G_2-orbifolds with ADE-singularities

HABILITATIONSSCHRIFT

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Chapter 1

Introduction

A G_2-structure is a 3-form ϕ on a 7-dimensional manifold M that satisfies certain algebraic conditions. On a manifold with a G_2-structure there exists a natural metric that is induced by ϕ. If ϕ is parallel with respect to the Levi-Civita connection, the holonomy is a subgroup of the exceptional Lie group G_2. If the holonomy acts irreducibly on the tangent space, we call (M, ϕ) a G_2-manifold.

The group G_2 was one of the last groups on Berger's list of possible holonomy groups, for which it was unknown if metrics with these holonomy groups actually exist. The first local examples of metrics with holonomy G_2 have been constructed by Bryant [12], the first complete but non-compact ones by Bryant and Salamon [13] and the first compact examples by Joyce [36]. The idea of Joyce for the construction of G_2-manifolds was to divide a torus T^7 that carries a flat G_2-structure by a finite group Γ that preserves the G_2-structure. The singularities of T^7/Γ can be resolved and it is possible to show by several steps the existence of a metric with holonomy G_2 on the resolved manifold.

Another idea for the construction of G_2-manifolds has been proposed by Donaldson and was carried out in detail by Kovalev [43]. The starting point of this construction are two asymptotically cylindrical (ACyl) Calabi-Yau manifolds W_i with $i = 1, 2$ that approach $D_i \times S^1 \times (0, \infty)$ at infinity, where the D_i are K3 surfaces. $W_1 \times S^1$ and $W_2 \times S^1$ can be glued together after cutting off the cylindrical ends. The glueing map interchanges the circle factors of the ends and acts as an isometry with certain properties on the D_i. The manifold that we obtain by this method is called a twisted connected sum and carries a metric with holonomy G_2. By the twisted connected sum method several authors [17, 43, 44] constructed a large number of G_2-manifolds.

Most of the known examples of compact G_2-manifolds are smooth. In this thesis, we study G_2-orbifolds with a certain kind of singularities, namely ADE-singularities. This means that at the singular points the orbifolds are locally diffeomorphic to $\mathbb{R}^3 \times \mathbb{C}^2/\Delta$, where Δ is a finite subgroup of $SU(2)$, which can be embedded in to G_2. The finite subgroups of $SU(2)$ have been classified by Klein [41]. For any finite subgroup Δ of $SU(2)$ there exists an associated affine Dynkin diagram. This relation is called the McKay correspondence. By deleting one node we obtain a Dynkin diagram of a simple finite-dimensional Lie algebra. The diagrams that can be obtained this way are exactly the A-, D- and E-series from the classification of the simple finite-dimensional Lie algebras, which is the reason why one of the names for singularities of type $\mathbb{C}^2/\Delta$ is ADE-singularities. There are two motivations to study G_2-orbifolds with ADE-singularities. One of them is mathematical and the other one is physical.

An important object in the theory of G_2-manifolds is the moduli space $\mathcal{M}$ of parallel G_2-structures on a fixed manifold M. Joyce [36] has proven that this moduli space is a smooth manifold of dimension $b^3(M)$. Beside this fact very little is known about the geometry of $\mathcal{M}$. It is conjectured that the boundary of $\mathcal{M}$ consists of singular G_2-manifolds. More explicitly, Karigiannis [39] conjectures that the moduli space of parallel G_2-structures with a conical singularity is a boundary component of the moduli space of parallel G_2-structures on a desingularized manifold. Halverson and Morrison [32] conjecture that the boundary components of $\mathcal{M}$ correspond to G_2-manifolds whose singular sets are of codimension 4, 6 and 7. This conjecture implies that for any parallel G_2-structure ϕ on an orbifold with ADE-singularities there exists a one-parameter family of smooth G_2-structures that converges in a suitable sense to ϕ. A proof of the conjecture from [32] and even of the weaker conjecture on the G_2-structures converging to an ADE-singularity is probably very hard. For example, a resolution of an ADE-singularity in the above sense is a step in the proposed construction of G_2-manifolds in the unpublished paper [40]. Although the conjectures on the moduli space of G_2-structures are not yet proven, constructing examples of G_2-orbifolds with ADE-singularities is an important step to better understand the shape of $\mathcal{M}$.

G_2-manifolds do not only play a role in pure mathematics but also in theoretical physics. In M-theory, spacetime is often modeled as $\mathbb{R}^{1,3} \times M^7$, where M^7 is a G_2-manifold that is too small to be observed at low energies. If M^7 is smooth, the quantum field theory on the 4-dimensional Minkowski space that we can observe does not fit to our observations. More precisely, we obtain a field theory with gauge group $U(1)^{b^2(M)}$ in the low-energy limit, but the weak and the strong interaction are described by Yang-Mills theories with a non-abelian gauge group. Moreover, the existence of chiral fermions cannot be explained by this hypothesis. If M^7 is not smooth but has conical singularities, the existence of chiral fermions can be shown [3, 4, 9]. Moreover, M-theory compactified on a G_2-orbifold with ADE-singularities yields, at least if some topological conditions on the singular locus are satisfied, a four-dimensional quantum field theory whose gauge group is determined by the Dynkin diagram that is associated to the singularity by the McKay correspondence.

Up to now, there are only few examples of G_2-orbifolds with ADE-singularities known. In [3], the author considers a class of non-compact G_2-orbifolds that can be obtained by dividing a complete G_2-manifold of cohomogeneity one by a subgroup of $SU(2)$. Moreover, Joyce [36] constructs a torus quotient with A_1-singularities along 12 three-tori which is the starting point for the construction of a smooth G_2-manifold. Further examples of torus quotients with A_n-singularities can be found in [10]. In this thesis, we construct several examples of compact G_2-orbifolds of ADE-singularities. One of our priorities is to construct G_2-orbifolds with as many as possible types of ADE-singularities. Our results can be summed up as follows.

Let S be a K3 surface with a hyper-Kähler metric and let T^3 be a flat torus. Since the holonomy of $S \times T^3$ is $Sp(1)$, which can be embedded into G_2, $S \times T^3$ carries a parallel G_2-structure ϕ. If Γ is a group that acts freely on $S \times T^3$ and preserves ϕ, $(S \times T^3)/\Gamma$ carries a parallel G_2-structure, too. If we allow S to have singularities or Γ to act non-freely, we obtain G_2-orbifolds with ADE-singularities. We find examples of such quotients that have E_8-singularities along 2 submanifolds of type $T^3/\mathbb{Z}_2^2$ and A_1-singularities along 3 submanifolds of type $T^3/\mathbb{Z}_2^2$. Let $G_1, \ldots, G_n$ be a set of connected Dynkin diagrams that can be obtained by deleting some nodes from the union of 2 diagrams of type E_8 and 3 of type A_1. There exist further quotients of type $(S \times T^3)/\Gamma$ whose singular locus consists of n submanifolds that are isometric to the flat manifold $T^3/\mathbb{Z}_2^2$ with singularities of type G_i along them.

Another class of examples is constructed with help of twisted connected sums. Kovalev and Lee [44] find many examples of smooth G_2-manifolds by applying the twisted connected sum construction to a pair of suitable ACyl Calabi-Yau manifolds W_1 and W_2. At least one of them is constructed by a procedure that is described in [44] from of a K3 surface S that admits a non-symplectic involution. We modify this construction such that S is a K3 surface with singularities. We obtain a series of G_2-orbifolds that have singularities along one or several three-spheres. The number and type of the singularities is described by two copies of an arbitrary subdiagram of E_8.

Finally, we consider quotients of a torus with a flat G_2-structure by a finite group with ADE-singularities. A simple algebraic criterion restricts the possible ADE-singularities of a torus quotient to A_1, A_2, A_3, A_5, D_4, D_5 and E_6. We construct examples with all possible ADE-singularities. The singular locus consists of three-tori T^3 or quotients of T^3 by a finite group. We resolve the orbifold singularities by the methods of Joyce [36] and obtain smooth G_2-manifolds. At the end of this thesis, we modify our construction and obtain torus quotients that have ADE-singularities but also more complicated singularities of type $\mathbb{R} \times \mathbb{C}^3/\Delta$, where Δ is a discrete subgroup of $SU(3)$. We resolve the singularities of a particular torus quotient of that type and obtain a smooth G_2-manifold with Betti numbers $(b^1, b^2, b^3) = (0, 22, 45)$. Since no other G_2-manifolds with these Betti numbers can be found in the literature, our example is new.

This thesis is organized as follows. In Chapter 2 we discuss the finite subgroups of $SU(2)$ and the corresponding singularities. The following chapter is about K3 surfaces. We introduce the various moduli spaces of K3 surfaces and sum up the most important facts about non-symplectic involutions. For our construction of G_2-orbifolds we need K3 surfaces with ADE-singularities that admit a non-symplectic involution. There is not much information in the literature about this specialized topic. Therefore, we prove some results, especially

Theorem 3.5.10 and Corollary 3.5.11 that cannot be found in the literature. In Chapter 4, we introduce the reader to G_2-manifolds, their construction and their relation to physics. Since Joyce's method involves torus quotients with singularities as an intermediate step and the understanding of the twisted connected sum construction requires some knowledge about K3 surfaces, we place this chapter after our introduction to K3 surfaces. In Chapter 5 we construct our examples of G_2-orbifolds with ADE-singularities. Section 5.1 deals with quotients of a K3 surface and a torus by a finite group. In Section 5.2, we study twisted connected sums and Section 5.3 is about the torus quotients. Moreover, it contains the smooth G_2-manifolds that we obtain by the resolution of the singularities. At the end of the thesis, we provide a short outlook on possible subjects of future research.

Chapter 2

The ADE-classification

We require the G_2-orbifolds that we study in this thesis to have a certain kind of singularities, namely ADE-singularities. Moreover, M-theory compactified on a G_2-orbifold yields a four-dimensional quantum field theory with a certain gauge group as its low-energy limit. The type of the singularities and the gauge group are related by the McKay correspondence. Therefore, we introduce the most important facts on the ADE-classification and related concepts in this section. Our starting point is the classification of finite subgroups of $SU(2)$.

Theorem 2.0.1. *(Felix Klein [41]) Let Γ be a finite subgroup of $SU(2)$ and let $\tau : SU(2) \to SO(3)$ be the standard double cover. Then Γ is conjugate either to a cyclic group that is generated by*

$$\begin{pmatrix} \exp\left(\frac{2\pi i}{n}\right) & 0 \\ 0 & \exp\left(-\frac{2\pi i}{n}\right) \end{pmatrix}$$

with $n \in \mathbb{N}$ or it is up to conjugation the preimage of the dihedral, tetrahedal, octahedral or icosahedral subgroup of $SO(3)$ with respect to τ.

Remark 2.0.2.

1. In fact, Felix Klein classified the finite subgroups of $SL(2,\mathbb{C})$. Since for any finite $\Gamma \subset SL(2,\mathbb{C})$ there exists a Γ-invariant Hermitian form on $\mathbb{C}^2$, both problems are equivalent.

2. We supplement the name of any of the subgroups of $SO(3)$ with the word "binary" when we consider its preimage with respect to τ, e.g. the preimage of the dihedral group is called the binary dihedral group.

The above groups have the following presentations:

Group	Presentation	Order
Cyclic group	$\langle x \vert x^n = 1\rangle$	n
Binary dihedral group	$\langle x, y \vert x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$	$4n$
Binary tetrahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^3\rangle$	24
Binary octahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^4\rangle$	48
Binary icosahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^5\rangle$	120

Since we will need to do many explicit calculations that involve the finite subgroups of $SU(2)$, we will describe each of them in terms of unit quaternions, complex 2×2-matrices and real 4×4-matrices. We identify $i, j, k \in \mathbb{H}$ with the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As real matrices, i, j and k can therefore be written as

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

The cyclic group of order n is generated by $\exp\left(\frac{2\pi i}{n}\right) \in \mathbb{H}$. Equivalently, the generator can be written as

$$\begin{pmatrix} \exp\left(\frac{2\pi i}{n}\right) & 0 \\ 0 & \exp\left(-\frac{2\pi i}{n}\right) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) & 0 & 0 \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ 0 & 0 & -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} \tag{2.2}$$

The binary dihedral group with $4n$ elements is generated by $\exp\left(\frac{\pi i}{n}\right)$ and j. The corresponding 2×2- and 4×4-matrices can be found in (2.1) and (2.2). Since $\exp\left(\frac{\pi i}{n}\right)$ is of order $2n$, it is easy to see that those matrices generate indeed a group with $4n$ elements. The binary tetrahedral group consists of all quaternions

$$\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

where all sign combinations are allowed. A family of generators is for example $(g, h) = (\frac{1}{2}(1 + i + j + k), \frac{1}{2}(1 + i + j - k))$. These elements can be written as

$$\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}$$

or

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

By playing around with the above matrices we see that the binary tetrahedral group consists of the following elements:

$$g^l, g^l h, g^l hg, g^l hg^2 \tag{2.3}$$

where $l \in \{0, \ldots, 5\}$. The binary octahedral group consists of all elements of the binary tetrahedral group together with

$$\frac{1}{\sqrt{2}}(\pm\epsilon_1 \pm \epsilon_2)\,,$$

where $\epsilon_1 \neq \epsilon_2$ are taken from the set $\{1, i, j, k\}$ and all sign combinations are allowed. A set of generators is given by $\frac{1}{2}(1+i+j+k)$ and $\frac{1}{\sqrt{2}}(1+i)$ and the second generator can be written as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Let $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ be an even permutation of $(1, i, j, k)$. The binary icosahedral group consist of all elements of the binary tetrahedral group together with all unit quaternions of type

$$\frac{1}{2}(0 \cdot \epsilon_0 \pm 1 \cdot \epsilon_1 \pm \phi^{-1} \cdot \epsilon_2 \pm \phi \cdot \epsilon_3)$$

where $\phi = \frac{1}{2}(\sqrt{5}+1)$ is the golden ratio and all sign combinations are allowed. A set of generators is given by $\frac{1}{2}(1+i+j+k)$ and $\frac{1}{2}(\phi + \phi^{-1}i + j)$. The second generator can be written as

$$\frac{1}{2}\begin{pmatrix} \phi+\phi^{-1}i & i \\ i & \phi-\phi^{-1}i \end{pmatrix} \quad \text{or} \quad \frac{1}{2}\begin{pmatrix} \phi & -\phi^{-1} & 0 & -1 \\ \phi^{-1} & \phi & 1 & 0 \\ 0 & -1 & \phi & \phi^{-1} \\ 1 & 0 & -\phi^{-1} & \phi \end{pmatrix}$$

Good references for the generating matrices and the presentations of the finite subgroups of $SU(2)$ are [4], [15], [18].

Remark 2.0.3. G_2 has a subgroup that is isomorphic to $SU(2)$ and acts irreducibly on the subspace $\mathrm{span}(e_4, e_5, e_6, e_7)$ of the irreducible representation $\mathbb{R}^7$ of G_2. Later on, we will embed G_2 into $GL(7,\mathbb{R})$ in such a way that we can identify $\mathrm{span}(e_4, e_5, e_6, e_7)$ with $\mathbb{C}^2$ by the map $ae_4 + be_5 + ce_6 + de_7 \mapsto (a + bi, c + di)$.

There is a relationship between the above groups and a certain kind of Dynkin diagrams which is known as the *McKay correspondence* [49, 50]. Let G be a finite subgroup of $SU(2)$ and V be its representation on $\mathbb{C}^2$ that is induced by the embedding $G \subset SU(2) \subset GL(2,\mathbb{C})$. Moreover, let $V_1, \ldots, V_n$ be the set of all non-trivial finite-dimensional irreducible representations of G. The tensor product decomposition

$$V_i \otimes V = \bigoplus_{j=1}^{n} m_{ij} V_j$$

defines non-negative integers m_{ij}. For each G we define a directed multi-graph. Its nodes are $V_1, \ldots, V_n$ and there shall be m_{ij} edges from V_i to V_j. It is called the *McKay quiver of G*. Now, let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. We define an infinite-dimensional Lie algebra $\hat{\mathfrak{g}}$. The underlying vector space of $\hat{\mathfrak{g}}$ shall be

$$\big(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus \mathbb{C}c$$

with the Lie bracket that is defined by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \kappa(x, y) n \delta_{n+m,0} \cdot c\,, \qquad [x, c] = 0$$

where $x, y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$ and κ is the Killing form on $\mathfrak{g}$. A Lie algebra of this kind is called an *affine Lie algebra*. It is a special type of a Kac-Moody algebra. We can assign to any affine Lie-algebra a Cartan matrix and a Dynkin diagram in the usual way. These diagrams are called *affine Dynkin diagrams*. We denote the affine Dynkin diagram of $\hat{\mathfrak{g}}$ by the Dynkin diagram of $\mathfrak{g}$ with a tilde. It turns out that the Dynkin diagram of $\hat{\mathfrak{g}}$ is the Dynkin diagram of $\mathfrak{g}$ together with an additional node and edges that connect the node to the Dynkin diagram of $\mathfrak{g}$. We refer the reader to [37] for further details. An (affine) Dynkin diagram is called *simply laced* if between two nodes there is either no vertex or only one in each direction. The simply laced affine Dynkin diagrams are precisely $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. The statement of the McKay correspondence is as follows:

Theorem 2.0.4. *(see [49, 50])The McKay quiver of any finite subgroup of $SU(2)$ is a simply laced affine Dynkin diagram. Moreover, this correspondence defines a bijection between the finite subgroups of $SU(2)$ and the set of all simply laced affine Dynkin diagrams. In detail, we have*

Group	*Dynkin diagram*
Cyclic group of order n	$\widetilde{A}_{n-1}$
Binary dihedral group with $4n$ elements	$\widetilde{D}_{n+2}$
Binary tetrahedral group	$\widetilde{E}_6$
Binary octahedral group	$\widetilde{E}_7$
Binary icosahedral group	$\widetilde{E}_8$

Convention 2.0.5. For reasons of simplicity we often refer to the finite subgroups of $SU(2)$ by the Dynkin diagram that we obtain by deleting the additional node from the affine Dynkin diagram. For example, we denote the binary tetrahedral group by E_6 as long as it is clear from the context that we talk about a discrete group instead of a Lie group.

There is a further link between the finite subgroups of $SU(2)$ and the simply laced Dynkin diagrams that involves isolated singularities of complex surfaces. Let G be a finite subgroup of $SU(2)$. The quotient $\mathbb{C}^2/G$ has a singularity at the origin and is smooth elsewhere. Isolated singularities of complex surfaces with a neighborhood that can be identified with a ball around the origin of $\mathbb{C}^2/G$ are called *du Val singularities* or *ADE-singularities*. The spaces $\mathbb{C}^2/G$ are actually isomorphic to zero sets of certain polynomials that can be found in the table below.

A_n	$x^2 + y^2 + z^{n+1} = 0$
D_n	$x^2 + y^2 z + z^{n-1} = 0$
E_6	$x^2 + y^3 + z^4 = 0$
E_7	$x^2 + y^3 + yz^3 = 0$
E_8	$x^2 + y^3 + z^5 = 0$

We search for crepant resolutions of these singularities, i.e. for resolutions $\pi : X \to \mathbb{C}^2/G$ such that $\pi^*\mathcal{K}_{\mathbb{C}^2/G} = \mathcal{K}_X(= \mathcal{O}_X)$ where $\mathcal{K}$ denotes the canonical bundle. $\mathbb{C}^2/G$ has a unique crepant resolution that can be described as the composition of several blow-ups. Its exceptional divisor is the union of several $\mathbb{CP}^1$s with self-intersection -2. More precisely, the number of $\mathbb{CP}^1$s is the same as the rank k of the finite-dimensional simple Lie algebra $\mathfrak{g}$ that is associated to G by the McKay correspondence. $H^2(X,\mathbb{Z})$ is spanned by the k copies of $\mathbb{CP}^1$ that form the exceptional divisor and the intersection form on $H^2(X,\mathbb{Z})$ is given by the Cartan matrix of $\mathfrak{g}$.

In Section 4.5 we will see that the correspondence between the ADE-singularities and the simply laced Dynkin diagrams also appears in M-theory. The gauge group that we obtain from M-theory compactified on a G_2-orbifold with a singularity of type A_n, D_n or E_n is a compact Lie group with the corresponding Dynkin diagram.

The complex manifold X that we obtain by resolving $\mathbb{C}^2/G$ carries a family of hyper-Kähler metrics. Since those metrics will be important for the description of the moduli space of K3 surfaces and Joyce's construction of G_2-manifolds, we introduce the most important facts on this topic. First of all, we recall what a hyper-Kähler metric is.

Definition 2.0.6. A *hyper-Kähler manifold* is a $4n$-dimensional Riemannian manifold (M, g) together with three linearly independent complex structures I, J and K such that

1. the complex structures satisfy the quaternion multiplication relation $IJK = -1$,
2. g is a Hermititan metric with respect to I, J and K and
3. the 2-forms ω_I, ω_J and ω_K defined by $\omega_I(X, Y) = g(I(X), Y)$ etc. are Kähler forms.

The data $(g, \omega_I, \omega_J, \omega_K)$ on M are called the *hyper-Kähler structure*.

Remark 2.0.7.

1. The holonomy of the metric on a $4n$-dimensional hyper-Kähler manifold is a subgroup of $Sp(n)$. Conversely, any Riemannian manifold whose holonomy is a subgroup of $Sp(n)$ carries a hyper-Kähler structure. Since $Sp(1) \cong SU(2)$, any Calabi-Yau manifold of complex dimension 2 is also hyper-Kähler.
2. The set of all parallel complex structures on a hyper-Kähler manifold whose holonomy equals $Sp(n)$ is a 2-sphere

$$\{\alpha_I I + \alpha_J J + \alpha_K K | \alpha_I^2 + \alpha_J^2 + \alpha_K^2 = 1\}$$

   A metric with holonomy $Sp(n)$ determines a hyper-Kähler structure that is unique up to an action of an element of $SO(3)$ on the triple (I, J, K).

Moreover, we need the following definitions.

Definition 2.0.8. Let G be a finite subgroup of $SU(2)$ and let (I_0, J_0, K_0, g_0) be the hyper-Kähler structure on $\mathbb{C}^2/G$ that is induced by left-multiplication with i, j and k on $\mathbb{H} \cong \mathbb{C}^2$ and the Hermitian metric $|z_1|^2 + |z_2|^2$. We define a radial coordinate $r : \mathbb{C}^2/G \to [0, \infty)$ by $r := (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$. Let (M, I, J, K, g) be a complete hyper-Kähler manifold of real dimension 4 such that there exists a compact subset $K \subset M$, an $R \in [0, \infty)$ and a diffeomorphism $\pi : M \setminus K \to \{x \in \mathbb{C}^2/G | r(x) > R\}$ such that

$$\nabla_0^k(\pi_*(g) - g_0) = O(r^{-(k+4)}) \quad \text{and} \quad \nabla_0^k(\pi_*(L) - L_0) = O(r^{-(k+4)}) \,,$$

where ∇_0 is the Levi-Civita connection of g_0, $k \in \mathbb{N}_0$ is arbitrary, $L \in \{I, J, K\}$ is a complex structure on M and L_0 is the corresponding complex structure on $\mathbb{C}^2/G$. In this situation, (M, I, J, K, g) is called an *Asymptotically Locally Euclidean (ALE) hyper-Kähler 4-manifold.*

For each finite subgroup $G \subset SU(2)$ there exists an ALE hyper-Kähler metric on the crepant resolution X of $\mathbb{C}^2/G$ and the moduli space of these metrics is known. In the case $G = \mathbb{Z}_2$, that metric has a particularly simple description. Let $\pi : X \to \mathbb{C}^2/\mathbb{Z}_2$ denote the crepant resolution. It can be shown that X is biholomorphic to $T^*\mathbb{CP}^1$. We define a function $f : X \setminus \pi^{-1}(0) \to \mathbb{R}$ by

$$f(r(x)) := \sqrt{r^4 + 1} + 2\ln(r) - \ln\left(\sqrt{r^4 + 1} + 1\right)$$

$\omega_I := i\partial\overline{\partial} f$ is a Kähler form on $X \setminus \pi^{-1}(0)$ with respect to the complex structure I. It can be smoothly extended to all of X and the metric $g(X, Y) = -\omega_I(I(X), Y)$ is an ALE hyper-Kähler metric on $T^*\mathbb{CP}^1$. (X, g) is called the *Eguchi-Hanson space.* It is named after Eguchi and Hanson [23] who found the metric, but it is also a special case of Calabi's construction of a hyper-Kähler metric on $T^*\mathbb{CP}^n$ [14] that was discovered in parallel. We can easily construct further ALE hyper-Kähler metrics on $T^*\mathbb{CP}^1$ either by rescaling the metric or by choosing a different complex structure on $\mathbb{C}^2/\mathbb{Z}_2$ in the definition of the Kähler form. The set of all ALE hyper-Kähler metrics that we obtain this way can be identified with $\mathbb{R}^3 \setminus \{0\}$. The point $0 \in \mathbb{R}^3$ can be identified with the hyper-Kähler orbifold $\mathbb{C}^2/\mathbb{Z}_2$. It can be shown that any ALE hyper-Kähler metric that is asymptotic to $\mathbb{C}^2/\mathbb{Z}_2$ is isometric to one of the members of this 3-parameter family.

The above statements have been generalized to the other cyclic subgroups of $SU(2)$ [25, 34]. The final result that deals with all finite subgroups of $SU(2)$ has been proven by Kronheimer [46, 47] and can also be found in [36, Theorem 7.2.3]. We present the theorem without proof below.

Theorem 2.0.9. *Let G be a non-trivial finite subgroup of $SU(2)$ and let Δ be the Dynkin diagram of the simple finite-dimensional Lie algebra $\mathfrak{g}$ that is associated to G. We define a real vector space V whose basis are the non-trivial finite-dimensional irreducible representations of G. Due to the McKay correspondence, V can be identified with a real subspace of maximal dimension of a Cartan subalgebra of $\mathfrak{g}$. Let Σ be the set of roots of $\mathfrak{g}$. We define a set*

$$U := \left\{(\alpha_1, \alpha_2, \alpha_3) \in V \otimes \mathbb{R}^3 \middle| \forall \delta \in \Sigma : \delta(\alpha_1)^2 + \delta(\alpha_2)^2 + \delta(\alpha_3)^2 \neq 0\right\} .$$

There exists a continuous family of ALE hyper-Kähler manifolds or orbifolds that are asymptotic to $\mathbb{C}^2/G$. This family can be identified with $V \otimes \mathbb{R}^3$. The hyper-Kähler space X_α with $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in V \otimes \mathbb{R}^3$ has the following properties.

1. *If $\alpha \in U$, X_α is an ALE hyper-Kähler manifold that is diffeomorphic to the crepant resolution of $\mathbb{C}^2/G$.*

2. *If $\alpha \notin U$, X_α is an ALE hyper-Kähler orbifold.*

3. *Let Y be an ALE hyper-Kähler manifold or orbifold that is asymptotic to $\mathbb{C}^2/G$. Then there exists an $\alpha \in V \otimes \mathbb{R}^3$ such that Y is isometric to X_α.*

Remark 2.0.10. If $\alpha \notin U$, the singular set of X_α can be described explicitly. The set of all $\delta \in \Sigma$ with $\delta(\alpha_i) = 0$ for $i = 1, 2, 3$ is a root system. The connected components of its Dynkin diagram describe the type of the singular points. We assume for example that G is of type E_8 and that the smaller Dynkin diagram has two connected components of type D_5 and A_2. This can happen if all α_i are in the kernel of suitable $\delta \in \Sigma$, but not in the kernel of any other simple root. In this situation, X_α has exactly two singular points. One of them has a D_5-singularity and the other one has an A_2-singularity. The reason behind this correspondence is that the volume of any $\mathbb{CP}^1$ that spans $H^2(X_\alpha, \mathbb{Z})$ is given by $\left(\sum_{i=1}^{3} \delta(\alpha_i)^2\right)^{\frac{1}{2}}$. If $\delta(\alpha_i) = 0$ for $i = 1, 2, 3$, the sphere shrinks to a point and we obtain an ADE-singularity. As a special case, the space X_0 is isometric to $\mathbb{C}^2/G$.

Chapter 3

K3 surfaces

3.1 Introduction to lattices

The second cohomology $H^2(S,\mathbb{Z})$ of a K3 surface S together with the intersection form is a lattice. Since we have to work with $H^2(S,\mathbb{Z})$ and its various sublattices, we need some concepts from lattice theory. The content of this section can be found in any reference on this subject, for example in [22], in [11, Chapter I.2] or in [19].

Definition 3.1.1. 1. A *lattice* is a free abelian group L of finite rank together with a symmetric bilinear form $\cdot : L \times L \to \mathbb{Z}$. We write x^2 for $x \cdot x$. The *rank* of a lattice is the same as the rank of the underlying abelian group. L is called *even* if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. Let $(e_1, \dots, e_n)$ be a basis of L. The $n \times n$-matrix with coefficients $e_i \cdot e_j$ is called the *Gram matrix of* L. L is called *unimodular* if there exists a basis $(e_1, \dots, e_n)$ of L with $|\det(e_i \cdot e_j)_{i,j=1,\dots,n}| = 1$.

2. The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a real vector space. The bilinear form on L can be extended to an $\mathbb{R}$-bilinear form on $L \otimes_{\mathbb{Z}} \mathbb{R}$. Terms as *non-degenerate lattice* and *signature of a lattice* will always be defined with respect to the extended form.

3. Let L and L' be lattices and let $\cdot_L$ and $\cdot_{L'}$ be the corresponding bilinear forms. A *lattice isomorphism of L and L'* is a bijective $\mathbb{Z}$-linear map $\phi : L \to L'$ with $x \cdot_L y = \phi(x) \cdot_{L'} \phi(y)$ for all $x, y \in L$. If $L = L'$, ϕ is called an *automorphism*. We denote the group of all automorphisms of L by $\mathrm{Aut}(L)$.

4. The *direct sum $L \oplus L'$ of L and L'* is the direct sum of the underlying groups together with the bilinear form

$$(x_1, x_2) \cdot_{L \oplus L'} (y_1, y_2) := x_1 \cdot_L y_1 + x_2 \cdot_{L'} y_2 \,.$$

Remark 3.1.2. The number $|\det(e_i \cdot e_j)_{i,j=1,\dots,n}|$ from the above definition is independent of the choice of the basis $(e_1, \dots, e_n)$.

Later, when we consider non-symplectic involutions of K3 surfaces, we will need the notion of a primitive sublattice.

Definition 3.1.3.
1. An element x of a lattice L is called *primitive* if there exists no natural number $k > 1$ and $y \in L$ such that $x = k \cdot y$.
2. A sublattice $K \subset L$ is called *primitive* if the quotient L/K has no torsion.
3. A lattice N is *primitively embedded* in L if L has a primitive sublattice that is isomorphic to N.

In particular, $x \in L$ is primitive if and only if the sublattice that is generated by x is a primitive sublattice. The dual of a lattice L is defined as

$$L^* := \{\phi : L \to \mathbb{Z} | \phi \text{ is } \mathbb{Z}\text{-linear}\} .$$

From now on, we assume that L is a non-degenerate lattice. Let $(e_1, \ldots, e_n)$ be a basis of L. We denote the matrix $(e_i \cdot e_j)_{i,j=1,\ldots,n}$ by A and the dual basis of L^* by $(e_1^*, \ldots, e_n^*)$. The dual bilinear form on L^* is given by $(e_i^* \cdot e_j^*)_{i,j=1,\ldots,n} = A^{-1}$. It takes its values in $\mathbb{Q}$ but not necessarily in $\mathbb{Z}$. The map $\imath : L \to L^*$ that is defined by $\imath(x)(y) := x \cdot y$ is an injection. The quotient group $L^*/\imath(L)$ is called the *discriminant group of* L.

Lemma 3.1.4. *Let L be a lattice and $(e_1, \ldots, e_n)$ be a basis of L. The discriminant group of L is a finite group of order $|\det(e_i \cdot e_j)_{i,j=1,\ldots,n}|$. The minimal number $\ell(L)$ of generators of the discriminant group satisfies $\ell(L) \leq \mathrm{rank}\, L$.*

The invariant $\ell(L)$ allows us to formulate a theorem on primitive embeddings of lattices that can be found in [19] or [54]. Kovalev and Lee [44] use this theorem for their construction of compact G_2-manifolds by twisted connected sums. Since the theorem will be important for us, too, we include it in this section.

Theorem 3.1.5. *Let K be an even non-degenerate lattice of signature (k_+, k_-) and L be an even unimodular lattice of signature (l_+, l_-). We assume that $k_+ \leq l_+$ and $k_- \leq l_-$ and that*

1. $2 \cdot \mathrm{rank}(K) \leq \mathrm{rank}(L)$ *or*
2. $\mathrm{rank}(K) + \ell(K) < \mathrm{rank}(L)$.

Then there exists a primitive embedding $i : K \to L$. If in addition $k_+ < l_+$ and $k_- < l_-$ and one of the following conditions holds

1. $2 \cdot \mathrm{rank}(K) \leq \mathrm{rank}(L) - 2$,

2. *rank*$(K) + \ell(K) \leq$ *rank*$(L) - 2$,

the embedding i is unique up to an automorphism of L.

Finally, we remark that there exists a canonical bilinear form on the discriminant group that is defined by

$$(x + \imath(L)) \cdot (y + \imath(L)) := x \cdot y + \mathbb{Z}$$

and takes its values in $\mathbb{Q}/\mathbb{Z}$.

3.2 Basic facts about K3 surfaces

In this section, we introduce some topological facts on K3 surfaces that we will need later on. The results of this and the following two sections are well-known. We refer the reader to [11, Chapter VIII], [36, Chapter 7.3] and references therein for a more detailed account.

Definition 3.2.1. A *K3 surface* is a compact, simply connected, complex surface with trivial canonical bundle.

In this section, we consider only smooth K3 surfaces. Later on, we allow ADE-singularities, too. The underlying manifold of any K3 surface is of a fixed diffeomorphism type. Therefore, the Hodge diamond and the intersection form on the second cohomology are the same for any K3 surface.

Theorem 3.2.2. *Let S be a K3 surface.*

1. *The Hodge numbers of S are determined by $h^{0,0}(S) = h^{2,0}(S) = 1$, $h^{1,0}(S) = 0$ and $h^{1,1}(S) = 20$.*

2. *The second integral cohomology $H^2(S, \mathbb{Z})$ together with the intersection form is an even unimodular lattice of signature $(3, 19)$. Up to isometries, the only lattice with these properties is*

$$L := 3H \oplus 2(-E_8)\,,$$

where H is the hyperbolic plane lattice with the bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

and $-E_8$ is the root lattice of E_8 together with the negative of the usual bilinear form.

These facts motivate the following definitions.

Definition 3.2.3. 1. The lattice L from the above theorem is called the *K3 lattice.*

2. A K3 surface S together with a lattice isometry $\phi : H^2(S, \mathbb{Z}) \to L$ is called a *marked K3 surface.*

3. Two marked K3 surfaces (S, ϕ) and (S', ϕ') are called *isomorphic* if there exists a biholomorphic map $f : S \to S'$ such that $\phi \circ f^* = \phi'$, where $f^* : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is the pull-back.

The first Chern class is a bijective map between the Picard group of holomorphic line bundles on a K3 surface S and $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$. Therefore, we introduce the following notions:

Definition 3.2.4. The lattice $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ is called the *Picard lattice* and its rank is called the *Picard number.* The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is called the *transcendental lattice.*

In Section 5.1 and 5.2 we construct G_2-orbifolds with help of certain K3 surfaces with ADE-singularities. These constructions are easier to carry out if the Picard number is large. Since $H^{2,0}(S)$ and $H^{0,2}(S)$ are one-dimensional, the maximal value of the Picard number is 20. An example of a K3 surface with Picard number 20 is the *Fermat quartic*:

$$\{[x : y : z : w] \in \mathbb{CP}^3 | x^4 + y^4 + z^4 + w^4 = 0\} .$$

Convention 3.2.5. In the literature, a K3 surface with maximal Picard number is often called singular and a compact, simply connected, complex surface with trivial canonical bundle that may admit ADE-singularities is often called a *Gorenstein K3 surface.* In this thesis, we use a different convention and call K3 surfaces with ADE-singularities singular.

Any K3 surface S admits a Kähler metric. Since S has trivial canonical bundle, there exists a unique Ricci-flat Kähler metric in each Kähler class. The holonomy group $SU(2)$ is isomorphic to $Sp(1)$. Therefore, the Ricci-flat Kähler metrics are in fact hyper-Kähler. When we talk about isomorphisms between K3 surfaces we usually mean biholomorphic maps with respect to fixed complex structures on the K3 surfaces. Another natural class of maps between K3 surfaces are isometries between K3 surfaces with hyper-Kähler metrics. It should be noted that there are isometries between K3 surfaces that are not holomorphic. For example, a matching between two K3 surfaces as it is defined on page 53 is not a holomorphic map with respect to the complex structures I_1 and I_2.

3.3 Moduli spaces of K3 surfaces

There exist several related moduli spaces whose points represent K3 surfaces with an extra structure. We denote them all by $\mathscr{K}3$ with an appropriate index. The first of them is the *moduli space of marked K3 surfaces* $\mathscr{K}3^m$ that is defined as the set of all marked K3 surfaces modulo isomorphisms. We describe $\mathscr{K}3^m$ in more detail below.

On any K3 surface S, there exists a global holomorphic $(2,0)$-form that we denote by $\omega_J + i\omega_K$, where ω_J and ω_K are 2-forms with real values. We denote the real and the imaginary part by the subscripts J and K since ω_J and ω_K are Kähler forms with respect to the additional complex structures that make S a hyper-Kähler manifold. Since $h^{2,0}(S) = 1$, the holomorphic form is unique up to multiplication with a complex constant. This fact motivates the following definition.

Definition 3.3.1. Let (S, ϕ) be a marked K3 surface. Moreover, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $L_{\mathbb{K}} := L \otimes_{\mathbb{Z}} \mathbb{K}$ and $\phi_{\mathbb{K}} : H^2(S, \mathbb{K}) \to L_{\mathbb{K}}$ be the $\mathbb{K}$-linear extension of ϕ. The complex line that is spanned by $\phi_{\mathbb{C}}([\omega_J + i\omega_K])$, where the square brackets denote the cohomology class, defines a point $p(S, \phi) \in \mathbb{P}(L_{\mathbb{C}})$, where $\mathbb{P}(L_{\mathbb{C}})$ is the projective space of all complex lines in $L_{\mathbb{C}}$. $p(S, \phi)$ is called the *period point of* (S, ϕ). This assignment defines a map $p : \mathscr{K}3^m \to \mathbb{P}(L_{\mathbb{C}})$, which is called the *period map for K3 surfaces*.

Since ω_J and ω_K are Kähler forms, they are positive and we have

$$[\omega_J + i\omega_K] \cdot [\omega_J + i\omega_K] = 0 \qquad \text{and} \qquad [\omega_J + i\omega_K] \cdot [\omega_J - i\omega_K] > 0\,,$$

where the dot denotes the extension of the intersection form to $H^2(S, \mathbb{C})$. We consider the set of all points in $\mathbb{P}(L_{\mathbb{C}})$ with these properties.

Definition 3.3.2. We denote the complex line that is spanned by $x \in L_{\mathbb{C}}$ by ℓ_x. The set

$$\Omega := \{\ell_x \in \mathbb{P}(L_{\mathbb{C}}) | x \cdot x = 0,\ x \cdot \overline{x} > 0\}$$

is called the *period domain*.

We reduce the target set of the period map such that from now on $p : \mathscr{K}3^m \to \Omega$. An important theorem that yields some information on the structure of $\mathscr{K}3^m$ is the following.

Theorem 3.3.3. *The period map for K3 surfaces is surjective.*

Our aim to describe $\mathscr{K}3^m$ is linked to the so called *Torelli problem for K3 surfaces*: What does the period point of a marked K3 surface tells us about the complex structure of S? Answers to this question are called *Torelli theorems*. The first of them is:

Theorem 3.3.4. *(Local Torelli theorem) The period map is a local isomorphism of complex manifolds.*

The deformation theory of Kodaira and Spencer tells us that $\mathscr{K}3^m$ is a complex analytic space. It follows from the above theorem that locally $\mathscr{K}3^m$ is in addition a complex manifold of dimension 20. Unfortunately, $\mathscr{K}3^m$ is not Hausdorff. Later on, we introduce a different moduli space that is Hausdorff. We proceed to the next Torelli theorem.

Theorem 3.3.5. *(Weak Torelli theorem) Let (S, ϕ) and (S', ϕ') be two marked K3 surfaces with the same period point. Then there exists a biholomorphic map $f : S \to S'$.*

We cannot conclude that the period domain and $\mathscr{K}3^m$ are isomorphic since the weak Torelli theorem does not state that $\phi \circ f^* = \phi'$. In fact, this is not true in general and $\mathscr{K}3^m$ and Ω are not isomorphic. In order to solve the Torelli problem and describe $\mathscr{K}3^m$ explicitly, we need some further definitions.

Definition 3.3.6.

1. Let S and S' be K3 surfaces. A lattice isometry $\psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is called a *Hodge-isometry* if its $\mathbb{C}$-linear extension preserves the Hodge decomposition $H^2(S, \mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$.

2. A class $x \in H^2(S, \mathbb{Z})$ is called *effective* if there exists an effective divisor D of S with $c_1(\mathcal{O}_S(D)) = x$. An effective class x is called *nodal* if $x \cdot x = -2$.

3. The connected component of the set $\{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0\}$ which contains a Kähler class is called the *positive cone* of S.

4. A Hodge-isometry $\psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is called *effective* if it maps the positive cone of S to the positive cone of S' and effective classes in $H^2(S, \mathbb{Z})$ to effective classes in $H^2(S', \mathbb{Z})$.

Remark 3.3.7. Since $H^{1,1}(S) = \overline{H^{1,1}(S)}$, $H^{1,1}(S)$ is a complex vector space. We denote its real part $H^{1,1}(S) \cap H^2(S, \mathbb{R})$ by $H^{1,1}(S, \mathbb{R})$. The restriction of the intersection form to $H^{1,1}(S, \mathbb{R})$ has signature $(1, 19)$. The set $\{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0\}$ thus has exactly two connected components. Exactly one of them contains a Kähler class and the definition of the positive cone therefore makes sense.

The following lemma will be helpful later on.

Lemma 3.3.8. *(See [11, p. 313]) Let S and S' be K3 surfaces and $\psi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ be a Hodge-isometry. If ψ maps at least one Kähler class of S to a Kähler class of S', then ψ is effective.*

With help of the terms that we have defined above, the weak Torelli theorem can be reformulated.

Theorem 3.3.9. *Let S and S' be two unmarked K3 surfaces. If there exists a Hodge-isometry $\psi : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$, S and S' are isomorphic.*

Furthermore, we are able to explain how the Torelli problem can be solved.

Theorem 3.3.10. *(Torelli theorem) Let S and S' be two unmarked K3 surfaces. If there exists an effective Hodge-isometry $\psi : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$, ψ is the pull-back of a unique biholomorphic map $f : S \to S'$.*

The converse of the above theorem is also true. If $f : S \to S'$ is a biholomorphic map, its pull-back is a Hodge-isometry. Moreover, if g is a Kähler metric on S', f^*g is a Kähler metric on S. Therefore, we can conclude with help of Lemma 3.3.8 that $f^* : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is an effective Hodge-isometry. Therefore, we have a one-to-one correspondence between biholomorphic maps $f : S \to S'$ and effective Hodge-isometries $\psi : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$. With help of an explicit description of the Kähler cone we are therefore able to describe $\mathscr{K}3^m$.

Theorem 3.3.11. *Let S be a K3 surface and let $\mathcal{C}_S \subset H^{1,1}(S, \mathbb{R})$ be its Kähler cone, i.e. the set of all cohomology classes representing a Kähler form. Then we have*

$$\mathcal{C}_S = \{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0 \text{ and } x \cdot d > 0 \text{ for all effective classes } d\}$$

This description can be simplified to

$$\mathcal{C}_S = \{x \in H^{1,1}(S, \mathbb{R}) | x \cdot x > 0 \text{ and } x \cdot d > 0 \text{ for all nodal classes } d\}$$

We introduce further terms that allow us to describe the Kähler cone more algebraically.

Definition 3.3.12.

1. Let $\ell_x \in \mathbb{P}(L_\mathbb{C})$. We define the *root system of ℓ_x* as

$$\triangle_x := \{d \in L | d \cdot d = -2,\ x \cdot d = 0\}.$$

2. We define the *Kähler chambers* of ℓ_x as the connected components of

$$\{z \in L_\mathbb{R} | z \cdot z > 0,\ z \cdot x = 0,\ z \cdot d \neq 0\ \forall d \in \triangle_x\}.$$

Theorem 3.3.13. *The subgroup of $Aut(L)$ that preserves ℓ_x acts transitively on the set of all Kähler chambers of ℓ_x. The image $\phi_\mathbb{R}(\mathcal{C}_S)$ of the Kähler cone of a marked K3 surface (S, ϕ) with period point x is one of the Kähler chambers of ℓ_x.*

Definition 3.3.14. We define the *augmented period domain* as

$$\widetilde{\Omega} = \{(\ell_x, C) | \ell_x \in \Omega,\ C \subset L_{\mathbb{R}} \text{ is a Kähler chamber of } \ell_x\}$$

and the *augmented period map* $\widetilde{p} : \mathscr{K}3^m \to \widetilde{\Omega}$ by

$$\widetilde{p}(S, \phi) := (p(S, \phi), \phi_{\mathbb{R}}(\mathcal{C}_S))\ .$$

Theorem 3.3.15. *The augmented period map $\widetilde{p} : \mathscr{K}3^m \to \widetilde{\Omega}$ is bijective.*

Remark 3.3.16. The automorphism group $\mathrm{Aut}(L)$ acts naturally on Ω. This action can be lifted to an action on $\widetilde{\Omega}$. The moduli space of unmarked K3 surfaces is diffeomorphic to the quotient $\widetilde{\Omega}/\mathrm{Aut}(L)$. It is a complex 20-dimensional complex orbifold which is Hausdorff.

Later on, when we search for matchings between K3 surfaces, we need to specify a Kähler class and not only the complex structure. Therefore, we need a further moduli space that takes this additional information into account.

Definition 3.3.17.

1. A *marked pair* is a pair of a marked K3 surface (S, ϕ) and a Kähler class $y \in H^{1,1}(S, \mathbb{R})$. We usually write a marked pair as (S, ϕ, y).
2. Two marked pairs (S, ϕ, y) and (S', ϕ', y') are called *isomorphic* if there exists a biholomorphic map $f : S \to S'$ that satisfies $\phi \circ f^* = \phi'$ and $f^*y' = y$.
3. The *moduli space of marked pairs* $\mathscr{K}3^{mp}$ is the set of all marked pairs modulo isomorphisms.

Moreover, we define the following two sets:

$$\begin{aligned} K\Omega &:= \{(\ell_x, y) \in \Omega \times L_{\mathbb{R}} | x \cdot y = 0, y \cdot y > 0\} \\ (K\Omega)^0 &:= \{(\ell_x, y) \in K\Omega | y \cdot d \neq 0 \ \forall d \in L \text{ with } d^2 = -2, x \cdot d = 0\} \end{aligned}$$

and the *refined period map*

$$p' : \mathscr{K}3^{mp} \to \Omega \times L_{\mathbb{R}} \tag{3.2}$$
$$p'(S, \phi, y) := (p(S, \phi), \phi_{\mathbb{R}}(y)) \tag{3.3}$$

Theorem 3.3.18. *p' takes its values in $(K\Omega)^0$. Moreover, it is a bijection between $\mathscr{K}3^{mp}$ and $(K\Omega)^0$. As a consequence, $\mathscr{K}3^{mp}$ is a real analytic Hausdorff manifold of dimension 60.*

Finally, we describe the moduli space of all hyper-Kähler metrics on K3 surfaces. The following lemma shows that $\mathscr{K}3^{mp}$ is closely related to that moduli space.

Lemma 3.3.19. *We denote the underlying 4-dimensional real manifold of a K3 surface by M. There is a one-to-one correspondence between the hyper-Kähler metrics on M together with a choice of a parallel complex structure and a marking on the one hand and marked pairs on the other hand.*

Proof. Let (S, ϕ, y) be a marked pair. We define ω_I as the unique Kähler form in the cohomology class y whose associated metric g is Ricci-flat, ω_J as the real part of the holomorphic volume form and ω_K as its imaginary part. $(g, \omega_I, \omega_J, \omega_K)$ defines a hyper-Kähler structure on M. Conversely, a hyper-Kähler metric together with the additional data from the lemma yields a marked pair in a canonical way. If we start with a marked pair, the normalized holomorphic volume form can be chosen as any element from the family $e^{i\phi}(\omega_J + i\omega_K)$ with $\phi \in \mathbb{R}$. We therefore obtain a family of hyper-Kähler structures that is parametrized by $U(1)$ instead of a unique hyper-Kähler structure. Conversely, if we start with a hyper-Kähler structure with a fixed parallel complex structure, we are free to choose the normalized holomorphic volume form from the same family. This proves that our correspondence is well-defined and bijective. $\square$

Moreover, there is a useful lemma on the isometries of a K3 surface that should be mentioned.

Lemma 3.3.20. *Let S_j with $j \in \{1,2\}$ be K3 surfaces together with hyper-Kähler metrics g_j and Kähler forms $\omega_I^{(j)}$, $\omega_J^{(j)}$ and $\omega_K^{(j)}$. Moreover, let $V_j \subset H^2(S_j, \mathbb{R})$ be the subspace that is spanned by $[\omega_I^{(j)}]$, $[\omega_J^{(j)}]$ and $[\omega_K^{(j)}]$.*

1. *Let $f : S_1 \to S_2$ be an isometry. The pull-back $f^* : H^2(S_2, \mathbb{Z}) \to H^2(S_1, \mathbb{Z})$ is a lattice isometry. Its $\mathbb{R}$-linear extension maps V_2 to V_1.*
2. *Let $\psi : H^2(S_1, \mathbb{Z}) \to H^2(S_2, \mathbb{Z})$ be a lattice isometry such that $\psi_{\mathbb{R}}(V_1) = V_2$. Moreover, $\psi_{\mathbb{R}}$ shall map the positive cone of S_1 to the positive cone of S_2. Then there exists an isometry $f : S_2 \to S_1$ such that $f^* = \psi$.*
3. *Let $f : S \to S$ be an isometry that acts as the identity on $H^2(S, \mathbb{Z})$. Then, f itself is the identity map. As a consequence, the isometry from 2. is unique.*

The first claim is obvious and the second is a consequence of the Torelli theorem and Lemma 3.3.19. The last claim follows from Proposition 11.3 in Chapter VIII in [11]. The Lemma 4.4.6 on matchings of K3 surfaces is a direct consequence of the above lemma.

Remark 3.3.21. If we had omitted the condition that $\psi_{\mathbb{R}}$ preserves the positive cone, the second part of our lemma would have been slightly more complicated. In that situation $\psi := -\mathrm{Id}_{H^2(S,\mathbb{Z})}$ would satisfy all conditions from the lemma. The corresponding isometry $f : S \to S$ would be the identity map, but it would have to be interpreted as an anti-holomorphic map between (S, I) and $(S, -I)$. The sign is necessary to map the Kähler form ω_I to $-\omega_I$.

Finally, we describe the moduli space $\mathscr{K}3^{hk}$ of all marked hyper-Kähler structures $(S, g, \omega_I, \omega_J, \omega_K, \phi)$. As a consequence of Theorem 3.3.18 and Lemma 3.3.20 we see that $\mathscr{K}3^{hk}$ is diffeomorphic to the *hyper-Kähler period domain*

$$\begin{aligned}\Omega^{hk} := \{(x,y,z) \in L_{\mathbb{R}}^3 | x^2 = y^2 = z^2 > 0,\ x \cdot y = x \cdot z = y \cdot z = 0, \\ \nexists\, d \in L \text{ with } d^2 = -2 \text{ and } x \cdot d = y \cdot d = z \cdot d = 0\}\,,\end{aligned}$$

see for example [36, p. 161]. $SO(3)$ acts from the left on the positive subspace of $L_{\mathbb{R}}$ that is spanned by x, y and z. $-\mathrm{Id}_{L_{\mathbb{R}}}$ acts as -1 on that space and transforms S to a K3 surface with the same metric and the opposite complex structure. Let $\phi : H^2(S,\mathbb{Z}) \to L$ be a marking and $\psi : L \to L$ be a lattice isomorphism. Since $\phi \circ \psi$ is a marking, too, $\mathrm{Aut}(L)$ acts from the right on Ω^{hk}. The moduli space of all unmarked hyper-Kähler metrics is thus diffeomorphic to the biquotient

$$\mathrm{Aut}(L) \backslash \Omega^{hk} / O(3)\,.$$

Replacing the triple (x, y, z) by $(\lambda x, \lambda y, \lambda z)$ with $\lambda > 0$ yields a hyper-Kähler metric that is rescaled by the factor λ. If we want to restrict ourselves to hyper-Kähler metrics with volume 1, we have to replace $O(3)$ by $O(3) \times \mathbb{R}^{>0}$ in the above formula.

3.4 Singular K3 surfaces

We discuss singular K3 surfaces and how they are related to the smooth ones. The results that we present in this section were originally proven in [6, 7, 42]. A short overview can also be found in [36, p.161 - 162]. As we will see, the theory of singular K3 surfaces is similar to the theory of ALE hyper-Kähler orbifolds.

Let (S, ϕ) be a marked K3 surface and let $d \in L$ be a lattice element with $d^2 = -2$ that represents an effective divisor. d can be interpreted as the cohomology class of a certain 2-cycle Z in S with self-intersection -2. We assume that S carries a hyper-Kähler structure $(g, \omega_I, \omega_J, \omega_K)$. It can be shown that Z is a sphere S^2 that is minimal with respect to g. Its area A is given by

$$A^2 = (\phi([\omega_I]) \cdot d)^2 + (\phi([\omega_J]) \cdot d)^2 + (\phi([\omega_K]) \cdot d)^2$$

If we move within the moduli space Ω^{hk} towards a point $(x, y, z) \in L_{\mathbb{R}}^3$ with

$$x \cdot d = y \cdot d = z \cdot d = 0\,,$$

the volume of the sphere shrinks to zero. In other words, we obtain a singularity. This is in fact the geometric meaning of the condition in the definition of Ω^{hk} that there shall be no $d \in L$ with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. We assume that there is exactly one $d \in L$ with this property. In this situation, we obtain the singularity by collapsing a single sphere with self-intersection -2 to a point. Since this is the reversal of blowing up an A_1-singularity, the K3 surface has an A_1-singularity at a single point. Next, we assume that there exists an arbitrary number of ds with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. We define the set

$$\widetilde{\Omega}^{hk} := \{(x, y, z) \in L^3_{\mathbb{R}} | x^2 = y^2 = z^2 > 0\,, x \cdot y = x \cdot z = y \cdot z = 0\}$$

and for any $\alpha = (x, y, z) \in \widetilde{\Omega}^{hk}$ we define

$$\mathcal{D}_\alpha := \{d \in L | d^2 = -2, x \cdot d = y \cdot d = z \cdot d = 0\}\,. \tag{3.4}$$

Let the cardinality of $\mathcal{D}_\alpha$ be greater than 1. By joining $d_1, d_2 \in \mathcal{D}_\alpha$ with $d_1 \neq d_2$ by $d_1 \cdot d_2$ edges, we obtain a graph G. G is the disjoint union of simply laced Dynkin diagrams. As we approach α, a set of 2-spheres whose intersection numbers are given by $d_i \cdot d_j$ collapses, which means that the Dynkin diagrams describe the type of the singularities. For example, if G consists of one Dynkin diagram of type E_8 and 2 isolated nodes, the singularities of the K3 surface are at 3 different points. At one of them we have a singularity of type E_8 and at the other ones we have A_1-singularities.

We see that the singular and the smooth K3 surfaces can be combined into a larger moduli space. The moduli space $\mathscr{K}3'^{hk}$ of all possibly singular hyper-Kähler structures on marked K3 surfaces is in fact diffeomorphic to $\widetilde{\Omega}^{hk}$. Since the intersection form has signature $(3, 19)$ there is a natural action of $O(3, 19)$ on that moduli space. We forget our particular choice of the complex structures I, J and K from the 2-sphere of all parallel complex structures and consider the moduli space of hyper-Kähler metrics instead of hyper-Kähler structures. We see that this space can in fact be identified with the non-compact symmetric space

$$O(3, 19)/(O(3) \times O(19))$$

or equivalently

$$SO_0(3, 19)/(SO(3) \times SO(19))\,,$$

where the subscript "0" denotes the identity component. The moduli space of hyper-Kähler metrics on unmarked K3 surfaces is the biquotient

$$\mathrm{Aut}(L) \backslash SO_0(3, 19)/(SO(3) \times SO(19))\,.$$

In some cases, it is actually possible to describe how the metric changes as we approach a point in the moduli space that describes a singular K3 surface. We illustrate this by the Kummer construction. Let $\Lambda \subset \mathbb{C}^2$ be a lattice of rank 4 with a basis (v_1, v_2, v_3, v_4). The quotient $\mathbb{C}^2/\Lambda$ is a torus. The map $v \mapsto -v$ transforms lattice vectors to other lattice vectors. Therefore, we have a well-defined action of $\mathbb{Z}_2$ on T^4. The quotient $T^4/\mathbb{Z}_2$ is simply connected and has trivial canonical bundle. It has 16 A_1-singularities at the points

$$\sum_{i=1}^{4} \tfrac{1}{2}\epsilon_i v_i + \Lambda \quad \text{with} \quad \epsilon_i \in \{0, 1\} \,.$$

After blowing up the singularities, we obtain a smooth K3 surface S. This is the so called Kummer construction of K3 surfaces. The blown up points become 16 rational curves with self-intersection -2. We denote the cohomology class of the exceptional divisor of a blow-up of a point $\sum_{i=1}^{4} \frac{1}{2}\epsilon_i v_i + \Lambda$ with $i := (\epsilon_1, \ldots, \epsilon_4) \in \{0, 1\}^4$ by e_i. The e_i span a sublattice of $H^2(S, \mathbb{Z})$ with $e_i \cdot e_j = -2\delta_{ij}$. In fact, the intersection of $H^2(S, \mathbb{Z})$ with $\text{span}_{\mathbb{R}}(e_i)_{i \in I}$ is a more complicated lattice K with a discriminant group of order 64 that can be embedded into L by a non-obvious map. For more details, we refer the reader to [52].

The hyper-Kähler metric on $T^4/\mathbb{Z}_2$ is simply the flat metric. Let $(g_k)_{k \in \mathbb{N}}$ be a sequence of smooth hyper-Kähler metrics that converge to the flat metric on $T^4/\mathbb{Z}_2$. Informally speaking, the metrics g_k converge to the flat metric far away from the singularities and near the singularities the g_k approach the Eguchi-Hanson metric. The precise statement and its proof can be found in [48].

3.5 Non-symplectic automorphisms

The G_2-orbifolds in Section 5.1 and 5.2 will be constructed by means of non-symplectic involutions of K3 surfaces. In this section, we introduce the most important facts about this issue. The results that we present are proven by Nikulin [53, 54, 55]. Good summaries of these papers can be found in [8, 44]. Finally, we show some results on non-symplectic involutions of singular K3 surfaces that cannot be found in the literature.

Definition 3.5.1. Let S be a K3 surface. A *non-symplectic automorphism of order* n is a biholomorphic map $\rho : S \to S$ such that

1. $\rho^n = \text{Id}$, but $\rho^k \neq \text{Id}$ for all $k \in \{1, \ldots, n-1\}$.
2. The pull-back $\rho^* : H^{2,0}(S) \to H^{2,0}(S)$ is not the identity map.

A non-symplectic automorphism is called *purely non-symplectic* if $\rho^{*k} \neq \text{id}$ for all $k \in \{1, \ldots, n-1\}$. If $n = 2$, ρ is called a *non-symplectic involution.*

Remark 3.5.2. Since $H^{2,0}(S)$ is spanned by $\omega_J + i\omega_K$, a purely non-symplectic automorphism of order n can be defined by the relation $\rho^*(\omega_J + i\omega_K) = \zeta_n(\omega_J + i\omega_K)$ where ζ_n is an nth root of unity.

A non-symplectic automorphism of prime order p is automatically purely non-symplectic. In this situation, we have $p \in \{2, 3, \ldots, 19\}$. The classification of the non-symplectic automorphisms of prime order can be found in [8]. In this section, we restrict ourselves to the classification of non-symplectic involutions since only these will be used for our constructions of G_2-orbifolds. From now on, let S be a K3 surface and $\rho : S \to S$ be a non-symplectic involution. We define the *fixed lattice of* ρ by

$$L^\rho := \{x \in H^2(S, \mathbb{Z}) | \rho^* x = x\} .$$

L^ρ is a primitive sublattice of $H^2(S, \mathbb{Z})$. Since ρ^* acts as -1 on $H^{2,0}(S)$ and $H^{0,2}(S)$, L^ρ is a sublattice of the Picard lattice $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$. A K3 surface with a non-symplectic automorphism admits an integral Kähler class $x \in H^{1,1}(S)$ and is thus algebraic by the Kodaira embedding theorem. Moreover, it admits even an integral ρ-invariant Kähler class since $x + \rho^* x$ is ρ-invariant.

We choose a marking $\phi : H^2(S, \mathbb{Z}) \to L$ and abbreviate $\phi(L^\rho)$ by L^ρ. It can be shown that L^ρ is a primitive non-degenerate sublattice of L with signature $(1, t)$. A lattice with that kind of signature is called *hyperbolic*. We define an invariant r of L^ρ by $r = 1 + t$. L^ρ is *2-elementary* which means that $L^{\rho *}/L^\rho$ is isomorphic to a group of type $\mathbb{Z}_2^a$. The number $a \in \mathbb{N}^{\geq 0}$ is a second invariant of L^ρ. We define a third invariant by

$$\delta := \begin{cases} 0 & \text{if } x^2 \in \mathbb{Z} \text{ for all } x \in L^{\rho *} \\ 1 & \textit{otherwise} \end{cases}$$

Theorem 3.5.3. *(Theorem 4.3.2 in [55]) For each triple* $(r, a, \delta) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \{0, 1\}$ *there is up to isometries at most one even, hyperbolic, 2-elementary lattice with invariants* (r, a, δ).

We denote the lattice with invariants (r, a, δ) by $L(r, a, \delta)$. Let N be a non-degenerate lattice with signature $(1, r-1)$ such that there exists a primitive embedding of N into L. We assume that N^*/N is 2-elementary and that $N \subset L$ contains a Kähler class. Then there exists a unique involution ρ_N of L with fixed lattice N. ρ_N acts as -1 on $N^\perp$ and $N^\perp$ contains a positive plane P with an orthonormal basis (x, y). The Torelli theorem and the surjectivity of the period map guarantee that there exists a K3 surface S together with a non-symplectic involution ρ such that $\rho^* = \rho_N$ and $H^2(S, \mathbb{R}) \cap (H^{2,0}(S) \oplus H^{0,2}(S)) = P$. The period point of that K3 surface is the complex line that is spanned by $x + iy$.

It follows that the deformation classes of K3 surfaces with a non-symplectic involution can be classified in terms of triples (r, a, δ). Nikulin [55] has shown that there exist 75 possible triples that satisfy

$$1 \leq r \leq 20\,, \quad 0 \leq a \leq 11 \quad \text{and} \quad r-a \geq 0\,.$$

A figure with a graphical representation of all possible values of (r, a, δ) can be found in [55] and in [44]. We present a theorem that yields further information on the structure of L^ρ. In order to do this, we need some notation.

- Let L be a lattice with bilinear form $\cdot_L$. The lattice with the same underlying abelian group and the bilinear form $(x, y) \mapsto -x \cdot_L y$ is denoted by $-L$ and the lattice with the form $(x, y) \mapsto k(x \cdot_L y)$, where $k \in \mathbb{N}$, is denoted by $L(k)$.
- The lattice that is generated by a single element x with $x^2 = 1$ is denoted by $\mathbf{1}$.
- The root lattice of a Dynkin diagram of a simple Lie algebra will be denoted by the same name as the Dynkin diagram itself.

Theorem 3.5.4. *(cf. [44, 55]) The fixed lattice of a non-symplectic involution with $r > 1$ is always isometric to a direct sum $L_1 \oplus L_2$ where*

- *$L_1 \in \{H, H(2), \mathbf{1}(2) \oplus \mathbf{1}(-2)\}$ and*
- *L_2 is a direct sum $\bigoplus_{i=1}^n K_i$ with $i \in \mathbb{N}_0$ where the K_i are isometric to $-A_1$, $-D_{2k}$ with $k \in \mathbb{N}$, $-E_7$, $-E_8$, or $-E_8(2)$.*

In order to make the above results more tangible, we describe some explicit examples of involutions of L that are induced by non-symplectic automorphisms of a K3 surface. We write

$$L = H_1 \oplus H_2 \oplus H_3 \oplus (-E_8)_1 \oplus (-E_8)_2$$

in order to distinguish between the different summands. We choose a basis (v_1^i, v_2^i) of each H_i such that

$$v_1^i \cdot v_1^i = v_2^i \cdot v_2^i = 0\,, \quad v_1^i \cdot v_2^i = 1\,.$$

Moreover, we introduce involutions ρ_1^i with $i \in \{1, \ldots, 4\}$ of $(-E_8)_1 \oplus (-E_8)_2$. Let $x_1 \in (-E_8)_1$ and $x_2 \in (-E_8)_2$. We define

$$\begin{aligned} \rho_1^1(x_1, x_2) &:= (x_1, x_2)\,, & \rho_1^2(x_1, x_2) &:= (-x_1, x_2)\,, \\ \rho_1^3(x_1, x_2) &:= (-x_1, -x_2)\,, & \rho_1^4(x_1, x_2) &:= (x_2, x_1)\,. \end{aligned}$$

The isomorphism type and the invariants (r, a, δ) of the fixed lattices of ρ_1^i can be found in the table below.

i	Fixed lattice	r	a	δ
1	$2(-E_8)$	16	0	0
2	$-E_8$	8	0	0
3	0	0	0	0
4	$-E_8(2)$	8	8	0

We introduce further involutions ρ_2^j with $j = 1, 2, 3$ of $H_1 \oplus H_2 \oplus H_3$ that are defined as follows.

1. $\rho_2^1(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$ for all $x_i \in H_i$.
2. $\rho_2^2(x_1, x_2, x_3) = (x_2, x_1, -x_3)$ for all $x_i \in H_i$.
3. $\rho_2^3(v_1^1) = v_2^1$, $\rho_2^3(v_2^1) = v_1^1$, $\rho_2^3(v_1^2) = -v_2^2$, $\rho_2^3(v_2^2) = -v_1^2$, $\rho_2^3(v_1^3) = -v_1^3$, $\rho_2^3(v_2^3) = -v_2^3$.

The fixed lattices and invariants of the ρ_2^j are given by:

j	Fixed lattice	r	a	δ
1	H	2	0	0
2	$H(2)$	2	2	0
3	$\mathbf{1}(2) \oplus \mathbf{1}(-2)$	2	2	1

Let $\rho_2^j \oplus \rho_1^i$ with $1 \leq i \leq 4$ and $1 \leq j \leq 3$ be an arbitrary combination of the above involutions. The orthogonal complement of the fixed lattice always contains a positive plane. Let (x, y) be an orthonormal basis of that plane. It follows from the surjectivity of the period map and the Torelli theorem that there exist a K3 surface with period point $\ell_{x+i\cdot y}$ and that $\rho_2^j \oplus \rho_1^i$ is the pull-back of a non-symplectic involution of that K3 surface.

We thus have constructed 12 types of non-symplectic involutions. Let $K = K_1 \oplus \ldots \oplus K_l$ be a direct sum of even 2-elementary lattices. The rank r of K simply is $\sum_{i=1}^{l} \operatorname{rank}(K_i)$. The order of the discriminant group K^*/K is the same as $|\det G(K)|$ where $G(K)$ is the Gram matrix of K. Since K is 2-elementary, we have $|\det G(K)| = 2^a$. $G(K)$ is a diagonal block matrix

$$\begin{pmatrix} \boxed{G(K_1)} & & \\ & \ddots & \\ & & \boxed{G(K_l)} \end{pmatrix}$$

and thus we have $a = \sum_{i=1}^{l} \log_2 |\det G(K_i)|$. Our observation shows that the invariants r and a are additive. The Gram matrix of K^* is $G(K)^{-1}$. We determine the quadratic form on the K_i^* and check if it only takes integer values. The invariant δ of K is 0 if all of these quadratic forms are integral and it is 1 otherwise. In other words, we have $\delta = \max\{\delta_1, \dots, \delta_l\}$, where δ_i denotes the invariant of the lattice K_i. Therefore, we obtain the following triples (r, a, δ) for the 12 involutions:

(i,j)	(r,a,δ)	(i,j)	(r,a,δ)
$(1,1)$	$(18,0,0)$	$(3,1)$	$(2,0,0)$
$(1,2)$	$(18,2,0)$	$(3,2)$	$(2,2,0)$
$(1,3)$	$(18,2,1)$	$(3,3)$	$(2,2,1)$
$(2,1)$	$(10,0,0)$	$(4,1)$	$(10,8,0)$
$(2,2)$	$(10,2,0)$	$(4,2)$	$(10,10,0)$
$(2,3)$	$(10,2,1)$	$(4,3)$	$(10,10,1)$

By similar but more complicated methods it would be possible to describe all 75 types of non-symplectic involutions and their fixed lattices. Since we do not need that explicit description, we do not carry out this program. Next, we describe the moduli space of all K3 surfaces with a non-symplectic involution whose fixed lattice is of a given isomorphism type. In order to do this, we need the following concept.

Definition 3.5.5. (cf. Dolgachev [20]) Let N be a hyperbolic lattice and let $i : N \to L$ be a primitive embedding. We denote $i(N)$ by N, too.

1. A *marked ample N-polarized K3 surface* is a K3 surface S together with a marking $\phi : H^2(S, \mathbb{Z}) \to L$ such that $\phi^{-1}(N)$ is a sublattice of the Picard lattice. Moreover, $\phi^{-1}(N)$ shall contain an integral ample class, which is since S is a compact Kähler manifold, the same as an integral Kähler class.

2. Two marked ample N-polarized K3 surfaces (S, ϕ) and (S', ϕ') are called isomorphic if there exists a biholomorphic map $f : S \to S'$ such that $\phi' = \phi \circ f^*$.

3. We denote the moduli space that consists of all marked ample N-polarized K3 surfaces modulo isomorphisms by $\mathscr{K}3^m(N)$.

The moduli space of all marked K3 surfaces with a non-symplectic involution whose fixed lattice is isomorphic to $L(r, a, \delta)$ is the same as $\mathscr{K}3^m(L(r, a, \delta))$, which we abbreviate by $\mathscr{K}3^m(r, a, \delta)$. There is a nice explicit description of that moduli space.

Theorem 3.5.6. *(Corollary 3.2 in [20]) Let N be a hyperbolic lattice that can be primitively embedded into L. We denote the orthogonal complement of N in L by M and define the following sets:*

$$\begin{aligned}
\Omega_N &:= \{\ell_x \in \mathbb{P}(M_{\mathbb{C}}) | x \cdot x = 0, x \cdot \overline{x} > 0\} \\
\triangle(M) &:= \{d \in M | d^2 = -2\} \\
H_d &:= \{\ell_z \in \mathbb{P}(M_{\mathbb{C}}) | z \cdot d = 0\} \\
\Omega'_N &:= \Omega_N \setminus \textstyle\bigcup_{d \in \triangle(M)} (H_d \cap \Omega_N)
\end{aligned}$$

$\mathscr{K}3^m(N)$ is isomorphic to Ω'_M and the isomorphism is given by the period map. We define the group

$$\Gamma(N) = \{\sigma \in Aut(L) | \sigma(x) = x \quad \forall x \in N\}$$

The moduli space of all unmarked ample N-polarized K3 surfaces is isomorphic to $\Omega'_M/\Gamma(N)$.

Remark 3.5.7.

1. $\mathscr{K}3^m(N)$ is a projective complex variety of dimension $20 - \mathrm{rk}(N)$. If the orthogonal complement M contains a sublattice that is isomorphic to H, $\mathscr{K}3^m(N)$ is irreducible.
2. By assumption, the sublattice N contains a Kähler class. Therefore, $M \cap \phi(H^{1,1}(S))$ cannot contain a d with $d^2 = -2$. This is the reason why we have to remove the set $\bigcup_{d\in\triangle(M)}(H_d \cap \Omega_N)$ from Ω_N.

The topology of the fixed locus S^ρ of a non-symplectic involution $\rho : S \to S$ can be described in terms of the invariants r and a.

Theorem 3.5.8. *(cf. [44, 55]) Let $\rho : S \to S$ be a non-symplectic involution of a K3 surface and let (r, a, δ) be the invariants of the fixed lattice that we have defined above. The fixed locus of S^ρ of ρ is a disjoint union of complex curves.*

1. *If $(r, a, \delta) = (10, 10, 0)$, S^ρ is empty.*
2. *If $(r, a, \delta) = (10, 8, 0)$, S^ρ is the disjoint union of two elliptic curves.*
3. *In the remaining cases, we have*

$$S^\rho = C_g \cup E_1 \cup \ldots \cup E_k \,,$$

   where C_g is a curve of genus $g = \frac{22-r-a}{2}$ and the E_i are $k = \frac{r-a}{2}$ curves that are biholomorphic to $\mathbb{CP}^1$, i.e. they are rational curves.

Remark 3.5.9.

1. In the case $(r, a, \delta) = (10, 10, 0)$, the action of ρ on L can be identified with the map $\rho_2^2 \oplus \rho_1^4$ that interchanges H_1 and H_2 as well as $(-E_8)_1$ and $(-E_8)_2$ and acts as -1 on H_3. If S is smooth, the quotient S/ρ is a smooth complex manifold that is called an *Enriques surface*.
2. In the case $(r, a, \delta) = (10, 8, 0)$, the action of ρ on L can be identified with the map $\rho_2^1 \oplus \rho_1^4$ that interchanges $(-E_8)_1$ and $(-E_8)_2$, acts as the identity on H_1 and as -1 on H_2 and H_3.

3. For any $p \in S^\rho$ there exists a complex basis of T_pS such that $(d\rho)_p$ has the matrix representation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In Section 5.1 and 5.2, we need singular K3 surfaces that admit a non-symplectic involution. At the end of this chapter, we therefore study which kinds of ADE-singularities such a K3 surface may have. Let (S, ϕ) be a marked K3 surface with a hyper-Kähler metric and a distinguished complex structure. Moreover, let ℓ_{x+iy} be its period point and let z be the image of the Kähler class with respect to ϕ. We assume that S admits a non-symplectic involution ρ with invariants (r, a, δ) that leaves the metric invariant. This happens if and only if $\rho^* z = z$. We recall that the set

$$D := \{d \in L | d^2 = -2 \, , x \cdot d = y \cdot d = z \cdot d = 0\}$$

is a root system that determines the number and type of the singular points. We try to choose x, y and z in such a way that D is as large as possible. The embedding of L^ρ into L determines the action of ρ on L. Since ρ is non-symplectic, x and y have to be positive elements in the orthogonal complement of L^ρ. Our description of the moduli space $\mathscr{K}3^m(r, a, \delta)$ guarantees that any choice of $x, y \in L^{\rho\perp}$ with $x^2 = y^2 > 0$ and $x \cdot y = 0$ yields a period point of a (possibly singular) K3 surface with a non-symplectic involution with invariants (r, a, δ). Moreover, we can choose z as an arbitrary element of L^ρ with $z^2 = x^2$ and $z \cdot x = z \cdot y = 0$.

According to Theorem 3.5.4, L^ρ decomposes as a direct sum $L_1 \oplus L_2$ where L_1 is isomorphic to H, $H(2)$ or $\mathbf{1}(2) \oplus \mathbf{1}(-2)$. Since the rank of L_1 is only 2, Theorem 3.1.5 guarantees that the embedding of L_1 into L is unique up to an automorphism of L. We assume that L_1 is isomorphic to H. In this situation, L_1 can be embedded as H_1 into L. L_1 can be embedded into L as the lattice that is generated by $w_1 := v_1^1 + v_1^2$ and $w_2 := v_2^1 + v_2^2$ if it is isomorphic to $H(2)$. If L_1 is isomorphic to $\mathbf{1}(2) \oplus \mathbf{1}(-2)$, it can be embedded as the lattice that is generated by $w_1 := v_1^1 + v_2^1$ and $w_2 := v_1^2 - v_2^2$. The Kähler class z has to be an element of $L^\rho \otimes_{\mathbb{Z}} \mathbb{R}$. We label z with a subscript that depends on the isomorphism type of L_1 and choose

- $z_1 := v_1^1 + v_2^1$ if $L_1 \cong H$,
- $z_2 := v_1^1 + v_2^1 + v_1^2 + v_2^2$ if $L_1 \cong H(2)$,
- $z_3 := v_1^1 + v_2^1$ if $L_1 \cong \mathbf{1}(2) \oplus \mathbf{1}(-2)$.

Since $z_1^2 = 2$, $z_2^2 = 4$ and $z_3^2 = 2$, z is positive. We choose x_i and y_i as:

- $x_1 := v_1^2 + v_2^2$ and $y_1 := v_1^3 + v_2^3$ if $L_1 \cong H$,
- $x_2 := v_1^1 + v_2^1 - v_1^2 - v_2^2$ and $y_2 := 2(v_1^3 + v_2^3)$ if $L_1 \cong H(2)$,
- $x_3 := v_1^2 + v_2^2$ and $y_3 := v_1^3 + v_2^3$ if $L_1 \cong \mathbf{1}(2) \oplus \mathbf{1}(-2)$.

By a short calculation, we see that $x_j^2 = y_j^2 = z_j^2$ and that x_j, y_j and z_j are pairwise orthogonal. The orthogonal complement of $\mathrm{span}_{\mathbb{Z}}(x_j, y_j, z_j)$ is in all three cases

$$\mathrm{span}_{\mathbb{Z}}(v_1^1 - v_2^1, v_1^2 - v_2^2, v_1^3 - v_2^3) \oplus (-E_8)_1 \oplus (-E_8)_2 \,.$$

A K3 surface S with a hyper-Kähler structure that is determined by x_j, y_j and z_j thus has 3 singular points of type A_1 and 2 singular points of type E_8. Its Picard lattice is the direct sum of the above lattice and $\mathrm{span}_{\mathbb{Z}}(z_j)$ and S therefore has maximal Picard number.

We have to show that S admits a non-symplectic involution ρ that acts on x_j, y_j and z_j as desired and whose fixed lattice has the prescribed value of the invariants (r, a, δ). Since we have chosen z_j as an element of $L_1 \subset L^\rho$, we only have to show that the lattice $L_1 \oplus L_2$ can be primitively embedded into L such that $x_j, y_j \perp L_1 \oplus L_2$. If this is the case, we can define ρ^* as the identity on $L_1 \oplus L_2$ and as minus the identity on $(L_1 \oplus L_2)^\perp$. x_j and y_j are in the (-1)-eigenspace of ρ^* and the period point $x_j + iy_j$ defines a K3 surface that admits a non-symplectic involution with fixed lattice $L^\rho \cong L_1 \oplus L_2$. We have already chosen a primitive embedding of L_1 into L and defined x_j and y_j in such a way that $x_j, y_j \perp L_1$. The final step therefore is to embed L_2 in such a way into L that it is a sublattice of $N := (\mathrm{span}_{\mathbb{Z}}(x_j, y_j) \oplus L_1)^\perp$. N is isomorphic to

$$\mathbf{1}(-2) \oplus \mathbf{1}(-2) \oplus 2(-E_8)$$

if $j = 1$ or $j = 3$ and it is isomorphic to

$$\mathbf{1}(-4) \oplus \mathbf{1}(-2) \oplus 2(-E_8)$$

if $j = 2$. L_2 is a direct sum of k copies of $-A_1 = \mathbf{1}(-2)$ and larger summands that are isomorphic to $-D_{2l}$, $-E_7$, $-E_8$, or $-E_8(2)$. Since by assumption the invariants (r, a, δ) of $L_1 \oplus L_2$ are chosen from the allowed list of triples, there exists a primitive embedding of L_2 into L. Any summand K of L_2 that is not isometric to $-A_1$ is either embedded into $2(-E_8)$ or there exists a $w \in K$ with $w \notin 2(-E_8)$. First, we assume that K is isomorphic to $-D_{2l}$, $-E_7$ or $-E_8$. These lattices have a canonical basis $(w_1, \ldots, w_n)$ that consists of vectors with square -2. The matrix representation of the bilinear form with respect to this basis is the negative of the Cartan matrix of D_{2l}, E_7 or E_8. If a $w \in K$ with $w \notin 2(-E_8)$ exists, there has to exist a basis element w_i with $w_i \notin 2(-E_8)$, too. Therefore, we assume that there exists an i with $w = w_i$. If w was transversely embedded into the direct sum

$3H \oplus 2(-E_8)$, it would split as $w = w' + w''$ with $w' \in 3H$ and $w'' \in 2(-E_8)$. Since both $3H$ and $2(-E_8)$ are even, we would have $w^2 \leq -4$ which contradicts our assumption that $w^2 = -2$. Therefore, we necessarily have $w \in 3H$. Since the signature of $3H$ is $(3,3)$, there exist at most 3 basis elements $w_i \in 3H$. If such w_i existed, they would span a sublattice K' of K such that K can be written as an orthogonal sum $K' \oplus K''$. A brief look at the Cartan matrices shows us that such a splitting does not exist. Therefore, any $K \in \{-D_{2k}, -E_7, -E_8\}$ has to be embedded into $2(-E_8)$.

Next, we assume that K is isomorphic to $-E_8(2)$. Theorem 3.1.5 guarantees that the embedding of $-E_8(2)$ into L is unique up to an automorphism of L. The diagonal sublattice $\{(v,v) \in 2(-E_8) | v \in (-E_8)\}$ defines a particular embedding $\imath : -E_8(2) \to L$. Let $\psi : L \to L$ be an automorphism such that the actual embedding of K is given by $\psi \circ \imath$. We assume that $\psi(\imath(-E_8(2))) \not\subseteq 2(-E_8)$. In this situation, we have $\psi(2(-E_8)) \not\subseteq 2(-E_8)$, too, since $\imath(-E_8(2)) \subseteq 2(-E_8)$. This can only happen if there exists an $i \in \{1,2\}$ with $\psi((-E_8)_i) \not\subseteq 2(-E_8)$ where $(-E_8)_1$ and $(-E_8)_2$ are the two summands of $2(-E_8) \subseteq L$. In other words, $(-E_8)_i$ is embedded by ψ in such a way into L that it is not contained in $2(-E_8)$. Since we have excluded this possibility already in the previous paragraph, we again have $K \subseteq 2(-E_8)$.

Finally, we take a look at the summands of L_2 that are isomorphic to $-A_1$. $-A_1$ is generated by a single v with $v^2 = -2$. For the same reasons as before, v cannot be embedded transversely into $3H \oplus 2(-E_8)$. Therefore, v is either an element of $2(-E_8)$ or of $3H$. By a short calculation, we see that the only elements of $3H$ with square -2 are $\pm(v_1^i - v_2^i)$ with $i = 1,2,3$. All in all, we have shown that L_2 is necessarily primitively embedded into

$$K' := \mathrm{span}_{\mathbb{Z}}(v_1^1 - v_2^1, v_1^2 - v_2^2, v_1^3 - v_2^3) \oplus 2(-E_8)\,. \tag{3.5}$$

Since $L_1 \oplus L_2$ can be primitively embedded into L, L_2 can even be embedded into $(L_1)^\perp \cap K'$. We describe this lattice for all values of j:

j	$(L_1)^\perp \cap K'$
1	$\mathrm{span}_{\mathbb{Z}}(v_1^2 - v_2^2, v_1^3 - v_2^3) \oplus 2(-E_8)$
2	$\mathrm{span}_{\mathbb{Z}}(v_1^1 - v_2^1 - v_1^2 + v_2^2, v_1^3 - v_2^3) \oplus 2(-E_8)$
3	$\mathrm{span}_{\mathbb{Z}}(v_1^1 - v_2^1, v_1^3 - v_2^3) \oplus 2(-E_8)$

We see that for all $j \in \{1,2,3\}$ x_j and y_j are orthogonal to $(L_1)^\perp \cap K'$. Since L_2 is embedded into $(L_1)^\perp \cap K'$, x_j and y_j are orthogonal to L_2, too, and L_2 is a primitive sublattice of $(\mathrm{span}_{\mathbb{Z}}(x_j, y_j) \oplus L_1)^\perp$. All in all, we have proven the following theorem.

Theorem 3.5.10. *Let $(r,a,\delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0,1\}$ be a triple such that there exists a K3 surface with a non-symplectic involution with invariants (r,a,δ). Then there exists a K3 surface which has 3 singular points with A_1-singularities and 2 singular points with*

E_8-singularities and carries a hyper-Kähler metric that is invariant with respect to a non-symplectic involution with the same invariants.

Let G be a Dynkin diagram that can be obtained by deleting some nodes from the union of three Dynkin diagrams of type A_1 and two of type E_8. We investigate if there exists a K3 surface with a non-symplectic involution whose singularities are described by G. Let (r, a, δ) be an arbitrary triple of invariants of a non-symplectic involution. Moreover, let S be the singular K3 surface with a non-symplectic involution ρ with invariants (r, a, δ) that we have constructed in the proof of the above theorem. As usual, we denote the images of the cohomology classes of the 3 Kähler forms with respect to a marking by $x, y, z \in L$. The orthogonal complement of $\mathrm{span}(x, y, z)$ is the lattice K' from equation (3.5). We choose a basis $(w_1, \ldots, w_{19})$ of K' such that for $j \in \{1, 2, 3\}$ we have $w_j = v_1^j - v_2^j$. Moreover, $(w_4, \ldots, w_{11})$ and $(w_{12}, \ldots, w_{19})$ shall span $(-E_8)_1$ and $(-E_8)_2$ such that the bilinear form on $(-E_8)_i$ with $i = 1, 2$ has the standard form

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

There are 3 possibilities how the pull-back of ρ could act on a w_j:

1. w_j is an element of the 1-eigenspace of ρ^*.
2. w_j is an element of the (-1)-eigenspace of ρ^*.
3. w_j is not contained in any of the eigenspaces. In this situation, w_j is mapped by ρ^* to another class such that $(w_j, \rho^* w_j)$ is linearly independent. Since ρ is an involution we have $\rho^{*2} w_j = w_j$.

There exist $k_1, k_2, k_3 \in \mathbb{N}_0$ with $k_1 + k_2 + 2k_3 = 19$ and disjoint subfamilies $(w_{j_1}, \ldots, w_{j_{k_1}})$, $(w_{j_{k_1+1}}, \ldots, w_{j_{k_1+k_2}})$ and $(w_{j_{k_1+k_2+1}}, \ldots, w_{j_{k_1+k_2+k_3}})$ of $(w_1, \ldots, w_{19})$ such that

1. $w_{j_1}, \ldots, w_{j_{k_1}}$ are in the 1-eigenspace of ρ^*,
2. $w_{j_{k_1+1}}, \ldots, w_{j_{k_1+k_2}}$ are in the (-1)-eigenspace of ρ^*,
3. $(w_{j_{k_1+k_2+m}}, \rho^* w_{j_{k_1+k_2+m}})$ is linearly independent for all $m \in \{1, \ldots, k_3\}$ and

4. the union of all pairs from 3. spans the orthogonal complement of $\mathrm{span}(w_{j_1}, \ldots, w_{j_{k_1+k_2}})$.

We choose arbitrary k_1', k_2' and k_3' with $0 \leq k_i' \leq k_i$ and arbitrary subfamilies $(w_{j_1'}, \ldots, w_{j_{k_1'}'}) \subseteq (w_{j_1}, \ldots, w_{j_{k_1}})$, $(w_{j'_{k_1'+1}}, \ldots, w_{j'_{k_1'+k_2'}}) \subseteq (w_{j_{k_1+1}}, \ldots, w_{j_{k_1+k_2}})$ and $(w_{j'_{k_1'+k_2'+1}}, \ldots, w_{j'_{k_1'+k_2'+k_3'}}) \subseteq (w_{j_{k_1+k_2+1}}, \ldots, w_{j_{k_1+k_2+k_3}})$. Moreover, let $(\alpha_1, \ldots, \alpha_{k_1'})$, $(\beta_1, \ldots, \beta_{k_2'})$ and $(\gamma_1, \ldots, \gamma_{k_3'})$ be families of real numbers such that $(1, \alpha_1, \ldots, \alpha_{k_1'}, \beta_1, \ldots, \beta_{k_2'}, \gamma_1, \ldots, \gamma_{k_3'})$ is $\mathbb{Q}$-linearly independent. We replace z by

$$z' := z + \alpha_1 w_{j_1'} + \ldots + \alpha_{k_1'} w_{j'_{k_1'}} \, . \tag{3.6}$$

Moreover, we replace x by

$$x' := x + \beta_1 w_{j'_{k_1'+1}} + \ldots + \beta_{k_2'} w_{j'_{k_1'+k_2'}} + \gamma_1 (w_{j'_{k_1'+k_2'+1}} - \rho^* w_{j'_{k_1'+k_2'+1}}) + \ldots + \gamma_{k_3'} (w_{j'_{k_1'+k_2'+k_3'}} - \rho^* w_{j'_{k_1'+k_2'+k_3'}}) \tag{3.7}$$

and y by

$$y' := y + \beta_1 w_{j'_{k_1'+1}} + \ldots + \beta_{k_2'} w_{j'_{k_1'+k_2'}} + \gamma_1 (w_{j'_{k_1'+k_2'+1}} - \rho^* w_{j'_{k_1'+k_2'+1}}) + \ldots + \gamma_{k_3'} (w_{j'_{k_1'+k_2'+k_3'}} - \rho^* w_{j'_{k_1'+k_2'+k_3'}}). \tag{3.8}$$

x' and y' are still in the (-1)-eigenspace of ρ^* and z' is ρ^*-invariant. If the α_l, β_l and γ_l are sufficiently small, x', y' and z' are positive. We have

$$x'^2 = x^2 - 2 \sum_{l=1}^{k_1'} \alpha_l^2 \, , \quad y'^2 = y^2 - 2 \sum_{l=1}^{k_2'} \beta_l^2 - 4 \sum_{l=1}^{k_3'} \gamma_l^2 \, , \quad z'^2 = z^2 - 2 \sum_{l=1}^{k_2'} \beta_l^2 - 4 \sum_{l=1}^{k_3'} \gamma_l^2 \, ,$$

since x, y and z are orthogonal to K'. Since it is possible to choose α_l, β_l and γ_l such that

$$\sum_{l=1}^{k_1'} \alpha_l^2 = \sum_{l=1}^{k_2'} \beta_l^2 + 2 \sum_{l=1}^{k_3'} \gamma_l^2$$

we can assume that $x'^2 = y'^2 = z'^2$. If $k_1' = 0$ or $k_2' + k_3' = 0$, we can define $z' = \lambda z$ or $(x', y') = \mu(x, y)$ for appropriate $\lambda, \mu \in \mathbb{R}$ such that $x'^2 = y'^2 = z'^2$. $x' + iy'$ therefore is an element of the period domain and z' is an element of the Kähler cone if the α_l are sufficiently small. The triple (x', y', z') thus defines a new hyper-Kähler structure on S. Since x' and y' remain in the (-1)-eigenspace of ρ^*, S admits an involution with fixed lattice $L(r, a, \delta)$ that is non-symplectic with respect to the complex structure that is associated to the period point $\ell_{x'+iy'}$. Since z' is ρ^*-invariant, ρ^* is the pull-back of an isometry with respect to the new hyper-Kähler metric. The set

$$D' := \{d \in L | d^2 = -2, x' \cdot d = y' \cdot d = z' \cdot d = 0\}$$

is a root system that describes the number and type of the singular points of the new hyper-Kähler metric. By choosing the $w_{j'_l}$ appropriately, we can produce a large number of different singularities. Unfortunately, it is hard to describe D' in the general situation. Therefore, we prove a corollary for the case $k_3 = 0$ and give an example for the case $k_3 > 0$. If $k_3 = 0$, which means that any w_i is contained in an eigenspace of ρ^*, D' is spanned by the complement of $\{w_{j'_1}, \ldots, w_{j'_{k'_1+k'_2}}\}$ in $\{w_1, \ldots, w_{19}\}$. This follows from the fact that any $d \in L$ with $d^2 = -2$ has integer coefficients with respect to the basis $(x, y, z, w_1, \ldots, w_{19})$ and that the family that consists of the α_l and β_l is $\mathbb{Q}$-linearly independent. The Dynkin diagram of D' is obtained from the union of three A_1 and two E_8 by deleting the nodes with numbers $j'_1, \ldots, j'_{k'_1+k'_2}$. All in all, we have proven the following corollary for the case $k_3 = 0$.

Corollary 3.5.11. *Let $(r, a, \delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\}$ be a triple such that there exists a K3 surface with a non-symplectic involution with invariants (r, a, δ). Moreover, let S be the K3 surface with 3 points with A_1-singularities and 2 points with E_8-singularities that we have constructed in Theorem 3.5.10 and let ρ be the non-symplectic involution from the same theorem. We assume that ρ acts either as 1 or as -1 on the elements of the standard basis $(w_1, \ldots, w_{19})$ of the lattice K' that is defined by equation (3.5). Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be a set of connected Dynkin diagrams that can be obtained by deleting nodes from the union of three Dynkin diagrams of type A_1 and two of type E_8. Then there exists a K3 surface with a hyper-Kähler metric that admits an isometric non-symplectic involution with the same invariants and has k singular points of type $\mathcal{D}_1, \ldots, \mathcal{D}_k$.*

Example 3.5.12. Let ρ^* be defined as $\rho_2^2 \oplus \rho_1^4$. We recall that this choice of ρ makes S/ρ an Enriques surface. We have

$$\rho^* w_1 = w_2 \ , \quad \rho^* w_2 = w_1 \ , \quad \rho^* w_3 = -w_3 \ , \quad \rho^* w_k = w_{k+8} \ , \quad \rho^* w_{k+8} = w_k$$

for all $k \in \{4, \ldots, 11\}$. The elements w_1, w_2 and $w_4, \ldots, w_{19}$ are not contained in an eigenspace of ρ^*, and therefore we have $k_3 = 18$, but ρ^* maps one basis element to another basis element. We permute the w_i such that we have

$$\rho^* w_1 = -w_1 \ , \quad \rho^* w_k = w_{k+9} \ , \quad \rho^* w_{k+9} = w_k$$

for all $k \in \{2, \ldots, 10\}$. Let $x, y, z \in L$ be chosen as in Theorem 3.5.10. We choose $\epsilon \in \{0, 1\}$, a subfamily $(w_{i_1}, \ldots, w_{i_k})$ of $(w_2, \ldots, w_{10})$ and sufficiently small real numbers $(\beta, \gamma_{i_1}, \ldots, \gamma_{i_k})$

such that $(1, \beta, \gamma_{i_1}, \ldots, \gamma_{i_k})$ is $\mathbb{Q}$-linearly independent. As in the proof of the above corollary, we define

$$\begin{array}{rcl} x' & := & x + \epsilon\beta w_1 + \sum_{j=1}^{k} \gamma_{i_j}(w_{i_j} - w_{i_j+9}) \\ y' & := & y + \epsilon\beta w_1 + \sum_{j=1}^{k} \gamma_{i_j}(w_{i_j} - w_{i_j+9}) \\ z' & := & \lambda z \end{array}$$

where $\lambda \in \mathbb{R}$ is chosen such that $x'^2 = y'^2 = z'^2$. $(w_2, \ldots, w_{10})$ can be identified with the nodes of a Dynkin diagram of type $A_1 \cup E_8$. We determine the set D' and see that it is the root system that is spanned the complement of $(w_{i_1}, \ldots, w_{i_k}, w_{i_1+9}, \ldots, w_{i_k+9})$ in $(w_2, \ldots, w_{19})$ and, if $\epsilon = 1$, by w_1. The singular locus of the K3 surface that is defined by x', y' and z' therefore is described by two copies of an arbitrary subdiagram of $A_1 \cup E_8$ and ϵ additional points with an A_1-singularity.

Chapter 4

Introduction to G_2-manifolds

4.1 Basic facts about G_2-manifolds

In this chapter, we introduce the results about G_2-manifolds and -orbifolds that will be needed later on. In particular, the known construction methods for compact G_2-manifolds and the role of G_2-orbifolds in theoretical physics will be discussed. First of all, we define what a G_2-manifold is and introduce the most important facts about them. For a more thorough introduction, we refer the reader to the book of Dominic Joyce [36] and the article of Spiro Karigiannis on deformations of G_2- and Spin(7)-structures [38], which contains many useful explicit formulas. We define a G_2-manifold as a manifold that carries a positive stable 3-form. The notion of a stable form was introduced by Nigel Hitchin [35].

Definition 4.1.1. Let V be a real or complex vector space and $\alpha \in \bigwedge^k V^*$ with $k \in \{0, \ldots, \dim V\}$ be a k-form. α is called *stable* if the $GL(V)$-orbit of α is an open subset of $\bigwedge^k V^*$.

We are especially interested in the case of 3-forms on a 7-dimensional space.

Proposition 4.1.2. *(Reichel [56], Schouten [61]) Let V be a 7-dimensional real vector space. The action of $GL(7)$ on $\bigwedge^3 V^*$ has exactly to open orbits. Their union is a dense subset of $\bigwedge^3 V^*$. One orbit consists of all 3-forms that are stabilized by the group G_2 whose Lie algebra is the compact real form of $\mathfrak{g}_2^{\mathbb{C}}$. The other one consists of all 3-forms that are stabilized by the group $\widetilde{G}_2$ whose Lie algebra is the split real form of $\mathfrak{g}_2^{\mathbb{C}}$.*

We define a $GL(7)$-equivariant map from the set of stable 3-forms on V to $S^2(V^*)$ that allows us to test if a 3-form is from the first or the second orbit.

Definition 4.1.3. (Karigiannis [38]) Let $(v_1, \ldots, v_7)$ be an arbitrary basis of V and ϕ be a stable 3-form. We define a symmetric bilinear form g_ϕ by the following formula:

$$g_\phi(v,w) := 6^{-\frac{2}{9}} \frac{((v\lrcorner\phi)\wedge(w\lrcorner\phi)\wedge\phi)(v_1,\ldots,v_7)}{(\det(((v_i\lrcorner\phi)\wedge(v_j\lrcorner\phi)\wedge\phi)(v_1,\ldots,v_7))_{i,j=1,\ldots,7})^{\frac{1}{9}}} . \tag{4.1}$$

Moreover, the following equation defines a volume form vol on V:

$$(v\lrcorner\phi)\wedge(w\lrcorner\phi)\wedge\phi = 6g_\phi(v,w)\,\mathrm{vol} . \tag{4.2}$$

Remark 4.1.4. It can be proven that the denominator in equation (4.1) is non-zero for any stable form ϕ.

The signature of g_ϕ is constant on any orbit of $GL(7)$. On the orbit with stabilizer G_2 the signature is $(7,0)$. We call the forms from this orbit *positive* 3*-forms* or G_2*-forms*. On the other open orbit the signature is $(3,4)$. For explicit calculations it is useful to fix a positive 3-form on $\mathbb{R}^7$.

Convention 4.1.5. Let $(v_i)_{i=1,\ldots,n}$ be a basis of a vector space V. We denote its dual basis by $(v^i)_{i=1,\ldots,n}$ and abbreviate $v^{i_1}\wedge\ldots\wedge v^{i_k}$ by $v^{i_1\cdots i_k}$. Analogously, we write $dx^{i_1\cdots i_k}$ for $dx^{i_1}\wedge\ldots\wedge dx^{i_k}$.

Definition 4.1.6. The form

$$\phi_0 := dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \tag{4.3}$$

is called the *standard* G_2*-form on* $\mathbb{R}^7$.

g_{ϕ_0} is the standard metric $dx^1\otimes dx^1 + \ldots + dx^7\otimes dx^7$ on $\mathbb{R}^7$ and the volume form defined by ϕ_0 is $dx^{1234567}$. g_{ϕ_0} and vol determine the Hodge-star operator $*$ and we see that

$$*\phi_0 = -dx^{1247} - dx^{1256} - dx^{1346} + dx^{1357} + dx^{2345} + dx^{2367} + dx^{4567} . \tag{4.4}$$

Remark 4.1.7. Alternatively, the group G_2 can be defined as the automorphism group of the octonions $\mathbb{O}$. An automorphism of $\mathbb{O}$ is defined as an $\mathbb{R}$-linear bijective map $\varphi:\mathbb{O}\to\mathbb{O}$ that satisfies $\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,y\in\mathbb{O}$. Let $\langle .,.\rangle$ be the scalar product that makes $\mathbb{O}$ a normed division algebra, which means that we have $\langle xy,xy\rangle = \langle x,x\rangle\cdot\langle y,y\rangle$ for all $x,y\in\mathbb{O}$. The orthogonal complement of the unit element 1 is called the *imaginary space* $Im(\mathbb{O})$ *of* $\mathbb{O}$. Any automorphism φ of $\mathbb{O}$ satisfies $\varphi(1)=1$ and $\varphi(\mathrm{Im}(\mathbb{O}))=\mathrm{Im}(\mathbb{O})$. G_2 therefore acts on the 7-dimensional space $\mathrm{Im}(\mathbb{O})$. Let (i,j,k) be the standard basis of the imaginary quaternions and let ϵ be a unit octonion that is orthogonal to $\mathbb{H}\subset\mathbb{O}$. We identify the bases $(e_1,\ldots,e_7)$ of $\mathbb{R}^7$ and $(i,j,k,\epsilon,i\epsilon,j\epsilon,-k\epsilon)$ of $\mathrm{Im}(\mathbb{O})$ with each other. With this identification, the standard G_2-form is determined by $\phi_0(x,y,z)=\frac{1}{2}\langle xy-yx,z\rangle$ and its Hodge-dual by $*\phi_0(x,y,z,w) = -\frac{1}{2}\langle x(yz)-(xy)z,w\rangle$. The reason that we have defined ϕ_0 in such a way that we have to identify e_7 with $-k\epsilon$ instead of $k\epsilon$ is that the identification of $\mathrm{span}(e_4, e_5, e_6, e_7)$ with $\mathbb{C}^2$ from Remark 2.0.3 is more straightforward. By a direct calculation, we can see that any linear map that is the identity on $\mathrm{span}(e_1, e_2, e_3)$ and acts as an element of $SU(2)$ on $\mathrm{span}(e_4, e_5, e_6, e_7) = \mathbb{H}\epsilon$ leaves ϕ_0 invariant.

We proceed to G_2-structures on manifolds.

Definition 4.1.8. A *G_2-structure* on a 7-dimensional manifold M is a 3-form ϕ such that for all $p \in M$ ϕ_p is a positive form on T_pM.

A G_2-structure induces a Riemannian metric g on M. Whenever we talk about a metric on a manifold with a G_2-structure we refer to this one. Moreover, M carries a volume form that is determined by equation (4.2) and thus is orientable. G_2 acts on any tangent space T_pM as the stabilizer group of the 3-form ϕ_p. Therefore, G_2 also acts on local frames and we can interpret a G_2-structure as a G-structure with structure group G_2. For any representation of G_2, there is an associated vector bundle on M. The inclusion of G_2 into $SO(7)$ can be lifted to an inclusion of G_2 into $\mathrm{Spin}(7) \subset GL(8, \mathbb{R})$ since G_2 is simply connected. The bundle that is associated to this 8-dimensional representation is the spinor bundle. G_2 splits the spin representation into a 1- and a 7-dimensional part. The spinor bundle therefore splits into a subbundle with 7-dimensional fibers and another bundle that is isomorphic to the bundle that is associated to the trivial representation of G_2. Since the second bundle simply is $M \times \mathbb{R}$, there exists a nowhere vanishing spinor on M. The converse of this statement is also true.

Proposition 4.1.9. *(Theorem 3.2. in [26]) Let M be a 7-dimensional orientable manifold that admits a spin structure. Then M also admits a G_2-structure.*

The following proposition helps us to decide if a G_2-structure induces a metric with holonomy G_2.

Proposition 4.1.10. *Let (M, ϕ) be a manifold with a G_2-structure and let g be the metric that is induced by ϕ. The following statements are equivalent.*

1. *$\nabla^g \phi = 0$, where ∇^g is the Levi-Civita connection.*
2. *$d\phi = d * \phi = 0$.*
3. *Hol $\subset G_2$, where Hol is the holonomy group of the Levi-Civita connection.*

If any of the above statements is true, (M, g) is Ricci-flat.

Definition 4.1.11. In the situation of the above proposition, ϕ is called a *parallel or torsion-free G_2-structure.*

The equation $d * \phi = 0$ is in fact a non-linear partial differential equation since the metric and thus the Hodge-star operator depend non-linearly on ϕ. Therefore, one has to apply advanced analytical methods to show the existence of parallel G_2-structures on a manifold.

If a G_2-structure ϕ is parallel, $*\phi$ is obviously parallel, too. Since the action of G_2 leaves a one-dimensional subbundle of the spinor bundle invariant, a manifold with a parallel G_2-structure carries at least one parallel spinor. If the holonomy group is not all of G_2, its identity component is either trivial, $SU(2)$ or $SU(3)$. This follows from the classification of the holonomy groups. A manifold M with a parallel G_2-structure ϕ may carry 1, 2, 4 or 8 linearly independent parallel spinors. If (M, ϕ) carries only one parallel spinor, the holonomy is either G_2, $SU(3) \rtimes \Delta$ or $Sp(1) \rtimes \Delta$, where Δ is a discrete group such that the holonomy acts irreducibly on the tangent space. If (M, ϕ) carries 2 parallel spinors, it is a product of S^1 or $\mathbb{R}$ with a 6-dimensional manifold whose holonomy is either $SU(3)$ or $Sp(1) \rtimes \Delta$. In the second case, the holonomy shall act irreducibly on the 6-dimensional tangent space. If there are 4 parallel spinors, M is the product of a 4-dimensional non-flat hyper-Kähler manifold and a 3-dimensional flat manifold and if there are 8 parallel spinors it is covered by $\mathbb{R}^7$. There are several slightly different definitions of a G_2-manifold. We choose the following one.

Definition 4.1.12. A *G_2-manifold* is a 7-dimensional manifold with a parallel G_2-structure such that the holonomy group acts irreducibly on the tangent space.

If the underlying manifold is compact, it is particularly easy to decide if the holonomy is all of G_2 or just a subgroup.

Lemma 4.1.13. *Let M be a compact manifold with a parallel G_2-structure. The holonomy of the induced metric is all of G_2 if and only if $\pi_1(M)$ is finite.*

An important object in the theory of G_2-manifolds is the moduli space of parallel G_2-structures on a 7-dimensional manifold.

Definition 4.1.14. Let M be a 7-dimensional manifold that admits a parallel G_2-structure. We denote the set of all parallel G_2-structures on M by $\Xi(M)$. Any diffeomorphism of M acts on $\Xi(M)$ by its pull-back. We denote the group of all diffeomorphisms of M that are isotopic to the identity by $\mathcal{D}(M)$ and define the moduli space of all parallel G_2-structures on M as $\mathcal{M}_M := \Xi(M)/\mathcal{D}(M)$.

There is a natural projection map $\pi : \mathcal{M}_M \to H^3(M, \mathbb{R})$ that maps an orbit $\phi\mathcal{D}(M)$ to the cohomology class $[\phi]$. Joyce [36] has proven the following theorem.

Theorem 4.1.15. *The moduli space $\mathcal{M}_M$ has a differentiable structure such that π becomes a local diffeomorphism. In particular, $\mathcal{M}_M$ can be regarded as a finite-dimensional smooth manifold of dimension $b^3(M)$.*

It may be possible to add pieces to $\mathcal{M}_M$ that correspond to singular G_2-manifolds such that the geometry of the enlarged moduli space is better behaved. An analogy is the moduli space of hyper-Kähler metrics on a marked K3 surface that may have ADE-singularities. This moduli space can be identified with the symmetric space $SO_0(3,19)/(SO(3) \times SO(19))$ and therefore has a simpler description than the moduli space of smooth hyper-Kähler metrics on a K3 surface. Karigiannis [39] conjectures that the moduli space of parallel G_2-structures on a manifold with a suitable conical singularity is a boundary component of the moduli space of parallel G_2-structures on a smooth manifold that is a desingularization of the conical singularity. Moreover, Halverson and Morrison [32] conjecture that the boundary components of the moduli space of smooth parallel G_2-structures consist of singular G_2-manifolds whose singular loci have codimension 4, 6 or 7.

We will see that the singular set of a G_2-orbifold with ADE-singularities is a certain type of calibrated submanifold, called an associative submanifold. Therefore, we introduce the most important facts about calibrated submanifolds here at the end of this section. More detailed presentations of this topic can be found in the papers by Harvey and Lawson [29] and by McLean [51]. In the following, let (M, g) be a Riemannian manifold and V be an oriented subspace of a tangent space T_pM. The orientation of V together with the restriction of the metric determine a volume form vol_V on V.

Definition 4.1.16. Let (M, g) be a Riemannian manifold and φ be a closed k-form on M. φ is called a *calibration form* if for all $p \in M$ and all oriented k-dimensional subspaces V of T_pM, we have $\varphi|_V \leq \mathrm{vol}_V$.

Remark 4.1.17. Since the degree of φ equals the dimension of V, we have $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for an $\alpha \in \mathbb{R}$. The condition $\varphi|_V \leq \mathrm{vol}_V$ simply means that $\alpha \leq 1$.

Definition 4.1.18. Let (M, g) be a Riemannian manifold with a calibration form φ of degree k. A k-dimensional oriented submanifold N of M is called a *calibrated submanifold with respect to* φ if for all $p \in N$ we have $\varphi|_{T_pN} = \mathrm{vol}_{T_pN}$.

Lemma 4.1.19. *The volume of a compact calibrated submanifold is minimal within its homology class. In particular, calibrated submanifolds are minimal submanifolds.*

Being a minimal submanifold is a second order condition since it is equivalent to the vanishing of the mean curvature. The condition for a calibrated submanifold is of first order since it is only an algebraic condition on the tangent spaces. This fact makes calibrated submanifolds often easier to handle than minimal submanifolds.

Let (M, ϕ) be a G_2-manifold. For any $p \in M$ the form ϕ_p can be written with respect to a suitable basis of T_pM as (4.3) and $*\phi_p$ can be written as (4.4). With help of these explicit formulas, it is possible to show that ϕ and $*\phi$ are both calibration forms. Therefore, a G_2-manifold has two kinds of calibrated submanifolds.

Definition 4.1.20. Let (M, ϕ) be a G_2-manifold. 3-dimensional submanifolds of M that are calibrated with respect to ϕ are called *associative submanifolds* and 4-dimensional submanifolds that are calibrated with respect to $*\phi$ are called *coassociative submanifolds.*

The imaginary space of the octonions $\mathrm{Im}(\mathbb{O})$ together with the standard G_2-form is a flat G_2-manifold. The 3-dimensional linear subspaces of $\mathrm{Im}(\mathbb{O})$ that are associative submanifolds of $\mathrm{Im}(\mathbb{O})$ are given by $\varphi(\mathrm{Im}(\mathbb{H}))$, where φ is an arbitrary automorphism of $\mathbb{O}$. Analogously, the coassociative linear subspaces are of the form $\varphi(\mathbb{H}\epsilon)$. Up to conjugation, there is a unique subgroup of G_2 that is isomorphic to $SO(4)$. The set of all 3-dimensional or 4-dimensional calibrated subspaces of $\mathrm{Im}(\mathbb{O})$ can be identified with the quotient $G_2/SO(4)$.

In simple terms, an associative submanifold of a G_2-manifold is a 3-dimensional manifold whose tangent spaces can be identified with $\mathrm{Im}(\mathbb{H})$ and a coassociative submanifold is a manifold whose tangent spaces can be identified with $\mathbb{H}\epsilon$.

We finally remark that the deformations of a compact coassociative submanifold N have a smooth moduli space whose dimension is $b_2^+(N)$ where the "+" denotes the self-dual part of the cohomology. In the associative case, there may be obstructions for the extension of an infinitesimal deformation to an actual one. Therefore, we do not have an analogous result about the moduli space as in the coassociative case. These facts were proven first by McLean [51].

4.2 Introduction to orbifolds

Since the subject of this thesis are G_2-orbifolds, we provide the reader with a definition of an orbifold and discuss the similarities and differences between manifolds and orbifolds. An orbifold can be thought of as a topological space that is locally modeled on $\mathbb{R}^n/\Gamma$, where Γ is a finite group, instead of $\mathbb{R}^n$. Orbifolds have been defined for the first time by Satake [59, 60] under the name V-manifolds. There exist several slightly different definitions of this term. Our definition is close to the one from [59, 60].

Definition 4.2.1. An n-dimensional orbifold is a Hausdorff, second countable topological space X with an open cover $(U_i)_{i\in I}$ such that:

1. For any $p \in U_i \cap U_j$, where $i, j \in I$, there exists an U_k with $p \in U_k \subset U_i \cap U_j$.
2. For any U_i there exists a finite group $G_i \subset GL(n,\mathbb{R})$, an open G_i-invariant subset $V_i \subset \mathbb{R}^n$ and a continuous map $\pi_i : V_i \to U_i$ with $\pi_i(g.v) = \pi_i(v)$ for all $g \in G_i$ and $v \in V_i$ such that the induced map $\pi_i' : V_i/G_i \to U_i$ is a homeomorphism. We say U_i is *uniformized* by (G_i, V_i, π_i) and the tuple (G_i, V_i, π_i) is a *local uniformizing system.*
3. For any $U_i \subset U_j$ there exists an injective group homomorphism $\phi_{ij} : G_i \to G_j$ and an injective C^∞-map $\psi_{ij} : V_i \to V_j$ such that $\psi_{ij} : V_i \to \psi_{ij}(V_i)$ is a homeomorphism. Moreover, ψ_{ij} shall be ϕ_{ij}-equivariant, i.e. $\psi_{ij}(g.v) = \phi_{ij}(g).\psi_{ij}(v)$ for all $g \in G_i$ and $v \in V_i$ and it shall satisfy $\pi_j \circ \psi_{ij} = \pi_i$. The tuple (ϕ_{ij}, ψ_{ij}) is called an *injection.*
4. If we have two injections (ϕ_{ij}, ψ_{ij}) and (ϕ_{ij}', ψ_{ij}') with $\phi_{ij}, \phi_{ij}' : G_i \to G_j$ and $\psi_{ij}, \psi_{ij}' : V_i \to V_j$, there shall exist a unique $g \in G_j$ with $\psi_{ij}' = g.\psi_{ij}$.

Remark 4.2.2. 1. In the above definition, we have assumed that the G_i act faithfully on $\mathbb{R}^n$. This condition avoids some technical issues, but it is not necessary and other authors also allow non-faithful group actions.

2. A point $p \in U_i$ with the property that any $q \in \pi_i^{-1}(p)$ is stabilized only by the unit element $e \in G_i$ is called a *smooth point*. Otherwise, it is called a *singular point*. The set of all singular points of an orbifold is called the *singular locus*. The stabilizer group of q is called the *orbifold group of p*. Any point p of an orbifold has a neighborhood that is homeomorphic to a neighborhood of 0 in $\mathbb{R}^n/G$, where G is the orbifold group of p. All of these terms are independent of the choice of the local uniformizing system. With the notion of smooth maps between orbifolds that is introduced below we can choose these neighborhoods as diffeomorphic.

3. We assume from now on that the fixed point set of each $g \in G_i$ with $g \neq e$ is of codimension at least two. This ensures that the set of all smooth points is connected. Moreover, we have Poincare duality for compact, oriented orbifolds [36, p.133] under this assumption. All orbifolds that we consider will either be complex orbifolds, where the codimension is even, or G_2-orbifolds with singularities that have codimension 4 or 6. Therefore, our assumption will always be satisfied.

Example 4.2.3. Let M be a manifold and let G be a finite group acting smoothly, effectively and orientation preserving on M. The quotient M/G is an orbifold. Orbifolds of that kind are called *global quotients*. Although global quotient are orbifolds, there exist orbifolds that are not global quotients.

We define smooth maps between orbifolds equivalently to [59, 60].

Definition 4.2.4. Let X and X' be orbifolds and let $f : X \to X'$ be a continuous map. Moreover, let $(U_i)_{i\in I}$ and $(U'_i)_{i\in I'}$ be orbifold atlases with uniformizing systems (G_i, V_i, π_i) and (G'_i, V'_i, π'_i). Let $p \in X$ be an arbitrary point and let $i \in I$ and $j \in I'$ such that $p \in U_i$ and $f(p) \in U'_j$. A *lift of f* is a map $\widetilde{f}_{ij} : V_i \to V'_j$ is $\pi'_j \circ \widetilde{f}_{ij} = f \circ \pi_i$ and if for any $g \in G_i$ there exists an $g' \in G'_j$ such that $g'.\widetilde{f}_{ij}(v) = \widetilde{f}_{ij}(g.v)$ for all $v \in V_i$. f is called a smooth map if

1. for any $p \in X$ there exist $i \in I$ and $j \in I'$ such that $p \in U_i$, $f(p) \in U'_j$ and there exists a lift $\widetilde{f}_{ij} : V_i \to V'_j$.

2. for any injection $(\phi_{i\bar{i}}, \psi_{i\bar{i}})$ with $\psi_{i\bar{i}} : V_i \to V_{\bar{i}}$ there exists an injection $(\phi'_{j\bar{j}}, \psi'_{j\bar{j}})$ with $\psi_{j\bar{j}} : V'_j \to V'_{\bar{j}}$ such that we have $\psi_{j\bar{j}} \circ \widetilde{f}_{ij} = \widetilde{f}_{\bar{i}\bar{j}} \circ \psi_{i\bar{i}}$.

Other concepts from differential geometry can be generalized to orbifolds by similar methods. For example, it is possible to define orbifold vector bundles [60]. The key idea is to lift the group action of the G_i to the total space of the bundle. The tangent bundle of an orbifold is an orbifold vector bundle. The tangent space T_pX of an orbifold X is isomorphic to $\mathbb{R}^n/G$, where G is the orbifold group of $p \in X$. As in the case of manifolds, we can define sections, dual bundles and tensor products of bundles. Therefore, we can talk about tensor fields, differential forms, Riemannian metrics etc. on orbifolds. We can define the standard differential operators such as the exterior differential and the Levi-Civita connection on a Riemannian orbifold as usual.

Let g be a Riemannian metric on an orbifold X and let $p \in X$ be a point with orbifold group G. The metric on T_pX can be lifted to a G-invariant metric on $\mathbb{R}^n$. Therefore, G has to be a subgroup of $O(n)$. This is not a real restriction on the type of the singularities since for any finite group G acting on $\mathbb{R}^n$ there exists a G-invariant scalar product on $\mathbb{R}^n$. Moreover, it can be shown by the same arguments as in the smooth case that any orbifold admits a Riemannian metric. If X carries a complex structure and is of complex dimension n, the orbifold groups have to be embedded into $GL(n,\mathbb{C})$ in order to make the complex structure well-defined.

We have to be careful with the definition of the holonomy of an orbifold since it is not clear how the parallel transport can be defined for paths that pass through singular point. One possibility to avoid this problem is to define the holonomy of an orbifold as the holonomy of X/S where S is the singular locus. Since X/S may not be simply connected even if X is simply connected, a simply connected orbifold may have a holonomy group that is not connected [36, p.135] We are now able to define G_2-orbifolds.

Definition 4.2.5. Let X be a 7-dimensional orbifold. A G_2-structure on X is a 3-form ϕ on X such that for all $p \in X$ the form ϕ_p can be identified via a bijective linear map $\mathbb{R}^7 \to T_pX$ (or the projection of a linear map to a bijective map $\mathbb{R}^7/G \to T_pX$) with ϕ_0 (or with its projection to $\mathbb{R}^7/G$). A G_2-structure is called parallel if $\nabla^g\phi = 0$, where g is the metric that is induced by ϕ. (X, ϕ) is called a *G_2-orbifold* if ϕ is a parallel G_2-structure and the holonomy of g acts irreducibly on the tangent space.

The above definition forces all orbifold groups to be subgroups of G_2. If all orbifold groups are conjugate to a subgroup of $SU(2) \subset G_2$, we say that (X, ϕ) is a *G_2-orbifold with ADE-singularities*. The singular locus of a G_2-orbifold with ADE-singularities is a disjoint union of associative submanifolds since the fixed point set of any $g \in SU(2) \subset G_2$ that is not the identity is an associative subspace.

The de Rham cohomology of an orbifold is well-defined and can be calculated by the usual methods. For example, if the orbifold is a global quotient M/G, $b^k(M/G)$ is the number of G-invariant, linear independent, harmonic forms on M. If we define the first Chern class of a complex orbifold X in the usual way, we encounter some difficulties. Let n be the complex dimension of X and G be an orbifold group of a point $p \in X$. The determinant is a map $\det : \mathbb{C}^{n\times n} \to \mathbb{C}$. The image of G with respect to det is a cyclic group $\mathbb{Z}_k$. The fiber of the canonical bundle at p is isomorphic to $\mathbb{C}/\mathbb{Z}_k$ instead of $\mathbb{C}$. If we define $c_1(X)$ as usual as the cohomology class of a curvature form, we see that it is still well-defined, but it is an element of $H^2(X,\mathbb{Q})$ instead of $H^2(X,\mathbb{Z})$. With these modifications, the statement of the Calabi conjecture still makes sense for orbifolds and it can be proven by the same ideas as in the case of manifolds [7, 36]. We remark that all orbifold groups of the complex orbifolds in this thesis are embedded into $SL(n, \mathbb{C})$ such that we avoid the above technicalities entirely.

Most other theorems about the existence of geometric objects on orbifolds can be directly translated from the case of manifolds. The condition for the existence of an object usually is a partial differential equation. Locally, one can lift sections of vector bundles and differential operators from open subsets of $\mathbb{R}^n/G$ to G-invariant objects on open subsets of $\mathbb{R}^n$ and solve the problem there. Therefore, one can simply take the proof from the smooth case and replace the word "manifold" by "orbifold". In particular, the Theorems 4.4.3 and 4.4.8 on twisted connected sums that we need in Section 5.2 are also true for orbifolds. In addition to these two theorems, the only mathematical results for orbifolds that we use in this thesis are either the Calabi conjecture for orbifolds or theorems from algebraic geometry that hold for smooth varieties as well as for varieties with ADE-singularities.

4.3 Joyce's construction of G_2-manifolds

The first examples of compact G_2-manifolds have been constructed by Dominic Joyce [36]. The idea behind his construction is to resolve the singularities of a torus quotient T^7/Γ, where Γ is a discrete group that preserves the flat G_2-structure on the 7-dimensional torus T^7. In Section 5.3, we construct torus quotients that carry a G_2-structure, too. We will see that Γ can be chosen in such a way that we obtain a wide range of ADE-singularities. We will resolve the singularities of our torus quotients by the method of Joyce and obtain examples of smooth G_2-manifolds. Therefore, we review the construction of Joyce in this section. We focus on the general idea behind this construction and on those details that will become important later on. For a comprehensive treatment of this subject, we refer the reader to the book [36].

Let $T^7 = \mathbb{R}^7/\Lambda$, where Λ is a lattice of rank 7, be a torus. T^7 carries a flat G_2-structure that is induced by the 3-form (4.3). Let Γ be a finite group that preserves the G_2-structure. Any singular point of T^7/Γ has a neighborhood U that can be identified with $B_\epsilon(0)/\Delta$ where $B_\epsilon(0) = \{x \in \mathbb{R}^7 | \|x\| < \epsilon\}$ and Δ is a finite subgroup of G_2. We assume that each Δ can be embedded into the subgroup $SU(2)$ or $SU(3)$ of G_2. U can be identified with a neighborhood of zero in $\mathbb{R}^3 \times \mathbb{C}^2/\Delta$ or $\mathbb{R} \times \mathbb{C}^3/\Delta$, where Δ is a finite subgroup of $SU(2)$ or $SU(3)$. The singularities of T^7/Γ can therefore be resolved by means of complex geometry. For reasons of simplicity, we restrict ourselves to crepant resolutions of $\mathbb{C}^2/\Delta$ and $\mathbb{C}^3/\Delta$ although Joyce considers other resolutions, too. We introduce some facts about crepant resolutions.

Definition 4.3.1. Let Y be a singular algebraic variety over $\mathbb{C}$. A *resolution of* Y is a proper birational map $\pi : X \to Y$ such that X is a non-singular normal variety. A resolution is called *crepant* if it preserves the canonical bundle, i.e. $\pi^*\mathcal{K}_Y = \mathcal{K}_X$.

Remark 4.3.2. The above definition contains some technical terms from algebraic geometry that are defined in textbooks on this subjects [27, 33]. In the cases that we consider, X will be a complex manifold and π will be a holomorphic surjective map whose restriction $\pi|_{X\setminus\pi^{-1}(S)} : X \setminus \pi^{-1}(S) \to Y \setminus S$, where S is the singular locus of Y, is biholomorphic. Moreover, there shall exist local coordinates such that $\pi|_{X\setminus\pi^{-1}(S)}$ and its inverse can be written as rational functions. In this situation, the conditions of the above definition are satisfied.

Singularities of type $\mathbb{C}^n/\Delta$ with $n \in \{2, 3\}$ always allow a crepant resolution. If $n = 2$, the resolution $\pi : X \to \mathbb{C}^2/\Delta$ is the composition of several blow-ups of the singular locus. X is biholomorphic to the underlying complex manifold of the ALE hyper-Kähler manifold that is asymptotic to $\mathbb{C}^2/\Delta$ from the end of Chapter 2 with respect to one of the complex structures. This is in fact the only crepant resolution. If $n = 3$, a crepant resolution exists for all finite subgroups Δ of $SU(3)$, but it is not unique. The different resolutions are related by so called flops and all of them have the same Betti numbers. Explicit resolutions of $\mathbb{C}^3/\Delta$ for all discrete subgroups of $SU(3)$, or equivalently of $SL(3, \mathbb{C})$ are constructed by Roan [58].

As we have mentioned, the resolution of $\mathbb{C}^2/\Delta$ carries an ALE hyper-Kähler metric that is asymptotic to the standard hyper-Kähler metric on $\mathbb{C}^2/\Delta$. In the case $n = 3$, the resolution carries a Ricci-flat Kähler metric. This fact has been proven by Joyce [36], too. The Ricci-flat Kähler metric is asymptotic to $\mathbb{C}^3/\Delta$, but we have to be careful about the meaning of the word "asymptotic". We explain this with help of an example.

Example 4.3.3. Let Δ be the group that is generated by $\mathrm{diag}(-1, -1, 1)$ and $\mathrm{diag}(1, -1, -1)$ in $\mathbb{C}^{3\times3}$. Δ is isomorphic to $\mathbb{Z}_2^2$. The singular locus of $\mathbb{C}^3/\mathbb{Z}_2^2$ consists of the three axes $S_1 := \{(z, 0, 0)\Delta | z \in \mathbb{C}\}$, $S_2 := \{(0, z, 0)\Delta | z \in \mathbb{C}\}$ and $S_3 := \{(0, 0, z)\Delta | z \in \mathbb{C}\}$. Along each of the axes we have an A_1-singularity. S_1, S_2 and S_3 intersect in the point 0Δ. We blow up the axis S_1. S_2 and S_3 are lifted to complex curves that do not intersect in the blow-up since they intersect transversally in $\mathbb{C}^3/\Delta$. We obtain a smooth complex manifold by blowing up the lifts of S_2 and S_3. Since a blow-up preserves the canonical bundle, we have constructed a crepant resolution. We could have started with blowing up S_2 or S_3 instead of S_1 and would have obtained crepant resolutions, too. These three resolutions are in fact all crepant resolutions of $\mathbb{C}^3/\Delta$. Let $\pi : X \to \mathbb{C}^3/\mathbb{Z}_2^2$ be one of the resolutions. On X, there exists a smooth Ricci-flat Kähler metric g such that its restriction to the three sets $\{\pi^{-1}((z_1, z_2, z_3)\Delta) \in \mathbb{C}^3 | z_i = t\}$ approaches for $t \to \infty$ the ALE hyper-Kähler metric on the resolution of $\mathbb{C}^2/\mathbb{Z}_2$.

The above metric is not asymptotically locally Euclidean. If this was the case, g would approach the standard metric g_0 on $\{z \in \mathbb{C}^3/\Delta | \|z\| > R\}$. Since g is smooth and $\{z \in \mathbb{C}^3/\Delta | \|z\| > R\}$ has singularities, g cannot be an ALE metric. Instead we call the metric *quasi asymptotically locally Euclidean (QALE)*. A precise definition of QALE Ricci-flat Kähler metrics on crepant resolutions of $\mathbb{C}^3/\Delta$ can be found in [36].

We go back to the torus quotient T^7/Γ. We cover the singular locus of T^7/Γ by small open sets U_i that can be identified with open subsets of $\mathbb{R}^3 \times \mathbb{C}^2/\Delta$ or $\mathbb{R} \times \mathbb{C}^3/\Delta$. We resolve each singularity $\mathbb{C}^n/\Delta$ with $n \in \{2,3\}$ by a crepant resolution $\pi : X \to \mathbb{C}^n/\Delta$ such that the open subset of $\mathbb{R}^{7-2n} \times \mathbb{C}^n/\Delta$ is replaced by an open subset V_i of $\mathbb{R}^{7-2n} \times X$. The V_i can be chosen as $B_\epsilon(0) \times \{\pi^{-1}(z\Delta) | \|z\| < \epsilon\}$, where $B_\epsilon(0)$ is a small ball in $\mathbb{R}^{7-2n}$. The second factor of V_i is diffeomorphic to $Y_i^R := \{\pi^{-1}(z\Delta) | \|z\| < R\}$ for a any $R > 0$. We choose an QALE Ricci-flat Kähler metric g_i on each Y_i^R. If R is sufficiently large, we can make the difference between g_i and its asymptotic model arbitrarily small. We rescale g_i such that it fits on Y_i^ϵ. Since the holonomy of g_i is $SU(2)$ or $SU(3)$, we have a parallel G_2-structure $\phi_{i,R}$ on V_i. We can glue together the $\phi_{i,R}$ with the flat G_2-structure on the smooth part of T^7/Γ by a suitable intermediate 3-form such that we obtain a smooth G_2-structure ϕ with $d\phi = 0$ on the resolved manifold M. By choosing R sufficiently large, we can make $\|d * \phi\|$ arbitrarily small. Joyce [36] has proven a theorem that guarantees that ϕ can be deformed to a torsion-free G_2-structure if $\|d * \phi\|$ is small enough. If the fundamental group of the resolved manifold is finite, the holonomy of the induced metric on M is all of G_2.

Although the above picture is essentially correct, we have neglected some aspects. First, we have some freedom to choose the crepant resolutions. Second, the resolutions of the singularities on the different U_i have to satisfy certain consistency conditions such that we can glue them together. A choice of of suitable resolutions that satisfy those conditions is called *R-data* by Joyce. All in all, Joyce has proven a theorem that can be stated informally as follows.

Theorem 4.3.4. *Let T^7/Γ be a torus quotient that carries a G_2-structure that is flat on the smooth part of T^7/Γ. For any choice of R-data for the resolutions of the singularities, there exists a smooth parallel G_2-structure on the resolved manifold.*

At the end of this section, we discuss a few cases in order to show how different R-data influence the geometry of the resolved manifolds.

The simplest case is that T^7 contains a 3-dimensional submanifold N on which a finite subgroup Δ of $SU(2)$ acts trivially. Since any element of Γ acts as an isometry of the flat metric on T^7, it can be written as an affine transformation. Therefore, N has to be a 3-torus. Moreover, we assume that no $g \in \Gamma \setminus \Delta$ fixes a point of N. In this situation, the quotient T^7/Γ has an ADE-singularity along a submanifold that we denote by N, too, and no other connected component of the singular locus intersects N. We replace a neighborhood of the origin in the normal space of N with an ALE hyper-Kähler metric from Theorem 2.0.9 that is asymptotic to $\mathbb{C}^2/\Delta$. After that, we can apply the methods of Joyce in order to obtain a smooth G_2-manifold. We recall that the ALE hyper-Kähler metrics that are asymptotic to $\mathbb{C}^2/\Delta$ form a family of dimension $3 \cdot b^2(X)$, where X is the resolution of $\mathbb{C}^2/\Delta$. The Betti numbers of the G_2-manifold are independent of the choice of the hyper-Kähler metric.

Next, we assume that there is a 3-torus $N \subset T^7$ whose points are fixed by a group $\Delta \subset SU(2)$ and that there are further elements g_i of Γ that act non-trivially and freely on N. Moreover, we assume that N does not intersect with the fixed point sets of group elements outside of the group that is generated by Δ and the g_i. T^7/Γ has an ADE-singularity of type $\mathbb{C}^2/\Delta$

along a quotient $N' := N/\langle g_i\rangle$. On the fibers of the normal space of N there is a 2-sphere of complex structures that are invariant under Δ since $\Delta \subset SU(2)$. For reasons of simplicity we assume that the ADE-singularity is of type A_1. We resolve it by replacing it with an Eguchi-Hanson metric. We recall that we obtain different metrics if we choose different complex structures on $\mathbb{C}^2/\mathbb{Z}_2$. The g_i act as linear maps on the normal space of N and the ALE hyper-Kähler metric have to be chosen such that the metric on the resolution of the singularity along N' is well-defined. If a g_i acts as a $\mathbb{C}$-linear map on the normal space of N, it acts well-defined on the exceptional divisor $\mathbb{CP}^1$ of the blow-up. This ensures that the metric on the resolution is well-defined, too. Moreover, the action of g_i on $\mathbb{CP}^1$ preserves the orientation. If g_i acts orientation-preserving on N, too, it acts as the identity on the cohomology class of the exceptional divisor that we obtain by blowing up N as a subset of T^7/Δ. It may happen that g_i acts anti-holomorphically and thus orientation-reversing on $\mathbb{CP}^1$ or that it reverses the orientation of N. For each reversal of the orientation we have to multiply the sign of the action on the cohomology class by -1. If we blow up N' with respect to a different complex structure, the action of g_i on the new $\mathbb{CP}^1$ may change. Thus, the sign of the action on the cohomology class may change, too. Therefore, we may obtain G_2-manifolds with a different topology by choosing different resolutions of the singularity, even if there is only an A_1-singularity. If the ADE-singularity along N' is of another type, we have essentially the same but slightly more complicated picture.

Finally, we assume that we have a singular point with a neighborhood U that is diffeomorphic to a neighborhood of 0 in $\mathbb{R} \times \mathbb{C}^3/\mathbb{Z}_2^2$, where $\mathbb{Z}_2^2$ is defined as in Example 4.3.3. We have shown that there are 3 different crepant resolutions that can be obtained from each other by permuting the 3 axes in $\mathbb{C}^3/\mathbb{Z}_2^2$. Although all of these resolutions have the same Betti numbers, we may obtain different Betti numbers of the resolved G_2-manifold since the axis in $\mathbb{C}^3/\mathbb{Z}_2^2$ that is blown up first is glued to different parts of the singular locus of T^7/Γ.

At the end of Section 5.3, we resolve the singularities of our torus quotients and determine the Betti numbers of the resolved G_2-manifolds. We see that the blow-ups of the ADE-singularities with respect to different complex structures indeed yield different Betti numbers. Moreover, we encounter a singularity of type $\mathbb{C}^3/\mathbb{Z}_2^2$ and another one of type $\mathbb{C}^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ that we resolve by a particular crepant resolution that has a nice geometric interpretation in that context.

4.4 Twisted connected sums

Beside the resolution of torus quotients there is another method to construct compact G_2-manifolds, which is called the *twisted connected sum construction*. It has been proposed by Simon Donaldson and was worked out in detail by Alexei Kovalev [43]. Since some of our examples of G_2-orbifolds with ADE-singularities are constructed by this method, we describe how the twisted connected sum construction works. In short, it consists of the following three steps.

1. Construct two non-compact Calabi-Yau manifolds W_1 and W_2 that are asymptotic to cylinders $D_i \times S^1 \times (0, \infty)$ with $i = 1, 2$, where D_1 and D_2 are K3 surfaces.
2. Truncate the cylindrical ends of $W_1 \times S^1$ and $W_2 \times S^1$ and glue together the remaining parts by a map that interchanges the circle factors at the ends and whose projection to the K3 factors is an isometry $f : D_1 \to D_2$. f has to satisfy a certain condition that will be explained later on. The resulting manifold M is compact and has finite fundamental group.
3. Define a closed G_2-structure ϕ on M such that $d * \phi$ is small and deform it to a parallel G_2-structure.

The three steps will be described in more detail below. We start with the construction of the asymptotically cylindrical Calabi-Yau manifolds. The proofs of the following theorems and additional information on asymptotically cylindrical Calabi-Yau manifolds can be found in Haskins et al. [30] and in Kovalev [43].

Definition 4.4.1. (Definition 1.1. in [30]) A complete Riemannian manifold (W, g) is called *asymptotically cylindrical (ACyl)* if there exist a bounded domain $U \subset W$, a closed Riemannian manifold (X, h) and a diffeomorphism $\Phi : X \times [0, \infty) \to W \setminus U$ such that $\|\nabla^k(\Phi^* g - g_\infty)\| = O(e^{-\delta t})$ for all $k \in \mathbb{N}_0$ and a $\delta > 0$. ∇ denotes the Levi-Civita connection of the cylindrical metric $g_\infty := dt^2 + h$, where $t : X \times [0, \infty) \to [0, \infty)$ is the projection map.

We denote $t \circ \Phi^{-1}$ by t, too, and call it the *cylindrical coordinate*. We consider t only on $W \setminus U$ although it is possible to extend t smoothly to all of W by introducing a cut-off function on $X \times [0, \epsilon]$ and setting $t = 0$ on U. Finally, we call the connected components of $X \times [0, \infty)$ *cylindrical ends* and (X, h) the *cross section*.

Henceforth, we assume that (W, g) is an ACyl Calabi-Yau manifold. We identify the metric with help of the complex structure with the Kähler form ω. It follows from the Cheeger-Gromoll splitting theorem that any Ricci-flat ACyl manifold and thus (W, ω) has only one cylindrical end. There is the following theorem on ACyl Calabi-Yau manifolds.

Theorem 4.4.2. *(Theorem B in [30]) Let (W, ω) be a simply connected, irreducible ACyl Calabi-Yau manifold of complex dimension $n > 2$. Then the holonomy of (W, ω) is the whole group $SU(n)$. Moreover, the cross section is isometric to $(D \times S^1)/\mathbb{Z}_m$, where D is a compact Calabi-Yau manifold. $\mathbb{Z}_m$ is generated by an isometry that acts on $D \times S^1$ as $(x, \theta) \mapsto (\imath(x), \theta + \frac{2\pi}{m})$ where $\imath$ is an isometry of D that preserves the Kähler form and the holomorphic volume form of D but no other holomorphic form of positive degree.*

We assume that $m = 1$. Since $S^1 \times (0, \infty)$ is diffeomorphic to $\mathbb{C} \setminus \{0\}$, it is possible to glue together an ACyl Calabi-Yau threefold (W, ω) with $D \times \{z \in \mathbb{C} \mid |z| < 1\}$ along $D \times S^1 \times (1 - \epsilon, 1 + \epsilon)$ such that we obtain a compact complex manifold. If $m > 1$, we can glue in a copy of $(D \times \{z \in \mathbb{C} | |z| < 1\})/\mathbb{Z}_m$ and obtain a complex orbifold. The precise statement of this fact can be found in Theorem C in [30]. The converse of this theorem is also true. If $\overline{W}$ is a compact complex orbifold with certain properties, we obtain a manifold that admits an ACyl Ricci-flat Kähler metric by removing a suitable divisor.

Theorem 4.4.3. *(Theorem D in [30]) Let $\overline{W}$ be a compact Kähler orbifold of complex dimension $n \geq 2$. Moreover, let $\overline{D} \in |-\mathcal{K}_{\overline{W}}|$ be an effective divisor that satisfies the following conditions:*

1. *The complement $W := \overline{W} \setminus \overline{D}$ is a smooth manifold.*
2. *The normal bundle of $\overline{D}$ is biholomorphic to $(D \times \mathbb{C})/\mathbb{Z}_m$, where D is a connected compact complex manifold, the generator of $\mathbb{Z}_m$ acts as $(x, w) \mapsto (\imath(x), \exp(\frac{2\pi i}{m})w)$ on $D \times \mathbb{C}$ and $\imath$ denotes a complex automorphism of D of order m.*

Finally, let Ω be a meromorphic $(n, 0)$-form on $\overline{W}$ with a simple pole along $\overline{D}$ and let $\mathfrak{x}$ be a Kähler class on $\overline{W}$. Then there exists an ACyl Calabi-Yau metric on W such that its Kähler form ω is an element of the restriction $\mathfrak{x}|_W$ of $\mathfrak{x}$ to W and $\omega^n = i^{n^2}\Omega \wedge \overline{\Omega}$.

It follows from the adjunction formula that $\overline{D}$ has trivial canonical bundle. Therefore, the metric on the factor D of the cylindrical end converges to a Ricci-flat Kähler metric. From now on, we restrict ourselves to ACyl Calabi-Yau manifolds that are suitable for the twisted connected sum construction and thus make the following assumptions.

Assumption 4.4.4. In the situation of Theorem 4.4.3, let

1. the complex dimension n be 3,
2. the order of the group $\mathbb{Z}_m$ be 1,
3. the complex manifold D from the cross section $D \times S^1$ be not a torus and thus a K3 surface, and
4. the fundamental group of $W := \overline{W} \setminus \overline{D}$ be finite.

In order to construct ACyl Calabi-Yau threefolds with the desired cross section, we need a compact Kähler manifold $\overline{W}$ of complex dimension 3 with an anti-canonical K3 divisor whose normal bundle is holomorphically trivial. There are several methods to construct $\overline{W}$. Kovalev [43] constructed the first examples of twisted connected sums by choosing $\overline{W}$ as the blow-up of a Fano threefold along the self-intersection of an anti-canonical K3 divisor. This construction was generalized in Corti et al. [16, 17] to weak Fano threefolds. Kovalev and Lee [44] pursued a different idea. Their starting point was a quotient of $S \times \mathbb{P}^1$, where S is a K3 surface, by a certain involution. The quotient has A_1-singularities along a disjoint union of complex curves that can be resolved by a blow-up. After that, the authors obtain a smooth manifold $\overline{W}$ that satisfies the conditions from Theorem 4.4.3 and can be used as a building block of a twisted connected sum. We describe the constructions from [44] in more detail when we generalize it to orbifolds with ADE-singularities.

We describe how ACyl Calabi-Yau threefolds can be used to construct compact G_2-manifolds. Let W_1 and W_2 be ACyl Calabi-Yau threefolds with cross-sections $D_1 \times S^1$ and $D_2 \times S^1$, where D_1 and D_2 are K3 surfaces. In order to glue together $W_1 \times S^1$ and $W_2 \times S^1$, the K3 surfaces have to satisfy the following condition.

Definition 4.4.5. Let D_1 and D_2 be K3 surfaces with a hyper-Kähler metric. We denote the three complex structures on D_j by I_j, J_j and K_j. The corresponding Kähler forms shall be ω_I^j, ω_J^j and ω_K^j. As usual, $\omega_J^j + i\omega_K^j$ is the holomorphic volume form for I_j. D_1 and D_2 satisfy the *matching condition* if there exists a $\mathbb{Z}$-linear map $h : H^2(D_2, \mathbb{Z}) \to H^2(D_1, \mathbb{Z})$ that preserves the intersection form such that the $\mathbb{R}$-linear extension of h satisfies

$$h([\omega_I^2]) = [\omega_J^1]\,, \quad h([\omega_J^2]) = [\omega_I^1]\,, \quad h([\omega_K^2]) = -[\omega_K^1]\,.$$

The following lemma is a consequence of Lemma 3.3.20.

Lemma 4.4.6. *Let D_1 and D_2 be K3 surfaces that satisfy the matching condition. Then there exists an isometry $f : D_1 \to D_2$ such that the pull-back $f^* : H^2(D_2, \mathbb{R}) \to H^2(D_1, \mathbb{R})$ restricted to the cohomology with integer coefficients equals h and we have*

$$f^*\omega_I^2 = \omega_J^1\,, \quad f^*\omega_J^2 = \omega_I^1\,, \quad f^*\omega_K^2 = -\omega_K^1\,.$$

We denote the asymptotically cylindrical Ricci-flat Kähler metric on W_j by ω_j and the holomorphic volume form by Ω_j. $\phi_j := \omega_j \wedge d\theta_j + \mathrm{Im}(\Omega_j)$, where θ_j denotes the standard coordinate on the circle, is a parallel G_2-structure on $W_j \times S^1$. Let $T \in [0, \infty)$ be sufficiently large. We cut off the cylindrical ends of $W_1 \times S^1$ and $W_2 \times S^1$ at $t = T + 1$. For reasons of simplicity, we denote the truncated manifolds with boundary by $W_1 \times S^1$ and $W_2 \times S^1$, too.

The metric on D_j approaches for $t \to \infty$ a Ricci-flat Kähler metric that we denote by ω_I^j and the holomorphic volume form approaches a form $\omega_J^j + i\omega_K^j$. We define a parallel G_2-structure on $D_j \times S^1 \times S^1 \times [T, T+1]$ by

$$\omega_I^j \wedge d\theta_j + \omega_J^j \wedge d\theta_{3-j} + \omega_K^j \wedge dt + d\theta_{3-j} \wedge d\theta_j \wedge dt\,. \tag{4.5}$$

It is possible to define a closed G_2-structure $\widetilde{\phi}_j$ on $W_j \times S^1$ that coincides with ϕ_j on $(W_j \times S^1) \setminus (D_j \times S^1 \times S^1 \times [T-1, \infty))$ and with the form (4.5) on $D_j \times S^1 \times S^1 \times [T, T+1]$. On $D_j \times S^1 \times S^1 \times [T-1, T]$ we need to define $\widetilde{\phi}_j$ as a closed G_2-structure that is not necessarily coclosed but interpolates between the forms on the two parts. We assume that

D_1 and D_2 with their asymptotic hyper-Kähler structures structures satisfy the matching condition and that $f : D_1 \to D_2$ is an isometry with the properties from Lemma 4.4.6. We define a map

$$\begin{array}{rcl} F : D_1 \times S^1 \times S^1 \times [T, T+1] & \to & D_2 \times S^1 \times S^1 \times [T, T+1] \\ F(x, \theta_1, \theta_2, T+t) & := & (f(x), \theta_2, \theta_1, T+1-t) \end{array}$$

We have $F^* \widetilde{\phi}_2 = \widetilde{\phi}_1$. Therefore, we can glue together $W_1 \times S^1$ and $W_2 \times S^1$ along the collars $D_j \times S^1 \times S^1 \times [T, T+1]$ with help of the map F and obtain a compact manifold M with a well-defined closed G_2-structure. We call M the *twisted connected sum of* $W_1 \times S^1$ *and* $W_2 \times S^1$ and denote the G_2-structure on M by ϕ_T since it depends on the cut-off parameter T. The map F interchanges both circle factors. This ensures that M has finite fundamental group. The matching condition is exactly the condition on D_1 and D_2 that is required in order to make ϕ_T a well-defined G_2-structure. We can adjust the interpolating G_2-structure on $D_j \times S^1 \times S^1 \times [T-1, T]$ such that $\| d * \phi_T \|$ becomes arbitrarily small as $T \to \infty$.

The last step of the construction is to conclude from $\lim_{T \to \infty} \| d * \phi_T \| = 0$ that there also exists a parallel G_2-structure on M. Kovalev [43] provided a proof of this fact that uses glueing techniques for the solutions of elliptic PDEs. This proof builds upon earlier work of Floer [24] and Kovalev, Singer [45]. The above results can be summed up as follows.

Definition 4.4.7. Let $\overline{W}_1$ and $\overline{W}_2$ be compact Kähler manifolds of complex dimension three. Moreover, let D_1 and D_2 be effective divisors with $D_i \in |-\mathcal{K}_{\overline{W}_i}|$ for $i = 1, 2$ such that the D_i are K3 surfaces, their normal bundle is holomorphically trivial and $|\pi_1(\overline{W}_i \setminus D_i)| < \infty$. If in addition D_1 and D_2 satisfy the matching condition, we call $(\overline{W}_1, D_1)$ and $(\overline{W}_2, D_2)$ a *matching pair*.

Theorem 4.4.8. *Let $(\overline{W}_1, D_1)$ and $(\overline{W}_2, D_2)$ be a matching pair and let $W_i := \overline{W}_i \setminus D_i$. The twisted connected sum of $W_1 \times S^1$ and $W_2 \times S^1$ has finite fundamental group and admits a parallel G_2-structure. The induced metric thus has holonomy G_2.*

At the end of this section, we introduce some formulas for the topological invariants of a twisted connected sum M. Let $\overline{W}$ be a complex threefold that satisfies the conditions from Theorem 4.4.3 and Assumption 4.4.4. We have $h^{1,0}(\overline{W}) = h^{2,0}(\overline{W}) = h^{3,0}(\overline{W}) = 0$, see [44]. The only non-trivial information on the Hodge-diamond of $\overline{W}$ is therefore given by $h^{1,1}(\overline{W})$ and $h^{1,2}(\overline{W})$. The topology of a twisted connected sum M is determined by the topology of the matching pair $(\overline{W}_1, D_1)$ and $(\overline{W}_2, D_2)$ and by the isometry $f : D_1 \to D_2$. We present a result from [44] on the Betti numbers of M. Let $i \in \{1, 2\}$. D_i can be included into W_i as a part of the cylindrical end. Let

$$\imath_i : H^2(W_i, \mathbb{R}) \to H^2(D_i, \mathbb{R})$$

be the pull-back of this inclusion. We define a subspace

$$X := \imath_1(H^2(W_1,\mathbb{R})) \cap f^*\imath_2(H^2(W_2,\mathbb{R})) \subset H^2(D_1,\mathbb{R})$$

and introduce the numbers

$$d_i := \dim \ker \imath_i \quad \text{and} \quad n := \dim X \,.$$

There exist subspaces $X_1, X_2 \subset H^2(D_1,\mathbb{R})$ such that we have orthogonal sum decompositions

$$\begin{aligned} \imath_1(H^2(W_1,\mathbb{R})) &= X \oplus X_1 \\ f^*\imath_2(H^2(W_2,\mathbb{R})) &= X \oplus X_2 \end{aligned}$$

with respect to the intersection form of D_1. With this notation, we are able to state the following results.

Theorem 4.4.9. *(Theorem 2.5. in [44]) Let M be a twisted connected sum that is constructed from the matching pair $(\overline{W}_1, D_1)$ and $(\overline{W}_2, D_2)$. Then*

$$\begin{aligned} \pi_1(M) &= \pi_1(W_1) \times \pi_1(W_2) \\ b^2(M) &= n + d_1 + d_2 \end{aligned}$$

If $b^2(W_1) - d_1 + b^2(W_2) - d_2 \leq 22$ and X_1 is orthogonal to X_2 with respect to the intersection form, we have

$$b^3(M) = b^3(\overline{W}_1) + b^3(\overline{W}_2) + b^2(M) - 2n + 23 \,.$$

4.5 G_2-orbifolds and physics

The main application of G_2-manifolds and -orbifolds outside of pure mathematics is compactification of M-theory. Since M-theory is a vast subject, we can provide only a short overview. We refer the reader to the survey articles by Acharya [3], Acharya and Gukov [4] and by Duff [21] for a more detailed introduction to M-theory on G_2-manifolds. We also point out the recent publication by Halverson and Morrison [31] about M-theory on twisted connected sums. The enhancement of the gauge group that is the focus of this section is discussed by Acharya [1], [2], Barrett [10] and by Halverson and Morrison [32]. Phenomenological predictions of M-theory on G_2-manifolds can be found in [5].

M-theory is a candidate for a physical theory that allows us to quantize gravity and explains all the fields and interactions that can be observed in nature. It is defined on an 11-dimensional space-time M^{11} with signature $(10, 1)$ and its fundamental objects are

2- and 5-dimensional membranes, that are called M2- and M5-branes. M-theory can be compared to superstring theory whose fundamental objects are 1-dimensional strings. In fact, it is believed that the different versions of superstring theory arise as certain limits of M-theory. Unfortunately, the quantization of a theory with higher-dimensional objects is much harder than in superstring theory and not yet fully understood. In the limit where the volume of the branes shrinks to zero we obtain 11-dimensional supergravity. The fields that appear in 11-dimensional supergravity are the metric g, a 3-form C and a spinor. The action for the fields g and C is given by the Lagrangian

$$\mathcal{L} = \int_{M_{11}} R\,\mathrm{vol}_g - \frac{1}{2} G \wedge *G - \frac{1}{6} C \wedge G \wedge G\,,$$

where R is the scalar curvature and $G = dC$. It is the highest-dimensional theory that contains the Einstein-Hilbert action functional, which describes gravity, is supersymmetric and contains no fields with spin > 2. The last condition is necessary for the consistent quantization of the theory. The condition of supersymmetry means that the Lagrangian is locally invariant under a super Lie algebra that is the extension of the Poincare algebra. This is sufficient to determine all the missing terms in the Lagrangian that contain the spinor field. Since the world that we observe is 4-dimensional, we have to explain the physical meaning of the other 7 dimensions if we assume that M-theory or 11-dimensional supergravity is a description of nature. If the space-time can be written as

$$M^{11} = \mathbb{R}^{3,1} \times M^7$$

where $\mathbb{R}^{3,1}$ is Minkowski space and M^7 is a sufficiently small Riemannian manifold, an observer would not notice the presence of the other dimensions. In this context, "small" usually means that the volume of M^7 is approximately the 7th power of the Planck length of $10^{-35}\,m$. This ansatz for the space-time is called compactification and the quantum field theory that we observe on $\mathbb{R}^{3,1}$ is called the low-energy limit of M-theory compactified on M^7. For various theoretical and phenomenological reasons, we want the four-dimensional theory to be supersymmetric, too. The Poincare group in four dimensions may be enhanced by $\mathcal{N} \in \{1, 2, 4, 8\}$ odd generators. Physicists often assume that the field C vanishes in the vacuum state. In this situation, the amount of supersymmetry $\mathcal{N}$ is determined by the number of parallel spinors on M^7. Each odd generator maps a bosonic field to a fermionic field and vice versa. Supersymmetry therefore predicts that for any kind of elementary particle there exists a superpartner whose spin is shifted by $\frac{1}{2}$. The case $\mathcal{N} = 1$ is particularly interesting since it predicts the smallest number of particles beyond the standard model.

If we assume that $\mathcal{N} = 1$, the manifold M^7 has to be a G_2-manifold. In this case, the four-dimensional theory that we obtain as low-energy limit is a super Yang-Mills theory with gauge group $U(1)^{b^2(M^7)}$ and $b^3(M^7)$ complex scalar fields that are not charged under the gauge group. Unfortunately, this above theory is not compatible with the observations.

The standard model of particle physics contains so called chiral fermions that cannot be modeled by this ansatz. Moreover, the electro-weak and strong interaction are described by Yang-Mills theories with gauge groups $SU(2) \times U(1)$ and $SU(3)$ that are non-abelian.

If we allow M^7 to have suitable singularities, it is believed that M-theory compactified on M^7 is still well-defined. This is a difference to general relativity that requires a smooth underlying manifold. If M^7 has conical singularities, the field theory that we obtain in the low-energy limit contains chiral fermions [3, 4, 9]. Until now, no explicit examples of compact G_2-manifolds with conical singularities are known.

Our focus will be on G_2-orbifolds with ADE-singularities. These singularities yield field theories with a non-abelian gauge group in the low-energy limit. The idea behind this is as follows. A G_2-orbifold with an ADE-singularity looks locally like $\mathbb{R}^3 \times \mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is finite. Let $(X_t)_{t\in[0,\epsilon)}$ be a family of ALE hyper-Kähler metrics that converges for $t \to 0$ in the Gromov-Hausdorff limit against $\mathbb{C}^2/\Gamma$ as described in Chapter 2. $H^2(\mathbb{R}^3 \times X_t, \mathbb{Z})$ can be identified with the root lattice of the finite-dimensional simple Lie algebra $\mathfrak{g}$ that is associated to Γ by the McKay correspondence. The cohomology class of an M2-brane is an element x of $H^2(\mathbb{R}^3 \times X_t, \mathbb{Z})$. If $x \neq 0$, the M2-brane cannot contract to a point which means that it carries mass. It can be shown that there are $\dim \mathfrak{g}$ cohomology classes that correspond to a single M2-brane. We consider the limit $t \to 0$. Since there are several submanifolds of type $\mathbb{R}^3 \times S^2$ whose cohomology classes span $H^2(\mathbb{R}^3 \times X_t, \mathbb{Z})$ and that collapse to $\mathbb{R}^3$, the M2-branes are allowed to shrink arbitrarily in size, too. This means that they correspond to $\dim \mathfrak{g}$ massless fields on $\mathbb{R}^3 \times \mathbb{C}^2/\Gamma$. These fields interact in such a way that we obtain a Yang-Mills theory with a gauge group G whose Lie algebra is $\mathfrak{g}$. If we resolve the singularity by replacing $\mathbb{C}^2/\Gamma$ with X_t, the symmetry group is broken down to $U(1)^{\mathrm{rank}(\mathfrak{g})}$. Physicists have argued that this general picture is correct if M^7 is one of the following:

- $(\mathbb{C}^2 \times T^3)/\Gamma$ for certain groups Γ such that the quotient has ADE-singularities [1].
- $S \times \mathbb{R}^3$ where S is a K3 surface with ADE-singularities [3].
- $X \times S^1$ where X is a Calabi-Yau threefold with ADE-singularities [32].
- $T^7/\mathbb{Z}_2^3$ where $\mathbb{Z}_2^3$ acts on T^7 such that we obtain the torus quotient that we discuss at the beginning of Section 5.3.2 [32].

However, there are some difficulties that have to be taken into account.

- Let M^7 be a G_2-orbifold with an ADE-singularity along an associative submanifold A. We denote the rank of the Lie algebra that is associated to the singularity by r. If the holonomy of M^7 is the full group G_2, M^7 cannot locally be written as $X \times U$ where X is a complex orbifold and U is an open subset of $\mathbb{R}^{7-\dim X}$. Therefore, we cannot use arguments from complex geometry and it is not a priori clear that a family of smooth

G_2-manifolds M_t^7 exists that converges to M^7 in such a way that r $\mathbb{CP}^1$-bundles over A collapse fiberwise. In fact, one step of the proposed construction in [40], which is in fact the most difficult step, is to carry out this procedure for an A_1-singularity along a non-flat associative submanifold.

- The global geometry of A may influence the physical picture, too. For example, if we move along a curve in A, the holonomy of M^7 may induce non-trivial monodromies acting on the normal space of A. These are assumed to break the gauge symmetry to a smaller group [1]. In [32] a second torus quotient of type $T^7/\mathbb{Z}_2^3$ is considered that has A_1-singularities not only along submanifolds of type T^3 but also along $T^3/\mathbb{Z}_2$. It is not entirely clear from a physical point of view what the gauge group of the 4-dimensional theory that we obtain in the low-energy limit is. Moreover, the A_1-singularities along $T^3/\mathbb{Z}_2$ admit two kinds of resolutions that yield different values of b^2 and thus different gauge groups.

Since this thesis is primarily about mathematics and not physics, we are rather cautious about these issues. When we construct G_2-orbifolds with ADE-singularities, we will describe the singular locus but we make no predictions about the gauge group. The only exception is the torus quotient $T^7/\mathbb{Z}_2^3$ at the beginning of Section 5.3 since it is also discussed in [32].

Chapter 5

G_2-orbifolds with ADE-singularities

5.1 Quotients of the product of a K3 surface and torus

Let S be a K3 surface with a hyper-Kähler metric and let $T^3 = \mathbb{R}^3/\Lambda$, where $\Lambda \subset \mathbb{R}^3$ is a lattice of rank 3, be a torus with a flat metric. The metric on the product $S \times T^3$ has holonomy $Sp(1) \subset G_2$ and therefore $S \times T^3$ is a G_2-manifold or, if S is singular, a G_2-orbifold. In this section, we consider quotients $(S \times T^3)/\Gamma$ where Γ is a finite group that preserves the G_2-structure and leaves no non-trivial subspace of the tangent space invariant. Although the holonomy is $Sp(1) \rtimes \Delta$, where Δ is discrete, rather than G_2, there exists exactly on parallel spinor on $(S \times T^3)/\Gamma$. Therefore, we nevertheless obtain a four-dimensional field theory with $\mathcal{N} = 1$ by compactifying M-theory on $(S \times T^3)/\Gamma$. Quotients of this kind have been investigated in [57] and the material of this section is a continuation of that work. Let x_1, x_2, and x_3 be coordinates on T^3 such that $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$ is an orthonormal frame and let $\omega_1, \omega_2, \omega_3$ be the three Kähler forms on S. The 3-form

$$\phi := \omega_1 \wedge dx^1 + \omega_2 \wedge dx^2 + \omega_3 \wedge dx^3 + dx^1 \wedge dx^2 \wedge dx^3$$

is a parallel G_2-form whose associated metric is the product metric on $S \times T^3$. Its Hodge-dual is

$$*\phi = \mathrm{vol}_S + \omega_1 \wedge dx^2 \wedge dx^3 + \omega_2 \wedge dx^3 \wedge dx^1 + \omega_3 \wedge dx^1 \wedge dx^2 \ ,$$

where $\mathrm{vol}_S = \omega_i \wedge \omega_i$ with $i \in \{1, 2, 3\}$ is the volume form of S. Let Γ be a group that acts isometrically and orientation-preserving on T^3. The action of any $\gamma \in \Gamma$ can be written as

$$x\Lambda \mapsto (A^\gamma x + v^\gamma)\Lambda \ ,$$

where $v^\gamma \in \mathbb{R}^3$ and $A^\gamma = (A^\gamma_{ij})_{i,j=1,2,3} \in SO(3)$. Let $GL(\Lambda)$ be the subgroup of $GL(3,\mathbb{R})$ that preserves Λ. γ defines a well-defined map $T^3 \to T^3$ if and only if $A^\gamma \in SO(3) \cap GL(\Lambda)$. We assume that for any $\gamma \in \Gamma$ there exists an isometry of S whose pull-back acts on the three Kähler forms as

$$\omega_i \mapsto \sum_{j=1}^{3} A^\gamma_{ji}\omega_j \ .$$

In this situation, we can extend the action of Γ on T^3 to an action on $S \times T^3$ that preserves the G_2-structure. It is necessary that A^γ is an element of $SO(3)$ rather than $O(3)$. Otherwise, Γ would not preserve the summand $dx^1 \wedge dx^2 \wedge dx^3$ of ϕ. If S is non-singular and Γ acts freely on $S \times T^3$, the quotient $(S \times T^3)/\Gamma$ is a manifold. In order to obtain a G_2-orbifold with ADE-singularities, there are three possibilities:

1. S is singular, Γ acts freely.
2. S is non-singular, Γ does not act freely.
3. S is singular, Γ does not act freely.

We focus on the first case, but we also discuss the other cases. Let us assume that Γ acts freely on $S \times T^3$. The action of a $\gamma \in \Gamma$ has a fixed point if and only if the projections of the action to S and to T^3 both have a fixed point. If the action of γ on T^3 has a fixed point and Γ acts freely, the action of γ on S has to be fixed point free. Since there are only few automorphisms of K3 surfaces without fixed points, we do not consider this case further and assume from now on that the action of Γ on T^3 is free. A quotient N of a torus by an isometric free group action is called a compact Euclidean space form. In dimension 3, they are classified by Hantzsche and Wendt [28]. Although we do not work with the space forms themselves, their classification yields the group actions on the torus that we need. In the case where the group Γ that acts on T^3 preserves the orientation there are the following 6 space forms.

- N is a 3-dimensional torus $\mathbb{R}^3/\Lambda$, where Λ is a lattice, or equivalently, Γ is trivial.
- Λ is a direct sum $\Lambda_1 \oplus \Lambda_2$, where Λ_1 is generated by a translation and Λ_2 is generated by two translations. The torus splits as $\mathbb{R}/\Lambda_1 \times \mathbb{R}^2/\Gamma$. We introduce coordinates x^1 on the first factor and x^2, x^3 on the second factor. Moreover, we identify the circle $\mathbb{R}/\Lambda_1$ with $\mathbb{R}/\mathbb{Z}$. Λ_2 shall admit an isometry α' of order $n \in \{2,3,4,6\}$. We define $\alpha : T^3 \to T^3$ by $\alpha(x^1,x^2,x^3) := (x^1 + \frac{1}{n}, \alpha'(x^2,x^3))$ and N as $(\mathbb{R}^3/\Lambda)/\langle\alpha\rangle$.
- $N = (\mathbb{R}^3/\Lambda)/\Gamma$ where Γ is isomorphic to $\mathbb{Z}_2^2$. Let (b_1, b_2, b_3) be a basis of Λ. Γ can be defined as the group that is generated by

$$
\begin{array}{rcl}
\alpha_1((\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) + \Lambda) & := & (\frac{1}{2} + \lambda_1) b_1 - \lambda_2 b_2 + (\frac{1}{2} - \lambda_3) b_3 + \Lambda \\
\alpha_2((\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) + \Lambda) & := & -\lambda_1 b_1 + (\frac{1}{2} + \lambda_2) b_2 - \lambda_3 b_3 + \Lambda
\end{array}
\tag{5.1}
$$

For reasons of simplicity, we assume from now on that $\Lambda = \mathbb{Z}^3$ and that b_i is the standard unit vector e_i.

If N is a space form that is not of type $T^3/\mathbb{Z}_2^2$, we can split of a circle, i.e. $N = N' \times S^1$, where N' is a two-dimensional manifold. If the group Γ that acts on $S \times T^3$ does not act in this way on the torus factor, we can also split of a circle from $(S \times T^3)/\Gamma$ and its holonomy therefore does not act irreducibly on the tangent space. Since we are mainly interested in the case where the holonomy acts irreducibly, we assume from now on that Γ is isomorphic to $\mathbb{Z}_2^2$. The holonomy of the quotient orbifolds will be $Sp(1) \rtimes \mathbb{Z}_2^2$. Since the generators of Γ shall act on T^3 as in (5.1), it follows that the pull-backs of α_1 and α_2 have to act on the Kähler forms on S as:

$$
\begin{array}{ccc}
\alpha_1^* \omega_1 = \omega_1 & \alpha_1^* \omega_2 = -\omega_2 & \alpha_1^* \omega_3 = -\omega_3 \\
\alpha_2^* \omega_1 = -\omega_1 & \alpha_2^* \omega_2 = \omega_2 & \alpha_2^* \omega_3 = -\omega_3
\end{array}
\tag{5.2}
$$

In other words, α_1 acts on S as a non-symplectic involution, which is holomorphic with respect to the complex structure with period point $[\omega_2] + i[\omega_3]$. Analogously, α_2 is a non-symplectic involution for the complex structure with period point $[\omega_3] + i[\omega_1]$. A pair of such non-symplectic involutions can be constructed as follows. Let ψ_1 and ψ_2 be involutions of the K3 lattice L with fixed lattices L^{ψ_1} and L^{ψ_2}. ψ_1 and ψ_2 shall be chosen in such a way that each of the three spaces

$$
V_1 := L_{\mathbb{R}}^{\psi_1} \cap \left(L_{\mathbb{R}}^{\psi_2} \right)^{\perp}, \quad V_2 := \left(L_{\mathbb{R}}^{\psi_1} \right)^{\perp} \cap L_{\mathbb{R}}^{\psi_2}, \quad V_3 := \left(L_{\mathbb{R}}^{\psi_1} \oplus L_{\mathbb{R}}^{\psi_2} \right)^{\perp}
$$

contains a positive element $v_i \in V_i$. By our definition of the V_i, the v_i are pairwise orthogonal. Without loss of generality, we assume that $v_i^2 = 1$. Lemma 3.3.20 guarantees that there exist the following objects:

- A K3 surface S with a marking $\phi : H^2(S, \mathbb{Z}) \to L$,
- a hyper-Kähler structure $(g, \omega_1, \omega_2, \omega_3)$ on S such that $[\omega_i] = \phi^{-1}(v_i)$ and
- isometric involutions ρ_1 and ρ_2 of S such that $\phi \circ \rho_i^* \circ \phi^{-1} = \psi_i$.

In this situation, ρ_1 and ρ_2 act on the Kähler forms as in (5.2). We define $\alpha_1, \alpha_2 : T^3 \to T^3$ as in equation (5.1). The group that is generated by the $\rho_i \times \alpha_i$ with $i = 1, 2$ acts freely on

$S \times T^3$ and preserves the G_2-structure. If S has ADE-singularities, the quotient $(S \times T^3)/\Gamma$ has also ADE-singularities along one or more copies of $T^3/\langle \alpha_1, \alpha_2 \rangle$.

We determine the Betti numbers of the orbifolds $M := (S \times T^3)/\Gamma$ that we obtain by this method. First, we assume that S is smooth. After that, we show how the calculation can be modified for singular K3 surfaces. The cohomology of M with real coefficients simply is the Γ-invariant part of $H^*(S \times T^3, \mathbb{R})$. The first de Rham cohomology of $S \times T^3$ is spanned by the harmonic forms dx^1, dx^2 and dx^3 since $b^1(S) = 0$. We have chosen α_1 and α_2 in such a way that no coordinate direction is preserved by both maps. Therefore, we have $b^1(M) = 0$. $H^2(S \times T^3, \mathbb{R})$ is spanned by the forms $dx^1 \wedge dx^2$, $dx^1 \wedge dx^3$ and $dx^2 \wedge dx^3$ on the torus and by $H^2(S, \mathbb{R})$. None of the forms on the torus is Γ-invariant and the invariant part of $H^2(S, \mathbb{R})$ has the same dimension as $V_0 := L_{\mathbb{R}}^{\psi_1} \cap L_{\mathbb{R}}^{\psi_2}$. The Künneth formula yields

$$H^3(S\times T^3, \mathbb{R}) \cong H^3(T^3, \mathbb{R}) \oplus \big(H^2(T^3, \mathbb{R}) \otimes H^1(S, \mathbb{R})\big) \oplus \big(H^1(T^3, \mathbb{R}) \otimes H^2(S, \mathbb{R})\big) \oplus H^3(S, \mathbb{R}).$$

With $b^1(S) = b^3(S) = 0$ this can be simplified to

$$H^3(S \times T^3, \mathbb{R}) \cong H^3(T^3, \mathbb{R}) \oplus \left(H^1(T^3, \mathbb{R}) \otimes H^2(S, \mathbb{R})\right) .$$

$\rho_1 \times \alpha_1$ and $\rho_2 \times \alpha_2$ act linearly on the spaces $H^1(T^3, \mathbb{R})$ and $H^2(S, \mathbb{R})$ with eigenvalues 1 or -1. We denote the intersection of the eigenspace of $\rho_1 \times \alpha_1$ with eigenvalue ϵ_1 and the eigenspace of $\rho_2 \times \alpha_2$ with eigenvalue ϵ_2 by the superscript ϵ_1, ϵ_2. Since $H^1(T^3, \mathbb{R})^{1,1} = \{0\}$, we have

$$\begin{aligned}\left(H^1(T^3, \mathbb{R}) \otimes H^2(S, \mathbb{R})\right)^{\Gamma} = \; & H^1(T^3, \mathbb{R})^{1,-1} \otimes H^2(S, \mathbb{R})^{1,-1} \\ & \oplus H^1(T^3, \mathbb{R})^{-1,1} \otimes H^2(S, \mathbb{R})^{-1,1} \\ & \oplus H^1(T^3, \mathbb{R})^{-1,-1} \otimes H^2(S, \mathbb{R})^{-1,-1}\end{aligned}$$

where the superscript Γ denotes the subspace of all vectors that are invariant under Γ. Since the spaces $H^1(T^3, \mathbb{R})^{\epsilon_1, \epsilon_2}$ have dimension 1 if $(\epsilon_1, \epsilon_2) \neq (1, 1)$, $\left(H^1(T^3, \mathbb{R}) \otimes H^2(S, \mathbb{R})\right)^{\Gamma}$ has dimension $\dim V_1 + \dim V_2 + \dim V_3$. All in all, we obtain the following Betti numbers:

$$b^1(M) = 0 \tag{5.3}$$
$$b^2(M) = \dim V_0 \tag{5.4}$$
$$b^3(M) = 1 + \dim V_1 + \dim V_2 + \dim V_3 = 23 - \dim V_0 \tag{5.5}$$

Next, we assume that S carries a hyper-Kähler metric with ADE-singularities that we obtain by deforming a smooth hyper-Kähler metric on a K3 surface such that the area of

n complex curves shrinks to zero. This procedure yields in the Gromov-Hausdorff limit an orbifold whose Hodge number $h^{1,1}(S)$ decreases by n, i.e. $h^{1,1}(S) = 20 - n$. The other Hodge numbers remain the same. We assume that the cohomology class of any curve with vanishing volume is either contained in V_0 or in its orthogonal complement. We denote the number of cohomology classes in V_0 by n_0 and obtain:

$$b^1(M) = 0 \tag{5.6}$$
$$b^2(M) = \dim V_0 - n_0 \tag{5.7}$$
$$b^3(M) = 23 - \dim V_0 - n + n_0 \tag{5.8}$$

We assume that there exists a cohomology class x of a curve with vanishing volume that is not contained in $V_0 \cup V_0^{\perp}$. Since ρ_1 and ρ_2 are isometries, they map singular points to singular points. Let $\mathcal{O}$ be the orbit of x, e.g. $\mathcal{O} = \{x, \rho_1^* x, \rho_2^* x, \rho_1^* \rho_2^* x\}$ if neither ρ_1 nor ρ_2 leave x invariant. $\mathcal{O}$ spans a four-dimensional subspace W of $H^2(S,\mathbb{R})$. There is a one-dimensional subspace of W that is invariant under ρ_1 and ρ_2. Therefore, we have to subtract 1 from $b^2(M)$ and 3 from $b^3(M)$. If $\mathcal{O}$ contains two elements, we obtain analogous relations.

In order to construct examples of quotients of type $(S \times T^3)/\Gamma$, we take a look at the maps $\rho_1^i : 2(-E_8) \to 2(-E_8)$ with $i = 1, \ldots, 4$ and $\rho_2^j : 3H \to 3H$ with $j = 1, 2, 3$ that we have defined in Section 3.5. Since ρ_1^1 is the identity and ρ_1^3 is minus the identity, all maps ρ_1^i commute with each other with the only possible exception ρ_1^2 and ρ_1^4. We recall that $\rho_1^2(x_1, x_2) = (-x_1, x_2)$ and $\rho_1^4(x_1, x_2) = (x_2, x_1)$ for all $x_1 \in (-E_8)_1$ and $x_2 \in (-E_8)_2$. Therefore, ρ_1^2 and ρ_1^4 do not commute. We redefine ρ_2^1 as $\rho_2^1(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$. Up to a permutation of the signs, this is the same as the ρ_2^1 from Section 3.5. This redefinition makes it easier for us to define the hyper-Kähler structure on S with the desired properties. By a short calculation, we see that all ρ_2^j commute pairwise. Let $\psi_{ij} : L \to L$ be defined as the map $\rho_2^j \oplus \rho_1^i$. The 27 pairs $(\psi_{ij}, \psi_{i'j'})$ with $1 \leq i \leq i' \leq 4$, $(i, i') \neq (2, 4)$ and $1 \leq j < j' \leq 3$ consist of two commuting lattice isometries of L. Let (v_1^i, v_2^i) be the standard basis of H_i. We define the following three elements of $L_{\mathbb{R}}$.

- If $(j, j') = (1, 2)$, we define $x := \frac{1}{\sqrt{2}}(v_1^3 + v_2^3)$, $y := \frac{1}{2}(v_1^1 + v_2^1 + v_1^2 + v_2^2)$ and $z := \frac{1}{2}(v_1^1 + v_2^1 - v_1^2 - v_2^2)$.
- If $(j, j') = (1, 3)$, we define $x := \frac{1}{\sqrt{2}}(v_1^3 + v_2^3)$, $y := \frac{1}{\sqrt{2}}(v_1^1 + v_2^1)$ and $z := \frac{1}{\sqrt{2}}(v_1^2 + v_2^2)$.

x, y and z are pairwise orthogonal and satisfy $x^2 = y^2 = z^2 = 1$. Therefore, the triple (x, y, z) defines a marked K3 surface S with a hyper-Kähler structure. As usual, the three Kähler forms ω_1, ω_2, ω_3 can be chosen such that $\phi([\omega_i]) = x_i$, where $\phi : H^2(S, \mathbb{Z}) \to L$ denotes the marking. The maps ψ_{ij} and $\psi_{i'j'}$ act on x, y and z as prescribed by (5.2). We can deduce with help of Lemma 3.3.20 that there exist isometries ρ_1 and ρ_2 of S such that

$\phi \circ \rho_1^* \circ \phi^{-1} = \psi_{ij}$ and $\phi \circ \rho_2^* \circ \phi^{-1} = \psi_{i'j'}$. In particular, ρ_1 and ρ_2 act on the Kähler forms as in (5.2), too. The commutator $\rho_1^{-1} \circ \rho_2^{-1} \circ \rho_1 \circ \rho_2$ as well as the maps ρ_1^2 and ρ_2^2 act as the identity on L and thus are the identity map. Therefore, ρ_1 and ρ_2 are commuting involutions of S that are non-symplectic with respect to different complex structures. All in all, we obtain a G_2-orbifold M by dividing $S \times T^3$ by the group that is generated by $\rho_1 \times \alpha_1$ and $\rho_2 \times \alpha_2$. By a short calculation, we see that if $(j, j') = (2, 3)$ a triple (x, y, z) such that ψ_{ij} and $\psi_{i'j'}$ act as desired does not exist. In the case $(j, j') = (1, 2)$ the triple does only exist since we have redefined ρ_2^1. The number of possible pairs $(\psi_{ij}, \psi_{i'j'})$ therefore reduces to 18. The K3 surfaces S are singular. The orthogonal complement of $\mathrm{span}_{\mathbb{Z}}(x, y, z)$ is for all values of (i, j) and (i', j') given by

$$\mathrm{span}_{\mathbb{Z}}(v_1^1 - v_2^1, v_1^2 - v_2^2, v_1^3 - v_2^3) \oplus 2(-E_8)\,.$$

This is precisely the lattice K' from the proof of Theorem 3.5.10. Therefore, S has 3 singular points with A_1-singularities and 2 singular points with E_8-singularities. Furthermore, M has A_1-singularities along 3 copies of $T^3/\mathbb{Z}_2^2$ and E_8-singularities along 2 copies of $T^3/\mathbb{Z}_2^2$. Since there are no more $\mathbb{CP}^1$s in S left that can collapse, we call this singular set the *maximal singularity of* M.

We want to know if it is possible to obtain by our construction G_2-orbifolds with smaller singularities. $L_{\mathbb{R}}$ decomposes into the 4 spaces $V_0, \ldots, V_4$. If $i, i' \neq 4$ and $j, j' \neq 2$, any element of the standard basis $(w_1, \ldots, w_{19})$ of K' is contained in one of the V_k. We choose an arbitrary subfamily of $(w_1, \ldots, w_{19})$ and distribute it among the 4 spaces. We obtain 4 families $(w_{j_1^{\epsilon_1,\epsilon_2}}, \ldots, w_{j_{k_{\epsilon_1,\epsilon_2}}^{\epsilon_1,\epsilon_2}})$ with $\epsilon_1, \epsilon_2 \in \{1, -1\}$. The first superscript denotes the eigenvalue ϵ_1 of $w_{j_l^{\epsilon_1,\epsilon_2}}$ with respect to ψ_{ij} and the second superscript the eigenvalue with respect to $\psi_{i'j'}$. We redefine x, y and z as:

$$\begin{aligned}
x' &= x + \textstyle\sum_{l=1}^{k_{1,-1}} \alpha_l^{1,-1} w_{j_l^{1,-1}} \\
y' &= y + \textstyle\sum_{l=1}^{k_{-1,1}} \alpha_l^{-1,1} w_{j_l^{-1,1}} \\
z' &= z + \textstyle\sum_{l=1}^{k_{-1,-1}} \alpha_l^{-1,-1} w_{j_l^{-1,-1}}
\end{aligned}$$

As in the proof of Corollary 3.5.11, the $\alpha^{\epsilon_1,\epsilon_2}$ are chosen such that they are $\mathbb{Q}$-linearly independent and $x'^2 = y'^2 = z'^2 > 0$. Since $x' \in V_1$, $y' \in V_2$ and $z \in V_3$, (x', y', z') defines a hyper-Kähler structure on a K3 surface that satisfies the relations (5.2). The Dynkin diagram that describes the singularities is obtained from $3A_1 \cup 2E_8$ by removing the nodes that correspond to the $w_{j_l^{1,-1}}$, $w_{j_l^{-1,1}}$ and $w_{j_l^{-1,-1}}$. Since x', y' and z' have to be orthogonal to V_0 in order to satisfy (5.2), there is no way to get rid of the nodes that are represented by the $w_{j_l^{1,1}}$. This means that there is a *minimal singularity* that is described by the set of all $w_l \in V_0$ and cannot be resolved. All in all, we have proven that the singular locus of S can be chosen such that it is described by an arbitrary subdiagram of $3A_1 \cup 2E_8$ that contains the Dynkin diagram of all $w_l \in V_0$. In the case where there exists a w_l that is not contained in any of the V_k, the above proof can be easily modified. For example, we assume that there exists a w_l with $\psi_{i'j'}(w_l) = -w_l$ but $\psi_{ij}(w_l) \notin \{-w_l, w_l\}$. We take a look at the definition of the maps ρ_2^j and ρ_1^i and see that there exists an l' with $\psi_{ij}(w_l) = w_{l'}$. We can add a term of type $\beta(w_l + w_{l'})$ in the definition of x' or a term of type $\beta(w_l - w_{l'})$ in the definition of z'. By this method, we see that in the general case there exists a minimal singularity that is described by the Dynkin diagram that belongs to the root system

$$D_0 := \{d \in V_0 \cap L | d^2 = -2\} .$$

The description of all possible "intermediate" singularities would be rather complicated and would involve different cases. Therefore, we restrict ourselves to the maximal and minimal singularity.

We compute the Betti numbers of M in the case where the singularities of S are minimal in the sense that we have explained above. The vector space V_0 splits into $V_0' \oplus V_0''$ where $V_0' = V_0 \cap (3H \otimes \mathbb{R})$ and $V_0'' = V_0 \cap (2(-E_8) \otimes \mathbb{R})$. By taking a look at the definitions of the ρ_2^j we see that V_0' is always trivial. Depending on the values of j and j' we obtain for V_0'' and the Dynkin diagram of the minimal singularity:

(j, j')	V_0''	$\dim V_0$	D_0
$(1,1)$	$2(-E_8) \otimes \mathbb{R}$	16	$2E_8$
$(1,2)$	$(-E_8)_2 \otimes \mathbb{R}$	8	E_8
$(1,3)$	$\{0\}$	0	$\emptyset$
$(1,4)$	$(-E_8)(2) \otimes \mathbb{R}$	8	$\emptyset$
$(2,2)$	$(-E_8)_2 \otimes \mathbb{R}$	8	E_8
$(2,3)$	$\{0\}$	0	$\emptyset$
$(3,3)$	$\{0\}$	0	$\emptyset$
$(3,4)$	$\{0\}$	0	$\emptyset$
$(4,4)$	$(-E_8)(2) \otimes \mathbb{R}$	8	$\emptyset$

In the above table, $(-E_8)(2)$ denotes the diagonally embedded lattice $\{(x, x) \in 2(-E_8) | x \in -E_8\}$. We denote the G_2-orbifold that we obtain from a K3 surface with the minimal singularity by M_{min} and the orbifold that we obtain from the K3 surface with 2 E_8- and 3 A_1-singularities by M_{max}. We obtain for the Betti numbers:

(j,j')	$b^2(M_{min})$	$b^3(M_{min})$	$b^2(M_{max})$	$b^3(M_{max})$
$(1,1)$	0	7	0	4
$(1,2)$	0	15	0	4
$(1,3)$	0	23	0	4
$(1,4)$	8	15	0	4
$(2,2)$	0	15	0	4
$(2,3)$	0	23	0	4
$(3,3)$	0	23	0	4
$(3,4)$	0	23	0	4
$(4,4)$	8	15	0	4

In particular, b^2 and b^3 are independent of the choice of i and i'.

Remark 5.1.1. It is possible to modify our construction. Instead of α_1 and α_2 we consider the maps

$$\begin{aligned}\beta_1((x_1,x_2,x_3)+\mathbb{Z}^3) &:= (x_1+\tfrac{1}{4},-x_2+\tfrac{1}{4},-x_3)+\mathbb{Z}^3\\ \beta_2((x_1,x_2,x_3)+\mathbb{Z}^3) &:= (-x_1+\tfrac{1}{4},x_2+\tfrac{1}{2},-x_3+\tfrac{1}{2})+\mathbb{Z}^3\end{aligned}$$

Since the signs in front of the x_i are the same as in (5.1), the quotient $M=(S\times T^3)/\Gamma$, where Γ is generated by the $\rho_i\times\beta_i$ with $i=1,2$ and the ρ_i satisfy (5.2), is a G_2-orbifold. The group that is generated by β_1 and β_2 is isomorphic to $\mathbb{Z}_4\rtimes\mathbb{Z}_2$ and acts freely on T^3. Therefore, Γ is isomorphic to $\mathbb{Z}_4\rtimes\mathbb{Z}_2$, too, and the only singularities of M are induced by the singularities of S. Since

$$\beta_1^2((x_1,x_2,x_3)+\mathbb{Z}^3) := (x_1+\tfrac{1}{2},x_2,x_3)+\mathbb{Z}^3$$

the quotient $T^3/\langle\beta_1,\beta_2\rangle$ can be interpreted as the quotient of the torus $\mathbb{R}^3/\Lambda$ with $\Lambda=\mathrm{span}_{\mathbb{Z}}(\frac{1}{2}e_1,e_2,e_3)$ by a group that is isomorphic to $\mathbb{Z}_2^2$. We could combine any pair (ρ_1,ρ_2) of involutions of a K3 surface that satisfies (5.2) with β_1 and β_2 to obtain a G_2-orbifold, but we do not carry out this procedure explicitly for reasons of brevity.

At the end of this section, we show how to construct G_2-orbifolds of type $(S\times T^3)/\Gamma$ with ADE-singularities in the case where Γ does not act freely. First, we assume that S is smooth and after that we shortly touch the case where S has singularities. These are the second and the third case form page 61. The easiest way to obtain a non-free action is to modify the definition of α_1 such that

$$\alpha_1((x_1,x_2,x_3)+\mathbb{Z}^3) := (x_1,-x_2,\tfrac{1}{2}-x_3)+\mathbb{Z}^3$$

The fixed point set of α_1 consists of four circles, namely:

$$\{(x, \tfrac{1}{2}\epsilon_1, \tfrac{1}{4} + \tfrac{1}{2}\epsilon_2) + \mathbb{Z}^3 | x \in \mathbb{R}\}$$

with $\epsilon_1, \epsilon_2 \in \{0, 1\}$. The fixed point set of $\rho_1 \times \alpha_1$ is the Cartesian product of the above four circles and the fixed point set of the non-symplectic involution ρ_1. According to Theorem 3.5.8, the fixed locus of ρ_1 consists of disjoint complex curves. The differential of ρ_1 at a fixed point can be written as $\mathrm{diag}(1, -1) \in \mathbb{C}^{2\times 2}$. Since α_1 acts as -1 on the normal space of a circle of fixed points, $(S \times T^3)/\langle \rho_1 \times \alpha_1\rangle$ has A_1-singularities along several submanifolds that can be described as the product of a circle and a complex curve. The involution α_2 has no fixed points and maps each of the 4 circles to another one. Therefore, dividing $(S \times T^3)/\langle \rho_1 \times \alpha_1\rangle$ by $\rho_2 \times \alpha_2$ halves the number of the connected components of the singular locus. As in Theorem 3.5.8, let $C_g \cup E_1 \cup \ldots \cup E_k$, where the E_i are are rational curves and C_g is a curve that may have a higher genus, be the fixed locus of ρ_1. If S is smooth, the singular locus of M can therefore be written as

$$\left(\{(x, 0, \tfrac{1}{4}) + \mathbb{Z}^3 | x \in \mathbb{R}\}\langle \alpha_1, \alpha_2\rangle \cup \{(x, 0, \tfrac{3}{4}) + \mathbb{Z}^3 | x \in \mathbb{R}\}\langle \alpha_1, \alpha_2\rangle\right) \times (C_g \cup E_1 \cup \ldots \cup E_k) \ . \tag{5.9}$$

We show how to construct a large number of smooth K3 surfaces with two commuting involutions ρ_1 and ρ_2 that satisfy (5.2). Let (r_1, a_1, δ_1) and (r_2, a_2, δ_2) be two triples such that for each (r_i, a_i, δ_i) there exists a K3 surface with a non-symplectic involution whose invariants are (r_i, a_i, δ_i). We denote the lattice $L(r_1, a_1, \delta_1)$ by L_1 and $L(r_2, a_2, \delta_2)$ by L_2. Theorem 3.1.5 guarantees that $L_1 \oplus L_2$ can be primitively embedded into L if $2(r_1 + r_2) \leq 22$ or $r_1 + r_2 + a_1 + a_2 < 22$. We define ρ_1 as an involution of a marked K3 surface S that acts as the identity on L_1 and as -1 on $L_1^\perp$ and define ρ_2 as an analogous involution of S. Since L_1 and L_2 are orthogonally embedded, we have

$$V_0 = \{0\}\ , \quad V_1 = L_1 \otimes \mathbb{R}\ , \quad V_2 = L_2 \otimes \mathbb{R}\ , \quad V_3 = ((L_1 \oplus L_2) \otimes \mathbb{R})^\perp\ .$$

The lattice $L_1 \oplus L_2$ has signature $(2, r_1 + r_2 - 2)$. Let $x_i \in V_i$ with $i = 1, 2, 3$ be positive elements. By construction, the x_i are pairwise orthogonal. We rescale them such that $x_1^2 = x_2^2 = x_3^2$. Since $L_1 \oplus L_2$ is a primitive sublattice of L, there exists a basis $(w_1, \ldots, w_{22})$ of L such that $(w_1, \ldots, w_{r_1})$ is a basis of L_1, $(w_{r_1+1}, \ldots, w_{r_1+r_2})$ is a basis of L_2 and $(w_{r_1+r_2+1}, \ldots, w_{22})$ is a basis of $(V_1 \oplus V_2)^\perp$. We redefine x_1, x_2 and x_3 as

$$\begin{array}{rcl} x_1' & := & x_1 + \sum_{i=1}^{r_1} \alpha_i w_i \\ x_2' & := & x_2 + \sum_{i=r_1+1}^{r_1+r_2} \alpha_i w_i \\ x_3' & := & x_3 + \sum_{i=r_1+r_2}^{22} \alpha_i w_i \end{array}$$

As usual, we choose the α_i such that $(1, \alpha_1, \ldots, \alpha_{22})$ is $\mathbb{Q}$-linearly independent. Moreover, they should be chosen such that ${x_1'}^2 = {x_2'}^2 = {x_3'}^2 > 0$. We see that $\mathrm{span}(x_1, x_2, x_3)^\perp$ contains no $d \in L$ with $d^2 = -2$. Therefore, the triple (x_1, x_2, x_3) defines a smooth K3 surface with a marking $\phi : H^2(S, \mathbb{Z}) \to L$ and a hyper-Kähler structure such that $\phi([\omega_i]) = x_i$ for $i = 1, 2, 3$. We have defined the x_i in such a way that the maps ρ_i satisfy the relations (5.2). Any pair of triples (r_i, a_i, δ_i) with $2(r_1 + r_2) \leq 22$ or $r_1 + r_2 + a_1 + a_2 < 22$ therefore yields a quotient $M := (S \times T^3)/\langle \rho_1 \times \alpha_1, \rho_2 \times \alpha_2 \rangle$ with A_1-singularities along the set (5.9) where $C_g \cup E_1 \cup \ldots \cup E_k$ is the fixed locus of ρ_1. We compute the Betti numbers of M as in the case where Γ acts freely by counting the Γ-invariant forms of $S \times T^3$. Since $\dim V_0 = 0$, we obtain the Betti numbers

$$b^1(M) = 0\,, \quad b^2(M) = 0\,, \quad b^3(M) = 23\,.$$

In particular, M is 2-connected and the Betti numbers are independent of the choice of (r_1, a_1, δ_1) and (r_2, a_2, δ_2). As a by-product of our considerations, we have proven the following lemma.

Lemma 5.1.2. *Let (r_i, a_i, δ_i) with $i = 1, 2$ be two triples such that K3 surfaces with non-symplectic involutions with fixed lattices $L_i := L(r_i, a_i, \delta_i)$ exist. Moreover, there shall exist a primitive embedding of $L_1 \oplus L_2$ into the K3 lattice L. Then there exists a single marked K3 surface S with a smooth hyper-Kähler structure and a pair of commuting involutions ρ_i with fixed lattices L_i that act on the Kähler forms as described by (5.2).*

Finally, let $M := (S \times T^3)/\Gamma$ be constructed as above with help of the modified map α_1 and let S have singular points $p_i \notin C_g \cup E_1 \cup \ldots \cup E_k$. Each p_i yields an ADE-singularity along a suborbifold of type $T^3/\mathbb{Z}_2^2$. It is important that $p_i \notin C_g \cup E_1 \cup \ldots \cup E_k$. Otherwise, the suborbifolds in the singular locus of M that are induced by the fixed locus of α_1 and those that are induced by the singular points of S would intersect. At the intersection points, M would have singularities of type $\mathbb{C}^3/\Delta$ that are more complicated than ADE-singularities, which means that there exists no one-dimensional complex subspace of the tangent space on which Δ acts trivially. In the following section, we present an example of a singular K3 surface with a non-symplectic involution where we see how the case $p_i \in C_g \cup E_1 \cup \ldots \cup E_k$ can be avoided. Therefore, we do not construct explicit examples of quotient orbifolds of the above type at this point.

5.2 Twisted connected sums with ADE-singularities

In this section, we modify the method for the construction of compact G_2-manifolds that was developed by Kovalev and Lee [44] such that we obtain examples of G_2-orbifolds with ADE-singularities. First, we review the construction from [44] and then we show how it can be generalized to G_2-orbifolds. Let S be a K3 surface and let $\rho : S \to S$ be a non-symplectic involution. Moreover, let $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a holomorphic involution that has two distinct fixed points z_1 and z_2. $\rho \times \psi$ generates a group of order 2 that acts on $S \times \mathbb{CP}^1$. The quotient $(S \times \mathbb{P}^1)/\langle \rho \times \psi \rangle$ will be denoted by Z. As we have described in Theorem 3.5.8, the fixed locus of ρ is a disjoint union of complex curves $C_g \cup E_1 \cup \ldots \cup E_k$, where C_g is a curve of genus g and $E_1, \ldots, E_k$ are rational curves. The singular set of Z thus consists of two copies of $C_g \cup E_1 \cup \ldots \cup E_k$. Any singularity is of type $\mathbb{C}^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

In other words, we have A_1-singularities along the curves in the singular locus. These curves can be blown up and we obtain a smooth complex threefold $\overline{W}$. Let $p : S \times \mathbb{CP}^1 \to Z$ be the quotient map and let $\pi : \overline{W} \to Z$ be the blow-up. Moreover, let $z \in \mathbb{CP}^1 \setminus \{z_1, z_2\}$ be arbitrary. $D := \pi^{-1}(p(S \times \{z\}))$ is a K3 surface and a subvariety of $\overline{W}$. We prove that D is an anti-canonical divisor of $\overline{W}$. The point $\{z\}$ is a divisor of $\mathbb{CP}^1$ that corresponds to the hyperplane line bundle $\mathcal{H}$. We denote the canonical bundle of a complex variety X by $\mathcal{K}_X$. Moreover, we denote the kth power of a complex line bundle $\mathcal{L}$ with respect to the tensor product by $\mathcal{L}^k$. $\mathcal{K}_{\mathbb{CP}^1}$ is isomorphic to $\mathcal{H}^{-2}$. Therefore, there exists a meromorphic 1-form α on $\mathbb{CP}^1$ that has a pole of second order at z and is holomorphic elsewhere. Let $\omega_J + i \cdot \omega_K$ be the holomorphic volume form on S. $\beta := \alpha \wedge (\omega_J + i \cdot \omega_K)$ is a meromorphic 3-form on $S \times \mathbb{P}^1$ with a pole of second order along $S \times \{z\}$. The section

$$\beta \otimes (\rho \times \psi)^* \beta$$

of $\mathcal{K}^2_{S \times \mathbb{P}^1}$ is $(\rho \times \psi)$-invariant since $\rho \times \psi$ is an involution. Therefore, there exists a unique meromorphic section γ of $\mathcal{K}^2_Z$ with

$$p^* \gamma = \beta \otimes (\rho \times \psi)^* \beta \, .$$

γ has a pole of second order along $p(S \times \{z\})$, $\pi^* \gamma$ has a pole of second order along D and both sections are holomorphic elsewhere. Let z_1, z_2, z_3 be complex coordinates on an open subset $U \subset \overline{W}$. There exists a unique meromorphic function f_U on U such that we have

$$\pi^* \gamma = f_U \cdot (dz^1 \wedge dz^2 \wedge dz^3)^{\otimes 2}$$

on U. The two 3-forms

$$\eta_{\pm, U} := \pm \sqrt{f_U} \cdot dz^1 \wedge dz^2 \wedge dz^3$$

are meromorphic on U and have a simple pole along $U \cap D$. Let $(U_\alpha)_{\alpha \in I}$ be an atlas of $\overline{W}$. We show that the forms $\eta_{\pm, U_\alpha}$ can be glued together to a globally defined form η. Since η

has a simple pole along D, it follows that D is an anti-canonical divisor of $\overline{W}$. For any α from the index set I we define two sets

$$\begin{aligned} X_\alpha :=& \{\sqrt{f_{U_\alpha}(z_1,z_2,z_3)} \cdot dz^1 \wedge dz^2 \wedge dz^3 | (z_1,z_2,z_3) \in U_\alpha\} \\ & \cup \{-\sqrt{f_{U_\alpha}(z_1,z_2,z_3)} \cdot dz^1 \wedge dz^2 \wedge dz^3 | (z_1,z_2,z_3) \in U_\alpha\} \subset \mathcal{K}_{\overline{W}} \\ Y_\alpha :=& \{f_{U_\alpha}(z_1,z_2,z_3) \cdot (dz^1 \wedge dz^2 \wedge dz^3)^{\otimes 2} | (z_1,z_2,z_3) \in U_\alpha\} \subset \mathcal{K}^2_{\overline{W}} \end{aligned}$$

where $\mathcal{K}_{\overline{W}}$ and $\mathcal{K}^2_{\overline{W}}$ should be interpreted as the total spaces of the bundles. We add a point that represents $\infty \cdot dz^1 \wedge dz^2 \wedge dz^3$ to each fiber and make $\mathcal{K}_{\overline{W}}$ and $\mathcal{K}^2_{\overline{W}}$ into bundles with fiber $\mathbb{CP}^1$. This ensures that our definition makes sense if $U_\alpha \cap D$ is non-empty. We define the following submanifolds of $\mathcal{K}_{\overline{W}}$ and $\mathcal{K}^2_{\overline{W}}$:

$$X := \bigcup_{\alpha \in I} X_\alpha \quad \text{and} \quad Y := \bigcup_{\alpha \in I} Y_\alpha \, .$$

The map that sends $\pm\sqrt{f_{U_\alpha}(z_1,z_2,z_3)} \cdot dz^1 \wedge dz^2 \wedge dz^3$ to $f_{U_\alpha}(z_1,z_2,z_3) \cdot (dz^1 \wedge dz^2 \wedge dz^3)^{\otimes 2}$ defines a double cover $X \to Y$. Y is diffeomorphic to $\overline{W}$. The diffeomorphism is given by the restriction of the projection map $\mathcal{K}^2_{\overline{W}} \to \overline{W}$ to $Y \subset \mathcal{K}^2_{\overline{W}}$. In [44] it is shown that $\overline{W}$ is simply connected if ρ is not fixed-point free. Therefore, the covering space X has two connected components that are both diffeomorphic to $\overline{W}$. One of them can be used to define the global holomorphic 3-form η and we have finally shown that D is an anti-canonical divisor.

$S \times \{z\} \subset S \times \mathbb{CP}^1$ has a neighborhood that is biholomorphic to $S \times \{z \in \mathbb{C} | |z| < \epsilon\}$. In other words, $S \times \{z\}$ has trivial normal bundle. It is easy to see that $p(S \times \{z\}) \subset Z$ and $D \subset \overline{W}$ have trivial normal bundle, too.

We show that $\overline{W}$ admits a Kähler metric. $S \times \mathbb{P}^1$ is a Kähler manifold since any K3 surface is Kähler. Let g be a Kähler metric on $S \times \mathbb{P}^1$. $g' := g + (\rho \times \psi)^* g$ is a $(\rho \times \psi)$-invariant Riemannian metric. Moreover, it is Kähler since the set of all Kähler metrics is a cone. Since $S \times \mathbb{P}^1$ carries a $(\rho \times \psi)$-invariant Kähler metric, the quotient Z is Kähler, too. Let ω be a Kähler form on Z and let E be the exceptional divisor of the blow-up $\pi : \overline{W} \to Z$. There is a class in $H^{1,1}(\overline{W})$ that corresponds to E, that we denote by E, too. Moreover, there exists an $\epsilon > 0$ such that $\pi^*[\omega] - \epsilon E$ is a Kähler class (see [27, p.187] or [44, Proposition 4.1.]). All in all, the pair $(\overline{W}, D)$ satisfies all conditions from Theorem 4.4.3 and we have shown the following result.

Proposition 5.2.1. *(Proposition 5.1. in [44]) Let S be a K3 surface with a non-symplectic involution ρ that has invariants $(r,a,\delta) \neq (10,10,0)$. Moreover, let $(g,\omega_I,\omega_J,\omega_K)$ be a hyper-Kähler structure on S such that $\rho^*(\omega_J + i\omega_K) = -\omega_J - i\omega_K$ and $\rho^*\omega_I = \omega_I$. Finally, we define $\overline{W}$ and D as above. In this situation, $W := \overline{W} \setminus D$ admits an ACyl Ricci-flat Kähler metric whose holomorphic $(3,0)$-form is asymptotic to the form $(\omega_J + i\omega_K) \wedge (dt + id\theta)$*

on $D \times S^1 \times \mathbb{R}^{>0}$ and whose Kähler form is asymptotic to $\omega_I + dt \wedge d\theta$, where θ and t denote the usual coordinates on the circle S^1 and on $\mathbb{R}^{>0}$.

It is possible to calculate the values of the following invariants that determine the topology of the twisted connected sums.

Proposition 5.2.2. *(Proposition 4.3. in [44]) In the situation of the above proposition, the Hodge numbers of $\overline{W}$ are determined by*

- $h^{1,0}(\overline{W}) = h^{2,0}(\overline{W}) = h^{3,0}(\overline{W}) = 0,$
- $h^{1,1}(\overline{W}) = 3 + 2r - a,$
- $h^{1,2}(\overline{W}) = 22 - r - a.$

Moreover, the rank of the restriction map $\imath : H^2(\overline{W}, \mathbb{R}) \to H^2(D, \mathbb{R})$ is r and the kernel of the restriction of $\imath$ to $H^2(W, \mathbb{R})$ has dimension $2 + r - a$.

Kovalev and Lee [44] construct many examples of compact G_2-manifolds with help of the ACyl Calabi-Yau manifolds from Theorem 5.2.1. We describe one particular construction method in detail. Let S_i with $i \in \{1, 2\}$ be K3 surfaces with non-symplectic involutions ρ_i. We denote their fixed lattices by L_i and their invariants by (r_i, a_i, δ_i). Moreover, we choose hyper-Kähler structures $(g^i, \omega^i_1, \omega^i_2, \omega^i_3)$ with $i \in \{1, 2\}$ on the S_i. We construct ACyl Calabi-Yau manifolds W_1 and W_2 by the method that we have described above with (S_1, ρ_1) and (S_2, ρ_2) as input. Each W_i is asymptotic to $D_i \times S^1 \times \mathbb{R}^{>0}$ where D_i carries the same hyper-Kähler structure as S_i. Theorem 4.4.8 guarantees that $W_1 \times S^1$ and $W_2 \times S^1$ can be glued together to a twisted connected sum M that carries a metric with holonomy G_2 if S_1 and S_2 satisfy the matching condition.

We assume that $L_1 \oplus L_2$ can be primitively embedded into L. Let $\phi_i : H^2(S, \mathbb{Z}) \to L$ be markings such that $\phi_1(L_1)$ and $\phi_2(L_2)$ are orthogonal to each other and that $\phi_1(L_1) \oplus \phi_2(L_2)$ is a primitive sublattice of L. As usual, we identify $\phi_i(L_i)$ and L_i with each other. We denote $\phi_1([\omega^1_j]) \in L$ with $j \in \{1, 2, 3\}$ by x_j and $\phi_2([\omega^2_j])$ by y_j. Since ρ_1 and ρ_2 are non-symplectic, we have

$$
\begin{array}{ccc}
\rho_1^* x_1 = x_1 & \rho_1^* x_2 = -x_2 & \rho_1^* x_3 = -x_3 \\
\rho_2^* y_1 = y_1 & \rho_2^* y_2 = -y_2 & \rho_2^* y_3 = -y_3
\end{array}
$$

We assume that ρ_1 and ρ_2 act as follows on the Kähler classes x_j:

$$
\begin{array}{ccc}
\rho_1^* x_1 = x_1 & \rho_1^* x_2 = -x_2 & \rho_1^* x_3 = -x_3 \\
\rho_2^* x_1 = -x_1 & \rho_2^* x_2 = x_2 & \rho_2^* x_3 = -x_3
\end{array}
\tag{5.10}
$$

In this situation, we can define $y_1 := x_2$, $y_2 := x_1$, $y_3 = -x_3$ and the matching $h : H^2(S_2,\mathbb{Z}) \to H^2(S_1,\mathbb{Z})$ as $\phi_1^{-1} \circ \phi_2$. The relations (5.10) are the same as the relations (5.2) in Section 5.1. In Lemma 5.1.2 from that section, we have already shown that a smooth hyper-Kähler structure on a marked K3 surface exists such that (5.10) is satisfied. Therefore, we have proven the following theorem that is a part of Theorem 5.7. in [44].

Theorem 5.2.3. *Let (r_i, a_i, δ_i) with $i = 1, 2$ be triples such that non-symplectic involutions ρ_i of K3 surfaces S_i with invariants (r_i, a_i, δ_i) exist. We denote their fixed lattices by L_i. Moreover, we assume that $r_1 + r_2 \leq 11$ or $r_1 + r_2 + a_1 + a_2 < 22$ such that $L_1 \oplus L_2$ can be primitively embedded into the K3 lattice L. Let W_i be the ACyl Calabi-Yau manifolds that are constructed from the pairs (S_i, ρ_i) as in Theorem 5.2.1. In this situation, $W_1 \times S^1$ and $W_2 \times S^1$ can be glued together to a twisted connected sum M that is smooth and carries a metric with holonomy G_2 if the hyper-Kähler structures on the S_i are chosen appropriately. In this situation, the Betti numbers of M are determined by $b^1(M) = 0$, $b^2(M) = 4 + r_1 + r_2 - (a_1 + a_2)$, $b^3(M) = 115 - (r_1 + r_2) - 3(a_1 + a_2)$.*

Remark 5.2.4. The values of the Betti numbers can be obtained from Theorem 5.7. in [44] or from Theorem 4.4.9 in this thesis. It is crucial for the formulas for $b^k(M)$ that $L_1 \cap L_2 = \{0\}$. If $L_1 \otimes \mathbb{R}$ and $L_2 \otimes \mathbb{R}$ intersect in a subspace of dimension n and the spaces $V_i = L_i \otimes \mathbb{R} \cap (L_{3-i} \otimes \mathbb{R})^\perp$ with $i = 1, 2$ are orthogonal, we have to add n to b^2 and subtract $2n$ from b^3.

In the proof of Theorem 5.2.3, we considered a K3 surface with two commuting involutions that satisfy the relations (5.2) and obtained a pair of K3 surfaces that satisfy the matching condition. This idea does not rely on the fact that L_1 and L_2 are orthogonal. Whenever we have a K3 surface with two commuting involutions and a hyper-Kähler structure that satisfies (5.2), which is the same as (5.10), we can define a second K3 surface that satisfies the matching condition by $y_1 := x_2$, $y_2 := x_1$, $y_3 := -x_3$ and $h := \phi_1^{-1} \circ \phi_2$. This means that our examples from Section 5.1 can be used to construct matching K3 surfaces. As we have discussed in Section 4.2, the theorems on ACyl Calabi-Yau manifolds and twisted connected sums remain true in the orbifold case. Therefore, it is possible to construct ACyl Calabi-Yau orbifolds from singular K3 surfaces with a non-symplectic involution and G_2-orbifolds from singular K3 surfaces with two commuting involutions that satisfy (5.2). In the following, we assume that no singular point lies on a fixed curve of the involutions. This makes it easier to ensure that the G_2-orbifold has only ADE-singularities and to calculate its Betti numbers. We construct an explicit example where it is particularly easy to see that no singular point is on a fixed curve in order to show how our method works. As usual, we write the K3 lattice L as

$$H_1 \oplus H_2 \oplus H_3 \oplus (-E_8)_1 \oplus (-E_8)_2$$

We define two involutions ψ_i with $i = 1, 2$ of L that act as the identity on H_i, as -1 on $H_{3-i} \oplus H_3$ and interchange $(-E_8)_1$ and $(-E_8)_2$. The invariants of their fixed lattices are

$(10, 8, 0)$. Moreover, the ψ_i commute. We choose x_i with $i = 1, 2$ as $\frac{1}{2}(v_1^i + v_2^i) + \alpha(v_1^i - v_2^i)$ where α is irrational and sufficiently small. Moreover, we choose x_3 as $\lambda_1(v_1^3+v_2^3)+\lambda_2(v_1^3-v_2^3)$, where the λ_i are chosen $\mathbb{Q}$-linearly independent and such that $x_1^2 = x_2^2 = x_3^2 > 0$. The x_i define a hyper-Kähler structure on a K3 surface S such that the ψ_i are the pull-backs of involutions ρ_i of S that satisfy (5.10). The intersection of L with $\mathrm{span}(x_1, x_2, x_3)^\perp$ is spanned by the elements $(w_4, \ldots, w_{19})$ of the basis that we have introduced in the proof of Theorem 3.5.10. More explicitly, the w_j are the vectors with $w_j^2 = -2$ that generate both $(-E_8)$-lattices. This means that we have constructed a K3 surface with two E_8-singularities. Each ψ_i maps w_j with $j \in \{4, \ldots, 11\}$ to w_{j+8} and vice versa. Therefore, the two singular points are mapped to each other by the ρ_i. Let $(w_{j_1}, \ldots, w_{j_k})$ be a subfamily of $(w_4, \ldots, w_{11})$. By perturbing x_3 to

$$x_3' := x_3 + \sum_{l=1}^{k} \beta_l (w_{j_l} - w_{j_l+8})$$

with suitable $\beta_l \in \mathbb{R}$, we can achieve as in the proof of Corollary 3.5.11 and Lemma 5.1.2 that the hyper-Kähler structure still satisfies (5.10) but has a singular set that is described by two copies of an arbitrary subdiagram of E_8.

We consider a w_j with $j \in \{4, \ldots, 19\}$ that corresponds to a curve with vanishing volume. Since the invariants (r_i, a_i, δ_i) of L_i are $(10, 8, 0)$, the fixed locus of ρ_i consists of two elliptic curves C_1^i and C_2^i. The image of their cohomology classes c_1^i and c_2^i with respect to the marking are contained in $L_i = H_i \oplus E_8(-2)$. Moreover, we have $c_k^i \cdot c_k^i = 0$ for $k = 1, 2$ and C_1^i and C_2^i are linearly equivalent such that we have $c_1^i = c_2^i$, see for example [55]. Since $c_1^i \cdot c_1^i = 0$, it can be written as

$$y + \sum_{l=4}^{11} \beta_l (w_l + w_{l+8})$$

with $\beta_l \in \mathbb{Z}$, $y \in H_i$ and $y^2 = 4 \sum_{l=4}^{11} \beta_l^2$. It is possible to map c_1^i to $v_1^i \in H_i$ by reflecting it through the hyperplane that is orthogonal to $c_1^i - v_1^i$. Since reflections are lattice automorphisms and we can change the marking of the K3 surface by an automorphism of L, we can assume that $c_1^i = v_1^i$ and we have $c_1^i \cdot w_j = 0$. Therefore, we can assume that the curves with vanishing volume do not intersect the fixed curves.

We can now apply the orbifold version of Theorem 5.2.3 and obtain G_2-orbifolds with ADE-singularities. Let D_i with $i = 1, 2$ be the K3 factors of the cylindrical ends of $W_i \times S^1$. We denote the singular points of D_1 by $p_1, \ldots, p_n$ and the type of the singularity at p_j by Δ_j. Since the matching condition guarantees that there exists an isometry $f : D_1 \to D_2$, D_2 has singularities of the same type at the points $q_j := f(p_j)$. Any p_j (or q_j) is mapped by ρ_1 (or ρ_2) to another point with the same kind of singularity. Therefore, $\overline{W}_i$ has ADE-singularities along one or more $\mathbb{CP}^1$s. Their number and type is described by two copies of an arbitrary subdiagram of E_8. When we remove the divisor D_i, a singularity along $\mathbb{CP}^1$ becomes a singularity along $\mathbb{CP}^1 \setminus \{z\}$, which is diffeomorphic to the disc $B_1(0)$ with unit radius around $0 \in \mathbb{C}$. Thus, the ACyl Calabi-Yau orbifolds have singularities along massive tori $S^1 \times B_1(0)$ at their cylindrical ends. They are glued together such that their common boundary is a torus $S^1 \times S^1$. This means that we obtain an ADE-singularity along a 3-sphere.

Since $r_1 = r_2 = 10$ and $a_1 = a_2 = 8$ and the intersection of L_1 and L_2 has dimension 8, we obtain the Betti numbers $b^2 = 16$ and $b^3 = 31$ in the smooth case. We can check with help of the formulas in [44] that the condition $b^2(W_1) - d_1 + b^2(W_2) - d_2 \leq 22$ for the application of Theorem 4.4.9 is indeed satisfied. Let $2n$ be the number of collapsed $\mathbb{CP}^1$s in the singular K3 surface S_i. We have $b^2(S_i) = 22 - 2n$. In the orbifold case, we thus obtain slightly different formulas for the Betti numbers. We go through the proofs in [44] and see that most of the steps can be easily modified. The value of $h^{1,1}(\overline{W}_i)$ decreases by n since there are n classes in the fixed lattice of ρ_i that correspond to collapsing curves. The value of the Euler characteristic $\chi(\overline{W}_i)$ is $24 - 2n + 3\chi(C_1^i \cup C_2^i)$ instead of $24 + 3\chi(C_1^i \cup C_2^i)$. Therefore, $h^{1,2}(\overline{W}_i)$ remains the same. We take a look at the proof of Proposition 4.3.b in [44] and see that the formulas for d_1 and d_2 remain unchanged, too. The reason behind this is that d_i is defined as the dimension of the kernel of restriction map $H^2(W_i, \mathbb{R}) \to H^2(D_i, \mathbb{R})$ and we remove n classes from $H^2(W_i, \mathbb{R})$ that are not mapped to zero. Since only n, d_i and $b^3(\overline{W}_i)$ appear as summands in the formula for $b^2(M)$ and $b^3(M)$, we obtain the same Betti numbers as in the smooth case.

At the end of this section, we remark that there is a rather simple way to obtain G_2-orbifolds with ADE-singularities from twisted connected sums. Instead of blowing up all connected components of the singular locus of Z, it would be possible to blow up none or only some of them. After that, we could remove the divisor D and apply the orbifold version of Theorem 5.2.1 and glue together two $W_i \times S^1$ that are obtained by this method. The G_2-orbifold that we obtain this way has A_1-singularities along products of a circle and complex curves in $S_i \times \{z_j\}$ with $j = 1, 2$, where z_1 and z_2 are the fixed points of $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$.

5.3 Torus quotients and their resolutions

5.3.1 Basic facts

Another idea to construct G_2-orbifolds with ADE-singularities is to divide a torus T^7 with a flat G_2-structure by a discrete group Γ such that all isotropy groups can be embedded into $SU(2)$ acting on a four-dimensional subspace of the tangent space. The first G_2-manifold that is constructed in Joyce's book [36] actually is a resolution of a torus quotient of type $T^7/\mathbb{Z}_2^3$ that has A_1-singularities along 12 disjoint 3-tori. This construction is generalized by A. Barrett [10] to quotients by other groups. Moreover, he proves a classification result for torus quotients with ADE-singularities that satisfy certain conditions. We review the constructions of [36] and [10]. After that, we present our own examples. By relaxing the condition that the torus quotient has no singularities except ADE-singularities, we can modify our construction to obtain further G_2-orbifolds. We discuss the nature of the more complicated singularities and resolve them in order to obtain smooth G_2-manifolds. We introduce some basic facts about torus quotients with a G_2-structure. Let $(v_1, \ldots, v_7)$ be a basis of $\mathbb{R}^7$ and let

$$\Lambda := \{n_1 v_1 + \ldots + n_7 v_7 | n_1, \ldots, n_7 \in \mathbb{Z}\} .$$

We denote the group of translations by elements of Λ by Λ, too. The standard G_2-form ϕ_0 on $\mathbb{R}^7$ is invariant under translations. Therefore, any torus $\mathbb{R}^7/\Lambda$ carries a flat G_2-structure that we denote by ϕ_0, too. We denote $\mathbb{R}^7/\Lambda$ together with ϕ_0 simply by T^7 and keep in mind that different choices of Λ may yield different G_2-structures on the same underlying manifold. Our plan is to divide T^7 by a finite group Γ that preserves the G_2-structure but does not act freely. If all isotropy groups of this action leave a 3-dimensional subspace invariant and act as a subgroup of $SU(2)$ on the complement, T^7/Γ is a G_2-orbifold with ADE-singularities.

We call a diffeomorphism of a G_2-manifold that preserves the G_2-structure an *automorphism*. Since an automorphism has to preserve the metric and orientation, it is easy to see that the automorphisms of $\mathbb{R}^7$ are precisely the maps

$$x \mapsto Ax + v \tag{5.11}$$

with $A \in G_2$ and $v \in \mathbb{R}^7$. The automorphism group of $\mathbb{R}^7$ therefore is $G_2 \rtimes \mathbb{R}^7$. We denote the group of all linear maps with positive determinant that preserve Λ by $SL(\Lambda)$. If $\Lambda = \mathbb{Z}^7$, we have $SL(\Lambda) = SL(7, \mathbb{Z})$ and if Λ is a different lattice, $SL(\Lambda)$ is conjugate to $SL(7, \mathbb{Z})$ in $GL(7, \mathbb{R})$. Any automorphism of T^7 has to be of type

$$x + \Lambda \mapsto Ax + v + \Lambda \tag{5.12}$$

with $A \in G_2 \cap SL(\Lambda)$ and $v \in \mathbb{R}^7$. Therefore, the automorphism group of T^7 is $(G_2 \cap SL(\Lambda)) \rtimes T^7$. The flat G_2-structure ϕ_0 on T^7 induces a G_2-structure on any quotient T^7/Γ where $\Gamma \subset (G_2 \cap SL(\Lambda)) \rtimes T^7$ is a finite subgroup. We denote this G_2-structure also by ϕ_0. Let T^7/Γ be a G_2-orbifold with ADE-singularities and N be one of the associative submanifolds along which the singularities occur. Since N is the fixed point set of a group that consists of affine linear maps, it has to be a torus T^3. Let $p \in N$ and let Γ_p be the isotropy group at p. Γ_p acts trivially on the 3-dimensional tangent space of N. On the orthogonal complement it acts as a finite subgroup Γ'_p of $SU(2)$. Any element of Γ_p can be written in the form (5.12). Therefore, Γ'_p has to be conjugated to a subgroup of $SL(7, \mathbb{Z})$. An easy test to see if this is the case is to check if the trace of all elements of Γ'_p is an integer. This restricts the possible ADE-singularities of a torus quotient significantly. We take the explicit matrix representations from Section 2 and see that the only remaining groups are

$$A_1, A_2, A_3, A_5, D_3, D_4, D_5, E_6 \,. \tag{5.13}$$

We observe that in the case D_3 the discrete group is simply the cyclic group of order 4 that is generated by

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is conjugate to the generator of A_3 by a matrix in $SL(4, \mathbb{Z})$. Therefore, we obtain the same orbifold singularity as in the A_3-case. This is not surprising since the Dynkin diagrams A_3 and D_3 are isomorphic, too. Therefore, we will omit this group from now on from the above list.

5.3.2 Known examples

Our aim is to study if for all groups from the list (5.13) there exists at least one torus quotient with a singularity of the corresponding type along an associative submanifold. For the case A_1 this question is answered by the example of Joyce [36, p. 309ff.]. We will describe this example in detail since this will provide us with ideas that will be helpful for the other cases. We denote the standard coordinates of $\mathbb{R}^7$ by $x_1, \ldots, x_7$. Moreover, we define $T^7 := \mathbb{R}^7/\mathbb{Z}^7$ and three maps $\alpha, \beta, \gamma : T^7 \to T^7$ by

$$\begin{aligned} \alpha((x_1, \ldots, x_7) + \mathbb{Z}^7) &:= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) + \mathbb{Z}^7 \\ \beta((x_1, \ldots, x_7) + \mathbb{Z}^7) &:= (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7) + \mathbb{Z}^7 \\ \gamma((x_1, \ldots, x_7) + \mathbb{Z}^7) &:= (-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7) + \mathbb{Z}^7 \end{aligned}$$

Since the linear part of the above maps preserves Λ, α, β and γ are well-defined. We see that α, β and γ commute pairwise and that $\alpha^2 = \beta^2 = \gamma^2 = 1$. The group Γ that is generated by α, β and γ is therefore isomorphic to $\mathbb{Z}_2^3$. Finally, we can verify by a straightforward calculation that Γ preserves the standard G_2-structure ϕ_0 on T^7. The quotient T^7/Γ thus carries a flat G_2-structure.

We describe the set of all points $p \in T^7$ where the isotropy group Γ_p is non-trivial and as a next step the singular set of T^7/Γ. In order to do this we determine the fixed point sets $\mathrm{Fix}(g)$ of all elements $g \in \Gamma$. It is easy to see that

$$
\begin{array}{rcl}
\mathrm{Fix}(\alpha) & = & \{(x_1,\ldots,x_7)+\mathbb{Z}^7 \in T^7 | x_4,x_5,x_6,x_7 \in \frac{1}{2}\mathbb{Z}\} \\
\mathrm{Fix}(\beta) & = & \{(x_1,\ldots,x_7)+\mathbb{Z}^7 \in T^7 | x_2,x_3,x_7 \in \frac{1}{2}\mathbb{Z}; x_6 \in \frac{1}{4}+\frac{1}{2}\mathbb{Z}\} \\
\mathrm{Fix}(\gamma) & = & \{(x_1,\ldots,x_7)+\mathbb{Z}^7 \in T^7 | x_1,x_3 \in \frac{1}{2}\mathbb{Z}; x_5,x_7 \in \frac{1}{4}+\frac{1}{2}\mathbb{Z}\}
\end{array}
$$

The fixed point set of α is the disjoint union of 16 3-dimensional tori. They can be explicitly described as

$$\{(x_1,x_2,x_3,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,\tfrac{1}{2}\epsilon_3,\tfrac{1}{2}\epsilon_4)+\Lambda | x_1,x_2,x_3 \in \mathbb{R}\}$$

where $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{0,1\}$. β and γ have also 16 tori as fixed points. The fixed tori of β are given by

$$\{(x_1,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,x_4,x_5,\tfrac{1}{4}+\tfrac{1}{2}\epsilon_3,\tfrac{1}{2}\epsilon_4)+\Lambda | x_1,x_4,x_5 \in \mathbb{R}\}$$

and those of γ by

$$\{(\tfrac{1}{2}\epsilon_1,x_2,\tfrac{1}{2}\epsilon_2,x_4,\tfrac{1}{4}+\tfrac{1}{2}\epsilon_3,x_6,\tfrac{1}{4}+\tfrac{1}{2}\epsilon_4)+\Lambda | x_2,x_4,x_6 \in \mathbb{R}\}\ .$$

We see that there are no points that are fixed by two of the three elements simultaneously. We determine the other elements of Γ and obtain

$$
\begin{array}{rcl}
\alpha\beta((x_1,\ldots,x_7)+\mathbb{Z}^7) & = & (x_1,-x_2,-x_3,-x_4,-x_5,\frac{1}{2}+x_6,x_7)+\mathbb{Z}^7 \\
\alpha\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) & = & (-x_1,x_2,-x_3,-x_4,\frac{1}{2}+x_5,-x_6,\frac{1}{2}+x_7)+\mathbb{Z}^7 \\
\beta\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) & = & (-x_1,-x_2,x_3,x_4,\frac{1}{2}-x_5,\frac{1}{2}-x_6,\frac{1}{2}+x_7)+\mathbb{Z}^7 \\
\alpha\beta\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) & = & (-x_1,-x_2,x_3,-x_4,\frac{1}{2}+x_5,\frac{1}{2}+x_6,\frac{1}{2}-x_7)+\mathbb{Z}^7
\end{array}
$$

The fixed point equation for any $g \in \{\alpha\beta,\alpha\gamma,\beta\gamma,\alpha\beta\gamma\}$ forces at least one x_i with $i \in \{1,\ldots,7\}$ to satisfy $x_i+\mathbb{Z} = x_i+\frac{1}{2}+\mathbb{Z}$. Since this is impossible, the only elements with fixed points are 1, α, β and γ. In order to determine the singular set of T^7/Γ we need to know how the groups that are generated by two of the elements α, β and γ act on the fixed point set of the third element. β, γ, and $\beta\gamma$ permute the 16 tori in the fixed point set of α. Since the action of any of these elements contains a term $\frac{1}{2}\pm x_i$ with $i \in \{4,5,6,7\}$, they leave no torus fixed. In other words, the group $\langle\beta,\gamma\rangle$ acts freely on the 16 tori. We take a look at the action of α, γ and $\alpha\gamma$ on the coordinates x_6 and x_7 and can conclude that $\langle\alpha,\gamma\rangle$ acts freely on the set of 16 tori that are fixed by β. Finally, we consider the action of α, β and $\alpha\beta$ on x_5 and x_7 and see that $\langle\alpha,\beta\rangle$ acts freely on the fixed tori of γ, too. We are now able to describe the singular set S of T^7/Γ. Since only α, β and γ have fixed points, we have

$$S = (\mathrm{Fix}(\alpha)\dot\cup\mathrm{Fix}(\beta)\dot\cup\mathrm{Fix}(\gamma))/\Gamma$$

The union of the three fixed point sets consists of 48 tori. Since the groups $\langle\alpha,\beta\rangle$, $\langle\alpha,\gamma\rangle$ and $\langle\beta,\gamma\rangle$ consist of 4 elements and act freely, S consists of $\frac{48}{4}=12$ 3-tori. On a tubular neighborhood of any fixed torus, either α, β or γ acts as minus the identity on the other 4 coordinates. This means that we have singularities of type A_1 along each of the tori.

Our next step is to determine the Betti numbers of T^7/Γ. The harmonic k-forms on T^7 with the standard metric are precisely the monomials $dx^{i_1\cdots i_k}$ and their linear combinations. There is a one-to-one correspondence between the harmonic forms on T^7/Γ and the Γ-invariant forms on T^7. Any monomial dx^i or dx^{ij} is mapped by α, β or γ to $-dx^i$ or $-dx^{ij}$. Therefore, there are no harmonic 1- or 2-forms on T^7 that are invariant under Γ and we have $b^1(T^7/\Gamma)=b^2(T^7/\Gamma)=0$. The Γ-invariant harmonic 3-forms on T^7 are spanned by the 7 monomials that appear in the definition of ϕ_0. Therefore, we have $b^3(T^7/\Gamma)=7$. All in all, M-theory compactified on T^7/Γ yields an $\mathcal{N}=1$ supersymmetric field theory with gauge group $SU(2)^{12}$ and 7 complex scalar fields as low energy limit.

Joyce does not stop his investigation at this point and resolves the singularities of T^7/Γ in order to obtain a smooth G_2-manifold. He replaces a tubular neighborhood of each of the 12 singular tori by a product of an Eguchi-Hanson space and a 3-torus. Any of these replacements adds 1 to b^2 and 3 to b^3. This procedure yields a G_2-manifold with Betti numbers $b^1=0$, $b^2=12$ and $b^3=43$.

In this context, the work of A. Barrett [10] should be mentioned, too. The author classifies G_2-orbifolds of type T^7/Γ under the following assumptions:

1. The only singularities of T^7/Γ are ADE-singularities.
2. Let $g\in\Gamma$ be arbitrary. The action of g on T^7 can be written as $x\mapsto Ax+v$. There shall be a basis $(v_1,\ldots,v_7)$ of Λ such that A acts as the identity on $\mathrm{span}_{\mathbb{R}}(v_1,v_2,v_3)$ and preserves $\mathrm{span}_{\mathbb{R}}(v_4,v_5)$ as well as $\mathrm{span}_{\mathbb{R}}(v_6,v_7)$.
3. Let $x\in T^7$ be a fixed point of a $g\in\Gamma\setminus\{1\}$ and let $h\in\Gamma$ be a further group element that is not contained in the subgroup that is generated by g. Then h maps x to another connected component of the fixed locus of g.

Moreover, the author implicitly assumes that

4. $T^7=\mathbb{R}^7/\Lambda$, where the lattice Λ is generated by $\lambda_1e_1,\ldots,\lambda_7e_7$ with $\lambda_1,\ldots,\lambda_7\in\mathbb{R}^{>0}$. Moreover, the G_2-structure on T^7 shall be induced by the standard G_2-structure on $\mathbb{R}^7$.
5. Two bases $(v_1,\ldots,v_7)$ and $(w_1,\ldots,w_7)$ with the properties from assumption 2. are related to each other by a permutation of the basis elements.

If T^7/Γ has only ADE-singularities, the fixed point set of any $g\in\Gamma\setminus\{1\}$ is the union of disjoint 3-tori. If x is a fixed point of g, $h.x$ is a fixed point of hgh^{-1}. It follows from the third assumption that h maps fixed 3-tori to other fixed 3-tori with the same isotropy group. This guarantees that the singular set of the quotient T^7/Γ is the disjoint union of 3-tori, too. The author classifies the possible isomorphism types of Γ under the above assumptions. Moreover, he describes the physics that is associated with compactification of M-theory on a quotient T^7/Γ and the topology of the smooth manifolds that can be obtained by resolving the singularities. Some of the smooth G_2-manifolds that can be obtained by this method have $b^3 < 4$. The minimal value of b^3 in [36] is 4 and the examples in [43] and [44] all have large values of b^3 ($b^3 \geq 71$ or $b^3 \geq 35$). Therefore, at least the examples with $b^3 < 4$ are new.

Although our aim is to construct torus quotients with ADE-singularities, too, our approach is different. First of all, we do not attempt to classify all torus quotients of a certain kind. Our aim is rather to find examples with each kind of singularity from the list (5.13). Moreover, we drop the second and fifth assumption. These assumptions force the ADE-singularity to be of type A_n. The reason for this is that a group element $g \in \Gamma$ that acts as $\mathrm{diag}(\exp\left(\frac{2\pi i}{n}\right), \exp\left(-\frac{2\pi i}{n}\right))$ on a four-dimensional subspace $V \subset \mathbb{R}^7$ fixes two two-dimensional subspaces of V. However, any group of type D_k or E_k contains an element h that interchanges both spaces. Therefore, the bases from the second assumption that are associated to g and h can not be obtained from each other by a permutation of the coordinates and the fifth condition is thus violated. We relax the third assumption, too. As we will see, there will be elements h_i of our groups Γ that act non-trivially on 3-tori that are fixed by another $g \in \Gamma$. Therefore, our torus quotients will have singular sets of type $T^3/\langle h_i \rangle$.

5.3.3 Choice of the group action on T^7

We construct our examples of G_2-orbifolds with ADE-singularities by a single method that can be adapted to the different cases. In order to simplify our notation, we identify $\mathrm{span}(x_4, x_5, x_6, x_7)$ with $\mathbb{C}^2$ by the map $(x_4, x_5, x_6, x_7) \mapsto (x_4 + ix_5, x_6 + ix_7)$. We denote $x_4 + ix_5$ by z_1 and $x_6 + ix_7$ by z_2. With this notation, we are able to rewrite any linear map $\phi : \mathbb{R}^7 \to \mathbb{R}^7$ that preserves the splitting of $\mathbb{R}^7$ into $\mathrm{span}(e_1, e_2, e_3)$ and $\mathrm{span}(e_4, e_5, e_6, e_7)$ as a pair (ϕ', ϕ'') with $\phi' : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi'' : \mathbb{C}^2 \to \mathbb{C}^2$. The three maps α, β and γ from Joyce's example are exactly of this kind. In order to illustrate our idea, we consider the three $\mathbb{R}$-affine maps $\alpha'', \beta'', \gamma'' : \mathbb{C}^2 \to \mathbb{C}^2$. These maps are given by

$$
\begin{aligned}
\alpha''(z_1, z_2) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\beta''(z_1, z_2) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\
\gamma''(z_1, z_2) &= \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}i \\ \frac{1}{2}i \end{pmatrix}
\end{aligned}
$$

For our construction, we define α'', β'' and γ'' slightly different, namely as

$$\begin{aligned}
\alpha_i''(z_1,z_2) &:= M_i \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\beta''(z_1,z_2) &:= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\gamma''(z_1,z_2) &:= \begin{pmatrix} -\overline{z}_1 \\ -\overline{z}_2 \end{pmatrix}
\end{aligned}$$

where the matrices M_i are a set of generators of a finite subgroup of $SU(2)$. If there is only one generator, we omit the index i. Our idea is to choose the M_i as generators of an arbitrary group from the list (5.13) and to divide $\mathbb{C}^2$ by a lattice Λ' that is preserved by the ADE group and by β'' and γ''. The 7-dimensional lattice Λ that defines $T^7 = \mathbb{R}^7/\Lambda$ shall be $\text{span}_{\mathbb{Z}}(e_1,e_2,e_3) \oplus \Lambda'$. The action of α, β and γ on the T^3-factor shall be generalized such that we have

$$\begin{aligned}
\alpha'(x_1,x_2,x_3) &:= (x_1,x_2,x_3) \\
\beta'(x_1,x_2,x_3) &:= (x_1+v_1,-x_2+v_2,-x_3+v_3) \\
\gamma'(x_1,x_2,x_3) &:= (-x_1+w_1,x_2+w_2,-x_3+w_3)
\end{aligned}$$

where $(v_1,v_2,v_3)^T,(w_1,w_2,w_3)^T \in \mathbb{R}^3$. The group that is generated by β' and γ' shall act as $\mathbb{Z}_2^2$ on T^3. Since we have chosen T^3 as $\mathbb{R}^3/\mathbb{Z}^3$, we need $v_1,w_2 \in \frac{1}{2}\mathbb{Z}$ in order to have $\beta'^2 = \gamma'^2 = 1$. We have

$$\begin{aligned}
(\beta'\gamma')(x_1,x_2,x_3) &= (-x_1,-x_2,x_3) + (v_1+w_1,v_2-w_2,v_3-w_3) \\
(\gamma'\beta')(x_1,x_2,x_3) &= (-x_1,-x_2,x_3) + (-v_1+w_1,v_2+w_2,-v_3+w_3)
\end{aligned}$$

The condition $\beta'\gamma' = \gamma'\beta'$ implies that $v_3+w_3 \in \frac{1}{2}\mathbb{Z}$. We choose two different pairs (β_j',γ_j') with $j=1,2$ that yield topologically different G_2-orbifolds, namely

$$\begin{aligned}
(\beta_1')(x_1,x_2,x_3) &= (\tfrac{1}{2}+x_1,-x_2,\tfrac{1}{2}-x_3) \\
(\gamma_1')(x_1,x_2,x_3) &= (-x_1,\tfrac{1}{2}+x_2,-x_3) \\
\\
(\beta_2')(x_1,x_2,x_3) &= (x_1,-x_2,-x_3) \\
(\gamma_2')(x_1,x_2,x_3) &= (-x_1,\tfrac{1}{2}+x_2,-x_3)
\end{aligned}$$

We combine the maps on $\mathbb{C}^2/\Lambda'$ and $\mathbb{R}^3/\mathbb{Z}^3$ that we have defined above and obtain a family of maps $\alpha_i,\beta,\gamma : T^7 \to T^7$ that depends on one of the two possible choices of $(v_1,v_2,v_3)^T$ and $(w_1,w_2,w_3)^T$. Our definition of these maps is motivated by [1] where the author works with similar maps from $\mathbb{C}^2\times\mathbb{R}^3$ to itself to construct orbifolds of type $(\mathbb{C}^2\times T^3)/\Gamma$. Since the author assumes that there is only map α that is defined by the matrix $M = \text{diag}(e^{2\pi i/n},e^{-2\pi i/n})$, the orbifolds have A_{n-1}-singularities along $\{0\}\times T^3$. Since

$\mathbb{C}^2$ is not replaced by a torus, the orbifolds in [1] are non-compact. As before, we denote the group that is generated by the α_i, β and γ by Γ. Since we will choose Λ' in such a way that the linear part of α_i, β and γ preserves Λ, the three generators are well-defined maps $T^7 \to T^7$ and it makes sense to talk about the orbifold T^7/Γ. β'' and γ'' can be rewritten as

$$\begin{array}{rcl} \beta''(x_4, x_5, x_6, x_7) & = & (-x_4, -x_5, x_6, x_7) \\ \gamma''(x_4, x_5, x_6, x_7) & = & (-x_4, x_5, -x_6, x_7) \end{array}$$

With help of these expressions it is straightforward to see that ϕ_0 is invariant under β and γ. $SU(2)$ can be defined as the group that leaves all self-dual 2-forms on a 4-dimensional space invariant. The space of all self-dual 2-forms on $\mathbb{C}^2 = \mathrm{span}(e_4, e_5, e_6, e_7)$ is spanned by

$$\omega_1 := dx^{45} + dx^{67} , \qquad \omega_2 := dx^{46} - dx^{57} , \qquad \omega_3 := -dx^{47} - dx^{56} .$$

The standard G_2-form can be written as

$$\phi_0 = dx^{123} + dx^1 \wedge \omega_1 + dx^2 \wedge \omega_2 + dx^3 \wedge \omega_3 .$$

Since $\alpha' : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity map and the matrices M_i are elements of $SU(2)$, the pull-back of α preserves ϕ_0. Therefore, the standard G_2-structure is actually Γ-invariant and T^7/Γ thus carries a parallel G_2-structure. In order to check if T^7/Γ is a G_2-orbifold with ADE-singularities, we have to determine the fixed locus of all group elements of Γ. Therefore, we need to know the isomorphism type of Γ. We have already shown that β'_j and γ'_j commute and are of order 2. The same is true for β'' and γ'' and the group that is generated by β and γ is therefore isomorphic to $\mathbb{Z}_2^2$. By a short computation we see that

$$\begin{array}{rcl} \alpha''_i\beta''(z_1, z_2) & = & (-a_i z_1 + b_i z_2, -c_i z_1 + d_i z_2) \\ \beta''\alpha''_i(z_1, z_2) & = & (-a_i z_1 - b_i z_2, c_i z_1 + d_i z_2) \\ & & \\ \alpha''_i\gamma''(z_1, z_2) & = & (-a_i \overline{z_1} - b_i \overline{z_2}, -c_i \overline{z_1} - d_i \overline{z_2}) \\ \gamma''\alpha''_j(z_1, z_2) & = & (-\overline{a}_i \overline{z_1} - \overline{b}_i \overline{z_2}, -\overline{c}_i \overline{z_1} - \overline{d}_i \overline{z_2}) \end{array}$$

α' commutes with β'_j and γ'_j since we have defined it as the identity map. Therefore, α_i and β commute if and only if M_i is a diagonal matrix and α_i and γ commute if and only if $M_i \in SO(2) \subset SU(2)$. Unfortunately, both conditions are satisfied for all M_i only in the case where our finite subgroup of $SU(2)$ is $\{\mathrm{Id}, -\mathrm{Id}\}$. In this situation, Γ is isomorphic to $\mathbb{Z}_2^3$. In order to determine the isomorphism type of Γ in the other cases, we observe that

$$\begin{aligned} \beta''^{-1}\alpha_i''\beta'' &= \begin{pmatrix} a_i & -b_i \\ -c_i & d_i \end{pmatrix} \\ \gamma''^{-1}\alpha_i''\gamma'' &= \begin{pmatrix} \overline{a}_i & \overline{b}_i \\ \overline{c}_i & \overline{d}_i \end{pmatrix} \end{aligned} \tag{5.14}$$

The matrices on the right hand side are elements of $SU(2)$ since we assume that $M_i \in SU(2)$. Let Δ be the finite group that is generated by the M_i. Δ is either of type A_n, D_n or E_6. In the first case, Δ is generated by the matrix

$$M_1 = \begin{pmatrix} \exp\left(\frac{2\pi i}{n+1}\right) & 0 \\ 0 & \exp\left(-\frac{2\pi i}{n+1}\right) \end{pmatrix}$$

M_1 is mapped by conjugation with β'' to itself and by conjugation with γ'' to its inverse. Since Δ is preserved by conjugation with β'' and γ'', Γ is a semidirect product $\Delta \rtimes \mathbb{Z}_2^2$. The second generator

$$M_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

of D_n is mapped by conjugation with β'' or γ'' to its negative. Since the negative of the identity matrix is an element of the binary dihedral group, Γ is in this case a semidirect product, too. E_6 consists of all elements of D_4 and all matrices of type

$$\frac{1}{2}\begin{pmatrix} \epsilon_1 + i\epsilon_2 & -\epsilon_3 + i\epsilon_4 \\ \epsilon_3 + i\epsilon_4 & \epsilon_1 - i\epsilon_2 \end{pmatrix}$$

with $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{1, -1\}$. It is easy to see that conjugation with β'' or γ'' preserves the set of these matrices. Therefore, Γ is in any case a semidirect product $\Delta \rtimes \mathbb{Z}_2^2$ and the set of all elements of Γ is given by

$$\Delta \cup \{\sigma\beta|\sigma \in \Delta\} \cup \{\sigma\gamma|\sigma \in \Delta\} \cup \{\sigma\beta\gamma|\sigma \in \Delta\} \; ,$$

where we identify the finite subgroup Δ of $SU(2)$ with the group that is generated by the α_i and acts on T^7. Let $M \in SU(2)$ be an arbitrary matrix acting on $\mathbb{C}^2$. If M is not the identity matrix, there exists no non-trivial real subspace of $\mathbb{C}^2$ on which M acts trivially. This follows from the fact that the eigenvalues of M are $\exp(i\lambda)$ and $\exp(-i\lambda)$ with $\lambda \notin 2\pi\mathbb{Z}$. The set of all points on the torus $\mathbb{C}^2/\Lambda'$ that are fixed by a non-trivial subgroup of Δ therefore consist of isolated points. We denote these points by $p_1, \ldots, p_n$ and their isotropy groups by $\Delta_1, \ldots, \Delta_n \subset \Delta$. It may happen that a p_i is mapped by an element $g \in \Delta$ with $g \notin \Delta_i$ to another point p_j. The quotient $(\mathbb{C}^2/\Lambda')/\Delta$ therefore has at most n singular points. We denote them by $q_1, \ldots, q_m$ with $m \leq n$. Each q_i has a neighborhood that can be identified with an open subset of $\mathbb{C}^2$ divided by a finite group that we denote by Δ_i, too. Let σ be an element of the group Δ that is generated by the α_i. We have

$$\sigma((x_1, x_2, x_3, z_1, z_2) + \Lambda) = (x_1, x_2, x_3, az_1 + bz_2, cz_1 + dz_2) + \Lambda$$

with

$$M_\sigma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2) \,.$$

The above considerations allow us to describe the singular locus of T^7/Γ qualitatively. This analysis will be helpful later on when we investigate explicit examples. In the case $j = 1$, we have

$$\begin{array}{rcl} \sigma\beta((x_1, x_2, x_3, z_1, z_2) + \Lambda) & = & (\frac{1}{2} + x_1, -x_2, \frac{1}{2} - x_3, -az_1 + bz_2, -cz_1 + dz_2) + \Lambda \\ \sigma\gamma((x_1, x_2, x_3, z_1, z_2) + \Lambda) & = & (-x_1, \frac{1}{2} + x_2, -x_3, -a\overline{z}_1 - b\overline{z}_2, -c\overline{z}_1 - d\overline{z}_2) + \Lambda \\ \sigma\beta\gamma((x_1, x_2, x_3, z_1, z_2) + \Lambda) & = & (\frac{1}{2} - x_1, \frac{1}{2} - x_2, \frac{1}{2} + x_3, a\overline{z}_1 - b\overline{z}_2, c\overline{z}_1 - d\overline{z}_2) + \Lambda \end{array}$$

Any of the above maps contains a term of type $\frac{1}{2} + x_i$. Therefore, the only non-empty fixed loci are those of the $\sigma \in \Delta$. We have

$$\mathrm{Fix}(\sigma) = \bigcup_{(z_1^i, z_2^i) \in \mathrm{Fix}(M_\sigma)} \{(x_1, x_2, x_3, z_1^i, z_2^i) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\} \,,$$

where $\mathrm{Fix}(M_\sigma)$ denotes the set of all fixed points of the action of M_σ on $\mathbb{C}^2/\Lambda'$. The set of all points that are fixed by at least one element of Γ can therefore be written as the following union of 3-tori:

$$\bigcup_{i=1}^{n} T^3 \times \{p_i\} \quad \text{with} \quad T^3 = \mathbb{R}^3/\mathbb{Z}^3 \,.$$

Since Δ is a normal subgroup of Γ, we obtain T^7/Γ by dividing T^7/Δ by the group $\langle \beta, \gamma \rangle$, which is isomorphic to $\mathbb{Z}_2^2$. T^7/Δ is an orbifold with ADE-singularities of type Δ_i along the 3-tori $T^3 \times \{q_i\}$. If β and γ both fix q_i, the group $\langle \beta, \gamma \rangle$ acts freely and orientation-preserving on $T^3 \times \{q_i\}$. Therefore, $(T^3 \times \{q_i\})/\langle \beta, \gamma \rangle$ is a compact orientable Euclidean space form. In fact, it is the space form of type $T^3/\mathbb{Z}_2^2$ from Section 5.1. The harmonic k-forms on T^3 with the flat metric are spanned by the monomials $dx^{i_1 \cdots i_k}$. Since $\langle \beta, \gamma \rangle$ leaves no harmonic 1- or 2-form invariant, we have $b^1(T^3/\mathbb{Z}_2^2) = b^2(T^3/\mathbb{Z}_2^2) = 0$. In this situation, T^7/Γ has an ADE-singularity of type Δ_i along a submanifold that is diffeomorphic to $T^3/\mathbb{Z}_2^2$. If

β fixes q_i and γ maps q_i to a q_j with $i \neq j$, the tori $T^3 \times \{q_i\}$ and $T^3 \times \{q_j\}$ yield an ADE-singularity along a submanifold of type $T^3/\mathbb{Z}_2$. By similar arguments as above we see that $b^1(T^3/\mathbb{Z}_2) = b^2(T^3/\mathbb{Z}_2) = 1$. If γ fixes q_i and β maps q_i to a different point, we also obtain a singularity along $T^3/\mathbb{Z}_2$. If β and γ both do not fix q_i, the four tori in the orbit of $\langle \beta, \gamma \rangle$ yield a singularity of T^7/Γ along a 3-torus.

It should be mentioned that the division of T^3 by β or γ induces monodromies that may make the normal bundle of the singular submanifolds non-trivial. We assume that the singular associative submanifold A is isometric to $T^3/\langle \beta \rangle$ or $T^3/\langle \beta, \gamma \rangle$. Parallel transport with respect to the flat metric along the loop $c(t) : [0, \frac{1}{2}] \to A$ with $c(t) = (t, 0, \frac{1}{4})$ induces an endomorphism of the normal bundle that is given by the map β''. Analogously, parallel transport along $c(t) = (0, t, 0)$ induces an endomorphism that is given by γ'' if A is isometric to $T^3/\langle \gamma \rangle$ or $T^3/\langle \beta, \gamma \rangle$. The group that is generated by these endomorphisms is the monodromy group. As we have explained in Section 4.5, the gauge group of the four-dimensional field theory that we obtain as low-energy limit may be influenced by the monodromy group. Therefore, we will describe the monodromies along the singular submanifolds of our examples. We proceed to the case $j = 2$. The group elements that are not contained in Δ act on T^7 as

$$\begin{aligned}
\sigma\beta((x_1, x_2, x_3, z_1, z_2) + \Lambda) &= (x_1, -x_2, -x_3, -az_1 + bz_2, -cz_1 + dz_2) + \Lambda \\
\sigma\gamma((x_1, x_2, x_3, z_1, z_2) + \Lambda) &= (-x_1, \tfrac{1}{2} + x_2, -x_3, -a\overline{z}_1 - b\overline{z}_2, -c\overline{z}_1 - d\overline{z}_2) + \Lambda \\
\sigma\beta\gamma((x_1, x_2, x_3, z_1, z_2) + \Lambda) &= (-x_1, \tfrac{1}{2} - x_2, x_3, a\overline{z}_1 - b\overline{z}_2, c\overline{z}_1 - d\overline{z}_2) + \Lambda
\end{aligned}$$

The elements of type $\sigma\gamma$ have no fixed points. The fixed point set of $\sigma\beta$ is given by

$$\mathrm{Fix}(\sigma\beta) = \bigcup_{(z_1,z_2)\in \mathrm{Fix}(N_\sigma);\epsilon_1,\epsilon_2 \in \{0,1\}} \{(x_1, \tfrac{1}{2}\epsilon_1, \tfrac{1}{2}\epsilon_2, z_1, z_2) + \Lambda | x_1 \in \mathbb{R}\} \,,$$

where N_σ is the matrix

$$\begin{pmatrix} -a & b \\ -c & d \end{pmatrix}$$

that acts on $\mathbb{C}^2/\Lambda'$ and $\mathrm{Fix}(N_\sigma)$ is the fixed point set of this action. In particular, we have

$$\mathrm{Fix}(\beta) = \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{0,1\}} \{(x_1, \tfrac{1}{2}\epsilon_1, \tfrac{1}{2}\epsilon_2, \tfrac{1}{2}\epsilon_3, \tfrac{1}{2}\epsilon_4, x_6, x_7) + \Lambda | x_1, x_6, x_7 \in \mathbb{R}\}$$

if $\sigma = 1$. We remark that N_σ may have 1 as eigenvalue even if $\sigma \neq 1$ and the set $\mathrm{Fix}(N_\sigma)$ is thus not necessarily discrete. The fixed point set of $\sigma\beta\gamma$ is given by

$$\mathrm{Fix}(\sigma\beta\gamma) = \bigcup_{(w_1,w_2)\in\mathrm{Fix}(P_\sigma);\epsilon_1,\epsilon_2\in\{0,1\}} \{(\tfrac{1}{2}\epsilon_1, \tfrac{1}{4}+\tfrac{1}{2}\epsilon_2, x_3, w_1, w_2)+\Lambda | x_3 \in \mathbb{R}\} \,,$$

where $P_\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ is the $\mathbb{R}$-linear map with

$$P_\sigma(z_1, z_2) = (a\overline{z}_1 - b\overline{z}_2, c\overline{z}_1 - d\overline{z}_2)^T$$

and $\mathrm{Fix}(P_\sigma)$ is the fixed point set of the induced map $\mathbb{C}^2/\Lambda' \to \mathbb{C}^2/\Lambda'$. As before, it may happen that $\mathrm{Fix}(P_\sigma)$ has eigenvalue 1 for a $\sigma \neq 1$. We discuss the shape of the singular locus of T^7/Γ in more detail later on when we consider explicit examples. Here, we mention only one important point. Our modification of β yields 16 tori that are fixed by β. They intersect the tori that are fixed by a subgroup Δ_i of Δ in a circle. The normal space of any point of this circle can be identified with $\mathbb{C}^3/G$ where G is the larger subgroup of $SU(3)$ that is generated by all

$$\begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_i \quad \text{and} \quad \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} ,$$

where we obtain the first coordinate of $\mathbb{C}^3$ as $x_2 + ix_3$. Moreover, the non-empty fixed point sets of $\sigma\beta\gamma$ yield additional singularities. The physical interpretation of M-theory compactified on G_2-orbifolds with singularities of type $\mathbb{C}^3/G$ is beyond the scope of this thesis. We refer the reader to [32] for a discussion of the physical consequences of this kind of singularities. At the end of this section, we nevertheless consider some torus quotients with $j = 2$, mainly because the resolution of the singularities yields interesting smooth G_2-manifolds.

5.3.4 Explicit examples

We investigate the torus quotients T^7/Γ that can be obtained by choosing Δ from the list (5.13) of allowed subgroups of $SU(2)$. In particular, we determine the type of the singularities of T^7/Γ and the Betti numbers of the torus quotients. In this subsection, we restrict ourselves to the case $j = 1$ since only then the torus quotients are G_2-orbifolds with ADE-singularities. Two examples with $j = 2$ will be considered in the following subsection. Since the explicit calculations are slightly different for each choice of Δ, we study each case separately and start with A_1. In this situation, Δ is generated by a single element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{C}^{2\times 2} .$$

We have observed in the previous subsection that Γ is isomorphic to $\mathbb{Z}_2^3$. If we look only at the linear part of the maps α, β and γ and neglect the translation part, we see that our maps are the same as in Joyce's example up to a permutation of the coordinates. As we will see below, the different translation part influences the shape of the singular locus. We choose the lattice Λ that defines the torus simply as $\mathbb{Z}^7 = \mathrm{span}_{\mathbb{Z}}(e_1, \ldots, e_7)$ since $\mathrm{span}_{\mathbb{Z}}(e_4, e_5, e_6, e_7)$ is preserved by α''. The maps α, β and γ can be written as

$$\begin{array}{rcl} \alpha((x_1, \ldots, x_7) + \Lambda) & = & (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) + \Lambda \\ \beta((x_1, \ldots, x_7) + \Lambda) & = & (\frac{1}{2} + x_1, -x_2, \frac{1}{2} - x_3, -x_4, -x_5, x_6, x_7) + \Lambda \\ \gamma((x_1, \ldots, x_7) + \Lambda) & = & (-x_1, \frac{1}{2} + x_2, -x_3, -x_4, x_5, -x_6, x_7) + \Lambda \end{array}$$

Since $j = 1$, only the group elements in Δ may have fixed points. Therefore, the only non-empty fixed loci are $\mathrm{Fix}(1) = T^7$ and

$$\mathrm{Fix}(\alpha) = \bigcup_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{0,1\}} \{(x_1, x_2, x_3, \tfrac{1}{2}\epsilon_1, \tfrac{1}{2}\epsilon_2, \tfrac{1}{2}\epsilon_3 \tfrac{1}{2}, \epsilon_4) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\}$$

The above set consists of 16 disjoint tori. β and γ map each of these tori to itself since they leave $\mathrm{span}(e_1, e_2, e_3)$ invariant and act as $+1$ or -1 on the coordinates x_4, x_5, x_6 and x_7. The singular set of T^7/Γ therefore consists of 16 copies of $T^3/\langle\beta, \gamma\rangle$. α acts trivially on $T^3/\langle\beta, \gamma\rangle$ and the action of α'' makes the normal bundle of $T^3/\langle\beta, \gamma\rangle$ into a bundle with fibers $\mathbb{C}^2/\{\pm \mathrm{Id}\}$. All in all, T^7/Γ is an orbifold with A_1-singularities along 16 disjoint copies of $T^3/\langle\beta, \gamma\rangle$. The monodromies β'' and γ'' act trivially on the group $\{\pm \mathrm{Id}\}$.

We determine the Betti numbers of the quotient T^7/Γ. Since we are able to use many of our arguments for the other cases, we include all the details for this case. As at the beginning of this section, where we investigated Joyce's example, we determine the dimension of the space of all Γ-invariant harmonic k-forms. α, β and γ send each monomial $dx^{i_1 \cdots i_k}$ to itself or to $-dx^{i_1 \cdots i_k}$. The space of all Γ-invariant harmonic k-forms is thus spanned by monomials. Since the maps β and γ are contained in any of the groups Γ that we consider in this section, we start by searching for the forms that are invariant under β and γ and check the invariance under α afterwards. Since the three choices of β and γ coincide up to a translation, we obtain the same Betti numbers for any value of j. The only 1-forms that are invariant under β are dx^1, dx^6, dx^7 and their linear combinations. Since dx^1 and dx^6 are not γ-invariant, only dx^7 remains. dx^7 is not invariant under α. In fact, any group Δ contains an element that does not leave dx^7 invariant. Otherwise, the representation of Δ on $\mathbb{C}^2$ would split into a 1-dimensional and a 3-dimensional representation. Therefore, we have $b^1(T^7/\Gamma) = 0$ in all cases that we consider in this section. The vector space of all monomials dx^{ij} decomposes into

$$\begin{array}{rcl} V_1 & := & \mathrm{span}\{dx^{ij} | 1 \leq i < j \leq 3\} \\ V_2 & := & \mathrm{span}\{dx^{ij} | 1 \leq i \leq 3, 4 \leq j \leq 7\} \\ V_3 & := & \mathrm{span}\{dx^{ij} | 4 \leq i < j \leq 7\} \end{array}$$

Moreover, α, β and γ preserve this decomposition. It is therefore possible to consider the three spaces separately. The only 2-form from V_1 that is invariant under β is dx^{23}. Since dx^{23} is not γ-invariant, we obtain no Γ-invariant forms from V_1. By a short calculation we see that the space of all 2-forms in V_2 that are invariant under β and γ is spanned by

$$dx^{16},\ dx^{25},\ dx^{34}. \tag{5.15}$$

The β-invariant forms in V_3 are spanned by dx^{45} and dx^{67}. Since none of them is γ-invariant, the 2-forms from (5.15) and their linear combinations are the only $\langle\beta,\gamma\rangle$-invariant harmonic 2-forms. The space of all dx^{ijk} decomposes into

$$\begin{array}{lcl} W_1 & := & \mathrm{span}\{dx^{123}\} \\ W_2 & := & \mathrm{span}\{dx^{ijk} | 1 \leq i < j \leq 3, 4 \leq k \leq 7\} \\ W_3 & := & \mathrm{span}\{dx^{ijk} | 1 \leq i \leq 3, 4 \leq j < k \leq 7\} \\ W_4 & := & \mathrm{span}\{dx^{ijk} | 4 \leq i < j < k \leq 7\} \end{array}$$

and Γ preserves this splitting. We consider each of the W_i separately and see that the space of all 3-forms with constant coefficients that are invariant under β and γ is spanned by

$$dx^{123}, dx^{124}, dx^{135}, dx^{145}, dx^{167}, dx^{236}, dx^{246}, dx^{257}, dx^{347}, dx^{356}, dx^{456}\,. \tag{5.16}$$

α acts as -1 on V_2 and there remain no Γ-invariant 2-forms with constant coefficients on T^7. Therefore, we have $b^2(T^7/\Gamma) = 0$. The same is true for any finite subgroup of $SU(2)$ that contains minus the identity and thus for all groups from the list (5.13) with the only possible exception A_2. α acts as $(-1)^{l+1}$ on W_l. Therefore, the Γ-invariant harmonic 3-forms are spanned by

$$dx^{123}, dx^{145}, dx^{167}, dx^{246}, dx^{257}, dx^{347}, dx^{356}\,. \tag{5.17}$$

These are precisely the monomials that appear in the definition of ϕ_0. All in all, we obtain $b^3(T^7/\Gamma) = 7$. We jump to the case A_3 before we consider A_2. This has the advantage that we can choose Λ' again as $\mathrm{span}_{\mathbb{Z}}(e_4, e_5, e_6, e_7)$ since this lattice is invariant under the generator $\mathrm{diag}(i, -i)$ of Δ which can be written as the real matrix

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix}$$

The square of this matrix is obviously the α'' from the previous case. In our case A_3, the group Γ is generated by the map $\alpha : \mathbb{R}^7 \to \mathbb{R}^7$ which is given by

$$\alpha(x_1,\ldots,x_7) := (x_1,x_2,x_3,-x_5,x_4,x_7,-x_6)$$

and the usual maps β and γ. Let $[g,h] = ghg^{-1}h^{-1}$ be the commutator of two group elements. By a short calculation we see that $[\alpha,\beta] = [\beta,\gamma] = 1$. With help of the second of the equations (5.14) we obtain

$$\gamma''^{-1}\alpha''\gamma'' = -\alpha'' \, .$$

Since α''^2 is minus the identity, we have $[\alpha'',\gamma''] = \alpha''^2$ and we can conclude that $[\alpha,\gamma] = \alpha^2$. All in all, Γ is a semi-direct product $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, which consists of the 16 elements α^i, $\alpha^i\beta$, $\alpha^i\gamma$ and $\alpha^i\beta\gamma$ with $i \in \{0,1,2,3\}$. We have to determine the fixed point set of all of these elements. The only non-empty fixed point sets are those of 1, α, α^2 and α^3. Since $\alpha^3 = \alpha^{-1}$, the fixed point set of α^3 is the same as of α. Therefore, the only interesting cases are α and α^2. By a short calculation we obtain

$$\begin{array}{rcl} \mathrm{Fix}(\alpha) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2) + \Lambda | x_1,x_2,x_3 \in \mathbb{R}\} \\ \mathrm{Fix}(\alpha^2) & = & \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4) + \Lambda | x_1,x_2,x_3 \in \mathbb{R}\} \end{array}$$

There are 12 tori that are fixed by α^2 but not by α and 4 tori that are by fixed α. The map α acts on the set of the 12 tori and the action has 6 orbits that consist of two tori. The singular locus of the quotient T^7/Δ thus consists of 10 tori. 6 of them are projections of the tori that are fixed by α^2 only. Since α^2 acts as minus the identity on the coordinates (x_4,x_5,x_6,x_7) of the normal bundle, we have A_1-singularities along these tori. The other 4 tori are the projection of the tori that are fixed by α. Along these tori we have an A_3-singularity. As in the A_1-case, β and γ act freely on each torus. The singular locus of T^7/Γ therefore consists of 10 submanifolds that are isometric to $T^3/\mathbb{Z}_2^2$ with the flat metric. The monodromy group of each submanifold is generated by β'' and γ''. β'' acts trivially on Δ, but conjugation with γ'' yields an automorphism of Δ that sends α^i to α^{-i}.

We determine the Betti numbers of T^7/Γ. As before, we have $b^1(T^7/\Gamma) = b^2(T^7/\Gamma) = 0$. We take a look at the 3-forms that are invariant under the group Γ from the A_1-example and check if they are invariant under the map α that generates $\Delta \cong \mathbb{Z}_4$, too. We see that the harmonic 3-forms on T^7 that are invariant under the current group Γ are spanned by

$$dx^{123}, dx^{145}, dx^{167}, dx^{246} - dx^{257}, dx^{347} + dx^{356}$$

and therefore we have $b^3(T^7/\Gamma) = 5$. The next cases are A_2 and A_5. In the case A_2 the group Δ is generated by

$$\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \qquad \text{or equivalently by the real matrix} \qquad \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

In the case A_5, Δ is generated by

$$\begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \qquad \text{or the real matrix} \qquad \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Both groups contain a rotation around an angle of $\frac{2\pi}{3}$ in the (x_4, x_5)-plane and thus do not leave the lattice $\mathrm{span}_{\mathbb{Z}}(e_4, e_5, e_6, e_7)$ invariant. Therefore, we define the lattice Λ' as the sum of two hexagonal lattices instead.

$$\Lambda' := \mathrm{span}_{\mathbb{Z}} \left(e_4, \frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5, e_6, \frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7 \right)$$

Λ' is invariant with respect to Δ. The maps β'' and γ'' leave Λ' invariant, too. Therefore, T^7/Γ is a well-defined orbifold that carries a flat G_2-structure. We determine the isomorphism type of Γ. Let k be the order of the group Δ. Analogously to the case A_3, we see that

$$\alpha^k = \beta^2 = \gamma^2 = 1\,, \quad [\alpha, \beta] = [\beta, \gamma] = 1\,, \quad [\alpha, \gamma] = \alpha^2\,.$$

Therefore, Γ is isomorphic to $(\mathbb{Z}_k \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. Our next step is to determine the fixed locus of all elements of Γ. For the same reasons as before, only the powers of α may have a non-empty fixed locus. Let $v \in \mathbb{C}^2$. $v + \Lambda' \in \mathbb{C}^2/\Lambda'$ is a fixed point of a map $A' : \mathbb{C}^2/\Lambda' \to \mathbb{C}^2/\Lambda'$ that is induced by a linear map $A : \mathbb{C}^2 \to \mathbb{C}^2$ if and only if

$$A(v) - v \in \Lambda'\,.$$

In order to find the fixed points we therefore have to determine the preimage $(A-I)^{-1}(\Lambda') \subset \mathbb{C}^2$ and project it down to $\mathbb{C}^2/\Lambda'$. Let $(v_1, \ldots, v_4)$ be a basis of the lattice Λ' and let L be the 4×4-matrix whose columns are $(v_1, \ldots, v_4)$. $(A - I)^{-1}(\Lambda')$ is a lattice that is spanned by the columns of the matrix $(A - I)^{-1}L$. By computing the matrix $L^{-1}(A - I)^{-1}L$ we are able to write the generators of $(A - I)^{-1}(\Lambda')$ as a linear combination of the v_i.

We carry out this procedure for the case A_2. Since $\alpha''^2 = \alpha''^{-1}$, it suffices to consider the fixed locus of α''. $(\alpha'' - I)^{-1}$ has the following matrix representation with respect to the standard basis (e_4, e_5, e_6, e_7):

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{6} & 0 & 0 \\ -\frac{\sqrt{3}}{6} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{3}}{6} & -\frac{1}{2} \end{pmatrix}$$

The hexagonal lattice is generated by the two vectors $(1, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Therefore, the matrix L is given by

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} .$$

We obtain

$$L^{-1}(\alpha'' - I)^{-1}L = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} .$$

Since we are free to shift a vector $v \in \mathbb{C}^2$ by a lattice vector without changing $v + \Lambda'$, we conclude that the projection of $(\alpha'' - I)^{-1}(\Lambda')$ down to $\mathbb{C}^2/\Lambda'$ is generated by

$$\tfrac{2}{3}v_1 + \tfrac{2}{3}v_2 + \Lambda', \quad \tfrac{1}{3}v_1 + \tfrac{1}{3}v_2 + \Lambda', \quad \tfrac{1}{3}v_3 + \tfrac{1}{3}v_4 + \Lambda', \quad \tfrac{2}{3}v_3 + \tfrac{2}{3}v_4 + \Lambda',$$

which can be reduced to

$$\tfrac{1}{3}v_1 + \tfrac{1}{3}v_2 + \Lambda' = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \\ 0 \\ 0 \end{pmatrix} + \Lambda' \quad \text{and} \quad \tfrac{1}{3}v_3 + \tfrac{1}{3}v_4 + \Lambda' = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix} + \Lambda'$$

Therefore we have

$$\mathrm{Fix}(\alpha) = \bigcup_{\lambda_1, \lambda_2 \in \{0,1,2\}} \{(x_1, x_2, x_3, \tfrac{1}{2}\lambda_1, \tfrac{\sqrt{3}}{6}\lambda_1, \tfrac{1}{2}\lambda_2, \tfrac{\sqrt{3}}{6}\lambda_2) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\} .$$

T^7/Δ therefore is an orbifold with A_2-singularities along nine 3-tori. By a short calculation we see that β leaves any torus with $\lambda_1 = 0$ invariant, maps a torus with $\lambda_1 = 1$ to the corresponding torus with $\lambda_1 = 2$ and vice versa. We have

$$\gamma''(\tfrac{1}{2}\lambda_1, \tfrac{\sqrt{3}}{6}\lambda_1, \tfrac{1}{2}\lambda_2, \tfrac{\sqrt{3}}{6}\lambda_2) = (-\tfrac{1}{2}\lambda_1, \tfrac{\sqrt{3}}{6}\lambda_1, -\tfrac{1}{2}\lambda_2, \tfrac{\sqrt{3}}{6}\lambda_2)$$

Since the difference between $(\frac{1}{2}\lambda_1, \frac{\sqrt{3}}{6}\lambda_1, \frac{1}{2}\lambda_2, \frac{\sqrt{3}}{6}\lambda_2)$ and its image is a lattice vector, the map γ leaves any torus invariant. The singular locus of T^7/Γ therefore consists of three copies of $T^3/\mathbb{Z}_2^2$ that are the projections of the 3-tori with $\lambda_1 = 0$ and of three copies of $T^3/\langle\gamma\rangle$ that are the projections of a pair of tori with $\lambda_1 = 1$ and $\lambda_1 = 2$.

In the case A_5, α generates a group of order 6. Therefore, we have to consider the fixed point sets $\mathrm{Fix}(\alpha) = \mathrm{Fix}(\alpha^5)$, $\mathrm{Fix}(\alpha^2) = \mathrm{Fix}(\alpha^4)$ and $\mathrm{Fix}(\alpha^3)$. We can use the same method as before and see that $\alpha'' : \mathbb{C}^2/\Lambda' \to \mathbb{C}^2/\Lambda'$ has only $0 + \Lambda'$ as fixed point since the matrix $L^{-1}(\alpha'' - I)^{-1}L$ has only integer coefficients. α^2 is the same as α from the case A_2 and we obtain the same fixed point set. α''^3 is minus the identity and its fixed points are given by $\frac{1}{2}\Lambda'$. Since Λ' is generated by 4 vectors we have 16 distinct fixed points. All in all, the non-empty fixed point sets of Γ are given by

$$\begin{aligned}
\mathrm{Fix}(\alpha) &= \{(x_1, x_2, x_3, 0, 0, 0, 0) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\} = \mathrm{Fix}(\alpha^5) \\
\mathrm{Fix}(\alpha^2) &= \textstyle\bigcup_{\lambda_1,\lambda_2 \in \{0,1,2\}} \{(x_1, x_2, x_3, \frac{1}{2}\lambda_1, \frac{\sqrt{3}}{6}\lambda_1, \frac{1}{2}\lambda_2, \frac{\sqrt{3}}{6}\lambda_2) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\} = \mathrm{Fix}(\alpha^4) \\
\mathrm{Fix}(\alpha^3) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{0,1\}} \{(x_1, x_2, x_3, \frac{1}{2}\epsilon_1 + \frac{1}{4}\epsilon_2, \frac{\sqrt{3}}{4}\epsilon_2, \frac{1}{2}\epsilon_3 + \frac{1}{4}\epsilon_4, \frac{\sqrt{3}}{4}\epsilon_4) | x_1, x_2, x_3 \in \mathbb{R}\}
\end{aligned}$$

We introduce the notation

$$\begin{aligned}
T_{\lambda_1,\lambda_2} &:= \{(x_1, x_2, x_3, \tfrac{1}{2}\lambda_1, \tfrac{\sqrt{3}}{6}\lambda_1, \tfrac{1}{2}\lambda_2, \tfrac{\sqrt{3}}{6}\lambda_2) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\} \\
T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4} &:= \{(x_1, x_2, x_3, \tfrac{1}{2}\epsilon_1 + \tfrac{1}{4}\epsilon_2, \tfrac{\sqrt{3}}{4}\epsilon_2, \tfrac{1}{2}\epsilon_3 + \tfrac{1}{4}\epsilon_4, \tfrac{\sqrt{3}}{4}\epsilon_4) | x_1, x_2, x_3 \in \mathbb{R}\}
\end{aligned}$$

The torus $T_{0,0}$ is fixed by the group that is generated by α. We interpret λ_1 and λ_2 as elements of $\mathbb{Z}_3$. α^3 maps T_{λ_1,λ_2} to $T_{3-\lambda_1,3-\lambda_2}$. Therefore, $\mathrm{Fix}(\alpha^2) \setminus \mathrm{Fix}(\alpha)$ decomposes into 4 orbits:

$$\{T_{0,1}, T_{0,2}\}\,, \quad \{T_{1,0}, T_{2,0}\}\,, \quad \{T_{1,1}, T_{2,2}\}\,, \quad \{T_{1,2}, T_{2,1}\}\,.$$

Let $v_1, \ldots, v_4$ be the basis of Λ' that consists of the columns of L. α''^2 acts as an rotation around an angle of $\frac{2\pi}{3}$ on the (x_4, x_5)-plane. It maps v_1 to v_2 and v_2 to $v_2 - v_1$. Therefore, there is an orbit of $\langle\alpha''^2\rangle$ that consists of the elements

$$\tfrac{1}{2}v_1 + \Lambda'\,, \quad \tfrac{1}{2}v_2 + \Lambda'\,, \quad \tfrac{1}{2}(v_1 + v_2) + \Lambda'\,.$$

A similar argument can be made for the (x_6, x_7)-plane. We see that $\mathrm{Fix}(\alpha^3) \setminus \mathrm{Fix}(\alpha)$ decomposes into the following $\langle \alpha^2 \rangle$-orbits:

$$\begin{aligned}
&\{T_{0,0,0,1}, T_{0,0,1,0}, T_{0,0,1,1}\}\,, \quad \{T_{0,1,0,0}, T_{1,0,0,0}, T_{1,1,0,0}\}\,, \quad \{T_{0,1,0,1}, T_{1,0,1,0}, T_{1,1,1,1}\}\,, \\
&\{T_{0,1,1,0}, T_{1,0,1,1}, T_{1,1,0,1}\}\,, \quad \{T_{0,1,1,1}, T_{1,0,0,1}, T_{1,1,1,0}\}\,.
\end{aligned}$$

All in all, T^7/Δ has singularities along 10 distinct 3-tori. Along one of them we have an A_5-singularity, along 4 we have A_2-singularities and along 5 we have A_1-singularities. β and γ leave $T_{0,0}$ invariant. $T_{0,0}$ therefore yields an A_5-singularity along $T^3/\langle \beta, \gamma \rangle$ on T^7/Γ. As we have already seen in the case A_2, β maps T_{λ_1,λ_2} to $T_{3-\lambda_1,\lambda_2}$ and γ leaves T_{λ_1,λ_2} invariant. The orbits $\{T_{0,1}, T_{0,2}\}$ and $\{T_{1,0}, T_{2,0}\}$ therefore yield two A_2-singularities along $T^3/\langle \beta, \gamma \rangle$ and the orbits $\{T_{1,1}, T_{2,2}\}$ and $\{T_{1,2}, T_{2,1}\}$ yield an A_2-singularity along $T^3/\langle \gamma \rangle$. β acts as -1 on $\mathrm{span}(e_4, e_5)$ and as the identity on $\mathrm{span}(e_6, e_7)$. Therefore, it leaves each $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ invariant. We interpret the ϵ_i as elements of $\mathbb{Z}_2$. γ maps the torus $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ to $T_{\epsilon_1+\epsilon_2,\epsilon_2,\epsilon_3+\epsilon_4,\epsilon_4}$. We see that each of the 5 $\langle \alpha^2 \rangle$-orbits is invariant under γ. $\mathrm{Fix}(\alpha^3)$ therefore yields 5 A_1-singularities along submanifolds that are isometric to $T^3/\langle \beta, \gamma \rangle$.

We determine the Betti numbers b^2 and b^3 of T^7/Γ for $\Delta \in \{A_2, A_5\}$. None of the 2-forms (5.15) that are invariant under β and γ and no linear combination of them is invariant under the map α from the case A_2. Therefore, we have in both cases $b^2(T^7/\Gamma) = 0$. We recall that the space $W^{\beta,\gamma}$ of all 3-forms with constant coefficients that are preserved by β and γ is spanned by

$$dx^{123}, dx^{124}, dx^{135}, dx^{145}, dx^{167}, dx^{236}, dx^{246}, dx^{257}, dx^{347}, dx^{356}, dx^{456}\,.$$

The monomials that are invariant with respect to the map α from the case A_2 are

$$dx^{123}, dx^{145}, dx^{167}\,.$$

Any element of $W^{\beta,\gamma}$ that contains a term of type $\mu_1 dx^{124} + \mu_2 dx^{135} + \mu_3 dx^{236}$ is mapped by the pull-back of α to an element outside of that space. $\alpha^* dx^{456}$ is a 3-form that contains a multiple of dx^{457}. Since α preserves the splitting of the space of all 3-forms with constant coefficients into W_4 and its complement and dx^{456} is the only element of W_4 that is invariant under β and γ, any element of $W^{\beta,\gamma}$ that contains a $\mu\, dx^{456}$ is mapped outside of $W^{\beta,\gamma}$. We finally consider the action of α on the spaces

$$\mathrm{span}\{dx^{246}, dx^{247}, dx^{256}, dx^{257}\} \quad \text{and} \quad \mathrm{span}\{dx^{346}, dx^{347}, dx^{356}, dx^{357}\}\,.$$

Both spaces are α-invariant and the action of α on both spaces has the matrix representation

$$\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}.$$

The above matrix has 1 as an eigenvalue with multiplicity 2. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The first eigenvector corresponds to $dx^{246} - dx^{257}$ from the first space or to $dx^{346} - dx^{357}$ from the second space. Only $dx^{246} - dx^{257}$ is invariant under $\langle \beta, \gamma \rangle$. The second eigenvector corresponds to $dx^{247} + dx^{256}$ or $dx^{347} + dx^{356}$. $dx^{347} + dx^{356}$ is $\langle \beta, \gamma \rangle$-invariant but $dx^{247} + dx^{256}$ is not invariant. All in all, there are 5 linearly independent Γ-invariant harmonic 3-forms and thus we have $b^3(T^7/\Gamma) = 5$ in the case A_2. In the case A_5, we obtain the group Γ by adding the additional generator

$$\alpha_2(x_1, \ldots, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)$$

to the group Δ from the case A_2. Since α_2 leaves all of the 5 forms from the case A_2 invariant, we obtain the same Betti numbers in the case A_5.

The remaining cases D_4, D_5 and E_6 will be discussed in parallel. In order to keep our presentation short, we restrict our discussion to the most important intermediate steps. The group Γ is in all three cases generated by β and γ and by two additional maps α_1 and α_2. In the case D_n, these maps are given by

$$\begin{aligned} \alpha_1(x_1, \ldots, x_7) &= (x_1, x_2, x_3, \cos\left(\tfrac{\pi}{n-2}\right)x_4 - \sin\left(\tfrac{\pi}{n-2}\right)x_5, \sin\left(\tfrac{\pi}{n-2}\right)x_4 + \cos\left(\tfrac{\pi}{n-2}\right)x_5, \\ &\quad \cos\left(\tfrac{\pi}{n-2}\right)x_6 + \sin\left(\tfrac{\pi}{n-2}\right)x_7, -\sin\left(\tfrac{\pi}{n-2}\right)x_6 + \cos\left(\tfrac{\pi}{n-2}\right)x_7) \\ \alpha_2(x_1, \ldots, x_7) &= (x_1, x_2, x_3, -x_7, x_6, -x_5, x_4) \end{aligned}$$

and Γ consists of the elements

$$\alpha_1^{i_1} \alpha_2^{i_2} \beta^{i_3} \gamma^{i_4}$$

with $i_1 \in \{0, \ldots, 2n-5\}$ and $i_2, i_3, i_4 \in \{0, 1\}$. In the case E_6, the generators of Δ are:

$$\begin{array}{rcl}\alpha_1(x_1,\ldots,x_7) & = & (x_1,x_2,x_3,\frac{1}{2}x_4-\frac{1}{2}x_5-\frac{1}{2}x_6-\frac{1}{2}x_7,\frac{1}{2}x_4+\frac{1}{2}x_5+\frac{1}{2}x_6-\frac{1}{2}x_7, \\ & & \frac{1}{2}x_4-\frac{1}{2}x_5+\frac{1}{2}x_6+\frac{1}{2}x_7,\frac{1}{2}x_4+\frac{1}{2}x_5-\frac{1}{2}x_6+\frac{1}{2}x_7) \\ \alpha_2(x_1,\ldots,x_7) & = & (x_1,x_2,x_3,\frac{1}{2}x_4-\frac{1}{2}x_5+\frac{1}{2}x_6-\frac{1}{2}x_7,\frac{1}{2}x_4+\frac{1}{2}x_5+\frac{1}{2}x_6+\frac{1}{2}x_7, \\ & & -\frac{1}{2}x_4-\frac{1}{2}x_5+\frac{1}{2}x_6+\frac{1}{2}x_7,\frac{1}{2}x_4-\frac{1}{2}x_5-\frac{1}{2}x_6+\frac{1}{2}x_7)\end{array}$$

A list of all elements of the group E_6 can be found in equation (2.3). The underlying set of Γ consists of

$$g\beta^{i_1}\gamma^{i_2}$$

with $g \in E_6$ and $i_1, i_2 \in \{0,1\}$. Our next step is to choose a lattice $\Lambda' \subset \mathbb{C}^2$ that is invariant under the group Δ that is generated by α_1 and α_2. Suitable choices for the lattices are

Group	Lattice Λ'
D_4	$\mathrm{span}_{\mathbb{Z}}(e_4,e_5,e_6,e_7)$
D_5	$\mathrm{span}_{\mathbb{Z}}\left(e_4,\frac{1}{2}e_4+\frac{\sqrt{3}}{2}e_5,e_7,\frac{1}{2}e_7+\frac{\sqrt{3}}{2}e_6\right)$
E_6	$\mathrm{span}_{\mathbb{Z}}(e_4,e_5,e_6,e_7)\cup((\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})+\mathrm{span}_{\mathbb{Z}}(e_4,e_5,e_6,e_7))$

The lattice in the case D_5 is the same as in the case A_2 except that we have interchanged the role of e_7 and e_6. This is necessary to ensure that Λ' is invariant under α_2. It is a little bit hard to see directly that the third lattice is invariant under the generators α_1 and α_2 of E_6. The mathematical reason behind this is that the lattice can be identified with the ring of the Hurwitz quaternions:

$$\{x_1+x_2i+x_3j+x_4k|x_1,\ldots,x_4\in\mathbb{Z} \ \text{ or } \ x_1,\ldots,x_4\in\mathbb{Z}+\tfrac{1}{2}\}$$

E_6 is precisely the group of units of this ring [15]. In particular, this implies that E_6 leaves the Hurwitz quaternions invariant. We determine the fixed locus of all group elements in oder to describe the singular locus of T^7/Γ. Since we assume that $j=1$, only the elements of Δ, i.e. those elements that do not contain a β or γ contribute to the fixed locus. We start with D_4. In this case, Δ consists of 8 elements. We determine the fixed point sets of all elements of Δ and obtain

$$\begin{array}{rcl}\mathrm{Fix}(1) & = & T^7 \\ \mathrm{Fix}(\alpha_1) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2)+\Lambda|x_1,x_2,x_3\in\mathbb{R}\} \\ \mathrm{Fix}(\alpha_1^2) & = & \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4)+\Lambda|x_1,x_2,x_3\in\mathbb{R}\} \\ \mathrm{Fix}(\alpha_1^3) & = & \mathrm{Fix}(\alpha_1) \\ \mathrm{Fix}(\alpha_2) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1)+\Lambda|x_1,x_2,x_3\in\mathbb{R}\} \\ \mathrm{Fix}(\alpha_1\alpha_2) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2)+\Lambda|x_1,x_2,x_3\in\mathbb{R}\} \\ \mathrm{Fix}(\alpha_1^2\alpha_2) & = & \mathrm{Fix}(\alpha_2) \\ \mathrm{Fix}(\alpha_1^3\alpha_2) & = & \mathrm{Fix}(\alpha_1\alpha_2)\end{array}$$

A torus $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4} := \{(x_1, x_2, x_3, \frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{2}\epsilon_3, \frac{1}{2}\epsilon_4) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\}$ is mapped by α_1 to $T_{\epsilon_2,\epsilon_1,\epsilon_4,\epsilon_3}$ and by α_2 to $T_{\epsilon_4,\epsilon_3,\epsilon_2,\epsilon_1}$. Therefore, we obtain the following isotropy groups:

$$\begin{array}{ll} \Delta & \text{if } \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 \\ \{1, \alpha_1, \alpha_1^2, \alpha_1^3\} & \text{if } \epsilon_1 = \epsilon_2,\ \epsilon_3 = \epsilon_4, \text{ but } \epsilon_1 \neq \epsilon_3 \\ \{1, \alpha_1\alpha_2, \alpha_1^2, \alpha_1^3\alpha_2\} & \text{if } \epsilon_1 = \epsilon_3,\ \epsilon_2 = \epsilon_4, \text{ but } \epsilon_1 \neq \epsilon_2 \\ \{1, \alpha_2, \alpha_1^2, \alpha_1^2\alpha_2\} & \text{if } \epsilon_1 = \epsilon_4,\ \epsilon_2 = \epsilon_3, \text{ but } \epsilon_1 \neq \epsilon_2 \\ \{1, \alpha_1^2\} & \text{otherwise} \end{array}$$

The set of all 16 tori decomposes into the following orbits with respect to the action of Δ:

$$\begin{array}{l} \{T_{0,0,0,0}\}\ , \quad \{T_{0,0,0,1}, T_{0,0,1,0}, T_{0,1,0,0}, T_{1,0,0,0}\}\ , \\ \{T_{0,0,1,1}, T_{1,1,0,0}\}\ , \quad \{T_{0,1,0,1}, T_{1,0,1,0}\}\ , \quad \{T_{0,1,1,0}, T_{1,0,0,1}\}\ , \\ \{T_{0,1,1,1}, T_{1,0,1,1}, T_{1,1,0,1}, T_{1,1,1,0}\}\ , \quad \{T_{1,1,1,1}\} \end{array}$$

All in all, T^7/Δ has singularities along 7 tori. Along 2 of them we have D_4-singularities, along 3 of them we have A_3-singularities and along 2 of them we have A_1-singularities. Since β'' and γ'' are both diagonal matrices with coefficients ± 1 along the diagonal, β'' and γ'' preserve all tori $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$. The singular locus of T^7/Γ therefore consists of 7 submanifolds of type $T^3/\langle \beta, \gamma \rangle$.

The next group that we consider is D_5, which consists of 12 elements. We introduce some new notation that allows us to present our results in a simple form. Let

$$(v_1, v_2, v_3, v_4) := (e_4, \tfrac{1}{2}e_4 + \tfrac{\sqrt{3}}{2}e_5, e_7, \tfrac{1}{2}e_7 + \tfrac{\sqrt{3}}{2}e_6)$$

be our basis of the lattice Λ' and let

$$L := \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

be the matrix whose columns can be identified with the v_i. Moreover, let $\mu_1, \ldots, \mu_4 \in \mathbb{R}$ be arbitrary and let $y := L(\mu_1, \ldots, \mu_4)^T$. We denote the torus

$$\{(x_1, x_2, x_3, y_1, y_2, y_3, y_4) + \Lambda | x_1, x_2, x_3 \in \mathbb{R}\}$$

by $T_{\mu_1,\mu_2,\mu_3,\mu_4}$. We keep im mind that $T_{\mu_1,\mu_2,\mu_3,\mu_4} = T_{\mu_1+q_1,\mu_2+q_2,\mu_3+q_3,\mu_4+q_4}$ for all $q_1, \ldots, q_4 \in \mathbb{Z}$. The fixed locus of any $g \in \Delta$ is a union of tori of this kind. We determine the fixed loci of all $g \in \Delta$ by computing the matrices $L^{-1}(g-I)^{-1}L$ with help of a computer algebra system. With help of these matrices we see that

$$\begin{array}{rcl}
\mathrm{Fix}(1) & = & T^7 \\
\mathrm{Fix}(\alpha_1) & = & T_{0,0,0,0} \\
\mathrm{Fix}(\alpha_1^2) & = & \bigcup_{\lambda_1,\lambda_2\in\{0,1,2\}} T_{\frac{1}{3}\lambda_1,\frac{1}{3}\lambda_1,\frac{1}{3}\lambda_2,\frac{1}{3}\lambda_2} \\
\mathrm{Fix}(\alpha_1^3) & = & \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}} T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4} \\
\mathrm{Fix}(\alpha_1^4) & = & \mathrm{Fix}(\alpha_1^2) \\
\mathrm{Fix}(\alpha_1^5) & = & \mathrm{Fix}(\alpha_1) \\
\mathrm{Fix}(\alpha_2) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}} T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}(\epsilon_1+\epsilon_2),\frac{1}{2}\epsilon_2} \\
\mathrm{Fix}(\alpha_1\alpha_2) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}} T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1} \\
\mathrm{Fix}(\alpha_1^2\alpha_2) & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}} T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1,\frac{1}{2}(\epsilon_1+\epsilon_2)} \\
\mathrm{Fix}(\alpha_1^3\alpha_2) & = & \mathrm{Fix}(\alpha_2) \\
\mathrm{Fix}(\alpha_1^4\alpha_2) & = & \mathrm{Fix}(\alpha_1\alpha_2) \\
\mathrm{Fix}(\alpha_1^5\alpha_2) & = & \mathrm{Fix}(\alpha_1^2\alpha_2)
\end{array}$$

We determine the isotropy groups of the fixed tori and see that

- $T_{0,0,0,0}$ has all of Δ as isotropy group.
- Any torus $T_{\frac{1}{3}\lambda_1,\frac{1}{3}\lambda_1,\frac{1}{3}\lambda_2,\frac{1}{3}\lambda_2}$ with $(\lambda_1,\lambda_2)\neq(0,0)$ has $\{1,\alpha_1^2,\alpha_1^4\}$ as isotropy group.
- Any torus $T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}(\epsilon_1+\epsilon_2),\frac{1}{2}\epsilon_2}$ with $(\epsilon_1,\epsilon_2)\neq(0,0)$ has $\{1,\alpha_2,\alpha_1^3,\alpha_1^3\alpha_2\}$ as isotropy group.
- Any torus $T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1}$ with $(\epsilon_1,\epsilon_2)\neq(0,0)$ has $\{1,\alpha_1\alpha_2,\alpha_1^3,\alpha_1^4\alpha_2\}$ as isotropy group.
- Any torus $T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_1,\frac{1}{2}(\epsilon_1+\epsilon_2)}$ with $(\epsilon_1,\epsilon_2)\neq(0,0)$ has $\{1,\alpha_1^2\alpha_2,\alpha_1^3,\alpha_1^5\alpha_2\}$ as isotropy group.
- The remaining 6 tori of type $T_{\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4}$ with $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}$ have $\{1,\alpha_1^3\}$ as isotropy group.

By a short calculation we see that α_1 maps the torus $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ to $T_{-\mu_2,\mu_1+\mu_2,-\mu_4,\mu_3+\mu_4}$ and that α_2 maps $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ to $T_{-\mu_3-\mu_4,\mu_4,\mu_1+\mu_2,-\mu_2}$. The set of all tori that are fixed by at least one $g\in\Delta$ therefore decomposes into the following orbits:

$$\begin{array}{l}
\{T_{0,0,0,0}\} \\
\left\{T_{0,0,\frac{1}{3},\frac{1}{3}}, T_{0,0,\frac{2}{3},\frac{2}{3}}, T_{\frac{1}{3},\frac{1}{3},0,0}, T_{\frac{2}{3},\frac{2}{3},0,0}\right\} \\
\left\{T_{\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}}, T_{\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}}, T_{\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3}}, T_{\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}}\right\} \\
\left\{T_{0,0,0,\frac{1}{2}}, T_{0,0,\frac{1}{2},\frac{1}{2}}, T_{0,0,\frac{1}{2},0}, T_{\frac{1}{2},\frac{1}{2},0,0}, T_{\frac{1}{2},0,0,0}, T_{0,\frac{1}{2},0,0}\right\} \\
\left\{T_{0,\frac{1}{2},0,\frac{1}{2}}, T_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}, T_{\frac{1}{2},0,\frac{1}{2},0}\right\} \\
\left\{T_{0,\frac{1}{2},\frac{1}{2},0}, T_{\frac{1}{2},\frac{1}{2},0,\frac{1}{2}}, T_{\frac{1}{2},0,\frac{1}{2},\frac{1}{2}}\right\} \\
\left\{T_{0,\frac{1}{2},\frac{1}{2},\frac{1}{2}}, T_{\frac{1}{2},\frac{1}{2},\frac{1}{2},0}, T_{\frac{1}{2},0,0,\frac{1}{2}}\right\}
\end{array}$$

The singular locus of T^7/Δ thus consists of seven 3-tori. Along 1 of them there is an D_5-singularity, along 3 of them we have A_3-singularities, along 2 of them we have A_2-singularities, and along 1 of them we have an A_1-singularity. β maps the torus $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ to $T_{-\mu_1,-\mu_2,\mu_3,\mu_4}$ and γ maps $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ to $T_{-\mu_1-\mu_2,\mu_2,\mu_3+\mu_4,-\mu_4}$. We see that β and γ act on each of the singular tori of T^7/Δ and thus T^7/Γ has singularities along seven submanifolds that are isometric to $T^3/\langle\beta,\gamma\rangle$.

Our last case is $\Delta = E_6$. We recall that E_6 consists of the elements ${\alpha_1''}^k$, ${\alpha_1''}^k\alpha_2''$, ${\alpha_1''}^k\alpha_2''\alpha_1''$ and ${\alpha_1''}^k\alpha_2''{\alpha_1''}^2$ where $k \in \{0,\ldots,5\}$ and

$$\alpha_1'' = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_2'' = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

We have to determine the fixed locus of all elements of Δ. As in the previous case, we do this by computing all matrices $F_g := L^{-1}(g-I)^{-1}L$ with $g \in \Delta$ where we choose L as

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

As in the previous case, we define v_1, v_2, v_3 and v_4 as the columns of L and $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ as

$$\{(x_1,x_2,x_3,y_1,y_2,y_3,y_4) + \Lambda | x_1,x_2,x_3 \in \mathbb{R}\}\,.$$

where $(y_1, y_2, y_3, y_4)^T = L(\mu_1, \mu_2, \mu_3, \mu_4)^T$. We take a look at the matrices F_g and see that several of them have only integer coefficients. This means that the only $v + \Lambda'$ that are mapped to itself by g are those with $v \in \Lambda'$. Therefore, we conclude that

$$\begin{aligned}\mathrm{Fix}(\alpha_1) &= \mathrm{Fix}(\alpha_1^5) = \mathrm{Fix}(\alpha_2) = \mathrm{Fix}(\alpha_1^5\alpha_2) = \mathrm{Fix}(\alpha_1^4\alpha_2\alpha_1) = \mathrm{Fix}(\alpha_1^5\alpha_2\alpha_1)\\ &= \mathrm{Fix}(\alpha_1^3\alpha_2\alpha_1^2) = \mathrm{Fix}(\alpha_1^4\alpha_2\alpha_1^2) = T_{0,0,0,0}\end{aligned}$$

$\alpha_1''^3$ is minus the identity. Its fixed points are $\frac{1}{2}\epsilon_1 v_1 + \ldots + \frac{1}{2}\epsilon_4 v_4 + \Lambda'$ where $\epsilon_1, \ldots, \epsilon_4 \in \{0, 1\}$. More explicitly

$$\mathrm{Fix}(\alpha_1^3) = \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{0,1\}} T_{\frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{2}\epsilon_3, \frac{1}{2}\epsilon_4}\,. \tag{5.18}$$

There are further 6 matrices whose coefficients are elements of $\frac{1}{2}\mathbb{Z}$. From these matrices we obtain

$$\begin{aligned}\mathrm{Fix}(\alpha_1\alpha_2) &= \mathrm{Fix}(\alpha_1^4\alpha_2) = \mathrm{Fix}(\alpha_2\alpha_1) = \mathrm{Fix}(\alpha_1^3\alpha_2\alpha_1) = \mathrm{Fix}(\alpha_1^2\alpha_2\alpha_1^2) = \mathrm{Fix}(\alpha_1^5\alpha_2\alpha_1^2)\\ &= \bigcup_{\epsilon_1,\epsilon_2 \in \{0,1\}} T_{\frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{2}(\epsilon_1+\epsilon_2), 0}\end{aligned} \tag{5.19}$$

The remaining 8 matrices have coefficients in $\frac{1}{3}\mathbb{Z}$. By carefully comparing the lattices that are spanned by their column vectors we see that

$$\begin{array}{rcl}\mathrm{Fix}(\alpha_1^2) &=& \mathrm{Fix}(\alpha_1^4)\\ &=& \bigcup_{\lambda_1,\lambda_2\in\{0,1,2\}} T_{\frac{1}{3}\lambda_1, \frac{1}{3}\lambda_2, \frac{2}{3}\lambda_2, \frac{1}{3}(\lambda_1+2\lambda_2)}\\ \mathrm{Fix}(\alpha_1^2\alpha_2) &=& \mathrm{Fix}(\alpha_1\alpha_2\alpha_1^2)\\ &=& \bigcup_{\lambda_1,\lambda_2\in\{0,1,2\}} T_{\frac{1}{3}\lambda_1, \frac{1}{3}(\lambda_1+\lambda_2), \frac{1}{3}(\lambda_1+2\lambda_2), \frac{1}{3}\lambda_2}\\ \mathrm{Fix}(\alpha_1^3\alpha_2) &=& \mathrm{Fix}(\alpha_1\alpha_2\alpha_1)\\ &=& \bigcup_{\lambda_1,\lambda_2\in\{0,1,2\}} T_{\frac{1}{3}\lambda_1, \frac{1}{3}\lambda_2, \frac{2}{3}\lambda_1, \frac{1}{3}(\lambda_1+\lambda_2)}\\ \mathrm{Fix}(\alpha_1^2\alpha_2\alpha_1) &=& \mathrm{Fix}(\alpha_2\alpha_1^2)\\ &=& \bigcup_{\lambda_1,\lambda_2\in\{0,1,2\}} T_{\frac{1}{3}\lambda_1, \frac{2}{3}\lambda_1, \frac{1}{3}\lambda_2, \frac{1}{3}(2\lambda_1+\lambda_2)}\end{array} \tag{5.20}$$

α_i acts on the indices of a torus $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ by the matrix $L^{-1}\alpha_i''L$. We have

$$L^{-1}\alpha_1''L = \begin{pmatrix} 0 & -1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 1 & 0\\ 1 & 1 & -1 & 1\end{pmatrix} \quad \text{and} \quad L^{-1}\alpha_2''L = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 1\\ -1 & 0 & 1 & 0\\ 1 & -1 & -1 & 0\end{pmatrix}.$$

These matrices can be used to determine the orbits of the fixed tori with respect to Δ. We carry out the necessary calculations with help of a computer algebra system and obtain the following orbits:

$$\begin{aligned}
\mathcal{O}_1 &:= \{T_{0,0,0,0}\} \\
\mathcal{O}_2 &:= \Big\{T_{0,0,0,\frac{1}{2}}, T_{0,0,\frac{1}{2},0}, T_{0,\frac{1}{2},0,0}, T_{\frac{1}{2},0,0,0}, T_{0,0,\frac{1}{2},\frac{1}{2}}, T_{0,\frac{1}{2},0,\frac{1}{2}}, \\
&\qquad T_{\frac{1}{2},0,0,\frac{1}{2}}, T_{0,\frac{1}{2},\frac{1}{2},\frac{1}{2}}, T_{\frac{1}{2},0,\frac{1}{2},\frac{1}{2}}, T_{\frac{1}{2},\frac{1}{2},0,\frac{1}{2}}, T_{\frac{1}{2},\frac{1}{2},\frac{1}{2},0}, T_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\Big\} \\
\mathcal{O}_3 &:= \Big\{T_{0,\frac{1}{2},\frac{1}{2},0}, T_{\frac{1}{2},0,\frac{1}{2},0}, T_{\frac{1}{2},\frac{1}{2},0,0}\Big\} \\
\mathcal{O}_4 &:= \Big\{T_{0,\frac{1}{3},\frac{2}{3},\frac{1}{3}}, T_{0,\frac{2}{3},\frac{1}{3},\frac{2}{3}}, T_{\frac{1}{3},0,0,\frac{1}{3}}, T_{\frac{2}{3},0,0,\frac{2}{3}}, T_{\frac{1}{3},\frac{2}{3},\frac{1}{3},0}, T_{\frac{2}{3},\frac{1}{3},\frac{2}{3},0}, T_{\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}}, T_{\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3}}\Big\} \\
\mathcal{O}_5 &:= \Big\{T_{0,\frac{1}{3},\frac{2}{3},\frac{2}{3}}, T_{0,\frac{2}{3},\frac{1}{3},\frac{1}{3}}, T_{\frac{1}{3},0,\frac{2}{3},\frac{1}{3}}, T_{\frac{2}{3},0,\frac{1}{3},\frac{2}{3}}, T_{\frac{1}{3},\frac{2}{3},0,\frac{2}{3}}, T_{\frac{2}{3},\frac{1}{3},0,\frac{1}{3}}, T_{\frac{1}{3},\frac{1}{3},\frac{1}{3},0}, T_{\frac{2}{3},\frac{2}{3},\frac{2}{3},0}\Big\} \\
\mathcal{O}_6 &:= \Big\{T_{0,\frac{1}{3},0,\frac{1}{3}}, T_{0,\frac{2}{3},0,\frac{2}{3}}, T_{\frac{1}{3},0,\frac{2}{3},\frac{2}{3}}, T_{\frac{2}{3},0,\frac{1}{3},\frac{1}{3}}, T_{\frac{1}{3},\frac{1}{3},\frac{2}{3},0}, T_{\frac{2}{3},\frac{2}{3},\frac{1}{3},0}, T_{\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3}}, T_{\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}}\Big\} \\
\mathcal{O}_7 &:= \Big\{T_{0,0,\frac{1}{3},\frac{1}{3}}, T_{0,0,\frac{2}{3},\frac{2}{3}}, T_{\frac{1}{3},\frac{2}{3},0,\frac{1}{3}}, T_{\frac{2}{3},\frac{1}{3},0,\frac{2}{3}}, T_{\frac{1}{3},\frac{2}{3},\frac{2}{3},0}, T_{\frac{2}{3},\frac{1}{3},\frac{1}{3},0}, T_{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}}, T_{\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3}}\Big\}
\end{aligned}$$

Each element of an orbit with cardinality l has an isotropy group of order $\frac{24}{l}$. This fact together with the calculations that we have made in order to determine the fixed tori allow us to determine the isomorphism types of all isotropy groups. We conclude that any element of the orbit

- $\mathcal{O}_1$ has isotropy group E_6,
- $\mathcal{O}_2$ has isotropy group A_1,
- $\mathcal{O}_3$ has isotropy group D_4,
- $\mathcal{O}_4$, $\mathcal{O}_5$, $\mathcal{O}_6$ or $\mathcal{O}_7$ has isotropy group A_2.

The quotient T^7/Δ therefore has an E_6-singularity along one torus, a D_4-singularity along one torus, A_2-singularities along four tori and an A_1-singularity along one torus. We consider the action of β and γ on the singular tori. β and γ act on the indices of a torus $T_{\mu_1,\mu_2,\mu_3,\mu_4}$ by the matrices

$$L^{-1}\beta''L = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L^{-1}\gamma''L = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

β and γ preserve the orbits $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. β interchanges the orbits $\mathcal{O}_4$ and $\mathcal{O}_7$ as well as $\mathcal{O}_5$ and $\mathcal{O}_6$. Analogously, γ interchanges the orbits $\mathcal{O}_4$ and $\mathcal{O}_5$ as well as $\mathcal{O}_6$ and $\mathcal{O}_7$. The orbit of the action of $\langle\beta,\gamma\rangle$ on the set $\{\mathcal{O}_4,\mathcal{O}_5,\mathcal{O}_6,\mathcal{O}_7\}$ thus is the whole set. Therefore, the singular locus of T^7/Γ consists of three copies of $T^3/\langle\beta,\gamma\rangle$ with an E_6-, D_4- and A_1-singularity along them and a 3-torus with an A_2-singularity.

Finally, we determine the Betti numbers of the last three examples. The group D_4 contains the cyclic group A_3 of order 4 as a subgroup. Any Γ-invariant k-form on T^7 therefore has to be one of the invariant forms from the case A_3. Analogously, A_5 is a subgroup of D_5 and A_3 is a subgroup of E_6. We take a look at the invariant harmonic 3-forms in the case A_3 and see that the subspace of those forms that are additionally invariant under D_4 is spanned by

$$dx^{123}, dx^{145} + dx^{167}, dx^{246} - dx^{257}, dx^{347} + dx^{356}\,. \tag{5.21}$$

Therefore, we have $b^3(T^7/\Gamma) = 4$ in the case D_4. Since D_4 is a subgroup of E_6 and the above forms are invariant under all of $SU(2)$, we have $b^3(T^7/\Gamma) = 4$ in the case E_6, too. We proceed to the case D_5. The 3-forms with constant coefficients that are invariant under the group Γ from the case A_5 are spanned by

$$dx^{123}, dx^{145}, dx^{167}, dx^{246} - dx^{257}, dx^{347} + dx^{356}\,.$$

The subspace of those 3-forms among them that are invariant under the second generator of the D_k-groups is again spanned by (5.21). Therefore, we have $b^3(T^7/\Gamma) = 4$ in all three cases. We sum up our results.

Theorem 5.3.1. *Let $\Lambda \subset \mathbb{R}^7$ be a lattice of rank 7, $T^7 := \mathbb{R}^7/\Lambda$ be a 7-dimensional torus, and Γ be a discrete group of isometries of T^7 that preserves the standard G_2-structure on T^7 that is given by (4.3). T^7/Γ is an orbifold that carries a flat G_2-structure. If T^7/Γ has only ADE-singularities, any orbifold group is one of the following:*

$$A_1,\ A_2,\ A_3,\ A_5,\ D_4,\ D_5,\ E_6$$

As usual, we denote the finite subgroup $\Delta \subset SU(2)$ and the Dynkin diagram that is obtained from Δ by the McKay correspondence by the same name. We also refer by the name of the Dynkin diagram to the type of the singularity. For any of the above groups Δ there exists a torus quotient T^7/Γ that carries a flat G_2-structure and has only ADE-singularities such that at least one of them is of type Δ. Particular examples for each Δ can be constructed as follows. Λ shall be the lattice $span_{\mathbb{Z}}(e_1,e_2,e_3) \oplus \Lambda'$ where

$$\Lambda' = \begin{cases} span_{\mathbb{Z}}(e_4, e_5, e_6, e_7) & \textit{in the cases } A_1,\ A_3 \textit{ and } D_4 \\ span_{\mathbb{Z}}\left(e_4, \frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5, e_6, \frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7\right) & \textit{in the cases } A_2 \textit{ and } A_5 \\ span_{\mathbb{Z}}\left(e_4, \frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5, e_7, \frac{1}{2}e_7 + \frac{\sqrt{3}}{2}e_6\right) & \textit{in the case } D_5 \\ span_{\mathbb{Z}}(e_4, e_5, e_6, e_7) \cup \left((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + span_{\mathbb{Z}}(e_4, e_5, e_6, e_7)\right) & \textit{in the case } E_6 \end{cases}$$

Moreover, we identify span(e_4, e_5, e_6, e_7) *with* $\mathbb{C}^2$ *by* $z_1 := x_4 + ix_5$ *and* $z_2 := x_6 + ix_7$. *The group* Γ *is generated by the following maps*

$$\begin{aligned} \alpha_i((x_1, x_2, x_3, z_1, z_2) + \Lambda) &:= (x_1, x_2, x_3, a_i z_1 + b_i z_2, c_i z_1 + d_i z_2) + \Lambda \\ \beta((x_1, x_2, x_3, z_1, z_2) + \Lambda) &:= (\tfrac{1}{2} + x_1, -x_2, \tfrac{1}{2} - x_3, -z_1, z_2) + \Lambda \\ \gamma((x_1, x_2, x_3, z_1, z_2) + \Lambda) &:= (-x_1, \tfrac{1}{2} + x_2, -x_3, -\overline{z_1}, -\overline{z_2}) + \Lambda \end{aligned}$$

where the matrices $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ *shall generate the group* Δ *that is embedded into* $SU(2)$ *as in Section 2. In this situation,* T^7/Γ *is a* G_2*-orbifold with ADE-singularities. We define the following maps*

$$\begin{aligned} \beta'(x_1, x_2, x_3) &:= (\tfrac{1}{2} + x_1, -x_2, \tfrac{1}{2} - x_3) \\ \gamma'(x_1, x_2, x_3) &:= (-x_1, \tfrac{1}{2} + x_2, -x_3) \end{aligned}$$

The singular locus of T^7/Γ *consists of several disjoint flat submanifolds that are diffeomorphic to* $P \in \{T^3, T^3/\langle\beta\rangle(\cong T^3/\langle\gamma\rangle), T^3/\langle\beta,\gamma\rangle\}$ *with a fixed type of ADE-singularity along each copy of* P. *We denote the singular locus as the sum of terms* $n_\Xi \cdot (\Xi, P)$ *where* Ξ *is a simply laced Dynkin diagram and* $n_\Xi \cdot (\Xi, P)$ *means that we have singularities of type* Ξ *along* n_Ξ *copies of* P. *In each of our cases, we have* $b^1(T^7/\Gamma) = b^2(T^7/\Gamma) = 0$. *The singular locus and the value of* $b^3(T^7/\Gamma)$ *can be found in the table below.*

Δ	*Singular locus*	b^3
A_1	$16 \cdot (A_1, T^3/\langle\beta,\gamma\rangle)$	7
A_2	$3 \cdot (A_2, T^3/\langle\beta,\gamma\rangle) + 3 \cdot (A_2, T^3/\langle\gamma\rangle)$	5
A_3	$6 \cdot (A_1, T^3/\langle\beta,\gamma\rangle) + 4 \cdot (A_3, T^3/\langle\beta,\gamma\rangle)$	5
A_5	$1 \cdot (A_5, T^3/\langle\beta,\gamma\rangle) + 1 \cdot (A_2, T^3/\langle\gamma\rangle) +$ $2 \cdot (A_2, T^3/\langle\beta,\gamma\rangle) + 5 \cdot (A_1, T^3/\langle\beta,\gamma\rangle)$	5
D_4	$2 \cdot (D_4, T^3/\langle\beta,\gamma\rangle) + 3 \cdot (A_3, T^3/\langle\beta,\gamma\rangle) +$ $2 \cdot (A_1, T^3/\langle\beta,\gamma\rangle)$	4
D_5	$1 \cdot (D_5, T^3/\langle\beta,\gamma\rangle) + 3 \cdot (A_3, T^3/\langle\beta,\gamma\rangle) +$ $2 \cdot (A_2, T^3/\langle\beta,\gamma\rangle) + 1 \cdot (A_1, T^3/\langle\beta,\gamma\rangle)$	4
E_6	$1 \cdot (E_6, T^3/\langle\beta,\gamma\rangle) + 1 \cdot (D_4, T^3/\langle\beta,\gamma\rangle) +$ $1 \cdot (A_2, T^3) + 1 \cdot (A_1, T^3/\langle\beta,\gamma\rangle)$	4

The quotient $(\mathbb{C}^2/\Lambda')/\Delta$ has trivial canonical bundle. The resolution of its singularities has trivial canonical bundle, too, and thus is a K3 surface. The torus quotients from this subsection can therefore be considered as quotients of type $(S \times T^3)/\langle \beta, \gamma \rangle$, where S is a singular K3 surface with a flat metric and β and γ are non-symplectic involutions with respect to appropriate complex structures on S. We point out that it is not obvious to see if a singular K3 surface that is described by its period point is a quotient of a flat torus. The main goal of this section was to prove that torus quotients with a flat G_2-structure and a wide range of ADE-singularities exist. The easiest way to achieve this goal was to work with lattices and groups acting upon them rather than with the theory of K3 surfaces as in Section 5.1.

It is possible to obtain smooth G_2-manifolds from our singular examples by the method of Joyce [36]. In the case $j = 1$ that we have considered, T^7/Γ has only ADE-singularities. We recall that an ADE-singularity can be resolved by a sequence of k blow-ups, where k is the number of nodes of the associated Dynkin diagram. Since the G_2-manifolds that we obtain this way are quotients of type $(S \times T^3)/\mathbb{Z}_2^2$, where S is a smooth K3 surface, their holonomy is $Sp(1) \rtimes \mathbb{Z}_2^2$.

We use equation (5.3) to determine the Betti numbers of the manifolds M_Δ that we obtain by resolving the torus quotients T^7/Γ, where Γ is constructed from the ADE-group Δ. Let V_0 be the subspace of $H^2(S,\mathbb{R})$ that is invariant under β and γ. Equation (5.3) states that $b^2(M_\Delta) = \dim V_0$ and $b^2(M_\Delta) + b^3(M_\Delta) = 23$. $H^2(S,\mathbb{R})$ is spanned by the pull-backs of the six 2-forms with constant coefficients on $\mathrm{span}(e_4, e_5, e_6, e_7)$ and by the cohomology classes of the exceptional divisors. There is no 2-form with constant coefficients that is invariant under β'' and γ''. Moreover, γ'' fixes in most cases the singular points and thus acts non-trivially on the exceptional divisors. γ'' is an anti-holomorphic map with respect to the complex structure that is defined by $z_1 = x_4 + ix_5$ and $z_2 = x_6 + ix_7$. Therefore, it acts orientation-reversing on the exceptional divisors and as -1 on the corresponding cohomology classes. The only exception is $\Delta = E_6$, where it acts trivially on the exceptional divisors that we obtain by blowing up the A_2-singularity. Therefore, we obtain the following Betti numbers:

	b^1	b^2	b^3
M_{A_1}	0	0	23
M_{A_2}	0	0	23
M_{A_3}	0	0	23
M_{A_5}	0	0	23
M_{D_4}	0	0	23
M_{D_5}	0	0	23
M_{E_6}	0	2	21

Alternatively, we can introduce a complex structure on $\mathbb{C}^2/\Lambda'$ that makes γ'' holomorphic and β'' anti-holomorphic. For example, we could introduce the holomorphic coordinates

$w_1 = x_4 + ix_7$ and $w_2 = x_5 + ix_6$. After that, we resolve the singularities by blow-ups with respect to the new complex structure. Since in the cases A_2 and A_5 there are exceptional divisors on which β acts trivially, we obtain different Betti numbers:

	b^1	b^2	b^3
M_{A_1}	0	0	23
M_{A_2}	0	6	17
M_{A_3}	0	0	23
M_{A_5}	0	2	21
M_{D_4}	0	0	23
M_{D_5}	0	0	23
M_{E_6}	0	2	21

There is a third complex structure on $\mathbb{C}^2/\Lambda'$ that makes $\beta''\gamma''$ holomorphic and β'' as well as γ'' anti-holomorphic. In that case, we obtain the same Betti numbers as in the first case.

5.3.5 More complicated examples

We proceed to the case $j = 2$. As usual, let Δ be the finite subgroup of $SU(2)$ that is generated by the maps α_i. We recall that T^7/Γ has singularities of type $\mathbb{R} \times \mathbb{C}^3/\Delta'$, where the group Δ' is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$$

and by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An open subset of $\mathbb{C}^3/\Delta'$ can be identified with a hypersurface in T^7/Γ by $z_1 := x_2 + ix_3$, $z_2 := x_4 + ix_5$ and $z_3 := x_6 + ix_7$. Δ and Δ' are both subgroups of $SU(3)$. There are further singularities that arise from the fixed points of $\sigma\beta\gamma$ with $\sigma \in \Delta$. Since the action of any element $\sigma\beta\gamma \in \Gamma$ preserves the x_3-direction, the orbifold groups are subgroups of $\{g \in G_2 | g(e_3) = e_3\}$. This group is isomorphic to $SU(3)$, too. Therefore, the singularities of T^7/Γ can be resolved by the methods of Joyce [36] and we obtain smooth G_2-manifolds. We describe the singular locus of the orbifold T^7/Γ with $j = 2$ and $\Delta = A_1$. In this case, T^7 is simply defined as $\mathbb{R}^7/\mathbb{Z}^7$ and Γ is generated by three maps α, β and γ that are defined as

$$\begin{aligned}
\alpha((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7)+\mathbb{Z}^7\\
\beta((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (x_1,-x_2,-x_3,-x_4,-x_5,x_6,x_7)+\mathbb{Z}^7\\
\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (-x_1,\tfrac{1}{2}+x_2,-x_3,-x_4,x_5,-x_6,x_7)+\mathbb{Z}^7
\end{aligned}$$

Up to a permutation of the coordinates, this torus quotient can be found in Joyce [36, Ch. 12.5], where it is the starting point for the construction of a number of smooth G_2-manifolds. We show how to calculate the Betti numbers of the resolved G_2-manifold in our particular case, where we only use crepant resolutions. These calculations will be helpful later on when we consider our second example. α, β and γ generate an abelian group that is isomorphic to $\mathbb{Z}_2^3$. We have

$$\begin{aligned}
\alpha\beta((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (x_1,-x_2,-x_3,x_4,x_5,-x_6,-x_7)+\mathbb{Z}^7\\
\alpha\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (-x_1,\tfrac{1}{2}+x_2,-x_3,x_4,-x_5,x_6,-x_7)+\mathbb{Z}^7\\
\beta\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (-x_1,\tfrac{1}{2}-x_2,x_3,x_4,-x_5,-x_6,x_7)+\mathbb{Z}^7\\
\alpha\beta\gamma((x_1,\ldots,x_7)+\mathbb{Z}^7) &= (-x_1,\tfrac{1}{2}-x_2,x_3,-x_4,x_5,x_6,-x_7)+\mathbb{Z}^7
\end{aligned}$$

We determine the fixed locus of all group elements and see that

$$\begin{aligned}
\mathrm{Fix}(1) &= T^7\\
\mathrm{Fix}(\alpha) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(x_1,x_2,x_3,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4)+\mathbb{Z}^7|x_1,x_2,x_3\in\mathbb{R}\}\\
\mathrm{Fix}(\beta) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(x_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4,x_6,x_7)+\mathbb{Z}^7|x_1,x_6,x_7\in\mathbb{R}\}\\
\mathrm{Fix}(\gamma) &= \emptyset\\
\mathrm{Fix}(\alpha\beta) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(x_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,x_4,x_5,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4)+\mathbb{Z}^7|x_1,x_4,x_5\in\mathbb{R}\}\\
\mathrm{Fix}(\alpha\gamma) &= \emptyset\\
\mathrm{Fix}(\beta\gamma) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,\frac{1}{4}+\frac{1}{2}\epsilon_2,x_3,x_4,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4,x_7)+\mathbb{Z}^7|x_3,x_4,x_7\in\mathbb{R}\}\\
\mathrm{Fix}(\alpha\beta\gamma) &= \textstyle\bigcup_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,\frac{1}{4}+\frac{1}{2}\epsilon_2,x_3,\frac{1}{2}\epsilon_3,x_5,x_6,\frac{1}{2}\epsilon_4)+\mathbb{Z}^7|x_3,x_5,x_6\in\mathbb{R}\}
\end{aligned}$$

We denote the tori from $\mathrm{Fix}(\alpha)$ by $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, the tori from $\mathrm{Fix}(\beta)$ by $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, the tori from $\mathrm{Fix}(\alpha\beta)$ by $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, the tori from $\mathrm{Fix}(\beta\gamma)$ by $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ and the tori from $\mathrm{Fix}(\alpha\beta\gamma)$ by $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$. The set of all points in T^7 where at least two tori from the sets of all $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ and $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ intersect, consists precisely of the 64 circles

$$p_{\epsilon_1,\ldots,\epsilon_6} := \{(x_1,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,\tfrac{1}{2}\epsilon_3,\tfrac{1}{2}\epsilon_4,\tfrac{1}{2}\epsilon_5,\tfrac{1}{2}\epsilon_6)+\mathbb{Z}^7|x_1\in\mathbb{R}\}$$

with $\epsilon_1,\ldots,\epsilon_6\in\{0,1\}$. At each of the above circles, exactly three tori, namely $R_{\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$, $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ and $T_{\epsilon_1,\epsilon_2,\epsilon_5,\epsilon_6}$ intersect. The set of all points in T^7 where at least two tori from the sets of all $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, and $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ intersect, consists of the circles

$$q_{\epsilon_1,\ldots,\epsilon_6} := \{\tfrac{1}{2}\epsilon_1,\tfrac{1}{4}+\tfrac{1}{2}\epsilon_2,x_3,\tfrac{1}{2}\epsilon_3,\tfrac{1}{2}\epsilon_4,\tfrac{1}{2}\epsilon_5,\tfrac{1}{2}\epsilon_6)+\mathbb{Z}^7|x_3\in\mathbb{R}\}$$

with $\epsilon_1, \ldots, \epsilon_6 \in \{0,1\}$. At each of these circles, exactly three tori, namely $R_{\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$, $P_{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5}$ and $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_6}$ intersect. A torus of type $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ or $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ cannot intersect a torus of type $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ or $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$. In order to determine the singular locus, we have to describe the action of Γ on the union of all tori and circles. We see that

1. (a) β acts on $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $p_{\epsilon_5,\epsilon_6,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
   (b) γ acts on $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an fixed point free involution.
   (c) $\beta\gamma$ acts on $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $q_{\epsilon_5,\epsilon_6,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
2. (a) α acts on $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $p_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
   (b) γ maps $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $S_{1-\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$.
   (c) $\alpha\gamma$ maps $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $S_{1-\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$.
3. (a) α acts on $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $p_{\epsilon_1,\epsilon_2,\epsilon_5,\epsilon_6,\epsilon_3,\epsilon_4}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
   (b) γ maps $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $T_{1-\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$.
   (c) $\alpha\gamma$ maps $T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $T_{1-\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$.
4. (a) α acts on $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $q_{\epsilon_1,\epsilon_2,\epsilon_5,\epsilon_3,\epsilon_4,\epsilon_6}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
   (b) γ maps $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $P_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4}$.
   (c) $\alpha\gamma$ maps $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $P_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4}$.
5. (a) α acts on $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ as an involution with fixed circles $q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_5,\epsilon_6,\epsilon_4}$, where $\epsilon_5, \epsilon_6 \in \{0,1\}$.
   (b) γ maps $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $Q_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4}$.
   (c) $\alpha\gamma$ maps $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to $Q_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4}$.
6. (a) α, β and $\alpha\beta$ act trivially on $p_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$.
   (b) γ, $\alpha\gamma$, $\beta\gamma$ and $\alpha\beta\gamma$ map $p_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ to $p_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$.
7. (a) α, $\beta\gamma$ and $\alpha\beta\gamma$ act trivially on $q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$.
   (b) β, $\alpha\beta$, γ and $\alpha\gamma$ map $q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ to $q_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$.

Let $p : T^7 \to T^7/\Gamma$ be the projection map. p maps

1. the 16 tori $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ bijectively to suborbifolds $\widetilde{R}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ of T^7/Γ. Since $R_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ contains 8 circles that are fixed either by β or by $\beta\gamma$ and two of them are mapped to each other by γ, $\widetilde{R}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ is an orbifold of type $T^3/\mathbb{Z}_2^2$ with four singular circles.

2. the 16 tori $S_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ ($T_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, $P_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ or $Q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$) to 8 suborbifolds $\widetilde{S}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ ($\widetilde{T}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, $\widetilde{P}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ or $\widetilde{Q}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$) of T^7/Γ. Since all of the 3-tori in T^7 contain 4 circles that are fixed by α, the suborbifolds of T^7/Γ are of type $T^3/\mathbb{Z}_2$ with four singular circles.

3. the 64 circles $p_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ (or $q_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$) to 32 circles $\widetilde{p}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ (or $\widetilde{q}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$). $\widetilde{p}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ is identified with $\widetilde{p}_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ and $\widetilde{q}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ is identified with $\widetilde{q}_{\epsilon_1,1-\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$.

All in all, the singular locus of T^7/Γ consists of 16 suborbifolds of type $T^3/\mathbb{Z}_2^2$ and 32 suborbifolds of type $T^3/\mathbb{Z}_2$. Since any point of the fixed tori in T^7 is invariant under exactly one element of Γ of order 2, there is an A_1-singularity along each of the suborbifolds. Moreover, there are 32 circles $\widetilde{p}_{\epsilon_1,*,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ where three suborbifolds, namely $\widetilde{R}_{\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$, $\widetilde{S}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$ and $\widetilde{T}_{\epsilon_1,\epsilon_2,\epsilon_5,\epsilon_6}$ intersect. These circles are also singular circles of the three intersecting suborbifolds and along each circle there is a singularity of type $\mathbb{R}\times\mathbb{C}^3/A_1'$. Finally, there are 32 circles $\widetilde{q}_{\epsilon_1,*,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$ where $\widetilde{R}_{\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}$, $\widetilde{P}_{\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5}$ and $\widetilde{Q}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_6}$ intersect. Along the circles, there is a singularity of type $\mathbb{R}\times\mathbb{C}^3/A_1'$ with respect to a different complex structure. We recall that in the case $j=1$ and $\Delta = A_1$ the singular locus consists of 16 submanifolds of type $T^3/\mathbb{Z}_2^2$. It is surprising that by simply changing the translation part of β we obtain a singular locus that is much larger and contains intersecting suborbifolds.

Our next step is to resolve the singularities of T^7/Γ and to determine the Betti numbers of the smooth G_2-manifold M'_{A_1} that we obtain this way by the methods of [36]. This will enable us to determine the Betti numbers in the second case, too. First, we blow up the suborbifolds in T^7/Γ that are induced by the fixed locus of α. This transforms $(\mathbb{C}^2/\Lambda')/\Delta$ into a Kummer surface S. We recall that in the language of complex geometry a blow-up replaces an A_1-singularity by the projective space of all complex lines that pass through the singularity. β and γ act on the exceptional divisor $\mathbb{CP}^1$, or more precisely on the $\mathbb{CP}^1$-bundle over an suborbifold that consists of fixed points of α, by $[z_1 : z_2] \mapsto [-z_1 : z_2]$ and $[z_1 : z_2] \mapsto [-\overline{z_1} : -\overline{z_2}]$. Therefore, β and γ can be lifted to involutions $\widetilde{\beta}$ and $\widetilde{\gamma}$ of $T^3 \times S$. These involutions can be factorized as $\beta' \times \widetilde{\beta}''$ and $\gamma' \times \widetilde{\gamma}''$, where β' and γ' act on T^3 as before and $\widetilde{\beta}''$ and $\widetilde{\gamma}''$ act on S. By blowing up the A_1-singularities along the fixed point sets of $\widetilde{\beta}$ and $\widetilde{\gamma}$ in $(T^3\times S)/\langle\widetilde{\beta},\widetilde{\gamma}\rangle$ we obtain the smooth G_2-manifold M'_{A_1}. At a point with a singularity of type $\mathbb{R}\times\mathbb{C}^3/\mathbb{Z}_2^2$, our procedure describes precisely the crepant resolution from Section 4.3. We remark that we introduce a torsion-free G_2-structure only after we have finished all the blow-ups since it is still an unproven conjecture that a torsion-free G_2-structure exists on the resolution of suitable quotients of a product of a K3 surface and a torus. In principle, it would be possible to resolve all singularities of T^7/Γ in a single step and to compute the Betti numbers of M'_{A_1} from the Betti numbers of the resolutions of $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/\mathbb{Z}_2^2$. Due to the complicated global structure of the singular locus this calculation would be very difficult. Instead we determine the Betti numbers of $(T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle$ and its resolution in two steps. The fixed point sets of all elements of $\langle \widetilde{\beta}, \widetilde{\gamma} \rangle$ are given by

$$\begin{array}{rcl} \mathrm{Fix}(1) & = & T^3 \times S \\ \mathrm{Fix}(\widetilde{\beta}) & = & \bigcup_{\epsilon_1,\epsilon_2 \in \{0,1\}} \{((x_1, \frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2) + \mathbb{Z}^3), z) | x_1 \in \mathbb{R}, z \in \mathrm{Fix}(\widetilde{\beta}'')\} \\ \mathrm{Fix}(\widetilde{\gamma}) & = & \emptyset \\ \mathrm{Fix}(\widetilde{\beta}\widetilde{\gamma}) & = & \bigcup_{\epsilon_1,\epsilon_2 \in \{0,1\}} \{((\frac{1}{2}\epsilon_1, \frac{1}{4} + \frac{1}{2}\epsilon_2, x_3) + \mathbb{Z}^3), z) | x_3 \in \mathbb{R}, z \in \mathrm{Fix}(\widetilde{\beta}''\widetilde{\gamma}'')\} \end{array}$$

Since γ' maps pairs of circles in the fixed point sets of β' and $\beta'\gamma'$ to each other, the singular locus of $(T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle$ consists of two copies of $S^1 \times \mathrm{Fix}(\widetilde{\beta}'')$ and of two copies of $S^1 \times \mathrm{Fix}(\widetilde{\gamma}'')$. Since $\widetilde{\beta}$ and $\widetilde{\gamma}$ are involutions, we have A_1-singularities along the singular sets. The most important step for the computation of the Betti numbers is to determine the topology of $\mathrm{Fix}(\widetilde{\beta}'')$ and $\mathrm{Fix}(\widetilde{\gamma}'')$. β'' is a holomorphic map on $\mathbb{C}^2$ and its pull-back maps the holomorphic volume form $dz^1 \wedge dz^2$ to its negative. $\widetilde{\beta}''$ is thus a non-symplectic involution on S. Let $\pi : S \to (\mathbb{C}^2/\Lambda')/\Delta$ be the resolution map. $\mathrm{Fix}(\widetilde{\beta}'')$ is the union of

$$\pi^{-1}\left(\bigcup_{\epsilon_1,\epsilon_2 \in \{0,1\}} \{(\tfrac{1}{2}\epsilon_1, \tfrac{1}{2}\epsilon_2, x_6, x_7) + \Lambda' | x_6, x_7 \in \mathbb{R}\}\Delta \right) \quad \text{and}$$

$$\pi^{-1}\left(\bigcup_{\epsilon_1,\epsilon_2 \in \{0,1\}} \{(x_4, x_5, \tfrac{1}{2}\epsilon_1, \tfrac{1}{2}\epsilon_2) + \Lambda' | x_4, x_5 \in \mathbb{R}\}\Delta \right)$$

since $z \in \mathbb{C}^2/\Lambda'$ is a fixed point if $\beta''(z) \in z\Delta = \{-z, z\}$. Each of the four tori $\{(\frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2, x_6, x_7) + \Lambda' | x_6, x_7 \in \mathbb{R}\}$ in $\mathbb{C}^2/\Lambda'$ intersects the four tori $\{(x_4, x_5, \frac{1}{2}\epsilon_1, \frac{1}{2}\epsilon_2) + \Lambda' | x_4, x_5 \in \mathbb{R}\}$. By dividing $\mathbb{C}^2/\Lambda'$ by Δ the tori become simply connected orbifolds of type $T^2/\mathbb{Z}_2$. The blow-up transforms the $T^2/\mathbb{Z}_2$ into simply connected complex curves, which means $\mathbb{CP}^1$s. Moreover, the $\mathbb{CP}^1$s do not intersect since the tangent spaces of the tori at the intersection points are spanned by $(1, 0)$ and $(0, 1) \in \mathbb{C}^2$ and the blow-up replaces the intersection point with the set of all complex directions. $\widetilde{\beta}''$ is thus a non-symplectic involution whose fixed locus consists of 8 rational curves. $\beta''\gamma''$ acts on $\mathrm{span}(e_4, \ldots, e_7)$ as

$$\beta''\gamma''(x_4, x_5, x_6, x_7) = (x_4, -x_5, -x_6, x_7) .$$

$\mathrm{Fix}(\widetilde{\beta}''\widetilde{\gamma}'')$ is the union of

$$\pi^{-1}\left(\bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_4,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,x_7)+\Lambda'|x_4,x_7\in\mathbb{R}\}\Delta\right)\quad\text{and}$$
$$\pi^{-1}\left(\bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(\tfrac{1}{2}\epsilon_1,x_5,x_6,\tfrac{1}{2}\epsilon_2)+\Lambda'|x_5,x_6\in\mathbb{R}\}\Delta\right).$$

The tori $\{(x_4,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,x_7)+\Lambda'|x_4,x_7\in\mathbb{R}\}$ and $\{(\frac{1}{2}\epsilon_1,x_5,x_6,\frac{1}{2}\epsilon_2)+\Lambda'|x_5,x_6\in\mathbb{R}\}$ are complex curves with respect to the complex structure that is defined by $w_1:=x_4+ix_7$ and $w_2:=x_5+ix_6$ but not with respect to the standard complex structure. At the intersection points, the tangent spaces of the tori are the same up to multiplication by i with respect to the standard complex structure. Therefore, the rational curves that we obtain by the blow-up still intersect. $\widetilde{\gamma}''$ is a non-symplectic involution with respect to the complex structure that is defined by (w_1,w_2) whose fixed locus is a single complex curve. It follows from Theorem 3.5.8 that the Euler characteristic of the fixed locus of a non-symplectic involution is $2r-20$. As in the previous subsection, $H^2(S,\mathbb{R})$ is spanned by the pull-backs of the 2-forms on $\mathbb{C}^2/\Lambda'$ with constant coefficients and by the cohomology classes of the blow-ups of the 16 singular points. γ'' acts as $+1$ on 2 of the forms on $\mathbb{C}^2/\Lambda'$ and as -1 on the other 4 forms. Again, we use an argument from the previous subsection and see that γ'' acts anti-holomorphically and thus orientation-reversing on the 16 rational curves that we obtain by blowing up the singular points. Therefore, we have $r=2$ and $\chi(\mathrm{Fix}(\widetilde{\gamma}''))=-16$ and $\mathrm{Fix}(\widetilde{\gamma}'')$ is thus a complex curve of genus 9. Its Betti numbers are $b^0=1$ and $b^1=18$. We obtain the same values for the Betti numbers of T^7/Γ as in the case $j=1$ and $\Delta=A_1$ since the spaces of invariant harmonic forms are not changed by our modification of the translation part. The resolution of the singular set that is induced by the fixed point set of α adds 16 cycles that are diffeomorphic to $(T^3\times\mathbb{CP}^1)/\mathbb{Z}_2^2$ where $\mathbb{Z}_2^2$ acts as the group $\langle\beta,\gamma\rangle$ in the previous subsection. We therefore obtain by the same arguments:

$$\begin{aligned}
b^1((T^3\times S)/\langle\widetilde{\beta},\widetilde{\gamma}\rangle)&=b^1(T^7/\Gamma)=0\\
b^2((T^3\times S)/\langle\widetilde{\beta},\widetilde{\gamma}\rangle)&=b^2(T^7/\Gamma)=0\\
b^3((T^3\times S)/\langle\widetilde{\beta},\widetilde{\gamma}\rangle)&=b^3(T^7/\Gamma)=23
\end{aligned}$$

Another argument why the above values are correct is that we can apply equation (5.3) and the subspace of $H^2(S,\mathbb{R})$ that is invariant under $\widetilde{\beta}''$ and $\widetilde{\gamma}''$ has dimension 0. We are finally able to determine the Betti numbers of M'_{A_1}. The singular set in $(T^3\times S)/\langle\widetilde{\beta},\widetilde{\gamma}\rangle$ that is diffeomorphic to $S^1\times\mathrm{Fix}(\widetilde{\beta}'')$ can be considered as the product of a circle and a union of complex curves with respect to the standard complex structure. Analogously, $S^1\times\mathrm{Fix}(\widetilde{\gamma}'')$ is the product of a circle and a complex curve with respect to the second complex structure. We can therefore apply methods from complex geometry to compute the Betti numbers of

M'_{A_1}. In [27, p.605], the following formula for the cohomology of a blow-up $\pi : \widetilde{M} \to M$ of a complex variety M along a subvariety X is proven.

$$H^*(\widetilde{M}, \mathbb{Z}) = \pi^* H^*(M, \mathbb{Z}) \oplus H^*(E, \mathbb{Z}) / \pi^* H^*(X, \mathbb{Z}) \; ,$$

where E denotes the exceptional divisor. This formula can be easily modified such that the circle factor is taken into account, too. The G_2-manifold M'_{A_1} that we obtain by the resolution map $\pi : M'_{A_1} \to (T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle$ therefore satisfies:

$$b^k(M'_{A_1}) = b^k((T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle) + b^k(\pi^{-1}(S)) - b^k(S) \; , \tag{5.22}$$

where S is the singular set. The exceptional divisor E that we obtain by blowing up $\mathrm{Fix}(\widetilde{\beta}'')$ in S is a $\mathbb{CP}^1$-bundle over $\mathrm{Fix}(\widetilde{\beta}'')$. Its Betti numbers are determined by

$$b^0(E) = b^0(\mathrm{Fix}(\widetilde{\beta}'')) \; , \quad b^1(E) = b^1(\mathrm{Fix}(\widetilde{\beta}'')) \; , \quad b^2(E) = 2 \cdot b^0(\mathrm{Fix}(\widetilde{\beta}'')) \; .$$

By multiplying E with a circle we obtain

$$\begin{aligned} & b^0(S^1 \times E) = b^0(\mathrm{Fix}(\widetilde{\beta}'')) \; , \quad b^1(S^1 \times E) = b^0(\mathrm{Fix}(\widetilde{\beta}'')) + b^1(\mathrm{Fix}(\widetilde{\beta}'')) \; , \\ & b^2(S^1 \times E) = 2 \cdot b^0(\mathrm{Fix}(\widetilde{\beta}'')) + b^1(\mathrm{Fix}(\widetilde{\beta}'')) \; . \end{aligned}$$

Of course, we have the same formulas for $\mathrm{Fix}(\widetilde{\gamma}'')$. All in all, we obtain

$$\begin{aligned} b^1(M'_{A_1}) &= b^1((T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle) = 0 \\ b^2(M'_{A_1}) &= b^2((T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle) + 2b^0(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^0(\mathrm{Fix}(\widetilde{\gamma}'')) = 0 + 2 \cdot 8 + 2 \cdot 1 = 18 \\ b^3(M'_{A_1}) &= b^3((T^3 \times S)/\langle \widetilde{\beta}, \widetilde{\gamma} \rangle) + 2b^0(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^0(\mathrm{Fix}(\widetilde{\gamma}'')) \\ &\quad + 2b^1(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^1(\mathrm{Fix}(\widetilde{\gamma}'')) = 23 + 2 \cdot 8 + 2 \cdot 1 + 2 \cdot 0 + 2 \cdot 18 = 77 \end{aligned}$$

An example of a G_2-manifold with $(b^2, b^3) = (18, 77)$ can be found in Table 12.2 in [36] and this pair indeed belongs to the smooth manifold that is obtained by resolving the singularities by our method. At the end of this section, we show how to construct new G_2-manifolds with help of the groups Γ that we have constructed. For reasons of brevity, we carry out this procedure only for the case $j = 2$ and $\Delta = A_3$. Since we have already described the kind of resolutions that we are using and it is not necessary to know the precise shape of the singular locus in order to determine the Betti numbers of M'_{A_3}, we do not describe T^7/Γ in detail but start with the calculations that are necessary to compute $b^k(M'_{A_3})$. First, we need to determine the fixed loci of β'' and $\beta''\gamma''$ acting on the quotient $(\mathbb{C}^2/\Lambda')/\Delta$. The fixed loci of the action of $\alpha^i\beta$ and $\alpha^i\beta\gamma$ on $\mathbb{C}^2/\Lambda'$ are given by

$$\begin{array}{rcl}
\mathrm{Fix}(\beta'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,x_6,x_7)+\Lambda'|x_6,x_7\in\mathbb{R}\} \\
\mathrm{Fix}(\alpha''\beta'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2)+\Lambda'\} \\
\mathrm{Fix}(\alpha''^2\beta'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_4,x_5,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2)+\Lambda'|x_4,x_5\in\mathbb{R}\} \\
\mathrm{Fix}(\alpha''^3\beta'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_2)+\Lambda'\}
\end{array}$$

and

$$\begin{array}{rcl}
\mathrm{Fix}(\beta''\gamma'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(x_4,\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,x_7)+\Lambda'|x_4,x_7\in\mathbb{R}\} \\
\mathrm{Fix}(\alpha''\beta''\gamma'') & = & \{(x_4,x_4,x_6,x_6)+\Lambda'|x_4,x_6\in\mathbb{R}\} \\
\mathrm{Fix}(\alpha''^2\beta''\gamma'') & = & \bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\{(\frac{1}{2}\epsilon_1,x_5,x_6,\frac{1}{2}\epsilon_2)+\Lambda'|x_5,x_6\in\mathbb{R}\} \\
\mathrm{Fix}(\alpha''^3\beta''\gamma'') & = & \{(x_4,-x_4,x_6,-x_6)+\Lambda'|x_4,x_6\in\mathbb{R}\}
\end{array}$$

The fixed locus of β'' on $(\mathbb{C}^2/\Lambda')/\Delta$ is the set of all $z\Delta$ with $z\in\mathbb{C}^2/\Lambda'$ and $\beta''(z)\in\{z,\alpha''(z),\alpha''^2(z),\alpha''^3(z)\}$. In other words, we have $z\in\bigcup_{i=0,\ldots,3}\mathrm{Fix}(\alpha''^i\beta'')$. We denote the fixed locus on $(\mathbb{C}^2/\Lambda')/\Delta$ by $\mathrm{Fix}(\beta'')$, too, and obtain

$$\mathrm{Fix}(\beta'')=\bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\big(\{((x_4,x_5,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2)+\Lambda')\Delta|x_4,x_5\in\mathbb{R}\}\cup\{((\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,x_6,x_7)+\Lambda')\Delta|x_6,x_7\in\mathbb{R}\}\big)$$

The subvarieties with $(\epsilon_1,\epsilon_2)=(0,1)$ and $(\epsilon_1,\epsilon_2)=(1,0)$ are mapped to each other by Δ. Therefore, the above set consists of six complex subvarieties. They intersect in points of type $((\frac{1}{2}\epsilon_1,\frac{1}{2}\epsilon_2,\frac{1}{2}\epsilon_3,\frac{1}{2}\epsilon_4)+\Lambda')\Delta$. These are exactly the singularities of $(\mathbb{C}^2/\Lambda')/\Delta$ that are resolved by one or more blow-ups. Since the tangent spaces of the subvarieties at the intersection points are different complex lines, they do not intersect after the blow-up. The fixed locus of $\widetilde{\beta''}$ thus consists of six disjoint rational curves. Analogously, the fixed locus of γ'' acting on $(\mathbb{C}^2/\Lambda')/\Delta$ is given by

$$\mathrm{Fix}(\beta''\gamma'')=\bigcup_{\epsilon_1,\epsilon_2\in\{0,1\}}\big(\{((x_4,\tfrac{1}{2}\epsilon_1,\tfrac{1}{2}\epsilon_2,x_7)+\Lambda')\Delta|x_4,x_7\in\mathbb{R}\}\cup\{((x_4,x_4,x_6,x_6)+\Lambda')\Delta|x_4,x_6\in\mathbb{R}\}\big)$$

This is the union of five complex subvarieties with respect to the modified complex structure that we have introduced in the previous example. The first four of them are disjoint and the fifth one intersects each of the other subvartieties. Again, they intersect at singular points that are blown up. The tangent spaces of the first four varieties are spanned by $(1,i)$ with respect to the standard complex structure and the tangent space of the fifth one is spanned by $(1+i, 1+i)$. These spaces cannot be mapped to each other, neither by multiplication with a complex number nor by an element of Δ. Therefore, the intersections vanish after the blow-up. The fixed locus of $\widetilde{\gamma}''$ thus consists of five rational curves. We have all information at hand that is necessary to compute the Betti numbers and obtain by the same methods as in the previous example:

$$\begin{aligned}
b^1((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) &= 0 \\
b^2((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) &= 0 \\
b^3((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) &= 23
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
b^1(M'_{A_3}) &= b^1((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) = 0 \\
b^2(M'_{A_3}) &= b^2((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) + 2b^0(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^0(\mathrm{Fix}(\widetilde{\gamma}'')) = 0 + 2\cdot 6 + 2\cdot 5 = 22 \\
b^3(M'_{A_3}) &= b^3((T^3 \times S)/\langle\widetilde{\beta}, \widetilde{\gamma}\rangle) + 2b^0(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^0(\mathrm{Fix}(\widetilde{\gamma}'')) \\
&\quad + 2b^1(\mathrm{Fix}(\widetilde{\beta}'')) + 2b^1(\mathrm{Fix}(\widetilde{\gamma}'')) = 23 + 2\cdot 6 + 2\cdot 5 + 2\cdot 0 + 2\cdot 0 = 45
\end{aligned}$$

Finally, we show that M'_{A_3} has finite fundamental group. The torus quotient T^7/Γ with $j = 2$ and $\Delta = A_3$ can be obtained by dividing the torus quotient with $j = 2$ and $\Delta = A_1$ by $\mathbb{Z}_2$. Since it is shown in [36] that the second torus quotient is simply connected, the fundamental group of T^7/Γ is either trivial or $\mathbb{Z}_2$. The resolution of the singularities does not change the fundamental group [36, Proposition 12.1.3]. Therefore, the fundamental group of M'_{A_3} is finite and the holonomy of the metric on M'_{A_3} is the whole group G_2.

Remark 5.3.2. The group Γ that we have used for the construction of M'_{A_3} is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2^2$. Since Joyce does not consider torus quotients by groups of this isomorphism type, our construction of M'_{A_3} has not been carried out in [36]. To the authors best knowledge, there appear no G_2-manifolds with Betti numbers $(b^2, b^3) = (22, 45)$ in the existing literature [17, 36, 43, 44]. Therefore, M'_{A_3} seems to be a new example of a G_2-manifold.

Chapter 6

Conclusion and Outlook

The aim of this thesis was to construct G_2-orbifolds with various kinds of ADE-singularities. The examples that we have found show that we can obtain singularities of nearly any kind that we want. In Section 5.1 and 5.2 we constructed G_2-orbifolds with an ADE-singularity of type E_8 along an associative submanifold. We have shown that in certain cases the singularity can be partially resolved such that the remaining singularity can be described by any subdiagram of E_8, namely

$$A_1, A_2, A_3, A_4, A_5, A_6, A_7, D_4, D_5, D_6, D_7, E_6, E_7 \, .$$

In Section 5.3, we constructed torus quotients with singularities of type

$$A_1, A_2, A_3, A_5, D_4, D_5, E_6 \, .$$

Moreover, we have shown that these are the only singularities that are possible for torus quotients. There are several ways how to proceed further.

First of all, it is possible to construct further examples of smooth G_2-manifolds and of G_2-orbifolds with ADE-singularities by modifying our constructions. In [36, Ch. 12.5], Joyce resolves the torus quotient that is isometric to our example with $j = 2$ and $\Delta = A_1$ by a more general method than ours. Beside the crepant resolution, there are further resolutions of the singularities that yield many different pairs of Betti numbers (b^2, b^3) of the resolved G_2-manifolds. Even the crepant resolution of $\mathbb{C}^3/\mathbb{Z}_2^2$ is not unique. Although Joyce constructs are large set of different G_2-manifolds, he states on page 317 of [36] that he had to choose a special ansatz since the set of all possible R-data is probably too big to investigate without the help of a computer. Therefore, it is likely that further G_2-manifolds can be constructed by resolving this torus quotient. Moreover, we could start with a torus quotient where $j = 2$ and Δ is another allowed finite subgroup of $SU(2)$. Since torus quotients of this kind have not been considered in [36], it is possible to obtain even more compact

G_2-manifolds by resolving the singularities of these quotients. Although this program is beyond the scope of this thesis, it may be an interesting subject of future research.

In Section 5.2, we constructed twisted connected sums by means of singular K3 surfaces with a non-symplectic involution. We restricted ourselves to a single kind of involution in order to show that our method works. It would be interesting to study systematically if non-symplectic involutions with a different invariant lattice could also be the starting point of our construction.

The first examples of twisted connected sums in [43] were constructed from threefolds $\overline{W}$ that are a certain blow-up of a Fano threefold. By starting with a Fano threefold with ADE-singularities and then applying the twisted connected sum construction, we could find further examples of G_2-orbifolds with ADE-singularities, too. In order to carry out this procedure it has to be studied if the Fano threefolds and their K3 divisors can be deformed in such a way that the K3 surfaces at the cylindrical ends satisfy the matching condition.

There are also more theoretical questions that fit into the context of this thesis. A motivation to investigate G_2-orbifolds with ADE-singularities is the conjecture from [32] which states that they arise as boundary components of the moduli space of parallel smooth G_2-structures. Although a proof of this conjecture seems to be out of reach for now, it might be possible to show for some of our particular examples that a one-parameter families of smooth G_2-manifolds exists that converges to our G_2-orbifolds with ADE-singularities such that a set of $\mathbb{CP}^1$-bundles over the singular locus collapses.

Finally, other kinds of singularities of G_2-manifolds, namely those of codimension 6 or 7 are an interesting object of research. Conical singularities have codimension 7 and G_2-manifolds with conical singularities are an active field of research, see for example [39]. However, there is not much work done on singularities of codimension 6. In particular, it is unknown if orbifold singularities of type $\mathbb{R} \times \mathbb{C}^3/\Delta$ are the only singularities of codimension 6 that a G_2-manifold may have and if G_2-orbifolds with singularities of type $\mathbb{R} \times \mathbb{C}^3/\Delta$ exist for any discrete subgroup of $SU(3)$.

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