

A VIETORIS-SMALE MAPPING THEOREM FOR THE HOMOTOPY OF HYPERDEFINABLE SETS

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ABSTRACT. Results of Smale (1957) and Dugundji (1969) allow to compare the homotopy groups of two topological spaces X and Y whenever a map $f : X \rightarrow Y$ with strong connectivity conditions on the fibers is given. We can apply similar techniques to compare the homotopy of spaces living in different categories, for instance an abelian variety over an algebraically closed field, and a real torus. More generally, working in o-minimal expansions of fields, we compare the o-minimal homotopy of a definable set X with the homotopy of some of its bounded hyperdefinable quotients X/E . Under suitable assumption, we show that $\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$ and $\dim(X) = \dim_{\mathbb{R}}(X/E)$. As a special case, given a definably compact group, we obtain a new proof of Pillay’s group conjecture “ $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ ” largely independent of the group structure of G . We also obtain different proofs of various comparison results between classical and o-minimal homotopy.

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1. INTRODUCTION

Let M be a sufficiently saturated o-minimal expansion of a field. We follow the usual convention in model theory [TZ12] to work in a sufficiently saturated structure, so we assume that M is κ -saturated and κ -strongly homogeneous for κ a sufficiently big uncountable cardinal (this can always be achieved going to an elementary extension). A set $X \subseteq M^k$ is **definable** if it is first-order definable with parameters from M , and it is **type-definable** if it is the intersection of a **small**

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family of definable sets, where “small” means “of cardinality $< \kappa$ ”. The dual notion of \forall -**definable** set is obtained by considering unions instead of intersections. The hypothesis that M has field operations ensures that every definable set can be definably triangulated [vdD98].

We recall that, given a definable group G , there is a normal type-definable subgroup G^{00} , called **infinitesimal subgroup**, such that G/G^{00} , with the **logic topology** [Pil04], is a real Lie group [BOPP05]. If in addition G is **definably compact** [PS99], we have $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ [HPP08], namely the o-minimal dimension of G equals the dimension of G/G^{00} as a real Lie group. These results were conjectured in [Pil04] and are still known as **Pillay’s conjectures**.

As an illustration, when G is an abelian variety over an algebraically closed field \mathbb{K} , we can view \mathbb{K} as $M[\sqrt{-1}]$ for some real closed subfield $M \subseteq \mathbb{K}$ and consider G as a definably compact group over M . In this case, assuming M sufficiently saturated, G/G^{00} is Lie isomorphic to a real torus $\mathbb{R}^n/\mathbb{Z}^n$ with n equal the dimension of G as a variety over M , namely twice its dimension over \mathbb{K} . Notice that the subgroup G^{00} will in general depend on the choice of $M \subseteq \mathbb{K}$, but G/G^{00} is in any case a real torus of dimension n .

It was later proved that if G is definably compact, then G is **compactly dominated** by G/G^{00} [HP11]. This means that for every definable subset D of G , the intersection $\mathfrak{p}(D) \cap \mathfrak{p}(D^c)$ has Haar measure zero (hence in particular it has empty interior) where $\mathfrak{p} : G \rightarrow G/G^{00}$ is the projection and D^c is the complement of D . Special cases were proved in [BO04] and [PP07].

The above results establish strong connections between definable groups and real Lie groups. The proofs are complex and based on a reduction to the abelian and semisimple cases, with the abelian case depending in turn on the study of the fundamental group and on the counting of torsion points [EO04]. A series of results of P. Simon [Sim15, Sim14, Sim13] provides however a new proof of compact domination which does not rely on Pillay’s conjectures or the results of [EO04]. More precisely, [Sim14] shows that fsg groups in o-minimal theories admit a smooth left-invariant measure, and [Sim15] contains a proof of compact domination for definable groups admitting a smooth measure (even in a NIP context). The fact that definably compact groups in o-minimal structures are fsg is proved in [HPP08, Thm. 8.1].

Our main theorem sheds new light on the connections between compact domination and Pillay’s conjectures, and concerns the topology of certain hyperdefinable sets X/E , where E is a bounded type-definable equivalence relation on a definable set X . Under a suitable contractibility assumption on the fibers of $\mathfrak{p} : X \rightarrow X/E$ (12.1), we obtain a homotopy comparison result between X and X/E , and in particular an isomorphism of homotopy groups

$$\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$$

in the respective categories. Similar results apply locally, namely replacing X/E with an open subset $U \subseteq X/E$ and X with its preimage $\mathfrak{p}^{-1}(U) \subseteq X$, thus obtaining

$$\pi_n^{\text{def}}(\mathfrak{p}^{-1}(U)) \cong \pi_n(U).$$

For the full result see Theorem 11.8 and Theorem 12.2.

From these local results and a form of “topological compact domination” (13.2) we shall deduce that

$$\dim(X) = \dim_{\mathbb{R}}(X/E),$$

namely the dimension of X in the definable category equals the dimension of X/E in the topological category (Theorem 13.3). This yields a new proof of “ $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ ” for compactly dominated groups which does not depend on the counting of torsion points (for in fact it does not depend on the group structure!).

Some comparison results between classical and o-minimal homotopy established in [DK85, BO02, BO09] also follow (see Corollary 12.3). In particular, if $X = X(M) \subseteq M^k$ is a closed and bounded \emptyset -semialgebraic set and $\text{st} : X \rightarrow X(\mathbb{R})$ is the standard part map, we can take $E = \ker(\text{st})$ and deduce

$$\pi_n^{\text{def}}(X) \cong \pi_n(X(\mathbb{R})).$$

This work can be considered as a continuation of the line of research initiated [BM11]: while in that paper we focused on the fundamental group, here we manage to encompass the higher homotopy groups and more generally homotopy classes $[X, Y]$ of maps $f : X \rightarrow Y$ in the relevant categories.

We have tried to make this paper as self-contained as possible. The proofs of the homotopy results are somewhat long but elementary and all the relevant notions are recalled as needed. The paper is organized as follows.

In Section 2 we recall the notions of definable space and definable manifold, the main example being a definable group G .

In Section 3 we introduce the logic topology on the quotient X/E of a definable set X by a bounded type-definable equivalence relation E .

In Section 4 we recall the notion of “normal triangulation” due to Baro [Bar10], and we show how to produce normal triangulations satisfying some additional properties.

In sections 5 and 6 we illustrate some of the analogies between the standard part map and the map $G \rightarrow G/G^{00}$, where G is a definably compact group and G/G^{00} has the logic topology.

These analogies are further developed in Section 7, where we discuss various versions of “compact domination”.

In Sections 8,9 we work in the category of classical topological spaces and we establish a few results for which we could not find a suitable reference. In particular in Section 8 we show that given an open subset U of a triangulable space, any open covering of U has a refinement which is a good cover.

In Section 10 we recall the definition of definable homotopy.

Sections 11,12 and 13 contain the main results of the paper, labeled Theorem A (11.8), Theorem B (12.2), and Theorem C (13.3), respectively, as the titles of the corresponding sections.

In Theorem A we prove that there is a homomorphism $\pi_n^{\text{def}}(X) \rightarrow \pi_n(X/E)$ from the definable homotopy groups of X and the homotopy groups of X/E , under a suitable assumption on E . We actually obtain a more general result of which this is a special case.

In Theorem B we strengthen the assumptions to obtain an isomorphism: $\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$. Since the standard part map can be put in the form $\mathfrak{p} : X \rightarrow X/E$ for a suitable E , some known comparison results between classical and o-minimal homotopy will follow.

Finally, in Theorem C we add the assumption of “topological compact domination” to obtain $\dim(X) = \dim_{\mathbb{R}}(X/E)$ and we deduce $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ and some related results.

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2. DEFINABLE SPACES

A fundamental result in [Pil88] establishes that every definable group G in M has a unique group topology, called **t-topology**, making it into a **definable manifold**. This means that G has a finite cover U_1, \dots, U_m by t-open sets and for each $i \leq m$ there is a definable homeomorphism $g_i : U_i \rightarrow U'_i$ where U'_i is an open subset of some cartesian power M^k with the topology induced by the order of M . The collection $(g_i : U_i \rightarrow X_i)_{i \leq m}$ is called an atlas and g_i is called a local chart.

Definable manifolds are special cases of **definable spaces** [vdD98]. The notion of definable spaces is defined through local charts $g_i : U_i \rightarrow U'_i$, like definable manifolds, with the difference that now U'_i is an arbitrary definable subset of M^k , not necessarily open. In particular every definable subset X of M^k , with the topology induced by the order, is a definable space (with the trivial atlas consisting of a single local chart), but not necessarily a definable manifold.

Remark 2.1. It should be stressed that all definable spaces we consider in this paper are also definable sets (e.g. a definable group), so the notion of definable subset of a definable space does not depend on the choice of the charts, only the topology does.

We collect in this section a few results on definable spaces which will be needed in the sequel. They depend on the saturation assumptions on M . The results are easy and well known to the experts but the proofs are somewhat dispersed in the literature.

Lemma 2.2. *Let $(A_i : i \in I)$ be a small downward directed family of definable open subsets of a definable space X (where “small” means $|I| < \kappa$). Then $\bigcap_{i \in I} A_i$ is open.*

Proof. Let $x \in \bigcap_{i \in I} A_i$ and fix a definable fundamental family $(B_\varepsilon : \varepsilon > 0)$ of neighbourhoods of x decreasing with ε (for example take B_ε to be the points of X at distance $< \varepsilon$ from x in a local chart). Since A_i is open in X , there is $\varepsilon_i > 0$ such that $B_{\varepsilon_i} \subseteq A_i$. By saturation, we can find an $\varepsilon > 0$ in M such that $\varepsilon < \varepsilon_i$ for each $i \in I$. It follows that $B_\varepsilon \subseteq \bigcap_{i \in I} A_i$, so x is in the interior of the intersection. \square

Lemma 2.3. *Let $(X_i : i \in I)$ be a small downward directed family of definable subsets of a definable space. Then $\overline{\bigcap_{i \in I} X_i} = \bigcap_{i \in I} \overline{X_i}$.*

Proof. The inclusion “ \subseteq ” is trivial. For the “ \supseteq ” direction let $x \in \bigcap_{i \in I} \overline{X_i}$ and suppose for a contradiction that $x \notin \overline{\bigcap_{i \in I} X_i}$. Then there is an open neighbourhood U of x disjoint from $\bigcap_{i \in I} X_i$. By saturation there is $i \in I$ such that U is disjoint from X_i , hence $x \notin \overline{X_i}$, a contradiction. \square

Lemma 2.4. *Let $(X_i : i \in I)$ be a small downward directed family of definable subsets of the definable space X . Suppose that $H := \bigcap_{i \in I} X_i$ is clopen. Then for every $i \in I$ there is $j \in I$ such that $\overline{X_j} \subseteq \text{int}(X_i)$.*

Proof. Fix $i \in I$. Since H is open, $H \subseteq \text{int}(X_i)$. Using the fact that H is also closed, we have $H = \overline{H} = \overline{\bigcap_{i \in I} X_i} = \bigcap_{i \in I} \overline{X_i}$ (by Lemma 2.3). The latter intersection is included in $\text{int}(X_i)$, hence by saturation there is $j \in I$ such that $\overline{X_j} \subseteq \text{int}(X_i)$. \square

3. LOGIC TOPOLOGY

Let X be a definable set and consider a type-definable equivalence relation $E \subseteq X \times X$ of **bounded index** (namely of index $< \kappa$) and let $\mathfrak{p} : X \rightarrow X/E$ be the natural map. We put on X/E the **logic topology**: a subset $O \subseteq X/E$ is open if and only if its preimage in X is \forall -definable, or equivalently $C \subseteq X/E$ is closed if

and only if its preimage in X is type-definable. This makes X/E into a compact Hausdorff space [Pil04]. We collect here a few basic results, including some results from [Pil04, BOPP05], which will be needed later.

Proposition 3.1. *For every definable set $C \subseteq X$, $\mathfrak{p}(C)$ is closed in X/E .*

Proof. By definition of logic topology, we need to show that $\mathfrak{p}^{-1}(\mathfrak{p}(C))$ is type-definable. By definition, x belongs to $\mathfrak{p}^{-1}(\mathfrak{p}(C))$ if and only if $\exists y \in C : xEy$. Since E is type-definable, xEy is equivalent to a possibly infinite conjunction $\bigwedge_{i \in I} \varphi_i(x, y)$ of formulas over some small index set I , and we can assume that every finite conjunction of the formulas φ_i is implied by a single φ_i . By saturation it follows that we can exchange \exists and \bigwedge_i , hence $\mathfrak{p}^{-1}(\mathfrak{p}(C)) = \{x : \bigwedge_i \exists y \varphi_i(x, y)\}$, a type-definable set. \square

Proposition 3.2. *Let $C \subseteq U \subseteq X/E$ with U open and C closed. Then there is a definable set $D \subseteq X$ such that $\mathfrak{p}^{-1}(C) \subseteq D \subseteq \mathfrak{p}^{-1}(U)$.*

Proof. This is an immediate consequence of the fact that if a type-definable set is contained in a \bigvee -definable set, there is a definable set between them. \square

Proposition 3.3. *Let $y \in X/E$ and let $D \subseteq X$ be a definable set containing $\mathfrak{p}^{-1}(y)$. Then y is in the interior of $\mathfrak{p}(D)$. Moreover, there is an open neighbourhood U of y such that $\mathfrak{p}^{-1}(U) \subseteq D$.*

Proof. By Proposition 3.1, $Z = \mathfrak{p}(X \setminus D)$ is a closed set in X/E which does not contain y . Hence the complement O of Z is an open neighbourhood of y contained in $\mathfrak{p}(D)$. We have thus proved the first part. For the second part note that, since X/E is compact Hausdorff, it is in particular a normal topological space. We can thus find a fundamental system of open neighbourhoods U_i of y such that $\{y\} = \bigcap_i U_i = \bigcap_i \overline{U}_i$. Each $\mathfrak{p}^{-1}(\overline{U}_i)$ is type-definable, and their intersections is contained in the definable set D , so there is some $i \in \mathbb{N}$ such that $\mathfrak{p}^{-1}(\overline{U}_i) \subseteq D$. \square

For our last set of propositions we assume that X is a definable space, possibly with a topology different from the one inherited from its inclusion in M^k .

Proposition 3.4. *Assume that X is a definable space and each fiber of $\mathfrak{p} : X \rightarrow X/E$ is a downward directed intersection of definable open subsets of X . Then $\mathfrak{p} : X \rightarrow X/E$ is continuous.*

Proof. By Lemma 2.2 the preimage of any point is open, hence the preimage of every set is open. \square

Proposition 3.5. *Assume $\mathfrak{p} : X \rightarrow X/E$ is continuous and let C be a definable subset of X . Then $\mathfrak{p}(\overline{C}) = \overline{\mathfrak{p}(C)}$.*

Proof. It suffices to observe that $\mathfrak{p}(\overline{C}) \subseteq \overline{\mathfrak{p}(C)} \subseteq \mathfrak{p}(C)$ where the first inclusion holds because \mathfrak{p} is continuous and the second by Proposition 3.1. \square

4. TRIANGULATION THEOREMS

The triangulation theorem [vdD98] is a powerful tool in the study of o-minimal structures expanding a field. In this section we review some of the relevant results and we prove a specific variation of the normal triangulation theorem of [Bar10] for simplexes with real algebraic vertices.

Simplicial complexes are defined as in [vdD98]. They differ from the classical notion because simplexes are open, in the sense that they do not include their faces. As in [vdD98], the vertices of a simplicial complex are concrete points, namely they have coordinates in the given o-minimal structure M (expanding a field). More

precisely, given $n + 1$ affinely independent points $a_0, \dots, a_n \in M^k$, the (open) n -**simplex** $\sigma_M = (a_0, \dots, a_n) \subseteq M^k$ determined by a_0, \dots, a_n is the set of all linear combinations $\sum_{i=0}^n \lambda_i a_i$ with $\lambda_0 + \dots + \lambda_n = 1$ and $0 < \lambda_i < 1$ (with $\lambda_i \in M$). If we go to a bigger model $N \succeq M$, we write σ_N for the set defined by the same formulas but with λ_i ranging in N . We omit the subscript if there is no risk of ambiguity. A **closed simplex** is defined similarly but with the weak inequalities $0 \leq \lambda_i \leq 1$. In other words a closed simplex is the closure $\bar{\sigma} = \text{cl}(\sigma)$ of a simplex σ , namely the union of a simplex and all its faces.

A **simplicial complex** is a finite collection P of (open) simplexes, with the property that for all $\sigma, \theta \in P$, $\bar{\sigma} \cap \bar{\theta}$ is either empty or the closure of some $\delta \in P$ (a common face of the two simplexes). We shall say that P is a **closed simplicial complex** if whenever it contains a simplex it contains all its faces. In this case we write \bar{P} for the collection of all closures $\bar{\sigma}$ of simplexes σ of P and we call $\bar{\sigma}$ a closed simplex of P .

The **geometrical realization** $|P|$ of a simplicial complex P is the union of its simplexes.

We shall often assume that P is defined over \mathbb{R}^{alg} , namely its vertices have real algebraic coordinates, so that we can realize P either in M or in \mathbb{R} . In this case, we write $|P|_M$ or $|P|_{\mathbb{R}}$ for the geometric realization of P in M or \mathbb{R} respectively. Notice that a simplicial complex is closed if and only if its geometrical realization is closed in the topology induced by the order of M .

If $L \subseteq P$ is a subcomplex of P , and $\sigma \in P$, we define $|\sigma|_{\mathbb{R}} = \sigma \cap |L|_{\mathbb{R}}$ and $|\sigma|_M = \sigma \cap |L|_M$. To keep the notation uncluttered, we simply write $\sigma|_L$ when the model is clear from the context.

Definition 4.1. A triangulation of a definable set $X \subseteq M^m$ is a pair (P, ϕ) consisting of a simplicial complex P defined over M and a definable homeomorphism $\phi : |P|_M \rightarrow X$. We say that the triangulation ϕ is compatible with a subset S of X if S is the union of the images of some of the simplexes of P .

Fact 4.2 (o-minimal triangulation theorem [vdD98]). *Every definable set $X \subseteq M^m$ can be triangulated. Moreover, if S_1, \dots, S_l are finitely many definable subsets of X , there is a triangulation $\phi : |P|_M \rightarrow X$ compatible with S_1, \dots, S_l .*

Now, suppose that we have a triangulation $\phi : |P|_M \rightarrow X$ and we consider finitely many definable subsets S_1, \dots, S_l of X . The triangulation theorem tells us that there is another triangulation $\psi : |P'|_M \rightarrow X$ compatible with S_1, \dots, S_l , but it does not say that we can choose P' to be a subdivision of P , thus in general $|P'|_M$ will be different from $|P|_M$. This is going to be a problem if we want to preserve certain properties. For instance suppose that ϕ is a definable homotopy (namely its domain $|P|_M$ has the form $Z \times I$ where $I = [0, 1]$). The triangulation theorem does not ensure that ψ can be taken to be a definable homotopy as well.

The “normal triangulation theorem” of Baro [Bar10] is a partial remedy to this defect: it ensures that we can indeed take P' to be a subdivision of P , hence in particular $|P'|_M = |P|_M$, although ψ will not in general be equal to ϕ . The precise statement is given below. It suffices to consider the special case when $X = |P|$ and ϕ is the identity.

Definition 4.3. Let P be an (open) simplicial complex in M^m and let S_1, \dots, S_l be definable subsets of $|P|$. A **normal triangulation** of P is a triangulation (P', ψ) of $|P|$ satisfying the following conditions:

- (1) P' is a subdivision of P ;
- (2) (P', ψ) is compatible with the simplexes of P ;
- (3) for every $\tau \in P'$ and $\sigma \in P$, if $\tau \subseteq \sigma$ then $\psi(\tau) \subseteq \sigma$.

From (3) it follows that the restriction of ψ to a simplex $\sigma \in P$ is a homeomorphism onto σ and ψ is definably homotopic to the identity on $|P|$ [Bar10]. The definable homotopy is the obvious one given by $\psi_t(x) = (1-t)x + t\psi(x)$ (using the convexity of the simplexes).

Fact 4.4 (Normal triangulation theorem [Bar10]). *If S_1, \dots, S_l are finitely many definable subsets of $|P|$, there exists a normal triangulation of P compatible with S_1, \dots, S_l .*

Since we are particularly interested in triangulations where the vertices of the simplicial complex have real algebraic coordinates, we prove the following proposition, which guarantees that the normal triangulation of a real algebraic simplicial complex can be also chosen to be real algebraic.

Proposition 4.5. *Let P be a simplicial complex in M^k defined over \mathbb{R}^{alg} and let L be a subdivision of P . Then there is a subdivision L' of P such that:*

- (1) L' is defined over \mathbb{R}^{alg} ;
- (2) there is a simplicial homeomorphism $\psi : |L| \rightarrow |L'|$ which fixes all the vertices of L with real algebraic coordinates.

Proof. Since L is a subdivision of P , we have an inclusion of the zero-skeleta $|P^0| \subseteq |L^0| \subseteq M^k$. For each $v \in |L^0|$, let $v_1, \dots, v_k \in M$ be the coordinates of v . The idea is that the combinatorial properties of the pair (P, L) (namely the properties invariant by isomorphisms of pairs of abstract complexes) can be described, in the language of ordered fields, by a first order condition $\varphi_{L,P}(\bar{x})$ on the coordinates \bar{x} of the vertices. We then use the model completeness of the theory of real closed fields to show that $\varphi_{L,P}(\bar{x})$ can be satisfied in the real algebraic numbers.

The details are as follows. For each $v \in |L^0|$ we introduce free variables x_1^v, \dots, x_k^v and let \bar{x}^v be the k -tuple x_1^v, \dots, x_k^v . Finally let \bar{x} be the tuple consisting of all these variables x_i^v as v varies. We can express in a first order way the following conditions on \bar{x} :

- (1) If $\sigma = (v_0, \dots, v_n) \in L$, then $\sigma(\bar{x}) := (\bar{x}^{v_0}, \dots, \bar{x}^{v_n})$ is n -simplex, namely $\bar{x}^{v_0}, \dots, \bar{x}^{v_n}$ are affinely independent;
- (2) If σ_1 and σ_2 are open simplexes of L with common face τ , then $\text{cl}(\sigma_1(\bar{x})) \cap \text{cl}(\sigma_2(\bar{x})) = \text{cl}(\tau(\bar{x}))$;
- (3) If σ_1 and σ_2 are open simplexes of L with no face in common, then $\text{cl}(\sigma_1(\bar{x})) \cap \text{cl}(\sigma_2(\bar{x})) = \emptyset$;
- (4) If $\sigma \subseteq \tau$ with $\sigma \in L$ and $\tau \in P$, then $\sigma(\bar{x}) \subseteq \tau(\bar{x})$.

These clauses express the fact that the collection $\sigma(\bar{x})$ as σ varies in L is a simplicial complex $L(\bar{x})$ (depending on the value of \bar{x}) isomorphic to L . Similarly we can define $P(\bar{x})$ and express the fact that $L(\bar{x})$ is a subcomplex of $P(\bar{x})$. Our desired formula $\phi_{P,L}(\bar{x})$ is the conjunction of these clauses together with the conditions $x_i^v = v_i$ whenever v_i is real algebraic. By definition $\phi_{P,L}(\bar{x})$ holds in M if we evaluate each variable x_i^v as the i -th coordinate of the vector v . By the model completeness of the theory of real closed fields, the formula can be satisfied by a tuple \bar{a} of real algebraic numbers. The map sending each v to \bar{a}^v induces the desired isomorphism $\psi : L \rightarrow L' = L(\bar{a})$. \square

Later we shall need the following.

Proposition 4.6. *Let P be a simplicial complex, let X be a definable space and let $f : |P|_M \rightarrow X$ be a definable function. Let $\mathcal{V} = \{V_i : i \in I\}$ be a small family of \vee -definable sets $V_i \subseteq X$ whose union covers the image of f . Then there is a subdivision P' of P and a normal triangulation (P', ϕ) of P such that for every $\sigma \in P'$, $(f \circ \phi)(\sigma_M)$ is contained in some V_i . Moreover, if P is defined over \mathbb{R}^{alg} , we can take P' defined over \mathbb{R}^{alg} .*

Proof. By saturation of M there is a finite set $J \subseteq I$ such that $\text{Im}(f) \subseteq \bigcup_{i \in J} V_i$. Again by saturation there are definable subsets $U_i \subseteq V_i$ for $i \in J$ such that $\text{Im}(f) \subseteq \bigcup_{i \in J} U_i$. By Fact 4.4 there is a subdivision P' of P and a normal triangulation (P', ϕ) of P compatible with the definable sets $f^{-1}(U_i)$, for $i \in J$. Thus for $\sigma \in P'$, there is $i \in J$ such that $\phi(\sigma_M) \subseteq f^{-1}(U_i)$, namely $(f \circ \phi)(\sigma_M) \subseteq U_i$.

The “moreover” part follows from Proposition 4.5. Indeed, if P is over \mathbb{R}^{alg} , we first obtain (P', ϕ) as above. If P' is over \mathbb{R}^{alg} we are done. Otherwise, we take a subdivision P'' of P over \mathbb{R}^{alg} and a simplicial isomorphism $\psi : P'' \rightarrow P'$, and replace (P', ϕ) with $(P'', \phi \circ \psi)$. \square

5. STANDARD PART MAP

Let $X = X(M) \subseteq M$ be a definable set and suppose $X \subseteq [-n, n]$ for some $n \in \mathbb{N}$. Then there is a map $\text{st} : X \rightarrow \mathbb{R}$, called **standard part**, which sends $a \in X$ to the unique $r \in \mathbb{R}$ satisfying the same inequalities $p < x < q$ with $p, q \in \mathbb{Q}$.

More generally, let X be a definable subset of M^k and assume $X \subseteq [-n, n]^k$ for some $n \in \mathbb{N}$. We can then define $\text{st} : X \rightarrow \mathbb{R}^k$ component-wise, namely $\text{st}((a_1, \dots, a_k)) := (\text{st}(a_1), \dots, \text{st}(a_k))$.

Now let

$$E := \ker(\text{st}) \subseteq X \times X$$

be the type-definable equivalence relation induced by st , namely aEb if and only if $\text{st}(a) = \text{st}(b)$. There is a natural bijection $\text{st}(X) \cong X/E$ sending $\text{st}(a)$ to the class of a modulo E , so in particular E has bounded index. The next two propositions are probably well known but we include the proof for the reader’s convenience.

Proposition 5.1. *The natural bijection $\text{st}(a) \mapsto a/E$ is homeomorphism*

$$\text{st}(X) \cong X/E$$

where X/E has the logic topology and $\text{st}(X) \subseteq \mathbb{R}^k$ has the euclidean topology.

Proof. Every closed subset C of \mathbb{R}^k can be written as the intersection $\bigcap_i C_i$ of a countable collection of closed \emptyset -semialgebraic sets C_i (where “ \emptyset -semialgebraic” means “semialgebraic without parameters”). We then have $\text{st}(a) \in C$ if and only if $a \in \bigcap C_i(M)$. This shows that the closed sets $C \subseteq \text{st}(X) \subseteq \mathbb{R}^k$ in the euclidean topology correspond to the sets whose preimage in X is type-definable, and the proposition follows. \square

Thanks to the above result we can identify $\text{st} : X \rightarrow \text{st}(X)$ and $\mathfrak{p} : X \rightarrow X/E$ where $E = \ker(\text{st})$. The next proposition shows that these maps are continuous.

Proposition 5.2. *The preimage of any point of $y \in \text{st}(X)$ under $\text{st} : X \rightarrow \text{st}(X)$ is open in $X \subseteq M^k$. In particular, the standard part map is continuous (as the preimage of every subset is open).*

Proof. Let $a \in X$ and let $r := \text{st}(a) \in \mathbb{R}^k$. Then $\text{st}^{-1}(r) = \bigcap_{n \in \mathbb{N}} \{b \in X : |b - r| < 1/n\}$. This is a small intersection of relatively open subsets of X , so it is open in X by Lemma 2.2. \square

Remark 5.3. If $X = X(M) \subseteq M^k$ is \emptyset -semialgebraic, we may interpret the defining formula of X in \mathbb{R} and consider the set $X(\mathbb{R}) \subseteq \mathbb{R}^k$ of real points of X . If we further assume that X is closed and bounded, then $X \subseteq [-n, n]^k$ for some $n \in \mathbb{N}$, so we can consider the standard part map $\text{st} : X \rightarrow \mathbb{R}^k$. It is easy to see that in this case $\text{st}(X)$ coincides with $X(\mathbb{R})$, so we can write $X(\mathbb{R}) = \text{st}(X) \cong X/E$.

Our next goal is to study the fibers of $\text{st} : X \rightarrow X(\mathbb{R})$. We need the following.

Definition 5.4. Given a simplicial complex P and a point $x \in |P|$ (not necessarily a vertex), the **open star** of x with respect to P , denoted $\text{St}(x, P)$, is the union of all the simplexes of P whose closure contains x . Note that $\text{St}(x, P)$ is an open neighbourhood of x in $|P|$.

The next result will be needed later (in the proof of Proposition 8.4).

Proposition 5.5. *Given $x, y \in |P|$, if $\text{St}(x, P) \cap \text{St}(y, P)$ is non-empty, then there is $z \in |P|$ such that $\text{St}(x, P) \cap \text{St}(y, P) = \text{St}(z, P)$.*

Proof. Let $\sigma \in P$ be a simplex of minimal dimension included in $\text{St}(x, P) \cap \text{St}(y, P)$ and let $z \in \sigma$. We claim that z is as desired. To this aim it suffices to show that, given $\theta \in P$, we have $\theta \subseteq \text{St}(x, P) \cap \text{St}(y, P)$ if and only if $\theta \subseteq \text{St}(z, P)$.

For one direction assume $\theta \subseteq \text{St}(x, P) \cap \text{St}(y, P)$. Then $\bar{\theta} \cap \bar{\sigma}$ is non-empty, as the intersection contains both x and y . It follows that there is a simplex $\delta \in P$ such that $\bar{\theta} \cap \bar{\sigma} = \bar{\delta}$. Notice that δ is included in $\text{St}(x, P) \cap \text{St}(y, P)$ since its closure contains x and y . Since σ was of minimal dimension contained in this intersection, it follows that $\delta = \sigma$. But then $z \in \sigma \subseteq \bar{\theta}$, hence $\theta \subseteq \text{St}(z, P)$.

For the other direction, assume $\theta \subseteq \text{St}(z, P)$, namely $z \in \bar{\theta}$. Since $z \in \sigma$, it follows that $\sigma \subseteq \bar{\theta}$ and therefore $\bar{\sigma} \subseteq \bar{\theta}$. But x, y are contained in $\bar{\sigma}$, so they are contained in $\bar{\theta}$, witnessing the fact that $\theta \subseteq \text{St}(x, P) \cap \text{St}(y, P)$. \square

The following result depends on the local conic structure of definable sets.

Proposition 5.6. *Let X be closed and bounded \emptyset -semialgebraic set and let $\text{st} : X(M) \rightarrow \text{st}(X) = X(\mathbb{R})$ be the standard part map. Then for every $y \in X(\mathbb{R})$, the preimage $\text{st}^{-1}(y)$ is an intersection of a countable decreasing sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ of definably contractible open subsets of X .*

Proof. By the triangulation theorem (Fact 4.2), there is a simplicial complex P over \mathbb{R}^{alg} and a \emptyset -definable homeomorphism $f : X \rightarrow |P|_M$. In this situation, $f_{\mathbb{R}} : X(\mathbb{R}) \rightarrow P(\mathbb{R})$ is a homeomorphism and $\text{st}(f(x)) = f_{\mathbb{R}}(\text{st}(x))$. Thus we can replace X with P and assume that X is the realization of a simplicial complex. Therefore, we now have a closed simplicial complex $X(\mathbb{R})$ over the reals, which is thus locally contractible. More precisely, given $y \in X(\mathbb{R})$ we can write $\{y\}$ as an intersection $\bigcap_{i \in \mathbb{N}} S_i$ where S_i is the open star of y with respect to the i -th iterated barycentric subdivision of P . The preimage $\text{st}^{-1}(y)$ can then be written as the corresponding intersection $\bigcap_{i \in \mathbb{N}} S_i(M)$ interpreted in M , and it now suffices to observe that each $S_i(M)$ is an open star, hence it is definably contractible (to any of its points). \square

Our next goal is to show that, much of what we said about the standard part map, has a direct analogue in the context of definable groups, with $\mathfrak{p} : G \rightarrow G/G^{00}$ in the role of the standard part.

6. DEFINABLE GROUPS

Let G be a definable group in M and let $H < G$ be a type-definable subgroup of bounded index. We may put on the coset space G/H the logic topology, thus obtaining a compact topological space. In this context we have a direct generalization of Proposition 5.2.

Fact 6.1 ([Pil04, Lemma 3.2]). *Every type-definable subgroup $H < G$ of bounded index is clopen in the t -topology of G . In particular, the natural map $\mathfrak{p} : G \rightarrow G/H$ is continuous, where G has the t -topology and the coset space G/H has the logic topology.*

If we further assume that H is normal, then G/H is a group and we may ask whether the logic topology makes it into a topological group. This is indeed the case [Pil04]. Some additional work shows that in fact G/H is a compact real Lie group [BOPP05]. In the same paper the authors show that G admits a smallest type-definable subgroup $H < G$ of bounded index (see [She08] for a different proof), which is denoted G^{00} and called the **infinitesimal subgroup**. When G is **definably compact** in the sense of [PS99], the natural map $p : G \rightarrow G/G^{00}$ shares many of the properties of the standard part map.

Definition 6.2. Let us recall that a definable set $B \subseteq X$ is called a **definable open ball** of dimension n if B is definably homeomorphic to $\{x \in M^n : |x| < 1\}$; a **definable closed ball** is defined similarly, using the weak inequality \leq ; we shall say that B is a **definable proper ball** if there is a definable homeomorphism f from \overline{B} to a definable closed ball taking ∂B to the definable sphere S^{n-1} .

In analogy with Proposition 5.6, the following holds.

Fact 6.3. [Ber09, Theorem 2.2] *Let G be a definably compact group of dimension n and put on G the t -topology of [Pil88]. Then there is a decreasing sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ of definably contractible subsets of G such that $G^{00} = \bigcap_{i \in \mathbb{N}} S_i$.*

The proof in [Ber09, Theorem 2.2] depends on compact domination and the sets S_n are taken to be “cells” in the o-minimal sense. For later purposes we need the following strengthening of the above fact, which does not present difficulties, but requires a small argument.

Corollary 6.4. *In Fact 6.3 we can arrange so that, for each $i \in \mathbb{N}$ $\overline{S}_{i+1} \subseteq S_i$ and S_i is a definable proper ball of dimension $n = \dim(G)$.*

Proof. By Lemma 2.4 we can assume that $\overline{S}_{i+1} \subseteq S_i$ for every $i \in \mathbb{N}$. Since M has field operations, a cell is definably homeomorphic to a definable open ball (first show that it is definably homeomorphic to a product of intervals). In general it is not true that a cell is a definable proper ball, even assuming that the cell is bounded [BF09]. However by shrinking concentrically S_i via the homeomorphism, we can find a definable proper n -ball C_i with $\overline{S}_{i+1} \subseteq C_i \subseteq S_i$. To conclude, it suffices to replace S_i with the interior of C_i . \square

7. COMPACT DOMINATION

A deeper analogy between the standard part map and the projection $p : G \rightarrow G/G^{00}$ is provided by Fact 7.1 and Fact 7.2 below.

Fact 7.1 ([BO04, Cor. 4.4]). *Let X be a closed and bounded \emptyset -semialgebraic subset of M^k and let D be a definable subset of X . Then $\text{st}(D) \cap \text{st}(D^c) \subseteq \mathbb{R}^k$ has Lebesgue measure zero.*

The above fact was used in [BO04] to introduce a finitely additive measure on definable subsets of $[-n, n]^k \subseteq M^k$ ($n \in \mathbb{N}$) by lifting the Lebesgue measure on \mathbb{R}^k through the standard part map. In the same paper it was conjectured that, reasoning along similar lines, one could try to introduce a finitely additive invariant measure on definably compact groups (the case of the torus being already handled thanks to the above result). When [BO04] was written, Pillay’s conjectures from [Pil04] were still open, and it was hoped that the measure approach could lead to a solution. A first confirmation to the existence of invariant measures came from [PP07], but only for a limited class of definable group. A deeper analysis led to the existence of invariant measures in every definable compact group [HPP08] and to the solution of Pillay’s conjectures, as discussed in the introduction. Finally,

the following far reaching result was obtained, which can be considered as a direct analogue to Fact 7.1.

Fact 7.2 ([HP11]). *Let G be a definably compact group and consider the projection $\mathfrak{p} : G \rightarrow G/G^{00}$. Then for every definable set $D \subseteq G$, $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}})$ has Haar measure zero.*

In the terminology introduced in [HPP08] the above result can be described by saying that G is **compactly dominated** by G/G^{00} . Perhaps surprisingly, when the above result was obtained, Pillay’s conjectures had already been solved, so compact domination did not actually play a role in its solution. In hindsight however, as we show in the last part of this paper (Section 13), compact domination can in fact be used to prove “ $\dim(G) = \dim_{\mathbb{R}}(G/G^{00})$ ”, as predicted by Pillay’s conjectures (the content of Pillay’s conjectures also includes the statement that G/G^{00} is a real Lie group).

To prepare the ground, we introduce the following definition. In the rest of the section E is a type-definable equivalence relation of bounded index on a definable set X .

Definition 7.3. We say that X is **topologically compactly dominated** by X/E if for every definable set $D \subseteq X$, $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}})$ has empty interior, where $\mathfrak{p} : X \rightarrow X/E$ is the projection.

Since “measure zero” implies “empty interior”, topological compact domination holds both for the standard part map (taking $E = \ker(\text{st})$) and for definably compact groups.

Notice that Definition 7.3 can be given for definable sets in arbitrary theories, not necessarily o-minimal, so it is not necessary that X carries a topology. However in the o-minimal case a simpler formulation can be given, as in Corollary 7.5 below.

We first recall some definitions. Let X be a definable space. We say that a type-definable set $Z \subseteq X$ is **definably connected** if it cannot be written as the union of two non-empty open subsets which are relatively definable, where a relatively definable subset of Z is the intersection of Z with a definable set.

Following [vdD98], we distinguish between the frontier and the boundary of a definable set, and we write $\partial D := \overline{D} \setminus D$ for the **frontier**, and $\delta D := \overline{D} \setminus D^{\circ}$ for the **boundary**, where D° is the interior.

A basic result in o-minimal topology is that the dimension of the frontier of D is less than the dimension of D . Here we shall however be concerned with the boundary, rather than the frontier.

Proposition 7.4. *Let X be a definable space. Assume that \mathfrak{p} is continuous and each fiber of $\mathfrak{p} : X \rightarrow X/E$ is definably connected. Then for every definable set $D \subseteq X$, $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}}) = \mathfrak{p}(\delta D)$.*

Proof. We prove $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}}) \subseteq \mathfrak{p}(\delta D)$. So let $y \in \mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}})$. If for a contradiction $\mathfrak{p}^{-1}(y) \cap \delta D = \emptyset$, then $\mathfrak{p}^{-1}(y) \cap D^{\circ}$ and $\mathfrak{p}^{-1}(y) \cap (D^{\mathfrak{c}})^{\circ}$ are both non-empty. Since they are relatively definable in $\mathfrak{p}^{-1}(y)$ and open, we contradict the hypothesis that $\mathfrak{p}^{-1}(y)$ is definably connected. The opposite inclusion is easy, using the fact that $\mathfrak{p}(A) = \mathfrak{p}(\overline{A})$ (Proposition 3.5). \square

In the light of the above proposition, topological compact domination takes the following form.

Corollary 7.5. *Assume that X is a definable space, $\mathfrak{p} : X \rightarrow X/E$ is continuous, and each fiber of \mathfrak{p} is definably connected. Then X is topologically compactly dominated by X/E if and only if the image $\mathfrak{p}(Z)$ of any definable set $Z \subseteq X$ with empty interior, has empty interior.*

Proof. Suppose that the image of every definable set with empty interior has empty interior. Given a definable set $D \subseteq X$, we want to show that $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}})$ has empty interior. This follows from the inclusion $\mathfrak{p}(D) \cap \mathfrak{p}(D^{\mathfrak{c}}) \subseteq \mathfrak{p}(\delta D)$ (Proposition 7.4) and the fact that δD has empty interior.

Conversely, assume topological compact domination and let Z be a definable subset of X with empty interior. By Proposition 3.5, $\mathfrak{p}(\delta Z) \subseteq \mathfrak{p}(\overline{Z}) \cap \mathfrak{p}(\overline{Z^{\mathfrak{c}}}) = \mathfrak{p}(Z) \cap \mathfrak{p}(Z^{\mathfrak{c}})$, so $\mathfrak{p}(\delta Z)$ has empty interior. \square

8. GOOD COVERS

By a **triangulable space** we mean a compact topological space which is homeomorphic to a polyhedron, namely to the realization $|P|_{\mathbb{R}}$ of a closed finite simplicial complex over \mathbb{R} .

Definition 8.1. Let \mathcal{U} be an open cover of a topological space Y . We say that \mathcal{U} is a **good cover** if every finite intersection $U_1 \cap \dots \cap U_n$ of open sets $U_1, \dots, U_n \in \mathcal{U}$ is contractible.

Our aim is to show that open subsets of a triangulable spaces have enough good covers. We are going to use barycentric subdivisions holding a subcomplex fixed, as defined in [Mun84, p. 90]. We need the following observation.

Remark 8.2. Let P be a closed (finite) simplicial complex and let L be a closed subcomplex. Let P_i be the i -th barycentric subdivision of P holding L fixed. Then for every real number $\varepsilon > 0$ there is $i \in \mathbb{N}$ such that for every closed simplex $\bar{\sigma}$ of P_i , either $\bar{\sigma}$ has diameter $< \varepsilon$ or $\bar{\sigma}$ lies inside the ε -neighbourhood of some simplex of L .

Definition 8.3. Let \mathcal{U} be an open cover of a topological space Y . Given $A \subseteq Y$ we recall that the star of A with respect to \mathcal{U} , denoted $\text{St}_{\mathcal{U}}(A)$, is the union of all $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$. We say that \mathcal{U} **star refines** another cover \mathcal{V} if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $\text{St}_{\mathcal{U}}(U) \subseteq V$. We define $\text{St}(\mathcal{U})$ to be the cover consisting of the sets $\text{St}_{\mathcal{U}}(U)$ as U ranges in \mathcal{U} .

Proposition 8.4. *Let O be an open subset of a triangulable space Y (not necessarily a manifold). Then every open cover \mathcal{V} of O has a locally finite refinement \mathcal{U} which is a good cover.*

Proof. We can assume that Y is the geometric realization $|P|$ (over \mathbb{R}) of a finite simplicial complex P .

Since Y is a metric space, so is $O \subseteq Y$. In particular O is paracompact, and therefore \mathcal{V} has a locally finite star-refinement $\mathcal{W} \prec \mathcal{V}$. We plan to show that O is the realization of an infinite simplicial complex L with the property that each closed simplex of L is contained in some element of \mathcal{W} . Granted this, by Proposition 5.5 we can take \mathcal{U} to be the open cover consisting of the sets $\text{St}(x, L)$ for $x \in O$.

To begin with, note that we can write O as the union $O = \bigcup_{n \in \mathbb{N}} C_n$ of an increasing sequence of compact sets in such a way that every compact subset of O belongs to C_n for some n (it suffices to define C_n as the set of points of O at distance $\geq 1/2^n$ from the frontier of O).

Since C_0 is compact, by the Lebesgue number lemma there is some $\varepsilon_0 > 0$ such that every subset of C_0 of diameter $< \varepsilon_0$ is contained in some element of \mathcal{W} . Now let P_0 be an iterated barycentric subdivision of P with simplexes of diameter $< \varepsilon_0$ and let L_0 be the largest closed subcomplex of P_0 with $|L_0| \subseteq C_0$. Notice that every closed simplex of L_0 is contained in some element of \mathcal{W} .

Starting with P_0, L_0, ε_0 we shall define by induction on $i \in \mathbb{N}$ a subdivision P_i of P_0 , a subcomplex L_i of P_0 and a positive element $\varepsilon_i \in M$. At stage i we assume

that P_i, L_i, ε_i have already been defined, L_i is the biggest closed subcomplex of P_i with $|L_i| \subseteq C_i$, and every closed simplex of L_i is contained in some element of \mathcal{W} . We continue as follows. The complex P_{i+1} will be of the form $P_i[n]$, where $P_i[n]$ is the n -th iterated barycentric subdivision of P_i holding the subcomplex L_i fixed. We next show how to choose the value of n . By the Lebesgue number lemma there is some $\varepsilon > 0$ with $\varepsilon < \varepsilon_i/2$ such that every closed subset of C_i of diameter $< \varepsilon$ is contained in some element of \mathcal{W} . By taking a smaller value for ε if necessary, we can also assume, that the closed ε -neighbourhood of any closed simplex $\bar{\sigma}$ of L_i is contained in some element of \mathcal{W} . Let ε_{i+1} be such an ε . By Remark 8.2 there is some n_0 such that for every $n \geq n_0$ and for every closed simplex $\bar{\sigma}$ of $P_i[n]$ either $\bar{\sigma}$ is contained in the ε_{i+1} -neighbourhood of some $\lambda \in L_i$, or the diameter of $\bar{\sigma}$ is less than ε_{i+1} . In both cases, if $\bar{\sigma}$ is included in C_i , then it is contained in some element of \mathcal{W} . We now define $P_{i+1} = P_i[n_0]$ and we let L_{i+1} be the biggest closed subcomplex of P_{i+1} with $|L_{i+1}| \subseteq C_{i+1}$.

Having finished the construction, notice that L_i is a subcomplex of L_{i+1} , because both are subcomplexes of P_{i+1} and $|L_i| \subseteq |L_{i+1}|$. We can thus consider the infinite simplicial complex $L := \bigcup_{i \in \mathbb{N}} L_i$. By construction each closed simplex of L is contained in some $W \in \mathcal{W}$, so in particular $|L| \subseteq O$. To finish the proof it suffices to show that $|L| = O$. The strategy is to show that the sets $|L_i|$ are not too small. More precisely, we claim that if $x \in O$ is such that its closed ε_{i+1} -neighbourhood is contained in C_{i+1} , then $x \in |L_{i+1}|$. Since $\varepsilon_i \rightarrow 0$, this will immediately imply $\bigcup_i |L_i| = O$, as desired. To prove the claim, consider the unique (open) simplex $\sigma \in P_{i+1}$ containing x . By definition of P_{i+1} , either $\bar{\sigma}$ has diameter $< \varepsilon_{i+1}$ or it is included in the ε_{i+1} -neighbourhood of $|L_i|$. In both cases $\bar{\sigma}$ is included in $|L_{i+1}|$, concluding the argument. \square

9. HOMOTOPY

Recall that two continuous maps $f_0, f_1 : Z \rightarrow Y$ between topological spaces are **homotopic** if there is a continuous function $H : Z \times [0, 1] \rightarrow Y$ such that $H(z, 0) = f_0(z)$ and $H(z, 1) = f_1(z)$ for every $z \in Z$.

Given base points $z_0 \in Z$ and $y_0 \in Y$ and a function $f : Z \rightarrow Y$, we write $f : (Z, z_0) \rightarrow (Y, y_0)$ if f sends z_0 to y_0 . Given an homotopy H between two maps $f_0, f_1 : (Z, z_0) \rightarrow (Y, y_0)$ we say that H is a homotopy relative to z_0 if $H(z_0, t) = y_0$ for all $t \in I$, where $I = [0, 1]$.

Definition 9.1. If Z and Y are topological spaces, we let $[Z, Y]$ denote the set of all homotopy classes of continuous functions from Z to Y . Given base points $z_0 \in Z$ and $y_0 \in Y$, we let $[(Z, z_0), (Y, y_0)]$, or simply $[Z, Y]_0$, denote the set of all homotopy classes of continuous functions $f : (Z, z_0) \rightarrow (Y, y_0)$ relative to z_0 . The n -th homotopy group is defined as

$$\pi_n(Y) := [S^n, Y]_0$$

where S^n is the n -th sphere and we put on $\pi_n(Y)$ the usual group operation if $n > 0$ (see [Hat02] for the details).

In the rest of this section we work in the classical category of topological spaces and we give a sufficient condition for two maps to be homotopic. Later we shall need to adapt the proofs to the definable category, but with additional complications.

Definition 9.2. Given a collection \mathcal{U} of subsets of a set O and two functions $f, g : Z \rightarrow O$, we say that f and g are \mathcal{U} -**close** if for any $z \in Z$ there is $U \in \mathcal{U}$ such that both $f(z)$ and $g(z)$ are in U .

The following definition is adapted from [Dug69, Note 4].

Definition 9.3. Let $f : Z \rightarrow Y$ be a function between two sets Z and Y . Let P be a collection of sets whose union $\bigcup P$ includes Z , and let \mathcal{U} be a collection of subsets of Y . We say that f is (\mathcal{U}, P) -**small** if for every $\sigma \in P$ the image $f(\sigma \cap Z)$ is contained in some $U \in \mathcal{U}$.

Lemma 9.4. Let \mathcal{U} be a locally finite good cover of a topological space Y and let L be a closed subcomplex of a closed simplicial complex P defined over \mathbb{R} . Let $P^{(0)}$ be the 0-skeleton of P and let $f : |L \cup P^{(0)}|_{\mathbb{R}} \rightarrow Y$ be a $(\mathcal{U}, \overline{P})$ -small continuous map (recall that \overline{P} is the collection of all closures of simplexes of P). Then f can be extended to a $(\mathcal{U}, \overline{P})$ -small continuous map $f' : |P|_{\mathbb{R}} \rightarrow Y$ with the property that, for all $U \in \mathcal{U}$ and for every closed simplex $\bar{\sigma}$ of P , if $f(\bar{\sigma}|_{L \cup P^{(0)}}) \subseteq U$, then $f'(\bar{\sigma}) \subseteq U$.

Proof. Reasoning by induction we can assume that f' is already defined on $|L \cup P^{(k)}|$, where $P^{(k)}$ is the k -skeleton of P , and we only need to extend it to $|L \cup P^{(k+1)}|$. Let $\sigma \in P^{(k+1)}$. We can identify $\bar{\sigma}$ with the cone over its boundary $\partial\sigma$, so that every point of $\bar{\sigma}$ is determined by a pair (t, x) with $t \in [0, 1]$ and $x \in \partial\sigma$. Let U_1, \dots, U_n be the elements of \mathcal{U} containing $f'(\bar{\sigma}|_{L \cup P^{(k)}})$ (notice that $n > 0$ by the inductive hypothesis), let V be their intersection, and let $\phi : [0, 1] \times V \rightarrow V$ be a retraction of V to a point. We extend f' to $\bar{\sigma}$ sending $(t, x) \in \bar{\sigma}$ to $\phi(t, f'(x)) \in V$. Note that if $f'(\bar{\sigma}|_{L \cup P^{(k)}}) \subseteq U \in \mathcal{U}$, then U is one of the U_i , and since by construction $f'(\bar{\sigma}) \subseteq V = \bigcap_i U_i$, we get $f'(\bar{\sigma}) \subseteq U$. \square

Proposition 9.5. Let \mathcal{U} be a locally finite good cover of a topological space Y , let P be a closed simplicial complex and let $f, g : |P|_{\mathbb{R}} \rightarrow Y$ be two continuous maps. Assume that f and g are \mathcal{U} -close. Then, f and g are homotopic.

Proof. Since f and g are \mathcal{U} -close, the family $\mathcal{V} = \{f^{-1}(U) \cap g^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of $|P|_{\mathbb{R}}$. By the Lebesgue number lemma (since we work over \mathbb{R}) there is an iterated barycentric subdivision P' of P such that every closed simplex of P' is contained in some element of \mathcal{V} . Then, by construction, for every $\sigma \in P'$ there is $U \in \mathcal{U}$ such that $f(\bar{\sigma})$ and $g(\bar{\sigma})$ are contained in U .

Let now $I = [0, 1]$ and consider the simplicial complex $P' \times I$ with the standard triangulation (as in [Hat02, p. 112, Proof of 2.10]). Consider the subcomplex $P' \times \{0, 1\}$ of $P' \times I$ and note that it contains the 0-skeleton of $P' \times I$. Define $f \sqcup g : |P' \times \{0, 1\}|_{\mathbb{R}} = |P' \times \{0, 1\}|_{\mathbb{R}} \rightarrow Y$ as the function which sends $(x, 0)$ to $f(x)$ and $(x, 1)$ to $g(x)$. Note that $f \sqcup g$ is $(\mathcal{U}, \overline{P' \times I})$ -small. Since \mathcal{U} is a good cover, by Lemma 9.4 we can extend it to a $(\mathcal{U}, \overline{P' \times I})$ -small function $H : |P' \times I|_{\mathbb{R}} \rightarrow Y$. This map is a homotopy between f and g . \square

10. DEFINABLE HOMOTOPIES

Given a definable set Z and a \vee -definable set Y , we say that a map $f : Z \rightarrow Y$ is definable if it takes values in a definable subset Y_0 of Y and is definable as a function from Z to Y_0 . We can adapt Definition 9.1 to the definable category as follows.

Definition 10.1. If Z is a definable space and Y is a \vee -definable set, we let $[Z, Y]^{\text{def}}$ denote the set of all equivalence classes of definable continuous maps f from Z to Y modulo definable homotopies. Similarly we write $[Z, Y]_0^{\text{def}}$ when we work with pointed spaces and homotopies relative to the base point $z_0 \in Z$. The n -th o-minimal homotopy group is defined as

$$\pi_n^{\text{def}}(Y) := [S^n, Y]_0^{\text{def}}$$

where S^n is the n -th sphere in M . If $n > 0$ we put on $\pi_n^{\text{def}}(Y)$ a group operation in analogy with the classical case.

In [BO02] it is proved that if Y is a \emptyset -semialgebraic set, $\pi_1^{\text{def}}(Y) \cong \pi_1(Y(\mathbb{R}))$, so in particular $\pi_1^{\text{def}}(Y)$ is finitely generated. This has been generalized to the higher homotopy groups in [BO10]. We shall later give a self-contained proof of both results. By the same arguments we obtain the following result of [BMO10]: given a definably compact group G there is a natural isomorphism $\pi_n^{\text{def}}(G) \cong \pi_n(G/G^{00})$. The new proofs yield a stronger result: if $\mathfrak{p} : G \rightarrow G/G^{00}$ is the projection, for every open subset O of G/G^{00} , there is an isomorphism $\pi_n^{\text{def}}(\mathfrak{p}^{-1}(O)) \cong \pi_n(O)$. This was so far known for $n = 1$ [BM11]. Notice that $\mathfrak{p}^{-1}(O)$ is \vee -definable, whence the decision to consider \vee -definable sets in Definition 10.1. With the new approach we obtain additional functoriality properties and generalizations, as it will be explained in the rest of the paper.

11. THEOREM A

As above, let $X = X(M)$ be a definable space, and let $E \subseteq X \times X$ be a definable equivalence relation of bounded index. In this section we work under the following assumption.

Assumption 11.1 (Assumption A). *X/E is a triangulable topological space and the natural map $\mathfrak{p} : X \rightarrow X/E$ is continuous.*

The fact that X/E is triangulable allows us to apply the results of Section 8 regarding the existence of good covers. Note that the continuity of \mathfrak{p} is not a vacuous assumption because X/E has the logic topology, not the the quotient topology. By the results in Section 5 and Section 6 the assumption is satisfied in the special case $X/E = G/G^{00}$ (where G is a definably compact group) and also when X is a closed and bounded \emptyset -semialgebraic set and $E = \ker(\text{st})$.

We shall prove that there is a natural homomorphism

$$\pi_n^{\text{def}}(X) \rightarrow \pi_n(X/E).$$

This will be obtained as a consequence of a more general result concerning homotopy classes. The following definition plays a crucial role in the definition of the homomorphism, and exploits the analogies between the projection $\mathfrak{p} : X \rightarrow X/E$ and the standard part map.

Definition 11.2. Let $O \subseteq X/E$ be an open subset. Let \mathcal{U} be an open cover of $O \subseteq X/E$, and let P be a closed simplicial complex defined over \mathbb{R}^{alg} . Consider a definable continuous map $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ and let $\text{st} : |P|_M \rightarrow |P|_{\mathbb{R}}$ be the standard part map. We say that a continuous map $f^* : |P|_{\mathbb{R}} \rightarrow O$ is an \mathcal{U} -**approximation** of f if $\mathfrak{p} \circ f$ and $f^* \circ \text{st}$ are \mathcal{U} -close, namely the two paths from the upper-left to the lower-right corner of the following diagram represent maps which are \mathcal{U} -close.

$$\begin{array}{ccc} |P|_M & \xrightarrow{\text{st}} & |P|_{\mathbb{R}} \\ f \downarrow & & \downarrow f^* \\ \mathfrak{p}^{-1}(O) & \xrightarrow{\mathfrak{p}} & O \end{array}$$

We say that f is \mathcal{U} -**approximable** if it has a \mathcal{U} -approximation.

In general, given f and \mathcal{U} , we cannot hope to find f^* which is a \mathcal{U} -approximation of f . However we shall prove that, given \mathcal{U} , every definable continuous function f is definably homotopic to a \mathcal{U} -approximable map.

Definition 11.3. Given a collection \mathcal{U} of open subsets of X/E let $\mathfrak{p}^{-1}(\mathcal{U})$ be the collection of consisting of the \vee -definable open sets $\mathfrak{p}^{-1}(U) \subseteq X$ as U varies in \mathcal{U} .

Notice that $f : |P|_M \rightarrow X$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small (Definition 9.3) if and only if $(\mathfrak{p} \circ f) : |P|_M \rightarrow X/E$ is (\mathcal{U}, P) -small. The next lemma shows that in this situation we can ignore the difference between closed and open simplexes. Recall that $\overline{P} = \{\bar{\sigma} : \sigma \in P\}$. We have:

Lemma 11.4. *Let \mathcal{U} be a collection of open subsets of X/E and let $f : |P|_M \rightarrow X$ be a definable continuous map. Then f is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small if and only if it is $(\mathfrak{p}^{-1}(\mathcal{U}), \overline{P})$ -small.*

Proof. Let $\sigma \in P$. Since f is continuous, $f(\bar{\sigma}) \subseteq \overline{f(\sigma)}$ and by Proposition 3.5 we have $\mathfrak{p}(\overline{f(\sigma)}) = \mathfrak{p}(f(\sigma))$, so if f is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small, it is also $(\mathfrak{p}^{-1}(\mathcal{U}), \overline{P})$ -small. \square

The following lemma shows that small maps are approximable.

Lemma 11.5. *Let \mathcal{V} be a locally finite good open cover of O and let $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ be a $(\mathfrak{p}^{-1}(\mathcal{V}), P)$ -small map. Then there exists a \mathcal{V} -approximation $f^* : |P|_{\mathbb{R}} \rightarrow O$ of f .*

Proof. Define f^* on the zero-skeleton of P by $f^{*(0)}(\text{st}(v)) = \mathfrak{p}(f(v))$ for any vertex v of $|P|_M$ (since v has coordinates in \mathbb{R}^{alg} we can identify $\text{st}(v) \in |P|_{\mathbb{R}}$ with v). Since f is $(\mathfrak{p}^{-1}(\mathcal{V}), P)$ -small, $f^{*(0)}$ is (\mathcal{V}, P) -small and therefore, by Lemma 9.4 (and Lemma 11.4), we can extend $f^{*(0)}$ to a (\mathcal{V}, P) -small map $f^* : |P|_{\mathbb{R}} \rightarrow O$. We claim that f^* is a \mathcal{V} -approximation of f . Indeed, fix a point $z \in |P|_M$ and let $\sigma = \sigma_M \in P$ be a simplex containing z . Since f is $(\mathfrak{p}^{-1}(\mathcal{V}), P)$ -small, there is a $V \in \mathcal{V}$ such that $\mathfrak{p} \circ f(\sigma) \subset V$, so in particular $\mathfrak{p} \circ f(\sigma_M^{(0)}) = f^{*(0)}(\sigma_{\mathbb{R}}^{(0)}) \subseteq V$. By Lemma 9.4, we also have $f^*(\sigma_{\mathbb{R}}) \subseteq V$. Since $\text{st}(\bar{\sigma}_M) = \bar{\sigma}_{\mathbb{R}}$, both $\mathfrak{p} \circ f(z)$ and $f^*(\text{st}(z))$ are in V . \square

The next lemma shows that every map f is homotopic to a small (hence approximable) map f' .

Lemma 11.6. *Let $O \subseteq X/E$ be an open subset of X/E . Given a definable map $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ and an open cover \mathcal{U} of O , we can find a subdivision P' of P and a normal triangulation (P', ϕ) of $|P|$ such that $f' = f \circ \phi$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P')$ -small. Moreover if P is defined over \mathbb{R}^{alg} , we can take P' defined over \mathbb{R}^{alg} . Notice that f' is homotopic to f (as ϕ is homotopic to the identity).*

Proof. By Proposition 4.6. \square

Lemma 11.7. *Let \mathcal{U} be a star-refinement of a good cover of O . Any two \mathcal{U} -approximations of $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ are homotopic.*

Proof. Let f_1^* and f_2^* be two \mathcal{U} -approximations of f . Then f_1^* and f_2^* are $\text{St}(\mathcal{U})$ -close and since \mathcal{U} star-refines a good cover they are homotopic by Proposition 9.5. \square

We are now ready to state the main result of this section.

Theorem 11.8 (Theorem A). *Assume 11.1. Let P be a closed simplicial complex defined over \mathbb{R}^{alg} .*

- (1) *For each open set $O \subseteq X/E$, there is a map*

$$\mathfrak{p}_O^P : [|P|_M, \mathfrak{p}^{-1}(O)]^{\text{def}} \rightarrow [|P|_{\mathbb{R}}, O].$$

- (2) *The maps \mathfrak{p}_O^P are natural with respect to inclusions of open sets. More precisely, let $U \subseteq V$ be open sets in X/E . Then, we have the following*

commutative diagram:

$$\begin{array}{ccc}
[[P|_M, \mathfrak{p}^{-1}(U)]^{\text{def}} & \xrightarrow{\mathfrak{p}_U^P} & [[P|_{\mathbb{R}}, U] \\
\downarrow i_{\mathfrak{p}^{-1}(U)} & & \downarrow i_U \\
[[P|_M, \mathfrak{p}^{-1}(V)]^{\text{def}} & \xrightarrow{\mathfrak{p}_V^P} & [[P|_{\mathbb{R}}, V]
\end{array}$$

where the vertical arrows are induced by the inclusions.

- (3) By the triangulation theorem, the same statements continue to hold if we replace everywhere $|P|$ by a \emptyset -semialgebraic set.
- (4) The results remain valid replacing all the homotopy classes with their pointed versions as in Definition 9.1 and Definition 10.1.
- (5) In particular, for all $O \subseteq X/E$ there is a natural map $\mathfrak{p}_O^{S_n} : \pi_n^{\text{def}}(\mathfrak{p}^{-1}(O)) \rightarrow \pi_n(O)$ which is a group homomorphism when $n > 0$. When $O = X/E$ we obtain a homomorphism $\pi_n^{\text{def}}(X) \rightarrow \pi_n(X/E)$.

In the rest of the section we fix a closed simplicial complex P in M defined over \mathbb{R}^{alg} and we prove Theorem 11.8. We shall define a map $\mathfrak{p}_O^P : [[P|_M, \mathfrak{p}^{-1}(O)]^{\text{def}} \rightarrow [[P|_{\mathbb{R}}, O]$ determined by the following property: if \mathcal{U} is a star-refinement of a good cover of O , f is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small and f^* is a \mathcal{U} -approximation of f , then $\mathfrak{p}_O^P([f]) = [f^*]$. A word of caution is in order: we are not claiming that if f is \mathcal{U} -approximable and f^* is an approximation of f , then $\mathfrak{p}_O^P([f]) = [f^*]$. We are only claiming that this will be the case if f is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small, which is a stronger property than being \mathcal{U} -approximable. The reason for the introduction of this stronger property, is that we are not able to show that if two definably homotopic maps are \mathcal{U} -approximable, then their approximations are homotopic. We can do this only if the maps are $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small. The formal definition is the following.

Definition 11.9 (Definition of the map \mathfrak{p}_O^P). Let $O \subseteq X/E$ be an open set. Let \mathcal{U} be an open cover of O which is a star-refinement of a good cover and let $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ be a definable map. By Lemma 11.6 there is a subdivision P' of P and a normal triangulation (P', ϕ) of $|P|$ such that $f' = f \circ \phi$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P')$ -small. By Lemma 11.5 f' has a \mathcal{U} -approximation f'^* . We shall see (Lemma 11.11 below) that the homotopy class $[f'^*]$ does not depend on the choice of P', ϕ and f'^* , so we can define $\mathfrak{p}_O^P([f]) = [f'^*]$.

To prove that the definition is sound we need the following.

Lemma 11.10. Let $f_0, f_1 : |P|_M \rightarrow X$ be definable maps and let f_0^* and f_1^* be \mathcal{U} -approximations of f_0, f_1 respectively. If f_0, f_1 are $\mathfrak{p}^{-1}(\mathcal{U})$ -close, then f_0^* and f_1^* are $\text{St}(\mathcal{U})$ -close.

Proof. Let $y \in |P|_{\mathbb{R}}$ and let $x \in |P|_M$ be such that $\text{st}(x) = y$. By definition of approximation $f_0^*(y)$ is \mathcal{U} -close to $\mathfrak{p}(f_0(x))$, which by hypothesis is \mathcal{U} -close to $\mathfrak{p}(f_1(x))$, which in turn is \mathcal{U} -close to $f_1^*(y)$. We deduce that $f_0^*(y)$ is $\text{St}(\mathcal{U})$ -close to $f_1^*(y)$. \square

We can now finish the proof that Definition 11.9 is sound.

Lemma 11.11. Let \mathcal{U} be a star-refinement of a good cover of O . Let $f_0, f_1 : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ be definably homotopic definable continuous maps and let (P_0, ϕ_0) and (P_1, ϕ_1) be two normal triangulations of P such that $f_0 \circ \phi_0$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P_0)$ -small and $f_1 \circ \phi_1$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P_1)$ -small. Now let $(f_0 \circ \phi_0)^*$ and $(f_1 \circ \phi_1)^*$ be \mathcal{U} -approximations of $f_0 \circ \phi_0$ and $f_1 \circ \phi_1$ respectively. Then $(f_0 \circ \phi_0)^*$ and $(f_1 \circ \phi_1)^*$ are homotopic.

Proof. First note that $f_0 \circ \phi_0$ and $f_1 \circ \phi_1$ are definably homotopic, because so are f_0 and f_1 and ϕ_0, ϕ_1 are both definably homotopic to the identity. Let $H : |P \times I|_M \rightarrow \mathfrak{p}^{-1}(O)$ be a definable homotopy between $f_0 \circ \phi_0 = H_0$ and $f_1 \circ \phi_1 = H_1$. Let P' be a common refinement of P_0 and P_1 (for the existence see for instance [Muk15, Cor. 9.5.8]).

Now let (T, ψ) be a normal triangulation of $P' \times I$ such that $H \circ \psi$ is $(\mathfrak{p}^{-1}(\mathcal{U}), T)$ -small. Notice that T induces two subdivisions P'_0 and P'_1 of P' such that $(P'_0 \times 0) \cup (P'_1 \times 1)$ is a subcomplex of T . Notice that both $f_0 \circ \phi_0$ and $f_1 \circ \phi_1$ are $(\mathfrak{p}^{-1}(\mathcal{U}), P')$ -small, because the smallness property is preserved by refining the triangulations. Moreover, the restriction of ψ to the subcomplex $(P'_0 \times 0) \cup (P'_1 \times 1)$ induces two normal triangulations (P'_0, ψ_0) and (P'_1, ψ_1) of P' , namely $\psi_0(x) = y$ if and only if $\psi(x, 0) = (y, 0)$, and similarly for ψ_1 . By the properties of normal triangulations, for each $\sigma \in P'$, we have $\psi_0(\sigma) = \sigma = \psi_1(\sigma)$, so $f_0 \circ \phi_0 \circ \psi_0$ and $f_1 \circ \phi_1 \circ \psi_1$ are also $(\mathfrak{p}^{-1}(\mathcal{U}), P')$ -small.

Now let $(H \circ \psi)^* : |P \times I|_{\mathbb{R}} \rightarrow O$ be a \mathcal{U} -approximation of $H \circ \psi$. Then $(H \circ \psi)^*$ is a homotopy between two maps, which are easily seen to be \mathcal{U} -approximations of $f_0 \circ \phi_0 \circ \psi_0$ and $f_1 \circ \phi_1 \circ \psi_1$ (the two maps induced by $H \circ \psi$ by restriction), so we may call them $(f_0 \circ \phi_0 \circ \psi_0)^*$ and $(f_1 \circ \phi_1 \circ \psi_1)^*$ respectively. Since ψ_0 fixes the simplexes of P' and $f_0 \circ \phi_0$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P')$ -small, we have that $f_0 \circ \phi_0 \circ \psi_0$ is \mathcal{U} -close to $f_0 \circ \phi_0$ (because any point of $|P|_M$ belongs to some $\sigma \in P'$ which is mapped into some element of $\mathfrak{p}^{-1}(\mathcal{U})$ by both maps). By Lemma 11.10 it follows that $(f_0 \circ \phi_0 \circ \psi_0)^*$ is $\text{St}(\mathcal{U})$ -close to $(f_0 \circ \phi_0)^*$ hence homotopic to it. Similarly $(f_1 \circ \phi_1 \circ \psi_1)^*$ is homotopic to $(f_1 \circ \phi_1)^*$ and composing the homotopies we obtain the desired result. \square

Lemma 11.12. *Points (1) and (2) of Theorem 11.8 hold.*

Proof. We have already proved that \mathfrak{p}_O^P is well defined and we need to establish the naturality with respect to inclusions of open sets $U \subseteq V \subseteq X/E$. Let $f : |P|_M \rightarrow \mathfrak{p}^{-1}(U) \subseteq \mathfrak{p}^{-1}(V)$ be a continuous definable map and notice that $C = \mathfrak{p} \circ f(|P|_M)$ is a closed set. By Theorem 11.8(1), there are open covers \mathcal{U} of U and \mathcal{V} of V which star-refine a good cover of U and V respectively. We can further assume that \mathcal{V} refines $\mathcal{U} \cup \{C^{\mathbb{G}}\}$. By Lemma 11.6 there is a definable homeomorphism $\psi : |P|_M \rightarrow |P|_M$ definably homotopic to the identity such that $f' := f \circ \psi$ is \mathcal{V} -approximable (and clearly definably homotopic to f). Since $\psi(|P|_M) = |P|_M$, we have $\mathfrak{p} \circ f'(|P|_M) = C$. Let $f'^* : |P|_{\mathbb{R}} \rightarrow V$ be a \mathcal{V} -approximation of f' . Then by definition $\mathfrak{p}_V^P([f]) = [f'^*]$. Now fix some $x \in |P|_M$, and using the definition of \mathcal{V} -approximation find $V' \in \mathcal{V}$ such that both $(f'^* \circ \text{st})(x) \in V'$ and $(\mathfrak{p} \circ f')(x) \in V'$. Notice that the latter implies that V' cannot be contained in $C^{\mathbb{G}}$, hence it is contained in some element of \mathcal{U} . This shows that f'^* has image contained in U and is a \mathcal{U} -approximation of f' . It follows that $i_U \circ \mathfrak{p}_U^P([f]) = \mathfrak{p}_V^P \circ i_{\mathfrak{p}^{-1}(U)}([f]) = [f'^*]$. \square

Lemma 11.13. *Theorem 11.8(3) holds, namely we can work with \emptyset -semialgebraic sets instead of simplicial complexes.*

Proof. If Z is a \emptyset -semialgebraic set, there is a \emptyset -definable homeomorphism $f : |P| \rightarrow Z$ where P is a simplicial complex P with real algebraic vertices. We have induced bijections $f_M^* : [Z(M), \mathfrak{p}^{-1}(U)]^{\text{def}} \simeq [|P|_M, \mathfrak{p}^{-1}(U)]^{\text{def}}$ and $f_{\mathbb{R}}^* : [Z_{\mathbb{R}}, U] \simeq [|P|_{\mathbb{R}}, U]$. The results now follows from the previous points of the theorem. \square

Lemma 11.14. *Theorem 11.8(4) holds, namely we can fix a base point and work with relative homology.*

Proof. It suffices to notice that all the constructions in the proofs can equivalently be carried out for spaces with base points. \square

Lemma 11.15. *Theorem 11.8(5) holds, namely for any open set $O \subseteq X/E$ there is a well defined map*

$$\mathfrak{p}_O^P : \pi_n^{\text{def}}(\mathfrak{p}^{-1}(O)) \rightarrow \pi_n(O)$$

which is a group homomorphism for $n > 0$.

Proof. We have already proved that there is a natural map $\mathfrak{p}_O^{S^n} : \pi_n^{\text{def}}(\mathfrak{p}^{-1}(O)) \rightarrow \pi_n(O)$. We need to check that this map is a group homomorphism. To this end, let S^{n-1} be the equator of S^n . Recall that, given $[f], [g] \in \pi_n^{\text{def}}(\mathfrak{p}^{-1}(O))$, where $f, g : S^n \rightarrow \mathfrak{p}^{-1}(O)$, the group operation $[f] * [g]$ is defined as follows. Consider the natural map $\phi : S^n \rightarrow S^n/S^{n-1} = S^n \vee S^n$, and let $[f] * [g] = [(f \vee g) \circ \phi]$, where $f \vee g$ maps the first S^n using f , and the second using g . A similar definition also works for $\pi_n(O)$. Now, we have to check that $\mathfrak{p}_O^{S^n}([f] * [g]) = \mathfrak{p}_O^{S^n}([f]) * \mathfrak{p}_O^{S^n}([g])$.

By the triangulation theorem we can identify S^n with the realization of a simplicial complex P defined over \mathbb{R}^{alg} and, modulo homotopy and taking a subdivision, we can assume that f and g are $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small where \mathcal{U} is an open cover of O star-refining a good cover. Let f^* and g^* be \mathcal{U} -approximations of f and g respectively, so that $\mathfrak{p}_O^{S^n}([f]) = [f^*]$ and $\mathfrak{p}_O^{S^n}([g]) = [g^*]$. Now it suffices to observe that $f^* \vee g^*$ is a \mathcal{U} -approximation of $f \vee g$. \square

The proof of Theorem 11.8 is now complete.

12. THEOREM B

In this section we work under the following strengthening of 11.1.

Assumption 12.1. *X/E is a triangulable topological space and each fiber of $\mathfrak{p} : X \rightarrow X/E$ is the intersection of a decreasing sequence of definably contractible open sets.*

By Proposition 3.4 the assumption implies in particular that \mathfrak{p} is continuous, so we have indeed a strengthening of 11.1. The above contractibility hypothesis was already exploited in [BM11, Ber09, Ber07] and is satisfied by the main examples discussed in Section 5 and Section 6.

Theorem 12.2 (Theorem B). *Assume that $\mathfrak{p} : X \rightarrow X/E$ satisfies 12.1 and let P be a closed simplicial complex defined over \mathbb{R}^{alg} . Then the map $\mathfrak{p}_O^P : [P|_M, \mathfrak{p}^{-1}(O)]^{\text{def}} \rightarrow [P|_{\mathbb{R}}, O]$ in Theorem 11.8 is a bijection and similarly for pointed spaces. Thus in particular*

$$\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$$

and more generally we have a natural isomorphism $\pi_n^{\text{def}}(\mathfrak{p}^{-1}(O)) \cong \pi_n(O)$ for every open subset $O \subseteq X/E$ and every $n > 0$.

Recall that if $X = X(M) \subseteq M^k$ is a closed and bounded \emptyset -semialgebraic and $\text{st} : X \rightarrow X(\mathbb{R})$ is the standard part map, we can identify $\mathfrak{p} : X \rightarrow X/E$ with $\text{st} : X \rightarrow X(\mathbb{R})$ and deduce the following result of [BO09].

Corollary 12.3. *If $X = X(M) \subseteq M^k$ is a closed and bounded \emptyset -semialgebraic and $\text{st} : X \rightarrow X(\mathbb{R})$ is the standard part map, then*

$$\pi_n^{\text{def}}(X) \cong \pi_n(X(\mathbb{R}))$$

and similarly $[P|_M, X]^{\text{def}} \cong [P|_{\mathbb{R}}, X(\mathbb{R})]$ for every closed simplicial complex P defined over \mathbb{R}^{alg} .

In the rest of the section we prove Theorem 12.2. The main difficulty is the following. The homotopy properties of a space are essentially captured by the nerve of a good cover, but unfortunately it is not easy to establish a correspondence between good covers of X/E in the topological category and good covers of X in the

definable category. One can try to take the preimages $\mathfrak{p}^{-1}(U)$ in X of the open sets U belonging to a good cover of X/E , but these preimages are only \mathbb{V} -definable, and if we approximate them by definable sets, we lose some control on the intersections. We shall show however, that we can perform these approximations with a controlled loss of the amount of “goodness” of the covers. Granted all this, the idea is to lift homotopies from X/E to X , with an approach similar to the one of [Sma57, Dug69], namely we start with the restriction of the relevant maps to the 0-skeleton, and we go up in dimension.

In the rest of the section fix an open set $O \subseteq X/E$. We need the following.

Lemma 12.4. *Let \mathcal{V} be an open cover of O . Then there is a refinement \mathcal{W} of \mathcal{V} such that for every $W \in \mathcal{W}$ there is $V \in \mathcal{V}$ and a definably contractible definable set $B \subseteq X$ such that $\mathfrak{p}^{-1}(W) \subseteq B \subseteq \mathfrak{p}^{-1}(V)$.*

Proof. Let $y \in O$. By our assumption $\mathfrak{p}^{-1}(y)$ is a decreasing intersection $\bigcap_{i \in \mathbb{N}} B_i(y)$ of definably contractible definable sets $B_i(y)$. Now let $V(y) \in \mathcal{V}$ contain y and note that $\mathfrak{p}^{-1}(V(y))$ is a \mathbb{V} -definable set containing $\mathfrak{p}^{-1}(y) = \bigcap_{i \in \mathbb{N}} B_i(y)$. By logical compactness $B_n(y) \subseteq \mathfrak{p}^{-1}(V(y))$ for some $n = n(y) \in \mathbb{N}$. By Proposition 3.3 we can find an open neighbourhood $W(y)$ of y with $\mathfrak{p}^{-1}(W(y)) \subseteq B_n(y)$. We can thus define \mathcal{W} as the collection of all the sets $W(y)$ as y varies in O . \square

Corollary 12.5. *Let \mathcal{V} be an open cover of O . Then there is a refinement \mathcal{W} of \mathcal{V} with the following property: every definable continuous map $f : |\partial\sigma|_M \rightarrow \mathfrak{p}^{-1}(W) \subseteq \mathfrak{p}^{-1}(O)$ whose domain is the boundary of a definable simplex and whose image is contained in $\mathfrak{p}^{-1}(W)$ for some $W \in \mathcal{W}$, can be extended to a definable continuous map $F : |\bar{\sigma}|_M \rightarrow \mathfrak{p}^{-1}(O)$ on the whole closed simplex $|\bar{\sigma}|_M$ with image contained in $\mathfrak{p}^{-1}(V)$ for some $V \in \mathcal{V}$.*

Proof. Let \mathcal{V} and \mathcal{W} be as in Lemma 12.4. By hypothesis, and by the property of \mathcal{W} , we have that $f(|\partial\sigma|_M) \subseteq \mathfrak{p}^{-1}(W) \subseteq B \subseteq \mathfrak{p}^{-1}(V)$ for some definably contractible set B and some $V \in \mathcal{V}$. Then, f can be extended to a definable map on $|\bar{\sigma}|_M$ with image contained in $B \subseteq \mathfrak{p}^{-1}(V)$ [vdD98, 8, 3,10]. \square

Definition 12.6. If \mathcal{W} and \mathcal{V} are as in Corollary 12.5, we say that \mathcal{W} is **semi-good** within \mathcal{V} .

Lemma 12.7. *For any open cover \mathcal{U} of O and any $n \in \mathbb{N}$, there is a refinement \mathcal{W} of \mathcal{U} such that, given a n -dimensional closed simplicial complex P , a closed subcomplex L , and a $(\mathfrak{p}^{-1}(\mathcal{W}), P)$ -small definable continuous map $f : |L \cup P^{(0)}|_M \rightarrow \mathfrak{p}^{-1}(O)$, there is a $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small definable continuous map $F : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ extending f .*

Proof. Reasoning by induction, it suffices to show that given $k < n$ and an open cover \mathcal{U} of O , there is a refinement \mathcal{W} of \mathcal{U} such that, given a n -dimensional closed simplicial complex P and a $(\mathfrak{p}^{-1}(\mathcal{W}), P)$ -small definable map $f : |L \cup P^{(k)}|_M \rightarrow \mathfrak{p}^{-1}(O)$, there is a $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small definable map $F : |L \cup P^{(k+1)}|_M \rightarrow \mathfrak{p}^{-1}(O)$ extending f .

To this aim, consider three open covers $\mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$ of O such that \mathcal{V} is a star-refinement of \mathcal{U} and $\mathcal{W} \prec \mathcal{V}$ is semi-good within \mathcal{V} . Let $\sigma \in P^{(k+1)}$ be a $(k+1)$ -dimensional closed simplex such that $\bar{\sigma}$ is not included in the domain of f . Since $|\partial\sigma|_M \subseteq |\sigma \cap P^{(k)}|_M \subseteq \text{dom}(f)$ and f is $(\mathfrak{p}^{-1}(\mathcal{W}), P)$ -small, there is $W \in \mathcal{W}$ such that $f(|\partial\sigma|_M) \subseteq f(|\sigma \cap P^{(k)}|_M) \subseteq \mathfrak{p}^{-1}(W)$. By the choice of \mathcal{W} , there is $V_\sigma \in \mathcal{V}$ such that we can extend $f|_{\partial\sigma}$ to a map $F_\sigma : |\bar{\sigma}|_M \rightarrow \mathfrak{p}^{-1}(V_\sigma)$ and define $F : |L \cup P^{(k+1)}|_M \rightarrow \mathfrak{p}^{-1}(O)$ as the union of f and the various F_σ for $\sigma \in P^{(k+1)}$. Notice that the union is a well defined continuous function since two distinct F_σ coincide with f on their common domain.

It remains to prove that $F : |L \cup P^{(k+1)}|_M \rightarrow \mathfrak{p}^{-1}(O)$ is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small. To this aim let $\tau \in P$ be any simplex. By our hypothesis there is $W \in \mathcal{W}$ such that $f(|\bar{\tau} \cap P^{(k)}|_M) \subseteq \mathfrak{p}^{-1}(W)$. Now let $V \in \mathcal{V}$ contain W . By construction each face σ of τ belonging to $L \cup P^{(k+1)}$ is mapped by F into $\mathfrak{p}^{-1}(V_\sigma)$ for some $V_\sigma \in \mathcal{V}$. Moreover V_σ intersects W , so it is included in $\text{St}_\mathcal{V}(W)$. The latter depends only on τ and not on σ and is contained in some $U \in \mathcal{U}$. We have thus shown that $\bigcup_\sigma V_\sigma$ is contained in some $U \in \mathcal{U}$, thus showing that F is $(\mathfrak{p}^{-1}(\mathcal{U}), P)$ -small. \square

Definition 12.8. Let \mathcal{U} be an open cover of O . If \mathcal{W} is as in Lemma 12.7 we say that \mathcal{W} is n -good within \mathcal{U} . If the only member of \mathcal{U} is O (or if the choice of \mathcal{U} is irrelevant), we simply say that \mathcal{W} is n -good.

Lemma 12.9. Let $n \in \mathbb{N}$ and let \mathcal{W} be an $n + 1$ -good cover of O . If P is an n -dimensional simplicial complex and $f, g : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ are definable continuous functions such that for every $\sigma \in P$ there is $W \in \mathcal{W}$ such that $f(\sigma)$ and $g(\sigma)$ are contained in $\mathfrak{p}^{-1}(W)$, then f and g are definably homotopic.

Proof. Let $I = [0, 1]$ and consider the simplicial complex $P \times I$ (of dimension $n + 1$) with the standard triangulation (as in [Hat02, p. 112, Proof of 2.10]). Consider the subcomplex $P \times \{0, 1\}$ of $P \times I$ and note that it contains the 0-skeleton of $P \times I$. Define $f \sqcup g : |P \times \{0, 1\}|_M \rightarrow \mathfrak{p}^{-1}(O)$ as the function which sends $(x, 0)$ to $f(x)$ and $(x, 1)$ to $g(x)$. Note that $f \sqcup g$ is $(\mathfrak{p}^{-1}(\mathcal{W}), P \times I)$ -small by hypothesis. By Lemma 12.7 we can extend it to a definable continuous function $H : |P \times I|_M \rightarrow \mathfrak{p}^{-1}(O)$. This map is a homotopy between f and g . \square

Lemma 12.10. Let $n \in \mathbb{N}$. Let \mathcal{V} be an open covering of O which is a star refinement of a $n + 1$ -good cover \mathcal{W} . Given an n -dimensional simplicial complex P and definable continuous maps $f, g : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$, if there is a map $f^* : |P|_\mathbb{R} \rightarrow O$ which is a \mathcal{V} -approximation of both f and g , then f and g are definably homotopic.

Proof. Let P' be an iterated barycentric subdivision of P such that for each $\sigma \in P'$ there is $V \in \mathcal{V}$ such that $f^*(\bar{\sigma}) \subseteq V$. We claim that for each $\sigma \in P'$, there is a $W \in \mathcal{W}$ such that $\mathfrak{p} \circ f(\sigma)$, $\mathfrak{p} \circ g(\sigma)$ (and $f^* \circ \text{st}(\sigma)$) are in W . Given this claim, we can conclude using Lemma 12.9.

To prove the claim, fix a $\sigma \in P'$ and let $V \in \mathcal{V}$ be such that $f^*(\sigma) \subseteq V$. Since f^* is \mathcal{V} -approximation of f , for each $x \in \sigma$ there is $V_x \in \text{St}(\mathcal{V})$ such that $\mathfrak{p} \circ f(x)$ and $f^* \circ \text{st}(x)$ are in V_x , and similarly there is a V'_x such that $\mathfrak{p} \circ g(x)$ and $f^* \circ \text{st}(x)$ are in V'_x . Since V intersects both V_x and V'_x , $\text{St}_\mathcal{V}(V)$ contains both $\mathfrak{p} \circ f(x)$ and $\mathfrak{p} \circ g(x)$, and since $\text{St}(\mathcal{V})$ refines \mathcal{W} , there is $W \in \mathcal{W}$ with the same property. \square

Lemma 12.11. Let $n \in \mathbb{N}$. There is an open cover \mathcal{W} of O such that, given an n -dimensional simplicial complex P and definable continuous maps $f, g : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$, if f^* and g^* are \mathcal{W} -approximations of f and g respectively, and $G : |P \times I|_\mathbb{R} \rightarrow O$ is a homotopy between f^* and g^* , then there is a definable homotopy $H : |P \times I|_M \rightarrow \mathfrak{p}^{-1}(O)$ between f and g .

Proof. Let \mathcal{U} be an $n + 1$ -good covering of O , let \mathcal{V} be such that $\text{St}(\mathcal{V})$ is a star refinement of \mathcal{U} and let $\mathcal{W} \prec \mathcal{V}$ be $n + 1$ -good within \mathcal{V} . Let T be a barycentric subdivision of $|P \times I|_\mathbb{R}$ such that G is (\mathcal{W}, T) -small. Let $H^{(0)} : T^{(0)} \rightarrow \mathfrak{p}^{-1}(O)$ be such that $\mathfrak{p} \circ H^{(0)} = G \circ \text{st}$ on the vertices of T . For each simplex $\sigma \in T$ there is $W \in \mathcal{W}$ such that $G(|\bar{\sigma}|_\mathbb{R}) \subseteq W$, hence $H^{(0)}(|\sigma^{(0)}|_M) \subseteq W$. Using Lemma 12.7 we can extend $H^{(0)}$ to a $(\mathfrak{p}^{-1}(\mathcal{V}), T)$ -small definable continuous map $H : |T|_M \rightarrow \mathfrak{p}^{-1}(O)$. If $\mathbf{x} = (x, 0) \in |P \times 0|_M$ is a vertex of T , then $(f^* \circ \text{st})(x) = (G \circ \text{st})(\mathbf{x}) = (\mathfrak{p} \circ H)(\mathbf{x})$ by construction. Since moreover $f^* \circ \text{st}$ and $\mathfrak{p} \circ H$ are (\mathcal{V}, T) -small, it follows that $f^* \circ \text{st}$ and $\mathfrak{p} \circ H|_0$ are $\text{St}(\mathcal{V})$ -close, hence f^* is a $\text{St}(\mathcal{V})$ -approximation of both f (by hypothesis) and of $H|_0$. We can then conclude using Lemma 12.10 that f and

$H_{|_0}$ are homotopic, and, similarly, that $H_{|_1}$ is homotopic to g . Composing the homotopies, we can finally prove that f is homotopic to g . \square

Lemma 12.12. *Let \mathcal{U} be an open cover of O . Let $f^* : |P|_{\mathbb{R}} \rightarrow O$ be a continuous map. Then, we can find a map $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$ such that f^* is a \mathcal{U} -approximation of f .*

Proof. Let $n = \dim(P)$, let \mathcal{V} be a star-refinement of \mathcal{U} and let \mathcal{W} be n -good within \mathcal{V} . Consider an iterated baricentric subdivision P' of P such that f^* is (\mathcal{W}, P') -small. Let $f^{(0)} : P'^{(0)} \rightarrow \mathfrak{p}^{-1}(O)$ be such that $\mathfrak{p} \circ f^{(0)} = f^* \circ \text{st}$ on the vertices of P' . Then we can apply Lemma 12.7 to extend $f^{(0)}$ to a (\mathcal{V}, P') -small map $f : |P|_M \rightarrow \mathfrak{p}^{-1}(O)$. Now notice that $\mathfrak{p} \circ f$ and $f^* \circ \text{st}$ are $\text{St}(\mathcal{V})$ -close (since they are (\mathcal{V}, P') -small and they coincide on the vertices), and therefore f^* is a $\text{St}(\mathcal{V})$ -approximation of f , so also a \mathcal{U} -approximation. \square

We can now finish the proof of the main result of this section.

Proof of Theorem 12.2. First we prove the injectivity. Suppose that $\mathfrak{p}_O^P([f]) = \mathfrak{p}_O^P([g])$. Let \mathcal{W} be as in Lemma 12.11. Choosing a different representative of the homotopy classes we can assume without loss of generality that f and g are $(\mathfrak{p}^{-1}(\mathcal{W}), P)$ -small, $\mathfrak{p}_O^P([f]) = [f^*]$ and $\mathfrak{p}_O^P([g]) = [g^*]$, where f^* and g^* are \mathcal{W} -approximation of f and g respectively. By definition $[f^*] = [g^*]$, that is f^* and g^* are homotopic. We can now apply Lemma 12.11 to find a definable homotopy between f and g , and so $[f] = [g]$.

The surjectivity is immediate from Lemma 12.12. \square

13. THEOREM C

In this section we work under the following strengthening of 12.1, where we consider definable proper balls (Definition 6.2) instead of definably contractible sets.

Assumption 13.1. *X/E is a triangulable manifold, X is a definable manifold, and each fiber of $\mathfrak{p} : X \rightarrow X/E$ is the intersection of a decreasing sequence of definable proper balls.*

We also need:

Assumption 13.2 (Topological compact domination). *The image under $\mathfrak{p} : X \rightarrow X/E$ of a definable subset of X with empty interior, has empty interior.*

Both assumptions are satisfied by $\mathfrak{p} : G \rightarrow G/G^{00}$ for any definably compact group G (see section 7 and Corollary 6.4).

Theorem 13.3 (Theorem C). *Under Assumptions 13.1 and 13.2, $\dim(X) = \dim_{\mathbb{R}}(X/E)$.*

To prove the theorem the idea is to exploit the following link between homotopy and dimension: given a manifold Y and a punctured open ball $U := A \setminus \{y\}$ in Y , the dimension of Y is the least integer i such that $\pi_{i-1}(U) \neq 0$.

Proposition 13.4. *Under Assumption 13.1, $\dim(X) \geq \dim_{\mathbb{R}}(X/E)$.*

Proof. Let $n = \dim(X)$ and $N = \dim_{\mathbb{R}}(X/E)$. Fix $x \in X$ and let $y = \mathfrak{p}(x)$. Let B_0 be an open definable ball containing $\mathfrak{p}^{-1}(y)$. Since X/E is a manifold, there is a decreasing sequence of proper balls $A_i \subseteq X/E$ such that $y = \bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i \in \mathbb{N}} \overline{A_i}$. Now $B_0 \supseteq \mathfrak{p}^{-1}(y) = \bigcap_{i \in \mathbb{N}} \mathfrak{p}^{-1}(\overline{A_i})$ and $\mathfrak{p}^{-1}(\overline{A_i})$ is type-definable (because $\overline{A_i}$ is closed), so there is some $i \in \mathbb{N}$ with $\mathfrak{p}^{-1}(\overline{A_i}) \subseteq B_0$. Let $A = A_i$ and observe that $\mathfrak{p}^{-1}(A)$ is \bigvee -definable and contains the type definable set $\mathfrak{p}^{-1}(y)$. Since the latter is

a decreasing intersection of definable proper balls, there is some definable proper ball B_1 such that

$$x \in \mathfrak{p}^{-1}(y) \subseteq B_1 \subseteq \overline{B_1} \subseteq \mathfrak{p}^{-1}(A) \subseteq B_0.$$

Now let $f : S^{n-1} \rightarrow \partial B_1 = \overline{B_1} \setminus B_1$ be a definable homeomorphism (whose existence follows by the hypothesis that the ball is proper). By fixing base points, we can consider the homotopy class $[f]$ as a non-zero element of $\pi_{n-1}^{\text{def}}(B_0 \setminus x)$ (namely f is not definably homotopic to a constant in $B_0 \setminus x$).

A fortiori, $0 \neq [f] \in \pi_{n-1}^{\text{def}}(\mathfrak{p}^{-1}(A) \setminus \mathfrak{p}^{-1}(y))$, because if f is homotopic to a constant within a smaller space, it remains so in the larger space. Now observe that $\mathfrak{p}^{-1}(A) \setminus \mathfrak{p}^{-1}(y) = \mathfrak{p}^{-1}(A \setminus y)$ and by Theorem 12.2 we have $\pi_{n-1}^{\text{def}}(\mathfrak{p}^{-1}(A \setminus y)) \cong \pi_{n-1}(A \setminus y)$.

We conclude that $\pi_{n-1}(A \setminus y) \neq 0$, and since A is an open ball in the manifold X/E this can happen only if $n \geq N$. \square

So far we have not used the full strength of the assumption, namely the topological compact domination. The proof $\dim(X) \leq \dim_{\mathbb{R}}(X/E)$ is more subtle, because spaces of different dimension can have the same homotopy, for instance S^1 and $S^1 \times \mathbb{R}$. Keeping this in mind, suppose that X/E is a two-dimensional manifold. How can we exclude the possibility that X is a three dimensional definable manifold? The informal idea is the following. Take an open subset $O \subseteq X/E$ which looks like a thickened circle, namely an annulus, embedded in the two-dimensional manifold X/E . Its preimage $\mathfrak{p}^{-1}(O)$ would then look like a thickened cylinder in the three-dimensional definable manifold X (although it would only be \mathbb{V} -definable). We can now “kill the homotopy” of $\mathfrak{p}^{-1}(O)$ gluing to it a two-dimensional definable disk (as we would do with the ordinary cylinder $S^1 \times \mathbb{R}$) and use Theorem B to show that the image of the disk in X/E would kill the homotopy of the annulus O . This can only happen if the image of the disk fills the hole of the annulus, so in particular it has non-empty interior in X/E . We have thus violated compact domination, since the disk has empty interior in its three-dimensional ambient space. In the actual proof we have to do some more work in order to approximate \mathbb{V} -definable sets with definable sets without losing the homotopy information.

Proposition 13.5. *Under Assumptions 13.1 and 13.2, $\dim(X) \leq \dim_{\mathbb{R}}(X/E)$.*

Proof. As before, let $n = \dim(X)$ and $N = \dim_{\mathbb{R}}(X/E)$. Let $A_0 \subseteq X/E$ be an open N -ball, namely a set homeomorphic to $\{|x| \in \mathbb{R}^N : |x| < 1\}$. Let $A_1 \subseteq A_0$ be the image of $\{|x| \in \mathbb{R}^N : |x| < 1/2\}$ under the homeomorphism and note that $0 \neq \pi_{N-1}(A_0 \setminus \overline{A_1})$ and $A_0 \setminus \overline{A_1}$ is a deformation retract of $A_0 \setminus \{y\}$ for every $y \in A_1$.

By Theorem 12.2, we have $0 \neq \pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0 \setminus \overline{A_1}))$, so there is a map $f : S^{N-1} \rightarrow \mathfrak{p}^{-1}(A_0 \setminus \overline{A_1})$ of pointed spaces with $0 \neq [f] \in \pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0 \setminus \overline{A_1}))$.

Since A_0 is a ball, we have $\pi_{N-1}(A_0) = 0$ and, by Theorem 12.2, $\pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0)) = 0$ as well. In particular $[f] = 0$ when seen as an element of $\pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0))$. This is equivalent to say that f can be extended to a definable map $F : D \rightarrow \mathfrak{p}^{-1}(A_0)$, where $D = S^{N-1} \times I$ and F is a definable homotopy (relative to the base point) between f and a constant map.

Notice that $\dim(F(D)) \leq \dim(D) = N$. Now assume for a contradiction that $N < \dim(X)$. Then $\dim(F(D)) < \dim(X)$, and therefore $F(D)$ has empty interior in X . By topological compact domination $(\mathfrak{p} \circ F)(D)$ has empty interior in X/E , so in particular there is some $y \in A_1$ such that $y \notin (\mathfrak{p} \circ F)(D)$.

It follows that the image of F is disjoint from $\mathfrak{p}^{-1}(y)$, namely F takes values in $\mathfrak{p}^{-1}(A_0) \setminus \mathfrak{p}^{-1}(y) = \mathfrak{p}^{-1}(A_0 \setminus y)$ and witnesses the fact that f is null-homotopic when seen as a map into $\mathfrak{p}^{-1}(A_0 \setminus y)$.

We can now reach a contradiction as follows. Since $A_0 \setminus \overline{A_1}$ is a deformation retract of $A_0 \setminus \{y\}$, the inclusion induces an isomorphism $\pi_{N-1}(A_0 \setminus \overline{A_1}) \cong \pi_{N-1}(A_0 \setminus y)$. By the functoriality part in Theorem 12.2, there is an induced isomorphism $\pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0 \setminus \overline{A_1})) \cong \pi_{N-1}^{\text{def}}(\mathfrak{p}^{-1}(A_0 \setminus y))$. Moreover, this isomorphism sends the homotopy class of f to the homotopy class of f itself, but seen as a map with a different codomain. This is absurd since f was not null-homotopic as a map to $\mathfrak{p}^{-1}(A_0 \setminus \overline{A_1})$, while we have shown that it is null-homotopic as a map to $\mathfrak{p}^{-1}(A_0 \setminus y)$. \square

As a corollary we obtain.

Corollary 13.6 ([EO04]). *Let G be an abelian definably compact and definably connected group of dimension n . Then $\pi_1^{\text{def}}(G) \cong \mathbb{Z}^n$ and $G[k] \cong (\mathbb{Z}/k\mathbb{Z})^n$, where $G[k]$ is the k -torsion subgroup.*

Proof. By [BOPP05], G/G^{00} is a compact abelian connected Lie group and by the previous result its dimension is n . It follows that G/G^{00} is isomorphic to an n -dimensional torus, so $\pi_1(G/G^{00}) \cong \mathbb{Z}^n$ and, by Theorem 12.2, $\pi_1^{\text{def}}(G) \cong \mathbb{Z}^n$ as well.

To determine the k -torsion two approaches are possible. The first is to argue as in [EO04], namely to observe that $G[k] \cong \pi_1^{\text{def}}(G)/k\pi_1^{\text{def}}(G)$ and $\pi_1^{\text{def}}(G) \cong \mathbb{Z}^n$. Alternatively we can use the fact that G^{00} is divisible [BOPP05] and torsion free [HPP08], so G and G/G^{00} have isomorphic torsion subgroups. Since G/G^{00} is a torus of dimension n , its torsion is known and we obtain the desired result. \square

Notice that in [EO04] both the isomorphism $\pi_1^{\text{def}}(G) \cong \mathbb{Z}^n$ and the determination of the k -torsion of G is proved directly without using G/G^{00} , while our argument is a reduction to the case of the classical tori.

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