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Multi-user Rate-Based Flow Control

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Multi-user Rate-Based Flow Control

Eitan Altman* and Tamer Başar†

Abstract

Flow and congestion control allow the users of a telecommunication network to regulate the traffic it sends into the network according to the quality of service that they require. The flow control may be performed by the network, as is the case in ATM networks (the Available Bit Rate transfer capacity), or by the users themselves, as is the case in the Internet (TCP/IP). We consider in this paper both cases using optimal control techniques. For the first case, we obtain a formulation of a dynamic team problem. The second case is handled by dynamic non-cooperative game techniques; we establish the existence and uniqueness of a Nash Equilibrium, and compute the corresponding performance measures and equilibrium (dynamic) policies. We further show that when the users update their policies in a greedy manner, not knowing *a priori* the utilities of the other players, their policies converge to the Nash equilibrium.

Keywords: Multi-user rate-based flow control; High speed networks; Linear-quadratic control; Linear-quadratic differential games; Nash equilibria.

1 Introduction

We consider M users that share a common bottleneck queue in a telecommunications network. The input flow of information from the users is controlled so as to achieve the best quality of service. As is the case in many proposed flow-control schemes [2, 11], we assume that there is some target value of the queue length which the users try to track; this value, and the control policies are chosen so as to avoid the buffer to be full (in order to minimize losses), and on the other hand, to avoid the queue to empty, in which case there is loss in the potential throughput. A second objective of each user is related to the input rate: we assume that some fraction of the available bandwidth is allocated to each user, and the user tries to minimize deviation from this allocated bandwidth.

The flow control is typically performed *dynamically*: some feedback information on the congestion or on round trip delay is used to update the input rate. For example, in the Tahoe version of TCP/IP congestion control [14], congestion is detected through losses or through time-out mechanisms. In the Vegas version of TCP/IP [7] the available bandwidth is also used as feedback information (it is obtained through estimation of round-trip delays). More detailed queue length information may also

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be available as feedback information. In the Available Bit Rate (ABR) transfer capacity of ATM, both queue length information as well as information on the rate may be conveyed from the switches to the sources through special information cells that are called RM (Resource Management) cells.

Flow control is often performed in a decentralized way in telecommunication networks: each user controls its own flow. This is the case in the Internet, see [14], or in some best-effort type traffic in ATM (the Unspecified Bit Rate transfer capacity, see [1]). This gives rise to a non-cooperative dynamic game framework: each user has its own objectives, but the action of the different users influence the quality of service of other users. Controllers that have been implemented in large scale, such as the TCP/IP, have typically been designed using heuristic techniques based on growing experience (and on simulation studies); however, they did not involve a game-theoretical related analysis.

On the other hand, there has been some work on the use of non-cooperative game theoretical techniques to design simple flow controls. The problem of choosing fixed rates (non-state dependent) has been investigated in [5, 6, 8, 10, 17, 19]; existence and uniqueness of the Nash equilibrium have been proven and convergence of synchronous and asynchronous implementations of the flow control have been obtained. For state-dependent flow control, only structural results of the Nash equilibria have been proven [12, 16]. Another type of a non-cooperative flow control occurs when a central controller is interested in some globally optimal performance measure for the entire network, but the objectives (utilities) of the users are private information. In that case, the game occurs in the revealing policy of these utilities. The design of a (non-state dependent) flow control that induces a truthful revealing policy for all users has been obtained in [9, 18].

In this paper we study state-dependent decentralized control and game problems. For both the non-cooperative problem (that occurs when the controls are at end-points) as well as the cooperative team problem (occurring when the controllers are in the switches, as is the case in the Available Bit Rate transfer capacity of ATM [1]), we obtain explicitly the optimal (unique) controllers and the associated values. We then study asynchronous and synchronous update algorithms that the users might implement in order to compute their policies *on line*, since in practice, a user may not have access to full information on the utilities of other users, and thus, may not be able to construct its own Nash equilibrium solution *off line*. Instead, it is natural to assume that such a user would follow a "greedy approach" of optimizing from time to time its response against the current policies of other users. We present three such algorithms, and show that they converge to the unique equilibrium policy. Presentation of some numerical results complement the study.

2 The model

It is assumed that the network has linearized dynamics (for the control of queue length), and all performance measures (such as throughput, delays, loss probabilities, etc.) are determined essentially by a bottleneck node. Both these assumptions admit theoretical as well as experimental justifications; see, [2].

Let $q(t)$ denote the queue length at a bottleneck link, and $s(t)$ denote the total effective service rate available at that link. We assume that each user is assigned a fix proportion of the available bandwidth; the traffic of source m at that link at time t has an available bandwidth of $a_m s(t)$. Unless otherwise stated, we assume that $\alpha := \sum_{m=1}^{M} a_m = 1$. We let $s(t)$ be arbitrary, but assume that the controllers have perfect measurements of it. Let $r_m(t)$ denote the (controlled) rate of source m at time t, $m \in \mathcal{M} := \{1, \ldots, M\}$, and $u_m(t) := r_m(t) - a_m s(t)$ be its shifted version.

Consider the following idealized dynamics for the queue length:

$$\frac{dq}{dt} = \sum_{m=1}^{M} (r_m - a_m s) = \sum_{m=1}^{M} u_m \,, \tag{1}$$

which we call *idealized* because the end-point effects have been ignored. The objectives of the flow controllers are *(i)* to ensure that the bottleneck queue size stays around some desired level $\bar{Q}$, and *(ii)* to achieve good tracking between input and output rates. In particular, the choice of $\bar{Q}$ and the variability around it have direct impact on loss probabilities and throughput. We therefore define a shifted version of q:

$$x(t) := q(t) - \bar{Q} \,,$$

in view of which (1) now becomes

$$\frac{dx}{dt} = \sum_{m=1}^{M} u_m \,. \tag{2}$$

An appropriate local cost function that is compatible with the objectives stated above would be the one that penalizes variations in $x(t)$ and $u_m(t)$ around *zero* — a candidate for which is the weighted quadratic cost function. Associated with user m is a positive constant c_m appearing in its immediate cost as described below.

We shall first consider two non-cooperative scenarios, formulated as non-cooperative differential games, in which each user minimizes its own individual cost function. Then we shall study two cooperative scenarios, formulated as team control problems, in which all users have a common objective.

We fix an initial state $x(0)$, and assume that the actions of controller m ($m \in \mathcal{M}$) are determined by a control policy $\mu_m \in \mathcal{U}_m$, where

$$u_m(t) = \mu_m(t, x_{[0,t]}), \qquad t \in [0, \infty) \,.$$

Here, μ_m is taken to be piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument. We denote the class of all such policies for user m by $\mathcal{U}_m$ It will soon turn out that it will be sufficient to restrict attention to a subclass of $\mathcal{U}_m$, comprising policies that are linear in the current value of x.

The two non-cooperative cases are defined as follows:

- N1: the individual cost to be minimized by controller m $(m \in \mathcal{M})$ is

$$J_m^{N1}(u) = \int_0^\infty \left(|x(t)|^2 + \frac{1}{c_m} |u_m(t)|^2 \right) dt \,. \tag{3}$$

- N2: the individual cost to be minimized by controller m $(m \in \mathcal{M})$ is

$$J_m^{N2}(u) = \int_0^\infty \left(\frac{1}{M} |x(t)|^2 + \frac{1}{c_m} |u_m(t)|^2 \right) dt \,. \tag{4}$$

Note that in case N2 the "effort" for keeping the deviations of the queue length from the desired value is split equally between the users. The precise formulation of N1 and N2 is in terms of Nash equilibria: We seek a multi-policy $\mu^* := (\mu_1^*, ..., \mu_M^*)$ such that no user has an incentive to deviate from, i.e.

$$J_m^{N1}(\mu^*) = \inf_{\mu_m \in \mathcal{U}_m} J_m^{N1}([\mu_m | \mu_{-m}^*]) \tag{5}$$

where $[\mu_m | \mu_{-m}^*]$ is the policy obtained when for each $j \neq m$, player j uses policy μ_j^*, and player m uses μ_m. The definition for N2 is similar.

The two cooperative (team) cases are defined as follows:

- T1: the global (team) cost to be minimized is

$$J^{T1}(u) = \int_0^\infty \left(|x(t)|^2 + \sum_{m=1}^{M} \frac{1}{c_m} |u_m(t)|^2 \right) dt \,. \tag{6}$$

- T2: the global cost to be minimized is

$$J^{T2}(u) := \int_0^\infty \left(\frac{1}{M} |x(t)|^2 + \sum_{m=1}^{M} \frac{1}{c_m} |u_m(t)|^2 \right) dt \,. \tag{7}$$

Let $\bar{c} := \sum_{m=1}^{M} c_m$.

Theorem 1 *For the noncooperative case Ni $(i = 1, 2)$, there exists a Nash equilibrium given by*

$$\mu_{Ni,m}^*(x) = -\beta_m^{Ni} x \,, \; m = 1, ..., M, \tag{8}$$

where β_m^{Ni} is given by

$$\beta_m^{N1} = \overline{\beta}^{(N1)} - \sqrt{\overline{\beta}^{(N1)^2} - c_m} \quad and \qquad \beta_m^{N2} = \overline{\beta}^{(N2)} - \sqrt{\overline{\beta}^{(N2)^2} - \frac{c_m}{M}}, \tag{9}$$

for cases N1 and N2, respectively; where $\overline{\beta}^{(Ni)} := \sum_{m=1}^{M} \beta_m^{Ni}, 1 = 1, 2$, are the unique solutions of

$$\overline{\beta}^{(N1)} = \frac{1}{M-1} \sum_{m=1}^{M} \sqrt{(\overline{\beta}^{(N1)})^2 - c_m} \quad and \quad \overline{\beta}^{(N2)} = \frac{1}{M-1} \sum_{m=1}^{M} \sqrt{(\overline{\beta}^{(N2)})^2 - \frac{c_m}{M}} = \frac{\overline{\beta}^{(N1)}}{\sqrt{M}}. \tag{10}$$

Moreover,

$$\beta_m^{N1} = \beta_m^{N2}\sqrt{M}. \tag{11}$$

The costs accruing to user m, under the two Nash equilibria above, are given by

$$J_m^{N1}(\mu_{N1}^*) = \frac{\beta_m^{N1}}{c_m} x^2 \quad and \quad J_m^{N2}(\mu_{N2}^*) = \frac{\beta_m^{N2}}{c_m} x^2 = \frac{1}{\sqrt{M}} J_m^{N1}(u_{N1}^*).$$

For each case, (8) is the unique equilibrium among stationary policies and is time-consistent. In particular,
(i) for the symmetric case $c_m = c_j =: c$ for all $m, j \in \mathcal{M}$,

$$\beta_m^{N1} = \sqrt{\frac{c}{2M-1}}, \qquad and \qquad \beta_m^{N2} = \sqrt{\frac{c}{M(2M-1)}}, \quad \forall m \in \mathcal{M}\,; \tag{12}$$

(ii) in the case of $M = 2$, with general c_m's, we have for $m = 1, 2,\ j \neq m$,

$$\beta_m^{N1} = \left[-\frac{2c_j - c_m}{3} + 2\frac{\sqrt{c_1^2 - c_1c_2 + c_2^2}}{3} \right]^{1/2}, \qquad \beta_m^{N2} = \frac{\beta_m^{N1}}{\sqrt{2}},$$

and if moreover, $c_1 = c_2 = c$ then $\beta_m^{N1} = \sqrt{c/3}$, $\beta_m^{N2} = \sqrt{c/6}$.

Proof: We prove the result only for the case $N1$; its counterpart for case $N2$ can be proven similarly. Also for ease of notation, we drop all superscripts pertaining to the case considered.

We first choose a candidate solution of the form (8) for each player, and consider the optimal response of player m to the fixed policy μ_j, $j \neq m$ of the other players. Let $\beta_{-m} := \sum_{j \neq m} \beta_j$. Player m is faced with a linear-quadratic optimal control problem with the dynamics

$$\frac{dx}{dt} = u_m - \beta_{-m} x\,,$$

and cost $J_m^{N1}(u)$ that is strictly convex in u_m. By a standard result in optimal control [3], there exists a unique optimal response for player m, of the form $u_m = -c_m P_m x$, where P_m is the unique positive solution of the Riccati equation

$$-2\beta_{-m}P_m - P_m^2 c_m + 1 = 0. \tag{13}$$

The optimal cost to player m is then $J_m^{N1} = P_m x^2$. Denoting $\beta'_m = c_m P_m$, we obtain from (13)

$$\beta'_m = f_m(\beta_{-m}) := -\beta_{-m} + \sqrt{\beta_{-m}^2 + c_m}. \tag{14}$$

A necessary and sufficient condition for μ to be in equilibrium is then that $\beta' = \beta$, or equivalently, that

$$\overline{\beta}^2 = \beta_{-m}^2 = c_m. \tag{15}$$

This yields the expressions (9)-(10) ((10) is obtained by summing (9) over $m \in \mathcal{M}$). The fact that (10) admits a unique solution follows from the fact that the left-hand side minus the right-hand side of (10) is strictly decreasing in $\overline{\beta}$ over the interval $[\max_m \sqrt{c_m}, \infty)$, it is positive at $\overline{\beta} = \max_m \sqrt{c_m}$ and it tends to $-\infty$ as $\beta \to \infty$. (i) and (ii) are obtained by solving for $\overline{\beta}$ from (10). ∎.

Corollary 1 β_m^{Ni}, $i = 1, 2$, *in (9) can be approximated by*

$$\beta_m^{N1} \sim \frac{c_m}{\sqrt{2\overline{c}}}, \qquad \beta_m^{N2} \sim \frac{c_m}{\sqrt{2M\overline{c}}}$$

when

$$\overline{c} >> \max_m c_m. \tag{16}$$

In that case, the Nash equilibrium costs per user m are approximated by

$$J_m^{N1}(\mu_{N1}^*) \sim \frac{x^2}{\sqrt{2\overline{c}}}, \qquad J_m^{N2}(\mu_{N2}^*) = \frac{x^2}{\sqrt{2M\overline{c}}}$$

which are independent of m.

Proof: We again consider only N1; the result for N2 is then follows from (11). We again drop the supercripts identifying the two cases. Assume that (16) holds. Then, (9) can be written as:

$$\beta_m = \overline{\beta}\left[1 - \sqrt{1 - \frac{c_m}{\overline{\beta}^2}}\right] = \frac{c_m}{2\overline{\beta}^2}\left[1 + O\left(\frac{c_m}{\overline{\beta}^2}\right)\right], \tag{17}$$

where $O(\cdot)$ is a function that satisfies $\lim_{x\to 0} O(x) = 0$. The right hand side of the expression for $\overline{\beta}^{N1}$ in (10) can be written as:

$$\begin{aligned}\frac{\overline{\beta}}{M-1}\sum_{m=1}^{M}\sqrt{1-\frac{c_m}{\overline{\beta}^2}} &= \frac{\overline{\beta}}{M-1}\sum_{m=1}^{M}\left[1-\frac{c_m}{2\overline{\beta}^2}\left(1+O\left(\frac{c_m}{\overline{\beta}^2}\right)\right)\right] \\ &= \frac{\overline{\beta}}{M-1}\left[M-\frac{\overline{c}}{2\overline{\beta}^2}\left(1+O\left(\frac{\max_m c_m}{\overline{\beta}^2}\right)\right)\right].\end{aligned}$$

Substituting this in (10) yields

$$\overline{\beta} = \sqrt{\frac{\overline{c}}{2}}\left[1+O\left(\frac{\max_m c_m}{\overline{\beta}^2}\right)\right]. \tag{18}$$

A solution of (18) (and thus of (10)) is

$$\overline{\beta} = \sqrt{\frac{\overline{c}}{2}}\left[1+O\left(\frac{\max_m c_m}{\overline{c}}\right)\right]. \tag{19}$$

Since, by Theorem 1, the solution of (10) is unique, this is indeed the required expression for $\overline{\beta}$. Substituting this into (17) yields the approximation for β_m. ∎

Theorem 2 *Consider cases T1 and T2. For each case, there exists a unique optimal policy which is stationary and is given by*

$$\mu_m^{Ti} = -\beta_m^{Ti} x, \tag{20}$$

where

$$\beta_m^{T1} = \frac{c_m}{\sqrt{\overline{c}}}, \qquad \beta_m^{T2} = \frac{c_m}{\sqrt{M\overline{c}}}, \qquad m = 1, ..., M. \tag{21}$$

The optimum team values are given by

$$J^{T1} := \inf_{\mu} J^{T1}(\mu) = \frac{x^2}{\sqrt{\overline{c}}}, \qquad J^{T2} := \inf_{\mu} J^{T2}(\mu) = \frac{x^2}{\sqrt{M\overline{c}}}. \tag{22}$$

Proof: Consider case T1. The form of (20) follows from standard results on linear quadratic control [3]. To compute the β_j's, we note that agent j is faced with a linear-quadratic optimal control problem with dynamics

$$\frac{dx}{dt} = u_m - \beta_{-m} x,$$

and where the cost to be minimized is

$$\int_0^\infty \left(|x(t)|^2 + \sum_{j \neq m} \frac{\beta_j^2}{c_j} |x(t)|^2 + \frac{1}{c_m} |u_m(t)|^2 \right) dt \, .$$

The optimal control of player m is $u_m = -c_m P x = -\beta_m x$ where P (which is independent of m) is the unique positive solution of the Riccati equation

$$-2\beta_{-m} P - P^2 c_m + 1 + \sum_{j \neq m} \frac{\beta_j^2}{c_j} = 0,$$

which is

$$P = -\frac{\beta_{-m}}{c_m} + \frac{1}{c_m} \sqrt{\beta_{-m}^2 + c_m + c_m \sum_{j \neq m} \frac{\beta_j^2}{c_j}}.$$

As $P = \beta_m / c_m$, this yields

$$\overline{\beta}^2 = \beta_{-m}^2 + c_m + c_m \sum_{j \neq m} \frac{\beta_j^2}{c_j}.$$

With $\overline{c} := \sum_{m=1}^M c_m$ and $\overline{c}_{-m} := \sum_{j \neq m}^M c_j$, this yields

$$\overline{c}^2 P^2 = \overline{c}_{-m}^2 P^2 + c_m + c_m \sum_{j \neq m} c_j P_j.$$

Solving this equation for P yields $P^2 = 1/\overline{c}$, from which (21) and (22) follow. The case for case T2 follows by an appropriate rescaling of the c_i's. ∎

3 Greedy decentralized algorithms

Although the Nash equilibrium is a natural solution concept for the non-cooperative decentralized control problem formulated here, its computation might yet require some coordination (and thus centralization and cooperation) between the users, since it involves the individual utilities of all players, as captured by the constants c_m, $m \in \mathcal{M}$. In practice, however, these utilities are typically private information, and communicating these might result in unacceptable additional complexity. It is thus natural to investigate whether simple greedy "best response" algorithms could be used for updating the users' control policies in a decentralized way, thus avoiding the need for communication, coordination, and computation of the Nash equilibrium. We show in this section that this is indeed the case, and moreover, sequences generated by such algorithms converge to the Nash equilibrium.

A greedy "best response" algorithm is defined by the following four conditions [4]:

- (i) Each user updates from time to time its policy by computing the best response against the most recently announced policies of the other users.

- (ii) The time between updates is sufficiently large, so that the control problem faced by a user when it updates its policy is well approximated by the original infinite horizon problem.

- (iii) The order of updates is arbitrary, but each user performs updates infinitely often.

- (iv) When the nth update occurs, a subset $K_n \subset \{1, \ldots, M\}$ of users simultaneously update their policies.

We shall consider the following particular cases, based on specific choices of K_n:

- **Parallel update algorithm (PUA):** $K_n = \{1, ..., M\}$ for all n.

- **Round robin algorithm (RRA):** K_n is a singleton for all n and equals $(n+k) \bmod M + 1$, where k is an arbitrary integer.

- **Asynchronous algorithm (AA):** K_n is a singleton for all n and is chosen at random.

We assume that the initial policy used by each user is linear of the form of (8). Let $\beta^{(n)}$ be the value corresponding to the end of the nth iteration. As in (14), the optimal response at each step n is obtained by

$$\beta_m^{(n)} = \begin{cases} f_m(\beta_{-m}^{(n-1)}) & \text{if } m \in K_n \\ \beta_m^{(n-1)} & \text{otherwise} \end{cases}, \tag{23}$$

where f_m is defined in (14).

We present a numerical analysis of the above three algorithms in Section 5. Below we study the convergence of the PUA algorithm.

Theorem 3 *Consider PUA.*
(i.a) Let $\beta_k^{(1)} = 0$ for all k. Then $\beta_k^{(2n)}$ monotonically increase in n and $\beta_k^{(2n+1)}$ monotonically decrease, for every player k, and thus, the following limits exist:

$$\hat{\beta}_k := \lim_{n\to\infty} \beta_k^{(2n)}, \qquad \tilde{\beta}_k := \lim_{n\to\infty} \beta_k^{(2n+1)}.$$

(i.b) Assume that $\hat{\beta}_k = \tilde{\beta}_k$ (defined as above, with $\beta_k^{(1)} = 0$ for all k). Consider now a different initial condition satisfying either $\beta_k^{(1)} \le \beta_k$ for all k, where β_k is as given in (9), or $\beta_k^{(1)} \ge \beta_k$ for all k. Then for all k,

$$\lim_{n\to\infty} \beta_k^{(n)} = \beta_k.$$

(ii) Global convergence: If

- (ii.a) $M = 2$, and either $\beta_k^{(1)} \leq \beta_k$ for all k, or $\beta_k^{(1)} \geq \beta_k$ for all k; or if

- (ii.b) $\beta_k^{(1)}$ and $c := c_k$ are the same for all k,

then β^n converges to the unique equilibrium β^*.

(iii) Local convergence: For arbitrary c_k, there exists some neighborhood V of β where β is given in (9), such that if $\beta_k^{(1)} \in V$ then $\beta^{(n)}$ converges to the unique equilibrium β^*.

Proof: (i) We note that for all m, f_m is decreasing in its argument:

$$\frac{df_m(\beta_{-m})}{d\beta_{-m}} < 0 \tag{24}$$

and is nonnegative.

Consider first the case when $\beta_k^{(1)} = 0$ for all k. Since f_k is nonnegative, $\beta_k^{(3)} \geq \beta_k^{(1)} = 0$ for all k so that $\beta_{-k}^{(3)} \geq \beta_{-k}^{(1)} = 0$ for all k. (24) then implies that $\beta_k^{(4)} \leq \beta_k^{(2)} = 0$. By an inductive argument it then follows that $\beta_k^{(2n)}$ is an increasing sequence and $\beta_k^{(2n+1)}$ is a decreasing sequence in n, for all k. This leads to the result (i) for the initial condition $\beta_k^{(1)} = 0$.

Denote by $\beta_k^{(n)}(0)$ the above sequence, obtained with the initial condition $\beta_k^{(1)} = 0$. Consider now an arbitrary initial condition satisfying $\beta_k^{(1)} \leq \beta_k$ for all k. This condition on $\beta_k^{(1)}$ implies that $\beta_{-k}^{(1)} \leq \beta_{-k}$, and hence by (24), $\beta_k^{(2)} \geq \beta_k$. Proceeding by induction, we get for all integers n and for all k,

$$\beta_k^{(2n)} \leq \beta_k, \qquad \beta_k^{(2n+1)} \geq \beta_k. \tag{25}$$

On the other hand, since $\beta_k^{(1)} \geq 0 = \beta_k^{(1)}(0)$, we have by (24): $\beta_k^{(2)} \leq \beta_k^{(2)}(0)$, and thus $\beta_k^{(3)} \geq \beta_k^{(3)}(0)$. Proceeding by induction, we get for all integers n and for all k

$$\beta_k^{(2n)} \leq \beta_k^{(n2)}(0), \qquad \beta_k^{(2n+1)} \geq \beta_k^{(2n+1)}(0). \tag{26}$$

Combining (25) with (26) establishes (ii.b).

(ii.a) Let $M = 2$, and assume first that $\beta_k^{(1)} = 0$, $k = 1, 2$. Then it follows that both $(\hat{\beta}_1, \tilde{\beta}_2)$ as well as $(\tilde{\beta}_1, \hat{\beta}_2)$ (defined in the first part of the Theorem) are in Nash equilibrium (since for any integer n, $\beta_1^{(n+1)}$ is the optimal response against $\beta_2^{(n)}$ and $\beta_2^{(n+1)}$ is the optimal response against $\beta_1^{(n)}$). Since the Nash equilibrium is unique, we have $\tilde{\beta} = \hat{\beta}_2$. The proof of (ii.a) (for nonzero initial condition as well) now follows from part (i.b) of the Theorem.

(ii.b) Assume first that $\beta_k^{(1)} = 0$ for all k. Due to the symmetry, $\overline{\beta}^{(n)} := \sum_k \beta_k^{(n)}$ is given by

$$\overline{\beta}^{(n+1)} = -(M-1)\overline{\beta}^{(n)} + \sqrt{(M-1)^2(\overline{\beta}^{(n)})^2 + M^2 c}.$$

Hence, $\hat{\beta} := \sum_m \hat{\beta}_m$ and $\tilde{\beta} := \sum_m \tilde{\beta}_m$ satisfy

$$\hat{\beta} = -(M-1)\tilde{\beta} + \sqrt{(M-1)^2(\tilde{\beta})^2 + M^2 c}, \qquad \tilde{\beta} = -(M-1)\hat{\beta} + \sqrt{(M-1)^2(\hat{\beta})^2 + M^2 c}.$$

This implies that both $\hat{\beta} = \tilde{\beta}$, satisfying

$$\beta = (M-1)^2\beta + \sqrt{(M-1)^2\left[-(M-1)^2\beta + \sqrt{(M-1)^2\beta^2 + M^2 c}\right]^2 + M^2 c}.$$

Taking the square of both sides of this equation yields after some simplifications

$$\beta^2 - 2\beta^2(M-1)^2 - M^2 c = -2\beta(M-1)\sqrt{(M-1)^2 + M^2 c}.$$

Squaring again and simplifying, we get

$$\beta^4[-1 + 4(M-1)^2] + 2M^2 c\beta^2 - M^4 c^2 = 0,$$

whose only positive solution is

$$\overline{\beta} = \left\{\frac{M^2 c}{2M-1}\right\}^{1/2},$$

which corresponds to the Nash solution obtained in (12). Hence $\hat{\beta}_k = \tilde{\beta}_k$ for all k. (ii.b) is now established by applying part (i.b).

(iii) Let $\Delta\beta_k^{(n)} := \beta_k^{(n)} - \beta_k$ for all integers n and for all players k. The proof is established by showing that there exists some neighborhood V of β such that $f = (f_1, ..., f_M)$ is a contraction on V, where f is defined in (14) (and is used in (23)). In other words, we have to show that there exists some matrix B whose eigenvalues are in the interior of the unit disk, such that

$$\Delta\beta^{(n+1)} = B\Delta\beta^{(n)} + o(\Delta\beta^{(n)}), \tag{27}$$

where $o(\cdot)$ is some function satisfying $\lim_{x\to\beta} o(x)/x = 0$, and $\Delta\beta$ is an M-dimensional vector whose kth component is $\Delta\beta_k$. Using (14)-(23), B is obtained as follows. The mk-th entry of B is given by

$$B_{mk} = \left.\frac{\partial f_m(\beta')}{\partial \beta'_k}\right|_{\beta'=\beta} = -b_m\,\delta_{km}, \tag{28}$$

where

$$b_m := -1 + \frac{\beta_{-m}}{\sqrt{\beta_{-m}^2 + c_m}}. \tag{29}$$

and δ_{km} is the Dirac delta function. A sufficient condition for all eigenvalues of B to be in the interior of the unit disk is that $\sum_{k \neq m} |b_k| < 1$, which follows directly from Gersgorin's Theorem (see [13] p. 344). This condition is indeed satisfied, as $\sum_{k \neq m} |b_k| = \beta_{-m}/\overline{\beta}$. ∎

Remark 1 *(Rate of convergence)*
Numerical experimentation has shown that the PUA algorithm has a very slow rate of convergence to the Nash equilibrium. To illustrate this (analytically), consider the symmetric case, where $c_m = 1$ *for all players, and the initial* $\beta_m^{(1)}$ *are the same. Combining (27), (28) and (29) yields*

$$\Delta\beta_k^{(n+1)} = -(M-1)\left(1 - \frac{(M-1)\beta_k}{\sqrt{(M-1)^2[\beta_k]^2 + 1}}\right)\Delta\beta_k^{(n)} + o(\Delta\beta_k^{(n)}).$$

Substituting $\beta_k = \sqrt{\frac{c}{2M-1}}$ *from (12), we obtain*

$$\Delta\beta_k^{(n+1)} = -\frac{M-1}{M}\Delta\beta_k^{(n)} + o(\Delta\beta_k^{(n)}).$$

Thus the difference between $\beta_k^{(n)}$ *and its limit* β_k *decrease by a multiplicative factor of* $(M-1)/M$*, approximately, in the vicinity of the equilibrium, and it changes sign at each iteration.* ∎

4 Multi-type traffic

We consider in this section the following extension of the model described in Section 2. We assume that source m $(m \in \mathcal{M})$ may have several types (say s_m) of possible traffic, with different kinds of requirements on the performance measures. Associated with type i_m traffic of user m, we have i-independent positive constants, $c_m(i)$, appearing in the immediate cost (instead of the constants c_m we had before). Typically, traffic requiring higher quality of service (QoS) might have a larger $c_m(i)$, which reflects the fact that it might require lower loss probabilities and higher throughput. It could be receiving a higher priority from the network in the sense that larger variations in $u_m(i)$ will be tolerated so as to achieve the required QoS.

The occurrence of these different types of traffic is governed by a continuous-time Markov jump process taking values in a finite state space $\mathcal{S}$; an element θ in $\mathcal{S}$ describes a possible traffic constellation of the s different sources.

The controlled rate matrix (of transitions within $\mathcal{S}$) is

$$\Lambda = \{\lambda_{ij}\}, \qquad i, j \in \mathcal{S},$$

where the λ_{ij}'s are real numbers such that for any $i \neq j$, $\lambda_{ij} \geq 0$, and for all $i \in \mathcal{S}$, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. Fix some initial state i_0 of the Markov chain $\mathcal{S}$. Consider the class of policies $\mu_m \in \mathcal{U}_m$ for controller m, whose elements are of the form

$$u_m(t) = \mu_m(t, x_{[0,t]}; \theta_{[0,t]}), \qquad t \in [0, \infty) .$$

Here, μ_m is taken to be piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument.

Theorem 4 *Consider the non-cooperative framework, where user m minimizes the cost*

$$J_m(\mu) := E^{\mu} \left[\int_0^{\infty} \left(|x(t)|^2 + \frac{1}{c_m(\theta(t))} |u_m(t)|^2 \right) dt \,\middle|\, x(0) = x_0, \theta(0) = i_0 \right] . \tag{30}$$

- *(i) There exists a Nash equilibrium given by*

$$\mu_m(x,i) = -\beta_m(i)\, x\,,\ m = 1, ..., M, \tag{31}$$

where

$$\beta_m(i) = c_m(i) P_m(i), \qquad \beta_{-m} := \sum_{j \neq m} \beta_j, \tag{32}$$

and $\{P_m(i)\ i \in \mathcal{S},\, m \in \mathcal{M}\}$ is a solution of the coupled set of equations

$$-2\beta_{-m}(i) P_m(i) - P_m(i)^2 c_m(i) + 1 + \sum_{j \in \mathcal{S}} \lambda_{ij} P_m(j) = 0, \tag{33}$$

$i \in \mathcal{S}, m = 1, ..., M.$

- *(ii) $\beta_m(i)$ is bounded by*

$$0 \leq \beta_m(i) \leq \frac{c_m(i)}{\min_j \sqrt{c_m(j)}} . \tag{34}$$

- *(iii) The cost for player m, corresponding to the Nash equilibrium above is given by*

$$J_m(x,i) = \frac{\beta_m(i)}{c_m(i)} x^2.$$

Proof: We first show that a Nash equilibrium exists among policies of the form (31). To see that, first note that since J_m is positive-quadratic in x and u_m, and x is linear in u_m for each fixed μ_k, $k \in \mathcal{M}$, $k \neq m$, the costs $J_m^{N1}(x,i;\mu)$ are strictly convex in μ_m for each user. This implies, in particular, that the cost of user m is strictly convex in β_m. We shall show, next, that β can be restricted without any loss of generality to a compact set.

Assume now that all players other than player m use policies of the form (31), for which the bound (34) is satisfied. Then player m is faced with a linear quadratic control problem with jump parameters, and has an optimal response given by (31)-(33) (see [15]). We show that this response also satisfies the bound (34). (32) and (33) imply that

$$-2\beta_{-m}(i)\beta_m(i) - \beta_m(i)^2 + c_m(i) + \sum_{j\in\mathcal{S}} \frac{\beta_m(j)}{c_m(j)} c_m(i) = 0,$$

which yields

$$\beta_m(i)[2\overline{\beta}(i) - \beta_m(i)] = c_m(i) + \sum_{j\in\mathcal{S}} c_m(i)\frac{\beta_m(j)}{c_m(j)}. \tag{35}$$

Let i^* be the state for which $\beta_m(i)/c_m(i)$ is maximized. We have

$$\sum_{j\in\mathcal{S}} \lambda_{i^*j}\frac{\beta_m(j)}{c_m(j)} \leq 0,$$

so that (35) yields

$$\frac{\beta_m(i^*)}{c_m(i^*)}[2\overline{\beta}(i) - \beta_m(i)] \leq 1.$$

This implies that

$$\beta_m(i^*) \leq \sqrt{c_m(i^*)}.$$

From the definition of i^* we have, for each $i \in \mathcal{S}$,

$$\frac{\beta_m(i)}{c_m(i)} \leq \frac{\beta_m(i^*)}{c_m(i^*)} \leq \frac{1}{\sqrt{c_m(i^*)}}$$

from which we obtain

$$0 \leq \beta_m(i) \leq \frac{c_m(i)}{\sqrt{c_m(i^*)}}.$$

This implies that the optimal response of player m also satisfies (34).

Consider a constrained version of the game in which player k is restricted to use $\beta_k(i) \in [0, B_k(i)]$, where $B_k(i)$ is an arbitrary constant larger than the bound $c_k(i)/\min_j \sqrt{c_k(j)}$. Since the cost of a typical player k in this constrained game is strictly convex in the policy of that player, and since the policy space is now compact and convex, there exists a Nash equilibrium β^* by a standard existence result for static games (see [4], Theorem 4.3, p.179). This turns out to be, however, an equilibrium in the unconstrained game as well. Indeed, consider the original unconstrained game, and assume that β^* is not in Nash equilibrium (for an initial state $i \in \mathcal{S}$). Then there exists some player m whose optimal response against $\beta_k^*, k \neq m$, to be denoted β_m', satisfies (34), and $\beta_m'(j) < \beta_m^*(j)$. Since β_m' is feasible in the constrained game and performs better than β_m^* in the constrained game, this contradicts the fact that β^* is Nash for the constrained game. Hence, this establishes the existence of a Nash equilibrium of the form (31) for the original game, which satisfies (32) and (33). ∎

5 Numerical results

We present in this section results of a numerical study on the behavior of the three greedy decentralized algorithms presented in Section 3. As pointed out in Remark 1, the rate of convergence of PUA becomes slower as the number of users become large. We have therefore tested the convergence of the algorithms for two cases: M moderate ($M = 4$) and M relatively large ($M = 10$). We have focussed on the symmetric case $c_m = 1$, for all m, and started the iterations with zero initial conditions. All three algorithms converged.

Figures 1 and 2 depict a comparison between the convergences of PUA and RRA, for $M = 10$ and $M = 4$, respectively. In both figures we have focussed on user number 1, and have computed its β after each cycle, i.e. each time all users have updated their policies. The PUA is seen to converge quite slowly: it takes around 40 cycles for convergence, and it takes longer as the number of users grow. The RRA converges almost instantaneously to the Nash equilibrium.

Figures 3 and 4 pertain to AA for 4 and 10 users, respectively. The user that updates at a given iteration is chosen with equal probabilities, and the choices are independent. Thus, the time between updates of user m are geometrically distributed with parameter $1/M$. Figure 3 depicts the behavior of each of the users, whereas Figure 4 depicts the behavior of only players 1,3,5,7,9. The basic unit of the x axis is one iteration. We can see that there are iterations where no update occurs: this happens when the same user updates consecutively. The rate of convergence is seen to be faster than the PUA and slower than the RRA.

6 Concluding remarks

We present, in this section, some observations on the asymptotic behaviors of the various solutions obtained, as the number of users grow, as well as the relative rates of convergence of the proposed algorithms — all for the single-type traffic.

Asymptotic behavior for large M

In order to come up with meaningful comparisons between systems with different numbers of users, we have to make some unifying assumptions on the cost functions. Natural assumptions in this context are:

(A1) $\bar{c}$ is nondecreasing as the number of users increase,

(A2) $\bar{c}$ tends to infinity as $M \to \infty$.

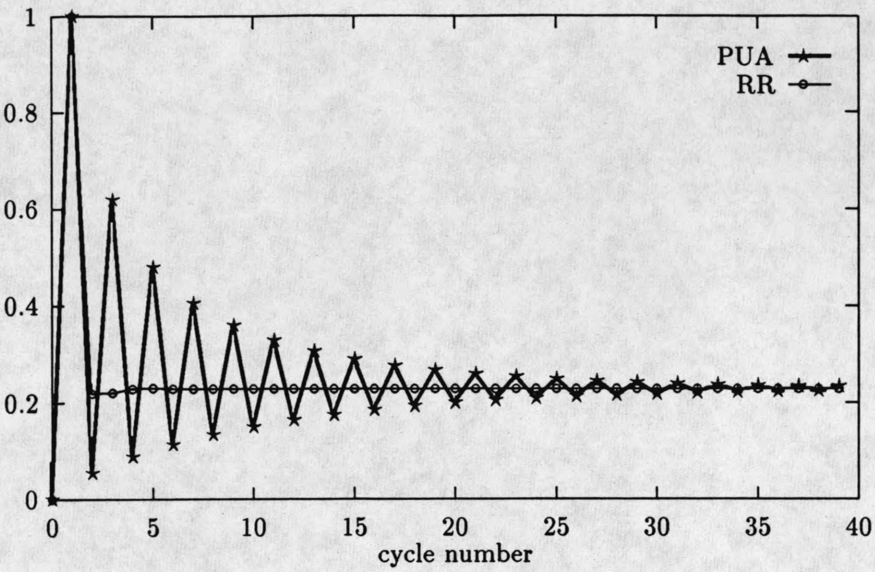

Figure 1: PUA versus RRA for $M = 10$

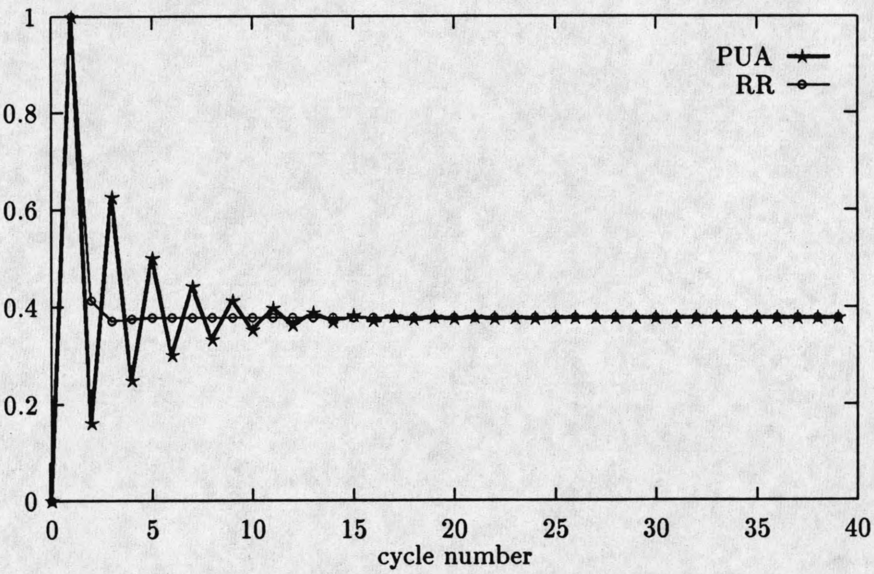

Figure 2: PUA versus RRA for $M = 4$

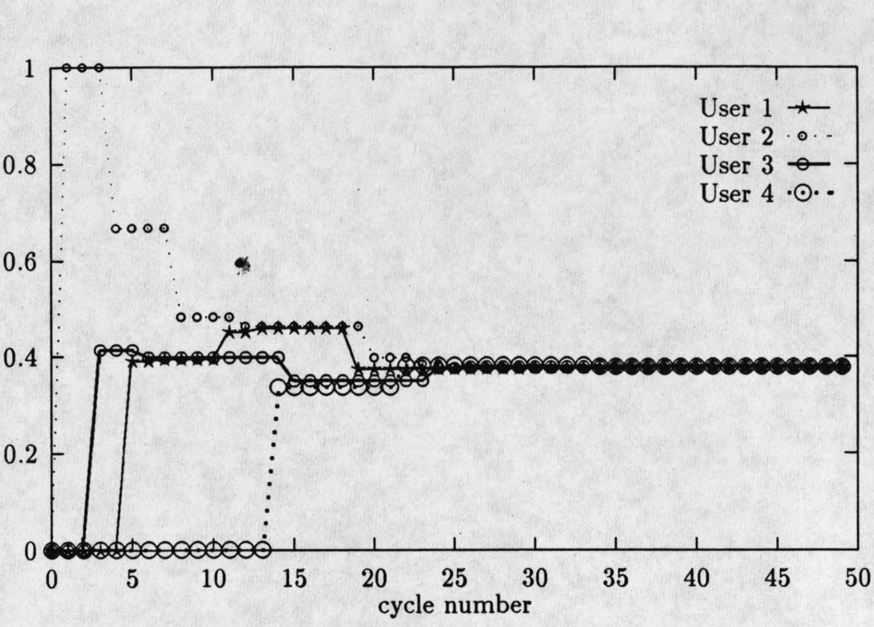

Figure 3: AA for $M = 4$

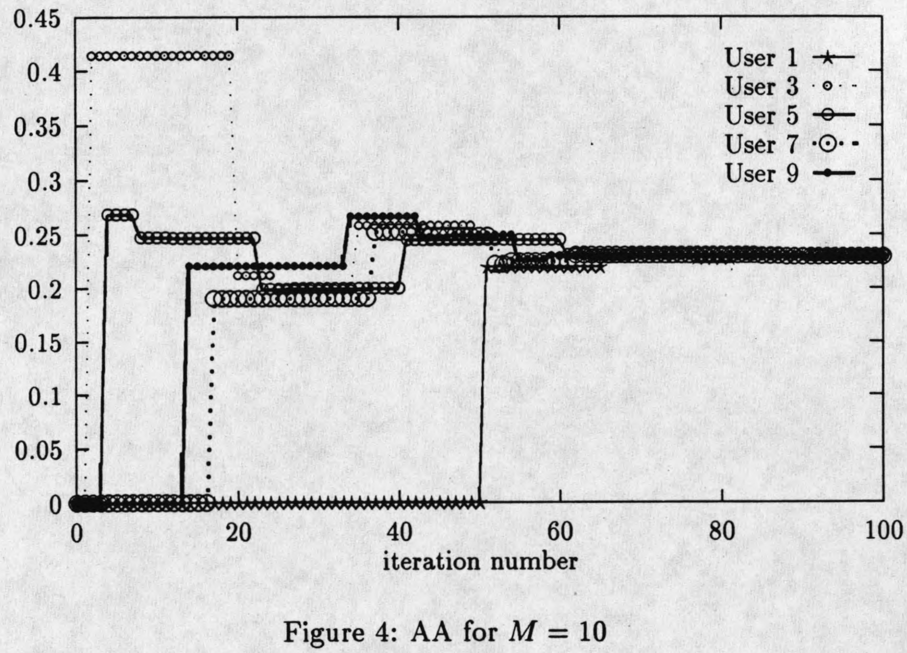

Figure 4: AA for $M = 10$

According to Theorem 2, (A1) implies that the equilibrium value J^{T2} goes to 0 as M grows to infinity. If (A2) also holds then J^{T1} goes to 0 as M grows to infinity. The fact that the values are very low (when M is large) might mean that the resources are underused in some sense: we have very small deviations from the target queue length which is obtained by minor effort on the part of the controllers (both the term that corresponds to x^2 and the term corresponding to u_m^2 are small when the total cost is low). This behavior is due to the fact that the dynamics of the queue length depend only on the sum of the u_m's; however, the total cost takes into account the sum of the *squares* of the u_m's. For the same (fixed) value of the sum of u_m's, the sum of the squares of the u_m's decreases as the number of users grow.

A problem that may arise due to this situation is that the users might choose non-optimal policies, which might cause an overall inefficient use of the network (for example, large variations of the sum of u_m's) and still have a correspondingly low cost. A way to circumvent this situation is to choose a network pricing policy that does not satisfy (A1). In other words, the individual costs $1/c_m$ may be chosen by the network according to the expected number of users; if the network is designed for a large number of users then the c_m should be smaller than those used in pricing in a network with a smaller expected number of users.

The situation is different, however, in the non-cooperative case: the total cost (summing over all users) need not go to zero, as can be seen from Corollary 1. Assume that $\bar{c}$ grows linearly with M, i.e. $\bar{c} \sim Mc$. In the case N1, the sum of the values of the individual users goes to infinity as M goes to infinity, whereas in case N2 it converges to a constant: $x^2/\sqrt{2c}$.

Yet, even in the non-cooperative case, we see that the cost *per user* tends to 0 as M tends to infinity, if $\bar{c} \sim Mc$. This again may result in the problems discussed above (for the team case), but a way to circumvent this situation is again to choose a network pricing policy for which $\bar{c}$ does not grow linearly in M. For example, if we choose $\bar{c}$ to be constant in M then the value per user in case N1 will tend to a constant as M goes to infinity. For case the N2, $\bar{c}$ has to be chosen to be decreasing (like $1/M$) in order to achieve this same behavior of the value.

Comparing values and gains (β_m's) in T1,T2, N1,N2

For both the cooperative and non-cooperative cases, we see that the β_m's are smaller when the part of the cost corresponding to the queue length variations is smaller: β_m in cases T1 and N1 are larger by a factor of $\sqrt{M}$ compared to the corresponding cases T2 and N2. The costs in cases T1 and N1 are, as can be expected, larger than those in the corresponding cases T2 and N2.

In order to make a comparison between the cooperative and non-cooperative values (of β), the proper quantities to compare are T1 on one hand, and the sum over all users of J_m^{N2}, on the other hand. This is because, for a fixed policy μ for all users, $J_m^{T1}(\mu) = \sum_m J_m^{N2}(\mu)$. A comparison of

Corollary 1 and Theorem 2 reveals that when $\overline{c} >> \max_m c_m$,

$$\sum_m J_m^{N2} \sim \sqrt{\frac{M}{2\overline{c}}} x^2 = \sqrt{\frac{M}{2}} J^{T1}.$$

As can be expected, the value is indeed higher under the non-cooperative mode of play.

For the symmetric case, where $c_m = c$ for all m, this conclusion is also confirmed:

$$\sum_m J_m^{N2} = \frac{1}{\overline{c}} \sqrt{\frac{M^2}{2M-1}} x^2 = \sqrt{\frac{M}{2-M^{-1}}} J^{T1}.$$

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