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# MATHEMATICAL INDUCTION AT THE TERTIARY LEVEL: LOOKING BEHIND APPEARENCES

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*The relevance of inductive proofs in Mathematics is beyond question and the research in Mathematics Education has widely documented the students' difficulties in understanding and applying mathematical induction, both at secondary school level and at university level. In this paper, we present a qualitative study involving third year Mathematics degree students aimed at investigating the solidity/fragility of mathematical induction comprehension. The results highlight that mathematical induction is a very hard topic also in this context, in which are involved mathematical competent students. We argue the need to design non-standard activities able to get the misconceptions emerge, in order to support a deep understanding of the topic.*

## INTRODUCTION AND THEORETICAL PERSPECTIVE

The process of reasoning called "mathematical induction" has had a long history as almost the whole of the mathematical concepts. We can already recognize the germ of mathematical induction (MI) in the Euclidean proof of the infinity of prime numbers, but the name "mathematical induction" was used for the first time in 1838 by Augustus De Morgan in his article "Induction (Mathematics)" (Cajory, 1918). From a didactical point of view, as Ernest (1984) underlines, the word "induction" introduces an ambiguity between the heuristic method for arriving at a conjecture (starting by a finite number of examples) and the mathematical rigorous form of deductive proof.

The relevance of MI in Mathematics is related to the foundation of natural numbers – the modern formal version of mathematical induction was stated in Peano Arithmetic (Peano, 1889) – and it is well summarized by Poincaré's opinion that MI is the "mathematical reasoning *par excellence*" (Poincaré, 1906). On the one hand Poincaré considers MI as the affirmation of a property of the mind itself, on the other hand he is aware of the complexity of a principle that "contains, condensed, in a unique formula, an infinity of syllogisms" (Poincaré, 1906). It exists a clear parallel between the relevance of MI in Mathematics and its relevance from an educational point of view. The students' understanding of MI allows to reflect on inductive reasoning and on the fundamental properties of natural numbers (Palla, Potari & Spyrou, 2011). This is one of the main reason why it has been argued that MI should be introduced in upper secondary school (NCTM, 2000).

The research in Mathematics Education has widely documented the (secondary school and university) students' difficulties in understanding and applying MI (Ernest, 1984;



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Harel, 2002; Nardi & Iannone, 2003; Palla, Potari & Spyrou, 2011). On the basis of the research results, we can identify three kinds of difficulties: the problematic relationship between inductive argumentation and proof by induction (this is related to the students' difficulty to conceptualise the need for a proof), technical difficulties in developing proof by MI and conceptual difficulties in understanding the structure and use of MI.

Pedemonte (2007) has carefully studied the cognitive difficulties related to the transition from an inductive argumentation to a proof by MI. She finds that students construct inductive proofs of a sentence only when they are able to generalize the process leading to the solution of the problem in inductive argumentation. Further upstream, this transition is problematic because several students see inductive argumentation as perfectly convincing, not recognizing the need for a rigorous proof by MI (Stylianides & Stylianides, 2009). Harel (2002) underlines that the standard instructional treatments introduce the formal principle of MI without ensuring that the students develop an intellectual need for it. On the other hand, Weber (2010) underlines that spurring students to see the limitations of empirical arguments can help to share with them the need for a rigorous proof, but it does not imply improving their comprehension and ability in producing mathematical proofs.

Nardi and Iannone (2003) have exactly studied the students' difficulties with MI when the necessity of proof seems to have been recognized. They carefully describe the technical difficulties emerging in the proof of the inductive step $P(n) \rightarrow P(n+1)$, especially in contexts where predicate $P$ involves non-symmetrical, one-way relationships such as algebraic inequalities. In some cases, also the explicit writing of the $P(n+1)$ term in the inductive step can be hard for undergraduate students.

For example, in the task: Prove that: for any $n\,(n > 0)$ positive numbers $x_1, x_2, x_3, \dots, x_n$, if $\prod_{i=1}^{n} x_i = 1$, then $\sum_{i=1}^{n} x_i \geq n$; a common mistake is the use of the same variables $x_1, x_2, x_3, \dots, x_n, x_{n+1}$ in the $P(n+1)$ part (Avital & Libeskind, 1978).

On the conceptual level, the pioneering studies of Ernest (1984) and of Fischbein and Engel (1989) discuss the difficulties related to the complex use of quantifiers in MI and to the awkward distinction in dealing with the inductive step $P(n) \rightarrow P(n+1)$ between the inductive hypothesis $P(n)$ and the thesis, generally expressed as $\forall n\, P(n)$. Also Sfard (1992), within her theory of reification, describes the conceptual difficulties related to MI. In the passage from the operational to the structural comprehension of MI, the learner reifies MI in an object – the global structure of natural numbers – that can be manipulated (Palla et al., 2011). In the passage from elementary to advanced mathematics, the level of complexity of proving by MI develops from the operational to the structural level, and the related difficulties from the technical level to the conceptual one.

An interesting general concept developed in the field of Mathematics Education – just in a research focused on students' understanding of MI – is the "fragility" of personal knowledge (Movshovitz-Hadar, 1991, pp. 41-42):

> A human being's knowledge is fragile as long as this person can be put in a cognitive conflict [...] The ultimate goal of the process of [...] mathematical learning in particular, is to develop one's understanding [...] that is robust enough to withstand any attempt to create a cognitive conflict, to inject contradictions into it.

Movshovitz-Hadar coins the term "knowledge fragility" to describe a knowledge "in the intermediate stage of development, when understanding is yet to be negotiated" (Movshovitz-Hadar, 1993, p. 266). In the context of mathematics, it seems particularly interesting to study this dimension (solidity/fragility) of knowledge also when the understanding of a complex concept has already been negotiated: in Sfard's term, when a conceptual understanding seems to have been developed.

In this framework, we developed a qualitative study involving a particular sample composed by third year Mathematics degree students (from three different cohorts). Our aim was to investigate the solidity/fragility of MI-comprehension of apparently *MI-competent* students, i.e. students that: i) surely recognize the need for a deductive proof for a mathematical statement; ii) seem to have no technical difficulties in complex algebraic manipulations and explicit writing of the $P(n+1)$ term in the inductive step; iii) seem to have developed a structural comprehension of MI.

## PROCEDURE AND RATIONALE

The work described in this report is based on the analysis of the qualitative data collected in three consecutive academic years during the course Elementary Mathematics from an Advanced Standpoint: Arithmetic. The course – held by the third author – is an optional course for the third year of the Mathematics degree and it deals with the topic of number systems. The reflection about natural numbers is an important part of the course program and, in particular, MI is the focus of five structured class meetings (CM in the following).

The first CM is used to get an idea of students' knowledge about the formulation of MI and its equivalent forms, and also to test their practice and ability in technical manipulations involved in the application of MI. For this aim, we use recursion problems that result insidious from a technical point of view. In particular, according to Nardi and Iannone's analysis (Nardi & Iannone, 2003), we use problems involving non-symmetrical, one-way relationships such as algebraic inequalities. Many of them were formerly faced by the students in the first-year exam of the Algebra course (see for example Figure 1).

Consider the following series: $\begin{cases} a_0 = 2;\ a_1 = 3;\ a_2 = 5 \\ a_{n+1} = a_n - a_{n-1} + 2a_{n-2}\ \forall\ n > 1 \end{cases}$

Prove that $a_{n+1} > a_n$, for each $n$.

Figure 1: Example of a task given in the first CM.

The remaining four CMs are organized into two pairs: the first CM of each pair is a problem solving section, the second CM takes place three days later and it is organized as a discussion group focused on the previous problem solving section.

The first pair of CMs deals with the relationship between conjecturing and proving. In particular, in these two meetings the aim is to observe the behaviour of the students when they are not successful in proving a conjecture achieved by empirical arguments: do they mistrust the developed conjecture or their capabilities in proving it by induction? During these CMs, two geometrical tasks are proposed to the students (Figure 2).

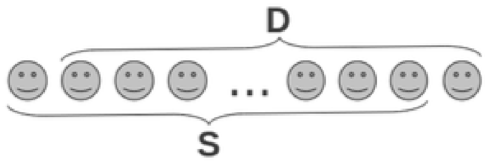
| | |
|---|---|
| Task 1: How does the number of diagonals of a convex polygon P vary, varying the number n of its sides? | Task 2: Determining the maximum number of pieces in which it is possible to divide a circle by n points on the circumference, joined by all the possible chords. |

Figure 2: Examples of tasks used in CM 2 and CM 3.

The second pair of CMs is focused on conceptual understanding of MI. Similar to what was done by Movshovitz-Hadar (1993), the first one of these meetings starts with a discussion about the following two related issues: a) to rate the difficulties of mathematical induction as proof method; b) to rate the expected success in solving a problem which calls for a proof by mathematical induction. At the end of the discussion, the "Children's eye colour" problem is proposed (see Figure 3).

***Theorem***: All newborns have eyes of the same colour.

**Proof**: For $n = 1$ the thesis is trivially true. Suppose $P(n)$ holds (that is, in every set of $n$ babies, all babies have eyes of the same colour). Consider now a set of $n + 1$ babies. The following graphical scheme proves that: if $P(n)$ holds, then $P(n + 1)$ holds.




What is wrong with this proof?

Figure 3: The children's eye colour problem.

The data of our study consist in the students' written responses to several mathematical tasks and in the notes taken by the third author during the class discussions following the work on the mathematical tasks. Forty-seven students participated overall in the three editions of the course. All of them had already encountered MI in two different courses of the first university year (that is, Analysis and Algebra) and they were used to prove rather complex arithmetical facts by MI, especially in the Algebra course.

## RESULTS AND DISCUSSION

**CM 1**. The first meeting, focused on tasks involving explicitly the use of induction to prove mathematical facts, is aimed at evaluating the technical competences of the sample participating in the study. The collected data confirm (for all the three involved cohorts) that the considered students have a stabilized familiarity with the use of MI. No technical difficulties in complex algebraic manipulations or in explicit writing of the $P(n+1)$ term in the inductive step emerge.

**CM 2 – CM 3**. In the second meeting the students have to face Task 1 and Task 2, which are similar at first glance and both belonging to a geometrical context, but actually very different. According to the terminology introduced by Harel (2002), the first task elicits a 'process pattern generalisation': all the students arrive quickly to the correct conjecture (the diagonals' number of a convex polygon of $n$ sides is $n(n\text{-}3)/2$) starting by empirical proves on polygons with few sides and then focusing on regularity of the process of adding a single vertex to a given polygons with $n$ vertexes (see Fig. 4). They observe that $n\text{-}2$ diagonals can be drawn from the new vertex $V_{n+1}$, and a new diagonal joins the two vertexes $V_1$, $V_n$ that in the n-side polygon were consecutive. The generalization of this process pattern coincides with the inductive step proof.

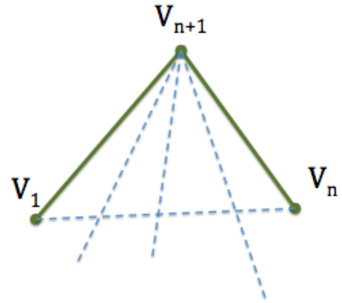



Figure 4: The typical student's sketch supporting the process pattern generalization.

Task 2, instead, elicits what Harel calls a 'result pattern generalization', that is a pattern based on regularity in the results. For $n$ varying from 1 to 5, the result is $2^{n\text{-}1}$, and this led all the students of the three cohorts to conjecture that the result holds for every $n$. In fact, they typically stop in their results' control process exactly when $n = 5$. The interesting aspect is that Task 2 has not a generalizable solution by simple algebraic formulas, since for $n = 6$ the maximum number of pieces is unexpectedly 31, instead of 32. It is only during CM3, that the failure in generating a proof leads to questioning the developed conjecture.

It is clear that the following clause of the didactic contract plays an important role in this situation: if students have to produce a generalization conjecture then the final statement can be express by a simple algebraic formula. On the other side, the interesting phenomenon is the comeback to the empirical arguments after the first failures of the proof by induction. Students spent a lot of time for the case n = 6, that is anything but trivial to graphically manage. The students that rightly finding 31 as the maximum number of pieces convince themselves that they get an inaccurate and wrong drawing or, alternatively, they believe that the disposition of the points on the circumference does not maximize the number of pieces. Then, they try with a bigger drawing to avoid the chords' overlapping or with a new arrangement of the points on the circumference.

This behaviour shows how also for those students who are undoubtedly aware that proving a statement for a finite number of cases is not enough to prove it for infinite cases, the trust in a beautiful conjecture, based on a finite number of arguments, strongly survives.

**CM 4 – CM 5**. The students' answers to questions a) and b) show an interesting overlap between the shared high confidence in their own capabilities in applying MI and the judgment about MI as an absolutely simple proof method. This phenomenon occurs despite the fact that many students can remember how had been hard to manage MI in the first years' courses of the master degree. This phenomenon highlights the issue – which we only sketch – regarding the poor attitude of high-achievers in Mathematics in understanding the difficulties that they do not have anymore. To the question "which difficulties can characterize a proof by induction?", the collected answers refers only only the particular tasks that: i) request some "specific tricks" (for example, specific minorizations) to carry out the inductive step; or ii) require to consider *particular subsequent terms* (for example conjecture involving sequences in which the even terms are defined by formulas different from the ones for odd terms).

The most interesting part of the two last meetings is doubtlessly the discussion about the Children's eye colour problem. This problem is well-known in different formulations, that can all be summarized in the (obviously false) result that every equivalence relation in a countable set is a total relation. The version used in this study is especially meaningful, since it is really evident that the conclusion is false.

There are few students (under the 10% of the whole sample) highlighting the error in the proof, that is, grasping the no validity of the induction step for every $n$: the underlying reasoning in the presented drawing (Figure 3) clearly does not work for $n = 2$. Indeed, in this case, set S and set D have empty intersection and it is not possible to use the relation's transitivity to conclude that the only element in S is equivalent to the element in D.

The explanations of the students which do not notice the error highlight the knowledge fragility with respect to MI and let the main misconceptions emerge.

In particular, from the analysis of the three cohorts' collected answers, we can classify the explanations regarding what does not work in the presented proof (reminding that the conclusion is surely false) into three different and interesting categories.

The first explanation *contests* the field of application of MI, in terms of: the considered set: "*MI is applicable to numerical sets, not to different sets (such as sets of persons)*"; the considered property: "*MI is not applicable to physical characteristics, only to numerical properties*"; not recognizable order: "*MI is applicable to events' sequences, but in this case the events are not subsequent*".

These argumentations can be easily demolished by a reformulation of the problem's statement, with $P(n)$, to be proved, becoming: "In every set of $n$ newborns there is only an eye colour".

The second category of explanation regards the conceptual difficulty related to the use of quantifiers in MI. What is contested, in fact, is that the inductive step hypothesis is used in two different sets with n elements: "*In the proof the inductive step is applied to different sets*"; "*In the proof I have to suppose that the property holds for a specific set of n newborns, not for all the sets of n newborns*".

The third category of explanation regards the already discussed conceptual complexity about the difference between inductive hypotheses and thesis. There is the inclination to think that MI presumes that the premise $P(n)$ of the implication $P(n) \rightarrow P(n+1)$ is true and so to wrongly believe that MI use as hypotheses what has to be proved: "*In the proof of the theorem about the eye colour $P(n)$ is supposed true, but it has not been proved*".

As Fischbein e Engel (1989, p. 282) affirm:

> The induction step requires a proof on its own (as a temporarily autonomous implicative statement). The idea that one has to prove an implication $p \rightarrow q$ for which the problem of the objective truth of each of the two components, $p$ and $q$, is totally irrelevant (in the realm of the induction step) seems to be intuitively unacceptable. This situation is complicated by the fact that the antecedent $p$ includes the theorem to be proven.

## CONCLUSIONS

The results of our study lead to some considerations. First of all, we can observe how MI remains a very hard topic, also in the case where there is a shared awareness of the necessity of proving by a deductive process the empirical conjectures and also when students have excellent technical capabilities. More in general, we would like to highlight that, to evaluate the knowledge fragility/solidity, it is necessary to develop and to propose activities able to create 'crisis' moments' for the students, activities which undermine their convictions. Tasks like the Children's eye colour problem seem to be adequate for pursuing this aim in relation to MI. So, it could be interesting to project other activities of this kind, not only for MI, but, more in general for other basic mathematical concepts. Indeed, we are convinced that finding out the knowledge fragility about some issues is surely interesting from the point of view of the research but it is also a crucial point for the educational practice.

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