CHARACTERISTIC IDEALS AND SELMER GROUPS

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Abstract. Let A be an abelian variety defined over a global field F of positive characteristic p and let $\mathcal{F}/F$ be a $\mathbb{Z}_p^{\mathbb{N}}$-extension, unramified outside a finite set of places of F. Assuming that all ramified places are totally ramified, we define a pro-characteristic ideal associated to the Pontrjagin dual of the p-primary Selmer group of A. To do this we first show the relation between the characteristic ideals of duals of Selmer groups for a $\mathbb{Z}_p^d$-extension $\mathcal{F}_d/F$ and for any $\mathbb{Z}_p^{d-1}$-extension contained in $\mathcal{F}_d$, and then use a limit process. Finally, we give an application to an Iwasawa Main Conjecture for the non-noetherian commutative Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(\mathcal{F}/F)]]$ in the case A is a constant abelian variety.

1. Introduction

Let F be a global function field of characteristic p and $\mathcal{F}/F$ a $\mathbb{Z}_p^{\mathbb{N}}$-extension unramified outside a finite set of places, whose Galois group we denote by Γ. We take an abelian variety A defined over F and let S_A be a finite set of places of F containing exactly the primes of bad reduction for A and those which ramify in $\mathcal{F}/F$. For any extension v of some place of F to the algebraic closure $\overline{F}$ and for any finite extension E/F, we denote by E_v the completion of E with respect to v and, if $\mathcal{L}/F$ is infinite, we put $\mathcal{L}_v := \cup E_v$, where the union is taken over all finite subextensions of $\mathcal{L}$. We define the p-part of the *Selmer group* of A over E as

$$Sel(E) := Sel_A(E)_p := Ker\left\{ H^1_{fl}(X_E, A[p^\infty]) \longrightarrow \prod_v H^1_{fl}(X_{E_v}, A)[p^\infty] \right\}$$

(where H^1_{fl} denotes flat cohomology, $X_E := Spec(E)$ and the map is the product of the natural restrictions at all places v of E). For infinite algebraic extensions we define the Selmer groups by taking direct limits on all the finite subextensions. For any algebraic extension K/F, let $\mathcal{S}(K)$ denote the Pontrjagin dual of $Sel(K)$ (other Pontrjagin duals will be indicated by the symbol $^\vee$).

For any infinite p-adic Lie extension $\mathcal{L}/F$, let $\Lambda(\mathcal{L}) := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{L}/F)]]$ be the associated Iwasawa algebra: we recall that, if $\mathrm{Gal}(\mathcal{L}/F) \simeq \mathbb{Z}_p^d$, then $\Lambda(\mathcal{L}) \simeq \mathbb{Z}_p[[t_1, .., t_d]]$ is a Krull domain. It is well known that $\mathcal{S}(\mathcal{L})$ is a $\Lambda(\mathcal{L})$-module and its structure has been described in several recent papers (see, e.g., [14] for $\mathrm{Gal}(\mathcal{L}/F) \simeq \mathbb{Z}_p^d$ and [5] for the non abelian case). When $\mathcal{S}(\mathcal{L})$ is a finitely generated module over a noetherian abelian Iwasawa algebra, it is possible to associate to $\mathcal{S}(\mathcal{L})$ a characteristic ideal which is a key ingredient in Iwasawa Main Conjectures. We are interested in the definition of the analogue of a characteristic ideal in $\Lambda(\mathcal{F})$ for $\mathcal{S}(\mathcal{F})$ (a similar result providing a *pro-characteristic ideal* for the Iwasawa module of class groups is described in [4]).

If R is a noetherian Krull domain and M a finitely generated torsion R-module, the structure theorem for M provides an exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} R/\mathfrak{p}_i^{e_i} R \longrightarrow Q \longrightarrow 0 \tag{1.1}$$

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where the $\mathfrak{p}_i$'s are height 1 prime ideals of R and P and Q are pseudo-null R-modules (i.e., torsion modules with annihilator of height at least 2). With this sequence one defines the *characteristic ideal* of M as

$$Ch_R(M) := \prod_{i=1}^{n} \mathfrak{p}_i^{e_i}$$

(if M is not torsion, we put $Ch_R(M) = 0$, moreover note that M is pseudo-null if and only if $Ch_R(M) = (1)$). In commutative Iwasawa theory characteristic ideals provide the algebraic counterpart for the p-adic L-functions associated to Iwasawa modules (such as duals of Selmer groups).

We fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i\in\mathbb{N}}$ for $\Gamma := \mathrm{Gal}(\mathcal{F}/F)$ and, for any $d \geqslant 0$, we let $\mathcal{F}_d \subset \mathcal{F}$ be the fixed field of $\{\gamma_i\}_{i>d}$. Then we have $\Lambda(\mathcal{F}) = \varprojlim \Lambda(\mathcal{F}_d)$ and $\mathcal{S}(\mathcal{F}) = \varprojlim \mathcal{S}(\mathcal{F}_d)$. Note that the filtration $\{\mathcal{F}_d\}$ of $\mathcal{F}$ is uniquely determined once the γ_i have been fixed, but we allow complete freedom in their initial choice. Put $t_i := \gamma_i - 1$: the subring $\mathbb{Z}_p[[t_1, \ldots, t_d]]$ of $\Lambda(\mathcal{F})$ is isomorphic to $\Lambda(\mathcal{F}_d)$ and, by a slight abuse of notation, the two shall be identified in this paper. In particular, for any $d \geqslant 1$ we have $\Lambda(\mathcal{F}_d) = \Lambda(\mathcal{F}_{d-1})[[t_d]]$. Let $\pi^d_{d-1} \colon \Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_{d-1})$ be the canonical projection, denote its kernel by $I^d_{d-1} = (t_d)$ and put $\Gamma^d_{d-1} := \mathrm{Gal}(\mathcal{F}_d/\mathcal{F}_{d-1})$.

Our goal is to define an ideal attached to $\mathcal{S}(\mathcal{F})$ in the non-noetherian Iwasawa algebra $\Lambda(\mathcal{F})$: we will do this via a limit of the characteristic ideals $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))$. Thus we need to study the relation between $\pi^d_{d-1}\big(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))\big)$ and $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$. A general technique to deal with this type of descent and ensure that the limit does not depend on the filtration has been described in [4, Theorem 2.13]. That theorem is based on a generalization of some results of [11, Section 3] (which directly apply to our algebras $\Lambda(\mathcal{F}_d)$, even without the generalization to Krull domains provided in [4]) and can be applied to the $\Lambda(\mathcal{F})$-module $\mathcal{S}(\mathcal{F})$. In our setting [4, Theorem 2.13] reads as follows

Theorem 1.1. *If, for every $d \gg 1$,*

1. *the t_d-torsion submodule of $\mathcal{S}(\mathcal{F}_d)$ is a pseudo-null $\Lambda(\mathcal{F}_{d-1})$-module, i.e.,*

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)_{t_d}) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) = (1) \ ;$$

2. $Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/t_d) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I^d_{d-1}) \subseteq Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1}))$,

then the ideals $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))$ form a projective system (with respect to the maps π^d_{d-1}).

In Section 2 we show that if $\mathcal{S}(\mathcal{F}_e)$ is $\Lambda(\mathcal{F}_e)$-torsion, then $\mathcal{S}(\mathcal{F}_d)$ is $\Lambda(\mathcal{F}_d)$-torsion for all $d \geqslant e$ and use [4, Proposition 2.10] to provide a general relation

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) \cdot \pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \cdot J_d \tag{1.2}$$

(see (2.9) where the extra factor J_d is more explicit). Then we move to the totally ramified setting, i.e., extensions in which all ramified primes are assumed to be totally ramified (an example are the extensions obtained from F by adding the $\mathfrak{a}^n$-torsion points of a normalized rank 1 Drinfeld module over F). In this setting, using some techniques and results of K.-S. Tan ([15]), we check the hypotheses of Theorem 1.1 using equation (1.2), and obtain (see Corollary 3.8 and Definition 3.9)

Theorem 1.2. *Assume all ramified primes in $\mathcal{F}/F$ are totally ramified. Then, for $d \gg 0$,*

$$\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1}))$$

and the pro-characteristic ideal

$$\widetilde{Ch}_{\Lambda(\mathcal{F})}(\mathcal{S}(\mathcal{F})) := \varprojlim_{d} Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) \subseteq \Lambda(\mathcal{F})$$

is well defined.

As an application, we use a deep result of Lai - Longhi - Tan - Trihan [9] to prove an Iwasawa Main Conjecture for constant abelian varieties in our non-noetherian setting (see Theorem 3.10).

2. General $\mathbb{Z}_p$-descent for Selmer groups

To be able to define characteristic ideals we need the following

Theorem 2.1. (**Tan**) *Assume that A has good ordinary or split multiplicative reduction at all ramified places of the finite set S_A. Then, for any d and any $\mathbb{Z}_p^d$-extension $\mathcal{L}/F$ contained in $\mathcal{F}$, the group $\mathcal{S}(\mathcal{L})$ is a finitely generated $\Lambda(\mathcal{L})$-module.*

Proof. In this form the theorem is due to Tan ([14, Theorem 5]). See also [3, Section 2] and the references there. □

If there is a place v ramified in $\mathcal{L}/F$ and of supersingular reduction for A, then the module $\mathcal{S}(\mathcal{L})$ is not finitely generated over $\Lambda(\mathcal{L})$ by [15, Proposition 1.1 and Theorem 3.10]. In order to obtain a nontrivial relation between the characteristic ideals, we need no ramified supersingular primes and something more than just Theorem 2.1, so we make the following

Assumptions 2.2.

1. All places ramified in $\mathcal{F}/F$ are of ordinary reduction.
2. There exists an $e > 0$ such that $\mathcal{S}(\mathcal{F}_e)$ is a torsion $\Lambda(\mathcal{F}_e)$-module.

Remarks 2.3.

1. Hypothesis **2** is satisfied in many cases: for example when $\mathcal{F}_e$ contains the arithmetic $\mathbb{Z}_p$-extension of F (proof in [15, Theorem 2], extending [12, Theorem 1.7]) or when $Sel(F)$ is finite and A has good ordinary reduction at all places which ramify in $\mathcal{F}_e/F$ (easy consequence of [14, Theorem 4]).
2. Our goal is an equation relating $\pi_{d-1}^d(Ch_{\Lambda(\mathcal{F}_d)}\mathcal{S}(\mathcal{F}_d))$ and the characteristic ideal of $\mathcal{S}(\mathcal{F}_{d-1})$. If the above assumption **2** is not satisfied for any e, then all characteristic ideals are 0 and there is nothing to prove.

In this section we also assume that none of the ramified prime has trivial decomposition group in $\mathrm{Gal}(\mathcal{F}_1/F)$. In Section 3 we shall work in extensions in which ramified places are totally ramified, so this assumption will be automatically verified. Anyway this is not restrictive in general because of the following

Lemma 2.4. *If $d \geqslant 2$, one can always find a $\mathbb{Z}_p$-subextension $\mathcal{F}_1/F$ of $\mathcal{F}_d/F$ in which none of the ramified places splits completely.*

Proof. See [4, Lemma 3.1] □

Consider the diagram

$$\begin{array}{ccccc} Sel(\mathcal{F}_{d-1}) & \hookrightarrow & H^1_{fl}(\mathcal{X}_{d-1}, A[p^\infty]) & \twoheadrightarrow & \mathcal{G}(\mathcal{X}_{d-1}) \\ \downarrow a^d_{d-1} & & \downarrow b^d_{d-1} & & \downarrow c^d_{d-1} \\ Sel(\mathcal{F}_d)^{\Gamma^d_{d-1}} & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty])^{\Gamma^d_{d-1}} & \longrightarrow & \mathcal{G}(\mathcal{X}_d)^{\Gamma^d_{d-1}} \end{array} \tag{2.1}$$

where $\mathcal{X}_d := Spec(\mathcal{F}_d)$, the vertical maps are induced by (global) restrictions and $\mathcal{G}(\mathcal{X}_d)$ is the image of the product of the (local) restriction maps

$$H^1_{fl}(\mathcal{X}_d, A[p^\infty]) \longrightarrow \prod_w H^1_{fl}(\mathcal{X}_{d,w}, A)[p^\infty] ,$$

with w running over all places of $\mathcal{F}_d$ where $\mathcal{X}_{d,w} := Spec(\mathcal{F}_{d,w})$ with $\mathcal{F}_{d,w}$ the completion of $\mathcal{F}_d$ at w.

Lemma 2.5. *Assume that no ramified place is totally split in $\mathcal{F}_1/F$. For any $d \geqslant 2$, the Pontrjagin dual of* $\operatorname{Ker} c^d_{d-1}$ *is a finitely generated torsion* $\Lambda(\mathcal{F}_{d-1})$*-module.*

Proof. For any place v of F we fix an extension to $\mathcal{F}$, which by a slight abuse of notation we still denote by v, so that the set of places of $\mathcal{F}_d$ above v will be the Galois orbit $\operatorname{Gal}(\mathcal{F}_d/F)\cdot v$. For any field L let $\mathcal{P}_L$ be the set of places of L. We have an obvious injection

$$\operatorname{Ker} c^d_{d-1} \hookrightarrow \prod_{u\in\mathcal{P}_{\mathcal{F}_{d-1}}} \operatorname{Ker}\left\{ H^1_{fl}(\mathcal{X}_{d-1,u},A)[p^\infty] \longrightarrow \prod_{w|u} H^1_{fl}(\mathcal{X}_{d,w},A)[p^\infty] \right\} \tag{2.2}$$

(the map is the product of the natural restrictions r_w). By the Hochschild-Serre spectral sequence, we get

$$\operatorname{Ker} r_w \simeq H^1(\Gamma^d_{d-1,w}, A(\mathcal{F}_{d,w}))[p^\infty] \tag{2.3}$$

where $\Gamma^d_{d-1,w}$ is the decomposition group of w in Γ^d_{d-1}. Those kernels really depend only on the place u of $\mathcal{F}_{d-1}$ lying below w (for any w_1, w_2 dividing u we obviously have $\operatorname{Ker} r_{w_1} \simeq \operatorname{Ker} r_{w_2}$). Hence for any v of F and any $u \in \mathcal{P}_{\mathcal{F}_{d-1}}$ dividing it, we fix a $w(u)$ of $\mathcal{F}_d$ over u and define

$$\mathcal{H}_v(\mathcal{F}_d) := \prod_{u\in\operatorname{Gal}(\mathcal{F}_{d-1}/F)\cdot v} H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]\ .$$

Equation (2.2) now reads as

$$\operatorname{Ker} c^d_{d-1} \hookrightarrow \prod_{v\in\mathcal{P}_F} \mathcal{H}_v(\mathcal{F}_d)\ . \tag{2.4}$$

Obviously $\mathcal{H}_v(\mathcal{F}_d) = 0$ for all primes which totally split in $\mathcal{F}_d/\mathcal{F}_{d-1}$ and, from now on, we only consider places such that $\Gamma^d_{d-1,w(u)} \neq 0$.

Let $\Lambda(\mathcal{F}_{i,v}) := \mathbb{Z}_p[[\operatorname{Gal}(\mathcal{F}_{i,v}/F_v)]]$ be the Iwasawa algebra associated to the decomposition group of v in $\operatorname{Gal}(\mathcal{F}_i/F)$ and note that each $\operatorname{Ker} r_w$ is a $\Lambda(\mathcal{F}_{d-1,v})$-module. Moreover, we get an action of $\operatorname{Gal}(\mathcal{F}_{d-1}/F)$ on $\mathcal{H}_v(\mathcal{F}_d)$ by permutation of the primes $u \in \operatorname{Gal}(\mathcal{F}_{d-1}/F)\cdot v$ and an isomorphism

$$\mathcal{H}_v(\mathcal{F}_d) \simeq \Lambda(\mathcal{F}_{d-1}) \otimes_{\Lambda(\mathcal{F}_{d-1,v})} H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty] \tag{2.5}$$

(see also [15, Lemma 3.2], note that $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]$ is finitely generated over $\Lambda(\mathcal{F}_{d-1,v})$).

First assume that the place v is unramified in $\mathcal{F}_d/F$ (hence inert in $\mathcal{F}_d/\mathcal{F}_{d-1}$). Then $\mathcal{F}_{d-1,v} = F_v \neq \mathcal{F}_{d,v}$ and one has, by [10, Proposition I.3.8],

$$H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)})) \simeq H^1(\Gamma^d_{d-1,w(u)}, \pi_0(\mathcal{A}_{0,v}))\ ,$$

where $\mathcal{A}_{0,v}$ is the closed fiber of the Néron model of A over F_v and $\pi_0(\mathcal{A}_{0,v})$ is its set of connected components. It follows that $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]$ is trivial when v does not lie above S_A and that it is finite of order bounded by (the p-part of) $|\pi_0(\mathcal{A}_{0,v})|$ for the unramified places of bad reduction. Hence (2.4) reduces to

$$\operatorname{Ker} c^d_{d-1} \hookrightarrow \bigoplus_{v\in S'_A(d)} \mathcal{H}_v(\mathcal{F}_d) \tag{2.6}$$

(where $S'_A(d)$ is the set of primes in S_A which are not totally split in $\mathcal{F}_d/\mathcal{F}_{d-1}$) and, by (2.5), $\mathcal{H}_v(\mathcal{F}_d)^\vee$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module for unramified v.

For the ramified case the exact sequence

$$A(\mathcal{F}_{d,w(u)})[p] \hookrightarrow A(\mathcal{F}_{d,w(u)}) \xrightarrow{p} pA(\mathcal{F}_{d,w(u)})$$

yields a surjection

$$H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)})[p]) \twoheadrightarrow H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p] \ .$$

The first module is obviously finite, so $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p]$ is finite as well: this implies that $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]^\vee$ has finite $\mathbb{Z}_p$-rank. As a finitely generated $\mathbb{Z}_p$-module, $H^1(\Gamma^d_{d-1,w(u)}, A(\mathcal{F}_{d,w(u)}))[p^\infty]^\vee$ must be $\mathbb{Z}_p[[\Gamma_{d-1,v}]]$-torsion for any $d \geqslant 2$ (because of our choice of $\mathcal{F}_1/F$) and (2.5) shows once again that $\mathcal{H}_v(\mathcal{F}_d)^\vee$ is finitely generated and torsion over $\Lambda(\mathcal{F}_{d-1})$. □

Remark 2.6. One can go deeper in the details and compute those kernels according to the reduction of A at v and the behavior of v in $\mathcal{F}_d/F$. We will do this in Section 3 but only for the particular case of a totally ramified extension (with the statement of a Main Conjecture as a final goal). See [15] for a more general analysis.

The following proposition provides a crucial step towards equation (1.2) (in particular it also takes care of hypothesis **2** of Theorem 1.1).

Proposition 2.7. *Assume that no ramified place is totally split in $\mathcal{F}_1/F$. Let e be as in Assumption 2.2.***2***. For any $d > e$, the module $\mathcal{S}(\mathcal{F}_d)/I^d_{d-1}$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module and $\mathcal{S}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module. Moreover, if $d > \max\{2, e\}$,*

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I^d_{d-1}) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \cdot Ch_{\Lambda(\mathcal{F}_{d-1})}((\mathrm{Coker}\, a^d_{d-1})^\vee) \ .$$

Proof. It suffices to prove the first statement for $d = e+1$, then a standard argument (detailed, e.g., in [8, page 207]) shows that $\mathcal{S}(\mathcal{F}_{e+1})$ is $\Lambda(\mathcal{F}_{e+1})$-torsion and we can iterate the process. From diagram (2.1) one gets a sequence

$$(2.7) \qquad (\mathrm{Coker}\, a^{e+1}_e)^\vee \hookrightarrow (Sel(\mathcal{F}_{e+1})^{\Gamma^{e+1}_e})^\vee \longrightarrow \mathcal{S}(\mathcal{F}_e) \twoheadrightarrow (\mathrm{Ker}\, a^{e+1}_e)^\vee \ .$$

By the Hochschild-Serre spectral sequence, it follows

$$\mathrm{Coker}\, b^{e+1}_e \hookrightarrow H^2(\Gamma^{e+1}_e, A[p^\infty](\mathcal{F}_{e+1})) = 0$$

(because Γ^{e+1}_e has p-cohomological dimension 1). Therefore there is a surjective map

$$\mathrm{Ker}\, c^{e+1}_e \twoheadrightarrow \mathrm{Coker}\, a^{e+1}_e$$

and, by Lemma 2.5, $(\mathrm{Coker}\, a^{e+1}_e)^\vee$ is $\Lambda(\mathcal{F}_e)$-torsion. Hence Assumption 2.2.**2** and sequence (2.7) yield that

$$(Sel(\mathcal{F}_{e+1})^{\Gamma^{e+1}_e})^\vee \simeq \mathcal{S}(\mathcal{F}_{e+1})/I^{e+1}_e$$

is $\Lambda(\mathcal{F}_e)$-torsion. To conclude note that (for any d) the duals of

$$\mathrm{Ker}\, a^d_{d-1} \hookrightarrow \mathrm{Ker}\, b^d_{d-1} \simeq H^1(\Gamma^d_{d-1}, A[p^\infty](\mathcal{F}_d)) \simeq A[p^\infty](\mathcal{F}_d)/I^d_{d-1}$$

are finitely generated $\mathbb{Z}_p$-modules (hence pseudo-null over $\Lambda(\mathcal{F}_{d-1})$ for any $d \geqslant 3$). Taking characteristic ideals in the sequence (2.7), for large enough d, one finds

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I^d_{d-1}) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \cdot Ch_{\Lambda(\mathcal{F}_{d-1})}((\mathrm{Coker}\, a^d_{d-1})^\vee) \ .$$

□

Remark 2.8. In [12, Theorem 1.7], the authors prove that $\mathcal{S}(F^{(p)})$ is a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(F^{(p)}/F)]]$-module (where $F^{(p)}$ is the arithmetic $\mathbb{Z}_p$-extension of F). The first part of the proof above provides a more direct approach to the generalization of this result given in [15, Theorem 2].

Whenever $\mathcal{S}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module, [4, Proposition 2.10] yields

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}})\cdot\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)))=Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I^d_{d-1})\ . \tag{2.8}$$

If $d>\max\{2,e\}$, equation (2.8) turns into

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}})\cdot\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)))=Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1}))\cdot J_d\ , \tag{2.9}$$

where $J_d:=Ch_{\Lambda(\mathcal{F}_{d-1})}((\mathrm{Coker}\,a^d_{d-1})^\vee)$.
Therefore, whenever we can prove that $\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}$ is a pseudo-null $\Lambda(\mathcal{F}_{d-1})$-module (i.e., hypothesis **1** of Theorem 1.1), we immediately get

$$\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)))\subseteq Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \tag{2.10}$$

and Theorem 1.1 will provide the definition of the pro-characteristic ideal for $\mathcal{S}(\mathcal{F})$ in $\Lambda(\mathcal{F})$ we were looking for.

3. $\mathbb{Z}_p$-descent for totally ramified extensions

The main examples we have in mind are extensions satisfying the following

Assumption 3.1. The (finitely many) ramified places of $\mathcal{F}/F$ are totally ramified.

In what follows an extension satisfying this assumption will be called a *totally ramified extension.* A prototypical example is the $\mathfrak{a}$-cyclotomic extension of $\mathbb{F}_q(T)$ generated by the $\mathfrak{a}$-torsion of the Carlitz module ($\mathfrak{a}$ an ideal of $\mathbb{F}_q[T]$, see, e.g., [13, Chapter 12]). As usual in Iwasawa theory over number fields, most of the proofs will work (or can be adapted) simply assuming that ramified primes are totally ramified in $\mathcal{F}/\mathcal{F}_e$ for some $e\geqslant 0$, but, in the function field setting, one would need some extra hypothesis on the behaviour of these places in $\mathcal{F}_e/F$ (as we have seen with Lemma 2.4, note that in totally ramified extensions any $\mathbb{Z}_p$-subextension can play the role of $\mathcal{F}_1$).
A relevant example for the last case is the composition of a $\mathfrak{a}$-cyclotomic extension and of the arithmetic $\mathbb{Z}_p$-extension of $\mathbb{F}_q(T)$ (with the second one playing the role of $\mathcal{F}_1$). Note that Assumption 2.2.**2** is verified in this case with $e=1$, thanks to [12, Theorem 1.7], hence our next results hold for all these extensions as well.

Let $v\in S_A$ be unramified in $\mathcal{F}/F$, then it is either totally split or it is inert in just one $\mathbb{Z}_p$-extension $\mathcal{F}_{d(v)}/\mathcal{F}_{d(v)-1}$ and totally split in all the others. Since $|S_A|$ is finite we can fix an index d_0 such that all unramified places of S_A are totally split in $\mathcal{F}/\mathcal{F}_{d_0}$.

Theorem 3.2. *Assume $\mathcal{F}/F$ is a totally ramified extension, then, for any $d>\max\{d_0,2\}$, we have*

$$Ch_{\Lambda(\mathcal{F}_{d-1})}((\mathrm{Coker}\ a^d_{d-1})^\vee)\ =(1)\ .$$

Proof. The proof of Proposition 2.7 shows that the $\Lambda(\mathcal{F}_{d-1})$-modules $(\mathrm{Coker}\ a^d_{d-1})^\vee$ and $(\mathrm{Ker}\,c^d_{d-1})^\vee$ are pseudo-isomorphic for $d\geqslant 3$. Moreover, by the proof of Lemma 2.5 (recall, in particular, equation (2.6)), we know that $(\mathrm{Ker}\,c^d_{d-1})^\vee$ is a quotient of $\displaystyle\bigoplus_{v\in S'_A(d)}\mathcal{H}_v(\mathcal{F}_d)^\vee$. Hence we only consider the contributions of the places of S_A which are not totally split in $\mathcal{F}/F$. By equation (2.5), we have (for a fixed w dividing v)

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{H}_v(\mathcal{F}_d)^\vee)=\Lambda(\mathcal{F}_{d-1})\otimes_{\Lambda(\mathcal{F}_{d-1,v})}Ch_{\Lambda(\mathcal{F}_{d-1,v})}(H^1(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p^\infty]^\vee)\ . \tag{3.1}$$

We also saw that, for a ramified prime v, $\mathcal{H}_v(\mathcal{F}_d)^\vee$ (which is $H^1(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p^\infty]^\vee$, because v is totally ramified) is finitely generated over $\mathbb{Z}_p$, hence pseudo-null over $\Lambda(\mathcal{F}_{d-1,v})=\Lambda(\mathcal{F}_{d-1})$ for $d\geqslant 3$.

We are left with the unramified (not totally split) primes in S_A. Assume v is inert in an extension $\mathcal{F}_r/\mathcal{F}_{r-1}$ ($r \leqslant d_0$ by definition), then

$$\Lambda(\mathcal{F}_{r-1,v}) \simeq \mathbb{Z}_p \quad \text{and} \quad \Lambda(\mathcal{F}_{d,v}) \simeq \mathbb{Z}_p[[t_r]] \text{ for any } d \geqslant r\ .$$

Since (again by Lemma 2.5) $H^1(\Gamma^r_{r-1,w}, A(\mathcal{F}_{r,w}))[p^\infty]$ is finite and $H^1(\Gamma^d_{d-1,w}, A(\mathcal{F}_{d,w}))[p^\infty] = 0$ for any $d \geqslant r$, we have

$$Ch_{\Lambda(\mathcal{F}_{r-1,v})}(H^1(\Gamma^r_{r-1,w}, A(\mathcal{F}_{r,w}))[p^\infty]^\vee) = (p^{\nu(v)})$$

for some $\nu(v)$ depending on $|\pi_0(\mathcal{A}_{0,v})|$, and

$$Ch_{\Lambda(\mathcal{F}_{d-1,v})}(H^1(\Gamma^d_{d-1,w}, A(\mathcal{F}_{d,w}))[p^\infty]^\vee) = (1) \text{ for any } d \geqslant d_0+1 \geqslant r+1\ .$$

These local informations and (3.1) yield the theorem. $\square$

Now we deal with the other extra term of equation (2.9), i.e., $Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}})$. Note first that, taking duals

$$(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}})^\vee \simeq \mathcal{S}(\mathcal{F}_d)^\vee/(\gamma_d - 1) = Sel(\mathcal{F}_d)/(\gamma_d - 1)\ ,$$

so we work on the last module.
From now on we put $\gamma := \gamma_d$ and we shall need the following two lemmas: the first is [15, Proposition 4.4] (we provide the proof for completeness), while the second generalizes [15, Proposition 4.2].

Lemma 3.3. *We have*

$$H^1_{fl}(\mathcal{X}_d, A[p^\infty]) = (\gamma - 1)H^1_{fl}(\mathcal{X}_d, A[p^\infty])\ .$$

Proof. Since

$$H^1_{fl}(\mathcal{X}_d, A[p^\infty]) = \varinjlim_{K\subset\mathcal{F}_d\,,\ [K:F]<\infty} \varinjlim_{m} H^1_{fl}(X_K, A[p^m])\ ,$$

an element $\alpha \in H^1_{fl}(\mathcal{X}_d, A[p^\infty])$ belongs to some $H^1_{fl}(X_K, A[p^m])$. Now let $\gamma^{p^{s(K)}}$ be the largest power of γ which acts trivially on K, and define a $\mathbb{Z}_p$-extension K_∞ with $\mathrm{Gal}(K_\infty/K) = \overline{\langle\gamma^{p^{s(K)}}\rangle}$ and layers K_n. Take $t \geqslant m$, consider the restrictions

$$H^1_{fl}(X_K, A[p^m]) \to H^1_{fl}(X_{K_t}, A[p^m]) \to H^1_{fl}(\mathcal{X}_{K_\infty}, A[p^m])$$

and denote by x_t the image of x. Now x_t is fixed by $\mathrm{Gal}(K_t/K)$ and $p^m x_t = 0$, so x_t is in the kernel of the norm $N^{K_t}_K$, i.e., x_t belongs to the (Galois) cohomology group

$$H^1(K_t/K, H^1_{fl}(\mathcal{X}_{K_\infty}, A[p^m]) \hookrightarrow H^1(K_\infty/K, H^1_{fl}(\mathcal{X}_{K_\infty}, A[p^m])\ .$$

Let Ker^2_m be the kernel of the restriction map $H^2_{fl}(X_K, A[p^m]) \to H^2_{fl}(\mathcal{X}_{K_\infty}, A[p^m])$, then, from the Hochschild-Serre spectral sequence, we have

$$Ker^2_m \to H^1(K_\infty/K, H^1_{fl}(\mathcal{X}_{K_\infty}, A[p^m]) \to H^3(K_\infty/K, A(K_\infty)[p^m]) = 0 \tag{3.2}$$

(because the p-cohomological dimension of $\mathbb{Z}_p$ is 1). To get rid of Ker^2_m note that, by [7, Lemma 3.3], $H^2_{fl}(X_K, A) = 0$. Hence, the cohomology sequence arising from

$$A[p^m] \hookrightarrow A \overset{p^m}{\twoheadrightarrow} A\ ,$$

yields an isomorphism $H^2_{fl}(X_K,A[p^m])\simeq H^1_{fl}(X_K,A)/p^m$. Consider the commutative diagram (with $m_2\geqslant m_1$)

$$\begin{array}{ccc} H^1_{fl}(X_K,A)/p^{m_1} & \xrightarrow{\sim} & H^2_{fl}(X_K,A[p^{m_1}]) \\ \downarrow{\scriptstyle p^{m_2-m_1}} & & \downarrow \\ H^1_{fl}(X_K,A)/p^{m_2} & \xrightarrow{\sim} & H^2_{fl}(X_K,A[p^{m_2}]) \ . \end{array}$$

An element of $H^1_{fl}(X_K,A)/p^{m_1}$ of order p^r goes to zero via the vertical map on the left as soon as $m_2\geqslant m_1+r$, hence the direct limit provides $\varinjlim_m H^1_{fl}(X_K,A)/p^m=0$ and, eventually, $\varinjlim_m Ker^2_m=0$ as well. By (3.2)

$$0=\varinjlim_m H^1(K_\infty/K,H^1_{fl}(\mathcal{X}_{K_\infty},A[p^m])=H^1(K_\infty/K,H^1_{fl}(\mathcal{X}_{K_\infty},A[p^\infty])\ ,$$

which yields

$$H^1_{fl}(\mathcal{X}_{K_\infty},A[p^\infty])=(\gamma^{p^{s(K)}}-1)H^1_{fl}(\mathcal{X}_{K_\infty},A[p^\infty])=(\gamma-1)H^1_{fl}(\mathcal{X}_{K_\infty},A[p^\infty])\ .$$

We get the claim by taking the direct limit on the finite subextensions K. ☐

Definition 3.4. For any finite extension L/F we define the *Tate module of the Selmer group* of L to be

$$T_p(Sel(L)):=\varprojlim_n Sel_A(L)_{p^n}=\varprojlim_n \mathrm{Ker}\left\{H^1_{fl}(X_L,A[p^n])\to\prod_v H^1_{fl}(X_{L_v},A)[p^n]\right\}\ .$$

For any infinite extension $\mathcal{L}/F$ the Tate module $T_p(Sel(\mathcal{L}))$ is defined via inverse limit on the finite subextensions with respect to the corestriction maps.

Lemma 3.5. *Let $\mathcal{F}/F$ be a totally ramified extension, then $T_p(Sel(\mathcal{F}_d))\sim_{\Lambda(\mathcal{F}_d)}0$ for any $d\geqslant 2$, where $\sim_{\Lambda(\mathcal{F}_d)}$ means pseudo-isomorphic $\Lambda(\mathcal{F}_d)$-modules.*

Proof. We recall that $F^{(p)}$ is the arithmetic $\mathbb{Z}_p$-extension of F and we denote by $F^{(p)}_n$ its layers. Let F_n denote the layers of the $\mathbb{Z}^d_p$-extensions $\mathcal{F}_d$: note that $\mathrm{Gal}(F_n/F)\simeq(\mathbb{Z}/p^n)^d$ and $\mathcal{F}$ and $F^{(p)}$ are disjoint. By [12, Theorem 1.7], $Sel(F_nF^{(p)})$ is a finitely generated torsion $\Lambda(F^{(p)})$-module and this implies that the $\mathbb{Z}_p$-coranks of $Sel(F_nF^{(p)}_t)$ are bounded (see, e.g., the proof of [1, Corollary 4.14]). Moreover for any $s\geqslant t$, the restriction maps

$$Sel_A(F_nF^{(p)}_t)_{p^m}\longrightarrow Sel_A(F_nF^{(p)}_s)_{p^m}$$

have finite kernels (embedded in $H^1(\mathrm{Gal}(F^{(p)}_s/F^{(p)}_t),A[p^m](F_nF^{(p)}_s))$, by the analogue of diagram (2.1)) of order bounded by $|H^1(\mathrm{Gal}(F^{(p)}/F),A[p^\infty](F_nF^{(p)}))|$, which is finite by [2, Lemma 3.4]. Hence the inverse limit of those kernels (with respect to multiplication by powers of p) is 0 and the restriction map between Tate modules is injective.
Let t be such that the corank of $Sel(F_nF^{(p)}_t)$ is maximal: then any $\alpha\in T_p(Sel(F_nF^{(p)}_s))$ $(s\geqslant t)$ is represented by a torsion element modulo (the image of) $T_p(Sel(F_nF^{(p)}_t))$. The diagram

$$\begin{array}{ccc} T_p(Sel(F_nF^{(p)}_t)) & \xhookrightarrow{res^{n,t}_{n,s}} & T_p(Sel(F_nF^{(p)}_s)) \\ \downarrow{\scriptstyle p^{s-t}} & & \downarrow{\scriptstyle cor^{n,s}_{n,t}} \\ T_p(Sel(F_nF^{(p)}_t)) & = & T_p(Sel(F_nF^{(p)}_t)) \end{array}$$

shows that

$$\begin{aligned}\bigcap_{s>t} cor^{n,s}_{n,t}\left(T_p(Sel(F_nF_s^{(p)}))\right) &\subseteq \bigcap_{s>t} cor^{n,s}_{n,t}\left(T_p(Sel(F_nF_s^{(p)}))_{tor}\right) + p^{s-t}T_p(Sel(F_nF_t^{(p)})) \\ &= \bigcap_{s>t} cor^{n,s}_{n,t}\left(T_p(Sel(F_nF_s^{(p)}))_{tor}\right)\ .\end{aligned}$$

Via the Kummer sequence one has $T_p(Sel(F_nF_s^{(p)}))_{tor} \subseteq A[p^\infty](F_nF_s^{(p)})$, hence

$$T_p(Sel(F_nF^{(p)})) = \varprojlim_s T_p(Sel(F_nF_s^{(p)})) \subseteq \varprojlim_s T_p(Sel(F_nF_s^{(p)}))_{tor} \subseteq \varprojlim_s A[p^\infty](F_nF_s^{(p)})\ .$$

Now using the layers $F_nF_n^{(p)}$ for the $\mathbb{Z}_p^{d+1}$-extension $\mathcal{F}_dF^{(p)}/F$, the formula

$$cor^{m,m}_{n,n} = cor^{n,m}_{n,n} \circ cor^{m,m}_{n,m}$$

and the previous computation, one has that

$$T_p(Sel(\mathcal{F}_dF^{(p)})) = \varprojlim_n T_p(Sel(F_nF_n^{(p)})) \subseteq \varprojlim_n A[p^\infty](F_nF_n^{(p)})$$

is a finitely generated $\mathbb{Z}_p$-module.
To conclude just note that the restriction maps

$$Sel(F_n)_{p^m} \longrightarrow Sel(F_nF_n^{(p)})_{p^m}$$

have kernels whose $\mathbb{Z}_p$-corank is bounded by the corank of $H^1(\mathrm{Gal}(F^{(p)}/F), A[p^\infty](\mathcal{F}_dF^{(p)}))$ (note that, by [15, Proposition 2.11] this is often finite). Taking limits we have that

$$T_p(Sel(\mathcal{F}_d)) = \varprojlim_n \varprojlim_m Sel(F_n)_{p^m}$$

is a finitely generated $\mathbb{Z}_p$-module as well, hence $\Lambda(\mathcal{F}_d)$-pseudo-null for $d \geqslant 2$. □

Now we are ready to deal with the module $\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}$.

Theorem 3.6. *Assume $\mathcal{F}/F$ is a totally ramified extension. For any $d \geqslant 3$ we have*

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) = (1)\ .$$

Proof. Consider the following diagram

$$(3.3)\qquad \begin{array}{ccccccc} Sel(\mathcal{F}_d) & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty]) & \xrightarrow{\phi_d} & \mathcal{H}^1(\mathcal{X}_d, A) & \twoheadrightarrow & Coker(\phi_d) \\ \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} \\ Sel(\mathcal{F}_d) & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty]) & \xrightarrow{\phi_d} & \mathcal{H}^1(\mathcal{X}_d, A) & \twoheadrightarrow & Coker(\phi_d) \end{array}$$

(where $\mathcal{H}^i(\mathcal{X}_d, A) := \prod_w H^i_{fl}(\mathcal{X}_{d,w}, A)[p^\infty]$ and the surjectivity of the second vertical arrow comes from the previous lemma). Inserting $\mathcal{M}(\mathcal{F}_d) := Im(\phi_d)$, we get two diagrams
(3.4)

$$\begin{array}{ccccc} Sel(\mathcal{F}_d) & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty]) & \xrightarrow{\phi_d} & \mathcal{M}(\mathcal{F}_d) \\ \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} \\ Sel(\mathcal{F}_d) & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty]) & \xrightarrow{\phi_d} & \mathcal{M}(\mathcal{F}_d) \end{array} \quad \begin{array}{ccccc} \mathcal{M}(\mathcal{F}_d) & \hookrightarrow & \mathcal{H}^1(\mathcal{X}_d, A) & \twoheadrightarrow & Coker(\phi_d) \\ \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} \\ \mathcal{M}(\mathcal{F}_d) & \hookrightarrow & \mathcal{H}^1(\mathcal{X}_d, A) & \twoheadrightarrow & Coker(\phi_d) \end{array}\ .$$

From the snake lemma sequence of the first one, we obtain the isomorphism

$$(3.5)\qquad \mathcal{M}(\mathcal{F}_d)^{\Gamma^d_{d-1}}/Im(\phi_d^{\Gamma^d_{d-1}}) \simeq Sel(\mathcal{F}_d)/(\gamma-1)$$

(where $\phi_d^{\Gamma_{d-1}^d}$ is the restriction of ϕ_d to $H^1_{fl}(\mathcal{X}_d, A[p^\infty])^{\Gamma_{d-1}^d}$). The snake lemma sequence of the second diagram (its "upper" row) yields an isomorphism

$$\mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d}/\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \simeq Coker(\phi_d)^{\Gamma_{d-1}^d} \ . \tag{3.6}$$

The injection $\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \hookrightarrow \mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d}$ induces an exact sequence

$$\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d}/Im(\phi_d^{\Gamma_{d-1}^d}) \hookrightarrow \mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d}/Im(\phi_d^{\Gamma_{d-1}^d}) \twoheadrightarrow \mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d}/\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$$

(with a little abuse of notation we are considering $Im(\phi_d^{\Gamma_{d-1}^d})$ as a submodule of $\mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d}$ via the natural injection above) which, by (3.5) and (3.6), yields the sequence

$$Sel(\mathcal{F}_d)/(\gamma - 1) \hookrightarrow Coker(\phi_d^{\Gamma_{d-1}^d}) \twoheadrightarrow Coker(\phi_d)^{\Gamma_{d-1}^d} \ . \tag{3.7}$$

Now consider the following diagram

$$\begin{array}{ccccccc}
H^1(\Gamma_{d-1}^d, A[p^\infty]) & \hookrightarrow & H^1_{fl}(\mathcal{X}_{d-1}, A[p^\infty]) & \longrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty])^{\Gamma_{d-1}^d} & \longrightarrow & 0 \\
\downarrow {\scriptstyle \phi_{d-1}^d} & & \downarrow {\scriptstyle \phi_{d-1}} & & \downarrow {\scriptstyle \phi_d^{\Gamma_{d-1}^d}} & & \downarrow \\
\mathcal{H}^1(\Gamma_{d-1}^d, A) & \hookrightarrow & \mathcal{H}^1(\mathcal{X}_{d-1}, A) & \longrightarrow & \mathcal{H}^1(\mathcal{X}_d, A)^{\Gamma_{d-1}^d} & \twoheadrightarrow & \mathcal{H}^2(\Gamma_{d-1}^d, A)
\end{array}$$

where:

- the vertical maps are all induced by the product of restrictions;
- the horizontal lines are just the Hochschild-Serre sequences for global and local cohomology;
- the 0 in the upper right corner comes from $H^2(\Gamma_{d-1}^d, A[p^\infty]) = 0$;
- the surjectivity on the lower right corner comes from $\mathcal{H}^2(\mathcal{X}_{d-1}, A) = 0$, which is a direct consequence of [10, Theorem III.7.8].

This yields a sequence (from the snake lemma)

$$Coker(\phi_{d-1}) \to Coker(\phi_d^{\Gamma_{d-1}^d}) \to \mathcal{H}^2(\Gamma_{d-1}^d, A) = \prod_w H^2(\Gamma_{d-1,w}^d, A(\mathcal{F}_{d,w}))[p^\infty] \ . \tag{3.8}$$

The module $Coker(\phi_{d-1})$**.** The Kummer map induces a surjection $H^1(\mathcal{X}_{d-1}, A[p^\infty]) \twoheadrightarrow H^1(\mathcal{X}_{d-1}, A)[p^\infty]$ which fits in the diagram

$$\begin{array}{ccc}
H^1(\mathcal{X}_{d-1}, A[p^\infty]) & \xrightarrow{\phi_{d-1}} & \mathcal{H}^1(\mathcal{X}_{d-1}, A) \\
\downarrow & \nearrow {\scriptstyle \lambda_{d-1}} & \\
H^1(\mathcal{X}_{d-1}, A)[p^\infty] & &
\end{array}$$

(λ_{d-1} is again a product of restrictions). This yields surjective maps $Im(\phi_{d-1}) \twoheadrightarrow Im(\lambda_{d-1})$ and, eventually, $Coker(\lambda_{d-1}) \twoheadrightarrow Coker(\phi_{d-1})$. For any finite extension K/F we have a similar map

$$\lambda_K : H^1(X_K, A)[p^\infty] \to \mathcal{H}^1(X_K, A)$$

whose cokernel verifies

$$Coker(\lambda_K)^\vee \simeq T_p(Sel_{A^t}(K)_p)$$

(by [6, Main Theorem]), where A^t is the dual abelian variety of A and T_p denotes the p-adic Tate module.

Taking limits on all the finite subextensions of $\mathcal{F}_{d-1}$ (with respect to the corestriction maps) we find

$$Coker(\lambda_{d-1})^\vee \simeq T_p(Sel_{A^t}(\mathcal{F}_{d-1})_p) \sim_{\Lambda(\mathcal{F}_{d-1})} 0 \ ,$$

by Lemma 3.5.

The modules $H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p^\infty]$. If the prime splits completely in $\mathcal{F}_d/\mathcal{F}_{d-1}$, then obviously $H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p^\infty]=0$. If the place is ramified or inert, then $\Gamma^d_{d-1,w}\simeq\mathbb{Z}_p$. Consider the exact sequence

$$A(\mathcal{F}_{d,w})[p] \hookrightarrow A(\mathcal{F}_{d,w}) \stackrel{p}{\twoheadrightarrow} pA(\mathcal{F}_{d,w}) \ ,$$

which yields a surjection

$$H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w})[p]) \twoheadrightarrow H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p] \ .$$

The module on the left is trivial because $cd_p(\mathbb{Z}_p)=1$, hence $H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p]=0$ and this yields $H^2(\Gamma^d_{d-1,w},A(\mathcal{F}_{d,w}))[p^\infty]=0$.

The sequence (3.8) implies that $Coker(\phi_d^{\Gamma^d_{d-1}})$ is $\Lambda(\mathcal{F}_{d-1})$ pseudo-null for $d\geqslant 3$ and, by (3.7), we get $Sel(\mathcal{F}_d)/(\gamma-1)$ is pseudo-null as well. Therefore

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}) = Ch_{\Lambda(\mathcal{F}_{d-1})}((Sel(\mathcal{F}_d)/(\gamma-1))^\vee) = (1) \ .$$

□

Remark 3.7. Assuming $d\geqslant\max\{e+1,3\}$ (i.e., $\mathcal{S}(\mathcal{F}_{d-1})$ is torsion) and using [15, Proposition 4.2] to deal with the Tate module, in place of the more general but weaker Lemma 3.5, one actually gets $\mathcal{S}(\mathcal{F}_d)^{\Gamma^d_{d-1}}=0$.

A direct consequence of equation (2.9) and Theorems 3.2 and 3.6 is

Corollary 3.8. *Assume $\mathcal{F}/F$ is a totally ramified extension, then, for any $d\gg 0$ and any $\mathbb{Z}_p$-subextension $\mathcal{F}_d/\mathcal{F}_{d-1}$, one has*

$$\pi^d_{d-1}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \ . \tag{3.9}$$

The modules $\mathcal{S}(\mathcal{F}_d)$ verify the hypotheses of Theorem 1.1 (because of Proposition 2.7 and Theorem 3.6), so we can define

Definition 3.9. For a totally ramified extension $\mathcal{F}/F$, the *pro-characteristic ideal* of $\mathcal{S}(\mathcal{F})$ is

$$\widetilde{Ch}_{\Lambda(\mathcal{F})}(\mathcal{S}(\mathcal{F})) := \varprojlim_d Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) \subseteq \Lambda \ .$$

We remark that Definition 3.9 only depends on the extension $\mathcal{F}/F$ and not on the filtration of $\mathbb{Z}_p^d$-extension we choose inside it. Indeed with two different filtrations $\{\mathcal{F}_d\}$ and $\{\mathcal{F}'_d\}$ we can define a third one by putting

$$\mathcal{F}''_0 := F \quad \text{and} \quad \mathcal{F}''_n = \mathcal{F}_n\mathcal{F}'_n \quad \forall\, n\geqslant 1 \ .$$

By Corollary 3.8, the limits of the characteristic ideals of the filtrations we started with coincide with the limit on the filtration $\{\mathcal{F}''_n\}$ (see [4, Remark 3.11] for an analogous statement for characteristic ideals of class groups).

This pro-characteristic ideal could play a role in the Iwasawa Main Conjecture (IMC) for a totally ramified extension of F as the algebraic counterpart of a p-adic L-function associated to A and $\mathcal{F}$ (see [1, Section 5] or [3, Section 3] for similar statements but with Fitting ideals). Anyway, at present, the problem of formulating a (conjectural) description of this ideal in terms of a natural p-adic L-functions (i.e., a general non-noetherian Iwasawa Main Conjecture) is still wide open. However, we can say something if A is already defined over the constant field of F.

Theorem 3.10. [Non-noetherian IMC for constant abelian varieties] *Assume A/F is a constant abelian variety and let $\mathcal{F}/F$ be a totally ramified extension as above. Then there* *exists an element* $\theta_{A,\mathcal{F}}$ *interpolating the classical* L*-function* $L(A,\chi,1)$ *(where* χ *varies among characters of Gal*$(\mathcal{F}/F)$*) such that one has an equality of ideals in* $\Lambda(\mathcal{F})$

$$\widetilde{Ch}_{\Lambda(\mathcal{F})}(\mathcal{S}(\mathcal{F})) = (\theta_{A,\mathcal{F}}) \ . \tag{3.10}$$

Proof. This is a simple consequence of [9, Theorem 1.3]. Namely, the element $\theta_{A,L}$ is defined in [9, Section 7.2.1] for any abelian extension L/F unramified outside a finite set of places. It satisfies $\pi^d_{d-1}(\theta_{A,\mathcal{F}_d}) = \theta_{A,\mathcal{F}_{d-1}}$ by construction and the interpolation formula (too complicated to report it here) is proved in [9, Theorem 7.3.1]. Since A has good reduction everywhere, our results apply here and both sides of (3.10) are defined. Finally [9, Theorem 1.3] proves that $Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d)) = (\theta_{A,\mathcal{F}_d})$ for all d and (3.10) follows by just taking a limit. □

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