Nuclear Fusion Reaction Kinetics and Ignition Processes in Z Pinches

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Declaration

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Brian Daniel Appelbe
20 June 2011
Abstract

This thesis presents work on two topics related to nuclear fusion in plasmas. The first topic is the energy spectrum of products of fusion reactions in plasmas, called the production spectrum. The second is an investigation of the fusion reaction processes in high energy density Z pinch plasmas and the feasibility of ignition of such plasmas.

A method is presented for the derivation of production spectra for plasmas with various distributions of ion velocities. The method is exact, requiring the solution of a 5 dimensional integral and is suitable for both isotropic and anisotropic distributions. It is shown that many of the integrals can be solved analytically. The solutions are used to study the spectra of neutron energies produced by deuterium-deuterium and deuterium-tritium reactions. It is found that for maxwellian distributions of ions the neutron spectrum is asymmetric with a longer high energy tail when compared with gaussian approximations of the spectrum.

Deuterium and deuterium-tritium Z pinch plasmas are studied computationally using a hybrid code in which the fuel is modelled as a magnetohydrodynamic (MHD) fluid and fast ions are modelled as discrete particle-in-cell (PIC) particles. Using a Z pinch model in which the magnetic and thermal pressures are in equilibrium it is found that significant energy gain can be achieved for currents greater than $50 \, MA$. Deuterium gas puff experiments with a $15 \, MA$ current are also analysed computationally in order to determine the reaction mechanism. The results of MHD simulations in 3 dimensions are post-processed with a PIC code to model reactions occurring due to the acceleration of deuterium ions by large electric fields. It is found that reactions due to this beam-target mechanism represent a small fraction (0.0001) of the number of thermonuclear reactions.
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Dedicated to
Ciara

Giorraíonn beirt bóthar!
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Chapter 1

Introduction

The reliable production of low cost energy is one of the the most important objectives of the modern world. It is essential for the achievement and maintenance of a high standard of living. However, continuously rising demand for energy puts increasing pressure on our resources. It is predicted that global energy consumption will increase by 50% from current levels by the year 2035 [1]. Meeting this demand will put current energy resources under severe strain and so the development of new resources is an urgent requirement. Of the possible resources, that with arguably the greatest potential is controlled thermonuclear fusion. The fuels required for thermonuclear fusion are isotopes of hydrogen, deuterium and tritium, that are almost inexhaustible in supply. The production process for thermonuclear fusion is relatively clean. Unlike fossil fuels, there is no emission of greenhouse gases and, unlike nuclear fission, no long-life radioactive material is produced.

Thermonuclear fusion works by harnessing the large amount of kinetic energy (17.1 $MeV$) that is released when a deuterium and tritium ion undergo the fusion reaction

\[
D + T \rightarrow \alpha (3.5 MeV) + n (14.1 MeV).
\]

In order for the fusion reaction to occur the ions initially need some kinetic energy to overcome the repulsive coulomb barrier. This kinetic energy is imparted by heating a mass of deuterium-tritium fuel to temperatures in the $keV$ range, forming a plasma. If sufficient fusion reactions occur then the energy output will outweigh the energy
input required for the heating of the plasma. However, a hot, ionized gas will have a strong tendency to break apart and so plasma confinement is an important issue. The Lawson criterion [2] is a commonly used approximation for the confinement required to achieve thermonuclear ignition. It is given by

\[ n\tau \approx 2 \times 10^{14} \, cm^{-3}s, \]  

(1.2)

where \( n \) is the plasma number density and \( \tau \) is the confinement time. The two most well-known proposals for meeting this criterion take different approaches. Magnetic fusion energy (MFE) uses low density plasmas (\( n \sim 10^{14} \, cm^{-3} \)) that have very long confinement times (in theory, the plasma is in a steady state). The plasma is confined using toroidal magnetic fields in a tokamak device [3]. At the other end of the scale, inertial confinement fusion (ICF) seeks to achieve thermonuclear fusion by confining a very high density plasma (\( n \sim 10^{27} \, cm^{-3} \)) for time periods that are only a few hundreds of picoseconds long. It is hoped to achieve such conditions using laser irradiation of spherical capsules of deuterium-tritium fuel [4]. Energy from the laser causes the capsules to implode reaching high temperatures and densities.

Given the very different approaches to fusion of MFE and ICF it would seem obvious that some intermediate schemes that combine the more favourable elements of MFE and ICF should exist. This is indeed the case with a number of proposed approaches to fusion falling into the category of magneto-inertial fusion (MIF). Such schemes generally involve the addition of a magnetic field to an imploding mass of fuel. The presence of the magnetic field reduces the density required compared with that of ICF. It helps to confine charged particles within the plasma, thereby reducing the heat transport that occurs when such particles escape from hotter regions of the plasma. ICF relies on the plasma density alone to confine these particles. A particular scheme of interest in this thesis is the imploding Z pinch liner [5]. In this scheme, a solid cylindrical liner of radius 0.5 cm and height 1 cm is filled with deuterium-tritium fuel. A multi-mega-ampere current is passed through the liner giving rise to a large magnetic field in the azimuthal direction around the liner. The magnetic field is strong enough to cause the liner to implode onto the axis, heating
and compressing the fuel as it does so and giving rise to thermonuclear fusion. The expected density and confinement time for this scheme are $n \sim 10^{23} \text{cm}^{-3}$ and $10^{-9} \text{s}$, respectively. However, this approach to fusion has been much less thoroughly researched than MFE of ICF. One of the goals of this thesis is to investigate the viability of the Z pinch for controlled thermonuclear fusion and, in particular, the conditions leading to thermonuclear ignition. Ignition can be defined as that state in which the plasma becomes self-heating, that is, in which some of the energy from fusion reactions is absorbed by the plasma causing its temperature to increase. Crucial to ignition is the absorption by the plasma of the $3.5 \text{MeV}$ energy of the $\alpha$ particle. The presence of the magnetic field in the Z pinch approach to fusion helps to confine these $\alpha$ particles within the plasma. In this thesis we assess the ability of the magnetic field to confine the particles and calculate the Z pinch conditions (current, temperature, density and radius) required to achieve ignition and energy gain.

The imploding Z pinch liner is at an early stage of research. However, much work has been previously done on other forms of fusion Z pinches. This includes experiments examining the deuterium gas puff [6]. In this set up, rather than using a solid liner filled with fuel, the current is passed directly through jets of deuterium gas which then implode due to the resulting azimuthal magnetic field. Although the density of the gas puff that is reached at peak compression is relatively low at $n \sim 10^{20} \text{cm}^{-3}$ (in the Z pinch liner scheme the heavy liner aids compression), the deuterium gas puff is capable of producing a significant number ($3 \times 10^{13}$) of neutrons through the reaction

$$D + D \longrightarrow He^3 (0.82 \text{MeV}) + n (2.45 \text{MeV}) .$$

This result has added experimental evidence to the case for achieving thermonuclear ignition using Z pinches. However, a note of caution should be sounded. Previous experiments using deuterium gas puffs have produced neutrons not by the attainment of a hot plasma but, rather, by accelerating a small number of ions to high energies which collide with a relatively cold plasma. This so-called beam-target
production mechanism is undesirable as it is unlikely to lead to ignition of the bulk of the plasma. In this thesis we use computational methods to examine whether the neutrons produced in recent gas puff experiments (reported in [6]) are truly thermonuclear in origin rather than due to the acceleration of ion beams.

In order to be able to study these reaction effects in Z pinches, a detailed understanding of the relevant reaction kinetics in a plasma is required. As well as the total reaction rate, there are two other important elements to consider. The first of these is the absorption spectrum. For a given distribution of ion velocities in the plasma the absorption spectrum gives the distribution of those ions that participate in the reaction. Knowledge of the absorption spectrum tells us whether or not the reacting ions have energies similar to the mean energy of the plasma, as is desirable. The second element of nuclear reaction kinetics that we study is the production spectrum. This is the distribution of velocities of the products of the nuclear reaction. Knowledge of the absorption and production spectra is important for accurate modelling of burning plasmas and self-heating by $\alpha$ particles. In addition, knowledge of the production spectrum and how it relates to the underlying plasma conditions is essential for experimental diagnostics such as neutron spectroscopy. The studies of the absorption and production spectra are carried out in 0 dimensions and so the results are not particular to one scheme for nuclear fusion. In fact, this research may also have relevance to areas in astrophysics such as stellar evolution and nucleosynthesis. In this thesis we show how absorption and production spectra may be calculated and study the spectra for various plasma conditions. We focus particularly on production spectra. We establish a general method for the derivation of production spectra that requires the solution of a 5 dimensional integral. We show that for many ion distributions up to 4 of these integrals can be solved analytically. The resulting single or double integral expressions for the spectra can sometimes contain the product of several functions such as the exponential and the error functions. However, they are generally easy to compute and so we can study the exact shape of production spectra without having to resort to approximations. This allows the identification of a number of interesting features of the spectra that are reported here and have previously been published in [7].
1.1 Outline of thesis

It seems appropriate to consider the microscopic before moving up to the macroscopic. With this in mind the thesis is structured such that the discussion on absorption and production spectra appears first, followed by our study of Z pinch ignition and, finally, the reaction effects in deuterium gas puffs. We now outline the contents of each chapter, highlighting what represents, to the best of the author’s knowledge, original work.

In chapter 2 we focus on the production spectrum in fusion plasmas. Expressions for the production spectra for a number of different distributions of ions are derived and interesting features of the spectra shapes are highlighted. The absorption spectrum is also studied. A comparison with existing theory is carried out at the end of the chapter.

Chapter 3 contains a study of the interaction of a single ion with a maxwellian distribution of ions.∗ We review Chandrasekhar’s model for the slowing down of a single fast ion due to cumulative small angle scattering from particles in the maxwellian distribution. We also derive original expressions for the reactivity, absorption spectrum and production spectrum of a single ion in a maxwellian distribution.

Chapter 4 focuses on the problem of achieving ignition and energy gain in an imploding Z pinch. Results of a computational study of the ability of azimuthal magnetic fields to confine fast ions such as α particles are presented. It is shown that adequate radial confinement of the α particles can be achieved. Results are also presented showing the relationship between current and energy gain for an equilibrium Z pinch. It is calculated that significant energy gain can be achieved with currents of 50 MA.

Chapter 5 contains the results of a computational analysis of 15 MA deuterium gas puff experiments carried out on the Z machine at Sandia National Laboratories. This work calculates the number of neutrons that are produced due to a beam-target reaction mechanism (7 × 10⁸) compared with the thermonuclear reaction mechanism (3 × 10¹³). The number of secondary deuterium-tritium reactions occurring is also

*The maxwellian distribution is sometimes referred to as a thermal plasma. We use the terms interchangeably throughout.
calculated to be $5 \times 10^9$.

Chapter 6 contains conclusions and outlines future work that could be carried out to develop the results of the thesis.
Chapter 2

The absorption and production spectra in fusion plasmas

The nuclear fusion reaction rate in a plasma is of fundamental importance for controlled fusion as it determines the energy production rate. Calculation of the reaction rate for a given distribution of reactants can be done using a well-known procedure [8], [9]. A related but less-studied problem in plasma physics is the calculation of the absorption and production spectra. For a plasma in which nuclear reactions are occurring we can define the absorption spectrum as the distribution function (as a function of lab frame energy or velocity) of the reacting ions and the production spectrum as the distribution function of the products of reactions, whereas the reaction rate is simply the total number of reactions occurring (as a function of time).

Knowledge of the production spectrum is useful for diagnostics such as neutron emission spectroscopy [10] and charged particle probing by fusion products [11], [12]. The production spectrum is also required when calculating the interaction of fusion products with the plasma such as in alpha particle heating. Knowledge of the absorption spectrum tells us, for example, whether reacting ions are located near the thermal bulk of the plasma distribution or in the high energy tail.

It is important to emphasise that the absorption and production spectra are defined for particle energies or velocities in the laboratory (lab) frame and not in any
other frame. The lab frame is that in which particle energies may be experimentally measured. Whilst it is often easier to calculate the energies of particles taking part in a reaction in the centre of mass (CM) frame, we cannot measure the centre of mass energies of particles. This problem is best illustrated by reference to the cross-sections for common fusion reactions (shown in figure 2.1). The cross-section gives the probability that a pair of ions will react (the probability that one ion will tunnel through the coulomb barrier of the other ion). The cross-sections are usually determined experimentally and expressed as a function of the centre of mass kinetic energy of the pair

\[ E_r = \frac{1}{2} \mu v_r^2, \]  

(2.1)

where

\[ \mu = \frac{m_1 m_2}{m_1 + m_2}, \]  

(2.2)

is the reduced mass of the reactant pair and \( v_r \) is the relative velocity. Thus the lab frame energies of reacting particles do not determine the probability of reaction. It is only the relative velocity of the pair that matters. For example, as is well-known, the DT reaction is at a maximum at \( E_r = 64 \text{ keV} \), corresponding to \( v_r = 3.2 \times 10^6 \text{ ms}^{-1} \). However, the pair of reacting particles can have any lab frame energies as long as their relative velocity has this value. They may be travelling in opposite directions each with speed of \( 1.6 \times 10^6 \text{ ms}^{-1} \) or perhaps they are travelling in the same direction with speeds of \( 10^7 \text{ ms}^{-1} \) and \( 1.32 \times 10^7 \text{ ms}^{-1} \), respectively.

This problem also occurs when we are considering a distribution of reacting particles and not just a single pair of reacting ions. The concept of a Gamow peak is widely used in stellar evolution models, see for example [13] and [14], to determine the energy region in which most reactions occur. The Gamow peak is the product of a maxwellian distribution of ions and the coulomb barrier penetration factor (see figure 2.1). The Gamow peak shows that the majority of reacting pairs have a similar centre of mass kinetic energy that is much larger than the thermal energy of the distribution. However, this does not give any information on the lab frame energies of the reacting particles. It could be the case that the lab frame energies of the reactants have a broad distribution of energies. The absorption spectrum can
resolve this issue.

Figure 2.1: The top diagram shows cross-sections for the deuterium-deuterium and deuterium-tritium reactions as a function of centre of mass kinetic energy. The bottom diagram shows schematically the Gamow peak, also as a function of centre of mass kinetic energy. It is a product of the maxwellian distribution (curve $A$, $\propto \exp \left( -E/KT \right)$) and the coulomb barrier penetration factor (curve $B$, $\propto \exp \left( -bE^{-\frac{1}{2}} \right)$). The product curve $C$ is shown magnified. The peak of this curve is the Gamow peak. The majority of reacting pairs of ions have CM kinetic energies that are close to $E_0$.

In this chapter, we show how to calculate absorption and production spectra for nuclear fusion reactions. The spectra are calculated for a number of plasma distribution functions and interesting features are discussed. Our treatment is non-relativistic throughout except for section 2.6 in which we outline how the relativistic case may be tackled. The chapter is concluded with a review of previous work on the topic of production spectra. It seems appropriate to leave such a review until the
end since we are developing our solution from first principles. We focus on particular on the work of Brysk [15] which is one of the most widely cited works on production spectra from fusion plasmas. We compare our solution for the exact shape of the neutron spectrum produced by DD fusion with the gaussian approximation given by Brysk.

We note that throughout we use the following deuterium-tritium and deuterium-deuterium fusion reactions as examples for absorption and production spectra

\[ D + T \rightarrow \alpha (3.5 \text{ MeV}) + n (14.1 \text{ MeV}), \]
\[ D + D \rightarrow \begin{cases} 
T (1.01 \text{ MeV}) + p (3.02 \text{ MeV}), \\
He^3 (0.82 \text{ MeV}) + n (2.45 \text{ MeV}). 
\end{cases} \]

The energy values shown in parentheses are the nominal energies of the products. These are the kinetic energies that each product would have if the reactant particles were to have zero kinetic energy.

### 2.1 Nuclear reaction kinematics and reaction rates

Before deriving the absorption and production spectra we will conduct a review of the relevant nuclear reaction kinematics and reaction rates. We begin with the kinematics of a binary reaction of the form \( 1 + 2 \rightarrow 3 + 4 \). For a pair of reactants with lab frame velocities of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), the relative and centre of mass (CM) velocities of the pair of reactants are given by

\[ \mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2, \quad (2.3) \]
\[ \mathbf{v}_{\text{cm}} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}. \quad (2.4) \]

The centre of mass velocity is of interest because it is in a frame moving with this velocity that the total momentum of the reacting particles is zero. The particles collide with equal and opposite momenta. The axis along which the particles travel in the CM frame is given by the relative velocity vector. The relative velocity is
important as it is the quantity that, through (2.1) and the reaction cross-section, determines the probability of a reaction occurring. Assuming that a reaction occurs then the gain in kinetic energy of the products is given by the difference in rest mass between the reactants and products, \( Q = (m_1 + m_2) c^2 - (m_3 + m_4) c^2 \). Thus, the total kinetic energy in the CM frame after the reaction is given by \( Q + \frac{1}{2} \mu v_r^2 \).

Conservation of energy requires that we also include the kinetic energy of the reactants, which is given by the centre of mass kinetic energy. The total kinetic energy of the products in the CM frame is divided between particles 3 and 4 according to the ratio of their masses such that total momentum in the CM frame remains zero. The kinetic energy of particle 3 in the CM frame is

\[
\frac{1}{2} m_3 u_3^2 = \frac{m_3}{\eta} \left( Q + \frac{1}{2} \mu v_r^2 \right),
\]

where

\[
\eta = \frac{m_3 (m_3 + m_4)}{m_4},
\]

and \( u_3 \) is the magnitude of the velocity of particle 3 in the CM frame. Clearly, particles 3 and 4 need to be emitted in opposite directions in the CM frame in order to conserve momentum. However, it is possible that there exists a preferential angle of emission with respect to the relative velocity vector of the reactants. In such cases the reaction cross-section depends on this scattering angle as well as the relative velocity of the reactants. Therefore, the cross-section determines the probability of the products being emitted in any particular direction in the CM frame.

We will initially assume the cross-section is independent of the scattering angle (and a function only of the centre of mass kinetic energy). We return to the case of a cross-section that is dependent on the scattering angle in section 2.4. For reactions with cross-sections independent of the scattering angle we can assume that the products are emitted isotropically in the CM frame.

Isotropic emission in the CM frame means that the probability of the velocity vector \( \mathbf{u}_3 \) being in any unit solid angle is uniform (and has a value of \((4\pi)^{-1}\)). Therefore, we can arbitrarily select a direction for the vector \( \mathbf{u}_3 \). Finally, after
assigning a direction to particle 3, the particle velocity is transformed back into the lab frame from the CM frame using

\[ \mathbf{v}_3 = \mathbf{v}_{\text{cm}} + \mathbf{u}_3. \]  

(2.7)

This gives \( \mathbf{v}_3 \), the velocity of particle 3 in the lab frame. Throughout we denote vector quantities in bold (e.g. \( \mathbf{v}_1, \mathbf{v}_2 \)) while scalars are shown in normal typeface (e.g. \( v_r, u_3 \)). Particle velocities in the lab frame are denoted by \( \mathbf{v} \) with appropriate numeric subscript and particle velocities in the CM frame are denoted by \( \mathbf{u} \).

The reaction cross-section, denoted by \( \sigma(v) \) with units of \( m^{-2} \), is defined as the number of reactions per target nucleus per unit time when the target is hit by a unit flux (one particle per unit time per unit area) of projectile particles with velocity \( v \). If the target comprises of species type 1 with density \( n_1 \) and the projectile beam of species type 2 with density \( n_2 \) (the flux of projectile particles is then \( vn_2 \)) the number of reactions occurring per unit volume per unit time is given by

\[ R_{12} = n_1 n_2 v \sigma(v). \]  

(2.8)

This is the reaction rate. In a plasma there is a distribution of velocities of the target species and the projectile species. We denote the normalised distribution of target particles by \( f_1(\mathbf{v}_1) \) and of projectile particles by \( f_2(\mathbf{v}_2) \). Generalising (2.8) we can say that the number of reactions occurring per unit volume per unit time between particles of species 1 with velocity \( \mathbf{v}_1 \) and particles of species 2 with velocity \( \mathbf{v}_2 \) is

\[ R_{12}(\mathbf{v}_1, \mathbf{v}_2) = \frac{n_1 n_2}{1 + \delta_{12}} v_r \sigma(v_r)f_1(\mathbf{v}_1)f_2(\mathbf{v}_2)d^3\mathbf{v}_1 d^3\mathbf{v}_2, \]  

(2.9)

where \( v_r = |\mathbf{v}_1 - \mathbf{v}_2| \). The \( \delta_{12} \) term, defined by

\[ \delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \]  

(2.10)

*We note that it is more common in the literature to express the cross-section as \( \sigma(E_r) \) where \( E_r \) and \( v_r \) are related by (2.1). However, we use the form \( \sigma(v_r) \) as it is the velocity variables that we work with when deriving the absorption and production spectra.
is included to account for the case in which the target and projectile particles are
the same species, to avoid double-counting.

Expression (2.9) is commonly used to calculate the total reaction rate by integrating
over the six independent velocity variables as shown in, for example, [8], [9]. However,
we can also use this expression to calculate the absorption and production spectra
by taking into account the relations given in (2.1)-(2.7). This is done in section 2.2
for the absorption spectrum and in section 2.3 for the production spectrum.

2.2 The absorption spectrum

We firstly define the absorption spectrum as follows: Given a plasma containing
species 1 and 2 with velocity distributions \( f_1(v_1) \) and \( f_2(v_2) \), respectively, and
a binary reaction of the form \( 1 + 2 \rightarrow 3 + 4 \) the absorption spectrum of
species 1, denoted by \( R_{12}(v_1) \), is the number of particles of species 1 with velocity \( v_1 \)
that react with particles of species 2 per unit volume per unit time. It is the volumetric
loss rate of the species from the plasma as a function of the particle velocity.

The procedure for calculating the absorption spectrum is shown geometrically
in figure 2.2. Since the cross-section is a function of \( v_r \) only, we know that the
probability that a particle with velocity \( v_1 \) will react with a particle with velocity
\( v_2 \) is dependent on \( v_r \) only. Therefore, particle 1 has an equal probability of reacting
with any particle that is located on the surface of a sphere in velocity space that
is centered on \( v_1 \) and has radius \( v_r \) (technically the particles are located in a thin
shell of width \( dv_r \)). Multiplying the probability of reaction for a single particle on
this sphere by the number of particles on the sphere and then summing the results
for all spheres of all radii from 0 to \( \infty \) then gives the total probability that particle
1 will react.

More rigorously, we can describe the steps for calculating the absorption spec-
trum as follows:

1. Transform the independent variables of (2.9) from \((v_1, v_2)\) to \((v_1, v_r)\) using
the relation \( v_2 = v_1 - v_r \).
2. Transform the relative velocity into spherical co-ordinates using 
\[(v_{rx}, v_{ry}, v_{rz}) \rightarrow (v_r \sin \theta \cos \phi, v_r \sin \theta \sin \phi, v_r \cos \theta).\] The Jacobian determinant for this transformation is \(v_r^2 \sin \theta\) (see appendix B.1 for a discussion of the role of the Jacobian determinant).

3. Integrate over the variables \((v_r, \theta_r, \phi_r)\) to get \(R_{12}(v_1)\).

This procedure does not depend on the type of distribution functions being considered but the integrals may be easier for certain distributions than others. For reaction cross-sections that depend on the scattering angle as well as \(v_r\) the procedure is the same, we just integrate the differential cross-section over the scattering angle to get the total cross-section. That is, since the cross-section is given by \(\sigma(v_r, \theta_{cm})\) where \(\theta_{cm}\) is the scattering angle we can integrate over all scattering directions to get the total cross-section.

For arbitrary distribution functions, the first two steps above give

\[
R_{12}(v_1) = \frac{n_1n_2}{1 + \delta_{12}} f_1(v_1) \int_0^\infty \int_0^\pi \int_0^{2\pi} v_r^3 \sigma(v_r) \sin \theta_r f_2(v_1 - v_r) \sin \theta_r d\phi_r d\theta_r d\phi_r dv_r d^3v_1.
\] (2.11)

The absorption spectrum for species 2, \(R_{12}(v_2)\) can be found in the same manner by using the substitution \(v_1 = v_2 + v_r\).
2.2.1 Absorption spectrum for reactants with maxwellian distributions

We derive the absorption spectrum of species 1 for the case of reactants with maxwellian distributions of differing temperatures. Expressions for absorption spectra for a number of other reactant distributions of interest are given in appendix C. The maxwellian distributions of the reactants are defined by

\[
f_1(v_1) = \frac{m_1}{2\pi T_1^2} \frac{3}{2} \exp \left( -\frac{m_1}{2T_1} v_1^2 \right),
\]

\[
f_2(v_2) = \frac{m_2}{2\pi T_2^2} \frac{3}{2} \exp \left( -\frac{m_2}{2T_2} v_2^2 \right).
\]

Substituting these distributions into (2.11) with \(v_r\) defined in spherical coordinates gives

\[
R_{12}(v_1) = \frac{n_1 n_2}{1 + \delta_{12}} \left( \frac{m_1 m_2}{4\pi^2 T_1 T_2} \right)^{\frac{3}{2}} \exp \left( -\frac{m_1}{2T_1} v_1^2 \right) \times
\]

\[
\int_0^\infty \int_0^\pi \int_0^{2\pi} \nu_r^3 \sigma(v_r) \sin \theta_r \exp \left( -\frac{m_2}{2T_2} (v_1^2 + v_r^2 - 2v_1 v_r) \right) d\phi_r d\theta_r dv_r d^3 v_1,
\]

where

\[
v_1 \cdot v_r = v_{1x} v_r \sin \theta_r \cos \phi_r + v_{1y} v_r \sin \theta_r \sin \phi_r + v_{1z} v_r \cos \theta_r = v_1 v_r \cos \psi.
\]

The integrals over \(\phi_r\) and \(\theta_r\) may be solved using the identity given in appendix B.2 resulting in

\[
\int_0^\pi \int_0^{2\pi} \sin \theta_r \exp \left( \frac{m_2}{T_2} v_1 v_r \right) d\phi_r d\theta_r = \frac{4\pi T_2}{m_2 v_1 v_r} \sinh \left( \frac{m_2 v_1 v_r}{T_2} \right).
\]

If we rewrite

\[
2 \exp \left( -\frac{m_2}{2T_2} (v_1^2 + v_r^2) \right) \sinh \left( \frac{m_2 v_1 v_r}{T_2} \right) =
\]

\[
\exp \left( -\frac{m_2}{2T_2} (v_1 - v_r)^2 \right) - \exp \left( -\frac{m_2}{2T_2} (v_1 + v_r)^2 \right),
\]

(2.17)
then (2.14) becomes

\[
R_{12}(v_1) = \frac{n_1 n_2}{1 + \delta_{12}} \left( \frac{m_1^3 m_2}{4 \pi^2 T_1^3 T_2} \right)^\frac{3}{2} \frac{1}{v_1} \exp \left( -\frac{m_1}{2T_1} v_1^2 \right) \times \\
\int_0^\infty v_r^2 \sigma(v_r) \exp \left( -\frac{m_2}{2T_2} (v_1 - v_r)^2 \right) - \exp \left( -\frac{m_2}{2T_2} (v_1 + v_r)^2 \right) dv_r d^3v_1.
\]  

(2.18)

Without specifying the type of reaction, this is the most compact expression for the absorption spectrum for reactions in a maxwellian plasma. We illustrate (2.18) in figure 2.3 using the DD reaction for a number of plasma temperatures. At lower plasma temperatures the majority of reacting particles are located in the high energy tail of the maxwellian distribution. Because the mean free path of ions in the high energy tail is much longer than for those in the bulk of the maxwellian distribution, the reactivity of the plasma can be sensitive to inhomogeneities of the plasma temperature in the burn region at low plasma temperatures, [16]. If the scale length of the temperature gradient is of similar magnitude to the mean free path of the reacting particles then it is possible that particles could leave the burn region before reacting. At higher plasma temperatures the energy of reacting particles is much closer to the bulk of the maxwellian distribution and such an effect is unlikely to occur. The absorption spectra for DT reactions show similar behaviour to those of DD reactions.

We can find the mean energy of the absorption spectrum given by (2.18) by converting the spectrum from velocity to energy \( E_1 = \frac{1}{2} m_1 v_1^2 \) and using

\[
\langle E_1 \rangle = \frac{\int_0^\infty E_1 R_{12}(E_1) dE_1}{\int_0^\infty R_{12}(E_1) dE_1},
\]

(2.19)

where the expression in the denominator is simply the total reaction rate. These mean energies are shown in figure 2.4. Also shown in this figure are the Gamow peak energies for the reactions. These are calculated using an approximation given in [13]

\[
E_G = 6.2535(Z_1 Z_2 A_r T_k)^{\frac{1}{3}},
\]

(2.20)

where \( Z_1 \) and \( Z_2 \) are the atomic numbers of the reactants, \( A_r = \mu/m_p \) (\( m_p \) is the...
proton mass) and $T_k$ is the temperature of the plasma ions in keV. The quantity $E_G$ is in keV. Observing the results in figure 2.4 we can make the following statements: For the DD reaction, as temperature increases the mean relative kinetic energy of a pair of reacting particles in the CM frame (given by $E_G$) becomes much less than the mean energy of the individual reactants in the lab frame. The opposite is true for the DT reaction. At high plasma temperatures the mean CM frame relative kinetic energy of a reacting pair is greater than the mean lab frame energy of the reacting particles.

![Absorption spectra for DD reactions](image)

**Figure 2.3**: Absorption spectra for DD reactions for plasma ion temperatures of (a) 2 keV, (b) 10 keV, (c) 50 keV, (d) 100 keV. In each diagram the blue curve is the absorption spectrum while the green dashed curve shows the distribution of the reactant species. Note that in the graphs each curve is independently normalised such that the maximum value is 1. In reality, the absorption spectrum will be many orders of magnitude smaller, by a factor corresponding to the reactivity.
2.3 The production spectrum part I: Cross-sections that are a function of relative velocity only

We define the production spectrum as follows: Given a plasma containing species 1 and 2 with velocity distributions $f_1(v_1)$ and $f_2(v_2)$, respectively, and a binary reaction of the form $1 + 2 \rightarrow 3 + 4$ the production spectrum of species 3, denoted by $R_{12}(v_3)$, is the number of particles of species 3 with velocity $v_3$ that are produced due to reactions between particles of species 1 and species 2 per unit volume per unit time. It is the volumetric production rate of the species in the plasma as a function of the particle velocity.

In this section we consider cross-sections that depend on the relative velocity only and not the scattering angle. Therefore, we denote the cross-section by $\sigma(v_r)$. For the majority of reactions the cross-section depends on both the relative velocity and the scattering angle. However, a cross-section dependent only on the relative velocity is worth studying for two reasons

1. For certain plasma ion distributions of interest, such as a maxwellian in which both reactant species have the same temperature, the production spectrum...
depends on the total cross-section only. Therefore, we do not need to consider the dependence of the differential cross-section on the scattering angle.

2. For certain reactions including those relevant to nuclear fusion the cross-section is only weakly dependent on scattering angle.

In section 2.4 we consider cross-sections that include a scattering angle dependency.

Figure 2.5: A plot of velocity space illustrating the procedure for calculating the production spectrum. For a pair of reactants with lab frame velocities \( v_1 \) and \( v_2 \) the emitted particle 3 will have a velocity \( v_3 \) that lies somewhere on the surface of a sphere that is centered on \( v_{cm} \) and has radius \( u_3 \). Since the cross-section depends on \( v_r \) only and not the scattering angle \( \theta_{cm} \) there is a uniform probability of \( v_3 \) lying anywhere on the sphere.

In deriving the production spectrum we make use of the fact that the cross-section depends on the magnitude of \( v_r \) only and also that the emission of the product particles is isotropic in the CM frame. Figure 2.5 shows a series of velocity space sketches showing the reaction between two particles with velocities \( v_1 \) and \( v_2 \). This illustrates the procedure we use for deriving the production spectrum, starting with (2.9). We begin by switching the independent variables \( v_1 \) and \( v_2 \) to \( v_{cm} \) and \( v_r \) (shown in (b) of figure 2.5) using (2.4) to give

\[
R_{12}(v_{cm}, v_r) = \frac{n_1 n_2}{1 + \delta_{12}} v_r \sigma(v_r) f_1(v_{cm}, v_r) f_2(v_{cm}, v_r) d^3v_{cm} d^3v_r. \tag{2.21}
\]

Since the direction of \( v_r \) plays no further role in the procedure we can integrate over \( d\theta_r d\phi_r \) to eliminate these extraneous variables. We can next convert \( v_r \) to \( u_3 \) using (2.5). Using the assumption that the products are emitted isotropically in the CM
frame, we know that in the CM frame the produced particles of species 3 will lie uniformly on a sphere of radius $u_3$ ((c) in figure 2.5). This gives

$$R_{12}(v_{cm}, u_3) = \frac{\sin \theta_3}{4\pi} u_3 \eta \frac{n_1 n_2}{\mu (1 + \delta_{12})} \sigma(u_3) f_1(v_{cm}, u_3) f_2(v_{cm}, u_3) d^3 v_{cm} d^3 u_3. \quad (2.22)$$

The term $\sin \theta_3 (4\pi)^{-1}$ is introduced to account for the isotropic emission. It is the probability that a particle will be emitted in any given unit solid angle. The $\sin \theta_3$ factor (where $\theta_3$ is the polar angle of vector $u_3$ defined in spherical coordinates) arises from the definition of the solid angle differential in spherical coordinates, $d\Omega = \sin \theta d\theta d\phi$. The factor $u_3 \eta / (v_r \mu)$ is the Jacobian determinant (see appendix B.1) associated with the transformation of $dv_r$ to $du_3$. Next we can use (2.7) to convert $v_{cm}$ to $v_3$ and transform from the CM frame back to the lab frame (shown in (d) of figure 2.5). This gives

$$R_{12}(v_3, u_3) = \frac{\sin \theta_3}{4\pi} u_3 \eta \frac{n_1 n_2}{\mu (1 + \delta_{12})} \sigma(u_3) f_1(v_3, u_3) f_2(v_3, u_3) d^3 v_3 d^3 u_3. \quad (2.23)$$

Finally, integration of (2.23) over $d^3 u_3$ gives us an expression $R_{12}(v_3)$, the production spectrum for species 3. The production spectrum for species 4 may be obtained by switching the subscripts 3 and 4. It is preferable to work with velocity variables rather than energy variables as the velocity variables include a directional component and so are more suitable for anisotropic distributions. Once $R_{12}(v_3)$ has been obtained the energy spectrum can be determined by converting the velocity $v_3$ to kinetic energy. We can summarise the above procedure as follows:

1. Beginning with the expression (2.9) with the reactant distribution functions $f_1(v_1)$ and $f_2(v_2)$ defined for cartesian coordinates, transform the independent variables of (2.9) from $(v_1, v_2)$ to $(v_{cm}, v_r)$ using (2.3) and (2.4). The Jacobian determinant for this transformation is 1.

2. Transform the relative velocity into spherical co-ordinates using

$$(v_{rx}, v_{ry}, v_{rz}) \rightarrow (v_r \sin \theta_r \cos \phi_r, v_r \sin \theta_r \sin \phi_r, v_r \cos \theta_r).$$

The Jacobian determinant for this transformation is $v_r^2 \sin \theta_r$. 
3. Integrate over the variables \((\theta, \phi)\).

4. Transform \(v_r\) to \(u_3\) using (2.5). The limits of integration for \(du_3\) are \([v_Q, \infty]\) where \(v_Q = \sqrt{2Q/\eta}\). The Jacobian determinant for this transformation is \(u_3\eta/(v_r\mu)\).

5. Introduce the variables \(\theta_3\) and \(\phi_3\) to vectorize \(u_3\) and multiply the expression by a factor \(\sin \theta_3 (4\pi)^{-1} d\theta_3 d\phi_3\). This step is based on the assumption that products are isotropically emitted in the CM frame.

6. Transform the variable \(v_{cm}\) to \(v_3\) using the relation given by (2.7). The Jacobian determinant for this is 1.

7. Integrate over the variables \((u_3, \theta_3, \phi_3)\) to obtain \(R_{12}(v_3)\), the distribution function of \(v_3\) in cartesian coordinates.

We note that whilst we typically express the production spectrum in cartesian coordinates it is usually easier to convert this to spherical coordinates in order to get the spectrum shape along a particular line of sight.

2.3.1 Single temperature maxwellian distribution

We begin with the simplest case in which the reacting distributions are maxwellians of equal temperature

\[
f_i(v_i) = \left(\frac{m_i}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_i v_i^2}{2T}\right),
\]

where \(i = 1, 2\). Inserting the maxwellian distributions into (2.9) gives

\[
R_{12}(v_1, v_2) = \Lambda v_i \sigma(v_r) \exp\left(-\frac{1}{2T} \left(m_1 v_1^2 + m_2 v_2^2\right)\right) d^3v_1 d^3v_2,
\]

where

\[
\Lambda = \frac{n_1 n_2}{1 + \delta_{12}} \left(\frac{\sqrt{m_1 m_2}}{2\pi T}\right)^3.
\]
From (2.3) and (2.4) we have

\[ v_1 = v_{cm} + \frac{m_2}{m_1 + m_2} v_r, \quad (2.27) \]
\[ v_2 = v_{cm} - \frac{m_1}{m_1 + m_2} v_r. \quad (2.28) \]

We use these relations to transform the independent variables of (2.25) from \((v_1, v_2)\) to \((v_{cm}, v_r)\). This results in

\[ R_{12}(v_{cm}, v_r) = \Lambda v_r \sigma(v_r) \exp \left( -\frac{1}{2T} \left( (m_1 + m_2) v_{cm}^2 + \mu v_r^2 \right) \right) d^3 v_r d^3 v_{cm}. \quad (2.29) \]

Next we transform the relative velocity term \(v_r\) from cartesian coordinates to spherical coordinates using

\[ \begin{align*}
  v_{rx} &= v_r \sin \theta_r \cos \phi_r, \\
  v_{ry} &= v_r \sin \theta_r \sin \phi_r, \\
  v_{rz} &= v_r \cos \theta_r.
\end{align*} \quad (2.30) \]

It is a trivial transformation in this case since only the magnitude of the relative velocity appears in (2.29). Including the Jacobian determinant \(v_r^2 \sin \theta_r\) the result of the transformation is

\[ R_{12}(v_{cm}, v_r) = \Lambda \sin \theta_r v_r^3 \sigma(v_r) \exp \left( -\frac{1}{2T} \left( (m_1 + m_2) v_{cm}^2 + \mu v_r^2 \right) \right) d\phi_r d\theta_r dv_r d^3 v_{cm}, \quad (2.31) \]

and integration over the angular components \(d\phi_r d\theta_r\) gives

\[ R_{12}(v_{cm}, v_r) = 4\pi \Lambda v_r^3 \sigma(v_r) \exp \left( -\frac{1}{2T} \left( (m_1 + m_2) v_{cm}^2 + \mu v_r^2 \right) \right) dv_r d^3 v_{cm}. \quad (2.32) \]
The next step is the transformation of the relative velocity magnitude \( v_r \) to the magnitude of the velocity of the product in the CM frame \( u_3 \). From (2.5) we have

\[
\zeta = v_r(u_3) = \sqrt{\frac{2}{\mu} \left( \frac{\eta}{2} u_3^2 - Q \right)}.
\]  

(2.33)

We use \( \zeta \) to denote that the relative velocity is a function of \( u_3 \). The Jacobian determinant of the transformation is

\[
dv_r = u_3 \frac{\eta}{\mu} v_r \, du_3 ,
\]  

(2.34)

giving

\[
R_{12}(v_{cm}, u_3) =
4\pi \frac{\eta}{\mu} \Lambda u_3 \zeta^2 \sigma(\zeta) \exp \left( -\frac{1}{2T} \left( (m_1 + m_2) v_{cm}^2 + \mu \zeta^2 \right) \right) du_3 d^3v_{cm}.
\]  

(2.35)

We also note that since the limits of integration for \( v_r \) were \([0, \infty] \) the limits for \( u_3 \) will be \([v_Q = \sqrt{2Q/\eta}, \infty]\). We next need to vectorize the velocity of particle 3 in the CM frame (i.e. \( u_3 \rightarrow u_3 \)), such that the emission direction of a product particle can be specified. Since the reaction cross-section is independent of the particle emission angle (isotropic emission) this simply requires that we multiply (2.35) by a factor \( \sin \theta_3 \left(4\pi\right)^{-1} \, d\theta_3 d\phi_3 \) to obtain \( R_{12}(v_{cm}, u_3) \) with \( u_3 \) defined in spherical coordinates

\[
R_{12}(v_{cm}, u_3) =
\sin \theta_3 \frac{\eta}{\mu} \Lambda u_3 \zeta^2 \sigma(\zeta) \exp \left( -\frac{1}{2T} \left( (m_1 + m_2) v_{cm}^2 + \mu \zeta^2 \right) \right) d\phi_3 \, d\theta_3 \, du_3 d^3v_{cm}.
\]  

(2.36)

The final transformation is used to find the lab frame velocity \( v_3 \) of the product particles. This is done by expressing \( v_{cm} \) as a function of \( u_3 \) and \( v_3 \) using

\[
v_{cmx} = v_{3x} - u_3 \sin \theta_3 \cos \phi_3,
\]

\[
v_{cmy} = v_{3y} - u_3 \sin \theta_3 \sin \phi_3,
\]

\[
v_{cmz} = v_{3z} - u_3 \cos \theta_3,
\]  

(2.37)
which gives

\[ v_{cm}^2 = v_3^2 + u_3^2 - 2u_3 (v_{3x} \sin \theta_3 \cos \phi_3 + v_{3y} \sin \theta_3 \sin \phi_3 + v_{3z} \cos \theta_3) = v_3^2 + u_3^2 - 2v_3 \cdot u_3. \]  

(2.38)

The Jacobian for this transformation is 1. After this transformation (2.36) becomes

\[
R_{12}(v_3, u_3) = \sin \theta_3 \frac{\eta}{\mu} \Lambda u_3 \zeta^2 \sigma(\zeta) \exp \left( -\frac{\mu}{2T} \zeta^2 \right) \\
\exp \left( -\frac{(m_1 + m_2)}{2T} \left( v_3^2 + u_3^2 - 2v_3 \cdot u_3 \right) \right) d\phi_3 d\theta_3 du_3 d^3v_3. \tag{2.39}
\]

Finally, to determine the production spectrum we need to integrate over the variables \(d\phi_3 d\theta_3 du_3\). The angular portion of this integral is found using the relation given in appendix B.2

\[
\int_0^\pi \int_0^{2\pi} \sin \theta_3 \exp \left( \frac{(m_1 + m_2)}{T} v_3 \cdot u_3 \right) d\phi_3 d\theta_3 = \\
\frac{4\pi T}{(m_1 + m_2) v_3 u_3} \sinh \left( \frac{(m_1 + m_2)}{T} v_3 u_3 \right), \tag{2.40}
\]

where

\[
\sinh (x) = \frac{\exp (x) - \exp (-x)}{2}, \tag{2.41}
\]

is the hyperbolic sine. Taking this result we have the following expression for the production spectrum for maxwellian distributions of equal temperature

\[
R_{12}(v_3) = \frac{n_1 n_2}{1 + \delta_{12}} \frac{\eta \sqrt{m_1 m_2}}{(2\pi T)^2} \frac{1}{v_3} \int_{v_Q}^\infty \zeta^2 \sigma(\zeta) \exp \left( -\frac{\mu}{2T} \zeta^2 \right) \times
\left[ \exp \left( -\frac{m_1 m_2}{2\mu T} (v_3 - u_3)^2 \right) - \exp \left( -\frac{m_1 m_2}{2\mu T} (v_3 + u_3)^2 \right) \right] du_3 d^3v_3. \tag{2.42}
\]

where \(v_Q = \sqrt{2Q/\eta}\). As in the case of absorption spectra, for an unspecified reaction cross-section, the most compact form of the production spectrum is a single variable integral. This expression is also appropriate when considering intra-species reactions (in which the reactants have just one distribution function) by letting \(m_1 = m_2\) and \(\delta_{12} = 1\).

Since the production spectrum is isotropic (the right-hand side of (2.42) is in-
dependent of the direction of \( \mathbf{v}_3 \) we can easily express it in terms of energy rather than velocity. Using \( E_3 = \frac{1}{2} m_3 v_3^2 \), \( E_r = \frac{1}{2} \mu v_r^2 \) we obtain

\[
R_{12}(E_3) = \frac{n_1 n_2}{1 + \delta_{12} \pi T^2} \frac{\sqrt{2}}{m_3 \mu} \int_0^\infty \frac{E_r \sigma(E_r)}{\sqrt{E_r + Q}} \exp \left( -\frac{E_r}{T} \right) \times \\
\left[ \exp \left( -\frac{m_1 m_2}{m_3 \mu T} \left( \sqrt{E_3} - \sqrt{\frac{m_4}{m_3 + m_4} (E_r + Q)} \right)^2 \right) - \\
\exp \left( -\frac{m_1 m_2}{m_3 \mu T} \left( \sqrt{E_3} + \sqrt{\frac{m_4}{m_3 + m_4} (E_r + Q)} \right)^2 \right) \right] dE_r dE_3. \tag{2.43}
\]

\( E_3 \) is the lab frame kinetic energy of species 3.

Figure 2.6: Diagram (a) shows the DD neutron production spectrum from a thermal plasma. Diagram (b) shows the DT neutron production spectrum where the D and T distributions have the same temperature. The temperatures curves plotted are the same in (a) and (b). The long high energy tail is not as pronounced in the DT case as in the DD case.
2.3.2 Two temperature maxwellian distribution

We now consider the case in which the reactant species have maxwellian temperatures of differing temperatures. We use the same steps as before to determine the production spectrum. We define the reactant distributions by

\[ f_i(v_i) = \left( \frac{m_i}{2\pi T_i} \right)^{\frac{3}{2}} \exp \left( -\frac{m_i v_i^2}{2T_i} \right), \quad (2.44) \]

where \( i = 1, 2 \). Inserting the maxwellian distributions into (2.9) gives

\[ R_{12}(v_1, v_2) = \Lambda v_r \sigma(v_r) \exp \left( -\frac{m_1 v_1^2}{2T_1} - \frac{m_2 v_2^2}{2T_2} \right) d^3 v_1 d^3 v_2, \quad (2.45) \]

where

\[ \Lambda = \frac{n_1 n_2}{1 + \delta_{12}} \left( \frac{m_1 m_2}{4\pi^2 T_1 T_2} \right)^{\frac{3}{2}}. \quad (2.46) \]

Changing the independent variables from \((v_1, v_2)\) to \((v_{cm}, v_r)\) results in

\[ R_{12}(v_{cm}, v_r) = \Lambda v_r \sigma(v_r) \exp \left( -\alpha v_{cm}^2 - \beta v_r^2 - \mu \gamma v_{cm} v_r \right) d^3 v_r d^3 v_{cm}, \quad (2.47) \]

where we define the constants

\[ \alpha = \frac{m_1}{2T_1} + \frac{m_2}{2T_2}, \quad \beta = \frac{\mu}{m_1 + m_2} \left( \frac{m_2}{2T_1} + \frac{m_1}{2T_2} \right), \quad \gamma = \frac{1}{T_1} - \frac{1}{T_2}. \quad (2.48) \]

After converting \( v_r \) to spherical coordinates only the \( \mu \gamma v_{cm} v_r \) term of the exponential contains \( \theta_r \) and \( \phi_r \) and so integration over these angular variables is

\[ \int_0^\pi \int_0^{2\pi} \sin \theta_r \exp \left( -\mu \gamma v_{cm} v_r \right) d\phi_r d\theta_r = 4\pi \frac{\sinh \left( \mu \gamma v_r v_{cm} \right)}{\mu \gamma v_r v_{cm}}. \quad (2.49) \]

The solution to this integral is obtained using the identity given in appendix B.2. The expression in (2.47) is now reduced to a function of four independent variables

\[ R_{12}(v_{cm}, v_r) = \frac{4\pi \Lambda}{\mu \gamma v_{cm} v_r^2} \sigma(v_r) \exp \left( -\alpha v_{cm}^2 - \beta v_r^2 \right) \sinh \left( \mu \gamma v_r v_{cm} \right) dv_r d^3 v_{cm}. \quad (2.50) \]
We next transform $v$ to $u_3$ after which expression (2.50) becomes

$$R_{12}(v_{cm}, u_3) = \frac{4\pi \eta \Lambda}{\mu^2 \gamma v_{cm}} u_3 \zeta \sigma(\zeta) \exp \left(-\alpha v_{cm}^2 - \beta \zeta^2\right) \sinh (\mu \gamma \zeta v_{cm}) \, du_3 \, d^3 v_{cm}. \quad (2.51)$$

Vectorization of $u_3$ and transformation from the CM frame to the lab frame using (2.37) leads to

$$R_{12}(v_3, u_3) = \sin \theta_3 \frac{\eta \Lambda}{\mu^2 \gamma} u_3 \zeta \sigma(\zeta) \frac{\exp(-\beta \zeta^2 - \alpha \xi)}{\sqrt{\xi}} \sinh \left(\mu \gamma \zeta \sqrt{\xi}\right) \, d\phi_3 \, d\theta_3 \, du_3 \, d^3 v_3. \quad (2.52)$$

where

$$\xi = v_3^2 + u_3^2 - 2v_3 u_3. \quad (2.53)$$

The angular integrals over $d\phi_3 d\theta_3$ are solved using the identity given in appendix B.2. This reduces the integral to the following single integral

$$\int_0^\pi \int_0^{2\pi} \sin \theta_3 \frac{\exp(-\alpha \xi)}{\sqrt{\xi}} \sinh \left(\mu \gamma \zeta \sqrt{\xi}\right) \, d\phi_3 \, d\theta_3 = 2\pi \int_{-1}^1 \frac{\exp(-\alpha (v_3^2 + u_3^2 - 2v_3 u_3 t))}{\sqrt{v_3^2 + u_3^2 - 2v_3 u_3 t}} \sinh \left(\mu \gamma \zeta \sqrt{v_3^2 + u_3^2 - 2v_3 u_3 t}\right) \, dt. \quad (2.54)$$

Now using the substitution $x = \sqrt{v_3^2 + u_3^2 - 2v_3 u_3 t}$ we have

$$\frac{2\pi}{v_3 u_3} \int_a^b \frac{\exp(-\alpha x^2)}{\sqrt{x}} \sinh (\mu \gamma \zeta x) \, dx, \quad (2.55)$$

where $a = v_3 - u_3$, $b = v_3 + u_3$. Upon expanding the sinh function we get

$$\frac{\pi}{v_3 u_3} \int_a^b \left[ \exp(-\alpha x^2 + \mu \gamma \zeta x) - \exp(-\alpha x^2 - \mu \gamma \zeta x) \right] \, dx \quad (2.56)$$

and completing the squares of the exponential arguments gives

$$\frac{\pi}{v_3 u_3} \exp \left(\frac{(\mu \gamma \zeta)^2}{4\alpha}\right) \int_a^b \left[ \exp\left(-\alpha \left(x - \frac{\mu \gamma \zeta}{2\alpha}\right)^2\right) - \exp\left(-\alpha \left(x + \frac{\mu \gamma \zeta}{2\alpha}\right)^2\right) \right] \, dx \quad (2.57)$$
The antiderivative of functions of the form $\exp(-t^2)$ is the error function $\text{erf}(z)$ defined by

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) \, dt.$$ (2.58)

The error function is a well-known special function. A number of rational approximations suitable for its computation are contained in [17]. Using the error function definition we find

$$\int \exp \left( -\alpha \left( x - \frac{\mu \gamma \zeta}{2\alpha} \right)^2 \right) \, dx = -\sqrt{\pi} \frac{\alpha \exp \left( \frac{(\mu \gamma \zeta - x)}{2\alpha} \right)}{2\sqrt{\alpha}},$$ (2.59)

$$\int \exp \left( -\alpha \left( x + \frac{\mu \gamma \zeta}{2\alpha} \right)^2 \right) \, dx = \sqrt{\pi} \frac{\alpha \exp \left( \frac{(\mu \gamma \zeta + x)}{2\alpha} \right)}{2\sqrt{\alpha}},$$ (2.60)

and, therefore, the result for the integration over the angular variables is

$$\int_0^{\pi} \int_0^{2\pi} \sin \theta_3 \sinh \left( \mu \gamma \zeta \sqrt{\xi} \right) \exp \left( \left( \mu \gamma \zeta \right)^2 \right) \times$$

$$\left[ \text{erf} \left( \frac{\mu \gamma \zeta + 2\alpha (v_3 - u_3)}{2\sqrt{\alpha}} \right) - \text{erf} \left( \frac{2\alpha (v_3 - u_3)}{2\sqrt{\alpha}} \right) \right]$$

$$\left[ \text{erf} \left( \frac{-\mu \gamma \zeta + 2\alpha (v_3 + u_3)}{2\sqrt{\alpha}} \right) - \text{erf} \left( \frac{2\alpha (v_3 + u_3)}{2\sqrt{\alpha}} \right) \right].$$ (2.61)

Combining this result with (2.52) we find that the production spectrum that is obtained when the reactants have maxwellian distributions with different temperatures is given by

$$R_{12}(v_3) = \frac{\pi \gamma \eta}{2 \mu^2 \sqrt{\alpha} v_3} \int_{v_Q}^{\infty} \zeta \sigma(\zeta) \exp \left( \left( \frac{(\mu \gamma \zeta)^2}{4\alpha} - \beta \zeta^2 \right) \times$$

$$\left[ \text{erf} \left( \frac{\mu \gamma \zeta + 2\alpha (v_3 - u_3)}{2\sqrt{\alpha}} \right) - \text{erf} \left( \frac{-\mu \gamma \zeta + 2\alpha (v_3 - u_3)}{2\sqrt{\alpha}} \right) \right]$$

$$\left[ \text{erf} \left( \frac{-\mu \gamma \zeta + 2\alpha (v_3 + u_3)}{2\sqrt{\alpha}} \right) - \text{erf} \left( \frac{\mu \gamma \zeta + 2\alpha (v_3 + u_3)}{2\sqrt{\alpha}} \right) \right] \, du_3 d^3 v_3.$$ (2.62)

It should be noted that the production spectrum obtained in section 2.3.1 for when both reacting species have the same temperature ($T_1 = T_2$) is a limiting case of
Since the derivative of the error function is defined by
\[
\frac{d}{dz} \text{erf}(z) = \frac{2}{\sqrt{\pi}} \exp(-z^2),
\]
(2.63)
as \(T_2 - T_1 \to 0\) we have
\[
\lim_{\gamma \to 0} \frac{\text{erf}\left(\frac{\mu \gamma \zeta + 2\alpha (v_3 - u_3)}{2\sqrt{\alpha}}\right) - \text{erf}\left(\frac{-\mu \gamma \zeta + 2\alpha (v_3 - u_3)}{2\sqrt{\alpha}}\right)}{\gamma} = \frac{2\mu \zeta}{\sqrt{\pi \alpha}} \exp\left(-\alpha (v_3 - u_3)^2\right),
\]
(2.64)
\[
\lim_{\gamma \to 0} \frac{\text{erf}\left(\frac{\mu \gamma \zeta + 2\alpha (v_3 + u_3)}{2\sqrt{\alpha}}\right) - \text{erf}\left(\frac{-\mu \gamma \zeta + 2\alpha (v_3 + u_3)}{2\sqrt{\alpha}}\right)}{\gamma} = \frac{2\mu \zeta}{\sqrt{\pi \alpha}} \exp\left(-\alpha (v_3 + u_3)^2\right).
\]
(2.65)
Substituting these limiting values into (2.62) results in (2.42) with \(T_1 = T_2 = T\).

The shape of spectra produced by two maxwellians of differing temperatures is similar to that for equal temperatures. Two examples for the DT reaction are shown in figure 2.7 in which the spectra for differing temperatures may be seen as intermediate between spectra for equal temperatures at the higher and lower values.
Figure 2.7: Both diagrams show the spectra produced by the DT reaction for maxwellian distributions of differing temperature. The spectra are intermediate between spectra produced when the D and T distributions have equal temperatures at the higher and lower temperature values. In these diagrams all curves have been individually normalised to unity. Although not shown here the total neutron production value for the cases of differing temperatures will be similarly intermediate between the equal temperature cases.
2.3.3 Maxwellian distribution with bulk fluid velocity

The distribution function for a Maxwellian plasma moving with a bulk fluid motion is

\[ f_i(v_i) = \left(\frac{m_i}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_i}{2T}(v_i - v_f)^2\right), \tag{2.66} \]

where \( v_f \) is the fluid velocity. The fluid velocity causes the distribution to be shifted in phase space. We can derive the production spectrum by the usual means with the initial expression given by

\[ R_{12}(v_1, v_2) = \frac{n_1 n_2}{1 + \delta_{12}} \left(\frac{\sqrt{m_1 m_2}}{2\pi T}\right)^3 v_r \sigma(v_r) \exp\left(-\frac{1}{2T}(m_1 (v_1 - v_f)^2 + m_2 (v_2 - v_f)^2)\right) d^3v_1 d^3v_2. \tag{2.67} \]

Changing the independent variables to \((v_{cm}, v_r)\) results in

\[ R_{12}(v_{cm}, v_r) = \frac{n_1 n_2}{1 + \delta_{12}} \left(\frac{\sqrt{m_1 m_2}}{2\pi T}\right)^3 v_r \sigma(v_r) \exp\left(-\frac{1}{2T}((m_1 + m_2)(v_{cm} - v_f)^2 + \mu v_r^2)\right) d^3v_r d^3v_{cm}. \tag{2.68} \]

We can see that the effect of the fluid velocity is to shift the velocity of the CM frame and since \( v_3 \) is linearly dependent on \( v_{cm} \) we can expect our solution for \( v_3 \) to be shifted by an amount \( v_f \). The production spectrum we derive for a single-temperature plasma with bulk fluid velocity is

\[ R_{12}(v_3) = \frac{n_1 n_2}{1 + \delta_{12}} \sqrt{\frac{m_1 m_2}{2\pi T}} \frac{1}{|v_3 - v_f|} \int_{v_3}^{\infty} \xi^2 \sigma(\xi) \exp\left(-\frac{\mu}{2T}\xi^2\right) \times \left[\exp\left(-\alpha (|v_3 - v_f| - u_3)^2\right) - \exp\left(-\alpha (|v_3 - v_f| + u_3)^2\right)\right] du_3 d^3v_3. \tag{2.69} \]

where \( \alpha = (m_1 + m_2)/2T \). This spectrum is clearly a modification of (2.42) in which \( v_3 \) is replaced with \( |v_3 - v_f| \) (the production spectrum for a two-temperature Maxwellian plasma moving with bulk fluid velocity can be obtained by similarly modifying (2.62)). In figure 2.8 it is shown that the shifting of the production spectrum caused by a bulk fluid motion can have an appreciable effect on the neutron spectrum produced by DD reactions. This is particularly the case at lower plasma
temperatures at which the ratio of the energy shift caused by the fluid motion to the FWHM value of the spectrum is greater. We note that the energy shift itself is independent of temperature.

Figure 2.8: Diagram (a) shows the effect on the neutron spectrum produced by a 2 keV deuterium plasma of a fluid velocity of $10^5 \text{ms}^{-1}$. The spectra shown are for viewing angles parallel to the flow direction, one in the upstream direction and one in the downstream direction. Diagram (b) shows the location of the spectra peaks in both upstream and downstream directions for a range of fluid velocities. The difference between the peaks is greater than the FWHM of the spectrum for fluid velocities greater than $3 \times 10^5 \text{ms}^{-1}$.

An interesting application of the effect of bulk fluid motion on production spectra is the case of an imploding (or exploding) spherical shell. Such a model could be of use for determining the effect of implosion velocity on production spectrum broadening in ICF experiments, for example. In [18] it was suggested that when fluid
velocities that are equal to or greater than the thermal velocity exist in expanding or contracting spherical sources, the resulting broadening effect on the neutron spectrum becomes significant. We can determine the production spectrum for an imploding spherical shell of uniform temperature by integrating \( v_f \) in (2.69) over all fluid directions. This requires solution of the following integral

\[
\int_0^{2\pi} \int_0^\pi \frac{v_f^2 \sin \theta_f}{|v_3 - v_f|} \left[ \exp \left( -\alpha \left( |v_3 - v_f| - u_3 \right) \right) - \exp \left( -\alpha \left( |v_3 - v_f| + u_3 \right) \right) \right] d\phi_f d\theta_f
\]

(2.70)

where

\[
|v_3 - v_f| = \sqrt{v_3^2 + v_f^2 - 2v_3 v_f}.
\]

(2.71)

This can be simplified using the identity in appendix B.2 and the substitution

\[
x = \sqrt{v_3^2 + v_f^2 - 2v_3 v_f t},
\]

(2.72)

to give an integral similar to that in (2.61)

\[
2\pi \frac{v_f}{v_3} \int_{v_3-v_f}^{v_3+v_f} \left[ \exp \left( -\alpha \left( x - u_3 \right) \right) - \exp \left( -\alpha \left( x + u_3 \right) \right) \right] dx =
\]

\[
\frac{\pi \frac{1}{2} v_f}{\sqrt{\alpha} v_3} \left[ \text{erf} \left( \sqrt{\alpha} (u_3-v_3 + v_f) \right) - \text{erf} \left( \sqrt{\alpha} (u_3-v_3-v_f) \right) + \text{erf} \left( \sqrt{\alpha} (u_3 + v_3-v_f) \right) - \text{erf} \left( \sqrt{\alpha} (u_3 + v_3 + v_f) \right) \right].
\]

(2.73)

We then obtain the following expression for the production spectrum

\[
R_{12}^{\text{shell}}(v_3) = \frac{n_1 n_2}{1 + \delta_{12}} \eta \sqrt{\frac{\mu}{\pi (2T)^3 v_3}} \int_{v_3}^{\infty} \zeta^2 \sigma(\zeta) \exp \left( -\frac{\mu}{2T} \zeta^2 \right) \times
\]

\[
\left[ \text{erf} \left( \sqrt{\alpha} (u_3-v_3 + v_f) \right) - \text{erf} \left( \sqrt{\alpha} (u_3-v_3-v_f) \right) + \text{erf} \left( \sqrt{\alpha} (u_3 + v_3-v_f) \right) - \text{erf} \left( \sqrt{\alpha} (u_3 + v_3 + v_f) \right) \right] d\zeta d^3v_3.
\]

(2.74)

This expression gives the production spectrum from a spherical shell of uniform temperature imploding (or exploding) with velocity \( v_f \) in which the shell is treated as a point source. Clearly it is an idealised case in which scattering of the products which are emitted on the “far” side of the shell to the observer are not considered.
However, the model is still instructive when we apply it to the case of the neutrons emitted by an imploding deuterium shell as it shows significant broadening of the spectrum. Examples are shown in figure 2.9 for temperatures of $0.5 \, keV$ and $2.0 \, keV$. As the implosion velocity increases the spectrum becomes much wider with a broad, flat peak. In both cases, when the implosion velocity is similar to the thermal velocity of the plasma the FWHM value of the spectrum is over twice that of a plasma with zero fluid velocity. For higher implosion velocities this broadening increases exponentially. There is also a clear asymmetry in the spectra for higher implosion velocities. This is an amplification of the asymmetry in the spectrum of a stationary thermal plasma. The long high energy tail of this spectrum results in a spectrum for the imploding shell in which the peak is at a much higher energy value than the mean energy of the spectrum. The effects of implosion velocity and temperature are shown more detail in figure 2.10 in which the ratio of the FWHM of an imploding shell to the FWHM of a stationary plasma at equal temperature is shown as a contour plot for a range of temperatures and implosion velocities. The broadening factor is greatest for low temperature and high implosion velocity plasma shells. However, even in the temperature range $1 - 8 \, keV$ a broadening factor of 2 can occur for implosion velocities of the order $10^5 \, m \, s^{-1}$.
Figure 2.9: The neutron spectra produced by a uniform imploding shell of deuterium with $T = 0.5\,\text{keV}$ is shown in the top diagram for implosion velocities of $3 \times 10^5\,\text{m}\,\text{s}^{-1}$ and $6 \times 10^5\,\text{m}\,\text{s}^{-1}$. The spectrum produced by a stationary maximilian plasma with $T = 0.5\,\text{keV}$ is shown for comparison. The thermal velocity at this temperature is $v_{th} = \sqrt{3T/m_i} = 2.68 \times 10^5\,\text{m}\,\text{s}^{-1}$. The corresponding spectra for $T = 2\,\text{keV}$ are shown in the bottom diagram where $v_{th} = 5.36 \times 10^5\,\text{m}\,\text{s}^{-1}$. Note the asymmetry of the spectra, which is most obvious for the high fluid velocity spectra. This asymmetry is due to the long high energy tail of the spectrum when stationary.
Figure 2.10: The top diagram shows contours of the ratio of the FWHM neutron spectrum for an imploding deuterium shell of uniform temperature $T$ and implosion velocity $v_f$ to the FWHM of a stationary plasma of equal temperature. The lower diagram shows the FWHM values of the stationary plasma.
2.3.4 Bi-maxwellian distribution

The bi-maxwellian distribution function can be used to model plasmas in a magnetic field [19]. It is given by

\[
 f_i(v_i) = \left( \frac{m_i}{2\pi} \right)^{\frac{3}{2}} \frac{1}{T_\perp \sqrt{T_\parallel}} \exp \left( -\frac{m_i}{2T_\perp} \left( v_{ix}^2 + v_{iy}^2 \right) - \frac{m_i}{2T_\parallel} v_{iz}^2 \right),
\]

(2.75)

where \( T_\parallel \) and \( T_\perp \) represent plasma temperatures parallel and perpendicular, respectively, to the magnetic field, which we have taken to be in the z direction. The bi-maxwellian distribution has been used, for example, in the modelling of plasmas in ion cyclotron resonant heating (ICRH) in tokamaks, [20], [21]. We can derive the following expression for the production spectrum

\[
 R_{12}(v_3) = \frac{n_1n_2}{1 + \delta_{12}^2 T_\perp^2 T_\parallel} \left( \frac{m_1 + m_2}{2\pi} \right)^{\frac{3}{2}} \frac{\eta}{\sqrt{T_\parallel - 1}} \times
\]

\[
 \int_{v_3}^\infty u_3 \zeta \sigma(\zeta) \exp \left( -\frac{\mu}{2T_\perp} \zeta^2 \right) \text{erf} \left( \zeta \sqrt{\frac{\mu}{2} \left( \frac{1}{T_\parallel} \frac{1}{1} \right)} \right) \times
\]

\[
 \int_0^\pi \int_0^{2\pi} \sin \theta_3 \exp \left( -\frac{m_1m_2}{2\mu} \left( \frac{v_{cmx}^2 + v_{cmy}^2}{T_\perp} + \frac{v_{cmz}^2}{T_\parallel} \right) \right) d\phi_3 d\theta_3 d\mu_3 d^3v_3
\]

(2.76)

where \( v_{cmx}, v_{cmy} \) and \( v_{cmz} \) are given by (2.37). Here we have assumed that the two reactant distributions have equal \( T_\parallel \) and \( T_\perp \) temperatures. An analytic solution to the integral over \( d\phi_3 d\theta_3 \) is not known to exist and so we must express the production spectrum for a bi-maxwellian as a triple integral. We solve the integral over \( d\phi_3 d\theta_3 \) numerically using Lebedev quadrature [22], a method suited to finding the surface integral on a sphere.

Since the bi-maxwellian distribution is anisotropic, the spectrum it produces will also be anisotropic. However, although the shape of the spectrum will change with viewing angle, we note that the total number of product particles emitted is the same in every direction. This is illustrated in figure 2.11 in which we take the example of a deuterium plasma with \( T_\parallel = 20 \text{ keV} \) and \( T_\perp = 60 \text{ keV} \). It is interesting to note that the FWHM for the spectrum observed in the parallel direction agrees with the spectrum for a maxwellian plasma with \( T = 20 \text{ keV} \) to within 5% while the FWHM
of the spectrum observed in the perpendicular direction agrees with the spectrum for a maxwellian plasma with \( T = 60 \, kev \) to within 1%.

Figure 2.11: Diagram (a) shows the neutron production spectrum for a bi-maxwellian deuterium plasma with \( T_{\parallel} = 20 \, keV \) and \( T_{\perp} = 60 \, keV \) at 3 different viewing angles. The total emission in each direction is the same. Diagram (b) shows the spectra produced by maxwellian plasmas of \( T = 20 \, keV \) and \( 60 \, keV \) for comparison of FWHM values.
2.3.5 Beam-target plasma

We can derive the production spectrum for a beam-target interaction in which an ion beam collides with a thermal target. The target is a maxwellian distribution with temperature $T_1$. The beam is modelled as a maxwellian of temperature $T_2$ moving with beam velocity $v_b$ given by (2.66). The temperature $T_2$ is a measure of the average energy divergence of the beam. As $T_2 \to 0$ the beam becomes monoenergetic. The production spectrum produced by reactions between beam and target ions is

$$R_{12}(v_3) = \Lambda \frac{\eta}{\mu^2} \exp\left(\frac{m_2}{2T_2}v_b^2\right) \int_{v_3}^{\infty} u_3 \sigma(\zeta) \exp\left(-\frac{\mu}{m_1 + m_2} \left(\frac{m_2}{2T_1} + \frac{m_1}{2T_2}\right) \zeta^2\right) \times$$

$$\int_0^\pi \int_0^{2\pi} \sin \theta_3 \sinh \left(\frac{\mu \zeta \chi}{\chi}\right) \exp\left(-\alpha \xi + \frac{m_2}{T_2} v_{\text{cm}} \cdot v_b\right) \, d\phi_3 d\theta_3 d u_3 d^3 v_3, \quad (2.77)$$

where

$$\chi = \left[\left(\frac{1}{T_1} - \frac{1}{T_2}\right)^2 \xi + \frac{1}{T_2^2} v_b^2 + \frac{2}{T_2} \left(\frac{1}{T_1} - \frac{1}{T_2}\right) v_{\text{cm}} \cdot v_b\right]^{1/2}, \quad (2.78)$$

$$v_{\text{cm}} \cdot v_b = v_3 \cdot v_b - u_3 (v_{bx} \sin \theta_3 \cos \phi_3 + v_{by} \sin \theta_3 \sin \phi_3 + v_{bz} \cos \theta_3), \quad (2.79)$$

and $\Lambda$, $\zeta$ and $\xi$ are as defined in (2.46), (2.33) and (2.53), respectively. As in the bi-maxwellian case, the integral over $d\phi_3 d\theta_3$ must be done numerically. Expression (2.77) represents the beam-target component of the production spectrum. The total production spectrum of the plasma will also contain contributions from reactions where both reactant ions are in the target (called the target-target component, given by the production spectrum for a single temperature maxwellian (2.42)) and, if the divergence of the beam is significant, from reactions where both reactant ions are in the beam plasma (the beam-beam component, given by the production spectrum of a maxwellian moving with bulk fluid velocity (2.69)). The relative weighting of each component, given by the ratio of the number density of the beam to the target plasma, is important for determining the shape of the total production spectrum, as shown by the following example.

Beam-target plasmas can arise in the case of neutral beam heating in Tokamaks
and due to transient electric fields in deuterium Z pinches [24]. We can use (2.77) to study the neutron spectrum arising from such a beam-target interaction. As in the case of a plasma with bulk fluid motion the spectrum will be anisotropic with the greatest energy shift being in the beam direction. However, unlike the bulk fluid motion case, the number of neutrons emitted is also anisotropic, with the greatest number emitted in the beam direction.

The parameters that determine the beam-target neutron spectrum include the target temperature, beam energy and dispersion of the beam, the most important of which are the beam energy and target temperature. Examples of the spectrum for beam energies of 100 keV and 500 keV and target temperatures of 2 keV and 8 keV are shown in figure 2.12. For a given viewing angle the beam energy determines the energy shift of the spectrum peak. The target temperature determines the broadening of spectrum about the peak. Generally, for beams with energies in the range 100 – 1000 keV and target temperatures in the range 1 – 10 keV the most significant energy shift occurs between viewing angles of $\pi/2$ and $\pi/4$ to the beam direction.

The total neutron spectrum will include neutrons emitted by the target (the target-target component) as well as the beam-target component. Examples are shown in figure 2.13 for targets with temperatures of 2 keV and 8 keV. The total spectrum has the form of a double-peak with the main peak due to the target reactions and the second peak in the high energy tail arising due to the beam-target component. Generally, for target temperatures in the range of 1 – 10 keV a double peak may be apparent in the neutron spectrum if the ratio of beam density to target density is of the order of $10^{-3} – 10^{-4}$. If the ratio is higher the beam-target component will dominate the spectrum, while neutrons from the target will dominate if the ratio is lower.

In the case of deuterium gas puffs on the Z machine [25] the ratio of the target-target component of the neutron spectrum to the beam-target component remains unresolved. In these experiments the ion density and temperature at stagnation are $10^{26} m^{-3}$ and 6 keV, respectively. It has been estimated that the beam current required to generate the observed number of neutrons solely by beam-target reac-
tions is an unrealistically high $26 - 115\, MA$, [26]. By calculating the exact shape of the target-target and beam-target neutron spectra for the given temperature and density we find that for beam energies greater than $100\, keV$ an ion current density of over $10^{12}\, Am^{-2}$ is required for the beam-target spectrum to have an intensity of the same order of magnitude as the target-target spectrum. Given that the stagnated pinch radius is approximately $0.001\, m$ this value appears to be in agreement with [26]. We return to the issue of beam-target reactions in chapter 5 in which we use computational methods to determine the number of beam-target reactions occurring.

We conclude this section by observing that the clearest evidence of a beam-target component in the reacting plasma is likely to be seen as an anisotropy in the high energy tail of the spectrum rather than anisotropy in the total number of neutrons emitted.
Figure 2.12: The top diagram shows the neutron spectra for a deuterium beam-target interaction with a beam energy of 100 keV and target temperatures of 2 keV and 8 keV. The divergence of the beam is \( T_b = 0.2 \) keV. Spectra marked (a) are those that occur at viewing angle of \( \pi/2 \) with respect to the beam direction, (b) for \( \pi/4 \) and (c) for a viewing angle parallel to the beam. The same spectra are shown in the bottom diagram for a beam energy of 500 keV.
Figure 2.13: Two examples of the compound neutron spectrum (including beam-target neutrons and neutrons from reactions in the target) are shown. The top diagram is the spectrum for a target temperature of 2 keV, beam energy of 100 keV and density ratio $n_b/n_t = 0.0002$. In the bottom diagram these values are 8 keV, 500 keV and 0.0005, respectively. In both cases the curve marked (a) is the spectrum seen at a viewing angle of $\pi/4$ with respect to the beam direction while the curve marked (b) has a viewing angle of $3\pi/8$. 
2.4 The production spectrum part II: Cross-sections that are a function of relative velocity and scattering angle

In the previous section we assumed that the reaction products are emitted isotropically in the CM frame. In sketch (c) of figure 2.5 the probability of vector \( u_3 \) being in any given direction is uniform. This is the case when the reaction cross-section is independent of the scattering angle. However, for some nuclear reactions the cross-section depends on the scattering angle, \( \theta_{cm} \). This means that two reacting particles are more likely to emit their products in certain directions. In such cases the cross-section is expressed as \( \sigma(v_r, \theta_{cm}) \). The scattering angle is defined as the angle between the reactant particle 1 and product particle 3 in the CM frame. Therefore, we can express the scattering angle as a function of the angular components of \( v_r \) and \( u_3 \) as follows

\[
\theta_{cm} = \arccos \left( \frac{v_r \cdot u_3}{v_r u_3} \right).
\]  

(2.80)

Since \( \theta_{cm} \) is defined in terms of these variables, similar steps to those used in the previous section can be used for deriving expressions for the production spectrum. However, the cross-section is now a function of two variables and so for many distributions the production spectrum will require a numerical integration over these two variables. For example, we can derive the following expression for the production spectrum for a plasma with a two temperature maxwellian distribution (the derivation is given in appendix C.2)

\[
R_{12}(v_3) = 4\pi^2 \Lambda \eta \mu^2 \sqrt{\alpha} \int_{v_q}^{\infty} \zeta \exp \left( \frac{\mu^2 \gamma^2}{4\alpha} - \beta \right) \zeta^2 \int_a^b \sigma(\zeta, \theta_{cm}) \times
\left[ \exp \left( - (x - v_3 \sqrt{\alpha})^2 \right) - \exp \left( -(x + v_3 \sqrt{\alpha})^2 \right) \right] dx du_3 dv_3,
\]  

(2.81)

\(^\dagger\)Cross-sections that depend on scattering angle are often called differential cross-sections since it gives the probability of reaction per unit angle of emission. The notation \( d\sigma/d\Omega \) is also commonly used to denote the differential cross-section. However, here we will use the notation \( \sigma(v_r, \theta_{cm}) \) to emphasize those variables that the differential cross-section depends on.
where
\[ \theta_{cm} = \arccos \left( \frac{4\alpha^2 u_3^2 + \mu^2 \gamma^2 \zeta^2 - 4\alpha x^2}{4\alpha u_3 \mu \gamma \zeta} \right). \] (2.82)

The limits of integration for \( x \) are \( a = \sqrt{\alpha u_3} - \mu \gamma \zeta / (2\sqrt{\alpha}) \) and \( b = \sqrt{\alpha u_3} + \mu \gamma \zeta / (2\sqrt{\alpha}) \) and the constants \( \Lambda, \alpha, \beta \) and \( \gamma \) are defined in (2.46) and (2.48). The two remaining integrals need to be carried out numerically using data for the parameterised cross-sections (such data may be found in [27], see appendix A). We note that (2.81) gives the production spectrum as a function of \( v_3 \) and not, as is usually the case, \( v_3 \) (this is as a result of the derivation method). Thus, to find the intensity per unit solid angle of the production spectrum (2.81) needs to be divided by a factor of \( 4\pi \).

It is also shown in appendix C.2 that for a plasma with a single temperature maxwellian distribution the production spectrum is the same for cross-sections that are both dependent on and independent of the scattering angle. Therefore, (2.42) applies in both cases.

### 2.5 The production spectrum part III: A summary of the derivation method for non-relativistic plasmas

An intuitive description of the method for deriving the production spectrum was given at the beginning of section 2.3. In this section we give a more rigorous description of the transformation and integration required for finding the production spectrum for any given distributions of the reactants. It is applicable to cross-sections that are both dependent on and independent of the scattering angle. After the transformation, a 5 dimensional integral needs to be solved to obtain the production spectrum. The ease with which this integral can be solved will depend on the distributions under consideration. As we have shown in the cases considered in sections 2.3 and 2.4 a number of techniques exist to simplify the integral.
We begin with the usual expression

\[ R_{12} (v_1, v_2) = v_r \sigma (v_r, \theta_{cm}) f_1 (v_1) f_2 (v_2) d^3v_1 d^3v_2. \] (2.83)

Because of the scattering angle, this is actually a function of 8 variables \((v_1, v_2, \hat{u}_3)\) and not just \((v_1, v_2)\). We can define the required transformation as being from the 8 variables \((v_1, v_2, \hat{u}_3)\) to the 8 variables \((v_3, u_3, \hat{u}_3, \hat{v}_r)\) where \(\hat{u}_3\) and \(\hat{v}_r\) are two unit vectors.\(^\dagger\) The transformation relations are

\[
\begin{align*}
    v_1 &= v_3 - u_3\hat{u}_3 + \frac{\mu}{m_1} \zeta \hat{v}_r, \\
    v_2 &= v_3 - u_3\hat{u}_3 - \frac{\mu}{m_2} \zeta \hat{v}_r, \\
    \hat{u}_3 &= \hat{u}_3,
\end{align*}
\] (2.86)

where we recall that \(\zeta\) is simply a function of \(u_3\)

\[
\zeta = \sqrt{\frac{2}{\mu} \left( \frac{\eta}{2} u_3^2 - Q \right)}. \] (2.87)

The Jacobian matrix of partial derivatives (see appendix B.1), denoted by \(A\), for this transformation is shown in (2.92). The Jacobian determinant that results from it is

\[
\det (A) = \sin \theta_r \frac{\eta}{\mu} u_3 \zeta. \] (2.88)

We must also include the factor \(\frac{\sin \theta_3}{4\pi}\). This is a probability factor that does not come from the transformation and so is not included in the Jacobian determinant. It is the probability of emission of a particle in the CM frame in any unit solid angle.

To summarise, if we write 2.83 explicitly in terms of the 8 variables as

\[ R_{12} (v_1, v_2, \hat{u}_3) = |v_1 - v_2| \sigma (|v_1 - v_2|, \theta_{cm}) f_1 (v_1) f_2 (v_2) d^2\hat{u}_3 d^3v_1 d^3v_2, \] (2.89)

\(^\dagger\) The unit vectors \(\hat{u}_3\) and \(\hat{v}_r\) are each functions of two variables defining the direction of the vectors. These variables are \((\theta_3, \phi_3)\) and \((\theta_r, \phi_r)\), respectively. The unit vectors are

\[
\begin{align*}
    \hat{u}_3 &= (\sin \theta_3 \cos \phi_3, \sin \theta_3 \sin \phi_3, \cos \theta_3), \\
    \hat{v}_r &= (\sin \theta_r \cos \phi_r, \sin \theta_r \sin \phi_r, \cos \theta_r). \quad (2.84)
\end{align*}
\]
The absorption and production spectra in fusion plasmas then after the transformation and inclusion of the probability factor it becomes

\[ R_{12}(v_3, u_3, \hat{u}_3, \hat{v}_r) = \frac{\eta u_3 \zeta^2}{4\pi \mu} \sin \theta_3 \sin \theta_r \sigma(\zeta, \arccos(\hat{v}_r, \hat{u}_3)) \times \]

\[ f_1 \left( v_3 - u_3 \hat{u}_3 + \frac{\mu \zeta}{m_1} \hat{v}_r \right) f_2 \left( v_3 - u_3 \hat{u}_3 - \frac{\mu \zeta}{m_2} \hat{v}_r \right) d^2 \hat{v}_rd^2 \hat{u}_3du_3dv_3. \tag{2.90} \]

The production spectrum is then found by integrating over the 5 variables \((u_3, \hat{u}_3, \hat{v}_r)\)

\[ R_{12}(v_3) = \int_{v_Q}^{\infty} \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} R_{12}(v_3, u_3, \hat{u}_3, \hat{v}_r), \tag{2.91} \]

where \(v_Q = \sqrt{2Q/\eta}\).
\[
A = \begin{bmatrix}
1 & 0 & 0 & -\sin \theta_3 \cos \phi_3 + \frac{u_3}{m_1} \zeta \sin \theta_3 \cos \phi_r & -u_3 \cos \theta_3 \cos \phi_3 & u_3 \sin \theta_3 \sin \phi_3 & \frac{\mu_1}{m_1} \zeta \cos \theta_3 \cos \phi_r & -\frac{\mu_1}{m_1} \zeta \sin \theta_3 \sin \phi_r \\
0 & 1 & 0 & -\sin \theta_3 \sin \phi_3 + \frac{u_3}{m_1} \zeta \sin \theta_3 \sin \phi_r & -u_3 \cos \theta_3 \sin \phi_3 & -u_3 \sin \theta_3 \cos \phi_3 & \frac{\mu_1}{m_1} \zeta \cos \theta_3 \sin \phi_r & \frac{\mu_1}{m_1} \zeta \sin \theta_3 \cos \phi_r \\
0 & 0 & 1 & -\cos \theta_3 + \frac{u_3}{m_1} \zeta \cos \theta_r & u_3 \sin \theta_3 & 0 & -\frac{\mu_1}{m_1} \zeta \sin \theta_r & 0 \\
1 & 0 & 0 & -\sin \theta_3 \cos \phi_3 - \frac{u_3}{m_2} \zeta \sin \theta_3 \cos \phi_r & -u_3 \cos \theta_3 \cos \phi_3 & u_3 \sin \theta_3 \sin \phi_3 & -\frac{\mu_2}{m_2} \zeta \cos \theta_3 \cos \phi_r & \frac{\mu_2}{m_2} \zeta \sin \theta_3 \sin \phi_r \\
0 & 1 & 0 & -\sin \theta_3 \sin \phi_3 - \frac{u_3}{m_2} \zeta \sin \theta_3 \sin \phi_r & -u_3 \cos \theta_3 \sin \phi_3 & -u_3 \sin \theta_3 \cos \phi_3 & -\frac{\mu_2}{m_2} \zeta \cos \theta_3 \sin \phi_r & -\frac{\mu_2}{m_2} \zeta \sin \theta_3 \cos \phi_r \\
0 & 0 & 1 & -\cos \theta_3 - \frac{u_3}{m_2} \zeta \cos \theta_r & u_3 \sin \theta_3 & 0 & \frac{\mu_1}{m_1} \zeta \sin \theta_r & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
2.6 The production spectrum part IV: The relativistic case

In this section we outline how the production spectrum for relativistic plasmas may be determined. The method outlined in the previous section may be extended to the relativistic case by considering particle momentum in place of velocity and using a Lorentz transform between the lab and CM frame. For the relativistic case the initial transition rate equation, equivalent to (2.83) is [28]

\[
R_{12}(p_1, p_2, \hat{p}_3) = \sqrt{(p_{1i}p_{2i})^2 - m_1^2m_2^2c^8/\varepsilon_1\varepsilon_2} \sigma(W, \theta_{cm}) f_1(p_1) f_2(p_2) d^3p_3 d^3p_1 d^3p_2, \tag{2.93}
\]

where \(p_1\) and \(p_2\) represent the lab frame momenta of particles 1 and 2, respectively, and \(\hat{p}_3\) is the unit vector in the direction of the momentum of product particle 3 in the CM frame (quantities in the CM frame are denoted by a \(\prime\)), used to define the scattering angle. The term \(\sqrt{(p_{1i}p_{2i})^2 - m_1^2m_2^2c^8/\varepsilon_1\varepsilon_2} \sigma(W, \theta_{cm})\) represents the invariant cross-section. This reduces to its non-relativistic analogue, \(|v_1 - v_2| \sigma\), when \(v_1\) and \(v_2\) are parallel or antiparallel [29]. In this term \(p_{1i} = (\varepsilon_1, p_{1i}c)\) and \(p_{2i} = (\varepsilon_2, p_{2i}c)\) denote the momentum four-vectors of the reactant particles. The energy and momentum components of these four-vectors are not independent but are related by the invariant \(\varepsilon^2 - p^2c^2 = m^2c^4\).

The differential cross-section for the reaction has been denoted by \(\sigma(W, \theta_{cm})\) in (2.93). The scattering angle is the angle between the CM frame momenta vectors of particles 1 and 3, \(\theta_{cm} = \arccos(\hat{p}_1 \cdot \hat{p}_3)\). The variable \(W\) in the cross-section represents the CM energy of the pair of reactants (which, since we are considering relativistic interactions is the same for the products), given by \(W^2 = (p_{1i} + p_{2i})^2\). Differential cross-sections for many interactions, including nuclear fusion, Compton scattering of photons from free electrons and electron-positron pair production are defined in this way [30]. Therefore, obtaining the production spectra for these interactions may be carried out in a similar manner.

As in the non-relativistic case, deriving the production spectrum is carried out by
transforming the transition rate equation (2.93) into a form in which it is a function of the variables that determine the cross-section and integrating. Therefore, we transform from the set of variables \((p_{1i}, p_{2i}, \hat{p}_3', \hat{p}_1')\) to \((p_{3i}, W, \hat{p}_3', \hat{p}_1')\) and integrate over the 5 variables \((W, \hat{p}_3', \hat{p}_1')\) obtain the lab frame production spectrum of particle 3.\(^8\)

The transformations for this, obtained by eliminating the frame velocity from the Lorentz transform, are as follows

\[
\begin{align*}
\varepsilon_1 &= \frac{\alpha}{\omega} \varepsilon_1' + \frac{\lambda}{\omega} c (p_3 - p_3') \cdot \mathbf{p}_1', \\
\mathbf{p}_1 c &= \mathbf{p}_1 c + \left( \frac{\lambda}{\omega} \varepsilon_1' + \frac{\beta}{\omega} c (p_3 - p_3') \cdot \mathbf{p}_1' \right) (p_3 - p_3'), \\
\varepsilon_2 &= \frac{\alpha}{\omega} \varepsilon_2' - \frac{\lambda}{\omega} c (p_3 - p_3') \cdot \mathbf{p}_1, \\
\mathbf{p}_2 c &= -\mathbf{p}_1' c + \left( \frac{\lambda}{\omega} \varepsilon_2' - \frac{\beta}{\omega} c (p_3 - p_3') \cdot \mathbf{p}_1 \right) (p_3 - p_3'), \\
\hat{p}_3' &= \hat{p}_3',
\end{align*}
\]

(2.94)

where

\[
\begin{align*}
\alpha &= \varepsilon_3^2 (p_3 - p_3')^2 + c^2 [(p_3 - p_3') \cdot p_3]^2, \\
\beta &= \varepsilon_3^2 - \varepsilon_3 \varepsilon_3' - c^2 (p_3 - p_3') \cdot p_3', \\
\lambda &= \varepsilon_3 c (p_3 - p_3') \cdot p_3 - \varepsilon_3' c (p_3 - p_3') \cdot p_3', \\
\omega &= \varepsilon_3 \varepsilon_3' (p_3 - p_3')^2 + c^2 [(p_3 - p_3') \cdot p_3] [(p_3 - p_3') \cdot p_3],
\end{align*}
\]

and

\[
\begin{align*}
|\mathbf{p}_1'| &= \frac{c}{2W} \sqrt{\frac{W^4}{c^4} - 2W^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 c^4}, \\
|\mathbf{p}_3'| &= \frac{c}{2W} \sqrt{\frac{W^4}{c^4} - 2W^2(m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2 c^4}.
\end{align*}
\]

These represent the relativistic equivalent of (2.86). After these transformations a 5 dimensional integral needs to be solved but, as in the non-relativistic case, properties of the distribution functions such as isotropy should allow a number of the integrals

\(^8\)Variables \(\hat{p}_3'\) and \(\hat{p}_1'\) represent integrals over the solid angle while the limits of integration of \(W\) are \([(m_1 + m_2) c^2, \infty)\).
to be carried out analytically in certain cases. However, such work has yet to be carried out. In the literature review of relativistic production spectra in section 2.7.2 we identify potential applications for the transformation that we have outlined.

### 2.7 A review of work on production spectra

This section contains an overview of previous studies of production spectra. We describe the methods used by previous authors to derive production spectra and compare them with the method we have used. We consider the non-relativistic and relativistic production spectra separately. Work on non-relativistic production spectra has mostly considered the production of neutrons in fusion plasmas. On the other hand, work on relativistic spectra is primarily concerned with astrophysical processes and there appears to be little overlap between the two strands of literature.

#### 2.7.1 Non-relativistic production spectra review

The investigation of the neutron spectra produced by DD and DT reactions is a major part of the literature on non-relativistic production spectra. One of the early works is that by Lehner and Pohl [31] in which analytic expressions are derived for the production spectrum in a few special cases. These cases include the single temperature maxwellian distribution. Lehner and Pohl’s result for this distribution is similar to ours, given by (2.43). However, the method used by Lehner and Pohl is difficult to generalise as it is suitable only for isotropic distributions. This is because they use particle energy rather than velocity as the independent variable. Therefore, including particle direction (as is necessary in anisotropic cases) requires the introduction of variables to specify it. Furthermore, only cross-sections that are independent of the scattering angle are considered by Lehner and Pohl.

The most well-known work on non-relativistic production spectra is probably that of Brysk [15]. Brysk’s derivation of the production spectrum is based on the fitting of a gaussian distribution to the mean neutron energy. The production spec-
trum is given by

\[ f(E_n) = \exp \left( - (E_n - \langle E_n \rangle)^2 \frac{m_n + m_\alpha}{4m_n T \langle E_n \rangle} \right) dE_n, \tag{2.95} \]

where \( \langle E_n \rangle \) is the mean neutron energy given by

\[ \langle E_n \rangle = \frac{1}{2} m_n \langle V_{cm}^2 \rangle + \frac{m_\alpha}{m_n + m_\alpha} (Q + \langle K \rangle). \tag{2.96} \]

Here, \( \langle V_{cm}^2 \rangle \) is the mean square of the centre of mass velocity given by \( \langle V_{cm}^2 \rangle = \frac{3T}{2m} \) and \( \langle K \rangle \) is the mean relative kinetic energy of the reacting particles given by

\[ \langle K \rangle = \frac{\int_{0}^{\infty} K^2 \sigma(K) \exp \left( -\frac{K}{T} \right) dK}{\int_{0}^{\infty} K \sigma(K) \exp \left( -\frac{K}{T} \right) dK}. \tag{2.97} \]

Brysk only applied his method to single temperature maxwellian distributions for reaction cross-sections that do not depend on the scattering angle. Extending the method to other cases would appear cumbersome although some work has been attempted for bi-maxwellian distributions [32]. However, it is interesting to note that in obtaining the Brysk approximation of the production spectrum it is necessary to carry out numerical integrations over the reaction cross-section term (i.e. in the numerator and denominator of (2.97)). Obtaining the exact shape of the production spectrum also requires a numerical integral over the cross-section term. Therefore, calculating the approximate production spectrum may not be significantly easier than calculating its exact shape. The accuracy of the Brysk approximation is assessed in figure 2.14. The approximate and exact shapes of the DD neutron spectrum are plotted for a range of temperatures. It can be seen that the exact result differs noticeably from a gaussian at a few orders of magnitude below the maximum intensity. The spectrum for \( T = 150 \text{ keV} \) is also shown in figure 2.14 on a linear scale.

A more modern approach to production spectra in fusion plasmas involves the use of Monte-Carlo calculations. This method has been used, for example, in Tokamaks to study the neutron spectra resulting from ion cyclotron resonance heating (ICRH) [33], [34], neutral beam injection (NBI) [23] and knock-on reactions [35], [36].
The principal advantages of a Monte-Carlo approach are that it can be used when an analytic expression for the plasma distribution function is not known and its suitability for dealing with transport problems such as neutron scattering. However, ensuring accurate results from such a method requires paying close attention to statistics. Although we have not addressed the issue of transport of the production species in this chapter, the method for deriving the production spectrum that we have outlined provides an important basis for this. Knowledge of the exact form of the production spectrum is useful as it allows us to identify key features of the spectrum. For example, the beam-target production spectra studied in section 2.3.5 can be used to determine how the spectra varies with viewing angle.

![Figure 2.14: Diagrams (a)-(c) compare the exact shape of the DD neutron spectrum (blue curve) with the Brysk approximation (green dashed curve) given by [15] for plasma temperatures of 10 keV, 75 keV and 150 keV, respectively. As the temperature increases the neutron spectrum becomes more asymmetric with a long high energy tail. However, the Brysk approximation remains symmetric. The difference between the two curves becomes significant at high temperatures. Diagram (d) is the same plot as diagram (c) but with a linear y axis instead of logarithmic.](image-url)
2.7.2 Relativistic production spectra review

The literature on relativistic production spectra tends to have a wider scope than the non-relativistic case as it arises in a range of astrophysical contexts. These include, most notably, Compton and inverse Compton scattering and electron-positron pair production and annihilation.

Compton scattering of mono-energetic photons from maxwellian electrons has been studied in order to calculate the Compton scattering kernel for use in radiative transfer problems [37], [38]. Inverse Compton scattering of an anisotropic distribution of photons from hot electrons is also a widely studied problem (see, for example, [39], [40], [41] and [42]). It is analogous to the non-relativistic beam-target case that was studied in section 2.3.5. This work is concerned with understanding the high energy radiation emitted by active galactic nuclei. It is thought that the coronae above accretion disks in active galactic nuclei can contain very hot electrons. Photons emitted by the accretion disk are upscattered to high energies by the electrons [43]. Observations of compact X-ray and $\gamma$-ray sources have also led to calculations of the production spectra for electron-positron pair production [44], [45] and also pair annihilation [46], [47].

However, many different approaches have been used for calculating the production spectra in the above scenarios. The approaches tend to be specific to the geometry and distributions under consideration. It is thought that the method that we have outlined in section 2.6 can provide a general method for determining such production spectra. A general method for deriving production spectra was proposed by Baring [48]. However, the method is suitable only for isotropic distributions (it is a relativistic analogue of the approach used by Lehner and Pohl for non-relativistic spectra). Furthermore, the solution is not well-defined for products with 0 rest mass. Some suggestions for dealing with anisotropic distributions were given by Stepney and Guilbert [49] but the assumptions regarding the distributions make it unsuitable for general use. As we have shown in section 2.6, using the particle momenta as independent variables rather than energy allows for a universal approach to all these problems.
We also note that little attention has been previously paid to the relativistic production spectrum in fusion plasmas. Ballabio et al. [50] extended Brysk’s approach to include relativistic kinematics. However, the exact shape of the relativistic spectrum remains to be calculated. Therefore, it seems that there is much work to be done on relativistic production spectra.

2.8 Neutron diagnostics

This section summarises experimental procedures for measuring neutron spectra produced by fusion plasmas in the laboratory. We focus in particular on those diagnostics employed at the National Ignition Facility (NIF) as they are currently amongst the most advanced in operation in plasma physics.

2.8.1 Magnetic recoil spectrometry

The magnetic recoil spectrometer (MRS) provides one of the most accurate measurements of a neutron energy spectrum [51]. It consists of three main components. A hydrogenated (CH) or deuterated (CD) carbon foil positioned close (within \( \sim 30 \text{ cm} \)) to the imploded capsule, a focusing magnet located approximately \( 5 \text{ m} \) from the foil and an array of CR-39 detectors. Neutrons striking the foil produce protons (or deuterons). These are then focused on the detectors by the magnetic field. The field also causes energy dispersion with lower energy particles undergoing a greater deflection. Therefore, by measuring the energy spectrum of the recoil protons the energy spectrum of the neutrons can be inferred. The MRS has a detection efficiency of \( \sim 10^{-9} \), energy resolution of 3% and a dynamic range of \( \sim 10^5 \), allowing it to detect the neutron spectrum with a high resolution [51], [52]. The main purpose of the MRS at the NIF is to measure the number of downscattered neutrons and thereby calculating the areal-density \( (\rho R) \) of the deuterium-tritium capsule [53]. However, the accuracy with which it can measure neutrons of higher energy make it useful for studying many different features of the neutron spectrum.
2.8.2 Neutron time of flight system

Neutron time of flight detectors can be used to measure the energy spectrum of emitted neutrons. The time taken for the neutrons to reach the detector is measured and this can be used to determine the energy of the neutron. Neutrons are detected by either a scintillator and photomultiplier tube or chemical vapor deposition (CVD) diamond detector [54]. Temporal resolution of the detectors is $\sim 100\, ps$ [55]. This resolution sets the minimum distance for the location of the detectors from the target. A complicating factor is that accurate knowledge of the emission time of the neutrons is required in order to distinguish between high energy neutrons and those that are produced early in time. However, it is possible to measure accurate neutron energy spectra using this detector.

2.8.3 Neutron activation diagnostic

The neutron activation diagnostic can be used to measure the number of emitted neutrons by the activation of a witness foil [56]. The emitted neutron strikes the foil causing a reaction. Unstable isotopes of the activation element are produced which then decay, emitting a positron. Pair annihilation of the positron produces $\gamma$-rays that can then be detected. The choice of activation material determines the minimum neutron energy that can be detected. For example, a copper foil will undergo the reaction $^{63}Cu\,(n,2n)^{62}Cu$ with a threshold neutron energy of $10.9\, MeV$ while a carbon activation diagnostic will undergo the reaction $^{12}C\,(n,2n)^{11}C$ with threshold neutron energy of $22\, MeV$. Therefore, a neutron activation diagnostic employing both carbon and copper activation materials can determine the number of neutrons in the ranges $10.9 - 22\, MeV$ (which will mostly be neutrons produced by thermonuclear DT reactions) and greater than $22\, MeV$ (which will mostly be produced by DT reactions in which one or more of the reactants has been accelerated to high energy, for example by scattering from a neutron). Indium witness foils are suitable for detecting the neutrons produced by a DD reaction [57]. However, this diagnostic is unable to resolve the energy spectrum of neutrons.
2.8.4 Neutron imaging system

The purpose of the neutron imaging system is to determine the spatial distribution of emitted neutrons. This is useful for studying the implosion symmetry in ICF experiments, for example. Emitted neutrons pass through a penumbral or pin-hole aperture and are detected by capillary tubes filled with a liquid scintillator or “bubble detector” (an elastic polymer matrix supporting liquid droplets - neutrons striking the droplets form bubbles that last indefinitely and can be imaged [58]). Spatial resolution of $\sim 10 \mu m$ can be achieved [56]. Some energy resolution of the images is possible by “gating” the image at different times. This makes it possible to achieve separate spatial images of downscattered neutrons and those that have not undergone scattering.

Of the above diagnostics the MRS and neutron time of flight offer the best possibility of directly detecting features in the neutron spectrum that we have highlighted such as the long high energy tail produced by a thermal plasma or the effects of fluid velocity. In particular the range of intensities that can be detected by the MRS make it suitable for resolving the high energy tail at intensities a few orders of magnitude lower than the peak intensity. This offers the possibility of obtaining measurements of the plasma ion temperature that are more accurate than those obtained by measuring the FWHM of the spectrum. As an example we consider the DD neutron spectrum produced by thermal plasmas with temperatures of 10 keV and 12 keV. The reactivity of the 10 keV plasma is $6.02 \times 10^{-19} cm^3 s^{-1}$ and the reactivity of the 12 keV plasma is $9.18 \times 10^{-19} cm^3 s^{-1}$. Therefore, assuming both plasmas have the same density we see that the total neutron production rate will increase by a factor of 1.5 as temperature increases from 10 keV to 12 keV. The FWHM value of the neutron spectrum will increase from 0.263 MeV to 0.289 MeV, an increase of less than 10%. However, if we instead look at the number of neutrons that are produced with an energy of, for example, 2.9 MeV we find that this number increases by a factor of 3 when temperature increases from 10 keV to 12 keV. The intensity of neutrons at 2.9 MeV is approximately $10^{-3}$ that of the peak intensity. If we look
at higher neutron energies the sensitivity of intensity to plasma temperature is even greater. At $3.1 \text{MeV}$ the neutron intensity increases by a factor of 10 when temperature rises from $10 \text{keV}$ to $12 \text{keV}$, from an intensity of $\sim 10^{-6}$ to $\sim 10^{-5}$. Therefore, measuring the number of these high energy, lower intensity neutrons could give a more accurate indication of the plasma temperature. Such a calculation would not be possible using the gaussian approximation for the neutron spectrum as the gaussian curve does not accurately represent the tails of the spectrum. There is also the added benefit that neutrons with a higher energy are less likely to be affected by collisions and downscattering.
Chapter 3

Fast ion interactions with a thermal plasma

In this chapter, we study how a single ion interacts with a maxwellian distribution of ions. There are two principal interactions that we study:

1. The slowing down of a single fast ion by multiple small angle coulomb collisions with ions in the maxwellian distribution. This can be used as a model for the heating of thermal plasmas by fast ions such as $\alpha$ particles. In section 3.1 we describe a method originally developed by Chandrasekhar for gravitational interactions [59], [60]. The method was then applied to electrostatic interactions by Spitzer [61]. In section 3.2 we outline how the increase in thermal energy and kinetic energy of the maxwellian distribution can be calculated from the slowing down of the fast ion.

2. The nuclear reaction effects of a single ion in a maxwellian distribution. In sections 3.3-3.5 we study the total probability that the single ion will react (reactivity), the probability that it will react with a thermal ion of given energy (absorption spectrum) and the probability that the reaction will emit a product particle of a given energy (production spectrum). We employ methods used in chapter 2 and the results of these sections are, to the best of our knowledge, original.
The results achieved for these two interactions are useful for models of burning plasmas in which the bulk of the plasma is treated as a MHD fluid and one species of ions is modelled using discrete particles.

### 3.1 Derivation of fast ion stopping in thermal plasmas

Although our particular interest is in the interaction of a single ion with a thermal distribution of particles the model discussed can also be applied to distributions other than the maxwellian. Therefore, we will refer to 'field' particles as those that are part of the maxwellian distribution and the 'test' particle being the single ion of interest. It is assumed that the number of test particles is much less than the number of field particles and so we need not consider interactions between test particles. As a test particle moves through the field particles there are many possible interactions that can occur including small and large angle coulomb collisions, nuclear elastic scattering, fusion reactions (considered in sections 3.3-3.5), etc. Many of these can have an appreciable effect in fusion plasmas (see, for example, [36]), however, slowing down of the test particle and heating of the field particles occurs mainly through small angle coulomb collisions. Although a large angle coulomb collision can cause a significant change in the velocity of the test particle in a single collision it is much more likely that the velocity of the test particle will change due a large number of small angle coulomb collisions. This is due to the large number of field particles and the fact that the electrostatic interaction obeys an inverse square law. Thus, the large number of distant encounters outweighs the smaller number of close encounters.

The following discussion of the cumulative effect of small angle collisions is based on the work of Chandrasekhar found in [59], [60]. We begin by considering a single interaction between a field particle with velocity $v_f$ and a test particle with velocity $v_t$. By solving the equations of motion for the interaction we get the change in the
test particle velocity both parallel and perpendicular to its original motion

\[ \Delta v_t^\parallel = -\frac{2\mu}{m_t} [(v_t - v_f \cos \theta) \cos \psi + v_f \sin \theta \cos \Theta \sin \psi] \cos \psi, \quad (3.1) \]

\[ \Delta v_t^\perp = \pm \frac{2\mu}{m_t} \left[ (v_f^2 + v_t^2 - 2v_f v_t \cos \theta) \cos \psi + v_f \sin \theta \cos \Theta \sin \psi \right]^{\frac{1}{2}} \cos \psi, \quad (3.2) \]

where \( \theta \) is the angle between \( \mathbf{v}_f \) and \( \mathbf{v}_t \), \( \Theta \) is the angle between the 'fundamental plane' (the plane containing the vectors \( \mathbf{v}_f \) and \( \mathbf{v}_t \)) and the 'orbital plane' (the plane containing the relative velocity vector \( \mathbf{v}_r = \mathbf{v}_f - \mathbf{v}_t \) and the velocity vector of the test particle in the CM frame \( \mathbf{u}_t = \mathbf{v}_t - \mathbf{v}_{cm} \)) and \( \psi \) is given by

\[ \tan \psi = \frac{\mu b v_r^2}{Z_f Z_t e^2}. \quad (3.3) \]

The angle \( \psi \) is related to the CM frame scattering angle \( \chi \) of the test particle by \( \chi = \pi - 2\psi. \) Since the interaction force acts over an infinite range the scattering angle \( \chi \) is actually the angle between the asymptotes of the test particle velocity vectors before and after the interaction. In (3.3) \( Z_f \) and \( Z_t \) are charge numbers and \( b \) is the impact parameter between the two particles (the distance of the closest approach in the absence of forces). The \( \pm \) in the expression for \( \Delta v_t^\perp \) indicates the possible directions for this component of \( \Delta \mathbf{v}_t \). From the above expressions we can see that the interaction of the test particle with velocity \( \mathbf{v}_t \) with any field particle is a function of 5 variables. These are \( v_f, b, \theta, \Theta \) and \( \varphi \). The final variable \( \varphi \) does not appear explicitly in the above expressions. It represents the azimuthal angle for a system of coordinates in which \( \mathbf{v}_t \) is in the \( z \) direction and, together with \( v_f \) and \( \theta \), defines the distribution of field particles \( N(v_f, \theta, \varphi) \) per unit volume with velocity between \( v_f \) and \( v_f + dv_f \). For a thermal plasma \( N(v_f, \theta, \varphi) \) represents a maxwellian distribution.

---

*There are many similarities between the problem of production spectra from fusion reactions discussed in chapter 2 and coulomb collisions discussed here, since both are essentially two body interactions. In fact, using the relation giving by (3.3) means we can construct a coulomb scattering cross-section that is a function of the relative velocity and CM frame scattering angle of the two interacting particles only and so we could use the approach developed in chapter 2 to find the distribution of particles due to coulomb scattering. However, the problem of divergence of the integral over the impact parameter will remain. This is because coulomb scattering is inherently dependent on the spatial coordinates of the two particles but the fusion cross-section is not.
distribution.

The assumption of small angle collisions means that we can approximate the total change in velocity of the test particle in a time interval $\Delta t$ by summing the individual interactions of the test particle with each field particle

\[
\sum \Delta v_{||} = (\Delta v_{||})_1 + (\Delta v_{||})_2 + \ldots + (\Delta v_{||})_N ,
\]

(3.4)

\[
\sum \Delta v_{\perp} = (\Delta v_{\perp})_1 + (\Delta v_{\perp})_2 + \ldots + (\Delta v_{\perp})_N .
\]

(3.5)

For an isotropic distribution of the field particles we can use symmetry arguments to conclude that $\sum \Delta v_{\perp} = 0$ due to the $\pm$ sign in (3.1). However, the squares of the velocity components will not cancel out and so $\sum (\Delta v_{\perp})^2$ will be non-zero.

Now, since we are considering a smoothed distribution of field particles we can find these sums by integrating over the variables that describe a collision rather than summing discrete events. We use the terms $\langle \Delta v_{||} \rangle$, $\langle (\Delta v_{||})^2 \rangle$ and $\langle (\Delta v_{\perp})^2 \rangle$ in place of $\sum \Delta v_{||}$ and $\sum (\Delta v_{||})^2$ and $\sum (\Delta v_{\perp})^2$ to denote that the quantities are averaged over the distribution function. For example, $\langle \Delta v_{||} \rangle$ is given by

\[
\langle \Delta v_{||} \rangle = \Delta t \int_0^\infty dv_f \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_{b_{\text{min}}}^{b_{\text{max}}} db \int_0^{2\pi} d\Theta (v_r b \Delta v_{||} N (v_f, \theta, \varphi))
\]

(3.6)

The details for integrating this expression and those for $\langle (\Delta v_{||})^2 \rangle$ and $\langle (\Delta v_{\perp})^2 \rangle$ may be found in [59] and [60], the solutions requiring an approximation to the 'dominant terms' in the expression. We simply note here that, in theory, the integral over the impact parameter $b$ should have limits $[0, \infty]$ in order to include all interaction events. However, when this is the case the integral diverges at $\infty$. Physically, this can be explained by the assumption of the model that all interaction events occur instantaneously and that we can sum the effects of the individual interactions. This is not the case. For a pair of particles with a large impact parameter it takes a much longer time for the pair to move through the full scattering angle $\chi$ than for a pair of particles with a smaller impact parameter. Interactions of the test particle with field particles at a lower impact parameter will interrupt the scattering with the field particle at a higher impact parameter. Therefore, in practice a cut off $b_{\text{max}}$
is introduced to prevent an overestimation in the result. The lower bound $b_{\text{min}}$ is included because effects such as nuclear elastic scattering begin to dominate at very low values of the impact parameter. The results of the integrations are as follows

$$
\langle \Delta v_{\parallel} \rangle = -A_D \left( 1 + \frac{m_t}{m_f} \right) \frac{G (v_t/\bar{v}_f)}{\bar{v}_f^2}, \tag{3.7}
$$

$$
\langle (\Delta v_{\parallel})^2 \rangle = A_D \frac{G (v_t/\bar{v}_f)}{v_t}, \tag{3.8}
$$

$$
\langle (\Delta v_{\perp})^2 \rangle = A_D \frac{\text{erf} (v_t/\bar{v}_f) - G (v_t/\bar{v}_f)}{v_t}, \tag{3.9}
$$

where

$$
A_D = \frac{\Delta t Z_i^2 Z_f^2 e^4 n_f \ln \Lambda}{2 \pi m_t^2 c_0^2}, \tag{3.10}
$$

$$
G (x) = \frac{\text{erf} (x) - x \frac{\partial}{\partial x} \text{erf} (x)}{2 x^2}. \tag{3.11}
$$

In the above $\bar{v}_f$ is the mean velocity of the field particles. In the case of a maxwellian distribution this is the thermal velocity. The term $\ln \Lambda$ is known as the coulomb logarithm. It arises from the integration over the impact parameter. We can now say that the slowing down of the test particle in a time $\Delta t$ is given by $\langle \Delta v_{\parallel} \rangle$. This is called the coefficient of dynamical friction. Furthermore, the test particle may change (increase or decrease) its velocity by the diffusive coefficients $\sqrt{\langle (\Delta v_{\parallel})^2 \rangle}$ and $\sqrt{\langle (\Delta v_{\perp})^2 \rangle}$. Therefore, we can calculate the slowing down of a test particle by a field of particles using the above relations. To do this we use the computational method described in [62]. A plot of the function $G (x)$ is shown in figure 3.1. The function is greater at lower values of $x$. From this we can conclude that friction and diffusion of a test particle is greater when $v_t$ is closer to $\bar{v}_f$. For maxwellian distributions of electrons and ions at similar temperatures the thermal velocity of the electron species is much greater than that of the ion species (since $m_i \gg m_e$). Therefore, we can see that a fast test particle such as an $\alpha$ particle will initially be slowed down by the electron species and then, at lower velocities, by the ion species.
3.2 Conservation of momentum and energy

In order to ensure conservation of momentum and energy in the system as the test particle slows down it is necessary to adjust the momentum and energy of the background plasma. We assume that the energy gained by the plasma takes the form of either kinetic or thermal energy. If we are considering just a single test particle then the procedure is straightforward. When multiple test particles are present the changes in momentum and total energy of the fluid are given by

\[ \Delta p_f = -\Sigma_i \Delta p_i, \]
\[ \Delta E_f = -\Sigma_i \Delta E_i, \]

(3.12) \hspace{1cm} (3.13)

where \( i \) represents individual test particles and the \( f \) subscript indicates the background fluid plasma. As the change of momentum of the fluid is known the change in its kinetic energy may be calculated by

\[ \Delta K_f = \frac{(p_f + \Delta p_f)^2 - p_f^2}{2n_fm_f}, \]

(3.14)
where \( n_f \) is the fluid number density and \( m_f \) is the fluid particle mass. Finally, this allows the change in thermal energy of the fluid to be calculated by

\[
\Delta U_f = \Delta E_f - \Delta K_f. \tag{3.15}
\]

From the above equations we can see that when multiple test particles are present the thermal energy gain of the plasma will be much greater if these particles have an isotropic, or near-isotropic, distribution of momenta (in which the \( \Delta p_i \) vectors cancel), rather than unidirectional.

### 3.3 Reactivity of a fast ion in a thermal plasma

In this section the reactivity of a test particle in a background thermal plasma is derived. We begin with the expression

\[
R_{12}(v_1, v_2) = \frac{1}{1 + \delta_{12}} v_r \sigma (v_r) f_1(v_1) f_2(v_2) d^3v_1 d^3v_2. \tag{3.16}
\]

The first distribution function is that of the test particle. As it is the distribution of a single particle we use the Dirac delta function

\[
f_1(v_1) = \delta (v_1 - v_t), \tag{3.17}
\]

where \( v_t \) is the velocity of the test particle. Notes on the Dirac delta function and some of the properties that we make use of may be found in appendix D. The second distribution function is that of the background thermal plasma and is maxwellian

\[
f_2(v_2) = \left( \frac{m_2}{2 \pi T} \right)^{\frac{3}{2}} \exp \left( -\frac{m_2 v_r^2}{2T} \right). \tag{3.18}
\]

To determine the reactivity of the test particle we firstly follow the rules for deriving the absorption spectrum outlined in section 2.2. Using \( v_2 = v_1 - v_r \) and inserting
into (3.16) gives

\[
R_{12}(v_1, v_r) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} \delta(v_1 - v_t) v_r \sigma(v_r) \exp \left( -\frac{m_2 (v_1 - v_r)^2}{2T} \right) d^3v_1 d^3v_r \tag{3.19}
\]

Converting \(v_r\) to spherical coordinates and results in the following integral over \(\theta_r\) and \(\phi_r\):

\[
\int_0^\pi \int_0^{2\pi} \sin \theta_r \exp \left( -\frac{m_2 (v_1 - v_r)^2}{2T} \right) d\phi_r d\theta_r = 2\pi \frac{T}{m_2 v_1 v_r} \left( \exp \left( -\frac{m_2 (v_1 - v_r)^2}{2T} \right) - \exp \left( -\frac{m_2 (v_1 + v_r)^2}{2T} \right) \right), \tag{3.20}
\]

which reduces (3.19) to

\[
R_{12}(v_1, v_r) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} \delta(v_1 - v_t) \frac{v_r^2}{v_1} \sigma(v_r) \times \left( \exp \left( -\frac{m_2 (v_1 - v_r)^2}{2T} \right) - \exp \left( -\frac{m_2 (v_1 + v_r)^2}{2T} \right) \right) d^3v_1 dv_r. \tag{3.21}
\]

Finally, we need to integrate over \(d^3v_1 dv_r\) to obtain the reactivity of the test particle. The integral over \(d^3v_1\) is trivial due to the multivariable Dirac delta function. As a result of this integration we have

\[
\langle \sigma v \rangle = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} \frac{1}{v_t} \int_0^\infty v_r^2 \sigma(v_r) \left( \exp \left( -\frac{m_2 (v_t - v_r)^2}{2T} \right) - \exp \left( -\frac{m_2 (v_t + v_r)^2}{2T} \right) \right) dv_r. \tag{3.22}
\]

This equation is the reactivity of a test particle with velocity \(v_t\) in a background thermal plasma of temperature \(T\). It represents the probability of reaction of the test particle per unit time per unit density of the background plasma.

It is a common assumption that if \(v_t \gg \overline{v}_f\), where \(\overline{v}_f = \sqrt{2T/m_2}\) is the thermal velocity of the background plasma, the reactivity of the test particle may be approximated by

\[
\langle \sigma v \rangle \approx v_t \sigma(v_t). \tag{3.23}
\]
In order to assess the accuracy of this approximation both (3.22) and (3.23) are plotted as functions of test particle energy $E_t$ in figure 3.2. The top diagram shows the reactivity of a deuterium test particle with a deuterium background plasma with temperatures of 5 keV and 10 keV. The bottom diagram shows a tritium test particle in a deuterium plasma for the same temperatures. Cross-sections from [63] are used. For both reactions the approximate reactivity given by (3.23) deviates from the exact solution at lower test particle energies. The difference is more significant for the DT reaction. The difference between the exact and approximate reactivities increases as the background plasma temperature increases.

Figure 3.2: The reactivity of a test deuterium (top) and tritium (bottom) particle in a thermal deuterium plasma at temperatures of 5 keV and 10 keV. The approximate reactivity given by (3.23) is shown as a dashed curve.
3.4 Absorption spectrum of a fast ion in a thermal plasma

The probability of reaction between a given particle in the thermal plasma background and a test particle is determined by the absorption spectrum of the test particle. This can be derived as described in section 2.2 by using the Dirac delta function for the test particle distribution. We begin with

\[
R_{12}(v_1, v_2) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} v_r \sigma(v_r) \delta(v_1 - v_t) \exp \left( -\frac{m_2 v_t^2}{2T} \right) d^3v_1 d^3v_2. \tag{3.24}
\]

We seek the distribution of particles of species 2 that react with the test particle with velocity \(v_t\). We carry out the transformation \(v_1 = v_r + v_2\). This transformation causes the Dirac delta function to become \(\delta(v_r + v_2 - v_t)\). We proceed by converting \(v_r\) to spherical coordinates and integrating over \(dv_r dv_r d\phi_r\). This requires us to find the surface (see appendix D) defined by

\[
\begin{align*}
v_r \sin \theta_r \cos \phi_r + v_{2x} - v_{tx} &= 0, \\
v_r \sin \theta_r \sin \phi_r + v_{2y} - v_{ty} &= 0, \\
v_r \cos \theta_r + v_{2z} - v_{tz} &= 0.
\end{align*}
\tag{3.25}
\]

The solution to this set of equations is

\[
\begin{align*}
\theta_r &= \arctan \left( \frac{\sqrt{(v_{2x} - v_{tx})^2 + (v_{2y} - v_{ty})^2}}{v_{2z} - v_{tz}} \right), \\
\phi_r &= \arctan \left( \frac{v_{2y} - v_{ty}}{v_{2x} - v_{tx}} \right), \\
v_r &= \sqrt{\left( v_2 - v_t \right)^2}.
\end{align*}
\tag{3.26}
\]

Therefore, after integration (3.24) becomes

\[
R_{12}(v_2) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} \sqrt{(v_2 - v_t)^2} \sigma(\sqrt{(v_2 - v_t)^2}) \exp \left( -\frac{m_2 v_t^2}{2T} \right) d^3v_2. \tag{3.27}
\]
This is the absorption spectrum of species 2 as a function of velocity $v_2$. We can express this as a function of kinetic energy $E_2$ by firstly converting $v_2$ to spherical coordinates and integrating over the angular terms. Using the integral relation outlined in appendix B.1 for the angular integral and then converting $v_2$ to $E_2$ gives

$$R_{12}(E_2) = \sqrt{\frac{E_2}{\pi T^3}} \exp \left( -\frac{E_2}{T} \right) \int_{-1}^{1} \sqrt{a + bx} \sigma \left( \sqrt{a + bx} \right) dx dE_2, \quad (3.28)$$

where

$$a = \frac{2E_2}{m_2} + \frac{2E_t}{m_t},$$

$$b = -4\sqrt{\frac{E_2 E_t}{m_2 m_t}}.$$

The mean energy of the absorption spectrum is given by

$$E_{\text{mean}} = \frac{\int E_2 R_{12}(E_2) dE_2}{\int R_{12}(E_2) dE_2}. \quad (3.29)$$

An example of the mean energy of the absorption spectrum is shown for a 5 keV plasma in figure 3.3. It shows that as the reactivity of the test particle increases the mean of the absorption spectrum becomes closer to the mean energy of the plasma.

![Figure 3.3: The mean energy of the absorption spectrum of a 5 keV thermal deuterium plasma is plotted as a function of test particle energy. The curves represent a deuterium test particle (green) and a tritium test particle (blue).](image-url)
3.5 Production spectrum of a fast ion in a thermal plasma

Intuitively we expect the production spectrum of a test particle in a thermal plasma to be anisotropic, with products emitted parallel to the test particle velocity direction tending to have a higher energy than those emitted in the opposite direction. In this section we express the production spectrum as a single integral. The case of a test particle is unlike the other anisotropic production spectra cases considered in chapter 2 in that it can be expressed as a single integral even when the reaction cross-section is dependent on both relative velocity and scattering angle. We begin with the usual expression

$$R_{12}(v_1, v_2) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^3 v_r \sigma(v_r, \theta_{cm}) \delta(v_1 - v_t) \exp \left( -\frac{m_2^2 v_2^2}{2T} \right) d^3v_1 d^3v_2. \quad (3.30)$$

Using the transformations outlined in section 2.5 and remembering that the presence of the Dirac delta function causes the Jacobian determinant to be cancelled out\(^\dagger\) results in

$$R_{12}(v_3, u_3, \hat{v}_r) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^3 \sin \frac{\theta_3}{4\pi} \zeta \sigma(\zeta, \theta_{cm}) \delta_0 \left( v_3 - u_3 + \frac{\mu \zeta}{m_1} \hat{v}_r - v_t \right) \times$$

$$\exp \left( \frac{m_2^2}{2T} \left( v_3 - u_3 - \frac{\mu \zeta}{m_2} \hat{v}_r \right)^2 \right) d^2\hat{v}_r d^3u_3 d^3v_3, \quad (3.31)$$

where, as usual,

$$\zeta = \sqrt{\frac{2}{\mu} \left( \frac{\eta}{2} u_3^2 - Q \right)}. \quad (3.32)$$

We now need to integrate over the 5 variables \(u_3, \hat{u}_3, \hat{v}_r\). Following our usual notation \(\hat{u}_3\) and \(\hat{v}_r\) are unit vectors and so are functions of two variables (the polar

\(^\dagger\)See appendix D. The determinant arising from change of volume element cancels with the determinant arising from the reduction of the argument of the Delta function to a level set of zeroes using (D.9). The 0 subscript on the Delta function in (3.31) is included to denote that the argument of the Delta function is where the expression in parentheses after \(\delta_0\) is equal to 0. Also, we remind the reader that the \(\sin \theta_3(4\pi)^{-1}\) term is not a part of the Jacobian determinant and so must be included in (3.31).
and azimuthal angles) whilst \( u_3 \) is the magnitude of the vector \( \mathbf{u}_3 \) (i.e. \( u_3 = \mathbf{u}_3 \hat{\mathbf{u}}_3 \)).

The Dirac delta function is 3 dimensional and so integration over 3 of the 5 variables is trivial. We choose \( u_3 \) and \( \hat{\mathbf{v}}_r \) to be the variables that we integrate over. To find the result of this integration we need to solve

\[
\mathbf{v}_3 - u_3 \hat{\mathbf{u}}_3 + \frac{\mu}{m_1} \zeta \hat{\mathbf{v}}_r - \mathbf{v}_t = 0,
\]

for \( u_3 \) and \( \hat{\mathbf{v}}_r \). We solve firstly for \( u_3 \) by rearranging (3.33) and squaring both sides to get

\[
(v_3 - v_t)^2 + u_3^2 - 2u_3 (v_3 - v_t) \cdot \hat{\mathbf{u}}_3 = \frac{2\mu}{m_1^2} \left( \frac{\eta}{2} u_3^2 - Q \right),
\]

where we have used (3.32) for \( \zeta \). We now have a quadratic in \( u_3 \) whose roots are

\[
u_3 = \frac{(v_3 - v_t) \cdot \hat{\mathbf{u}}_3 \pm \sqrt{((v_3 - v_t) \cdot \hat{\mathbf{u}}_3)^2 + \left( \frac{m}{m_1^2} - 1 \right) \left( \frac{2\mu}{m_1^2} Q + (v_3 - v_t)^2 \right)}}{1 - \frac{\mu}{m_1^2}}.
\]

Of the 5 variables under consideration this is a function of \( \hat{\mathbf{u}}_3 \) only. Now, returning to (3.33) it is obvious that \( \hat{\mathbf{v}}_r \) can be expressed as

\[
\hat{\mathbf{v}}_r = \frac{u_3 \hat{\mathbf{u}}_3 - (v_3 - v_t)}{\frac{\mu}{m_1} \zeta},
\]

which is again a function of \( \hat{\mathbf{u}}_3 \) only. Now, returning these two relations we can integrate (3.31) over \( d^2 \hat{\mathbf{v}}_r du_3 \) to eliminate the Dirac delta function and get

\[
R_{12}(\mathbf{v}_3, \hat{\mathbf{u}}_3) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{\frac{3}{2}} \frac{\sin \theta_3}{4\pi} \zeta' \sigma_3(\zeta', \theta_3') \times \exp \left( -\frac{1}{2m_2 T} \left( (m_1 + m_2) \mathbf{v}_3 - m_1 \mathbf{v}_t - (m_1 + m_2) u_3 \hat{\mathbf{u}}_3 \right)^2 \right) d^2 \hat{\mathbf{u}}_3 d^3 \mathbf{v}_3.
\]

where we use \( \zeta' \) to denote that \( \zeta \) is a now a function of \( \hat{\mathbf{u}}_3 \) and similarly for \( \theta_3' \) \((\theta_3' = \arccos (\hat{\mathbf{v}}_r \cdot \hat{\mathbf{u}}_3)) \text{ with } \hat{\mathbf{v}}_r \text{ given by (3.36)). We must now integrate this expression over } d^2 \hat{\mathbf{u}}_3 \text{ (= } d\phi_3 d\theta_3 \text{) to obtain the production spectrum for the test particle. Thus, we}
have

\[ R_{12}(v_3) = \frac{1}{1 + \delta_{12}} \left( \frac{m_2}{2\pi T} \right)^{3/2} \exp \left( -\frac{1}{2m_2T} ((m_1 + m_2)v_3 - m_1v_t)^2 \right) \times \]

\[ \int_0^\pi \int_0^{2\pi} \sin \theta \sigma(\zeta', \theta'_{cm}) \times \]

\[ \exp \left( -\frac{m_1^2}{2\mu^2 T} u_3^2 + \frac{m_1}{\mu T} u_3 ((m_1 + m_2)v_3 - m_1v_t).\hat{u}_3 \right) d\phi_3 d\theta_3 d^3v_3. \] (3.38)

If we inspect (3.38) and the expressions for \( \zeta', \theta'_{cm} \) and \( u_3 \) we see that \( \theta_3 \) and \( \phi_3 \) terms occur only in the dot product of \( \hat{u}_3 \) with two vectors. These vectors are \((v_3 - v_t)\) and \(((m_1 + m_2)v_3 - m_1v_t)\). Since the integrals over \( \theta_3 \) and \( \phi_3 \) are over the entire surface area of the sphere we can make the transformations outlined in appendix B.3 such that

\[ (v_3 - v_t).\hat{u}_3 = |v_3 - v_t| \cos \theta_3, \] (3.39)

\[ ((m_1 + m_2)v_3 - m_1v_t).\hat{u}_3 = |(m_1 + m_2)v_3 - m_1v_t| \times \]

\[ (\cos \psi \cos \theta_3 + \sin \psi \sin \theta_3 \cos \phi_3), \] (3.40)

where

\[ \cos \psi = \frac{(v_3 - v_t) \cdot ((m_1 + m_2)v_3 - m_1v_t)}{|v_3 - v_t||((m_1 + m_2)v_3 - m_1v_t)|}. \] (3.41)

We can now integrate over \( \phi_3 \) as it appears in only 1 term in expression (3.38). We have

\[ \int_0^{2\pi} \exp \left( \frac{m_1}{\mu T} u_3 |(m_1 + m_2)v_3 - m_1v_t| \sin \psi \sin \theta_3 \cos \phi_3 \right) d\phi_3 = \]

\[ 2\pi I_0 \left( \frac{m_1}{\mu T} u_3 |(m_1 + m_2)v_3 - m_1v_t| \sin \psi \sin \theta_3 \right). \] (3.42)

where \( I_0 \) is the zero order modified Bessel function of the first kind. The modified Bessel function is a well-known special function whose value may be easily computed. Properties of this function are outlined in appendix B.4 as well as a proof of the integral identity we have just used.

We have now reduced the expression for a production spectrum to a single inte-
To make the integral neater we make the substitution $x = |\mathbf{v}_3 - \mathbf{v}_t| \cos \theta_3$. The limits of integration are now $[-|\mathbf{v}_3 - \mathbf{v}_t|, |\mathbf{v}_3 - \mathbf{v}_t|]$ and the expression for the production spectrum becomes

$$R_{12}(\mathbf{v}_3) = \frac{1}{1 + \delta_{12}} \frac{1}{2 \pi T} \left( \frac{m_2}{2\pi} \right)^2 \frac{1}{|\mathbf{v}_3 - \mathbf{v}_t|} \exp \left( -\frac{1}{2m_2T} \left((m_1 + m_2) \mathbf{v}_3 - m_1\mathbf{v}_t \right)^2 \right) \times$$
$$\int_{-|\mathbf{v}_3 - \mathbf{v}_t|}^{\mathbf{v}_3 - \mathbf{v}_t} \zeta' \sigma(\zeta', \theta'_\text{cm}) \exp \left( -\frac{m_1^2m_2\mu^2}{2\mu^2T} u_3^2 + \frac{m_1}{\mu T} \cos \psi \frac{|(m_1 + m_2) \mathbf{v}_3 - m_1\mathbf{v}_t|}{|\mathbf{v}_3 - \mathbf{v}_t|} u_3 x \right) \times$$
$$I_0 \left( \frac{m_1}{\mu T} \sin \psi \frac{|(m_1 + m_2) \mathbf{v}_3 - m_1\mathbf{v}_t|}{|\mathbf{v}_3 - \mathbf{v}_t|} u_3 \sqrt{|\mathbf{v}_3 - \mathbf{v}_t|^2 - x^2} \right) dx d^3\mathbf{v}_3. \quad (3.43)$$

Now, recalling (3.35), we note that there are two values for $u_3$. It is necessary to include both in the calculation of the production spectrum. We integrate (3.43) over each in turn and sum the result. Furthermore, only real values of $u_3$ are perm issable. Therefore, we impose the condition that

$$x^2 + \left( \frac{\eta \mu}{m_1^2} - 1 \right) \left( \frac{2 \mu}{m_1^2} Q + (\mathbf{v}_3 - \mathbf{v}_t)^2 \right) \geq 0, \quad (3.44)$$

which gives

$$x^2 \geq \left( 1 - \frac{\eta \mu}{m_1^2} \right) \left( \frac{2 \mu}{m_1^2} Q + (\mathbf{v}_3 - \mathbf{v}_t)^2 \right). \quad (3.45)$$

Let

$$a = |\mathbf{v}_3 - \mathbf{v}_t|, \quad (3.46)$$

$$b = + \sqrt{\left( 1 - \frac{\eta \mu}{m_1^2} \right) \left( \frac{2 \mu}{m_1^2} Q + (\mathbf{v}_3 - \mathbf{v}_t)^2 \right)}. \quad (3.47)$$

Now, assuming that $a \geq b$, there are two valid intervals of integration for $x$ in which $u_3$ is real. These are $[-a, -b]$ and $[b, a]$. However, if $b > a$ then $u_3$ is complex in the interval $[-a, a]$. Therefore, no product particle is emitted such that

$$\sqrt{\left( 1 - \frac{\eta \mu}{m_1^2} \right) \left( \frac{2 \mu}{m_1^2} Q + (\mathbf{v}_3 - \mathbf{v}_t)^2 \right)} > |\mathbf{v}_3 - \mathbf{v}_t|. \quad (3.48)$$
Rearranging the above gives

\[(v_3 - v_t)^2 < 2Q \frac{m_1^2 - \eta \mu}{m_1^2 \eta}, \quad (3.49)\]

The test particle cannot produce a particle with velocity \(v_3\) such that the above condition is true. We will return to this result shortly but firstly we give our final, complete expression for the production spectrum produced by a test particle in a thermal plasma

\[R_{12}(v_3) = \begin{cases} 
0, & \text{if } (v_3 - v_t)^2 < 2Q \frac{m_1^2 - \eta \mu}{m_1^2 \eta}, \\
\Lambda \int_{-a}^{-b} [F(u_{3+}) + F(u_{3-})] \, dx + \Lambda \int_{a}^{b} [F(u_{3+}) + F(u_{3-})] \, dx, & \text{otherwise}
\end{cases}, \quad (3.50)\]

where \(a\) and \(b\) are given by (3.46)-(3.47) and

\[\Lambda = \frac{1}{1 + \delta_{12}} \frac{1}{2(2\pi T)^{\frac{3}{2}}} \frac{1}{|v_3 - v_t|} \exp \left( -\frac{1}{2m_2 T} \left( (m_1 + m_2) v_3 - m_1 v_t \right)^2 \right), \quad (3.51)\]

\[F(u_{3\pm}) = \zeta' \sigma(\zeta', \theta'_{cm}) \exp \left( -\frac{m_1 m_2}{2\mu^2 T} u_{3\pm}^2 + \frac{m_1}{\mu T} \cos \psi \frac{|(m_1 + m_2) v_3 - m_1 v_t|}{|v_3 - v_t|} x_{u_{3\pm}} \right) \times \]

\[I_0 \left( \frac{m_1}{\mu T} \sin \psi \frac{|(m_1 + m_2) v_3 - m_1 v_t|}{|v_3 - v_t|} u_{3\pm} \sqrt{|v_3 - v_t|^2 - x^2} \right) , \quad (3.52)\]

\[u_{3\pm} = \frac{x \pm \sqrt{x^2 + \left( \frac{\mu}{m_1^2} - 1 \right) \left( \frac{2\mu}{m_1^2} Q + (v_3 - v_t)^2 \right)}}{1 - \frac{\mu}{m_1^2}}, \quad (3.53)\]

\[\zeta' = \sqrt{\frac{2}{\mu}} \left( \frac{\eta}{2} u_{3\pm}^2 - Q \right), \quad (3.54)\]

\[\theta'_{cm} = \arccos \left( \frac{u_{3\pm} - x}{\frac{\mu}{m_1} \zeta'} \right), \quad (3.55)\]

\[\cos \psi = \frac{(v_3 - v_t) \cdot ((m_1 + m_2) v_3 - m_1 v_t)}{|v_3 - v_t| |(m_1 + m_2) v_3 - m_1 v_t|}. \quad (3.56)\]

Although the above expression looks complicated it should be remembered that it
is just a single integral expression and so computation is straightforward. Examples of this production spectrum are shown in figures 3.4-3.6. The first two examples consider the neutron spectra produced by a single deuterium ion in a thermal deuterium plasma while the third considers the neutron spectra produced by a single tritium ion in a thermal deuterium plasma. It is interesting to note that even for a relatively low test particle energy, as in figure 3.5, there is significant anisotropy in the production spectra.

For all examples, the \((r, \theta)\) plots show that there is a central region in which no neutron is produced. This is due to the condition imposed by (3.49) to which we now return. The condition shows that there is a sphere in velocity space surrounding the velocity vector of the test particle in which no product particle can be produced\(^\dagger\).

We note that the right hand side of (3.49) is dependent only on the masses of the 4 particles participating in and the \(Q\) value of the reaction. This neat result is remarkable in that it is independent of the velocity \(v_2\) of the second reactant and, consequently, the plasma temperature. From this we make the following conclusion:

Given a binary reaction of the type \(1 + 2 \rightarrow 3 + 4\), if particle 1 has velocity \(v_1\) and mass \(m_1\) then particle 3 has some velocity \(v_3\) such that

\[
(v_3 - v_1)^2 \geq 2Q \frac{m_2^2 - \eta \mu}{m_1^2 \eta},
\]

where

\[
\mu = \frac{m_1 m_2}{m_1 + m_2},
\]

\[
\eta = \frac{m_3 (m_3 + m_4)}{m_4},
\]

\[
Q = (m_1 + m_2 - m_3 - m_4) c^2.
\]

\(\dagger\)From (3.49) we can obtain, in terms of energy,

\[
\frac{E_3}{m_3} + \frac{E_1}{m_1} - 2 \sqrt{\frac{E_3 E_1}{m_3 m_1} \cos \theta} = Q \frac{m_2^2 - \eta \mu}{m_1^2 \eta}.
\]

This equation defines the boundary in figures 3.4-3.6 with \(E_3\) being the radial coordinate and \(\theta\) the azimuthal. No neutrons are produced in the interior region.
of the reactant and product velocities. Clearly, if \( m_i^2 - \eta \mu < 0 \) then (3.58) is satisfied for all \( \mathbf{v}_3 \) and \( \mathbf{v}_1 \), there is no limit on the velocity of particle 3. If we express particle mass as \( m_i = A_i m_u + \Delta m_i \) where \( A_i \) is the atomic number of the species, \( m_u \) is one atomic mass unit and \( \Delta m_i \) is the atomic mass excess of the species then the condition \( m_i^2 - \eta \mu < 0 \) is

\[
1 - \frac{(A_3 + \Delta m_3/m_u)}{(A_1 + \Delta m_1/m_u)} \frac{(A_4 + \Delta m_4/m_u)}{(A_2 + \Delta m_2/m_u)} < 0,
\]

(3.62)

Now, if we assume that \( \Delta m_i m_u \ll 1 \) and using \( A_1 + A_2 = A_3 + A_4 \) this becomes

\[
1 - \frac{A_2 A_3}{A_1 A_4} < 0.
\]

(3.63)

Finally, since \( A_2 = A_3 + A_4 - A_1 \) we have

\[
\left(1 + \frac{A_3}{A_4}\right) \left(1 - \frac{A_3}{A_1}\right) < 0.
\]

(3.64)

Therefore we can conclude that if \( A_3 > A_1 \) then there is no limit on the velocity that particle 3 can have. However, if \( A_3 < A_1 \) then a cut off exists such that there is a region in velocity space in which particle 3 cannot be produced. This is the case for the neutrons and protons produced by the \( DD \) and \( DT \) fusion reactions. Table 3.1 gives the value of this cut off for these reactions. The final column of this table gives the energy of the product at the cut off for when the reactant has zero velocity in order to give an indication of the magnitude of the cut off. In all cases this value is greater than half the nominal energy of the product particle. It can increase exponentially as \( \mathbf{v}_1 \) increases, particularly if \( \mathbf{v}_3 \) is in the same direction as \( \mathbf{v}_1 \). The effect is most obvious for a fast reactant ion when the production spectrum is taken in the direction of travel of the ion, as can be seen in figures 3.4 and 3.6. Here, at energies greater than the cut off value the spectra increase rapidly to their peak value.
Table 3.1: Cut off values given by (3.49) for some common fusion reactions.

| Reaction       | p1 | p3 | Q   | |v3 − v1| | E3 (v1 = 0) |
|----------------|----|----|-----|-------|----------------|----------------|
| D + D → p + T  | D  | p  | 4.03| 1.96 × 10^7| 2.01           |
| D + D → n + He^3| D  | n  | 3.27| 1.77 × 10^7| 1.63           |
| D + T → n + He^4| D  | n  | 17.6| 4.10 × 10^7| 8.77           |
| T              | n  | 17.6| 4.73 × 10^7| 11.7           |

Figure 3.4: The top diagram shows an intensity plot in (r, θ) coordinates of the neutron spectrum produced by a deuterium test particle with energy 1 MeV travelling in the θ = 0° direction. The temperature of the deuterium background plasma is 50 keV. The radial coordinate is the neutron energy in units of MeV. The bottom diagram shows lineouts of this data at angles of 0°, 90° and 180°. The central region of the top diagram is that in which no neutron can be produced. The boundary of this region is given by (3.57). In the θ = 0° direction this boundary has a value of E_0° = 3.94 MeV. The peak of the spectrum in this direction is 4.14 MeV. In the θ = 180° direction the boundary is E_180° = 0.33 MeV.
Figure 3.5: The diagrams are similar to those of figure 3.4 with a deuterium plasma temperature of 10\,keV and deuterium test particle energy of 27\,keV. This particle energy corresponds to the mean value of the absorption spectrum for a 10\,keV deuterium plasma (see figure 2.4). The energy cut off boundary has values $E_{0^\circ} = 1.95\,MeV$ and $E_{180^\circ} = 1.35\,MeV$. 
Fast ion interactions with a thermal plasma

Figure 3.6: These diagrams show the neutron production spectrum produced by a triton test particle with energy $1.01\, MeV$ (the nominal energy of a triton produced in the $D + D \rightarrow T (1.01\, MeV) + p (3.02\, MeV)$ reaction) in a deuterium plasma with temperature $20\, keV$. The cut off energies are $E_{0^\circ} = 16.05\, MeV$ and $E_{180^\circ} = 8.10\, MeV$. The peak of the spectrum in the $\theta = 0^\circ$ direction is $17.22\, MeV$. 

Chapter 4

Ignition in deuterium-tritium Z pinches

In this chapter, we investigate the possibility of achieving ignition and energy gain in an imploding deuterium-tritium (DT) Z pinch. A DT Z pinch has a peak density several orders of magnitude lower than that of inertial confinement fusion (ICF). However, the presence of magnetic fields as large as $10^3 T$ and longer confinement times can facilitate thermonuclear ignition. The DT Z pinch is one of a number of schemes in which a magnetic field is used to compensate for a lower fuel density. These are generally referred to as magneto-inertial fusion (MIF) schemes or, sometimes, magnetized-target fusion (MTF). Section 4.1 contains a review of the topic of MIF, its historical development and current status, particularly with respect to Z pinches. In section 4.2, we study the ability of azimuthal magnetic fields, such as those generated in mega-ampere (MA) Z pinch implosions, to contain high energy $\alpha$ particles. This is a crucial requirement for ignition of DT Z pinches. Finally, in section 4.3, we investigate the ignition criteria and burn dynamics for a DT Z pinch in hydrodynamic equilibrium.

4.1 Introduction to Magneto-Inertial Fusion

The two primary approaches to achieving controlled thermonuclear fusion are magnetic confinement and ICF. Magnetic confinement is characterised by a low density
plasma with confinement due entirely to an external magnetic field. On the other hand ICF has a very high plasma density, does not contain an external magnetic field and achieves confinement through the inertia of the imploding mass. Parameters for the two methods are shown in table 4.1.

A very large region of the density-magnetic field parameter space exists between the two methods. Achieving fusion in an interim region of this space was first investigated in the 1980s. Initial studies focused on imposing an external magnetic field on an ICF target in order to reduce the driver energy required for ignition. The primary advantages of the magnetic field are that it reduces electron thermal conduction losses from the plasma and it confines fast ions such as $\alpha$ particles (thereby reducing the $\rho R$ required for ignition). A study by Lindemuth and Kirkpatrick [64] found that the fuel density and implosion velocity required for ignition could be significantly reduced compared to standard ICF while Jones and Mead [65] showed that a magnetic field could lead to much enhanced volumetric burn in which burn occurs simultaneously throughout the plasma.*

As the field evolved, MIF came to be viewed as an approach to fusion in its own right rather than a modification of ICF [66]. This was partly due to the unavoidable loss of symmetry that accompanies the imposition of a magnetic field on a spherically imploding mass. On the other hand, implosions in a cylindrical geometry do not suffer from such a problem. A magnetic field that is symmetric about the cylindrical axis will cause no loss of symmetry. This lead to the development of a number of new fusion schemes. For example, a two-stage system in which a plasma is magnetized by means of explosive flux compression and then imploded using a magnetically driven cylindrical liner was experimentally investigated in the early 1990s [67]. This scheme has become part of a larger class of schemes in which magnetic flux is embedded in a plasma hotspot that is then imploded, causing the magnetic flux to compress with it and increasing the magnetic field [68]. Other notable schemes currently being investigated include the solid-liner driven field-reversed configuration (FRC) [69] and plasma-liner driven MIF [70]. The FRC configuration requires a poloidal magnetic

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*This is in contrast to a propagating burn wave in which ignition initially occurs in a small, centralized hotspot. Jones and Mead concluded that the presence of a magnetic field inhibits a propagating burn wave. The nature of burn in a Z pinch is discussed in section 4.3.2.
field to be “frozen” into the plasma by reversing the direction of the externally applied axial magnetic field. The magnetized plasma is then compressed by an imploding solid liner on a $\mu$s time-scale. Parameters and scale lengths of the FRC are shown in row 3 of table 4.1.

Table 4.1: Scales required for ignition conditions in controlled thermonuclear fusion schemes.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$L$ ($m$)</th>
<th>$t$ ($s$)</th>
<th>$n$ ($m^{-3}$)</th>
<th>$B$ ($T$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MFE</td>
<td>$10^1$</td>
<td>-</td>
<td>$10^{21}$</td>
<td>$10^1$</td>
</tr>
<tr>
<td>ICF</td>
<td>$10^{-3}$</td>
<td>$10^{-9}$</td>
<td>$10^{12}$</td>
<td>0</td>
</tr>
<tr>
<td>MIF - FRC</td>
<td>$10^{-2}$</td>
<td>$10^{-5}$</td>
<td>$10^{24}$</td>
<td>$10^2$</td>
</tr>
<tr>
<td>MIF - Z pinch</td>
<td>$10^{-3}$</td>
<td>$10^{-9}$</td>
<td>$10^{28} - 10^{30}$</td>
<td>$10^3$</td>
</tr>
</tbody>
</table>

A recent development in solid-liner driven MIF has been the proposal and investigation of a solid-liner driven by a Z pinch implosion [5]. In this scheme (called “MagLIF”) a solid cylindrical liner of radius 0.5 cm is filled with a DT fuel. The DT gas is preheated with a laser and a small axial magnetic field is frozen into it. The liner then implodes as a Z pinch, heating and compressing the fuel as it does so, leading to thermonuclear burn. The convergence ratio ($= r_{\text{initial}}/r_{\text{final}}$) of the liner is estimated to be $10 - 15$. At peak compression there will be large magnetic fields in both the axial ($\sim 10^4 T$) and azimuthal ($\sim 10^{3-4} T$) directions acting to confine the plasma and fusion products. Parameters for this scheme are shown in row 4 of table 4.1.†

The MagLIF proposal and subsequent encouraging studies of the imploding liner stability [74] make relevant the question of ignition and burn of plasmas in Z pinch configurations. Previous hydrodynamic studies have suggested that for a DT Z

†We note that this is not the only proposed role of Z pinches in fusion. The use of wire array Z pinches as the driver for conventional ICF has also been investigated [71], [72], [73]. In this case an ICF capsule surrounded by CH$_2$ foam is placed at the centre of the wire array. The imploding wire array acts as a dynamic hohlraum, driving a radiating shock wave through the foam that heats the capsule.
pinch with a hotspot, a current of over $50\,MA$ is required for ignition [75], which agrees with the predictions given for MagLIF in [5]. This is approximately double the highest current that is presently possible on the world’s largest pulsed power generator, the Z machine at Sandia National Laboratories. Results in section 4.3 of this chapter for the ignition of an equilibrium DT pinch suggest that a current of at least $50 - 60\,MA$ is required for high energy gain.

For any MIF scheme there are three requirements that need to be met in order to achieve ignition. First, the DT fuel temperature must reach a point where sufficient reactions can occur (typically $5 - 10\,keV$, depending on the scheme). Secondly, the fusion products must deposit a sufficient fraction of their energy in order to self-heat the plasma and, thirdly, the plasma must be confined long enough for significant burn to occur. In the next section we focus on the second requirement. We study the confinement of $\alpha$ particles by the magnetic fields present in a Z pinch. This is an area which has previously received little attention. The issue of $\alpha$ confinement for magnetized target fusion in cylindrical geometry was addressed by Basko and co-workers [76], [77]. However, as they were considering a uniform axial magnetic field it seems that a new study is required for a magnetic field that is azimuthal, as in a Z pinch. We now proceed with such a study.

### 4.2 Fast ion dynamics in a Z pinch

The $\rho R$ of a Z pinch is several orders of magnitude lower than that of ICF. Therefore, $\alpha$ particles (and other fast ions) will have long mean free paths compared to the pinch radius and will deposit little of their energy on a single radial transit of the Z pinch. In order for the particles to heat the plasma it will be necessary for them to make a number of transits across the Z pinch. This requires the $\alpha$ particles to be well confined within the Z pinch by the magnetic field. A rudimentary approximation of the ability of the magnetic field to confine the particles can be obtained by calculating the larmor radius of the particle. If the larmor radius is less than the pinch radius then we expect the particle to be confined. Assuming a pinch radius of 0.5 cm and $\alpha$ particle velocity of $v_\alpha = 1.29 \times 10^7\,ms^{-1}$ then the minimum
magnetic field required to meet this condition is $107 \, T$. This magnetic field is quite easily achieved at the pinch edge by Z pinches with currents of the order of $1 \, MA$ or above. However, this approximation does not take into account the nature of the magnetic field in a Z pinch which is $0 \, T$ on axis and rises to $\sim 10^3 \, T$ at the pinch edge. Particles can have a very small larmor radius at the pinch edge which becomes very large as they approach the axis. Furthermore, as a particle crosses the pinch axis, the magnetic field that it experiences effectively switches direction due to the azimuthal nature of the field. This leads to the phenomenon of singular orbits in which particles switch their direction of rotation as the magnetic field they experience switches.\footnote{Singular orbit phenomena appear to have been identified at least as early as the 1950s \cite{78} when the effect of antiparallel magnetic field lines on the trajectory of a particle was studied. Singular orbit phenomena were first studied in the context of Z pinches by Haines \cite{79} with regard to ion currents in equilibrium Z pinches.} Singular orbits in 2 dimensions are illustrated in figure 4.1. Singular orbits can lead to significant particle drifts in both axial directions, drifts that are much greater than those due to $E \times B$ or $\nabla B$, for example. Particles near the axis with a large $v_z$ velocity component are particularly affected. Singular orbits also exist in 3 dimensions for particles that pass sufficiently close to the pinch axis, as illustrated in figure 4.2.

Figures 4.1 and 4.2 are produced using the following dimensionless form of the Lorentz equation

$$\frac{d^2 \hat{x}}{d\tau^2} = \hat{v} \times \hat{B},$$

$$\hat{B} = \left( |\hat{B}_\phi| \sin \phi, |\hat{B}_\phi| \cos \phi, 0 \right),$$

$$|\hat{B}_\phi| = \begin{cases} \frac{\hat{r}}{R}, & \text{if } \hat{r} \leq R, \\ \frac{R}{\hat{r}}, & \text{if } \hat{r} > R. \end{cases} \quad (4.1)$$

We have used the scaling $\tau = \sqrt{\lambda t}$ where

$$\lambda = \frac{q_\alpha \mu_0 I v_\alpha}{2\pi m_\alpha R}. \quad (4.2)$$

In this equation, $R$ is the pinch radius, $I$ is total pinch current and $\mu_0$ is the per-
meability of a free space. The diagrams correspond to $I = 25\, MA$, $R = 0.1\, cm$ and the equations are solved for a period of $4\, ns$. It is assumed that the current flows uniformly within the pinch and so the magnetic field rises linearly from $\hat{r} = 0$ to $\hat{r} = R$. The pinch radius $R$ is held constant over time. Clearly, in an imploding Z pinch this is not the case. However, as ignition and burn occur when the pinch has imploded onto the axis we need only consider fast ion motion in this stagnation phase during which the radius changes little.

![Figure 4.1: The trajectories of three particles in an azimuthal magnetic field generated by a Z pinch with uniform current density are shown. Both the green and blue trajectories represent particles with singular orbits in different axial directions. The red trajectory, representing a particle that does not undergo a singular orbit but only drift due to $\nabla B$, is included for comparison. The magnetic field is directed into the page for $x > 0$ and out of the page for $x < 0$. The diagram axes are in units of pinch radius.](image1)

![Figure 4.2: Singular orbits in 3 dimensions. The closer the particle passes to the pinch axis then the greater its axial drift due to singular orbits.](image2)

As can be seen from figure 4.1 the confinement of an $\alpha$ particle is strongly
dependent on both its initial radial location and velocity direction. Therefore, in order to gain a better understanding of the overall confinement of \( \alpha \) particles we solve (4.1) for 15000 particles, each with different initial conditions, again for a period of 4 ns. The final axial displacements of the particles are then recorded and the cumulative distribution of these displacements is plotted (see figure 4.3). As can be seen from this plot over two-thirds of the particles undergo a displacement of less than \( 2R \). Displacements are greater in the \(+z\) direction than in the \(-z\) direction since particles moving in the \(-z\) direction need to complete more than one half of a larmor orbit on each side of the axis (as illustrated by the blue curve in figure 4.1). A second curve is shown in figure 4.3 representing the results when both an axial and azimuthal magnetic field are present (as would be the case for the MagLIF concept discussed in section 4.1). The magnitude of the axial field is taken to be the same as the peak magnitude of the azimuthal field and so we use
\[
\hat{B} = \left( |B_\phi| \sin \phi, |B_\phi| \cos \phi, 1 \right)
\]
in (4.1). Clearly, axial confinement of the \( \alpha \) particles is much poorer when the axial magnetic field is introduced. This is because the axial field prevents particles from reaching the pinch edge where the azimuthal field has greatest effect. Instead the particles tend to stream along the axial magnetic field lines. The final curve in figure 4.3 represents the case where no axial field is present and the azimuthal field is generated by a skin current flowing in the Z pinch. Therefore, the magnetic field in (4.1) is 0 when \( \hat{r} \leq R \). The axial confinement of \( \alpha \) particles in this case is better than the case with an axial field even though \( \alpha \) particles are essentially unmagnetized except when they reach the pinch edge. The skin current effectively creates a magnetic wall at the pinch edge which causes the \( \alpha \) particles to bounce back inwards when they meet it.

Given that the typical height of a Z pinch is 2 cm we can see from figure 4.3 that there will be a significant number of fast ions in the pinch that will undergo vertical displacements similar to the pinch height, particularly in the case where an axial field is present. Axial losses of fast ions may be particularly problematic if the imploded pinch is not uniform in the axial direction but instead consists of a number of discrete dense regions, as can be caused by magneto-Rayleigh-Taylor instabilities of the imploding mass. The particles will then escape to less dense regions. Thus, we
can conclude that magnetic confinement of alpha particles is possible in the radial direction of a Z pinch but issues may arise regarding axial confinement.

Figure 4.3: The cumulative distribution for the axial displacement of $\alpha$ particles emitted at $z = 0$. The blue curve represents the case when only an azimuthal magnetic field is present. The red curve is for both an azimuthal and axial magnetic field and the green curve is for an azimuthal magnetic field generated by a skin current.

4.3 Ignition of equilibrium Bennett pinches

Thermonuclear burn occurs during the stagnation phase of a Z pinch. The stagnation phase is the period when the plasma is compressed on the axis, achieving its highest density and temperature, before “bouncing” outwards and fragmenting. The timescale for the stagnation phase is typically $\sim 5 \, ns$. We can model the stagnation phase of the Z pinch without having to model the implosion phase by using the Bennett equilibrium model [80]. This is based on the assumption that the thermal pressure of the plasma and the magnetic pressure due to the azimuthal magnetic field are equal and opposite. From this pressure balance, and assuming that the current density and temperature are uniform throughout the pinch we can derive the following relations for the dependence of magnetic field, thermal pressure and
temperature on the pinch current and radius

\[ B(r) = \frac{\mu_0 I r}{2\pi R^2}, \quad (4.3) \]

\[ P(r) = \frac{\mu_0 I^2}{4\pi^2 R^2} \left( 1 - \frac{r^2}{R^2} \right), \quad (4.4) \]

\[ T = \frac{\mu_0 I^2}{8\pi (1 + Z) N}, \quad (4.5) \]

where \( I \) is the total current in the pinch, \( R \) is the pinch radius, \( \mu_0 \) is the vacuum permeability, \( N \) is ion line density and \( Z \) is the ionization of the plasma. Since temperature is uniform and the pressure has a parabolic radial profile the ion density must also have a parabolic radial profile. Using (4.3)-(4.5) we can choose \( I, R \) and \( N \) to be the independent parameters defining a Bennett Z pinch.

We model the ignition and burn of the Bennett Z pinch in 2 dimensions \((r, \theta)\) using a hybrid approach. The DT fuel is treated as a single fluid and is modelled using the MHD code GORGON [81], [82]. It is assumed that the plasma is fully ionized and optically thin. Bremsstrahlung power losses that are proportional to the ion and electron densities and the square root of electron temperature are included [83]. The MHD model also includes resistivity and thermal conduction. A description of the model is given in appendix E. The temperature of the fuel allows us to calculate the DD and DT reaction rates and the energy spectrum of the reaction products. The density and energy of the DT fuel is reduced according to the number of reactions that occur and the absorption spectrum of the thermal plasma (as described in section 2.2.1), respectively.\(^5\) The charged reaction products \((T,p,He ^3,\alpha - \text{it is assumed that neutrons escape the system without interacting})\) are then modelled as discrete particles using standard PIC methods [86]. Collisions between the particles and the fluid, resulting in heating of the fuel, are calculated using the method outlined in sections 3.1 and 3.2.

\(^5\)Kinetic calculations have shown that the effect of fuel burn-up on the reaction rate is small for a 0 dimensional plasma [84], [85]. However, these calculations considered ICF plasmas in which fuel depletion is less likely to affect the stopping of \(\alpha\) particles because of the extremely high densities. The situation may be different in Z pinches in which density is lower and transport of the fusion products is highly non-local. Therefore, it is important to include the effects of fuel burn-up.
4.3.1 Results I: Ignition criteria

Fusion burn in Bennett Z pinches is investigated for a range of values of the parameters $I$, $R$ and $N$ for a burn time of 5 $ns$ with the goal of determining the minimum requirements for ignition. The initial temperature is given by (4.5). It is assumed that electron and ion temperatures are initially equal. It is found that in order for significant burn to occur an average density in the pinch of at least $10^{28} m^{-3}$ is required. At lower densities, even for high temperatures, the reaction rate is too low for a large number of reactions to occur in the 5 $ns$ period and the pinch radius required for energy gain to occur is unfeasible ($\sim 5 cm$). For $n \geq 10^{28} m^{-3}$ it is found that in order for self-heating of the plasma by $\alpha$ particles to occur an initial temperature of approximately 4 $keV$ is required. At this temperature the reactivity of the DT reaction rises rapidly with increasing temperature and so increasing $\alpha$ production can occur quickly.

Shown in figure 4.4 are two cases where ignition and significant yield occur for the least stringent conditions of $R$ and $N$. The average density in both cases is $7.5 \times 10^{28} m^{-3}$. Initial temperatures range from $0.4 - 6.6 keV$ for the $R = 1 \ mm$ case and $1.66 - 26.6 keV$ for the $R = 0.5 \ mm$ case. A smaller pinch radius is important for two reasons. First, it gives a larger initial temperature for the pinch, according to (4.5). Secondly, a smaller radius means that the density of energy deposited by the $\alpha$ particles is much greater. The particles are much more efficient at heating the central dense region of the pinch. Therefore, even though the pinch with $R = 0.5 \ mm$ has only one quarter of the total mass of the $R = 1 \ mm$ pinch, it gives a much higher energy yield, especially at higher current values.\footnote{The well-known $I^4$ scaling (discussed in section 5.1) for energy yield as a function of pinch current is not applicable to the results in figure 4.4. This is because the pinch density and temperature are scaled according to the Bennett equilibrium. A different theory was used by Anderson to obtain the $I^4$ scaling [87].} We can conclude that ignition can be achieved by 50 $MA$ currents provided that the DT fuel is compressed to sufficiently small radii ($r < 1 \ mm$) and high density ($n > 10^{28} m^{-3}$).

An example of a pinch producing high gain is shown in figure 4.5. The pinch has a current of 60 $MA$ and has an energy yield of 1400 $MJ/m$. Heating of the pinch occurs rapidly and the temperature reaches a peak of 45 $keV$ after 5 $ns$. In
comparison with ICF, this is a low temperature for an igniting plasma. However, because of the longer burn time and larger fuel mass it is more than sufficient to produce a large energy yield.

![Graph showing energy gain and yield for Bennett equilibrium pinches as a function of current with an average density of $7.5 \times 10^{28} \text{ m}^{-3}$. Energy gain is the ratio of total kinetic energy of reaction products to total initial thermal energy of the pinch.]

### 4.3.2 Results II: Burn dynamics

A well-known study of thermonuclear burn of magnetized target plasmas by Jones and Mead [65] showed that the presence of a magnetic field enhanced volumetric burn (in which burn occurs simultaneously throughout the plasma) but reduced the ability of propagating burn to occur (in which a burn wave spreads outwards from a central hotspot). This was due to reduction in thermal conduction and $\alpha$ transport caused by the magnetic field. Therefore, it was concluded that a magnetic
field was not always beneficial. The Jones and Mead study considered spherically-symmetric plasmas. Studies of thermonuclear burn in cylindrical geometry, such as that by Basko [76], have generally considered that magnetic fields are strong enough to locally confine α particles and so only volumetric burn may occur. However, no study of the influence of the azimuthal magnetic field of a Z pinch on burn dynamics appears to have been carried out.

We have already shown in section 4.2 that the magnetic field can confine the α particles within the Z pinch but allow them to travel non-locally. Therefore, the α particles are capable of depositing their energy throughout the Z pinch facilitating a propagating burn wave. We now consider this energy deposition for a Z pinch with the same conditions as in figure 4.5. Because of the uniform initial temperature of the plasma, significant α particles are initially produced throughout the Z pinch (with the highest production on the axis, due to the parabolic density profile). However, despite this we can see from figure 4.6 that the α particles as a whole spend a significant proportion of their time near to the pinch axis. Furthermore, the radial variation of the α density changes little during the burn time. This results in the majority of the α energy deposition and consequent temperature increases occurring near the axis, as shown in figures 4.7 and 4.7, respectively. The parabolic density profile used in the simulations also leads to greater heating occurring in
the central, denser region. However, the radial variation of $\alpha$ density clearly shows that a majority of $\alpha$ particles will reside in the central region. Therefore, we can conclude that the magnetic field does reduce the rate of burn propagation outward from the axis. This undesirable effect is, however, outweighed by the excellent radial confinement of $\alpha$ particles that is offered by the magnetic field.

Figure 4.6: For the conditions as given in figure 4.5 the radial profile of the normalised $\alpha$ density is shown after 0.1 ns and 5 ns of the simulation time. The radial variation does not change significantly with time. The normalising values are $7 \times 10^{23} \text{ m}^{-3}$ for the 0.1 ns curve and $9 \times 10^{26} \text{ m}^{-3}$ for the 5 ns curve.
Figure 4.7: For the conditions as given in figure 4.5 the radial profile of the total $\alpha$ heating power deposited in the fuel during the 5 ns burn time period is shown. Heating power occurs primarily near the pinch axis both because of the parabolic density profile and the fact that $\alpha$ density tends to be higher near the axis.

Figure 4.8: For the conditions as given in figure 4.5 the radial profile of the fuel ion temperature after 0.9 ns, 2.9 ns and 5.0 ns is shown.
Chapter 5

Neutron production in deuterium gas puffs

The results of the previous chapter have shown that thermonuclear ignition is possible for an “ideal” Z pinch with currents greater than 50 \( MA \). However, the many experiments that have been carried out in the past have not conclusively shown that such “ideal” conditions (sufficiently high temperatures and densities) could be achieved. In particular, there has been much debate about the mechanism causing nuclear reactions. It has been shown that in many experiments, particularly those at currents less than 3 \( MA \), reactions have been due to a beam-target mechanism in which large electric fields generated by instabilities accelerate ions to high energies. This is in contrast to the thermonuclear mechanism that we have assumed is responsible for the plasma burning in chapter 4. It is unlikely that the beam-target mechanism can have the required efficiency at high currents to cause ignition. Therefore, a key element of research into deuterium and deuterium-tritium Z pinches is determining the mechanism responsible for reactions.

The highest current Z pinches with nuclear reactions that have so far been investigated in the laboratory are 15 \( MA \) deuterium gas puffs which produced \( 3.7 \times 10^{13} \) neutrons [6]. MHD modelling of the experiments suggests that the majority of these neutrons were thermonuclear in origin. In this chapter, we further analyse these experiments in order to estimate the number of beam-target reactions that take place.
in the gas puff. We also estimate the number of secondary DT reactions that will occur in the pinch as this is a useful plasma diagnostic. The first two sections review the history of neutron production experiments in Z pinches and describe the deuterium gas puff experiments that we are studying. The final two sections describe our analysis of the reaction kinetics in the gas puff.

5.1 A review of neutron production in Z pinches

When research into controlled thermonuclear fusion began in the 1950s some of the earliest approaches involved Z pinches. Deuterium filled discharge tubes of length $45 - 90\,\text{cm}$ and radius $7.5\,\text{cm}$ were subjected to longitudinal electric fields which ionized the gas and generated a magnetic field. Although non-imploding, these Z pinches were dynamic. When magnetic pressure outweighed thermal pressure the column would collapse radially driving a shock towards the axis. The reflected shock from the axis would then cause the plasma to expand again. Several such bounces occurred before the column was disrupted due to instabilities. During this period a relatively large number of neutrons (up to $10^8$) were produced. It was assumed that the neutrons were being produced by a thermonuclear mechanism, that is, the bouncing pinch lead to heating of the plasma and the production of neutrons. Assuming such a mechanism, it can be shown that the neutron yield was proportional to $I^4$ [87], giving a very optimistic outlook for energy gain in the pinch. However, it was soon determined that the neutrons were not due to a high temperature plasma but rather the acceleration of deuterium ions to energies of $200\,\text{keV}$ [87], [88]. This result was confirmed experimentally by the measurement of anisotropic neutrons [89]. These beams were believed to be formed by the $m = 0$ ("sausage-type") instability [90]. This instability occurs when a force acting perpendicular to a plasma surface is in equilibrium with a magnetic field parallel to the surface. In the plasma column this instability causes narrow “neck” regions to form in the column and is accompanied by the generation of large electric fields in the axial direction. These
electric fields are responsible for the formation of deuterium beams. The beam-target production mechanism is undesirable for two reasons. First, achieving net energy gain by striking a target with a beam of deuterium ions is difficult since the coulomb scattering cross-section for the ions is much greater than the reaction cross-section and, secondly, the neutron yield due to beam-target reactions will not scale upwards as $I^4$.

Despite the disappointing result for discharge tubes, expectations of thermonuclear production in Z pinches were raised again in the 1980s when frozen deuterium fibres were compressed using a 0.5 $MA$ current on a timescale of 100 $ns$ producing $4 \times 10^9$ neutrons [24]. This experiment differed from the deuterium discharge tube experiments in that confinement of the plasma was inertial due to the implosion of the fibre rather than the magnetic confinement of the discharge tube experiment. However, as in the case of the discharge tube experiments, the output was significantly higher than would be expected from the estimated plasma temperature and the neutron production mechanism was again determined to be beam-target. The prospect for nuclear fusion in a Z pinch again seemed unlikely.

However, recent success in the development of plasma radiation sources in Z pinch plasmas have again changed this picture. Implosions of argon gas puffs using currents of 15 $MA$ have produced x-ray outputs of almost 300 $kJ$ in a 12 $ns$ pulse [92] suggesting that temperatures on the keV scale required for fusion can indeed be achieved in Z pinches. The success of these high current gas puff implosions prompted the investigation of the neutron production capabilities of a deuterium gas puff [26]. Experiments carried out on the Z machine in 2007 showed that the deuterium gas puff (the gas puff is described in detail in the next section) was capable of producing $\sim 3 \times 10^{13}$ neutrons, a record for neutron production in the laboratory [25], [6]. Even more encouraging is the fact that, unlike previous experiments, this measured output agrees closely with the estimated output assuming that the production mechanism is thermonuclear. However, this does not give conclusive evidence that the neutrons produced are thermonuclear rather than beam-target,

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*The exact mechanism for the formation of the electric field from the $m = 0$ instability is still a subject of debate. A detailed discussion can be found in [91].
particularly since large axial electric fields exist in the deuterium gas puff as in other Z pinches. PIC simulations suggest that the importance of the beam-target mechanism decreases as current increases [93], [94]. At currents greater than 10 $MA$ at least half the neutrons are thermonuclear in origin. In this chapter we calculate the number of neutrons produced by a beam-target mechanism using MHD to model the implosion dynamics of the gas puff and then post-process the results to estimate the number of deuterium ions that are accelerated to high energies and capable of undergoing beam-target reactions.

5.2 Deuterium gas puffs

The gas puff experiments that we study were carried out on the Z machine at Sandia National Laboratories and are reported in [26], [25] and [6]. The gas puff consists of two concentric annular nozzles, the inner nozzle spanning the radii $1 - 2 \, cm$ and the outer nozzle spanning $3 - 4 \, cm$. Jets from the nozzles are emitted across two aluminium wire meshes forming the anode and cathode that are $2 \, cm$ apart. The load thus consists of a double shell of deuterium gas that is $2 \, cm$ in height. Although the radial mass distribution varies in the vertical direction due to expansion of the jets the line density is approximately constant with a value of $405 \, \mu g/cm$. The Z machine provides a current at the load of $15.4 \, mA$ with a rise time of $160 \, ns$.

The gas puff implosion was simulated in 3 dimensions using the GORGON code (see appendix E). Density profiles at a number of time steps are shown in figure 5.1 and we now discuss the implosion dynamics with reference to this figure. As the rising current begins to ionize the deuterium the skin effect causes the current to flow predominantly in the outer shell. The global magnetic field begins to sweep this material radially inwards. This is accompanied by the formation of bubbles due to the Rayleigh-Taylor instability as can be seen at $130 \, ns$. The purpose of the inner shell is to mitigate this Rayleigh-Taylor effect by acting as a stabilizer when the imploding outer shell collides with it. After the collision the double shell implodes inwards with a velocity of $10^6 \, ms^{-1}$ as peak current approaches, sweeping up the low density material in the interior. The bulk of the plasma begins to arrive on
the axis after about 168 \(\text{ns}\) and its kinetic energy is converted to thermal energy. In the experiment, argon and chlorine dopants were added to the deuterium gas to provide spectroscopic data. From this, the electron temperature and ion density of the stagnated pinch were estimated to be 2.2 \(\text{keV}\) and \(2 \times 10^{20} \text{cm}^{-3}\), respectively, suggesting an ion temperature in the \(\text{keV}\) range. The GORGON simulations give a peak ion temperature of \(\sim 6 \text{keV}\), which is sufficiently large to allow significant thermonuclear production of neutrons. Stagnation of the pinch takes place in a period of about 10 \(\text{ns}\). As can be seen from the last two diagrams in figure 5.1, the stagnation is not uniform in the axial direction but instead consists of a number of discrete high density hot spots. This is due to the Rayleigh-Taylor instability. The stagnation of the upper part of the pinch at an earlier time (170 \(\text{ns}\)) than the lower part (175 \(\text{ns}\)) is due to the difference in the initial radial distributions of the mass. This is the so-called zippering effect and can be mitigated by tilting the nozzles inwards [95]. The minimum pinch radius at stagnation is 6 \(\text{mm}\). This is a relatively large value compared with that of the MagLIF concept discussed in the previous chapter in which the heavy liner aids compression.

During the stagnation phase of the simulated gas puff, a total of \(2.8 \times 10^{13}\) neutrons are produced (see figure 5.2). In the experiment, the neutrons were detected using indium activation foils (described in section 2.8.3) and neutron time of flight diagnostics (section 2.8.2). The measured neutron output was \(3.7 \times 10^{13}\). The error bars for these measurements were \(\pm 20\%\). The discrepancy between the simulation and experimental neutron measurements may be due to a second mechanism causing neutron production or an inaccuracy in the modelling of the gas puff implosion. We note that for the DD fusion reaction, the reactivity varies strongly with temperature in the \(1 - 10 \text{keV}\) temperature range (it increases by several orders of magnitude). Thus, small inaccuracies in temperature can result in a significant difference in neutron production. No ion temperature was inferred from the neutron measurements. However, neutron diagnostics were fielded both radially and axially and the measured neutron outputs in each direction agreed to within the margin of error. Although isotropic neutron emission is a necessary condition for thermonuclear production, it is still possible that more than one mechanism is responsible for
the neutron production.

Figure 5.1: Vertical cross-sections showing the ion number density (in units of $m^{-3}$) at a number of times during the 3 dimensional MHD simulation of the imploding gas puff. Note that each diagram has a different colourbar.
5.2.1 Investigating the pusher-stabilizer-radiator concept

In section 5.3 we evaluate the importance of the beam-target mechanism for neutron production but first we look at a potential method of improving the neutron yield from the deuterium gas puff. Efficient production of x-ray radiation from an argon double shell gas puff was one of the original motivations for experimenting with a deuterium gas puff. It was subsequently found that the x-ray yield from an argon gas puff could be doubled by including a central jet of argon on the axis [96]. This is the so-called “pusher-stabilizer-radiator” model in which the outer shell of argon initially implodes and is then stabilized against Rayleigh-Taylor instabilities when it collides with the inner shell. Finally, the imploding mass compresses the central jet causing it to heat and radiate. The central jet provides a target for the imploding mass. It was found that the majority of the radiation was from material that was originally in the central jet rather than in the shells.

We use 3 dimensional MHD simulations to investigate if the pusher-stabilizer-radiator scheme could increase the neutron yield for a deuterium gas puff. For these simulations the total mass of deuterium in the gas puff is the same as in the double shell simulations described above. However, the mass of the inner jet is reduced by 30% in order to form the central jet, as shown in figure 5.3. For this configuration
the total neutron production is calculated to be $4.2 \times 10^{13}$, a factor of 1.5 increase on the production when no central jet is present. A much more substantial increase occurs when the central jet is artificially preheated to a temperature of 1 keV just prior to compression by the imploding mass. In this case, the neutron production is $9.5 \times 10^{13}$. It is important that the heating occurs late in time (about 160 ns) in order to reduce thermal expansion of the central jet. The central jet then acts as a hotspot which is compressed by the imploding mass. Previous computational work has shown that the hotspot can help achieve ignition in DT Z pinches also [75].

Figure 5.3: The top diagram shows an $r_z$ cross-section of the gas puff density at $t = 0$ ns for the double shell configuration. The lower diagram shows the corresponding situation for the pusher-stabilizer-radiator configuration.

## 5.3 Beam-target reactions

Isotropy of the neutron emission in the deuterium gas puff experiments suggests that the neutrons were produced by a thermonuclear mechanism. However, the MHD simulations show that large electric fields exist in the stagnated pinch and so neutron production through a beam-target mechanism may also be occurring. In this section we investigate if these large electric fields can accelerate a significant number of deuterium ions and assess the reactivity of these ions.

The model we use to determine if ions are accelerated is that developed by Dreicer for the “electron runaway” phenomenon [97], [98]. In this model charged
particles are accelerated when the force acting on them due to an external electric field is greater than a frictional force caused by collisions with other particles. The frictional force acting on a particle is the product of the particle velocity and the coefficient of dynamical friction given by (3.7). A schematic sketch of the behaviour of this frictional force is shown in figure 5.4 as a function of the particle velocity. The frictional force is 0 for $v_t = 0$ and asymptotically approaches 0 as $v_t \to \infty$. Therefore, when an external field is imposed on the plasma there are two ranges of particle velocities where the force due to the electric field is greater than that due to friction. These are $v_t < v_a$ and $v_t > v_b$. Particles in these velocity ranges will be accelerated by the electric field. For particles with $v_t < v_a$ the acceleration will simply lead to $v_t > v_a$ where the friction force again dominates. However, particles with $v_t > v_b$ can be accelerated without interruption, causing them to “run away”. Due to the difference in particle mass, the frictional force on electrons is much less than on ions and so much more electrons can run away. However, if the electric field is large then a significant number of ions may also be accelerated. These ions can then become “beam” ions.

![Figure 5.4: The general form of the frictional force $F(v_t)$ due to collisions acting on a particle with velocity $v_t$ is shown. A given electric field $E$ establishes the cut-offs $v_a$ and $v_b$.](image)

We use the run away model together with data from the MHD simulations to calculate the number of accelerated ions in a post-processing method. The temperature and density of the plasma and the electric and magnetic fields in the pinch are outputted from the simulations at 3 different times over a $10 \text{ ns}$ interval during stagnation of the pinch (see figure 5.5). From these data sets the number of run away
ions at each location in the pinch can be calculated. Discrete particles corresponding to this number of ions are created and a particle tracking algorithm is then used to calculate their trajectories through the pinch during the 10 ns interval. The effect of collisions on the particles and their reactivity with ions in the deuterium plasma (the “target”) is calculated using the expressions described in sections 3.1 and 3.3, respectively. In this way the total number of beam-target reactions occurring can be estimated.

The total number of beam-target neutrons produced is $7 \times 10^8$, which is very low compared to the $2.8 \times 10^{13}$ that were produced by thermonuclear reactions in the simulation. The mean energy of the reacting “beam” ions is $\sim 500$ keV. The number of particles that are accelerated is $\sim 10^{13}$, and so less than $10^{-4}$ of beam ions undergo a reaction. An explanation for the low number of beam-target reactions can be seen in the plots of electric and magnetic fields and plasma density in figure 5.5. In the central column of stagnated plasma where density is highest the electric field is low. High electric fields exist in the region between Rayleigh-Taylor bubbles where the plasma density is close to the vacuum density. Therefore, accelerated ions are predominantly in this low density region of the plasma. Furthermore, the target plasma into which they are accelerated is low density also, reducing the probability of reaction.
Figure 5.5: Cross-sections of electric field ($E$), density ($\rho$) and magnetic field ($B$) at the start and end of the 10\,ns stagnation period during which beam-target reactions are calculated. Diagrams courtesy of J. Chittenden.

5.4 Secondary deuterium-tritium reactions

Approximately 50% of the DD reactions in the gas puff will produce a neutron through the $D + D \rightarrow He^3 + n$ branch. The other 50% will produce a tritium ion through the $D + D \rightarrow T + p$ branch. The nominal energy of the triton is 1.01\,MeV. These fast tritons will then travel through the plasma and may react with a deuterium ion to produce neutrons with a nominal energy of 14.1\,MeV,
sometimes called secondary neutrons. Since the probability of reaction is dependent on the density of the deuterium plasma, the ratio of secondary neutrons to primary neutrons can be used to diagnose the fuel areal density ($\rho R$) of the plasma. Such a procedure has been used in ICF, as described in [99] and [100].

In the gas puff experiments on the Z machine, a copper activation diagnostic was used to calculate the number of high energy neutrons produced. The minimum neutron energy required for activation is 10.9 MeV (see section 2.8.3) and so only neutrons produced by a DT reaction will be detected. In the experiments, a minimum yield of $4 \times 10^9$ high energy neutrons was required for detection. Since no measurable activity of $^{62}\text{Cu}$ was observed, it suggests that the number of secondary neutrons was less than this threshold value.

The number of secondary neutrons produced in the MHD simulations of the gas puff were calculated using a similar post-processing procedure to that outlined in the previous section. It was assumed that tritons were produced from thermonuclear reactions in the deuterium plasma. Therefore, the triton particles were emitted isotropically with an energy spectrum given by (2.43). The particles are then accelerated by the electric and magnetic fields and slowing down is calculated using the procedure outlined in section 3.1. Additionally, the probability of reaction of the tritons with deuterium ions is calculated using (3.22) from section 3.3. Therefore, the number of secondary neutrons that are produced can be calculated. This number was found to be $5 \times 10^9$, comparable to the minimum number required for detection in the experiment.

Using theory developed for ICF (described in [99]), the calculated ratio of primary to secondary neutrons and the electron temperature of the gas puff suggest that the $\rho R$ value of the deuterium plasma is 0.01 $kg m^{-2}$. The average $\rho R$ for the stagnated pinch in the MHD simulations is 0.007 $kg m^{-2}$. The similarity of the two values suggests that the reactivity of the tritons is not significantly increased by their magnetic confinement and that the tritons do not make multiple transitions across the pinch, as was outlined in section 4.2. An analysis of some individual triton trajectories suggest this to be the case. A lack of axial uniformity of the stagnated pinch means that the magnetic field is not able to contain them within
the hot dense regions of the stagnated pinch but instead they escape to regions with large magnetic field and low density where their reactivity is much lower. Such a sample trajectory is shown in figure 5.6.

These results suggest that, although the magnetic field in a multi-MA Z pinch is large enough to contain fast ions, if the stagnated pinch consists of a number of discrete high density regions rather than a uniform column then the fast ions will not be contained in high density regions. This may have implications for the ignition of DT Z pinches discussed in chapter 4.

Figure 5.6: A sample trajectory of a triton over 5 ns is shown in 3 dimensions in a deuterium density background. The triton quickly escapes the central high density region and is confined by the magnetic field in a low density region where its reactivity is very low. The density has units of kg m$^{-3}$. 
Chapter 6

Conclusions and future work

The work reported in this thesis has addressed a number of questions regarding nuclear fusion in plasmas. We have analysed the production spectrum for fusion plasmas and shown how thermal broadening, fluid motion and anisotropies such as ion beams affect the spectrum shape. We have also studied the conditions required for thermonuclear ignition and energy gain in a Z pinch and identified some of the key issues in the process. We now summarise the main conclusions of this work and outline the areas that have potential for further study.

We begin with chapter 2. This chapter contained a detailed examination of the fusion reaction kinetics in 0 dimensions. Calculations of the absorption spectrum showed that for DD and DT reactions in thermal plasmas the mean energy of reacting particles is higher by a factor of $1 - 3$ (see figure 2.4) than the plasma temperature. Whilst the results show that the majority of reacting ions have energies greater than the mean energy of ions in the plasma, it is also clear that these reacting ions tend not to be located in the extremes of the distribution tails (figure 2.3). Therefore, a reduction of the reaction rate due to tail depletion is unlikely to be an issue for burning plasmas.

Chapter 2 also demonstrated how the production spectrum may be obtained. For reactions relevant to nuclear fusion it was shown that the production spectrum from a thermal plasma is asymmetric with a long high energy tail. The deviation of the tails of the spectrum from a gaussian shape becomes significant at higher plasma
temperatures (as shown in figure 2.14). Other interesting results of the study of production spectra included the demonstrations that bulk fluid motion can have an appreciable effect on the production spectrum in certain parameter ranges (section 2.3.3) and the significant variation of the production spectrum with viewing angle for a beam-target plasma (section 2.3.5). There is much potential for the application of these results to experimental studies of plasmas, particularly in the area of neutron diagnostics. Improved techniques for neutron detection and measurement mean that some of the features of production spectra that we have highlighted may be detectable. In particular, it may be possible to use the magnetic recoil spectrometer (MRS) to calculate plasma temperature by measuring the number of neutrons at a particular value in the high energy tail of the production spectrum. As discussed in section 2.8 this metric is much more sensitive to plasma temperature than the FWHM value of the spectrum.

Results of our study of the production spectrum may also be useful for computational physics, particularly for the inclusion of accurate source terms in models of burning plasmas or neutron transport. The results for the interaction of a single ion with a thermal plasma, given in chapter 3, are suitable for inclusion in a hybrid model. These results, particularly that for the production spectrum (section 3.5), are also of interest for the study of “knock-on” reactions, in which ions that have been accelerated to high energy by large angle coulomb collisions react with thermal ions. The anisotropy of emission of the products of such reactions is clearly shown in figures 3.4-3.6.

Although the features of the production spectra that we have highlighted (such as long high energy tails for thermal plasmas) are primarily due to the distributions of reactants we note that the accuracy of our results for production spectra is constrained by the accuracy of the reaction cross-section. Existing cross-sections for the fusion reactions, which have been calculated using a mixture of experimental data and theoretical methods, are subject to uncertainties, particularly at lower ion energies [63]. Thus, it appears there is much scope for improving our understanding of nuclear fusion reaction kinetics by improved calculations of reaction cross-sections.

Finally, further work is needed to determine the relativistic production spectra.
An outline of how this may be achieved has been given in section 2.6. Given that relativistic production spectra are of interest in a number of areas of astrophysics it seems that a successful general solution could be of some value.

In chapter 4 we examined the feasibility of igniting a DT Z pinch. First, it was shown that the azimuthal magnetic field generated by currents on the $MA$ scale offers very good radial confinement of fast ions such as $\alpha$ particles (section 4.2). Such confinement is an important requirement for ignition. Secondly, a study of the Bennett pinch showed that ignition can be achieved when the stagnated Z pinch has parameters of $n \geq 10^{28} m^{-3}$, $r < 1 \text{mm}$, $T_i \geq 4 \text{keV}$ and a stagnation period of about $5 \text{ns}$ (section 4.3.1). Such parameters can be achieved with driver currents of about $50 - 60 MA$.

At these ignition parameter values the $\rho R$ of the plasma in the radial direction is about $0.0035 \text{g cm}^{-2}$. This is a few orders of magnitude less than the predicted $\rho R$ required for inertial confinement fusion suggesting that the presence of the magnetic field greatly helps confinement in burning Z pinches. However, our study of DT Z pinches was in 2 dimensions and so effects in the axial direction have been neglected. One of the most important of these would appear to be axial confinement of $\alpha$ particles. It was shown in section 4.2 that drifts due to singular orbits can be significant, potentially leading to end losses of $\alpha$ particles from the pinch. Furthermore, the study of fast tritium ions in a gas puff in section 5.4 showed that axial non-uniformities of the azimuthal magnetic field can affect radial confinement of fast ions, with ions “leaking” out where the field is weaker and getting trapped in a low density region. Therefore, it would seem appropriate to extend our study of igniting DT Z pinches to 3 dimensions and modelling the implosion phase in order to generate accurate profiles of the stagnation conditions rather than the idealised conditions considered here.

In addition, a better understanding of how the specific conditions of MagLIF, currently the most promising scheme for Z pinch fusion, affect fusion burn is needed. There are a number of ways in which the presence of a heavy liner surrounding the fuel could alter the burn dynamics that have been reported in this thesis. First, if the liner material penetrates into the stagnated DT fuel then it is possible that some
of the $\alpha$ particles will deposit their energy in the liner rather than the DT plasma. This will cause an undesirable loss of energy from the DT plasma. Secondly, the amount of current that will flow in the liner or that will penetrate into the fuel is uncertain and so, consequently, is the magnetic field profile. Whilst we know that the strength of the magnetic field will contain the $\alpha$ particles in the system, it is important to determine what profile is suitable for best confining them to the hot DT fuel. The resolution of these issues will help to optimize the MagLIF concept.

Chapter 5 studied the reaction kinetics in a deuterium gas puff. The results suggested that the majority of reactions occurring in the 15 $MA$ gas puff are thermonuclear in origin and that neutron production due to beam-target effects is low (section 5.3). This is an encouraging result as it shows that $MA$ Z pinches are capable of reaching the keV ion temperatures that are required for ignition. However, as was discussed in the previous paragraph, the study of fast tritium ions in section 5.4 demonstrated that magnetic confinement of fast ions may be compromised by the effect of instabilities on the magnetic field topology. These results highlight the importance of a uniform implosion for fusion Z pinches.

In summary, the work presented in this thesis has enhanced our understanding of nuclear reaction kinetics in fusion plasmas and identified some of the key processes in achieving thermonuclear ignition in an imploding Z pinch. It is hoped that these results will be of benefit in the pursuit of controlled thermonuclear fusion.
Appendix A

Nuclear fusion cross-sections

The expressions for reaction cross-sections used in this thesis are outlined in this appendix.

For total reaction cross-sections (i.e. cross-sections that are functions of only the relative velocity of the reactants) we use the parameterised cross-sections given by Bosch and Hale [63]. These were calculated using R-matrix theory. The cross-section is given by the expression

$$\sigma(E_r) = \frac{S(E_r)}{E_r \exp \left( \frac{B_G}{\sqrt{E_r}} \right)}$$

(A.1)

where $E_r$ is the relative kinetic energy of the pair of reactants. We recall from chapter 2 that this energy is related to the relative velocity by $E_r = \frac{1}{2} \mu v^2$. The term $B_G$ is the Gamow constant given by $B_G = \pi \alpha Z_1 Z_2 \sqrt{2 \mu c^2}$, where $\alpha = \frac{e^2}{\hbar c} = 1/137.036$ is the fine structure constant. Finally, $S(E_r)$ is the astrophysical S-factor which is approximated by the following Padé approximation

$$S(x) = \frac{A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4}{1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4}.$$  

(A.2)

The parameter values in this expression for the principal fusion reactions were calculated by Bosch and Hale and are shown in table A.1.

The differential cross-sections are functions of both relative velocity and scattering angle (the total cross-section is equal to the differential cross-section integrated
over its angular components). We use expressions for the differential cross-sections given by Drosg and Schwerer [27]. In this the differential cross-section is defined by

\[
\sigma(E_r, \theta_{cm}) = \frac{d\sigma(E_r, \theta_{cm})}{d\Omega} = \frac{d\sigma(E_r, 0)}{d\Omega} \sum_i A_i(E_r) P_i(\theta_{cm}), \tag{A.3}
\]

where \(P_i\) are Legendre polynomials, \(A_i\) are the reduced Legendre coefficients and \(\frac{d\sigma(E_r, 0)}{d\Omega}\) is the differential cross-section at a scattering angle of 0°. Both the differential cross-sections at \(\theta_{cm} = 0\) and \(A_i\) are functions of \(E_r\). Data tables for values of these variables for different values of \(E_r\) and \(\theta_{cm}\) can be found in [27].

**Table A.1: Parameters for fusion cross-sections. Data taken from Bosch and Hale [63].**

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>(D(d, p)T)</th>
<th>(D(d, n)^3He)</th>
<th>(T(d, n)^4He)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_G(\sqrt{keV}))</td>
<td>31.3970</td>
<td>31.3970</td>
<td>34.3827</td>
</tr>
<tr>
<td>A1</td>
<td>(5.5576 \times 10^4)</td>
<td>(5.3701 \times 10^4)</td>
<td>(6.927 \times 10^4)</td>
</tr>
<tr>
<td>A2</td>
<td>(2.1054 \times 10^2)</td>
<td>(3.3027 \times 10^2)</td>
<td>(7.454 \times 10^8)</td>
</tr>
<tr>
<td>A3</td>
<td>(-3.2638 \times 10^{-2})</td>
<td>(-1.2706 \times 10^{-1})</td>
<td>(2.050 \times 10^6)</td>
</tr>
<tr>
<td>A4</td>
<td>(1.4987 \times 10^{-6})</td>
<td>(2.9327 \times 10^{-5})</td>
<td>(5.2002 \times 10^4)</td>
</tr>
<tr>
<td>A5</td>
<td>(1.8181 \times 10^{-10})</td>
<td>(-2.5151 \times 10^{-9})</td>
<td>0.0</td>
</tr>
<tr>
<td>B1</td>
<td>0.0</td>
<td>0.0</td>
<td>(6.38 \times 10^1)</td>
</tr>
<tr>
<td>B2</td>
<td>0.0</td>
<td>0.0</td>
<td>(-9.95 \times 10^{-1})</td>
</tr>
<tr>
<td>B3</td>
<td>0.0</td>
<td>0.0</td>
<td>(6.981 \times 10^{-5})</td>
</tr>
<tr>
<td>B4</td>
<td>0.0</td>
<td>0.0</td>
<td>(1.728 \times 10^{-4})</td>
</tr>
<tr>
<td>Energy range (keV)</td>
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<td>0.5 – 4900</td>
<td>0.5 – 550</td>
</tr>
</tbody>
</table>
Appendix B

Notes on multiple integrals

This appendix contains a review of some integral identities used in chapter 2.

B.1 Change of variables in multiple integrals

Changing the variables of an integral is often carried out in order to make evaluation of the integral easier or, as is the case in chapter 2, to represent the integral as a function of different variables. When changing the variables of the integral it is necessary to ensure that the infinitesimal volume over which the integral is evaluated is correctly transformed. The following theorem, given in [101], describes how the change of variables in an integral can be carried out:

If the transformation \( x_1 = \phi_1 (y_1, \ldots, y_n) \), \( x_n = \phi_n (y_1, \ldots, y_n) \) represents a one-to-one mapping of the closed region \( R \) of \( x_1 \ldots x_n \)-space onto a region \( R' \) of \( y_1 \ldots y_n \)-space whose Jacobian determinant \( \left| \frac{\partial (x_1, \ldots, x_n)}{\partial (y_1, \ldots, y_n)} \right| \) is everywhere positive then the following transformation formula holds:

\[
\int \ldots \int_R f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = \int \ldots \int_{R'} f(\phi_1, \ldots, \phi_n) \left| \frac{\partial (x_1, \ldots, x_n)}{\partial (y_1, \ldots, y_n)} \right| \, dy_1 \ldots dy_n. \tag{B.1}
\]

The Jacobian determinant is the determinant of the matrix of all first-order partial
derivatives of the mapping under consideration

\[
\frac{\partial (x_1, \ldots, x_n)}{\partial (y_1, \ldots, y_n)} = \begin{bmatrix}
\frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n}
\end{bmatrix}
\]  \hspace{1cm} (B.2)

The most common application of the change of variables theorem is when converting from cartesian to spherical coordinates in which case the Jacobian determinant is

\[
\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix}
\sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\cos \theta & -r \sin \theta & 0
\end{vmatrix} = r^2 \sin \theta. \hspace{1cm} (B.3)
\]

It should be noted that the transformation formula remains valid if the Jacobian determinant is equal to 0 at a finite number of isolated points in the region \( R' \). This allows us to consider regions that contain the origin in the spherical coordinate system.

The change of variables theorem also applies to single variable integrals in which case the Jacobian matrix is a \( 1 \times 1 \) matrix whose determinant can be trivially calculated.

**B.2 Spherical integral identity I**

The integral identity

\[
\int_0^{2\pi} \int_0^\pi \sin \theta f (x_1 \sin \theta \cos \phi + x_2 \sin \theta \sin \phi + x_3 \cos \theta) \, d\phi d\theta = 2\pi \int_{-1}^1 f (Rt) \, dt,
\]

(B.4)
where \( R = \sqrt{x_1^2 + x_2^2 + x_3^2} \) may be proved by defining two vectors

\[
\begin{align*}
\mathbf{a} &= (x_1, x_2, x_3), \\
\mathbf{b} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\end{align*}
\]

Therefore, \( \mathbf{a} \cdot \mathbf{b} = x_1 \sin \theta \cos \phi + x_2 \sin \theta \sin \phi + x_3 \cos \theta = |\mathbf{a}| |\mathbf{b}| \cos \psi \), where \( \psi \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \). As the integral is over the entire spherical surface the axes can easily be rotated such that \( \theta \) and \( \psi \) are coincident (i.e. the \((x, y, z)\) coordinate system is rotated to \((x', y', z')\) such that \( \mathbf{a} \) lies on the \( z' \) axis). Thus we obtain

\[
\int_0^\pi \int_0^{2\pi} \sin \theta f(x_1 \sin \theta \cos \phi + x_2 \sin \theta \sin \phi + x_3 \cos \theta) \, d\phi \, d\theta \\
= \int_0^\pi \int_0^{2\pi} \sin \psi f(R \cos \psi) \, d\phi' \, d\psi \\
= 2\pi \int_0^\pi \sin \psi f(R \cos \psi) \, d\psi. \tag{B.5}
\]

Finally, the substitution \( t = \cos \psi \) gives the right-hand side of (B.4).

### B.3 Spherical integral identity II

An extension of the above identity occurs if we introduce a second vector constant such that

\[
\int_0^\pi \int_0^{2\pi} \sin \theta f(\mathbf{a} \cdot \mathbf{c}, \mathbf{b} \cdot \mathbf{c}) \, d\theta \, d\phi. \tag{B.6}
\]

with the vectors defined by

\[
\begin{align*}
\mathbf{a} &= (x_1, x_2, x_3), \\
\mathbf{b} &= (y_1, y_2, y_3), \\
\mathbf{c} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\end{align*}
\]

As we are integrating over the entire surface of a sphere we can rotate the axes such that one vector lies on the \( z \) axis. We choose \( \mathbf{a} \) to lie on the \( z \) axis and so the angle between \( \mathbf{a} \) and \( \mathbf{c} \) is \( \theta \). As in the previous identity we can use the vector dot product

\[
\mathbf{a} \cdot \mathbf{c} = x_1 \sin \theta \cos \phi + x_2 \sin \theta \sin \phi + x_3 \cos \theta = |\mathbf{a}| |\mathbf{c}| \cos \psi,
\]

where \( \psi \) is the angle between \( \mathbf{a} \) and \( \mathbf{c} \). As the integral is over the entire spherical surface the axes can easily be rotated such that \( \theta \) and \( \psi \) are coincident (i.e. the \((x, y, z)\) coordinate system is rotated to \((x', y', z')\) such that \( \mathbf{a} \) lies on the \( z' \) axis). Thus we obtain

\[
\int_0^\pi \int_0^{2\pi} \sin \theta f(\mathbf{a} \cdot \mathbf{c}, \mathbf{b} \cdot \mathbf{c}) \, d\phi \, d\theta \\
= \int_0^\pi \int_0^{2\pi} \sin \psi f(R \cos \psi) \, d\phi' \, d\psi \\
= 2\pi \int_0^\pi \sin \psi f(R \cos \psi) \, d\psi. \tag{B.5}
\]

Finally, the substitution \( t = \cos \psi \) gives the right-hand side of (B.4).
to represent the integral as
\[
\int_0^\pi \int_0^{2\pi} \sin \theta f \left( |a| c \cos \theta, |b| c \cos \sigma \right) d\theta d\phi.
\] (B.7)

Here \( \sigma \) represents the angle between vectors \( \mathbf{b} \) and \( \mathbf{c} \) and before we carry out the integration it is necessary to work out the dependence of \( \sigma \) on the integration variables \( \theta \) and \( \phi \).

The situation is illustrated in figure B.1. The left diagram shows the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \). By joining the ends of these vectors we can form the tetrahedron 1234, as shown in the right diagram. The angle \( \phi \) represents the azimuthal angle about the \( z \) axis of vector \( \mathbf{c} \). Again, since we are integrating over the entire sphere surface we can rotate the axes such that \( \phi = 0 \) corresponds to when the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) all lie in the same plane. Therefore, \( \phi \) is the angle between the planes 123 and 124. The angle \( \psi \), between vectors \( \mathbf{a} \) and \( \mathbf{b} \), is known as these vectors are fixed.

The angle \( \rho \), between \( |23| \) and \( |24| \), is dependent on \( \theta \) and \( \phi \) and can be resolved as follows: We can define a local coordinate system \( x_1y_1z_1 \) such that \( |12| \) is in the \( y_1 \) direction and 123 in the \( y_1z_1 \) plane, as shown in figure B.2. Therefore, unit vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) that are perpendicular to \( |12| \) and lie on 123 and 124, respectively, have coordinates
\[
\mathbf{n}_1 = (0, 0, 1),
\]
\[
\mathbf{n}_2 = (\sin \phi, 0, \cos \phi).
\]

We can define vectors \( \mathbf{p} \) and \( \mathbf{q} \) that have the same length and orientation as \( |23| \) and \( |24| \), respectively, as follows
\[
\mathbf{p} = (0, -|\mathbf{p}| \cos \omega_1, |\mathbf{p}| \sin \omega_1),
\]
\[
\mathbf{q} = (|\mathbf{q}| \sin \omega_2 \sin \phi, -|\mathbf{q}| \cos \omega_2, |\mathbf{q}| \sin \omega_2 \cos \phi).
\]

The dot product of these two vectors is
\[
|23| |24| \cos \rho = \mathbf{p} \cdot \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \omega_1 \cos \omega_2 + |\mathbf{p}| |\mathbf{q}| \sin \omega_1 \sin \omega_2 \cos \phi. \quad \text{(B.8)}
\]
Using the sine rule on 123 and 124 gives

\[
\sin \omega_1 = \frac{|b| \sin \psi}{|23|}, \quad \sin \omega_2 = \frac{|c| \sin \theta}{|24|}.
\] (B.9)

And so finally we can express \( \rho \) as a function of \( \theta \) and \( \phi \) as follows

\[
|23| |24| \cos \rho = \sqrt{|23|^2 - |b|^2 \sin^2 \psi} \sqrt{|24|^2 - |c|^2 \sin^2 \theta + |b| |c| \sin \psi \sin \theta \cos \phi}.
\] (B.10)

Now applying the cosine rule to triangles 123, 124, 234 and 134, respectively, gives

\[
|23|^2 = |a|^2 + |b|^2 - 2 |a| |b| \cos \psi, \quad (B.11)
\]

\[
|24|^2 = |a|^2 + |c|^2 - 2 |a| |c| \cos \theta, \quad (B.12)
\]

\[
|34|^2 = |23|^2 + |24|^2 - 2 |23| |24| \cos \rho, \quad (B.13)
\]

\[
\cos \sigma = \frac{|b|^2 + |c|^2 - |34|^2}{2 |b| |c|}.
\] (B.14)

Now, inserting (B.11) and (B.12) into the right hand side of (B.10) and simplifying gives

\[
|23| |24| \cos \rho = (|a| - |b| \cos \psi) (|a| - |c| \cos \theta) + |b| |c| \sin \psi \sin \theta \cos \phi. \quad (B.15)
\]

Finally, substituting (B.13) into (B.14) and using (B.15) gives

\[
\cos \sigma = \cos \psi \cos \theta + \sin \psi \sin \theta \cos \phi.
\] (B.16)

Therefore, the integral given in (B.6) can be transformed to

\[
\int_0^\pi \int_0^{2\pi} \sin \theta f (|a| |c| \cos \theta, |b| |c| (\cos \psi \cos \theta + \sin \psi \sin \theta \cos \phi)) \, d\theta d\phi.
\] (B.17)

This transformation can help simplify integrals of the form of (B.6).
Figure B.1: The vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) (left diagram) define a tetrahedron 1234 (right diagram). We seek to express the angle \( \sigma \) as a function of \( \theta \) and \( \phi \).

Figure B.2: The dependence of \( \rho \) on \( \theta \) and \( \phi \) can be determined by considering the dot product of vectors \( \mathbf{p} \) and \( \mathbf{q} \).

### B.4 An integral resulting in the modified Bessel function

We prove the definite integral

\[
\int_0^{2\pi} \exp(z \cos \theta) \, d\theta = 2\pi I_0(z), \quad (B.18)
\]

where \( I_0(z) \) is the modified Bessel function of zeroth order.
We begin with the complex function

\[ f(z) = \exp \left( \frac{t}{2} \left( z + \frac{1}{z} \right) \right). \]  

(B.19)

The Laurent series of the function about a point \( z_0 \) is given by

\[ f(z) = \sum_{n=-\infty}^{\infty} I_n(t) (z - z_0)^n, \]  

(B.20)

where

\[ I_n(t) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz. \]  

(B.21)

If we consider \( z_0 = 0 \) and the contour \(|z| = 1\) then letting \( z = \cos \theta + i \sin \theta, \; dz = izd\theta \) gives

\[ I_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(t \cos \theta)}{(\exp(i\theta))^n} d\theta. \]  

(B.22)

In particular, where \( n = 0 \) we have

\[ 2\pi I_0(t) = \int_0^{2\pi} \exp(t \cos \theta) d\theta. \]  

(B.23)

We can find an expression for \( I_0(t) \) as follows. First we expand \( f(z) \) as a product of two series

\[ \exp \left( \frac{t}{2} \left( z + \frac{1}{z} \right) \right) = \exp \left( \frac{t}{2} z \right) \exp \left( \frac{t}{2z} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{n+k}}{2^{n+k} n! k!} z^{n-k}. \]  

(B.24)

Now, from (B.21) we have for

\[ I_0(t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{n+k}}{2^{n+k} n! k!} \oint \frac{z^{n-k}}{z} dz, \]  

(B.25)

which, by means of the Cauchy residue theorem [102], is

\[ I_0(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{n+k}}{2^{n+k} n! k!} \text{Res} (g(z); 0), \]  

(B.26)
where \( g(z) = z^{n-k}/z \). The residue may be computed to give

\[
\text{Res} (g(z); 0) = \begin{cases} 
0, & \text{if } n \neq k, \\
1, & \text{if } n = k,
\end{cases}
\]

(B.27)

and so (B.26) becomes

\[
I_0(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n}n!n!}.
\]

(B.28)

This series can be used to compute the modified Bessel function of zeroth order. The above derivation may be generalised for order \( \nu \) where \( \nu \in \mathbb{R} \) to give [103]

\[
I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}k!\Gamma(k + \nu + 1)}.
\]

(B.29)

The modified Bessel functions for orders \( n = 0, 1, \ldots, 5 \) are shown in figure B.3.

![Figure B.3: The modified Bessel function of orders 0, 1, \ldots, 5.](image)
Appendix C

Miscellaneous absorption and production spectra expressions

This appendix lists the absorption and production spectra for a number of distributions of the plasma ions that are of interest but have not been considered in the main part of the thesis.

C.1 Absorption spectra

In each of the following cases the absorption spectra were calculated using the procedure outlined in section 2.2.

C.1.1 Maxwellian distributions with bulk fluid velocity

Two maxwellian distributions moving with bulk fluid velocities of $v_{f1}$ and $v_{f2}$ respectively, have the following distribution functions

$$f_1(v_1) = \left(\frac{m_1}{2\pi T_1}\right)^{\frac{3}{2}} \exp\left(-\frac{m_1}{2T_1} (v_1 - v_{f1})^2\right), \quad (C.1)$$

$$f_2(v_2) = \left(\frac{m_2}{2\pi T_2}\right)^{\frac{3}{2}} \exp\left(-\frac{m_2}{2T_2} (v_2 - v_{f2})^2\right). \quad (C.2)$$
The resulting absorption spectrum for particles in the distribution \( f_1 \) is

\[
R_{12}(v_1) = \frac{n_1 n_2}{1 + \delta_{12}} \left( \frac{m_1 m_2}{4\pi^2 T_1^2 T_2} \right)^{\frac{1}{2}} \frac{1}{|v_1 - v_{r2}|} \exp \left( -\frac{m_1}{2T_1} (v_1 - v_{r1})^2 \right) \int_0^\infty v_r^2 \sigma(v_r) \times \left[ \exp \left( -\frac{m_2}{2T_2} (|v_1 - v_{r2}| - v_r)^2 \right) - \exp \left( -\frac{m_2}{2T_2} (|v_1 - v_{r2}| + v_r)^2 \right) \right] dv_r d^3v_1. \quad (C.3)
\]

When \( v_{r1} = v_{r2} \), this spectrum is simply a shifting of the absorption spectrum for a stationary maxwellian given by (2.18). This expression can also give the absorption spectra for the beam-target model discussed in section 2.3.5. If \( v_{r1} = 0, v_{r2} \neq 0 \) then (C.3) is the absorption spectrum for particles in a thermal target with temperature \( T_1 \) reacting with a beam of particles with velocity \( v_{r2} \) and thermal spread \( T_2 \). If \( v_{r1} \neq 0, v_{r2} = 0 \) then we have the absorption spectrum for particles in the beam reacting with a stationary thermal target.

### C.1.2 Bi-maxwellian distribution

For a bi-maxwellian distribution in which both reactants have the same parallel and perpendicular temperature the reactant distributions are given by

\[
f_i(v_i) = \left( \frac{m_i}{2\pi} \right)^{\frac{3}{2}} \frac{1}{\sqrt{T_{\perp} T_{\parallel}}} \exp \left( -\frac{m_i}{2T_{\perp}} (v_{ix}^2 + v_{iy}^2) - \frac{m_i}{2T_{\parallel}} v_{iz}^2 \right), \quad (C.4)
\]

for \( i = 1, 2 \). The absorption spectrum for a bi-maxwellian distribution is

\[
R_{12}(v_1) = \frac{n_1 n_2}{1 + \delta_{12}} \frac{1}{\sqrt{T_{\perp} T_{\parallel}}} \left( \frac{m_1 m_2}{4\pi^2} \right)^{\frac{3}{4}} \exp \left( -\frac{m_1}{2T_{\perp}} (v_{ix}^2 + v_{iy}^2) - \frac{m_1}{2T_{\parallel}} v_{iz}^2 \right) \int_0^\infty v_r^3 \sin \theta_r \sigma(v_r) \exp \left( -\frac{m_2}{2} \left( \frac{(v_{1z} - v_r \cos \theta_r)^2}{T_{\parallel}} \right) \right) \times \exp \left( -\frac{m_2}{2} \left( \frac{(v_{1x} - v_r \sin \theta_r \cos \phi_r)^2 + (v_{1y} - v_r \sin \theta_r \sin \phi_r)^2}{T_{\perp}} \right) \right) d\phi_r d\theta_r dv_r d^3v_1, \quad (C.5)
\]

An analytic solution to the integral over \( d\phi_3 d\theta_3 \) is not known to exist and is best solved numerically using Lebedev quadrature [22].
C.2 Derivation of production spectrum for two temperature maxwellian distribution with cross-section that depends on scattering angle

In section 2.3.2 we derived the production spectrum for two maxwellian distributions of reactants of differing temperatures for a reaction cross-section that is independent of scattering angle. We now consider the case of a reaction cross-section that is a function of both the relative velocity of the reactants and the scattering angle. We follow the usual procedure for converting the independent variables of (2.9) from \((v_1, v_2)\) to \((v_{cm}, v_r)\) to give

\[
R_{12}(v_{cm}, v_r) = \Lambda v_r^3 \sin \theta_r \sigma(v_r, \theta_{cm}) \exp \left(-\alpha v_{cm}^2 - \beta v_r^2 - \mu \gamma v_{cm} \cdot v_r \right) d\phi_r d\theta_r dv_r d^3v_{cm}, \quad (C.6)
\]

with the constants \(\Lambda, \alpha, \beta\) and \(\gamma\) as defined in (2.46) and (2.48). We next convert \(v_r\) to spherical coordinates so that the magnitude of the vector, \(v_r\), is expressed independently of its direction

\[
R_{12}(v_{cm}, v_r) =
\Lambda v_r^3 \sin \theta_r \sigma(v_r, \theta_{cm}) \exp \left(-\alpha v_{cm}^2 - \beta v_r^2 - \mu \gamma v_{cm} \cdot \hat{v}_r \right) d\phi_r d\theta_r dv_r d^3v_{cm}. \quad (C.7)
\]

Here we use \(\hat{v}_r\) to denote a unit vector in the \(v_r\) direction. It has cartesian components \((\sin \theta_r \cos \phi_r, \sin \theta_r \sin \phi_r, \cos \theta_r)\). We can now transform \(v_r\) to \(u_3\), the CM frame velocity of the product particle using (2.33) and the Jacobian determinant given in (2.34) to get

\[
R_{12}(v_{cm}, u_3, \hat{v}_r) = \Lambda \frac{\eta}{\mu} u_3 \zeta^2 \sin \theta_r \times
\sigma(\zeta, \theta_{cm}) \exp \left(-\alpha v_{cm}^2 - \beta \zeta^2 - \mu \gamma v_{cm} \cdot \hat{v}_r \right) d\phi_r d\theta_r dv_r d^3v_{cm}. \quad (C.8)
\]

We next follow the usual procedure of vectorizing the CM frame velocity of the
product

\[ R_{12}(v_{\text{cm}}, u_3, \hat{v}_r) = \Lambda \frac{\eta}{\mu} u_3 \xi^2 \sin \theta_3 \sin \theta_r \times \]

\[ \sigma(\zeta, \theta_{\text{cm}}) \exp \left( -\alpha v_{\text{cm}}^2 - \beta \xi^2 - \mu \gamma \xi v_{\text{cm}} \cdot \hat{v}_r \right) d\phi_r d\theta_r d\phi_3 d\theta_3 d\phi_3 d\theta_3 d\phi_3 d\theta_3. \quad (C.9) \]

We can now transform from \( v_{\text{cm}} \) to \( v_3 \) using \( v_{\text{cm}} = v_3 - u_3 \) which results in

\[ R_{12}(v_3, u_3, \hat{v}_r) = \frac{\Lambda \eta}{4\pi \mu} u_3 \xi^2 \sin \theta_3 \sin \theta_r \sigma(\zeta, \theta_{\text{cm}}) \exp \left( -\alpha (v_3^2 + u_3^2) - \beta \xi^2 \right) \times \]

\[ \exp \left( (2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r) \cdot v_3 + \mu \gamma \xi u_3 \hat{u}_3 \cdot \hat{v}_r \right) d\phi_r d\theta_r d\phi_3 d\theta_3 d\phi_3 d\theta_3 d\phi_3 d\theta_3. \quad (C.10) \]

Here \( \hat{u}_3 \) denotes a unit vector in the \( u_3 \) direction with cartesian components \((\sin \theta_3 \cos \phi_3, \sin \theta_3 \sin \phi_3, \cos \theta_3)\) and, therefore, the scattering angle can be denoted by \( \theta_{\text{cm}} = \arccos(\hat{u}_3 \cdot \hat{v}_r) \). Clearly, in order to determine the production spectrum we need to integrate over (C.9) over \( d\phi_r d\theta_r d\phi_3 d\theta_3 d\phi_3 d\theta_3 \). However, this integration is much simplified if we note that the production spectrum is isotropic, depending on the magnitude of \( v_3 \) only, and not its direction. Therefore, we remove the angular variables for vector \( v_3 \) by integrating (C.10) over \( d\phi_3 d\theta_3 \) (where \((v_3, \theta_{v3}, \phi_{v3})\) denote the spherical coordinates of \( v_3 \))

\[ \int_0^\pi \int_0^{2\pi} \sin \theta_{v3} \exp \left( (2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r) \cdot v_3 \right) d\phi_{v3} d\theta_{v3} = 4\pi \frac{\sinh(v_3 |2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r|)}{v_3 |2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r|}. \quad (C.11) \]

The integral is solved using the identity given in B.2. Now,

\[ |2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r| = \sqrt{(2\alpha u_3 \hat{u}_3 - \mu \gamma \xi \hat{v}_r)^2} \]

\[ = \sqrt{4\alpha^2 u_3^2 + \mu^2 \gamma^2 \xi^2 - 4\alpha u_3 \mu \gamma \xi \cos \theta_{\text{cm}}}. \quad (C.12) \]
and so, after integration over $d\phi v_3 d\theta v_3$, (C.10) becomes

$$R_{12}(v_3, u_3, \hat{v}_r) = \Lambda \frac{\eta}{\mu} v_3 u_3 \zeta^2 \sin \theta_3 \sin \theta_r \sigma (\zeta, \theta_{cm}) \times \exp \left( -\alpha \left( v_3^2 + u_3^2 \right) - \beta \zeta^2 + \mu \gamma \zeta u_3 \cos \theta_{cm} \right) \times \sinh \left( \frac{2\sqrt{\alpha} v_3 \sqrt{\alpha u_3^2 + \frac{\mu^2 \gamma^2 \zeta^2}{4\alpha} - u_3 \mu \gamma \zeta \cos \theta_{cm}}}{2\sqrt{\alpha} \sqrt{\alpha u_3^2 + \frac{\mu^2 \gamma^2 \zeta^2}{4\alpha} - u_3 \mu \gamma \zeta \cos \theta_{cm}}} \right) d\phi_r d\theta_r d\phi_3 d\theta_3 du_3 dv_3. \quad (C.13)$$

Now, of the remaining integrations that need to be carried out in order to calculate the production spectrum we note that those over $d\phi_r d\theta_r d\phi_3 d\theta_3$ are simply integrals over the surfaces of two unit spheres, corresponding to $\hat{v}_r$ and $\hat{u}_3$, respectively. Therefore, applying the axes rotations of B.2 we can assume that $\theta_{cm}$ corresponds to the angle $\theta_r$ and, therefore, the integrals over $d\phi_r d\phi_3 d\theta_3$ are trivial, resulting in a factor of $8\pi^2$. Therefore, the production spectrum can be expressed now as a double integral over the cross-section variables

$$R_{12}(v_3) = 8\pi^2 \Lambda \frac{\eta}{\mu} \int_{v_3}^{\infty} \int_0^\pi v_3 u_3 \zeta^2 \sin \theta_{cm} \sigma (\zeta, \theta_{cm}) \times \exp \left( -\alpha \left( v_3^2 + u_3^2 \right) - \beta \zeta^2 + \mu \gamma \zeta u_3 \cos \theta_{cm} \right) \times \sinh \left( \frac{2\sqrt{\alpha} v_3 \sqrt{\alpha u_3^2 + \frac{\mu^2 \gamma^2 \zeta^2}{4\alpha} - u_3 \mu \gamma \zeta \cos \theta_{cm}}}{2\sqrt{\alpha} \sqrt{\alpha u_3^2 + \frac{\mu^2 \gamma^2 \zeta^2}{4\alpha} - u_3 \mu \gamma \zeta \cos \theta_{cm}}} \right) d\theta_{cm} du_3 dv_3. \quad (C.14)$$

We can now use the substitution $x = \sqrt{\alpha u_3^2 + \frac{\mu^2 \gamma^2 \zeta^2}{4\alpha} - u_3 \mu \gamma \zeta \cos \theta_{cm}}$ to obtain

$$R_{12}(v_3) = 4\pi^2 \Lambda \frac{\eta}{\gamma \mu^2 \sqrt{\alpha}} \int_{v_3}^{\infty} \int_0^\pi \zeta \exp \left( \left( \frac{\mu^2 \gamma^2}{4\alpha} - \beta \right) \zeta^2 \right) \int_a^b \sigma (\zeta, \theta_{cm}) \times \left[ \exp \left( -\left( x - v_3 \sqrt{\alpha} \right)^2 \right) - \exp \left( -\left( x + v_3 \sqrt{\alpha} \right)^2 \right) \right] dx du_3 dv_3, \quad (C.15)$$

where

$$\theta_{cm} = \arccos \left( \frac{4\alpha^2 u_3^2 + \mu^2 \gamma^2 \zeta^2 - 4\alpha x^2}{4\alpha u_3 \mu \gamma \zeta} \right), \quad (C.16)$$

and the limits of integration for $x$ are $a = \sqrt{\alpha u_3} - \mu \gamma \zeta / (2\sqrt{\alpha})$ and $b = \sqrt{\alpha u_3} + \mu \gamma \zeta / (2\sqrt{\alpha})$.
\( \mu \gamma \zeta (2\sqrt{\alpha}). \) This is the expression for the production spectrum for a plasma with maxwellian distributions of differing temperature. To find the intensity per unit solid angle of the production spectrum this expression needs to be divided by a factor of \( 4\pi \).

We can also find the production spectrum for a cross-section that depends on scattering angle for the case where the reactants have equal temperatures. With \( T_2 = T_1 = T \), we have \( \gamma = 0 \) and (C.14) becomes

\[
R_{12}(v_3) = 8\pi^2 \Lambda \eta \frac{\mu}{\sqrt{\alpha}} \int_{v_q}^\infty v_3 u_3 \zeta^2 \exp \left( -\alpha \left( v_3^2 + u_3^2 \right) - \beta \zeta^2 \right) \times \\
\frac{\sinh \left( v_3 \sqrt{4\alpha^2 u_3^2} \right)}{\sqrt{4\alpha^2 u_3^2}} \int_0^\pi \sin \theta_{cm} \sigma \left( \zeta, \theta_{cm} \right) d\theta_{cm} du_3 dv_3. \tag{C.17}
\]

Now since the scattering angle only appears in the differential cross-section we can integrate over this to get the total cross-section

\[
\int_0^\pi \sin \theta_{cm} \sigma \left( \zeta, \theta_{cm} \right) d\theta_{cm} = \sigma \left( \zeta \right), \tag{C.18}
\]

and so (C.17) becomes

\[
R_{12}(v_3) = 4\pi^2 \Lambda \eta \frac{\mu}{\sqrt{\alpha}} v_3 \int_{v_q}^\infty \zeta^2 \exp \left( -\beta \zeta^2 \right) \times \\
\sigma \left( \zeta \right) \left[ \exp \left( -\alpha \left( v_3 - u_3 \right)^2 \right) - \exp \left( -\alpha \left( v_3 + u_3 \right)^2 \right) \right] du_3 dv_3. \tag{C.19}
\]

This is equivalent to the production spectrum for when the cross-section does not depend on scattering angle, which is an interesting result.

\*We note that the term \( \mu^2 \gamma^2 / (4\alpha) - \beta \) contained in (C.15) can be reduced to the simpler expression \( -\frac{\gamma}{2} \left( m_1 m_2 / (m_1 T_2 + m_2 T_1) \right) \).
Appendix D

Multidimensional Dirac delta function

In this appendix we review the properties of the Dirac delta function that have been utilised in this thesis. Greater detail of the underlying theory may be found in [104] and [105].

D.1 The Dirac delta function for a single variable

The Dirac delta function is a type of generalised function. It may be viewed as an “infinitely thin” gaussian distribution with unit area under the curve given by the limit

\[
\lim_{n\to\infty} \frac{n}{\sqrt{\pi}} \exp \left( -n^2 x^2 \right). \tag{D.1}
\]

For a given “test” function \( f(x) \) it has the fundamental “sifting” property

\[
\int_{a-\epsilon}^{a+\epsilon} f(x) \delta(x-a) \, dx = f(a) \quad \forall \quad \epsilon \in (0, \infty). \tag{D.2}
\]

This relationship can be used to define the delta function. We consider the case where the argument of the delta function is a function \( g(x) \)

\[
\int_{-\infty}^{\infty} \delta(g(x)) \, dx = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -n^2 g(x)^2 \right) \, dx. \tag{D.3}
\]
Using the substitution \( u = g(x) \) we have

\[
\lim_{n \to \infty} \frac{n}{\sqrt{n}} \int_{-\infty}^{\infty} \frac{\exp(-n^2 u^2)}{|g'(x_i)|} du.
\]  

(D.4)

Therefore, we have

\[
\delta(g(x)) = \sum_i \delta(x - x_i) \frac{\delta(x - x_i)}{|g'(x_i)|},
\]  

(D.5)

where \( x_i \) are the solutions to \( g(x) = 0 \). Therefore, combining (D.2) and (D.5) gives

\[
\int_{-\infty}^{\infty} f(x) \delta(g(x)) \, dx = \sum_i \int_{-\infty}^{\infty} f(x) \frac{\delta(x - x_i)}{|g'(x_i)|} \, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}.
\]  

(D.6)

We can use (D.5) to establish an identity for the change of variable for integrals containing the Dirac delta function. Given a continuously differentiable function \( u = g(x) \)

\[
\int_{-\infty}^{\infty} f(u) \delta(u) \, du = \int_{-\infty}^{\infty} f(g(x)) \delta(g(x)) |g'(x)| \, dx = \sum_i \int_{-\infty}^{\infty} f(g(x)) \delta(x - x_i) \, dx.
\]  

(D.7)

\section*{D.2 The Dirac delta function for multiple variables}

The multi-dimensional Dirac delta function behaves in an analogous manner to the single variables case. It is defined by the sifting property

\[
\int \delta(r) f(r) dV = f(0),
\]  

(D.8)

where \( dV \) is a volume element in Cartesian coordinates. For more general coordinates systems we can use the following theorem. If \( g \) is a diffeomorphism (an invertible continuously differentiable function whose inverse is also continuously dif-
ferentiable) between open sets in $\mathbb{R}^n$ such that $y = g(x)$ then

$$\delta(g(x)) = \frac{\delta(x)}{|\det g'(x)|},$$

(D.9)

where $\det g'(x)$ is the determinant of the Jacobian matrix [106] (p. 136, Example 6.1.3). Therefore, we have the identity

$$\int_{\mathbb{R}^n} f(y) \delta(y) \, dy = \int_{g(\mathbb{R}^n)} f(x) \delta(x) \, dx.$$  

(D.10)

In practice (D.10) means that when converting expressions containing the Dirac delta function between different coordinate systems the Jacobian determinant arising from the transformation of the Dirac delta function cancels out that arising from the change of variables theorem thus ensuring that

$$\int_{-\infty}^{\infty} f(u) \delta(u) \, du,$$

(D.11)

is invariant. In particular, we can state that when converting from cartesian to spherical polar coordinates we have

$$\int \int \int f(x, y, z) \delta(x - x_1) \delta(y - y_1) \delta(z - z_1) \, dx \, dy \, dz = \int \int \int f(r, \theta, \phi) \delta(r - r_1) \delta(\theta - \theta_1) \delta(\phi - \phi_r) \, dr \, d\theta \, d\phi,$$

(D.12)

where $(x_1, y_1, z_1) = (r \sin \theta_1 \cos \phi_1, r \sin \theta_1 \sin \phi_1, r \cos \theta_1)$.

### D.3 Dirac delta functions on smooth manifolds

Given the continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ with $m \leq n$ the Dirac delta function can define a manifold in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f(x) \delta(g(x)) \, dx = \int_{g^{-1}(0)} \frac{f(x)}{|\det g'(x)|} \, d\sigma(x).$$

(D.13)
The integral on the right hand side is an $m$ dimensional integral over the manifold defined by $g(x) = 0$. 
Appendix E

An outline of the GORGON code

GORGON is a 3 dimensional resistive magnetohydrodynamic code [81], [82]. It models plasma as a single fluid but allows the ion and electron temperatures to differ. The equations for conservation of mass, momentum and energy that it solves are as follows

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \quad (E.1)
\]

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} = -\nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \nabla (p_i + p_e) + \mathbf{j} \times \mathbf{B}, \quad (E.2)
\]

\[
\frac{\partial U_i}{\partial t} = -\nabla \cdot (U_i \mathbf{v}) - p_i \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{q}_i + \Delta_{i\text{e}}, \quad (E.3)
\]

\[
\frac{\partial U_e}{\partial t} = -\nabla \cdot (U_e \mathbf{v}) - p_e \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{q}_e + \Delta_{e\text{i}} + \eta |\mathbf{j}|^2 - \Lambda. \quad (E.4)
\]

Here, \(\rho\) represents fluid density, \(\mathbf{v}\) is fluid velocity, \(\mathbf{j}\) is electron current and \(\mathbf{B}\) is magnetic field intensity. The terms \(U_{i,e}\) and \(p_{i,e}\) represent ion and electron internal energies and pressures, respectively. Internal energy and pressure are related by ideal gas equations of state. Bremsstrahlung power losses are denoted by \(\Lambda\) in the electron energy equation. The \(\eta |\mathbf{j}|^2\) term in this equation represents heating due to electrical resistance, where \(\eta\) is the resistivity. The ion and electron thermal fluxes are represented by

\[
\mathbf{q}_i = -\kappa_i \nabla T_i, \quad (E.5)
\]

\[
\mathbf{q}_e = -\kappa_e \nabla T_e, \quad (E.6)
\]
where $\kappa$ denotes conductivity. The transport coefficients $\eta$, $\kappa$ and the ion-electron energy exchange rate $\Delta_{ie} (= -\Delta_{ei})$ are those of Braginskii [107]. The electromagnetic equations solved by the code are the induction equation and Ampere’s law

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right), \quad (E.7)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (E.8)$$

where $\mu_0$ is the permeability of a free space. The electric and magnetic fields are related by Faraday’s law and Ohm’s law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (E.9)$$

$$\eta \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \quad (E.10)$$

These are combined to give the induction equation. The code is run on parallel processors using time-explicit solver routines.
Bibliography


