Embodied Conceptions of Mathematical Understanding in the Twentieth Century: the emergence of Zoltan P. Dienes's principles and their origin

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Keywords: Maverick Philosophy of Mathematics, Embodied Mathematics, concept formation, Zoltán Paul Dienes, Valéria Dienes, communication theoretic philosophy of mathematics, life-reform movements, Henri Bergson, Avant-garde, Constructivism

Abstract

Despite the growing interest in Embodied Mathematics, there is a dearth of historical studies investigating the emergence of thoughts that linked mathematical abstraction and human activity in the twentieth century. Comparatively little is known about what events of the past set the stage for this latest innovation in human-centered philosophies of mathematics: the study of mathematical understanding and meaning creation, having been seen as human activity, being the product of mathematical culture. Maverick Philosophies of Mathematics rarely enlist Z. P. Dienes among their forefathers although he was a legendary educator representing the 'maverick position' and a natural ally of the embodied conception. The paper explores the origin of Dienes's principles of learning and teaching mathematics in light of these two approaches.

Introduction

Zoltan Paul Dienes (1916-2014) is one of the earliest representatives of both the embodied conception of learning and teaching mathematics and the so called "maverick position" in the philosophy of mathematics. His efforts of changing the practice of mathematical education are based on an activity centered approach with a "maverick" philosophy of mathematics lurking in the background.[1] Dienes translates this approach plainly to the educational context with these words:

> Most of the time there is no substitute to doing mathematics. You learn mathematics not out of a textbook, but by engaging in mathematical activity. It is something like learning to swim or to skate or to ski. I doubt if many have learned these skills out of books! (Dienes, 1995 p. 14)

The view that mathematics is not a Platonic realm or a formal system but consists of human activities is expressed among others in (Pólya, 1945), (Lakatos, 1976), (Davis & Hersh, 1971), or (Hersh, 1997); its social aspects are discussed in (Ernest 1994, 1998) or (Löwe & Müller, 2010). Varieties of this non-homogeneous tradition and their theoretical background are reviewed in (Kitcher & Aspray, 1988), (Tall, 2004), (Buldt *et al.* 2008) or (Celluci, 2017).[2]

Both the representatives of the "embodied conceptions" of mathematics (Lakoff & Núñez, 2000) and of the "maverick position" see mathematics as the product of a cultural community and capture it from the point of view of human practice focusing on mathematical cognition, let it be educational, or research practice. The standard view in psychology of mathematics, however, is that the apparently new developments of Embodied Mathematics came about as the result of the cognitive psychological study of mathematical concept formation initiated by such psychologist as Piaget, Vygotsky or Bruner and the consequent line of research in cognitive-linguistics that paved the way for the embodied conceptions.

The first part of this paper amends this view exploring how the cognitive structure of mathematics came to the fore in Zoltan Paul Dienes's educational practice. Dienes's principles spell out how multiple embodiments through manipulatives contribute to tacit spatial-temporal embodied action and interaction leading to abstraction. They are followed by the cognitive linguistic aspects of meaning

[1] Teaching and learning aspects of mathematical practice are often neglected in the philosophy of mathematics in spite of the fact that Dienes's efforts for changing math education go along "hand in hand" with Pólya's mathematical heuristics. Cf. Benedek and Tuska (2018) forthcoming. For Dienes's own summary of his approach see (Tuska 2018) in this volume.

[2] The foundation of the *Association for the Philosophy of Mathematical Practice* in 2009, the PhiMSAMP workshps and the series of Cultures of Mathematics conferences (Bielefeld 2010, Greifswald 2011, Guangzhou 2012, Delhi 2015) including the 2017 "Enabling Mathematical Cultures" workshop held in Oxford, demonstrate a growing interdisciplinary interest and consensus in the philosophy of mathematical practice.

creation in terms of Embodied Mathematics. Tracing the origin of the crucial insights of these approaches, I interpret Dienes's principles in terms of his mother's, Valéria Dienes's communication theoretic conception of understanding that is rooted in Bergsonian process philosophy. Finally, I sketch the socio-cultural background of these momentous initiatives that are embedded into the life-reform, and artistic movements of early twentieth century European Avant-garde and Constructivism. Surprisingly, it turns out that these initiatives that addressed the study of conceptual development point towards a communication theoretic philosophy of mathematics paving the way for quasi-empirical theories of inter-subjective concept formation.

Dienes's Principles

As a widely known reference, Dienes's *Multibase Arithmetic Blocks* linked his accomplishments to the idea of "multiple embodiment". They put his name on the list of inventors of didactic mathematical tools, a list of names such as Maria Montessori (1870–1952), Cuisenaire (1891–1975), and Gattegno (1911–1988) including their precursors from pre-twentieth centuries. (Tuska, 2018)[3] These manipulatives that serve teaching arithmetic operations and place value in arbitrary base number systems (not just base 10 what happens to be the case most of the time) are only one of the many artifacts that Dienes used for structuring students' experience. Another well-known example is his set of *logic blocks* suitable to construct logic models, or functions as patterns of sequences. "Multiple embodiment" in these contexts usually means embodied *objects* of study that can be experienced in different forms and modalities. As teaching and learning tools, they play a crucial role in Dienes's didactic practice but it is important to record at the outset that Dienes did not mean the "multiple embodiment principle" as "embodiment" of some mathematics or mathematical concept in the form of physical objects.[4] He rather emphasized the cognitive process of the learning subject in directed learning situations what these tools augment by structuring and organizing the learners' experience. Spelling out the role of diverse manipulatives in this process (a conception that is sometime referred as the "Multiple Embodiment Principle")[5] he distinguished four main principles (1960, 1971):

(1) the *Constructivity Principle*
(2) the *Dynamic Principle*
(3) the *Perceptual Variability Principle* (or *Multiple Embodiment Principle*)
(4) the *Mathematical Variability Principle*

They are complemented by the *Function Principle* the *Interdisciplinary Principle* and the principles drawn from the nature of mathematics, i.e., *Abstraction*, *Generalization*, and the *Deep-end Principle* (see below).

Talking about "principles" it is reasonable to clarify what they are the principles of. By the chapter title of Dienes (1960) they are principles of "a theory of mathematics-learning", a formulation that prompted his readers and reviewers to put his works to the shelf of "learning theories" next to Piaget and Bruner. Their place was well selected considering that Dienes who had a degree in both psychology and mathematics contacted Piaget, built on his insights, worked with his colleagues at the *Institut Rousseau* in Geneva, and worked with Bruner at *Harvard*. He himself described his principles as a "skeleton" of a theory of learning mathematics but never devoted himself to elaborating a full-fledged *descriptive* psychological learning theory.[6] A plausible explanation for this is that he considered more important changing the way mathematics is taught than theorizing. He put his finger, however, on critical points of contemporary learning theories, contributed to them from his field of expertise, and spelled out his principles, among the firsts, as explanations of the growth of mathematical knowledge, more precisely, of

[3] Tuska (2018) provides a testimony about Polya who remarked that he used similar tools in Hungary when he was a student.

[4] We freely use the phrase that physical objects "have", or "represent" a structure, moreover, Dienes considered structures as the constitutive components of mathematical concepts. He was well aware, however, that the structures are not in the objects but "build up", as the title of his (1960) suggests, in the learner's mind. "There are no mathematical objects floating about in the real world of objects and events. There is no such thing as a »two« even though there are certainly two chairs or two children or two days." (Dienes 1995, p. 4)

[5] There are different versions in the literature concerning both the names and the order of Dienes's principles which are sometimes interpreted in overlapping terms.

[6] Learning theory, as we use the term today was his special field of interest intertwined with math education, a field that he embraced and contributed to. His position in the history of psychology and his role in the development of the psychology of mathematics deserve a separate study.

the development of children's mathematical cognition: the process of mathematical abstraction and generalization.[7] He underlined the presence and consideration of individual differences in this process but offered his principles as guiding theoretical insights based on the experiences of a practitioner of teaching and doing mathematics; "principles" that show the way how contentful abstract meaning can (and can be assisted to) emerge from structured human experiences. Nothing could be further from his democratic conception of teaching and learning than formulating norms, still, his principles were offered as theoretical considerations to be followed and checked in actual field work. Behind his principles of mathematical concept development there was a tremendous amount of experience and experiment that he cumulated from all over the world utilizing his "maverick" life-style. (Dienes 2003; Tuska 2018) He was well aware what kind of detailed psychological experiments were needed to justify a more complete theory of concept formation, especially in those days when empirical cognitive psychology was clearly in need of special "magnifying glasses" pointed to linguistic and mathematical abstraction. He himself participated in psychological experiments, initiated some of them, and contributed directly and indirectly by his materials to the work of the first mathematical laboratories. In this respect, he consistently followed his own principles: experimented in a "learning by doing" manner and relied on the variability of his wide spectrum of experiences obtained with an outstanding diversity of students. His principles may be utilized as independent heuristic devices for designing didactic situations and promoting joyful learning experiences and discoveries. They are proposed, however, as parts of an integral theory and their didactic order is not arbitrary.[8] They are based on the conviction that teaching should follow and augment the natural stages of children's conceptual development, a reason why we should explore and understand them.

Six stages of learning

Dienes extended Piaget's well known four stage process of conceptual development and restructured it in the form of a six stage process concerning the field of mathematical concept formation:

Stage 1) *Free Play*: a prerequisite, or starting point of higher abstractions obtaining concrete experiences about the environment, its objects and their relations, discovering *via* "trial and error" the particular situation

Stage 2) *Rule-based Games*: systematic discovery of regularities, rule-invention, learning to play by the rules, making distinctions between the initial state and the final state, the rules and conditions to be satisfied

Stage 3) *Comparartive Structuring*: discussion of the games, comparison of the rules, looking for communalities between rule-based game structures, disregarding from particular components. Searching for the "common core" of similar games as their structural (would be mathematical) content, introducing 'dictionaries' for common features

Stage 4) *Representation*: diagrammatic, visual or multi-modal expression of the abstracted features of the games extracting the essence of the communalities and mapping the rules and regularities of the concrete games to the representations

Stage 5) *Symbolization*: analysis of the representation, studying the gleanable properties of the games as classes of regularities, verbal description of the extracted rules introducing symbols for the 'map' that signify abstract components, checking the results of abstract rules in concrete games

Stage 6) *Formalization*: discovery of relations between the described and symbolized properties of the representation appending ways of deducing other properties, discovering descriptions that imply other descriptions and determining the rules of deduction. Making the first steps for selecting axioms, finding theorems and creating proofs.

He gave many concrete examples of these stages and of the role his principles play in them. (Dienes 1960, 2000) His slogan: "*Give me a mathematical structure and I'll turn it into a game*" expressed his explicit conviction that children's activities can be brought into play by means of inventing rules of mathematical games. Creating a "game" that matches the rules that are inherent in some piece of mathematics enhances the students' natural tendency for game based learning. His target groups (mostly

[7] The age of his the pupils he usualy worked with explains that he dealt much less with proof. He laid emphasis though on achieving the same results in different ways and on checking the results.

[8] I follow Dienes (1960) and (1995) with respect the order of the principles except the two variability principles that complement one another.

primary school aged pupils) required appropriate didactics but he proposed his principles as applicable for higher age groups as well because learning is a cyclical process that continues at higher levels of abstraction even if it involves other disciplinary and cognitive motivations.

The Function Principle

is related to the recognition that human activity exists in time and follows a temporal order, what gives the opportunity to solve the difficulty of "fitting mathematical hierarchies to children's developmental dynamics". In result of the problems of the set theoretic reform of the abstract "new mathematics" of the 1950ies and 60ies and the "natural revulsion on the part of the teachers against such programs" it was admitted that the psychological developmental hierarchies in no way correspond to the set theoretic construction of natural number, a "reminiscent of Bertrand Russell's Principia Mathematica, or of Hilbert's Grundlagen der Mathematics." (Dienes 1995, p. 3) It was discovered that

> [t]he concept of ordinal is formed much before that of cardinal. When a little child learns to count on his fingers, he will call his thumb "the one", the index finger is the "two", until he gets to the little finger which is "the five". It is the individual fingers that correspond to the numbers, not the "set of fingers" up to including the finger in question starting from the thumb! So the small child's one, two and three, is really a first, second and a third. (*Ibid.*)

> The realization that order came before amount in the child's experience, persuaded some of us to try to base mathematics learning not on the formalist conception due to Russel and Hilbert, but rather on considering "succession" as fundamental, and so taking the "laws of succession" and so the idea of the "successor function" as fundamental, upon which later concepts can be based. This gave rise to what came to known as the *functions principle.* (Dienes 1995, p. 4)

In chapter five of (1995) Dienes gives many examples of the application of the principle that gained extreme importance in functional languages such as LISP or LOGO; and he himself immediately used the latter to augment human instruction. Operations expressed by *succession* and *sequencing* of his logic blocks were used for the concrete construction of a function and he applied them in various lower forms of abstraction, for pattern recognition and generalization and showed how much can be achieved without introducing the idea of amount. He illustrated the different developmental stages and levels of abstraction as progressing from *implicit knowledge* towards *explicit knowledge*, and underlined his observation that many five year old children are able to play games based on their implicit knowledge of the rule of a sequential function, but they are unable to tell the rule. His examples showed how temporal sequences of events where each situation determines the next situation can be turned into spatial sequences (e.g. of logic blocks).

The Constructivity Principle

Learning begins with free interaction with the environment. It is, and it should be, situated. Spontaneous contact with concrete materials leads to new experiences constructed by the learner's own activities. This implies, especially for lower age groups, that informal play precedes rules. Active confrontation with the nature of objects and the environment gives the opportunity to perceive and memorize properties and relationships. These experiences may become consciously reflected.

> The most crucial stage is the one in which we realize that there is something common to a number of different experiences, and so these experiences belong together, they are placed in the same »mental box« for subsequent use. (Dienes 1995, p. 4)

These stored experiences will give the content, the psychological meaning of the "residual imagery", a "peg on which to hang the abstraction" (*ibid*, p. 5).[9]

> When we listen to someone speaking in a language we do not know, we only hear the sounds, but no images are being generated and we say we do not understand. This is what happens when we speak

[9] Dienes refers to a litarary analogy at this point, "if a child reads the sentence: »*The witch has taken the child to her lair*« some quite dramatic imagery is created at once. If this were not so, the authors of children's books would not sell a single copy." This is just one example of the paralells he drwas between the origin of the content of linguistic, artistic and mathematical expression.

> "mathematics" to a child, who has not been able to attach any psychological meaning to the noises that he hears from us, no imagery, residual or otherwise, is being evoked […] (*ibid*, p. 6)

The memories of the experiences of the child's activities using concrete material provide the base of the psychological meaning that "devolves in successive cycles" to higher abstractions if they are "devised to form a connection with the child's developmental process." (Dienes, 1971, p. 201)

The constructivity principle in Dienes's practice preceded the insight of *cognitive constructivism*[10] put in explicit form, e.g., by Anderson (1982, pp. 249-250):

> [N]ew knowledge is in large part "constructed" by the learner. Learners do not simply add new information to their store of knowledge. Instead, they must connect the new information to already established knowledge structures and construct new relationships among those structures. This process of building new relationships is essential to learning. It means that mathematical knowledge – both the procedural knowledge of how to carry out mathematical manipulations and the conceptual knowledge of mathematical concepts and relationships – is always at least partly "invented" by each individual learner.

For Dienes the most important aspect of this principle was that the content of elementary concepts, their preliminary residual memories were based on the learner's own *activity* requiring a considerable degree of freedom. The principle has two corollaries: 1) activity based construction of concepts emerges from concrete experience that precedes the formation of abstract concepts; 2) structuring activities by proper manipulatives in the environment makes conceptualization more effective because it utilizes the natural way of building cognitive structures. This insight, the starting point of *discovery learning*, was confirmed by Piaget's and Bruner's research. Piaget pointed out that children usually do not develop analytical thinking before age 12. Dienes observed, however, that there are children who are capable of analysis much sooner and distinguished "constructive" and "analytic" tendencies in his pupils' thinking. Accepting that constructive thinkers correspond to learners who develop from Piaget's "pre-operational" to "concrete operational state", he asked for, and provided, a distinctive definition for the formal operational level.

> I asked several of [Piaget's group of researchers] what it meant to be "operational". I was given a large number of answers, no two of them of the same. Then I posed the following question to Jean Piaget in person:
> "Is it so, Monsieur Piaget that a pre-operational child can operate on states, but is unable to operate on an operator to get another operator, whereas an operational child can also operate on an operator, without having to think of the intervening states?
> To my relief Jean Piaget agreed with my definition.

Accepting the *distinction between the pre-operational and the operational stages,* Dienes & Golding (1971) proposed the application of the

The Deep-end Principle

what stated that it was easier for the students to grasp the general concept first and concretize it later, and suggested to use this method for introducing concepts, in case of observed analytic tendencies. Dienes maintained this view even in (1995) emphasizing that concept formation remains activity based in both cases but connects to linguistic understanding in different ways.[11]

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[10] Cf. Sriraman, B., & Lesh, R. (2007). A conversation with Zoltan P. Dienes. *Mathematical thinking and learning*, 9(1), 59-75. "Sriraman: What are your thoughts on constructivism, which became popular when people in mathematics education started to rediscover Piaget and Vygotsky's writings in the 80's?
Dienes: My answer to that is simple really. These things were practiced in my classrooms long before people invented a word for it and I am sure there are people out there doing similar things for children. I do not care very much for isms, be it constructivism or behaviorism or any other ism. What really matters is that actual learning can take place with the proper use of materials, games, stories and such and that should be our focus."

[11] The relationship of linguistic and arithmetic concept formation is in the forefront of current research (see, e.g., Carey, 2011) but the relationship of concretization to generalization in arithmetic concept formation was addressed already by (Semadeni, 1984).

The Dynamic Principle

claims that the mental process of building mathematical structures consists of successive cyclic (or rather spiral) development that builds on lower levels of abstraction and repeats the same dynamic process. Informal play develops into structured play that reflects on concrete features and their relations discovering the followed rules, if it is supported by suitable (well prepared or carefully selected) environment with proper affordances[12]. Allied with directed game-based expectations this process leads to the recognition of rules. Experience in rule-following in different situations develops into an understanding of abstract rules applicable in similar games what continues in more sophisticated play. The reapplication of the same moves at higher levels leads to abstraction, extracting general concepts that supervene on the concrete games. The didactic assumption behind the principle is *game based instruction* that promotes the recognition of the *structure* of the game. Hence, the progressive nature of the dynamics depends and relies upon *reflection* on the structure of the games that serve as concrete *models* of abstract rules.[13]

The Perceptual Variability Principle

is often mentioned as the "Multiple Embodiment Principle". It is usually quoted as the variable use of diverse physical objects referring to the multiple physical "embodiment" of the "same" abstract properties in various objects, or to their relations in different objective or inter-subjective situations. It appears that the essence of the principle is the use of different objects, forms and physical installations (associated to the same the mental process). With respect to the mental development, the principle underlines the importance of the *repetition* of similar *activities* in varied physical circumstances with different objects in diverse architectures that are capable to exhibit the "same" properties. Consequently, it does not mean just the "embodiment" of the same structures in different materials. (Cf. fn. 4 above):

> Small children learn to tell one color from another, one shape from another shape […] This is *perceptual discrimination*. At a later stage, when they are able to abstract complex concepts, they learn *conceptual discrimination*. They can learn to tell one abstract structure from another, or tell when they are meeting the same structure dressed up a little differently. (Dienes 1995, p. 25)

One consequence of the Constructivity Principle is that the *preliminary experience of the pupils may be different*: it is dependent on their personal activities performed in former stages. Presenting a situation (having a structure that models a concept) in a variety of experiences helps to stabilize and contrast their essential features. Considering that Dienes promoted group work, it also harmonizes and brings to the same level the different approaches of the learners preparing common terminology that paves the way for common visual and symbolic representation. Given that their "residual images" and the bodily experiences of their personal activities differ, they project different memories to the same situation. Exposing them to similar experiences using a variety of materials means

> providing tasks which look quite different but have essentially the same conceptual structure [...] we can vary the perceptual representation, keeping the conceptual structure constant. For example, parallelograms can be drawn on paper, they can be made out of two congruent triangles, they can be traced out with pegs on a pegboard (Dienes, 1967, p. 13).

The point is that the learners must discover the common structure in different installations, so they develop a better understanding of the concept by making connections between these diverse experiences even though their memories and former activities have a *personal* character.

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[12] Taken together with the Constructivity Principle this is a common point of departure with M. Montessori.

[13] Dienes's descriptions of this dynamics lack the explicit reference to modeltheoretic considerations and do not use its terminology. They rather represent the same insights in a situational logic of game based activities.

The Mathematical Variability Principle

Meanwhile perceptual variability enhances the interiorization of relevant common features and promotes reflection on isomorphic structures;[14] mathematical variability uses systematic elimination of irrelevant properties and differences in order to contrast, as constant, the remaining structure as the essence of the general mathematical concept. The principle should be applied in *varied situations* (a difference compared with perceptual variability) methodically. The method consists of holding the relevant variables constant and systematically changing the *irrelevant* variables.

It is a recalcitrant problem of cognitive psychology that perceptual objectification disregards certain features of the objects (e. g., position, or intensity of color or shade) but distinguish others. It is a telling example of Dienes's early insights that he underlines the didactic importance of the variation of relevant and irrelevant attributes in two complementary didactic principles already in his (1960). In the 1950is and 60ies the problem of grading evidence was a central problem of inductive logic. By the 1970ies these investigations lead to the method of "relevant variables" exhibiting certain similarities with Dienes's principles.[15] The two problems in those days belonged, however, to different contexts: the first one to the constructive nature of perception, the second one to the context of inductive justification:

> "What is going on here [grading inductive support] is that the circumstances of the experiment are being varied in certain systematic ways in order to ascertain the effect of these variations on the outcome of the experiment" (Cohen, 1977 p. 130).

For Dienes the issue was guided concept formation not the justification of plausible hypotheses. Concept formation belonged for him, just as for Lakatos (1976) to the context of discovery. Induction was a heuristic tool for Pólya who demonstrated its importance in mathematical heuristics for higher age groups. Dienes introduced his principles at a more basic level, for lower age groups, insisting on the complementary application of his principles (cf. Dienes 1960, Ch. 2, p. 44; and 1963, Sections 4.1, 9.3). He approached them from the point of view of cognitive psychology as the grounding of meaningful perceptual concept formation that is based on the activity of the learning subject.

The Interdisciplinary Principle

extends multiple embodiment to other disciplines. The principle suggests using different "threads" of multimodal experiences that run through mathematics, language, music and bodily movement such as different type of jumps and hops. Such connecting threads are, for example, (i) the use of time, (ii) the use of form, (iii) the role of meaning, (iv) the use of symbols. (Dienes 1995, pp. 43-46)

Dienes gives examples of learning the modulo 5 system accompanying the count by bodily positions as it was used in Hungary for teaching music scales according to the Kodály method demonstrating that they can be used to associate numbers the modulo 5 as "belonging together" just as in case of the sounds of the transponed tunes of the music scale. Multiplication by 4 on the pentatonic scale changes each note into its "reflexion" with respect to tune D, and in the arm positions the reflexion is about the horizontally outstretched arms. (*Ibid*, pp. 9-12)

> Such extensions are not only valuable as aids to the abstraction process, but also help the learner realize that the mathematics he or she is learning turns up in all sorts of unexpected places, and so helps to give reason for studying mathematics. (*Ibid*, p. 9)

The principle is further elaborated in (Dienes 1973) linked to artistic threads. He also remarks that

> [t]he examples on interdisciplinary work to some extent illustrate the application of the function principle. The regular succession of rhythmical patterns in bars and musical phrases is a function in the sense of the cyclic recurrence of the pattern as the music unfolds. The relationship between the notes and the arm position is also a function: given a

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[14] A common special field of study both for Dienes and current embodied mathematics.

[15] See for example L.J. Cohen's analysis of Karl von Frisch's work on the behavior of bees (1950) in Cohen (1977, ch. 13, § 42-43)

> note the arm position is determined; given the arm position, the corresponding note is determined. (*Ibid,* p. 58)

Dienes's principles were tested in real life situations as well as in laboratory experiments. Recent research in cognitive psychology is revitalizing attempts to validate his principles experimentally. (Cf. e.g., Behr, 1976; Semadeni, 1984; Sriraman & English, 2005; Gningue, 2016; Sarı & Tertemiz, 2017).

Maverick Philosophy of Mathematics and Embodied Cognition

Before varieties of "Maverick Philosophy of Mathematics" appeared in the second half of the twentieth century, and recent theories of Embodied Cognition (EC) infested cognitive psychological praxis, most philosophers of mathematics were known to shy away from psychological explanations of what does "doing math" mean, just as from empirical studies of how mathematical ideas are actually created and understood. Mathematicians dealt with foundational issues and philosophers of mathematics with the onto-epistemological status of mathematical concepts.

Learning from experience was a central issue of post-war analytic philosophy of science while mathematics was the last bastille of aprioristic epistemology. The initially normative or critical, later only "evaluative" attitude of finding the right scientific methods (prior to grounding them in the empirical sciences which were supposed to use them), just gradually gave way to experimental research. Even in the field of linguistic analysis that traditionally belonged to the province of critical philosophy of language, descriptive epistemology was a novelty. Those who worked in the contexts of discovery and learning faced the charge of psychologism. The 1960ies brought considerable change in this respect due to various factors, mainly to changes in anthropology, cognitive psychology and philosophy of science. The Kuhnian revolution in philosophy of science was simultaneous with the Lakatosian criticism of traditional philosophies of mathematics. It took nearly another quarter of the century until Embodied Mathematics appeared on the map of the twentieth century conceptions of epistemology and philosophy of mathematics.

Embodied Conceptions of Mathematical Understanding

Although "Embodied Mathematics" (EM) is hallmarked by (Lakoff & Núñez, 2000): *Where Mathematics Comes From* addressing the issue of *How the Embodied Mind Brings Mathematics into Being*, a book that spurred immediate criticism, enthusiasm and controversies (Auslander, 2001; Goldin, 2001; Paulos, 2001; Henderson, 2002; Voorhees, 2004), it should be considered as a flower that grew out of a well prepared soil with a long history of cultivation.

The Emergence of Embodied Mathematics

Núñez et al. (1999) discussed Embodied Cognition (EC) "as grounding for situatedness and context in mathematics education" and extended theories of cognitive structures and processes to explaining how people (can) do mathematics. It was a special, but late contribution to the understanding of mathematical concept formation, a field that has been adversely affected by the unhealthy distance between math education and psychology. A telling symptom of this situation was the reaction of the participants of the Hamburg Conference at the UNESCO Institute for Education who scrutinized the Dienes compiled report of the *International Study Group for Learning Mathematics* for UNESCO (in 1966):

"It was pointed out that the findings of psychologists tend to be too vague, too general or insufficiently related to mathematical learning situations to be of very much use in influencing the form that mathematics teaching should take. Some participants had found that psychologists lacked the mathematical expertise to make a significant contribution to mathematics-teaching, and that the most useful contributions had come from mathematicians and practicising teachers. In defense of the inclusion of psychological material [into the report] the following points were made:

> (i) Although the theories of psychologists are in themselves insufficiently precise or specific, they provide a general framework within which particular solutions to the problems of mathematics learning may be found by those who are confronted with these problems.
>
> (ii) Whatever the degree of success that psychologists have enjoyed hitherto, mathematics-learning certainly brings with it problems of a psychological nature that need a closer examination than most teachers are equipped to provide.
>
> (iii) Perhaps, if mathematicians, teachers and psychologists were better acquainted with one another's disciplines, each kind of expert could contribute the better to the construction of theoretical models that would embody the wealth of all three disciplines." (p. iv)

Lakoff and Núñez (2000) approached mathematical conceptualization from the point of view of *linguistic* and *cognitive psychological* insights about the role of bodily experience in human cognition. Attempting to resolve the mystery of Platonic ideas in the theory of conceptual metaphors they not simply extended Lakoff's and Johnson's theories (1980) of linguistic understanding to mathematics. V*ia* the study of concept formation their work saturated the field of EM with insights about the relationship of our verbal cognitive capacities and the intuitive genesis of mathematical ideas with "grounding metaphors". The latter meant organizing a family of examples of

> "grounded inference-preserving cross-domain mapping — a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say arithmetic)". (Lakoff and Núñez, 2000: p. 26)

Their approach aimed at criticizing a kind of folk psychological and folk philosophical view of mathematics, a combination of attitudes of modern Mathematical Platonism *à la* Bernays (1935, cf. Voorhees, 2004) and Formalism, while they found natural allies in the "maverick philosophies of mathematics". By the time conceptions of EC "swept over the world" contributing to several factors (the "Sputnik effect", STEM reforms, etc.) that motivated the empirical study of mathematics education, the subfield of the psychology of mathematics education was well defined.[16] EM meant a new incentive for EC and *vice versa*, theories of EC stimulated empirical research concerning EM. Though the issue of "embodiment" faced questions about other, nonlinguistic, aspects of cognition at the time (Goldin 2001) Lakoff and Núñez (2000) preserved the essentially *cognitive linguistic* attitude of their approach, in spite of that, by Núñez (2008), they went beyond it involving other fields more decisively. In result of this, EM joined forces with the social aspects of human centered philosophies of mathematics (Ernest 1998, Tall 2004) that opened the way for a combination with some insights of *social constructivism.* Lakoff and Núñez, (2000) maintained, however, that mathematics "is grounded in bodily experience in the world... not purely subjective...", but it is "not a matter of mere social agreement..." (pp. 348-365). Núñez (2008, p. 336) endorsing the socio-cultural dimensions of mathematics defended "the idea that mathematics is *not just* the result of sociocultural practices."

With some retrospective simplification we can say that EM first underlined the experiential bases of everyday concept formation and showed what role expressions (colloquial and visual) play in mathematics, but as critics those days noted, it was able to say relatively little about the details of more complex, precise mathematical conceptualization (Voorhees, 2004). In the next decade EM moved in this direction: spelling out the details of the similarities of everyday, and mathematical abstraction (conceptualizations of time, the "container" notion of set, the meaning of axioms of geometry, etc.) applying "Mathematical Idea Analysis" (Núñez, 2008). Lakoff already in (1987, p. 364) stated that "mathematics is based on structures within the human conceptual system, structures that people used to comprehend ordinary experience".

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[16] We have the 42nd Conference and Annual Meeting of the International Group for the Psychology of Mathematics Education this year.

It can be argued that mathematical reasoning has no special facets; it uses no special faculties of the mind. One can even come to the far-fetched conclusion that most learning difficulties related to the acquisition of mathematical knowledge are the result of our way and style of teaching. It is plausible that human cognitive capacities change much more slowly than human cultures, including mathematics what has a life span only of several thousand years. This, latter point, however, is about phylogenetic *versus* ontogenetic human capacities disregarding communal, augmented communicative practices, (among others, ways of teaching) and their effects on our innate human capacities and their development.

The unfolding of EM showed that "embodied conceptions" require more careful formulation with respect to the stages of personal, historical and disciplinary development, their relationship and the levels of social, technological, and communal practice of doing mathematics:

> Embodiment theory offers an answer to the question of how meaning arises, and of how thought is related to action, emotion and perception. Embodiment theory proposes that meaning and cognition are deeply rooted in physical, embodied existence, on at least three levels: 1. Phylogenetic […] 2. Ontogenetic […] 3. Microgenetic […]. (Edwards, 2011, pp. 297-299)

The hallmark of EM turned out to be rather the message that mathematical understanding is based on a much wider (if not on the total) spectrum of our cognitive capacities and doing mathematics utilizes all our bodily and multimodal human experience.

It is easy to diagnose that EM, when it emerged at the turn of the century, paid relatively little attention to details of the *concrete process* of – personal and communal – *development* of mathematical abstraction, e.g., in Dienes's sense, including its developmental stages, or the didactic aspects of manipulative, game based learning. It was rather concerned with the *epistemology* of mathematics in terms of a theory of *meaning creation*. EM first intended to explain the origin of mathematics in the cognitive mechanisms that structure our embodied experience and gradually turned towards giving "an account of the peculiar collection of features that make mathematics so unique: precision, objectivity, rigor, generalizability, stability, and applicability to the real world." (Núñez, 2008, p. 336).

The theory of meaning EM is built upon was opposed to both the representationalist syntactic theories of the pre-WW1 and interwar period, the semantic tradition that already figured in Polish logic and linguistics before WW2 and grew into full blown model theory only after the war, just as to the denotational semantics of postwar linguistics and the information-processing view of the mind. It belonged, in many respect, to the historical lineage of continental epistemology with a special pedigree in the *phenomenological tradition* that is associated with such representatives of twentieth century philosophy as Bergson, Husserl, Merleau-Ponty or Derrida. Núñez et al (1999) recorded their historical position just before the turn of the century:

> When taken seriously, genuine embodiment entails a reconceptualization of the nature of cognition and of mathematics itself, with implications for teaching (Lakoff & Núñez, 1997). A first implication is that we must leave behind the myth of mind-free mathematics as being about eternal, timeless truths, a legacy of Plato and Descartes. From an embodied perspective, the notion of an objective mathematics, independent of human understanding no longer makes sense. […]

Before they quote Lave's and Wenger's claim that "there is no activity that is not situated" they give a summation of their

> "emphasis on comprehensive understanding involving the whole person rather than 'receiving' a body of factual knowledge about the world; on activity in and with the world; and on the view that agent, activity, and the world mutually constitute each other" (Lave & Wenger, 1991, p. 33).

The Origin of Embodied Conceptions

Bergson's main move against Cartesian dualism consists of rejecting the separation of the object of substantiality (*res extensa*) as opposed to the thinking subject (*res cogitans*). Merleau-Ponty's standpoint against Descartes' Cogito was also that "I am my body" and that the body has its own knowledge about the world. He built on Husserl's concept of the distinction of the *Leib* (the lived, animate body), and the

Körper (the inanimate, corporeal body as in "I have a body") and talked about the world which is given in perception as 'the concrete, inter-subjectively constituted lifeworld of immediate experience'. It is a world that is filled with familiar cultural and natural objects, the world of other people, the world in which the 'I' acts. This world of perception for both of them consists of all the natural objects and the 'world of culture'. Husserl's concept of lifeworld that he already used in *The Origin of Geometry* is a historical *and* structural account of 'geometrizing thought in practical activity.' Geometrical praxis is a cultural acquisition that has become a sedimented cultural practice rooted in intuition. It makes the geometrical context united with experiences of the practical and material world, a combination of two different praxes that are derived from one another.

In 1897, in his essay 'The Relativity of Space'. Poincaré argued that space is irreducibly relative to any position in which the 'I' would be situated and accounted for as a definite 'here'. That is, there is no absolute or fixed point in which the 'I' is situated in space. As Poincaré explained: "I am at a specific place on a certain day and I have to meet someone the next day at the same place, for example, at the Place de Panthéon, but this place will have rotated with the earth by the next day." Poincaré's point was the opening of the conflict of "subjective" phenomenological and the "objective" scientific accounts of space-time; a conflict that culminated in the famous Einsten-Bergson debate. (Canales, 2015)

Empirically supported theories of cognition, such as (Carey 2011) indicate that human reasoning, including mathematical problem solving, is based in tacit spatial-temporal *simulated action*. Recent implications of these findings for the philosophy and design of mathematical instruction originate in the above mentioned momentous initiatives. In case of Dienes's principles we can trace more personal links in this direction to those thinkers who, from the side of philosophy, influenced his thoughts.

Interpreting Dienes's principles in terms of his mother's communication theoretic conception of understanding

Valéria Dienes, mother of Zoltan P. Dienes and his brother Gedeon, a well-known dance theorist, studied in Paris and became a renowned student of Bergson in 1912 when the two competing views that set the stage for philosophical discussion about time and movement in the beginning of the century faced one another. She was a teacher and a philosopher, who first studied mathematics and music and later gave up her excellent scientific career for becoming a choreographer and dance teacher, the creator of the dance theory *orkesztika* (orchestics) and the founder of a dance school based on her theory. She was among the first females to graduate from a university in Hungary. Her university work focused on studying mathematics and physics. Lipót Fejér, the doyen of Hungarian mathematics of that period, was in love with Valéria. He happened to introduce Paul Dienes (Zoltan's father) to her, with the remark that Paul Dienes was a mathematical genius (Borus, 1978, p. 14) an introduction that led to their marriage.

Valéria began her doctoral studies of philosophy by listening to Bernát Alexander's lectures, the leading philosopher of Budapest those days, just like Pólya and many contemporary intellectuals did. She received her degree in the same ceremony as Paul Dienes in 1905 at Pázmány Péter University, Budapest. They exchanged engagement rings during that ceremony. Paul received his doctorate in mathematics. Valéria's doctorate was in philosophy as a major subject with a first minor in mathematics and a second minor in aesthetics. Her interest in aesthetics was based on her love of composing and playing music (Borus, 1978, pp. 15-18; Boreczky, 2013, p. 56). She continued her doctoral studies in Paris and saw Isadora Duncan's plays. Duncan was Paris' celebrity who brought free dance into Europe from the USA. It was an experience that turned out to make a lifelong influence on Valeria Dienes. Bergson's philosophy on time and movement met and melted in her consecutive accomplishments with Duncan's innovation of artistic expression that used the freely moving woman body for creating the tradition of twentieth century modern dance. Valeria Dienes's theoretical and practical work from that point on concentrated on dance interpretation and on the art of movement. (Boreczky, 2013) In her further works she elaborated the theory and practice of *orchestics* and developed, at same time as Laban invented his notations (Hutchinson 1970) the first symbolic language for the description of dance.

The analysis of the theoretical interplay between mother and her son deserves a separate study. It should be considered that after the Communist Republic of Hungary Valeria Dienes went to live in Nice, France, at a commune set up by Raymond Duncan, brother of dancer Isadora Duncan, while Zoltan's father became a professor of mathematics in England. The commune was a social life-reform experiment, where all the children were 'owned' in common (Dienes, 2003, p. 23). His mother eventually fled the commune with her children and made her way to England. Since the parents divorced, after a short return to Hungary, Zoltán lived with his father. They met even after WW2 but in result of the historical circumstances, Zoltan had relatively few contact with her mother (Dienes, 2003, p 97-98) although they corresponded even during the socialist period of Hungary. In spite of their physical distance one can detect striking similarities in their views about mathematics and about the expression and development of thought.

Picture 1. Zoltan and Valeria Dienes after WW2

Picture 2. Valéria Dienes as a young wife

Concerning the background of the Interdisciplinary Principle, just as the idea of multiple embodiment, there can be little doubt that Valeria Dienes influenced Zoltan's thought. The use of bodily movements, gestures, the analogies drawn between patterns, operations and transformations of melodies and mathematical structures the application and references to the Kodály method in (Dienes, 1995) all point to this direction. Valeria Dienes belonged to the same circles as Bartók and Kodály, and worked with their student, composer and choirmaster Lajos Bárdos (1899-1986). As a child, Zoltan certainly danced with other children in Raymond Duncan's commune where her mother took care of them already introducing plays and exercises into their life. Playful learning, a central method of Zoltan Dienes has the same central role in his teaching of mathematics as in the methodology of her mother's dance school. The symbolic language that she developed for coding choreographies is built on a geometric analysis of the human body and its affordances, just as Zoltán utilized them in his work with children. Time and temporal order plays a crucial role in the formulation of the Function Principle. Valeria Dienes got the right to translate Bergson's work into Hungarian, and translated *Time and Free Will: An Essay on the Immediate Data of Consciousness*. She wrote an excellent introductory study to the volume and became a determining interpreter of Bergson's work. The analysis of consciousness, just as the explanation of the relationship of Einstein's and Bergson's theories of time and motion show how competent a student and researcher she was. We may even find some cross-modulation between Zoltan's thoughts about multiple experiences and Bergson's theories of quantitative and qualitative multiplicity. Bergson's concept of duration contrasted with Einstein geometric representations of space-time also has some consequences for Zoltan Dienes's remarks about the primacy of students experience with bodily movements and sequentiality opposed to the early introduction of the geometric representation of the number line.

Perhaps the most interesting deep connection between mother's and son's thought is related to Valeria's work on *meaning creation* and her theory of *communicative understanding of dance*.

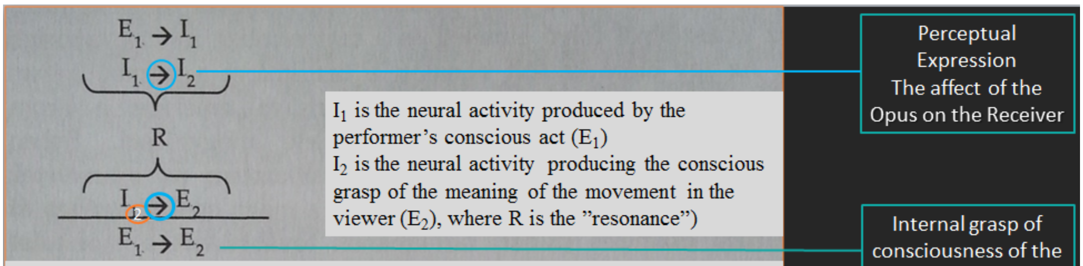

Fig. 1. The communicative way of meaning creation: how Consciousness-1 (E_1) creates Conciuosness-2 (E_2) in another person according to Valéria Dienes's *Ecologics*

It is in itself considerable a novelty of Valéria Dienes's theory of meaning (at the time when theories of meaning are busy with understanding Wittgenstein's *Tractatus*) that she does not talk about denotation or communicative transfer of "thoughts". The idea in a nutshell is that understanding emerges only from personal experience (a thought also present in Zoltan Dienes's conception.) We always have different personal ideas because understanding depends on our previous personal experiences. This crucial point is once again, a parallel with Zoltán's theory of the development of personal conceptual understanding.) The performer (in her case a dancer, but in Zoltan's case it can be another learner or a teacher) has a conscious thought, a mental act (E_1): the recognition of a relation or the "residual image" of an operation that he or she expresses by bodily movement, gestures or in some other form of neural activity, say oral, linguistic expression (I_1). (Words do not mean exactly the same for us for the same reason of having different personal experiences and memories as in Zoltan's approach.) The activity of the performer via Resonance (R) is perceived by the receiver and creates a neural activity in her (I_2) that creates the conscious grasp of meaning of the movement (of the performer) based on the receiver's former personal experiences and memories. So, Consciousness-1, that is the thinking process of person-1: (E_1) creates another Consciousness-2 (a thought in the receiver) that is based on her experiences (E_2). Resonance is meant partly as strictly physical resonance, (e.g. sound), but is also based on social and cultural developments that sequentialize the flow of physical events.

Translating all this to learning mathematics means that misunderstanding or missing certain meanings depend on the personal and cultural history of the participants of communication. If the student's personal history is missing certain components required to grasp (I_2) she will not understand the (full) meaning of (I_1). If her cultural background did not equip her with the ability to distinguish certain sequences she will not resonate to the movements, gestures or other expressions (say will not be able to distinguish the height of a triangle from its side.) There is no place here to spell out the consequences of Valéria Dienes's early communicative theory of meaning that also has an aesthetic interpretation, but reading Zoltan Dienes's works one can detect more and more signs that he clearly resonated to his mother' thoughts.

The socio-cultural background of life-reform, and artistic movements influencing the emergence of Dienes's principles

Valéria Dienes partially due to Bergson, partially due to the interest of young intellectuals also submerged in the "new science" of the turn of the century: psychology. Csaba Pléh actually describes her as a Bergsonian psychologist (Pléh, 1989):

> She had started her career as a characteristic Hungarian intellectual at the turn of century. […] She was part of those intellectual circles that were characterized with a combination of social responsibility, progressive social science and political reformism. She had lectures in the Galilei Circle, a Hungarian freethinker society, widely published in the review Huszadik Század (Twentieth Century) and the Galilei Füzetek (Galilei Monographs) the two leading organs of the new generation of social scientists.

They were characterized by

> [t]he combination of political, social and scientific progress, a sensitivity towards everything new in science and a passionate protest against all signs of oppression characterized this circle.

Hungary had an unusually vibrant intellectual and artistic life from the 1890's until the First World War and even in the beginning of the inter-war years. Not only the teachers of the famous "Fasori Gimnázium" (Mór Kármán, Manó Beke, László Rátz, and Gyula Kőnig, just to name a few key figures) that produced a series of "Martians" experimented with new methods, invented new approaches, (Szénássy, 1992, pp. 217-218) contemporary artists also belonged to the forefront of European art. The so called Hungarian Avant-garde meant that the leading artist were present in Vienna, Paris and Berlin and exhibited their work together with the top European impressionists, cubists and constructivists. László Mohole Nagy, Lajos Kassel and their circles created new interpretations of art. Constructivism that influenced different areas of art from music theory to painting and had a lasting effect even on postwar structuralism clearly influenced Zoltan Dienes's views. (Dienes, 1973). The emergence of this revolutionary culture determined the world view of Zoltán Dienes's parents. Valéria Dienes's work as indicated above went along with the contemporary development of the art of movement. Zoltán Dienes himself often refers to interdisciplinary connections between mathematics and the arts and used artistic methods and tools in his educational practice.

Conclusion

Zoltan Dienes played an important role in changing how mathematics was taught and learned in the twentieth century. As a representative of the maverick position in the philosophy of mathematics he drew many consequences in the field of education of the insight that mathematics is a human activity. He was faithful to his principles and considered doing mathematics and implementing his methods in order to change educational practice more important than spelling out the rich resources behind his theoretical insights and principles. It remains for us to reconstruct the historical origin of his methods but more importantly to reconstruct and revitalize them in everyday praxis. The reconstruction of the communication theoretic legacy of his mother in the context of mathematical education may shed new light to the way we think about "knowledge transfer" and may contribute to the implementation of her son's play-, and joyful attitude to communicating with children. The reconstruction requires separate studies dealing with different aspects of the mathematical, philosophical, artistic, and educational factors that played a role in the "magic" that the two Dienes, mother and son, was able to achieve in teaching as such. The present paper was only able to sketch the map that we should fill in with details.

Understanding is only a necessary condition of reconstruction. Reconstruction means construction in a different situation with the intention of preserving the "constructive" content and meaning of the legacy of our precursors. Being able to recreate what Dienes principles imply in practice, requires more than understanding his tricks: it calls for *doing* the magic!

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