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REACTOR KINETIC FORMULATION USING THE FINITE ELEMENT METHOD

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ABSTRACT

This research has the objective of solving the spatial Kinetic equations for two energy groups using the finite element method. In the methodology, was applied the direct method, such that matrix equations coefficients from spatial discretization was generated by finite element method. The formulation of the time-dependent problem was obtained by analytical integration of precursor concentration equation and using the Euler implicit scheme in the dynamic diffusion problem. A 2D example of a reactor static diffusion problem was solved using a linear triangular finite element. This solution was compared with the numerical benchmark solution, found in the literature, and the numerical results calculated by the finite difference methods. This comparison shows the capacity of the finite element method to obtain a precise solution.

1. INTRODUCTION

The major part of codes that solves the time-dependent diffusion equation did not take into account the interest, now in evidence, for the accelerator driven system (ADS), with high energy spallation neutrons or the molten-salt reactor concepts both brought by the GEN-IV innovative system. The objective of this work is to present an approximate method to the solution of space-time dependent neutron kinetic equation. besides that explained before, As pointed out by, Grossman and Hennart[4] this equation have been interest in reactor physics and reactor design since the early 1960s due to the inadequacy of the point reactor model for the analysis of large thermal light water reactor (LWR) as demonstrated by Yasinsky and Henry[9]. The point reactor power prediction are inaccurate in many cases even can underestimate the reactivity insertion and hence non conservative in safety sense. Examples of multidimensional space-energy dependent safety problems in LWR are:

- The analysis of consequences of the accidental ejection of a control rod on the radial and axial power distribution in pressurized water reactor (PWR);
- Power oscillation in the radial and azimuthal directions in a boiling water reactor (BWR) in unstable power-flow regions.

The model almost universally used in space-dependent dynamic is that of multi-group

diffusion theory coupling to the equation for delayed neutrons precursors. A lot of solution of this kind of problem have been developed and had been categorized as Spatial and Time integration method. The Spatial Methods are categorizing by Sutton and Aviles [8] as follow:

- Direct method;
- Space-time factorization methods;
- Modal and synthesis methods.

The direct methods could be classified into three groups:

- Finite difference method;
- Coarse mesh method;
- Nodal methods.

Others authors like Grossman and Hennart[4] also presents a review of more recent advances in nodal diffusion methods for space-time neutron kinetics. The space-time factorization methods have been developed as potentially less computationally intensive alternatives to the direct method. These methods involve a factorization of the time dependent flux into the product of two functions: one the amplitude function dependent only on the time variables, and the other the shape function dependent on space and energy as well as on time. Analytical and numerical work in this field are summarized in a review paper by Grossman and Hennart[4] that discuss only the direct space-time methods and Dahmani[3] that presents a mixed dual method based in the space-time factorization for reactor kinetics.

The modal and synthesis method appear to have been largely abandoned for the purpose of kinetic calculation and the discussions about this subject is presented in Sutton and Aviles[8]. To numerically advancing the space time diffusion group diffusion equation, along with their time-dependent delayed neutron precursor counterparts, through time a lot of methods can be used to do that:

- Theta method;
- Alternating Direction Implicit Method (ADI);
- Stiffness Confinement Method (SCM);
- Symmetric Successive Over-Relaxation (SSOR);
- Rosenbrock Generalized Runge-Kutta Method;
- Analytical treatment of the Delayed Neutron Precursors.

While Grossman and Hennart [4] adopt direct method and therefore factorization method as well as nodal and synthesis methods are outside of the scope of their paper, we adopt here a space-time factorization method using the Finite Element Method (FEM) for the spatial discretization. Instead of a classical finite-difference approach to this boundary value problem, the Finite Element Method (FEM) is chosen because it is quite versatile and allows in particular the treatment of any irregular geometry. The implementation of this method requires:

- Formulation of the boundary value problem in a week or variational form;
- Discretization of a given domain Ω into approximate sub-domains Ω^e;
- Replacement of infinite dimensional spaces by finite dimensional subspaces of functions

in a piecewise way, over the element;
- The formulation and solution of the linear algebraic problem for the coefficient in the expansion defined in the item before.

Finally some authors as Sutton and Aviles [8] presents summarized results of benchmark problems obtained using various methods for spatial and time integration methods.

2. SPACE-TIME NEUTRON KINETIC PROBLEM

As mentioned before, the theory used here is that of a multi-group diffusion theory coupled to the equation for delayed neutrons precursors. In real dynamics problems, especially those involving safety considerations, the coefficient in this system depend upon parameters such as temperature, void, and composition which themselves depend on the neutron power level. To simplify the problem, the coefficients are treated as a piecewise constant in the space with possible iterative updating in time in the course of a transient. Using the space time factorization method, a factorization of the time-dependent flux is composed into the product of two functions: one where the amplitude function dependent only on the time variables, and the other where the shape function dependent on space and energy as well as on time. In many problems, the shape function is only weakly time-dependent then, it may be not necessary to recalculate it at every time step. The computation of the amplitude function is relatively inexpensive giving to this method often accurate results using far less computer resources than direct methods.

2.1. Strong Formulation

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t}(r,t) = \nabla . D^g \nabla \phi_g - \Sigma_t^g \phi_g + \sum_{\substack{g'=1 \\ g' \neq g}}^{G} \Sigma_s^{g' \to g} \phi_{g'} + (1-\beta_{eff}) \chi_0^g \sum_{g'=1}^{G} \nu \Sigma_f^{g'} \phi_{g'} + \sum_{i=1}^{I} \lambda_i \chi_i^g C_i + s^g \quad g = 1, \dots, G \tag{2.1}$$

and

$$\frac{\partial C_i(r,t)}{\partial t} = \beta_i \sum_{g=1}^{G} \nu \Sigma_f^g \phi_g - \lambda_i C_i \quad i = 1, \dots, I \quad \forall (r,t) \epsilon \Omega x (0, \overline{T}) \tag{2.2}$$

where:

$\phi_g(x,t)$: neutron flux $[\mathrm{L}^{-2}\mathrm{t}^{-1}]$in group g, position r and time t.
$C_i(x,t)$: concentration of delayed neutron precursors $[\mathrm{L}^{-3}]$in group i, position x and time t.
v_g: mean velocity of neutrons in group g.
$D_g(x)$: diffusion coeficient in group g at the position x and time t.
$\Sigma^g(x,t)$: $removal\ cross\ section\ [L^{-1}] in\ group\ g\ at\ position\ x\ and\ time\ t.$
$\Sigma_s^{g' \to g}(x,t)$: $scattering\ cross\ section\ [L^{-1}]\ in\ group\ g\ at\ position\ x\ and\ time\ t.$
Σ_t^g: $total\ macroscopic\ cross\ section\ of\ group\ g.$

$v\Sigma_f^g$: $fission\ macroscopic\ cross\ section\ of\ group\ g.$
Σ_a^g: $absorption\ macroscopic\ cross\ section\ of\ group\ g.$
$\sum_s^{g\to g'}$: $scattering\ macroscopic\ cross\ section\ from\ group\ g\ to\ group\ g'.$
β_{eff}: $total\ effective\ fraction\ of\ delayed\ neutrons.$
χ_0^g: $spectrum\ of\ prompt\ neutrons\ in\ group\ g.$
χ_i^g: $spectrum\ of\ delayed\ neutrons\ in\ group\ g.$
v: $mean\ number\ of\ fission\ neutrons.$
$\Sigma_f^{g'}(x,t)$: $fission\ cross\ section\ [L^{-1}]\ in\ group\ g' at\ position\ x.$
λ_i: $decay\ constant\ [t^{-1}] of\ group\ i\ precursors.$
β_i : $fraction\ of\ delayed\ neutrons\ in\ group\ i.$
s^g: $external\ source\ of\ neutrons\ in\ group\ g.$

and where the cross-section relations are defined as:

$$\Sigma^{\mathrm{g}} = \Sigma_{\mathrm{t}}^{\mathrm{g}} - \Sigma_{\mathrm{s}}^{g'\to g} = \Sigma_{\mathrm{a}}^{\mathrm{g}} + \Sigma_{\mathrm{s}}^{g\to g'} \tag{2.3}$$

Following the work of Grossman and Hennart[4], the multi-group kinetics system can be written in reaction-diffusion form as a parabolic system:

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla . D\nabla \mathbf{u} + Q\mathbf{u} = \mathbf{S} \tag{2.4}$$

where **u** (x, t) is the column vector of the neutron flux and precursors density:

$$\mathbf{u} = \{\phi_{1,} \dots, \phi_{G,} C_{1,} \dots, C_I\}^T = \{\Phi\ C\}_{[(G+I)\times(G+I)]} \tag{2.5}$$

D is the diagonal matrix of the diffusion cross sections:

$$\boldsymbol{D} = diag[v_1 D_1, \dots,\ v_G D_G, 0, \dots, 0]^T_{[(G+I)\times(G+I)]} \tag{2.6}$$

and ***Q*** is the block matrix:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{\Gamma} \\ \boldsymbol{B} & \boldsymbol{\Lambda} \end{bmatrix}_{[(G+I)\times(G+I)]} \tag{2.7}$$

where ***H*** is the matrix of absorption, scattering and fission cross sections, ***Γ*** and ***B*** are the matrices of neutron precursors and **Λ** is the diagonal matrix of decay constants defined by the following expressions:

$$\boldsymbol{H} = -v_{g'}\Sigma_a^{g'}\delta_{gg'} + \left(1-\delta_{gg'}\right)v_g\Sigma_s^{g'\to g} + \ (1-\beta)v_g\chi_0^g\left(v\Sigma_f\ \right)^{g'} \quad [G\times G] \tag{2.8}$$

$$\boldsymbol{\Gamma} = v_g\chi_i^g\lambda_i \quad [G\times I] \tag{2.9}$$

$$\boldsymbol{\Lambda} = -diag\lambda_1, \dots., \lambda_I \quad [I\times I] \tag{2.10}$$

$$\boldsymbol{B} = \beta_i\left(v\Sigma_f\ \right)^{g'} \quad [I\times G] \tag{2.11}$$

The column vector of external group sources ***s*** is:

$$\boldsymbol{s} = [v_1 s_1 , \dots , v_G \ s_G, 0, \dots ,0]^T \qquad [(G+I)x1] \qquad (2.12)$$

For boundary conditions, we consider Dirichlet or zero flux conditions on a portion Γ_1 of the boundary Γ and Neumann or zero current conditions on the remainder Γ_2. Then, for boundary conditions, zero flux condition (Dirichlet conditions) is considered on portion Γ_1 of the boundary and zero current condition (Newmann condition) on the remainder Γ_2:

$$u(r_e) = 0 \ on \ \Gamma_1 \qquad (2.13)$$

$$\frac{\partial u}{\partial \bar{n}}(r_e) = 0 \ on \ \Gamma_2 \qquad (2.14)$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cup \Gamma_2 = \emptyset$ and $\bar{n}$ is the normal to Γ_2 .

2.2. Weak Formulation

The basic idea behind weak formulation, as opposed to strong ones related to the classical solution of equation, consists in multiplying it by some test function **v**:

$$\int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} - \nabla . \boldsymbol{D}\nabla \mathbf{u} + \boldsymbol{Q}\mathbf{u}\right). \mathbf{v} d\Omega = \int_\Omega \mathbf{s}. \mathbf{v} \, \mathrm{d}\Omega \qquad (2.15)$$

and integrating the streaming terms by parts:

$$\int_\Omega (\nabla . \boldsymbol{D}\nabla \mathbf{u}). \mathbf{v} d\Omega = \int_\Omega (\nabla . \boldsymbol{D}\nabla \mathbf{u}\mathbf{v}) \, \mathrm{d}\Omega + \int_\Omega \boldsymbol{D}\nabla \mathbf{u}. \nabla \mathbf{v} \, \mathrm{d}\Omega \qquad (2.16)$$

$$\int_\Omega (\nabla . \boldsymbol{D}\nabla \mathbf{u}\mathbf{v}) \, \mathrm{d}\Omega = \int_\Gamma \boldsymbol{D}\mathbf{v}\nabla \mathbf{u}. \mathbf{n} \, \mathrm{d}\Gamma \qquad (2.17)$$

to get:

$$\int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t}. \mathbf{v} + \boldsymbol{D}\nabla \mathbf{u}. \nabla \mathbf{v} + \boldsymbol{Q}\mathbf{u}. \mathbf{v}\right) d\Omega = \int_\Gamma \boldsymbol{D}\mathbf{v}\nabla \mathbf{u}. \mathbf{n} \, d\Gamma + \int_\Omega \mathbf{s}. \mathbf{v} \, \mathrm{d}\Omega \qquad (2.18)$$

As the flux is zero on Γ_2 and if it is imposed that **v** is zero on Γ_1 like **u** the equation can be written by:

$$\int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t}. \mathbf{v} + \boldsymbol{D}\nabla \mathbf{u}. \nabla \mathbf{v} + \boldsymbol{Q}\mathbf{u}. \mathbf{v}\right) d\Omega = \int_\Omega \mathbf{s}. \mathbf{v} \, \mathrm{d}\Omega \qquad (2.19)$$

3. NEUTRON DIFFUSION PROBLEM

3.1. Strong Formulation

The static diffusion equation can be obtained from the spatial kinetic equation (2.4) when the time flux variation and the delayed neutrons are not considered.

$$\nabla . \boldsymbol{D} \nabla \mathbf{u} + Q\mathbf{u} = \boldsymbol{s} \tag{3.1}$$

In this case, the static diffusion equation (3.1) using a discretization of 2 groups of neutron energies, presents the following form [2]:

$$-\nabla . D^1 \nabla \phi_1 + \sum^1 \phi_1 = \frac{1}{\lambda}\left(v \sum_f^1 \phi_1 + v \sum_f^2 \phi_2\right) \tag{3.2}$$

$$-\nabla . D^2 \nabla \phi_2 + \sum^2 \phi_2 = \sum_s^{1\to 2} \phi_1 \tag{3.3}$$

3.2. Weak Formulation

Using the same function for the test function and neutron flux, the weak formulation of the static diffusion equation can be presented in the following form:

$$\int_\Omega \nabla \phi_1 . D^1 \nabla \phi_1 \, d\Omega + \int_\Omega \phi_1 \left[\Sigma^1 \phi_1 - \frac{1}{\lambda}\left(v\Sigma_\mathrm{f}^1 \phi_1 - v\Sigma_\mathrm{f}^2 \phi_2\right)\right] d\Omega \tag{3.4}$$

$$\int_\Omega \nabla \phi_2 . D^2 \nabla \phi_2 d\Omega + \int_\Omega \phi_2 \left[\Sigma^2 \phi_2 - \Sigma_\mathrm{s}^{1\to 2} \phi_1\right] d\Omega \tag{3.5}$$

that can be rewritten by:

$$\int_\Omega \left(\nabla \phi_1 . D^1 \nabla \phi_1 + \phi_1 \Sigma^1 \phi_1\right) d\Omega = \int_\Omega \frac{1}{\lambda} \phi_1 \left[\left(v\Sigma_\mathrm{f}^1 \phi_1 - v\Sigma_\mathrm{f}^1 \phi_2\right)\right] d\Omega \tag{3.6}$$

$$\int_\Omega \left(\nabla \phi_2 . D^2 \nabla \phi_2 + \phi_2 \Sigma^2 \phi_2\right) d\Omega = \int_\Omega \phi_2 \Sigma_\mathrm{s}^{1\to 2} \phi_1 \, d\Omega \tag{3.7}$$

4. NUMERICAL SOLUTION

4.1. Neutron Diffusion Solution

4.1.1. Geometry and field approximation

In the Finite Element Approximation it is supposed that the geometry and the neutron flux are approximated by functions defined in sub-domains Ω^e that by juxtaposition defines the entire domain:

$$x^e = \sum_{i=1}^{m} N_i x_i^e \tag{4.1}$$

$$\phi^e = \sum_{i=1}^{m} N_i \phi_i^e \tag{4.2}$$

where N_i is the interpolation function defined for the node *i* and element *e*. Supposing to use the 2D-cartesian coordinate system and a linear triangle finite element, the following expressions can be obtained for the geometry:

$$x = \{N_1, N_2, N_3\} \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \end{Bmatrix} \tag{4.3}$$

$$\text{y} = \{N_1, N_2, N_3\} \begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \end{Bmatrix} \tag{4.4}$$

and for the neutron flux:

$$\phi^e = \{N_1 \quad N_2 \quad N_3\} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} \tag{4.5}$$

The flux gradient, using equation (4.5), is given as:

$$\nabla\phi^e = \begin{Bmatrix} \dfrac{\partial(\sum_{i=1}^{m} N_i\phi_i^e}{\partial x} \\ \dfrac{\partial(\sum_{i=1}^{m} N_i\phi_i^e}{\partial y} \end{Bmatrix} = \begin{Bmatrix} N_{1,x} & N_{2,x} & N_{3,x} \\ N_{1,y} & N_{2,y} & N_{3,y} \end{Bmatrix} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = \nabla N\Phi \tag{4.6}$$

Using the above equation in the diffusion equations (3.6) and (3.7), the following equation system is obtained:

$$\int_\Omega \Phi_1\left(\nabla N.\, D_1\nabla N + N\Sigma^1\, N\right)\Phi_1\, d\Omega = \int_\Omega \frac{1}{\lambda}\Phi_1 N\left[(v\Sigma_\text{f}^1 N - v\Sigma_\text{f}^2 N)\right]\Phi_1\, d\Omega \tag{4.7}$$

$$\int_\Omega \Phi_2\left(\nabla N.\, D_2\nabla N + N\Sigma^2\, N\right)\Phi_2\, d\Omega = \int_\Omega \Phi_2 N\Sigma_\text{s}^{1\to 2} N\Phi_1\; d\Omega \tag{4.8}$$

This equation system is solved, in the MEF program, by the Power Method via Gauss elimination (DIFM module) or conjugated gradient (DICG module). The adaptive remeshing in the MEF program that makes use of the TRIANGLE (TRIA module) and TETGEN (TETG module) free programs respectively in 2D and 3D, is presented by Jospin, Aghina and Sampaio [8].

4.2. Space-Time Neutron Kinetic Solution

4.2.1. Geometry and field approximation

The geometry is approximated by the following relation:

$$r_i^e = \sum_{i=1}^{m} N_i(r_i^e) \tag{4.9}$$

For the field approximation, the Finite Element method is used. Applying the Galerkin method, the test function and the field (neutron flux) are approximated by the same interpolation functions:

$$\mathbf{u}_l^e = \sum_{i=1}^{m} N_i(\mathbf{u}_l^e)_i \tag{4.10}$$

$$\mathbf{v}_l^e = \sum_{i=1}^{m} N_i(\mathbf{v}_l^e)_i \tag{4.11}$$

$$\dot{\mathbf{u}}_l^e = \sum_{i=1}^{m} N_i(\dot{\mathbf{u}}_l^e)_i \tag{4.12}$$

where $\boldsymbol{u}_l^e$ is the component l of the vector $\boldsymbol{u}$, N_i is the interpolation function associate to the node i and element e defined in the sub-domain Ω^{e} . Supposing this sub-domain is represented by a triangle with 3 nodes (m = 3) and the cartesian coordinates system is selected, the above relations take the following form in the matrix form:

$$\mathbf{N} = \{N_i \quad N_j \quad N_k\} \tag{4.13}$$

$$x_i^e = \mathbf{N}\begin{Bmatrix}(\mathbf{x}_l^e)_i\\(\mathbf{x}_l^e)_j\\(\mathbf{x}_l^e)_k\end{Bmatrix} \tag{4.14}$$

$$\mathbf{u}_i^e = \mathbf{N}\begin{Bmatrix}(\mathbf{u}_l^e)_i\\(\mathbf{u}_l^e)_j\\(\mathbf{u}_l^e)_k\end{Bmatrix} \tag{4.15}$$

$$\mathbf{v}_i^e = \mathbf{N}\begin{Bmatrix}(\mathbf{v}_l^e)_i\\(\mathbf{v}_l^e)_j\\(\mathbf{v}_l^e)_k\end{Bmatrix} \tag{4.16}$$

where i, j, k are the nodes of the triangle.

$$\boldsymbol{N}^G = \begin{bmatrix} \boldsymbol{N}_1^1 & \boldsymbol{N}_2^1 & \boldsymbol{N}_3^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_1^2 & \boldsymbol{N}_2^2 & \boldsymbol{N}_3^2 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{N}_1^G & \boldsymbol{N}_2^G & \boldsymbol{N}_2^G \end{bmatrix}_{G\mathrm{x}(3\mathrm{x}G)} \tag{4.17}$$

$$\boldsymbol{N}^I = \begin{bmatrix} \boldsymbol{N}_1^1 & \boldsymbol{N}_2^1 & \boldsymbol{N}_3^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_1^2 & \boldsymbol{N}_2^2 & \boldsymbol{N}_3^2 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{N}_1^I & \boldsymbol{N}_2^I & \boldsymbol{N}_2^I \end{bmatrix}_{I\text{x}(3\text{x}I)} \tag{4.18}$$

Grouping all the nodal variables together in the vector $\mathbf{u}_n$, the variable field u can be represented by:

$$\mathbf{u} = \mathbf{N}\mathbf{u}_n \tag{4.19}$$

where the interpolation matrix can be written by:

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}^G & \mathbf{0} \\ \mathbf{0} & \boldsymbol{N}^I \end{bmatrix}_{(G+I)\text{x}3(G+I)} \tag{4.20}$$

and the nodal variables by:

$$\mathbf{u}_n = \{\phi_1^1 \quad \phi_2^1 \quad \phi_3^1 \quad \phi_1^2 \quad \phi_2^2 \quad \phi_3^2 \quad \dots \quad \phi_3^G \quad C_1^1 \quad C_2^1 \quad C_3^1 \quad \dots \quad C_3^1\}_{3\text{x}(G+I)} \tag{4.21}$$

or:

$$\boldsymbol{u}_n = \{\boldsymbol{\Phi} \quad \mathbf{C}\}_{3\text{x}(G+I)} \tag{4.22}$$

Substituting this approximation into the integral formulation 2.19 results:

$$\sum_{e=1}^{ne} \int_{\Omega_e} (\mathbf{N}^T\mathbf{N}\mathbf{u}_n + \nabla\mathbf{N}\mathbf{u}_n + \mathbf{N}^T\mathbf{Q}\mathbf{N}\mathbf{u}_n) \,.\, \boldsymbol{u}_n d\Omega_e = \int_{\Omega_e} \mathbf{s}.\, \boldsymbol{u}_n d\Omega_e \tag{4.23}$$

That can be put in a matrix form:

$$\bar{\mathbf{A}}\dot{\mathbf{u}}_n + (\bar{\boldsymbol{D}} + \bar{\boldsymbol{Q}})\boldsymbol{u}_n = \bar{\mathbf{s}} \tag{4.24}$$

or:

$$\begin{bmatrix} \mathbf{A}^G & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^I \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{\Phi}} \\ \dot{\mathbf{C}} \end{Bmatrix} + \left(\begin{bmatrix} \bar{\mathbf{D}}^G & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{H} & \boldsymbol{\Gamma} \\ \mathbf{B} & \boldsymbol{\Lambda} \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{\Phi} \\ \mathbf{C} \end{Bmatrix} = \begin{Bmatrix} \mathbf{s}^G \\ \mathbf{0} \end{Bmatrix} \tag{4.25}$$

where:

$$\boldsymbol{A}^G = \int_{\Omega_e} \boldsymbol{N}^G . \boldsymbol{N}^G d\Omega_e \tag{4.26}$$

$$\boldsymbol{A}^I = \int_{\Omega_e} \boldsymbol{N}^I . \boldsymbol{N}^I d\Omega_e \tag{4.27}$$

$$\boldsymbol{D}^G = \int_{\Omega_e} \nabla \boldsymbol{N}^G . \bar{\boldsymbol{D}} \nabla \boldsymbol{N}^G d\Omega_e \tag{4.28}$$

$$\boldsymbol{H} = \int_{\Omega_e} \boldsymbol{N}^G . \bar{\boldsymbol{H}} \boldsymbol{N}^G d\Omega_e \tag{4.29}$$

$$\boldsymbol{\Gamma} = \int_{\Omega_e} \boldsymbol{N}^G . \bar{\boldsymbol{\Gamma}} \boldsymbol{N}^I d\Omega_e \tag{4.30}$$

$$\boldsymbol{B} = \int_{\Omega_e} \boldsymbol{N}^I . \bar{\boldsymbol{B}} \boldsymbol{N}^G d\Omega_e \tag{4.31}$$

$$\boldsymbol{\Lambda} = \int_{\Omega_e} \boldsymbol{N}^I . \bar{\Lambda} \boldsymbol{N}^I d\Omega_e \tag{4.32}$$

$$\boldsymbol{s}^G = \int_{\Omega_e} \boldsymbol{N}^G . \bar{\boldsymbol{s}} d\Omega_e \tag{4.33}$$

Using a local reference coordinate system (ξ, η, ζ) defined in the sub-domain Ω_e and the Gauss numerical integration, the above integrals can be written in the following form:

$$\boldsymbol{A}^G = \int_{\Omega_e} \boldsymbol{N}^G . \boldsymbol{N}^G det J_e d\xi d\eta d\zeta = \sum_{i=1}^{nimp} \boldsymbol{N}^G . \boldsymbol{N}^G det J_e^i w_e^i \tag{4.34}$$

$$\boldsymbol{A}^I = \int_{\Omega_e} \boldsymbol{N}^I \boldsymbol{N}^I det J_e d\xi d\eta d\zeta = \sum_{i=1}^{nimp} \boldsymbol{N}^I \boldsymbol{N}^I det J_e^i w_e^i \tag{4.35}$$

$$\boldsymbol{D}^G = \int_{\Omega_e} \nabla \boldsymbol{N}^G . \bar{\boldsymbol{D}} \nabla \boldsymbol{N}^G det J_e d\xi d\eta d\zeta = \sum_{i=1}^{nimp} \nabla \boldsymbol{N}^G . \bar{\boldsymbol{D}} \nabla \boldsymbol{N}^G det J_e^i w_e^i \tag{4.36}$$

where J_e is the Jacobian transformation from the reference coordinates (ξ, η, ζ) to the global coordinates x = (x, y, z), w^e is the weight of the numerical integration at point i and element e and $nimp$ the number of integration points.

4.2.2. Time discretization

4.2.2.1.Direct method-semi-implicit Euler method

Using the implicit Euler method, the space time kinetic equation can be put in the following form:

$$\dot{\mathbf{u}}_{t+\Delta t} = f(\mathbf{u}_{t+\Delta t}, t + \Delta t) \tag{4.37}$$

or in the matrix form:

$$\mathbf{A}\dot{\mathbf{u}}_{t+\Delta t} + (\mathbf{D} + \mathbf{Q})\mathbf{u}_{t+\Delta t} = \bar{\mathbf{s}} \tag{4.38}$$

The space-time kinetics result:

$$\mathbf{A}\frac{\mathbf{u}_{t+\Delta t}-\mathbf{u}_t}{\Delta t}+(\mathbf{D}+\mathbf{Q})\mathbf{u}_{t+\Delta t}=\bar{\mathbf{s}} \tag{4.39}$$

Taking into consideration the **A, D** and **Q** matrices definition, the above system can be separated in two equations:

$$\left(\frac{\mathbf{A}^G}{\Delta t}+\mathbf{D}^G+\mathbf{H}\right)\boldsymbol{\Phi}_{t+\Delta t}=\frac{\mathbf{A}^G}{\Delta t}\boldsymbol{\Phi}_t+\mathbf{s}^G-\boldsymbol{\Gamma}\mathbf{C}_{t+\Delta t} \tag{4.40}$$

$$\left(\frac{\mathbf{A}^I}{\Delta t}+\boldsymbol{\Lambda}\right)\mathbf{C}_{t+\Delta t}=\frac{\mathbf{A}^I}{\Delta t}\mathbf{C}_t+\mathbf{B}\boldsymbol{\Phi}_{t+\Delta t} \tag{4.41}$$

A solution of this system can be obtained in an approximated way using the stationary solution of the prompt neutron $\boldsymbol{\Phi}_{t+\Delta}$ (diffusion equation) defined in the equations 4.7 and 4.8:

$$(\boldsymbol{D}^G+\boldsymbol{H})\boldsymbol{\Phi}_{t+\Delta}=0 \tag{4.42}$$

that can be put in the following generalized eigenvalue problem and solved by the power method:

$$(\boldsymbol{D}^G+\boldsymbol{H}_a+\boldsymbol{H}_s)\boldsymbol{\Phi}=\boldsymbol{\Lambda}\boldsymbol{H}_f\boldsymbol{\Phi} \tag{4.43}$$

where the components of matrix $\boldsymbol{H}$ are defined by equation 2.8:

$$\boldsymbol{H}_a=-v_{g\prime}\Sigma_{\mathrm{a}}^{g\prime}\delta_{gg\prime} \tag{4.44}$$

$$\boldsymbol{H}_s=(1-\delta_{gg\prime})v_g\Sigma_{\mathrm{s}}^{g\prime\to g} \tag{4.45}$$

$$\boldsymbol{H}_f=(1-\beta)v_g\chi_0^g\left(v\Sigma_{\mathrm{f}}\right)^{g\prime} \tag{4.46}$$

A neutron flux discretization in two groups (G=2) produce the following equation system:

$$(\boldsymbol{D}^1+\boldsymbol{H}_a^1+\boldsymbol{H}_s^1)=\Lambda(\boldsymbol{H}_f^1+\boldsymbol{H}_f^2)\Phi^1 \tag{4.47}$$

$$(\boldsymbol{D}^2+\boldsymbol{H}_a^2)\Phi^2=\boldsymbol{H}_s^{1\to2}\Phi^1 \tag{4.48}$$

From the equation system 4.47 and 4.48 above, it is possible to obtain an estimate of the neutron flux $\boldsymbol{\Phi}_{t+\Delta t}$. In this case an approximated value of the coefficients delayed neutrons fraction $\boldsymbol{C}_{t+\Delta t}$ can be estimated by the equation 4.41. Then, it is possible to calculate the evolution of the variables: $\boldsymbol{\Phi}_{t+\Delta t}$ and $\boldsymbol{C}_{t+\Delta t}$. This system can be formulated in a simplified form by:

$$\boldsymbol{K}^G\Phi_{t+\Delta t}=\boldsymbol{F}^G \tag{4.49}$$

$$\boldsymbol{M}^I\boldsymbol{C}_{t+\Delta t}=\boldsymbol{R}^I \tag{4.50}$$

where:

$$\boldsymbol{K}^G=\frac{\boldsymbol{A}^G}{\Delta t}+\boldsymbol{D}^G+\boldsymbol{H} \tag{4.51}$$

$$\boldsymbol{M}^I = \frac{\boldsymbol{A}^I}{\Delta t} + \boldsymbol{\Lambda} \tag{4.52}$$

$$\boldsymbol{F}^G = \frac{\boldsymbol{A}^G}{\Delta t}\boldsymbol{\Phi}_t + \boldsymbol{s}^G - \boldsymbol{\Gamma}\boldsymbol{C}_{t+\Delta t} \tag{4.53}$$

$$\boldsymbol{R}^I = \frac{\boldsymbol{A}^I}{\Delta t}\boldsymbol{C}_t + \bar{\boldsymbol{B}}\boldsymbol{\Phi}_{t+\Delta t} \tag{4.54}$$

Putting in evidence the density of delayed neutron $\boldsymbol{C}_{t+\Delta t}$ in the equation 4.42 and substituting it into the equation 4.41 results:

$$\boldsymbol{C}_{t+\Delta t} = (\frac{\boldsymbol{A}^I}{\Delta t} + \boldsymbol{\Lambda})^{-1}[\frac{\boldsymbol{A}^I}{\Delta t}\boldsymbol{C}_t + \mathbf{B}\boldsymbol{\Phi}_{t+\Delta t}] \tag{4.55}$$

$$(\frac{\boldsymbol{A}^G}{\Delta t} + \boldsymbol{D}^G + \boldsymbol{H} + \boldsymbol{\Gamma}\left(\frac{\boldsymbol{A}^I}{\Delta t} + \boldsymbol{\Lambda}\right)^{-1}\boldsymbol{B}\Big)\boldsymbol{\Phi}_{t+\Delta t} = \boldsymbol{\Gamma}(\frac{\boldsymbol{A}^I}{\Delta t} + \boldsymbol{\Lambda})^{-1}\frac{\boldsymbol{A}^I}{\Delta t}\boldsymbol{C}_t + \frac{\boldsymbol{A}^G}{\Delta t}\boldsymbol{\Phi}_t + \boldsymbol{s}^G \tag{4.56}$$

where $\boldsymbol{C}_t$ and $\boldsymbol{\Phi}_t$ can be given from the stationary neutron diffusion solution.
Following the work of Sutton [8], the analytical solution of the equation 2.2 is given by:

$$C_i(t+\Delta t) = e^{-\lambda_i \Delta t}C_i(t) + \frac{\beta_i}{\lambda_i}\left(\frac{1-e^{-\lambda_{i\Delta t}}}{\lambda_{i\Delta t}} - e^{-\lambda_{i\Delta t}}\right)\sum_{g=1}^{G} v\Sigma_{\mathrm{f}}^{\mathrm{g}}(t)\phi^g(t) -$$
$$-\frac{\beta_i}{\lambda_i}[\frac{1-e^{-\lambda_i \Delta t}}{\lambda_i \Delta t} - 1]\sum_{g=1}^{G} v\,\Sigma_{\mathrm{f}}^{\mathrm{g}}(t+\Delta t)\phi^g(t+\Delta t) \tag{4.57}$$

Using the matrix form defined in equation 4.56 the solution system can be put in the following form:

$$\left(\frac{\boldsymbol{A}^G}{\Delta t} + \boldsymbol{D}^G + \boldsymbol{H} + \boldsymbol{\Gamma}\boldsymbol{P}_{t+\Delta t}\right)\boldsymbol{\Phi}_{t+\Delta t} = \bar{\boldsymbol{\Gamma}}\boldsymbol{T}^{\boldsymbol{T}}\boldsymbol{C}_t + \left(\frac{\boldsymbol{A}^G}{\Delta t} + \boldsymbol{\Gamma}\boldsymbol{P}_t\right)\boldsymbol{\Phi}_t + \boldsymbol{s}^G \tag{4.58}$$

where $\boldsymbol{\Phi}_{t+\Delta t}$ is the required solution of the dynamic diffusion equation and $\boldsymbol{\Phi}_t$ is the solution of the same equation at the preceding time. At the initial time this solution is identified as the stationary solution of the diffusion equation.

5. NUMERICAL EXAMPLES

The objective of this article is only to present a finite element formulation for the spatial kinetic problem. The example presented below has only the objective to show the precision this method have to represent a good solution at least in the case of static diffusion.

5.1. Seed Blanket Reactor Problem

In this example, we are interested to compare the precision of MEF program solutions with the solutions of the TWIGL problem presented by various authors and using different method like nodal methods, fine differences,…etc (Song and Kim[13] and Langenbuch[10]). The example uses a quarter of the core due to the symmetry of the reactor kinetics problem with

two neutron energy groups and one delayed neutron precursor. The spatial units are measured in cm.

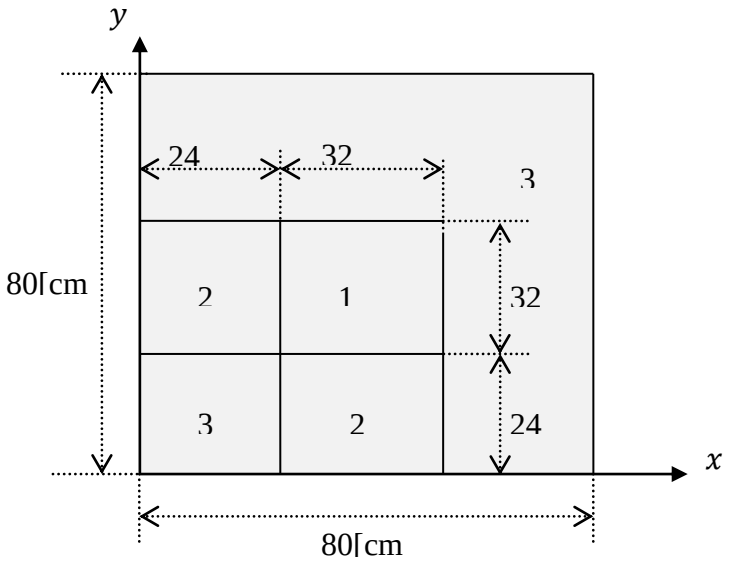

Figure 1: One quadrant of TWIGL reactor

The table 1 gives the two- group constants for TWIGL reactor.

Table 1-Group constants for the Seed-Blanket problem

Region	Group	D_g(cm)	$\sum_{ag}(cm^{-1})$	$v\sum_{fg}$	$\sum_{12}$
1	1 2	1.4 0.4	0.01 0.15	0.007 0.2	0.01
2	1 2	1.4 0.4	0.01 0.15	0.007 0.2	0.01
3	1 2	1.3 0.5	0.008 0.05	0.003 0.06	0.01

β_1	λ_1	$\frac{1}{v_1}$	$\frac{1}{v_2}$
0.0064	0.08	10^{-5}	10^{-7}

The discretization was performed using a two- dimensional triangle finite element with the same length for all the elements. The fast and thermal fluxes variations had been calculated by MEF-DIFU program using 25600 two-dimensional linear triangle finite elements. The contour fill of these fluxes are presented in figure 2 and the line diagram $x = y$ in the figure 3.

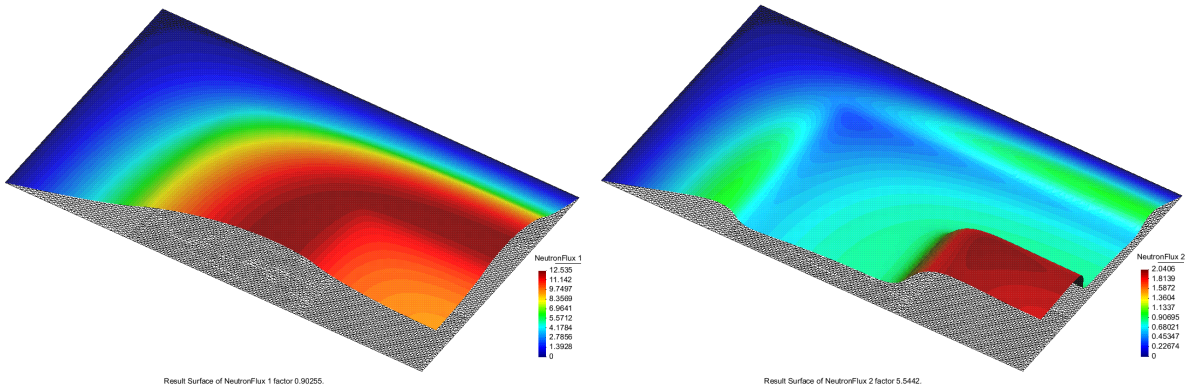

Figure 2: TWIGL fast and thermal flux

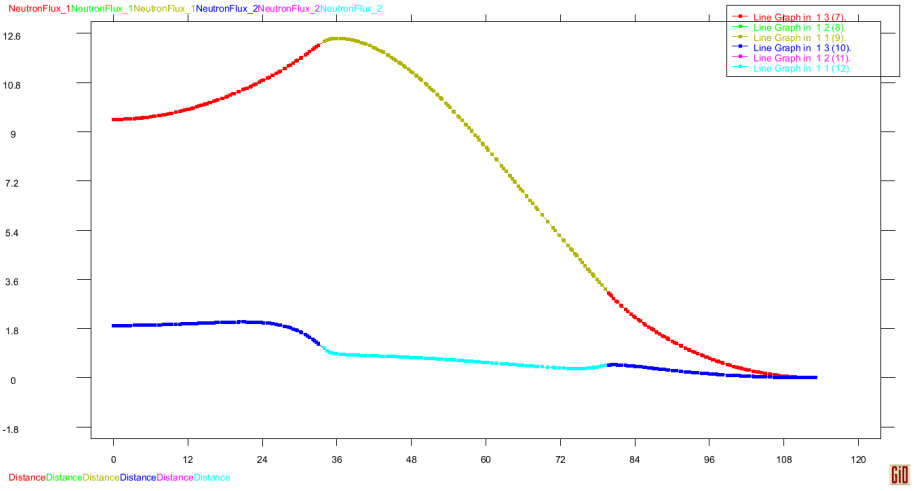

Figure 3: TWIGL fast and thermal flux

The table 2 presents a comparison among the results of the reactor criticality factor obtained by various authors and by MEF-DIFU program. The reference solution multiplication factor at the steady state presented by Hageman and Yasinsky [11] is 0.914193. It can be observed that the results of MEF-DIFU program present a good precision even with a low number of linear triangle finite elements.

Table 2: Multiplication factor value for the TWIGL

Zelmo[12] Finite Differences	MEF-DIFU Linear Triangle Finite Elements	Hageman & Yasinsky [11] Benchmark Multiplication Factor
0.91347226 (100 nodos)	0.913244 (100 F.E.)	0.914193
0.91299311 (400 nodos)	0.913397 (400 F.E.)	
0.91300493 (1600 nodos)	0.913294 (1600 F.E.)	
0.91269630 (6400 nodos)	0.913238 (6400 F.E.)	
0.91103587 (25600 nodos)	0.913215 (25600 F.E.)	

6. CONCLUSIONS

The neutron diffusion equation was formulated with Euler implicit scheme and the matrix equations coefficients from spatial discretization were generated by the finite element method. The solution of the time-dependent dynamic diffusion can be obtained by the solution of two equations. Using the analytical integration of precursor concentration is possible to reduce this system to only one equation. This method will be implemented in the MEF fortran program.

The static diffusion solution presented in the numerical results shows that accurate and efficient solutions are obtained by MEF-DIFU program using finite element. This can be observed from the comparison between this solution and the one presented by Lima [12] using finite difference. Then, a good solution as well as can be expected for the dynamic diffusion problem.

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