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Some Results on Sequential Spaces

Yoshio TANAKA*

Department of Mathematics

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Introduction

In this paper, we continue the work begun in [20] presenting some results on sequential spaces and also Fréchet spapces. Let X be a space. Then a subset A of X is called *sequentially closed* if and only if A contains all of its sequential limit points. Recall that a space X is a *sequential space* due to S.P. Franklin [4], if and only if every sequentially closed subset of X is closed. In [4], he proved that a sequential space is precisely a quotient image of a metric space.

Now unions of two sequential spaces need not be sequential even if under strong conditions. So, in section 1, we consider the sequentiality for unions $X = Y \cup Z$ with Y open in X. And we apply these results to compactifications of locally compact spaces. In [16] and [17], R. M. Stephenson, Jr. gave characterizations for spaces to be considered as closed subsets of all first countable spaces; and all symmetric space respectively. In section 2, we obtain some analogous results for sequential spaces and Fréchet spaces. Next, let X and Y be sequential spaces. Then not much can be said about the sequentiality of $X \times Y$ (cf. [4] [21] and [27]). In section 3, in terms of a certain set-theoretic axiom, we give a characterization for the product of sequential spaces X, Y to be sequential, where X and Y are closed images of some "nice" spaces. Finally, in view of products of sequential spaces, we consider the product of hereditarily isocompact spaces due to P. Bacon [2].

Throughout this paper, *spaces are assumed to be* T_1, *and also to be regular except section 2.*

1. Sequentiality for unions.

In general, a compact space which is a union of an open sequential subspace and a single point need not be sequential. Indeed, a compact linearly orderable space $[0, \omega_1]$ which is a union of an open first countable subspace $[0, \omega_1)$ and a single point ω_1 is not sequential. However, we have

Theorem 1.1. Let $X = Y \cup Z$ be a k-space with Y open or Z closed in X. Suppose that Y is sequential and Z is a sequential space in which every point is a G_δ-set. Then X is sequential if and only if every countably compact subset of Y is closed in X.

* Tokyo Gakugei University and Auburn University (1979–1980), U.S.A.

Proof. Since each countably compact subset of a sequential space is closed, the "only if" part is clear. So we prove the "if" part. First, we simplify the hypotheses step by step. If Y is open in X, then X – Y is closed in Z. Then X – Y is a sequential space in which every point is a G_δ-set. While, $X = Y \cup (X - Y)$ and X – Y is closed in X. Thus we assume that Z is closed in X and $Y \cap Z = \phi$. If Z is closed in X, since every open subset of a sequential space is sequential, similarly we may assume that Y is open in X with $Y \cap Z = \phi$. Thus, in any case, we can assume that Z is closed in X with $Y \cap Z = \phi$. Now, since X is a k-space, the proof will be completed, if we prove that every compact subset of X is sequential. Therefore, we can assume that $X = Y \cup Z$ is compact with Z closed and $Y \cap Z = \phi$. We remark that Z is first countable, because Z is a compact space each of whose points is G_δ. Let S be a quotient space obtained from X identifying the compact subset Z of X. Let $f: X \to S$ be a quotient map. Then f is a closed map such that each fiber is first countable. Thus, by [20; Theorem (A)], X is sequential if S is sequential. Hence we may assume that Z is a single point a. In the sequel, we can assume that $X = Y \cup \{a\}$ is a compact space and Y is a sequential space each of whose countably compact subsets is closed in X. Second, to prove X is sequentially compact, let $A = \{x_n; n \in N\}$ be an infinite subset of X with $x_n \neq a$. Let x be any accumulation point of A. If $x \in Y$, then $A - \{x\}$ is not closed in Y. Thus some subsequence of A converges to a point in Y. If A has no accumulation point in Y, then $\{a\} = \overline{A} - A$. Since A is countable and $\overline{A}$ is compact, the point a has a countable local base in $\overline{A}$. This implies that some subsequence of A converges to the point a. Therefore X is sequentially compact. Finally, to prove X is sequential, suppose F is a sequentially closed subset of X, but not closed in X. Then there exists a point $x_0 \in X$ with $x_0 \in \overline{F} - F$. If $x_0 \in Y$, then $x_0 \in \overline{F \cap Y}^{Y} - (F \cap Y)$. But $F \cap Y$ is sequentially closed in Y hence $F \cap Y$ is closed in Y. This is a contradiction. Thus $a = x_0 \notin F$. However, since X is sequentially compact, F is countably compact. Then $F \not\subset Y$ by the hypothesis, hence $a \in F$. This is a contradiction. Thus X is a sequential space.

A space X is called *isocompact* [2], if every closed countably compact subset of X is compact. All meta-compact spaces and all subparacompact spaces are isocompact.

Every countably compact subset of an isocompact sequential space is compact. Then, from Theorem 1.1 we have

Corollary 1.2. Let $X = Y \cup Z$ be a k-space with Y open or Z closed in X. Then X is sequential if Y is isocompact sequential and Z is first countable.

Recall that a space is *meta-Lindelöf* if every open covering has a point-countable open refinement.

Also recall that a space X has *countable tightness*, whenever if $x \in \overline{A}$ then $x \in \overline{D}$ for some countable $D \subset A$. It is well-known that every sequential space has countable tightness.

Let (MA) denote Martin's Axiom. Then we have

Theorem 1.3. ($2^{\omega} < 2^{\omega_1}$ or MA). Let $X = Y \cup Z$ be a k-space with Y open in X. Then X is sequential if Y is metal-Lindelöf sequential and Z is sequential.

Proof. As in the proof of Theorem 1.1, we can assume that X is compact and $Y \cap Z = \phi$. Let S be a quotient space obtained from X identifying Z. Thus, by Theorem 1.1, a compact space S is sequential. Hence S has countable tightness. Now, let $f:X \to S$ be a quotient map. Then f is a closed map such that each fiber has countable tightness. Thus, by [1, Theorem 6], X has countable tightness. This implies that $F \subset X$ is closed whenever $F \cap \bar{D}$ is closed in X for every countable subset D of X. Hence, to see X is sequential, it is sufficient to prove that the closure of each countable subset D of X is sequential. Put $\bar{D}' = \bar{D} \cap Y$. Since D′ is open in $\bar{D}$ and closed in Y, D′ is separable and meta-Lindelöf. Then D′ is Lindelöf. Thus, since D′ is locally compact, D′ is a σ-compact subset of Y. While, $\bar{D}$ is a union of a sequential space D′ and a closed sequential space $\bar{D} \cap Z$. Then the compact space $\bar{D}$ is a union of countably many compact sequential subspaces. Hence, under $2^{\omega} < 2^{\omega_1}$ or (MA), $\bar{D}$ is sequential by [15; Theorem 2]. Hence closure of each countable subset of X is sequential. This completes the proof.

Corollary 1.4. ($2^{\omega} < 2^{\omega_1}$) Let $X = Y \cup Z$ be a k-space such that Y is open sequential subspace and Y is sequential. Then X is sequential if Y is metacompact or normal subparacompact.

Proof. If Y is metacompact, this follows from Theorem 1.3. So we assume Y is normal subparacompact. By [10; Theorem 3], under $2^{\omega} < 2^{\omega_1}$, every separable normal space has a property that each uncountable subset has an accumulation point. While, every subparacompact space with this property is Lindelöf. Hence, if $2^{\omega} < 2^{\omega_1}$, every separable, normal subparacompact space is Lindelöf. Thus, by the proof of Theorem 1.3, X is sequential.

Recall that a space X is is *Fréchet* if, whenever $x \in \bar{A}$, then there exist $x_n \in A$ such that $x_n \to x$. Also a space X is a *k′-space* if, whenever $x \in \bar{A}$, then there exists a compact subset C of X such that $x \in \overline{C \cap A}$.

In general, a k-space X which is a union of open metric subspace and a single point need not be Fréchet (Cf. [5; Example 5.1]. However, if X is a k′-space, we have

Theorem 1.5. Let $X = Y \cup Z$ be a k′-space such that Y is open Fréchet subspace and Z is Fréchet. Then X is Fréchet if Y is meta-Lindelöf and each point of Z is a G_δ-set in Z.

Proof. Since X is k′, it suffices to prove that every compact subset of X is Fréchet. So, as in the proof of Theorem 1.1, we may assume that X is compact and $Y \cap Z = \phi$. Then, in view of the proof of Theorem 1.3, X has countable tightness, and for any countable subset D of X, $D'' = \bar{D} \cap Z$ is a G_δ-set in $\bar{D}$. Then, since each point of Z is a G_δ-set in Z, each point of D″ is a G_δ-set in $\bar{D}$. Since $\bar{D}$ is compact, each point of D″ has a countable local base in $\bar{D}$. While, $\bar{D} \cap Y$ is Fréchet and open in $\bar{D}$. Thus, that $\bar{D} = (\bar{D} \cap Y) \cup D''$ is Fréchet is straightforwards proved. Therefore D is Fréchet. Since X has countable tightness, by [12; Proposition 8.7], X is Fréchet.

By the following example, it is not easy to improve Theorem 1.5.

Example 1.6. Let Ψ be the Isbell-Mrówka space [6; 5I]. Here we shall describe the construction of Ψ. Let ξ be an infinite maximal pairwise almost disjoint collection of infinite subsets of natural numbers N and let $D = \{\omega_E ; E \in \xi\}$ be a new set of distinct points. Define $\Psi = D \cup N$ with each point of N isolated and neighborhoods of $\omega_E \in D$ these subsets of Ψ containing ω_E and all but finitely many points of E. Then Ψ is obviously a first countable, locally compact space. Moreover Ψ is isocompact. To prove this, let K be any countably compact subset of Ψ. Then, since D is a closed discrete subset of Ψ, $A = D \cap K$ is finite. So we may assume that $K = E_0 \cup A$, where E_0 is an infinite subset of N. If $E_0 \in \xi$, since K is countably compact, ω_{E_0} must belong to A. Thus K is compact. If $E_0 \notin \xi$, there exist $E_1 \in \xi$, such that $E_0 \cap E_1$ is infinite. Then $\omega_{E_1} \in A$. Thus it follows that K is compact if $E_0 - E_1$ is finite. So we can assume that $E_0 - E_1$ is infinite and belongs to ξ. But since A is finite, the progress must stop by finite step. This shows that Ψ is isocompact. Now, let Ψ^* be the one-point compactification of Ψ. Then Ψ^* is a compact sequential space, but it is not Fréchet [5; Example 7.1]. However Ψ^* is a union of an open subset Ψ, which is first countable isocompact, and a single point. We remark that Ψ^* is also a union of an open discrete subspace N and a compact, hereditarily paracompact Fréchet space $\Psi^* - N$.

Now, we apply some results to any compactification of a locally compact space. For the following theorem, we have (1) and (2) from Corollaries 1.2 and 1.4 respectively, and (3) from Theorem 1.5.

Theorem 1.7. Let C(X) be any compactification of a locally compact space X, and let $C^*(X) = C(X) - X$.

(1). Suppose $C^*(X)$ is first countable. Then C(X) is sequential if and only if X is isocompact sequential.

(2). ($2^{\omega} < 2^{\omega_1}$). Suppose that X is metacompact or normal subparacompact. Then C(X) is sequential if and only if so are X and $C^*(X)$.

(3). Suppose that X is meta-Lindelöf and $C^*(X)$ is first countable. Then C(X) is Fréchet if and only if so is X.

In view of the proof of [18; Theorem 15], we have the following.

Lemma 1.8. Let $X = Y \cup \{a\}$ be a countably compact space with Y normal Fréchet. Then X is Fréchet if and only if every countably compact subset of Y is closed in X.

From Theorem 1.7 and Lemma 1.8, we have

Corollary 1.9. Let X be a locally compact space, and let X^* be the one-point compactification of X.

(1). Then X^* is sequential if and only if X is isocompact sequential.

(2). If X is normal, then X^* is Fréchet if and only if X is isocompact Fréchet.

2. Sequential-closed spaces

Let P be a topological property. Following R.M. Stephenson, Jr. [17], a *(Hausdorff)* P-space is called *P-closed* if it is a closed subset of every *(Hausdorff)* P-space in which it can be embedded. In [16] and [17], he characterized Hausdorff first countable-closed spaces, and symmetrizable-closed spaces, etc. respectively.

In this section, we give analogous results for sequential-closed spaces and Fréchet-closed spaces.

Let X be a space. Then a subset G of X is called *sequentially open,* if $x_n \to x$ and $x \in G$, then G contains all but finitely many x_n's. Recall that a space X is sequential if and only if every sequentially open subset of X is open in X.

Proposition 2.1. Let X be a regular sequential space. Then X is a Hausdorff sequential-closed space if and only if X is countably compact.

Proof. The "if" part is easy, so we prove the "only if" part. Suppose X is not countably compact. Then there is a decreasing equence $\{F_n; n \in N\}$ of non-empty closed subsets of X such that $\cap F_n = \phi$. Let $p \notin X$ and let $Y = X \cup \{p\}$. To topologize Y, we refer [16; Lemma 2.10]. Put $\mathfrak{T}_Y = \{V; V \cap X$ is open in X, and if $V \ni p$ then V contains some $F_n\}$. Then it is easy to see that $(Y, \mathfrak{T}_Y)$ is a Hausdorff space. To prove Y is sequential, let G be not open in Y. If $G \not\ni p$, then G is not sequentially open in Y. So we assume $G \ni p$. If $G \cap X$ is not open, then G is not sequentially open in Y. If $G \cap X$ is open in X, then $G \not\supset F_n$ for all n. Thus there exist $x_n \in F_n - G$. Then $x_n \to p$, but $x_n \in G$. Hence G is not sequentially open. This shows that Y is sequential. Since X is sequential-closed, X is closed in Y. But X is a proper dense subset of Y. This is a contradiction. This completes the proof.

A space X is called *feebly compact* if every countable open filter base has an accumulation point in X. I do not know if every feebly compact subset of a Hausdorff Fréchet space is closed. However, in normal spaces, "feebly compact" is equivalent to "countably compact". So, using the concept of a feebly compact space, by the same way as in the proof of Proposition 2.1, we can prove the following.

Theorem 2.2. Let X be a normal Fréchet space. Then X is a normal Fréchet-closed space if and only if X is countably compact.

Since every regular space $Y \cup \{p\}$ is paracompact if Y is paracompact, then by Theorems 2.1 and 2.2 we have

Corollary 2.3. Let X be a Hausdorff paracompact sequential space (resp. Hausdorff paracompact Fréchet space). Then X is a Hausdorff paracompact sequential-closed space (resp. Hausdorff paracompact Fréchet-closed space) if and only if X is compact.

3. Products of sequential spaces

We shall begin with some definitions. Let X be a space. Then a decreasing sequence (A_n) in a space X is called a *k-sequence* [12], if $K = \cap A_n$ is compact and for every open subset V of X containing K, there exists A_n with $K \subseteq A_n \subseteq V$. According to E. Michael [12; Lemma 3.E.2], a space X is a *bi-k-space* if and only if for any filterbase $\mathfrak{T}$ accumulating at x in X, there exists a k-sequence (A_n) in X such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathfrak{T}$. It is known that every paracompact M-space, more generally every space of pointwise countable type is a bi-k-space.

Following [24], by $K(\omega)$ we mean the following property: For any k-sequence (A_n), some A_n is countably compact.

Lemma 3.1. Let $f_i: X_i \to Y_i$ $(i = 1,2)$ be closed maps such that X_1 is paracompact bi-k and Y_1 has countable tightness. If $Y_1 \times Y_2$ is a k-space, then Y_1 is bi-k or Y_2 has property $K(\omega)$.

Proof. From [24; Theorem 4.2], Y_2 has property $K(\omega)$, or Y_1 has the following condition: If (A_n) is a decreasing sequence with $\overline{A_n - \{y_1\}} \ni y_1$, then there exist $a_n \in A_n$ such that $\cup\{a_n;\ n \in N\}$ is not closed in Y_1. Thus, if Y_2 does not have property $K(\omega)$, Y_1 is bi-k by [12; Theorem 9.9 and Proposition 3.E.4].

A space Y is a *k_ω-space* [11], if it is the union of countably many compact subsets Y_n such that a subset A of Y is closed in Y whenever $A \cap Y_n$ is closed in Y for all n. We shall say that Y is a *locally k_ω-space,* if each point has a neighborhood whose closure is a k_ω-space.

The following is a generalization of [22; Theorem 4.3].

Lemma 3.2. Let $f_i:X_i \to Y_i$ $(i = 1.2)$ be closed maps such that each $\partial f_i^{-1}(y_i)$ is Lindelöf. Let X_i be paracompact bi-k spaces, and let Y_i have countable tightness. Then $Y_1 \times Y_2$ is a k-space if and only if one of the following properties holds.

(1). Y_1 or Y_2 is locally compact.

(2). Y_1 and Y_2 is locally k_ω-spaces.

(3). Y_1 and Y_2 are bi-k-spaces.

Proof. We have the "if" part from [3; 3,2], [11; (7.5) and 12; Proposition 3.E.4], respectively. So, we prove the "only if" part. From Lemma 3.1 one of the following properties holds.

(1). Y_1 or Y_2 is a bi-k-space with property $K(\omega)$.

(2). Y_1 and Y_2 have property $K(\omega)$.

(3). Y_1 and Y_2 are bi-k-spaces.

In case (1), by [24; Corollary 2.3] Y_1 or Y_2 is locally compact. In case (2), by [8; Lemma 2.8] Y_1 and Y_2 are locally k_ω. This completes the proof.

Let us assume α is an infinite cardinal. Then a space X is called *α-compact* if every subset of X of cardinality α has an accumulation point in X.

Next, by S_α, we denote the quotient space obtained from the disjoint union of α convergent sequences by identifying all the limit points.

Lemma 3.3. [26; Lemma 1.5]. Let $f:X \to Y$ be a closed map with X collectionwise normal and Y sequential. If Y contains no closed copy of S_α, then each $\partial f^{-1}(Y)$ is α-compact.

A space X is called *countably bi-sequential* [12; Lemma 4.D.2] if, whenever (A_n) is a decreasing sequence accumulating at $x \in X$, then there exist $x_n \in A_n$ such that $x_n \to x$. Such a space is also called strongly Fréchet.

As an application of Lemma 3.3, we have a generalization of [23; Theorem 2.2(2)].

Theorem 3.4. Let $f:X \to Y$ be a closed map with X collectionwise normal and Y sequential. Suppose that Z is Fréchet or a sequential space in which every point is G_δ. Then each $\partial f^{-1}(y)$ is 2^ω-compact or Z is countably bi-sequential if $Y \times Z$ is sequential, more generally it has countable tightness.

Proof. Suppose that some $\partial f^{-1}(y)$ is not c-compact, $c = 2^\omega$. Then, by Lemma 3.3, Y contains a closed copy of S_c. Then $S_c \times Z$ has countable tightness. If Z is Fréchet, by [26;

Corollary 1.3] Z is countably bi-sequential. So we assume that Z is a sequential space in which every point is G_δ. The proof will be completed if we prove Z is Fréchet. To see Z is Fréchet, suppose not. Following the proof of [5; Proposition 7.3], for $A \subseteq Z$, let A' be the set of all limit points of sequences in A. Since Z is not Fréchet, for some $B \subseteq Z$, $B' \neq \overline{B}$. Then $B' \neq \overline{B'}$, for $\overline{B'} = \overline{B}$. Since Z is sequential, there is a sequence $\{z_n; n \in N\}$ in B' converging to some $z_0 \notin B'$. We can assume that the z_n are distinct and $z_n \notin B$. Now, since z_0 is G_δ, there exist a decreasing sequence $\{V_n; n \in N\}$ of open subsets of Z with $\{z_0\} = \cap \overline{V}_n$, a subsequence $\{x_n; n \in N\}$ of $\{z_n: n \in N\}$, and also a sequence $\{x_{ni}; i \in N\}$ in B which converges to x_n such that $\{x_{ni}; i \in N\} \cup \{x_n\} \subseteq V_n - \overline{V}_{n+1}$ for each $n \in N$. Let $Z_0 = \{x_n; n \in N\} \cup \{x_{ni}; n,i \in N\} \cup \{z_0\}$. Then it is easy to see that Z_0 is closed in Z, hence is sequential. So, as is seen in the proof of [5; Proposition 7.3], $\sigma Z_0 = Z_0$ is the k_ω-space M of [5; Example 5.1], which is not locally compact. However, since $S_c \times Z$ has countable tightness, by [26; Proposition 1.1 (2)], each k_ω-subspace of Z is locally compact. This is a contradiction. Thus Z is Fréchet. This completes the proof.

In [25], under CH, we gave a characterization for the product of two closed images of metric spaces to be a k-space. Gary Gruenhage [7] proved this characterization (without CH) is equivalent to "the following set-theoretic axiom $BF(\omega_2)$ is *false*", which is weaker than CH.

$BF(\omega_2)$: If $F \subseteq \{f; f:N \to N$ is a function$\}$ has cardinality less than ω_2, then there is a function $g:N \to N$ such that $\{n \in N; f(n) > g(n)\}$ is finite for all $f \in F$.

Lemma 3.5. [7; Lemma 1]. $S_\omega \times S_{\omega_1}$ is not a k-space if and only if the set-theoretic axiom $BF(\omega_2)$ is false.

Now we are ready for the main result.

Theorem 3.6. The following are equivalent.

(a). $BF(\omega_2)$ is false.

(b). Let $f_i: X_i \to Y_i (i = 1,2)$ be closed maps such that X_i are paracompact bi-k and Y_i are sequential. Then $Y_1 \times Y_2$ is sequential if and only if one of the properties below holds.

(1). Y_1 or Y_2 is locally compact.

(2). Y_1 and Y_2 are locally k_ω.

(3). Y_1 and Y_2 are bi-k.

Proof. It is easy to see that S_ω is neither locally compact nor bi-k, and also S_{ω_1} is not locally k_ω. So, we have (b) $\Rightarrow$ (a) from lemma 3.5. Assume (a) holds. For the "if" part of (b), we do not use any axioms of set theory beyond ZFC. Indeed, $Y_1 \times Y_2$ is a k-space from the "if" part of Lemma 3.2. Thus, by [19; Theorem 2.2] $Y_1 \times Y_2$ is sequential. So, we need to prove the "only if" part of (b). If each $\partial f_1^{-1}(y_1)$ and each $\partial f_2^{-1}(y_2)$ are Lindelöf, then (1), (2) or (3) holds from Lemma 2.2. Thus we assume that some $\partial f_1^{-1}(y_1')$ is not Lindelöf. Since X_1 is paracompact, the subset $\partial f_1^{-1}(y_1')$ of X_1 is not ω_1-compact. Then, by Lemma 3.3, Y_1 contains a closed copy of S_{ω_1}. Since BF (ω_2) is false, by Lemma 3.5, Y_2 contains no closed copy of S_ω, for $S_{\omega_1} \times S_\omega$ is a k-space if Y_2 contains a closed copy of S_ω. Thus, by Lemma 3.3 each $\partial f_2^{-1}(y_2)$ is compact. Hence, by [12; Proposition 3.E.4], Y_2 is bi-k. On the other hand, since the subset $\partial f_1^{-1}(y_1')$ of X is not compact, by [12; Lemma 9.1 and Theorem

9.9], Y_1 is not bi-k. Hence, by Lemma 3.1, Y_2 must satisfy $k(\omega)$. Therefore, by [24, Corollary 2.3] Y_2 is locally compact.

Recall that a space X is of *pointwise countable type,* if each point of X has a k-sequence of open neighborhoods in X (equivalently, if each point of X is contained in a compact set of countable character in X).

It is easy to see that CH implies "$BF(\omega_2)$ is false." Then, from Theorem 3.6 we have the following, which is a generalization of [25; Theorem 1.1].

Corollary 3.7. [CH]. Let $f_i:X_i \to Y_i$ $(i = 1,2)$ be closed maps such that X_i are paracompact spaces of pointwise countable type and Y_i are sequential. Then $Y_1 \times Y_2$ is a sequential space if and only if one of the properties (1), (2) and (3) of Theorem 3.6 holds.

Remark 3.8. (1). As for the sequentiality for the product Y_2, without CH, we have

Proposition A. (Cf. [8; Theorem 2.11]). Let $f:X \to Y$ be a closed map such that X is paracompact bi-k and Y is sequential. Then Y^2 is sequential if and only if Y is bi-k or locally k_ω.

(2). As for the "Fréchetness" for the product $Y_1 \times Y_2$, we have the following from [13; Theorem 9.2] and [12; Theorems 3.D.3 and 9.9]. Also, cf. [12; Proposition 4.D.5].

Proposition B. Let $f_i:X_i \to Y_i (i = 1,2)$ be closed maps with X_i paracompact bi-sequential (for definition of bi-sequential spaces, see [12; Definition 3.D.1]). Then $Y_1 \times Y_2$ is Fréchet if and only if (1). Y_1 or Y_2 is discrete, or (2). Y_1 and Y_2 are bi-sequential.

In concluding this paper, we shall consider the product of hereditarily isocompact spaces. It is shown that [2; Theorem 2.12] the product of any collection of hereditarily isocompact spaces is isocompact. For the hereditarily isocompactness of products, we have the following in terms of the product of sequential spaces.

Theorem 3.9. ($2^\omega < 2^{\omega_1}$ or MA). Let $X_\gamma (\gamma \in \Gamma)$ be hereditarily isocompact spaces. Then ΠX_γ is hereditarily isocompact if and only if all but a countable number of spaces X_γ must be a single point. If each X_γ is especially hereditarily paracompact, then the assumption ($2^\omega < 2^{\omega_1}$ or MA) can be omitted.

Proof. If ΠX_γ is hereditarily isocompact, then each countably compact subset of ΠX_γ is closed in it. Thus, by [20; Lemma 2.1] all but a countable number of spaces X_γ must be a single point. So, we prove the "if" part. Let C be any countably compact subset of ΠX_γ ($\cong \prod_{i=1}^{\infty} X_{\gamma_i}$), and let $\Pi_i:\prod_{i=1}^{\infty} X_{\gamma_i} \to X_{\gamma_i}$ be projections. Then $X_i = \Pi_i(C)$ is a compact subspace of X_γ in which every countably compact subset is closed. Hence, by the main result of M. Ismail [9; Theorem 1.24] each X_i is sequential. Then by [14; Theorem 4.5] $\prod_{i=1}^{\infty} X_i$ is compact sequential. Since C is a countably compact subset of the sequential space $\prod_{i=1}^{\infty} X_i$, C is closed in $\prod_{i=1}^{\infty} X_i$. Hence C is compact. This shows that $\Pi X_\gamma (\cong \prod_{i=1}^{\infty} X_{\gamma_i})$ is hereditarily isocompact.

If each $X_\gamma (\gamma \in \Gamma)$ is hereditarily paracompact, then each X_γ is hereditarily normal and each countably compact subset of X_γ is closed in X_γ. Then, in view of [18; Theorem 15] we can see that every compact subset of X_γ is Fréchet, hence sequential. Thus, similarly ΠX_γ is hereditarily isocompact. That completes the proof.

Supplement. After preparing this paper, the author knew the following paper which has some relevance to section 1.

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