb{T}^n \) is the support of a finite polyhedral complex in \( \mathbb{T}^n \).
  - If we define the tropical variety \( V(f) \) defined by an ideal \( I \) in \( \mathbb{T}[x_1, ..., x_n] \) as
    \[ V(I) = \left\{ f \in \mathbb{T}^n \right\} \]
    then the variety \( V(I) \) is not always the support of a finite polyhedral complex (Example 5.14 in [1]).
  - Maclagan and Rincón defined tropical ideals in [1]. The tropical variety defined by a tropical ideal is the support of a finite polyhedral complex.

● Motivations
  - It is difficult to treat tropical ideals like classical one because they are not closed under the addition, multiplication or intersection. (We cannot even "generate" a tropical ideal from an arbitrary set of tropical polynomials.)

We want to make another “tropical ideals” such that
(a) The tropical variety defined by any of them is the support of a finite polyhedral complex.
(b) They are closed under the addition, multiplication and intersection.

● Notation
  - \( \mathbb{T} \): Tropical semifield
    - i.e. the semifield (\( R \cup \{-\infty\}, \oplus, \otimes \)), where
      \[ a \oplus b = \max \{a, b\} \ (\text{addition}) \]
      \[ a \otimes b = a + b \ (\text{multiplication}) \]
  - \( \mathbb{T}[x_1, ..., x_n] \): Tropical polynomial semiring


~Main Results~

● Definition
  - The tropical polynomial function semiring is the quotient semiring \( \mathbb{T}[x_1, ..., x_n]^\ast \), where \( \ast \) is defined as
    \[ f \ast g = f(a) \ast g(a) \quad \forall a \in \mathbb{R}^n \]
The tropical variety \( V(f) \) defined by a tropical polynomial \( f \in \mathbb{T}[x_1, ..., x_n] \) is
    \[ V(f) = \left\{ x \in \mathbb{T}^n \right\} \]
    The maximum of \( \{a(x)\} \) is attained at least twice or \( \{a(x)\} = -\infty \).

We may define the tropical variety \( V(\phi) \) just for a tropical polynomial function \( \phi \).

We denote by \( \{f\}_k \) the coefficient of the monomial \( x^k \) in a tropical polynomial \( f \).

Each tropical polynomial function \( \phi \) has a unique maximum representation.

An ideal \( I \) in \( \mathbb{T}[x_1, ..., x_n]^\ast \) is a tropical ideal if for any \( \phi, \psi \in I \) and any monomial \( x^k \) with \( \{\phi \}_k \neq \{\psi \}_k \neq -\infty \), there is a tropical polynomial \( h \) such that,
  1. the class of \( h \) is in \( I \),
  2. \( [h]_k = -\infty \),
  3. \( [h]_k \leq \{\phi \}_k \otimes [\phi]_k \) for all \( \nu, \) and
  4. \( [h]_k = \{\phi \}_k \otimes [\phi]_k \) if \( \{\phi \}_k \neq \{\phi \}_k \).

Maclagan and Rincón’s definition of tropical ideals

An ideal \( I \) in \( \mathbb{T}[x_1, ..., x_n] \) is a tropical ideal if for any \( f, g \in I \) and any monomial \( x^k \) with \( \{f \}_k \neq \{g \}_k \neq -\infty \), there is a tropical polynomial \( h \) such that,
  1. the class of \( h \) is in \( I \),
  2. \( [h]_k = -\infty \),
  3. \( [h]_k \leq \{f \}_k \otimes [g]_k \) for all \( \nu, \) and
  4. \( [h]_k = \{f \}_k \otimes [g]_k \) if \( \{f \}_k \neq \{g \}_k \).

~The minimum finishing time of projective networks~

A project network consists of some activities, where each activity can be started after all the preceding activities have finished.

In the above project network, let \( t_i \) be the time to complete the activity. Then the minimum finishing time of this project network is
\[ \max \{t_1 + t_2 + t_3, t_1 + t_2 + t_4, t_1 + t_2 + t_5 + t_6, t_1 + t_2 + t_3 + t_4 + t_5 + t_6 \} = t_1 + t_3 + t_5 + t_6 \]
(4-tropical notation), which is a tropical polynomial of \( t_1 \).

What kind of tropical polynomials can be realized as the minimum finishing time of a project network?

~Genera of tropicalization of curves~

Definition
  - \( P \)-polynomial \( f(t) = f(t_1, ..., t_n) \) has term extendability if, for any subset \( I \subset \{1, ..., n\} \) such that, for all \( f_i, j \in I \), there is a term of \( f(t) \) divisible by \( t_i t_j \), there is a term of \( f(t) \) divisible by \( \prod_{i \in I} t_i \).

Theorem 1
  - There is one-one correspondence between the set of \( P \)-polynomials \( f(t) = f(t_1, ..., t_n) \) having term extendability and the set of simple graphs with the vertex set \( \{1, ..., n\} \).

We denote by \( \mathcal{G}(f) \) the simple graph corresponding to \( f(t) \).

Theorem 2
  - Let \( f(t) \) be a \( P \)-polynomial of degree \( d \) having term extendability. Then \( f(t) \) is realizable if and only if there is a vertex coloring of \( \mathcal{G}(f) \) with the color set \( \{1, ..., d\} \) such that, if \( v_1, v_2, v_3 \) is colored by \( c_1, c_2, c_3 \) with \( c_1 < c_2 < c_3 \) and \( (v_1, v_2) \), \( (v_2, v_3) \) are adjacent respectively, then \( (v_1, v_3) \) is also adjacent.

Definition
  - A \( P \)-polynomial is a tropical polynomial \( f(t) \) such that
    1. the degree on each variable is exactly one,
    2. the coefficient of each term is a unity,
    3. no term is divisible by any other terms.
