

# Chapter 6

## Correlation Structures Corresponding to Forward Rates

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### 6.1 Introduction

In finance, there is a constant effort to model future prices of stocks, bonds, and commodities; the ability to predict future behaviour provides important information about the underlying structure of these securities. While it has become common to model a single stock using the Black-Scholes formulation, the modelling of bond prices requires one to simulate the change of interest rates as a function of their maturity, which requires one to model the movement of an entire yield curve. If one studies the spectral decomposition of the correlation matrix corresponding to the spot rates from this curve, then one finds that the top three components can explain nearly all of the data; in addition, this same structure is observed for any bond or commodity. In his 2000 paper, Ilias Lekkos [4] proposes that such results are an artifact due to the implicit correlation between spot rates, and that the analysis should instead be performed using forward rates. In this paper, we discuss the results obtained for the spectral structure of the correlation matrices of forward rates, and investigate a model for this associated structure. The paper is divided into four parts, covering forward rates background material, principal components analysis, yield curve modelling, and conclusions and research extensions.

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## 6.2 Background: Forward Rates

### 6.2.1 Spot Rates

Let us begin with a few definitions and concepts from financial mathematics that will be referred to throughout the paper. To model bond prices, one must know the yields for various maturities. These interest rates, as a function of maturity, constitute the yield curve and are referred to as spot rates. The spot rate  $R(T)$  gives the rate that must be paid when money is borrowed (or loaned) today for a time  $T$  years. Since each spot rate changes with time, we are interested in knowing the movement of the entire yield curve as time proceeds. When one studies a single stock, and assuming efficient markets, its movement may be predicted using the Black-Scholes formulation:

$$\frac{dS}{S} = \mu dt + \sigma dW$$

with  $S$  referring to the stock price,  $\mu$  its expected return, and  $\sigma$  its volatility.  $W$  is a Brownian motion representing the random movement of the stock. To study the movement of an entire yield curve, we may assume that each point moves as a Brownian motion. Since the correlation structures, and hence primary movements, of spot rates are well known and will be briefly mentioned in the next section, and since we are interested in studying the correlation structures of forward rates in this paper, let us now adapt the above formulations to focus on forward rates.

### 6.2.2 Forward Rates

A forward rate is the rate applied to borrow (or loan) money between two dates,  $T_1$  and  $T_2$ , determined today at time  $t$ ; we denote this as  $f(t, T_1, T_2)$ . In order that no-arbitrage conditions hold, we must have the following relationship between forward and spot rates:

$$e^{R_i T_i} e^{f(T_i, T_{i+1})(T_{i+1} - T_i)} = e^{R_{i+1} T_{i+1}}.$$

The formula simply states that the rate to borrow money starting from today to time  $T_{i+1}$  must be the same as the rate if one borrows from today until time  $T_i$ , and then from  $T_i$  to  $T_{i+1}$ . If this equation did not hold, one could borrow money at one rate and lend at another with no risk, thereby creating an arbitrage opportunity. For completeness, let us also define the instantaneous forward rate, which is the rate applied to borrow or lend money for an instant at time  $T_1$ , determined at time  $t$ , denoted  $f(t, T_1)$ .

In our work, we are interested in following the approach of Heath, Jarrow, and Morton [2] to model the entire forward rate curve directly. As an example of the type of changes that have taken place in forward rates historically, Figure 6.1 illustrates the movements of various forward rates as a function of time using data from the US.

As previously mentioned, while Black-Scholes is used to model a single stock, the modelling of an entire curve of forward rates will require more work. The formulation proposed by Heath, Jarrow, and Morton is a generalization of Black-Scholes; it is given by the formula:

$$\frac{df(t, T)}{f(t, T)} = \mu dt + \left( \sum_{i=1}^{\nu} \sigma_i(t, T) dW_i \right),$$



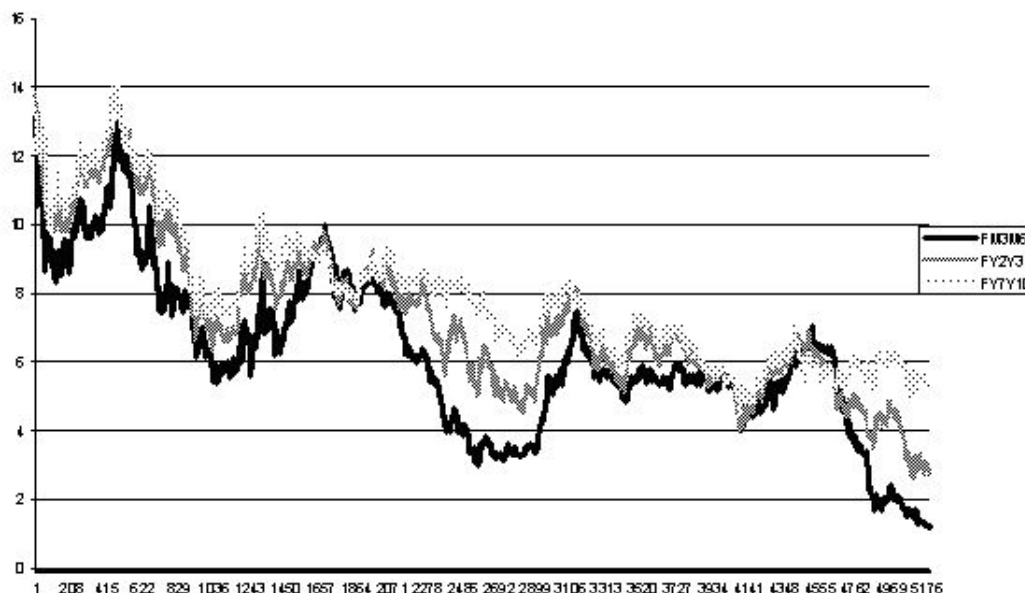


Figure 6.1: Historical Forward Rate Data for the US ( $x$ -axis : days;  $y$ -axis : percentile).

where the differential is taken with respect to time, so that

$$df(t, T) = f(t + h, T) - f(t, T).$$

The main question now arises as to what value of  $\nu$  should be used in the summation. Clearly, if we select  $\nu = 1$ , then we return to modelling a single quantity, which would incorrectly imply that the forward rates are completely correlated. If, however, we allow  $\nu$  to be the number of points on the curve, then we find that this computation is too costly, and we are not taking into account the fact that rates do indeed have a non-zero correlation. Our goal is to reduce the dimensionality by recovering most of the variances and covariances of the forward rates with a minimal number of components,  $\nu$ . This can be accomplished using principal components analysis.

### 6.3 Principal Components Analysis (PCA)

PCA is a statistical procedure that aims at taking advantage of the possible redundancy in multivariate data. It achieves that by transforming  $p$  (possibly) correlated variables into  $\nu$  uncorrelated ones. If the original variables are correlated, then the data is redundant and the observed behaviour can be explained by just  $\nu$  components of the original variables with  $\nu < p$ .

This procedure performs PCA on the selected dataset. A principal component analysis is concerned with explaining the variance-covariance structure of a high dimensional random vector through a few linear combinations of the original component variables. Consider a  $p$ -dimensional random vector  $X = (X_1, X_2, \dots, X_p)$ .  $\nu$  principal components of  $X$  are  $\nu$  (univariate) random variables  $Y_1, Y_2, \dots$ ,



$Y_\nu$  which are defined by

$$\left. \begin{aligned} Y_1 &= l_1 X = l_{11} X_1 + l_{12} X_2 + \dots + l_{1p} X_p, \\ Y_2 &= l_2 X = l_{21} X_1 + l_{22} X_2 + \dots + l_{2p} X_p, \\ &\vdots \\ Y_\nu &= l_\nu X = l_{\nu 1} X_1 + l_{\nu 2} X_2 + \dots + l_{\nu p} X_p, \end{aligned} \right\}$$

where the coefficient vectors  $l_1, l_2, \dots, l_\nu$  are chosen such that they satisfy the following conditions:

First Principal Component = Linear combination  $l_1 X$  that maximizes  $\text{Var}(l_1 X)$  and  $\|l_1\| = 1$ .

Second Principal Component = Linear combination  $l_2 X$  that maximizes  $\text{Var}(l_2 X)$  and  $\|l_2\| = 1$  and  $\text{Cov}(l_1 X, l_2 X) = 0$ .

$j^{\text{th}}$  Principal Component = Linear combination  $l_j X$  that maximizes  $\text{Var}(l_j X)$  and  $\|l_j\| = 1$  and  $\text{Cov}(l_i X, l_j X) = 0$  for all  $i < j$ .

This says that the principal components are those linear combinations of the original variables which maximize the variance of the linear combination and which have zero covariance (and hence zero correlation) with the previous principal components.

It can be proved that there are exactly  $p$  such linear combinations. However, typically, the first few of them explain most of the variance in the original data. So instead of working with all the original variables  $X_1, X_2, \dots, X_p$ , one typically performs PCA and uses only the first few principal components in subsequent analysis.

### 6.3.1 Spot Rates

We are interested in determining which components describing the movement of our curve can be used to explain most of the variance and covariance data while utilizing as few components as possible. In the case of spot rates, from the previous work in principle component analysis in this field, the results are well known. Let  $R_i$  denote a vector of yields for the day  $0 \leq i \leq N$  and define the matrix  $A$  so the column  $i$  of  $A$  is the vector  $R_i - R_{i-1}$ . One can then construct  $X = \text{cor}(A)$ , the correlation matrix formed from  $A$ . Note that  $[X]_{i,j}$  gives the correlation between the daily changes in rates with maturity  $T_i$  and maturity  $T_j$ . Calculating the eigenvalues and eigenvectors of this new matrix, one will find that the top three components are level, slope, and curvature. The first eigenvector, referred to as “level” can be interpreted as a parallel shift in the term structure, the second represents a change in the steepness, and the third is interpreted as a change in the curvature of the yield curve.

Using this process and obtaining the corresponding eigenvalues, we can compute the cumulative percentage of the first  $M$  eigenvalues, namely

$$\frac{\sum_{i=1}^M \lambda_i}{\sum_{i=1}^{\nu} \lambda_i},$$

where  $\nu$  is the total number of eigenvalues, as shown in Table 6.1. From the result of this principle component analysis process, we can see that the cumulative total of the top three components are already over 95% of original data, where we use US data as an example. These top three components represent the key movements of the yield curve for spot rates, their form is shown in Figure 6.2.



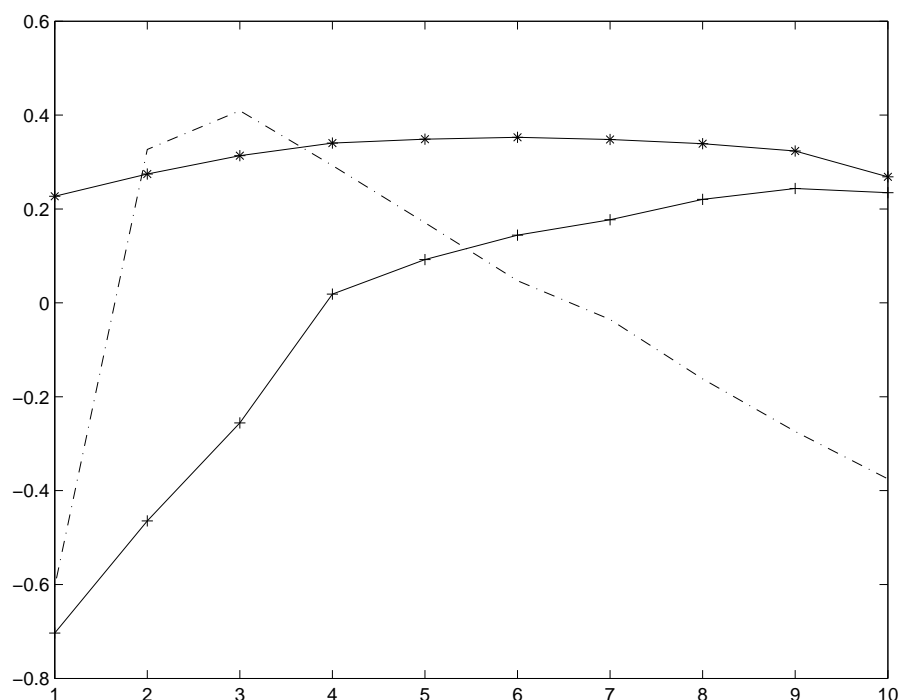


Figure 6.2: Top 3 eigenvectors representing key movements of spot rates in the US ( $x$ -axis : maturity;  $y$ -axis : eigenvalue component).

While the above graph was generated using US data, in fact we can get the same results regardless of the time period or the market used, and regardless of whether we consider bonds or commodities. In [4], Lekkos argued that such results are an artifact which arises due to the fact that spot rates are highly correlated by construction. He proposes that we should instead be working with forward rates, which although they may be correlated, are not correlated by construction. He claims that the resulting principal component analysis will yield much weaker results.

### 6.3.2 Forward Rates

As stated above, we are interested in investigating the results when principle component analysis is applied on the correlation matrix for forward rates instead of spot rates.

As before, we calculate the eigenvalues and eigenvectors of the correlation matrix, but for forward rates, we do indeed find that the decay of the eigenvalues is considerably slower, implying that it is not enough to only consider the top three components to adequately explain the movements of the curve. Figure 6.3 is a comparison of the eigenvalues obtained from the correlation matrices of spot and forward rates using 1982–2003 US data.

From this graph, we note that similarly to the top eigenvector for spot rates, the top component for forwards stands out considerably, although it is not as dominant, explaining less than 60% as compared to 80% for spots. If we consider the contribution of the top three components, we find that while these made up over 95% for spots, the total is now less than 80%, owing to the much slower decay of the

Spot Rate Data			
Eigenvalue		Individual variance (%)	Cumulative variance (%)
$\lambda_1$	8.0677	80.68	80.68
$\lambda_2$	1.1627	11.63	92.31
$\lambda_3$	0.2847	2.85	95.16
Forward Rate Data			
Eigenvalue		Individual variance (%)	Cumulative variance (%)
$\lambda_1$	5.7776	57.78	57.78
$\lambda_2$	0.9425	9.42	67.20
$\lambda_3$	0.6609	6.61	73.81

Table 6.1: Principle Component Analysis of US Data.

eigenvalues in the case of forwards.

It is also easy to verify that the first eigenvector in the case of forward rates is still a level movement and that the second still corresponds to slope. Yet, although the first two components can still explain a lot of the total variance, the remaining eigenvectors make up a substantial contribution, and their intuitive meaning, including that of the third eigenvector, is not so clear.

## 6.4 Yield Curve Modelling

### 6.4.1 Model Development and Implementation

Thus far, we have found that using forward rates instead of spot rates does not produce the same structure for the correlation matrix in which three exceptionally dominant components arise; in fact, the order of the later components may not even be the same as in the case of spot rates. How might we try to model the correlation matrix of the forward rates and its resulting spectral structure? In the case of spot rates, there is an existing model from [1] for the spot rates correlation matrix:

$$[X]_{i,j} = \rho^{|T_i - T_j|}$$

assuming that correlations,  $\rho$ , are high enough. Here  $T$  is maturity in years.

A comparison of the eigenvalue decay obtained using data and the above model is shown in Figure 6.3b. The circles represent the eigenvalues of the correlation matrix using spot rate data, while the squares stand for the eigenvalues of the modelled spot correlation matrix. We note that the two curves nearly coincide with each other; both of them exhibit a very fast decay and for each of them, the first three eigenvalues are very significant and explain over 95% of the behaviour of the correlation matrix; the other eigenvalues are insignificant and so the corresponding eigenvectors explain very little about the movement of spot rates. Thus, this model produces a good approximation to the spot rate correlation matrix. To propose a model in the case of forward rates, we can consider the relationship between the covariance matrix for spot and forward rates, namely:

$$\Omega_r = W\Omega_f W^T. \quad (6.1)$$



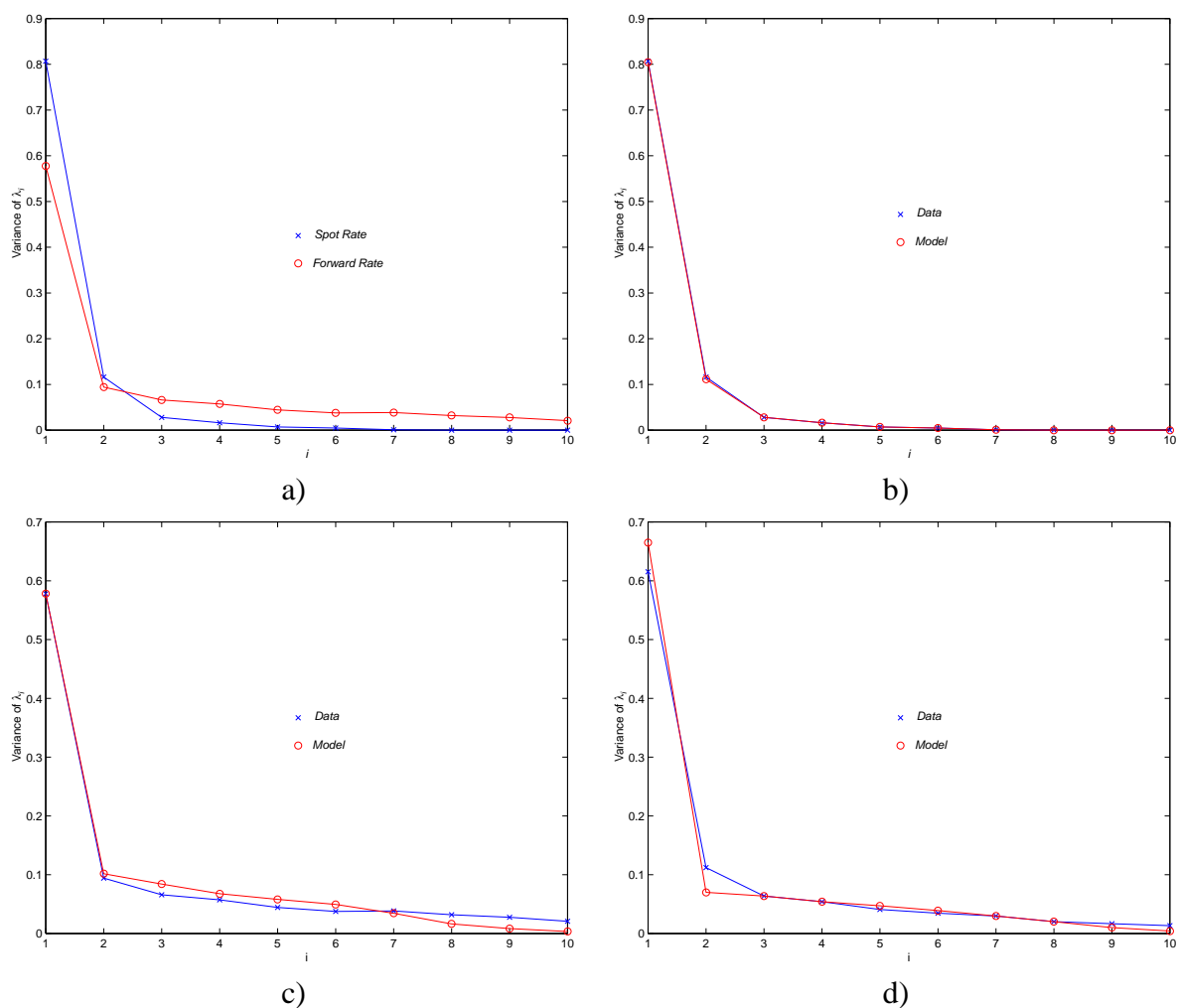


Figure 6.3: Variance structure of the eigenvalues. a) Spot rates versus forward rates for US data; b) Decay structure for US spot rates (data versus model); c) Decay structure for US forward rates (data versus model); d) Decay structure for European forward rates (data versus model).

Here,  $\Omega_r$  stands for the covariance matrix for the spot rates,  $\Omega_f$  stands for the covariance matrix for the forward rates, and  $W$  is a matrix of the weights of the forwards to the corresponding spot rates. However, we need to work with the correlation matrix. That means we need to find some way to convert this formula into a relationship between correlation matrices.

Given that the historical variance of the spots is pretty stable across tenors we have assumed constant variance when using formula (6.1) to transform the correlation matrices of the spots into correlation matrices of forwards.

Rearranging the resulting equation, we obtain a model for the forward rate correlation matrix. We may now compare the forwards eigenvalue decay from this modelled correlation matrix with that of the correlation matrix obtained from the data. Figures 6.3c and 6.3d both show such a comparison between model and data; Figure 6.3c illustrates results for 1982-2003 US data while Figure 6.3d presents 1998-2002 European data.

The circles represent the eigenvalues of the correlation matrix for the real data of the forward rates, while the squares stand for the eigenvalues of the modelled forward rates correlation matrix. From these two figures, we observe that for both markets, the model fits the data fairly well, but considerably worse than the fit that was obtained for the spots model earlier. To be specific, it seems that three components are no longer enough to adequately explain the correlation matrix; we may need to use more than five components. Indeed, it is also possible that the spots model, while it seemed to produce a good fit for spots data, is not an adequate foundation for our forwards model, which may be more sensitive to the exact nature of the spots correlation matrix; perhaps a more robust model for the spot rates is necessary when using it as a basis for forwards modelling.

### 6.4.2 Model Comparison Using Simulations

Since our ultimate goal is to predict forward rates which can then be used to predict bond prices, it is important to perform simulations to determine if using forward rates as we have implemented above, or spot rates (and subsequently computing forward rates) is indeed the best approach. While we know in the case of spot rates that it is sufficient to include the three top components, it still remains to determine how many eigenvectors are necessary when using forward rates. While we have performed some preliminary work for making such a comparison, simulations remain to be done to determine which method best predicts the variance of forward rates, and hence is a better model for predicting future values of forward rates.

## 6.5 Conclusion

As it is well known, the correlation matrices corresponding to spot rates contain a lot of structure. The fact that this structure is found across markets suggests the possibility that it is due to an artifact and not to any market-specific characteristics. In his work, I. Lekkos argued that forward rates should be the state variables in any such analysis since spot rates are correlated variables by construction. Using interest rate data from the US, Germany, United Kingdom and Japan he showed that the structure present in the correlation matrices when we use forward rates (as opposed to spot rates) is a lot weaker. In this work, we have analysed these type of matrices and found that the forward rate versions of parametric models that have been proposed for spot rate correlations do a fairly good job in describing the data. A lot of work remains to be done as far as understanding these matrices, their commonalities across markets and, of course, their modelling.





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