# Integrating decomposition methods with user preferences for solving many-objective optimization problems

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**Declaration**

I certify that except where due acknowledgement has been made, this work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and, any editorial work, paid or unpaid, carried out by a third party is acknowledged. I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

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- Mohammadi, A., Omidvar, M. and Li, X. (2013), "A New Performance Metric for User-preference Based Multi-objective Evolutionary Algorithms", in *Proceedings of Congress of Evolutionary Computation (CEC 2013)*, IEEE,2013, p.2825 - 2832.

- Mohammadi, A., Omidvar, M. and Li, X. (2012), "Reference Point Based Multi-objective Optimization Through Decomposition", in *Proceedings of Congress of Evolutionary Computation (CEC 2012)*, IEEE, 2012, p.1150 - 1157.

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# Abstract

This research aims to investigate methods to solve many-objective (a multi-objective problem with three or more objectives) optimization problems. To achieve this, we propose an algorithm combining user-preference and decomposition approaches. The main reasons that decomposition-based evolutionary multi-objective optimization (EMO) methods are employed in this research are: firstly, they suffer less from the selection pressure issue in comparison to dominance ranking as they rely on decomposition methods such as Weighted-sum, Tchebycheff and Penalty-based Boundary Intersection (PBI) to convert a multi-objective problem into a set of single-objective problems. Secondly, decomposition approaches employ a set of weight vectors which give us a reasonable control of solutions in the objective space. As user-preference approaches alleviate the scalability issue of many-objective problems, they are adopted in this research. User-preference methods can potentially save a considerable amount of computational resources by searching on more desired regions rather than the entire Pareto-optimal front. In this research, user-preference is defined in the form of one or more reference points or directions. The proposed algorithm outperforms R-NSGA-II which is one of popular dominance-based approaches on many-objective optimization problems.

Finding a diverse set of solutions is another major challenge for EMOs. The issue of solution diversity is of greater importance when dealing with many-objective problems. In this thesis, we propose an algorithm using a mechanism to update the weight vectors according to feedback that quantifies the uniformity of the solutions in the objective space. Two existing metrics and a newly developed metric are adopted as feedback mechanisms. These metrics allow us to assess the contribution of each solution towards improving the overall uniformity of the solution set in the objective space, and to use this information to update the weight vectors adaptively so that the overall uniformity is maintained. The overarching is to identify sparse areas in the objective space, and move the solutions from the denser to sparse areas. The newly developed metric uses the idea of electrostatic equilibrium to calculate the

direction in which each solution should move in order to improve the overall uniformity. As we use decomposition methods in this research, the availability of weight vectors gives us an explicit means of controlling the uniformity of solutions in the objective space.

Since existing metrics are neither sufficiently accurate nor scalable to measure the performance of user-preference based EMO algorithms, we develop a new performance metric to fill this gap. The proposed metric uses a composite front as a substitute for the Pareto-optimal front then a preferred region is defined on the composite front. Performances of the new metric are compared against a baseline which relies on knowledge of the Pareto-optimal front. One of the key advantages of the proposed metric is that it does not depend on prior knowledge of the Pareto-optimal front of a particular problem, which is most likely the case in real-world situations.

# Chapter 1

# Introduction

## 1.1 Motivation

The term optimization refers to finding one or more optimal solutions which correspond to the maximum or minimum values of one or more objectives [Schwefel, 1993; Ben-Tal and Nemirovski, 2002; Gill et al., 1981]. Optimization techniques have been applied to many problems such as engineering design and manufacturing industries [Coello et al., 1999]. There are different types of optimization problems. When there is only one objective, it is called single-objective optimization. The main goal of a single-objective optimization method is to find the best solution which corresponds to the maximum or minimum value of the objective function. When optimization involves more than one objective function and these objectives tend to conflict with each other, it is called multi-objective optimization. Multi-objective optimization problems are very important to both scientists and engineers as most real-world problems could be considered as multi-objective [Deb, 2001; Coello et al., 2006]. Some applications that use multi-objective optimization techniques are job scheduling [Xia and Wu, 2005], manufacturing the shape of turbine blades and aeroplane wings [Hughes, 2007; Takagi, 2001]. There are different methods used to solve optimization problems including classical methods [Miettinen, 1999; Laumanns et al., 2006; Miettinen, 1999; Benson, 1978; Keeney and Raiffa, 1993]. However, these methods are not effective to solve non-linear, non-convex and multi-objective optimization problems [Deb, 2001; Schwefel, 1993]. Evolutionary algorithms (EAs) [Back, 1996] are one of the alternatives to classical methods. EAs are based on Darwin's theory of evolution where the selection pressure allows fitter individuals to survive to the next generation. EAs evolve a population of potential solutions in successive iterations of the algorithm to find a set of candidate solutions.

In recent years, there has been a growing interest in the area of many-objective optimization [Hughes, 2005a]. That is a multi-objective optimization problem having three or more objectives, which is the main focus of this research. However, the performance of evolutionary multi-objective approaches degrades rapidly as the number of objectives increases [Ishibuchi et al., 2008a;b]. There are several challenges that Evolutionary multi-objective Optimization (EMO) algorithms are faced with when they are dealing with many-objective problems [Deb et al., 2006]. Firstly, visualizing the Pareto-front when there are more than three objectives is very difficult. It is challenging for the decision maker (DM) to get a visual sense of the solutions accurately and to be able to select a preference. Another challenge is related to the number of solutions required to approximate the Pareto-optimal front. In other words, the number of solutions increases exponentially with respect to the number of objectives. Finally, in the case of dominance-based approaches such as NSGA-II [Deb et al., 2002], SPEA [Zitzler and Thiele, 1999] and MOGA [Fonseca and Fleming, 1993] when the number of objectives increases, most of the solutions, even in the initial randomly generated population, are non-dominated to each other. This suggests that none of the objective functions can be improved in value without degrading some of the other objective values. As a result, there will not be enough selection pressure to propel the solutions towards the Pareto-optimal front [Ishibuchi et al., 2008b;a]. To overcome the scalability issue of Pareto dominance EMO approaches, we propose to investigate the following two strategies:

1. Replacing dominance ranking by using decomposition techniques.
2. Confining the search space and focus on specific parts of the Pareto-optimal front instead of finding the entire Pareto-optimal front, which helps to reduce the computational cost.

Decomposition strategies, which are borrowed from multi-criteria decision making [Hughes, 2005b], can alleviate the selection pressure problem imposed on dominance-based evolutionary algorithms. They convert a multi-objective problem into a set of single objective problems. In other words, since decomposition methods do not use dominance comparisons, they are scalable to a greater number of objectives. Decomposition methods use different scalarizing functions that assign a set of weights to the objective functions. When these scalarizing functions are used in conjunction with population-based metaheuristics, they can solve the resultant single objective problems with various weight values, resulting in obtaining solutions on different parts of the Pareto front. This makes the decomposition-based EMO algorithms less sensitive to the selection pressure issue. MOEA/D [Zhang and Li, 2007] and MOGLS [Ishibuchi and Murata, 1998] are two popular EMO algorithms that eliminate dominance ranking by employing scalarizing functions. One of the main reasons that we use decomposition methods in this research is that they provide an explicit means of controlling where a population converges by changing the weight vectors. This can also be used to control the distribution of solutions directly.

Adopting user-preference based approaches, e.g. a reference point, is another promising way of alleviating the scalability issue of EMO algorithms. Using a reference point allows us to save considerable computational resources by focusing the search on more desirable regions on the Pareto-optimal front instead of searching the entire Pareto-optimal front. In this approach the user may have one or more existing solutions that were obtained through various means and can be passed to the algorithm as a reference point(or points). For instance, in a car-buying decision-making multi-objective problem, there are two conflicting objectives: cost and comfort. A car with a price of $\$30,000$ and a comfort level of 60% can be passed to the algorithm as a reference point. There are various types of user-preference methods including *a priori* (where a user defines his/her preferences before the search process), *interactive* (where a user defines his/her preferences during the search process), and *a posteriori* (where a user defines his/her preferences after the search process) decision making. Some of the popular user-preference based EMO algorithms are R-NSGA-II [Deb et al., 2006] which uses reference points to incorporate the user preferences, RD-NSGA-II [Deb and Kumar, 2007b] which uses reference direction for the same purpose, LBS-NSGA-II [Deb and Kumar, 2007a] which uses a light beam approach to incorporate user preferences and PICEA [Wang et al., 2013] which is an example of *a posteriori* decision making. Since applying scalarization techniques to some extent alleviates the selection pressure issue of dominance-based approaches, and utilizing user-preference information reduces computational cost, this research combines both methods. In chapters 3 and 4, we develop user-preference decomposition based algorithms to solve many-objective optimization problems. One of the main advantages of this combination is that utilizing weight vectors of decomposition methods helps in handling user-preferences by guiding solutions towards the preferred region.

MOEA/D, which is the basis of our proposed approaches in this thesis, has been tested with two scalarizing functions: Tchebycheff and PBI. The Tchebycheff method works well on both convex and non-convex Pareto-optimal fronts, but does not result in a very uniform set of solutions on the Pareto-optimal front [Zhang and Li, 2007]. Penalty-based Boundary Intersection (PBI) is a variation of Normal Boundary Intersection (NBI) [Das and Dennis, 1998] that uses a simple penalty method to eliminate the need for direct handling of NBI's equality constraint. PBI generally produces more uniform solutions than Tchebycheff, but its convergence can be affected by its penalty parameter ($\theta$) which has not been well-studied. As a result, chapter 5 of this thesis presents a sensitivity analysis of the PBI penalty parameter.

Finding uniformly distributed solutions in the objective space is a major challenge in EMO. Two main difficulties of finding a uniform set of solutions on the Pareto front are as follows:

1. An accurate metric is needed to measure the uniformity of solutions in the objective space.
2. Most of the existing metrics measure the uniformity of the entire solution set, but they are unable to rank the individuals in terms of their contributions to the overall uniformity. In other words, most of the uniformity metrics face a credit assignment problem.

Several metrics [Deb et al., 2002; Pettie and Ramachandran, 2002; Silverman, 1986; Huang et al., 2005] have been proposed to address the uniformity issue. However, most of these metrics are not scalable to many-objective optimization problems [Purshouse and Fleming, 2007; Hughes, 2005b]. Maintaining the diversity of solutions becomes even more difficult in many-objective problems since the size of the search space grows exponentially. Therefore, the accuracy of current metrics degrades severely in high dimensional spaces.

As previously mentioned, one of the main features of decomposition methods is controlling the distribution of solutions by adjusting weight vectors. The following question arises when dealing with decomposition methods: How to generate a set of weight vectors for a given aggregation function in order to ensure a desired level of diversity among the solutions? Some approaches [Tan et al., 2013; Qi et al., 2014; Ma et al., 2014] replace the weight vector initialization method in MOEA/D (Simplex-lattice design) with a good lattice point (GLP) and other complex weight vector initialization methods to generate uniform solutions. However, most of those algorithms are not efficient in finding uniform solutions as they either use sophisticated methods which can be computationally expensive or they rely on the information of the Pareto-optimal front to generate a set of well distributed weight vectors. Since the relationship between the weight space and the objective space is not always linear, generating a uniform set of solutions from a uniform set of weight vectors is not always possible. Relying on Pareto-optimal front information to engineer weight vectors to obtain a uniform set of solutions is also not viable since this information is not always available. As a result, there is a need for a mechanism during the course of optimization to maintain this unique relationship between the weight vectors and solutions in objective space. In other words, weight vectors should be dynamically updated during the course of optimization with the aim of generating a uniform set of solutions in the objective space. To achieve this, in chapter 6 of this thesis a feedback mechanism is proposed. Some existing metrics [Van Moffaert et al., 2014; Ong et al., 2012] have been used to measure the uniformity of solutions and a new uniformity metric is also proposed.

Finally, since very few metrics exist [Veldhuizen and Lamont, 2000; Zitzler et al., 2003; Veldhuizen, 1999a] to measure the performance of user-preference approaches accurately, in chapter 7 we develop a metric to compare the performance of user preference-based algorithms fairly. The main novelties of this metric compared to existing metrics are: 1) It measures both convergence and diversity of the solutions in the preferred regions independent of any parameters such as nadir points; and 2) it is scalable when the number of objectives increases since it is independent of the knowledge of the Pareto-optimal front.

## 1.2 Research Objectives

This research will focus on addressing the following objectives:

1. To develop a novel method by combining the decomposition and user preference methods for better handling many objective optimization problems

2. To evaluate the effect of penalty parameters in PBI on the performance of user-preference and non-user-preference EMO algorithms

3. To design a new technique to improve the uniformity of solutions in the preferred region particularly when the shape of the Pareto-optimal front is complex or highly non-linear. To investigate a mechanism which can find a diverse set of solutions without the knowledge of the Pareto-optimal front

4. To define a new metric to evaluate the performance of different user-preference based algorithms

The next section presents techniques which are developed to address these research questions.

## 1.3 Methodology

Decomposition and user-preference are two main techniques adopted in this research to address the objectives stated above. In this research, we make use of user preference information through the form of reference points, which can be one of the existing solutions in the objective space. We use three different decomposition methods: Weighted-Sum [Miettinen, 1999], Tchebycheff [Miettinen, 1999], Boundary Intersection [Das and Dennis, 1998], and Penalty-based Boundary Intersection [Zhang and Li, 2007]. A set of multi-objective optimization benchmark functions with different Pareto-optimal shapes are used to evaluate the performance of the proposed algorithms. To measure the quality of solutions generated by the proposed approaches, both convergence and diversity aspects have been measured simultaneously and separately. Since existing metrics used in the field are not designed specifically to measure the quality of solutions in the preferred regions, the performance metrics take user-preference information into account. To assess the capability of the proposed methods in terms of finding optimal solutions, visualization tools have also been used. To determine whether the results are significantly different, a non-parametric statistical test is run on the results. In order to rank the algorithms, we used the Mann-Whitney-Wilcoxon (MWW) test with a Bonferroni correction only when the null hypothesis of Kruskal-Wallis was rejected under a 95% confidence interval.

## 1.4 Contributions

This thesis makes novel contributions to the field of EMO, particularly to many-objective optimization. The core of these contributions is about combining decomposition methods with user-preference models for solving many-objective optimization problems. In particular the key contributions are as follows:

1. The development of scalable decomposition and user-preference based multi-objective algorithms which can be applied to many-objective optimization problems. The proposed algorithm is less susceptible to the selection pressure, focused in the preferred region and converges on the Pareto-optimal front more rapidly.

2. The development of a more effective mechanism updating weight vectors and decoupling the population size and number of objectives. This makes the proposed approach more effective in higher numbers of objectives.

3. The development of a sensitivity analysis of penalty parameter in PBI. In order to do so, the effect of penalty parameter has been studied according to (a) the problems with and without user-preference information; (b) the convergence and uniformity of the solutions separately and simultaneously; and (c) the performance of the algorithm as the number of objectives increases.

4. The development of a mechanism to maintain the uniformity of solutions in user-preference based approaches by providing feedback from the behavior of solutions in the objective space to weight vectors. Dynamically adapting the weight vectors according to the solutions in the objective space during the course of optimization is the main novelty of this approach. To achieve this, a new uniformity metric is proposed, and the results are compared with those using existing metrics.

5. The development of a metric to measure the performance of user-preference based EMO algorithms so that we can compare preference-based EMO algorithms more fairly. The main idea has been borrowed from cardinality-based metrics. A composite front has been formed to replace the Pareto-optimal front and the preferred regions have been defined on it.

## 1.5 Overview of the Study and Organization

Chapter 2 first provides the basic definitions for multi-objective optimization with example problems. Next, we present a review of classical methods and evolutionary algorithms to solve multi-objective optimization problems. The literature review of decomposition and user-preference based EMO approaches are also presented. Finally, the performance metrics which are used in this thesis are described and reviewed.

In chapter 3, we propose a user-preference based evolutionary algorithm that relies on decomposition strategies to convert a multi-objective problem into a set of single-objective problems. The proposed approach is called R-MEAD. The use of a reference point allows the algorithm to focus the search on more preferred regions, which can potentially save a considerable amount of computational resources. Combining decomposition strategies with reference point approaches paves the way for the more effective optimization of many-objective problems.

In chapter 4, we propose a user-preference based evolutionary multi-objective algorithm that uses decomposition methods for solving many-objective problems. The newly proposed algorithm, R-MEAD2, improves the scalability of its previous version (R-MEAD) by adopting a Simplex-lattice design method for generating weight vectors. This makes the population size independent from the dimension size of the objective space. R-MEAD2 uses a uniform random number generator (RNG) to remove the coupling between dimension and the population size. It should be noted that a uniform set of weight vectors does not necessarily map to a uniform set of solutions in the objective space, especially on highly non-linear and complex Pareto-optimal fronts. This requires a feedback mechanism to adjust the weights in order to obtain a set of uniform solutions, which is the main topic of chapter 6.

As indicated previously, MOEA/D relies on decomposition methods such as weighted-sum, Tchebycheff and Penalty-based Boundary Intersection (PBI), to convert a multi-objective problem into a set of single-objective problems. It has been argued that PBI can generate a more uniform set of solutions than other decomposition methods. However, the main drawback of PBI is that it has a penalty parameter ($\theta$) which needs to be specified by the user. This penalty parameter can affect the convergence as well as the uniformity of solutions. Chapter 5 provides a comprehensive analysis of PBI's penalty parameter and its effect on a user-preference algorithm (R-MEAD2), and a non-user-preference algorithm (MOEA/D) has been conducted. Also in this chapter, the effect of $\theta$ on convergence, uniformity and the combination of convergence and uniformity have been analyzed.

Since generating a uniform set of solutions turns out to be a challenging task for multi-objective optimization problems, chapter 6 provides a strategy to tackle this. A feedback mechanism is developed to assess the uniformity of solutions in the objective space during the course of optimization. These feedback values are then used to dynamically adapt weight vectors for better solution uniformity in the objective space. The proposed method (UR-MEAD2) can adopt any uniformity metrics as a feedback mechanism. More specifically, adopted metrics are used to rank individuals in terms of their contributions towards improving the overall solution uniformity.

In chapter 7, we propose a metric for evaluating the performance of user-preference based EMO algorithms. It defines a preferred region based on the location of a user-supplied reference point. This metric uses a *composite front* which is a type of reference set and is used as a replacement for the Pareto-optimal front. This composite front is constructed by extracting the non-dominated solutions from the merged solution sets of all algorithms that are to be compared. A preferred region is then defined on the composite front based on the location of a reference point. Once the preferred region is defined, existing evolutionary multi-objective performance metrics can be applied with respect to that region.

# Chapter 2

# Literature Review

Since the late 1990s, the number of applications of multi-objective evolutionary algorithms (MOEAs) has grown considerably. The main reason behind this increase is the success of MOEAs in solving real-world problems. In recent years, many-objective optimization has become more popular. However, only a limited number of many-objective real-world applications exists since there is no efficient and effective technique to handle many-objective optimization problems. In this thesis, we have developed some user-preference Evolutionary Multi-objective Optimization algorithms (EMOs) which are able to solve many-objective optimization problems effectively. Before we get into the details of the proposed algorithms it is important to describe the background materials and literature related to the research presented in this thesis. First, in Section 2.1, some key concepts involving multi-objective and many-objective optimization are introduced. In Section 2.2, classical methods to solve multi-objective optimization problems are presented with their shortcomings identified to justify the use of Evolutionary Algorithms (EAs). Concepts of EAs with examples are described briefly in Section 2.3.1. Next, in Section 2.4, we present how EMOs are developed. Some issues that EMO algorithms are facing to solve many-objective optimization problems are also illustrated. Section 2.5 presents techniques found in the literature in tackling these issues. In this section, popular user-preference EMO algorithms are illustrated in detail. Finally in Section 2.6, existing performance metrics for user-preference and non-user-preference EMO algorithms in the literature are described.

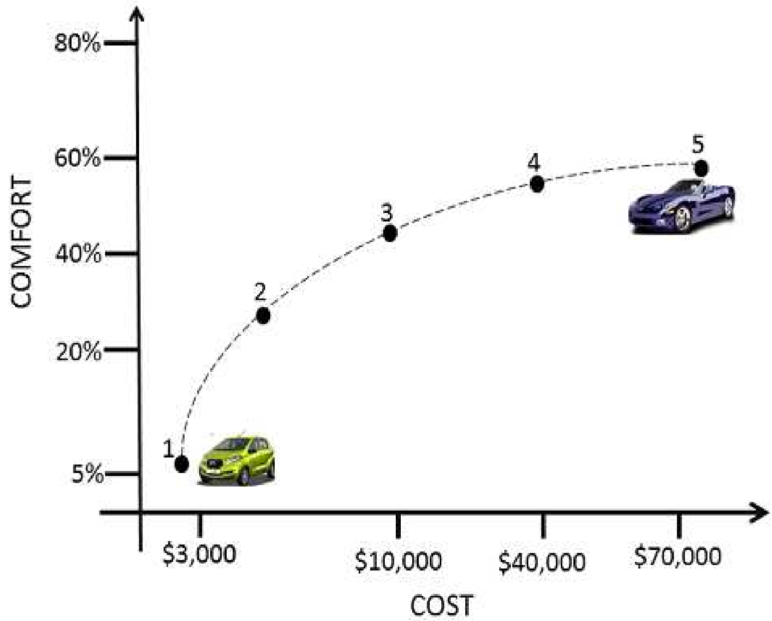



*Figure 2.1: An example of a multi-objective optimization problem*

## 2.1 Multi-objective Optimization

An optimization problem involving one objective is called a single-objective optimization. The main goal of a single-objective optimization method is to find the best solution, which corresponds to the maximum or minimum value of an objective function. When an optimization involves more than one objective function and these objectives tend to conflict with each other, it is called multi-objective optimization. In a multi-objective optimization problem due to the existence of conflicting objectives, there is often a set of trade-off solutions which is referred to as a Pareto-optimal front. Assuming minimization, a multi-objective optimization problem with $m$ objectives can be described as the following [Deb, 2001]:

$$\text{minimize } \mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})), \tag{2.1}$$

where $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^m$ is the objective vector and $f_i(\mathbf{x})$ is the $i$th objective function where $i \in \{1, \ldots, m\}$. Each decision vector $\mathbf{x} \in \mathbb{R}^n$ is defined as $(x_1, \ldots, x_n)$ where $n$ is the number of variables in the decision space.

Figure 2.1 shows an example of a multi-objective optimization problem, where one axis shows the price of a car ranging from $3,000 to $70,000. The second axis shows the comfort level of a car ranging from 5% to 80%. If the cost is the only objective of this problem, Solution 1 would be the optimal choice. As a result, there would be only one type of car on the road and manufacturers would not produce any expensive cars. In the same scenario, if comfort level is the only objective of this optimization problem, Solution 5 is the optimal choice. However, various other solutions (2, 3, 4) between these two extreme solutions provide a trade-off between comfort level and cost. As a result, none of these solutions can be said to be better than the others with respect to both objectives. These solutions are called non-dominated solutions and there is a set of such trade-off solutions. In Figure 2.1, all these solutions are joined in a form of curve. These solutions which are mapped from the decision spaced are called the Pareto-optimal front.

### 2.1.1 Dominance Relation

The dominance concept can be introduced to multi-objective optimization for comparison of two solutions [Deb, 2001]. In a minimization problem, $\mathbf{x}_1$ dominates $\mathbf{x}_2$ which is denoted as $\mathbf{x_1} \prec \mathbf{x_2}$ if:

$$\forall i \exists j \left( f_i(\mathbf{x_1}) \leq f_i(\mathbf{x_2}) \wedge f_j(\mathbf{x_1}) < f_j(\mathbf{x_2}) \right),$$

where $i, j \in \{1, \ldots, m\}$.

**Pareto-Optimal Set**: A non-dominated set is a set of solutions where no members dominate the others. A Pareto-optimal set is a non-dominated set where its members dominate all other possible solutions in the search space.

**Pareto-Optimal front**: The mapping of all possible Pareto-optimal solutions into the objective space form a curve (or surface) which is commonly referred to as a Pareto-optimal front. A Pareto-optimal front is said to be convex if and only if the connecting line between any two points on the Pareto-optimal front lies above it and non-convex otherwise.

### 2.1.2 Many-objective Optimization

In the past, most multi-objective problems have used two or three objectives. In recent literature, special attention has been given to problems with more objectives. A multi-objective problem with more than three objectives is commonly referred to as a many-objective problem [Fleming et al., 2005]. In this thesis, we mainly focus on many-objective optimization problems.

## 2.2 Classical Methods to Solve Multi-objective Problems

Classical methods [Branke et al., 2008] mainly use user-defined procedures to convert a multi-objective problem into a single objective problem. One of these methods is the weighted-sum approach [Miettinen, 1999] which uses the weighted sum of objectives to convert a multi-objective problem to a single-objective problem. The main drawback of the weighted-sum approach is that it is not applicable to non-convex problems. $\epsilon$-constraint [Haimes et al., 1971] is one possible replacement for weighted-sum. This method keeps one of the objectives and restricts the rest of the objectives within user-specified values. One of the main disadvantages of this method is being dependent on $\epsilon$ vector. Another classical method is Tchebycheff [Miettinen, 1999], which requires different weights for weighting objectives. The Penalty-Based Boundary Intersection Approach (PBI) [Zhang and Li, 2007] is another alternative to the weighted-sum approach. All three methods are explained in greater detail below.

### 2.2.1 The Weighted-Sum Approach

The weighted-sum method is one of the simplest and best-known strategies used to convert a multi-objective problem into a single-objective problem [Miettinen, 1999]. Although this approach is simple and easy to apply, choosing a weight vector that results in finding a solution near the user reference point is not straightforward. Choosing a weight value for each objective depends on its relative importance in the context of the actual problem. Moreover, in order to have a fair scaling of the objectives, they first need to be normalized [Deb, 2001]. A compound objective function is the sum of the weighted normalized objectives which is defined as follows:

$$\text{minimize } g^{\text{ws}} = \sum_{i=1}^{m} w_i f_i(\mathbf{x})\,, \tag{2.2}$$

where $0 \leq w_i \leq 1$, $m$ is the number of objectives and $w_i$ is the weight value for the $i$th objective function. It is customary to choose weights that add up to one. It has been proved that for any Pareto-optimal solution $\mathbf{x}^\star$ of a *convex* multi-objective problem, there exists a positive weight vector, such that $\mathbf{x}^\star$ is a solution to Equation (2.2) [Miettinen, 1999]. For any given set of weight values, Equation (2.2) will form a hyperplane in $\mathbb{R}^m$ for which the location is identified by the objective values which are subsequently dependent on the input vector $\mathbf{x}$, and the orientation of the plane is determined by the weight values $w_i$. In the special case of having two objectives, the $g^{\text{ws}}$ will take the form of a straight line for which the slope is

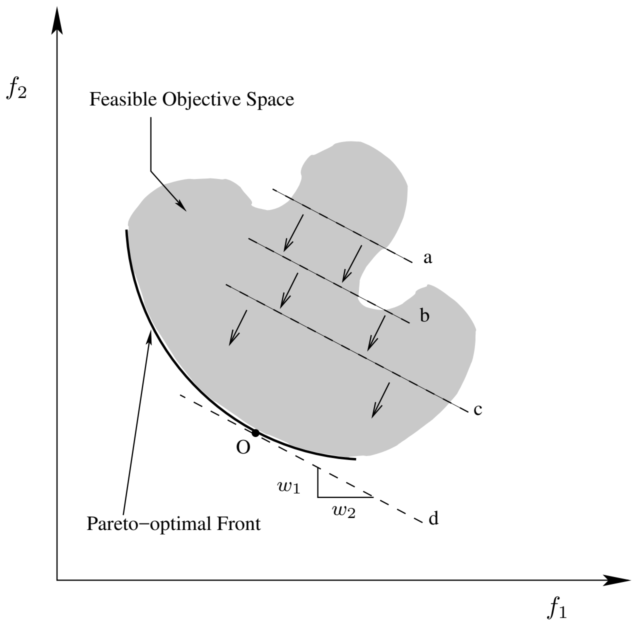



*Figure 2.2: Illustration of the weighted-sum approach on a convex Pareto-optimal front.*

determined by the weight vector as depicted in Figure 2.2. The effect of minimizing $g^{\mathrm{ws}}$ is to push this line as close as possible to the Pareto-optimal front until a unique solution is obtained. For example, in Figure 2.2 the solutions lie on the line 'a' and as they improve during the evolution, they move in the feasible region towards the Pareto-optimal front until an optimum solution ('O') is obtained. Lines 'a' through 'd' show how the improvement of solutions will result in the movement on the line (hyperplane in higher dimensions) until it becomes tangential to the Pareto-optimal front at point 'O'. A major advantage of the weighted-sum approach is its simplicity and effectiveness; however, it is less effective when dealing with non-convex Pareto-optimal fronts [Deb, 2001].

### 2.2.2 The Tchebycheff Approach

The Tchebycheff method [Miettinen, 1999] is formulated as follows:

$$\text{minimize } (g^{\mathrm{tch}}(\mathbf{x}, \mathbf{w}, \mathbf{z}^{\star}) = \max\{w_i|f_i(\mathbf{x}) - z_i^{\star}|\}), \tag{2.3}$$

where $i \in \{1, \ldots, m\}$, $m$ is the number of objectives, $\mathbf{z}^{\star} \in \mathbb{R}^m$ is the ideal point, and $\mathbf{w} = (w_1, \ldots, w_m)$ is a weight vector, which is positive. As shown in Figure 2.3, for each Pareto optimal point $x^{\star}$ which presented as black filled circle, there is at least a weight vector $\mathbf{w}$ (it is presented as an arrow in the figure) such that $x^\star$ is an optimal solution of Equation (2.3). The effect of the weight vector is also depicted in Figure 2.3.

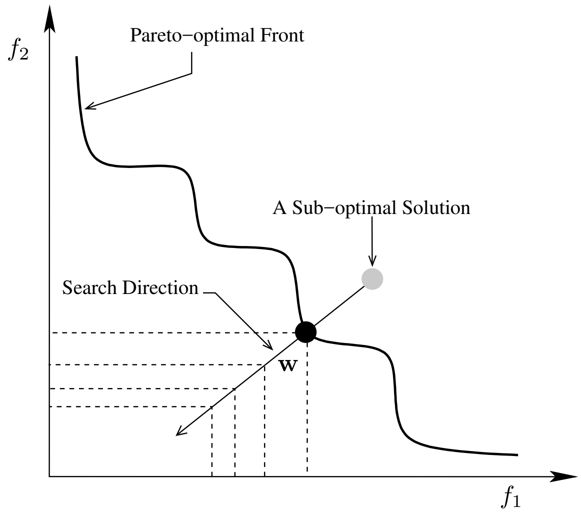



*Figure 2.3: Illustration of the Tchebycheff method.*

### 2.2.3 The Penalty-Based Boundary Intersection Approach

The Penalty-Based Boundary Intersection Approach (PBI) [Zhang and Li, 2007] is an improved version of a normal boundary intersection (NBI) [Das and Dennis, 1998]. Unlike NBI, which can only handle equality constraints, PBI [Zhang and Li, 2007] can handle both equality and inequality constraints. PBI is formulated as follows:

$$\text{minimize } g^{\text{pbi}}(\mathbf{x}, \mathbf{w}, \mathbf{z}^\star) = d_1 + \theta d_2, \tag{2.4}$$

$$\begin{gathered}\text{where } d_1 = \left\|(\mathbf{F}(\mathbf{x}) - \mathbf{z}^\star)^T \mathbf{w})\right\| / \|w\| \\ \text{and } d_2 = \|\mathbf{F}(\mathbf{x}) - (\mathbf{z}^\star + d_1 \mathbf{w})\|\,.\end{gathered}$$

As shown in Figure 2.4, $\mathbf{L}$ is a line passing through $\mathbf{z}^\star$ with the direction of $\mathbf{w}$ and $\mathbf{p}$ is the projection of $\mathbf{F}(\mathbf{x})$ on $\mathbf{L}$. The penalty parameter $\theta$ forces $\mathbf{F}(\mathbf{x})$ to be as close as possible to $\mathbf{L}$ (penalizing $d_2$).

The weighted-sum approach is the simplest and the most well known technique, which works well on convex optimization problems. However, non-convex Pareto-optimal fronts cannot be accurately approximated with this method [Deb, 2001]. The Tchebycheff method

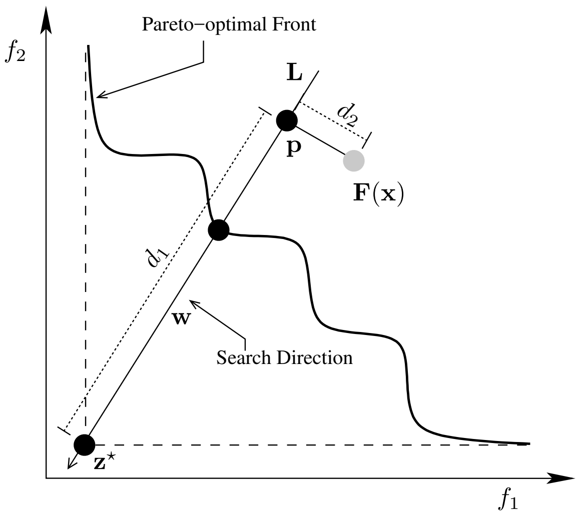



*Figure 2.4: Illustration of the PBI method.*

works well on both convex and non-convex Pareto-optimal fronts, but it does not result in a uniform set of solutions on the Pareto-optimal front [Zhang and Li, 2007]. PBI generates more uniform solutions compared to other decomposition methods. The Penalty parameter in PBI, which is called $\theta$, controls the uniformity of solutions to some extent. The convergence of the solutions can be affected by this penalty parameter.

## 2.3 Metaheuristics Algorithms

Metaheuristics are strategies that guide the search process to find the near-optimal solutions by exploring and exploiting the search space [Osman and Laporte, 1996]. Some of the properties of a metaheuristic algorithm are [Blum and Roli, 2003]: 1) they are usually non-deterministic and approximate; 2) they are not problem specific; 3) they use domain-specific knowledge in the form of heuristics that are controlled by a meta-level strategy. Some metaheuristics algorithms are introduced below.

### 2.3.1 Evolutionary Algorithms (EAs)

Evolutionary algorithms are one type of metaheuristics that use natural evolutionary principles to guide the search process and construct the optimization procedure. Evolutionary algorithms are an alternative approach to classical methods as they could overcome some of the most common difficulties of classical methods, being: 1. convergence on an optimal solution is very dependent on choosing an initial solution; 2. classical methods are not efficient in terms of handling parallel machines; and 3. classical methods are usually stuck in suboptimal solutions. The common theory behind evolutionary algorithms is that, given the population of individuals within a limited resource, competition for these resources causes natural selection, or "survival of the fittest" [Eiben and Smith, 2008]. The process of choosing better candidates for the next generation uses two operators: recombination and/or mutation. Mutation is an operator which is applied to one individual and which generates one new individual. Recombination is an operator which is applied to two or more individuals and generates one or more new offspring individuals. Therefore, the process of executing these two operators creates a set of new individuals (the offspring). The fitness values of these individuals are evaluated and competed with the parent individuals. This process will continue until individuals with sufficient quality are found.

There are two main forces in evolutionary systems according to Eiben and Smith [2008]:

1. Recombination and mutation can create necessary diversity in the population, which facilitates novelty.

2. Selection increases the quality of solutions in the population.

The combination of these two forces improves an individual's fitness values in the population. If the evolution is optimizing a fitness function, the optimal value is getting closer and closer over time. The main schema of an evolutionary algorithm can be defined as follows:

**Algorithm 2.1:** Evolutionary Algorithm

```
INITIALIZE population
EVALUATE each individual
while Termination Condition is Satisfied do
  SELECT parents;
  RECOMBINE pairs of parents;
  MUTATE the resulting offspring;
  EVALUATE new candidates;
  SELECT individuals for the next generation
end while
```

Different types of EAs can be generated by defining various components, procedures and operators. The most important components of an EA are: 1. Defining an individual; 2. Evaluating fitness functions; 3. Population; 4. Parent Selection Mechanism; 5. Variation Operators and 6. Survivor Selection Mechanism. To create a runnable algorithm, each component should be defined and its initial procedures should be specified.

### 2.3.2 Differential Evolution (DE)

Differential Evolution (DE) is another metaheuristic algorithm. Since we used it in our proposed approaches in this thesis, we explain its main properties here. DE was firstly introduced by Storn and Price [1997a]. It was a new heuristic approach for minimizing possibly nonlinear and nondifferentiable continuous space functions [Storn and Price, 1997a]. Given the population of solution vectors, by adding perturbation vector $\mathbf{p}$ (Equation 2.6) to an existing mutant vector $\mathbf{x}$, a new mutant vector $\mathbf{x}'$ (Equation 2.5) is generated,

$$\mathbf{x}' = \mathbf{p} + \mathbf{x} \tag{2.5}$$

$$\mathbf{p} = F(\mathbf{y} - \mathbf{z}) \tag{2.6}$$

where $\mathbf{p}$ is the scaled vector difference of the other two $\mathbf{y}$ and $\mathbf{z}$, which are randomly chosen population members (their values should not be exactly the same). Scaling factor $F$ is a real number greater than zero. $F$ controls the rate at which each population evolves. Crossover operation which is used in DE is mainly the same as a regular crossover operation that is used in evolutionary algorithms. However, crossover operator in DE has an extra parameter which is called crossover probability ($CR$). $CR \in [0, 1]$ specifies the chance that for any position in the parents currently undergoing crossover, the allele of the first parent will be included in the child.

In the main DE algorithm, population is like a list. This allows referencing to $i_{th}$ individual by its position $i \in \{1, \ldots, \mu\}$. The order of individuals in a population $P = (\mathbf{x}_1, ..., \mathbf{x}_\mu)$ is not related to its fitness values. In an evolutionary cycle, firstly a mutant vector population $M = (\mathbf{v}_1, ..., \mathbf{v}_\mu)$ is created, then for each mutant new vector $\mathbf{v_i}$ three different vectors are chosen from $P$, a base vector to be mutated and the other two vectors to define a perturbation vector. After making a mutant vector, by applying crossover to $v_i$ and $x_i$ a trial vector population $T = \langle \mathbf{u}_1, ..., \mathbf{u}_\mu \rangle$ is created where $\mathbf{u_i}$ is the result of applying crossover to $\mathbf{x_i}$ and $\mathbf{v_i}$. Finally, a deterministic selection is applied to each pair of individuals such as $\mathbf{x_i}$ and $\mathbf{u_i}$ . $\mathbf{x}_i$ will be selected in the next generation if and only if $f(\mathbf{x_i}) \leqslant f(\mathbf{u_i})$. In summary, there are three parameters in DE: $\mu$ (population size), $F$ scaling factor, $CR$ crossover probability. Over the years, many variants of DE have been developed and published, each of which focuses on different aspects of DE parameters [Storn and Price, 1997b; 1996; Storn, 1996] and [Das and Suganthan, 2011] have done a survey on DE from different aspects including: major variants, application to multi-objective, constrained, large scale, and uncertain optimization problems.

The below table shows the brief description of differential evolution

| | |
|---|---|
| Representation | Real-valued vectors |
| Recombination | Uniform crossover |
| Mutation | Differential mutation |
| Parent selection | Selection of the three different vectors |
| Survival selection | Deterministic elitist replacement (parent vs. child) |

## 2.4 Evolutionary Multi-objective Optimization (EMO) Algorithms

The history of EA approaches to multi-objective optimization begins with the vector-evaluated genetic algorithm (VEGA) which is proposed by Schaffer [1985]. In this approach, the population is first divided into sub-populations. Each sub-population receives a fitness value based on the objective function. However, recombination and parent selection happens globally. VEGA manages to approximate the Pareto front after a few generations. The main disadvantage of VEGA is that there is not enough diversity in the population. One alternative solution to tackle this issue was proposed by Kursawe [1990]. In this approach, diversity is maintained by using a niching strategy. In other words, they use an elimination of crowding strategy to remove solutions which are close to each other. Below we describe some of the popular EMO approaches.

### 2.4.1 Non-elitist Approaches

Generally speaking non-elitist approaches do not use any elite-preserving operator. The multi-objective genetic algorithm (MOGA) which was proposed by Fonseca and Fleming [1993] was one of the first non-elitist algorithms. MOGA uses genetic algorithms to solve a multi-objective optimization problem. Decision maker (DM) inputs are also used as a step in the evolutionary process. In other words, the interaction between DM and a genetic algorithm leads to the selection of satisfactory solutions to the problem. The non-dominated sorting genetic algorithm (NSGA), which is another none-elitist approach, was first proposed by Srinivas and Deb [1994]. NSGA is similar to MOGA in many ways. However, the population is divided into a number of fronts, each with equal domination, for assigning fitness. The procedure of the algorithm is as follows: Firstly, the algorithm searches for individuals which have not been labeled as related to a previous front. Secondly, individuals in the current front will be labeled and the front counter will be increased until all the individuals have been labeled. The fitness of each individual in a front is calculated based on the number of all solutions in the lower front. Solutions with the same rank are assigned the same fitness value. Algorithm 2.2 explains non-dominated sorting.

**Algorithm 2.2:** Non-dominated sorting Algorithm

```
for each p ∈ P do
  S_p=∅ // set of individuals p dominate
  n_p = 0 // counter for number of individuals that dominate p
  for each q ∈ P do
    if p ≺ q then
      S_p = S_p ⋃ {q}
    else if q ≺ p then
      n_p = n_p + 1
    end if
  end for
  if n_p == 0 then
    P_rank = 0
    F_1 = F_1 ⋃ {p}
  end if
end for
i = 1 // initialize the front counter
while F_i ≠ ∅ do
  Q = ∅ // set of individuals of the next front
  for each p ∈ F_i do
    for each q ∈ S_p do
      n_q = nq − 1
      if n_q = 0 then
        q_rank = i + 1
        Q = Q ⋃ {q}
      end if
    end for
  end for
  i = i + 1
  F_i = Q
end while
return F_i
```

For each individual $p$, there are two variables: 1) $n_p$ is the number of individuals that dominate $p$ and 2) $S_p$ is the set of solutions which $p$ dominates. $p_{rank}$ indicates the dominated front that $p$ belongs to. In the above algorithm in line 11, $p$ belongs to the first front and in line 22, $q$ belongs to the next front. A niched Pareto genetic algorithm (NPGA) was first proposed by Horn et al. [1994]. The main modification in this algorithm concerned tournament selection which was based on two criteria: firstly, dominance comparison and secondly, the number of similar solutions already present in the new population.

### 2.4.2 Elitist Approaches

Although non-elitist approaches can perform well in a number of test problems, there are some issues. Firstly, their performance is heavily dependent on choosing parameters. Secondly, they can also potentially lose good solutions. Elitist approaches were developed to address the issues of non-elitist approaches. NSGA-II [Deb et al., 2002] is an elitist approach which uses the idea of non-dominated sorting. NSGA-II differs from NSGA in two main aspects: 1) *Density Estimation*. To estimate the density of each point in relation to other points, crowding distance metric is defined. This metric indicates how far individuals are from each other. To achieve this for each point, the average distance of two points on either side of that point along each objective is calculated. This value is used to estimate the perimeter of the cuboid which is formed by the nearest neighbours as the vertices. For the purpose of calculating the crowding distance metric, individuals in the population should be sorted based on the objective function value. The smaller the value of a crowding distance indicates that the individual is in a dense area. 2) *Crowded-Comparison Operator*. This operator$\prec_n$ directs solutions towards diversity on the Pareto-optimal front in different stages of the algorithm. Each individual has two attributes: (a) non-dominate rank ($i_{rank}$) and (b) crowding distance ($i_{distance}$). The operator is defined as below: $i \prec_n j$ if ($i_{rank} < j_{rank}$) or ($i_{rank}$=($j_{rank}$ and $i_{distance}$) $> j_{distance}$) If two solutions have different non-domination ranks, the lower rank is preferred. If two solutions are in the same front, the one that is in a less dense area is proffered. Strength Pareto evolutionary algorithm (SPEA-2) [Zitzler et al., 2001] and Pareto achieved evolutionary strategy (PAES) [Knowles and Corne, 1999] are two other popular elitist algorithms. They both use a fixed size archive. Non-dominated points which are discovered during the search process are kept in the archive. The archive is updated based on dominance information and the number of archive points close to a new solution.

### 2.4.3 Decomposition-Based EMO Algorithms

Although elitist approaches remedy the issues raised by non-elitist approaches, they suffer from the selection pressure issue and cannot scale well in higher objective space. In contrast, decomposition approaches are less susceptible to the selection pressure problems and they can solve many-objective optimization problems more effectively. Because of their highly desirable properties, we have used decomposition-based approaches in this research to handle many-objective optimization problems.

Decomposition approaches convert a multi-objective problem into a single-objective problem which is then optimized using an Evolutionary Algorithm. As mentioned before in section 2.2, three widely used decomposition approaches are Weighted-Sum [Miettinen, 1999], Tchebycheff [Miettinen, 1999] and Penalty-based boundary intersection (PBI) [Zhang and Li, 2007]. Some popular decomposition-based EMO algorithms which combine evolutionary algorithms with decomposition methods are described here:

**MOGLS** was first proposed by Ishibuchi and Murata [1998] and was improved by Jaszkiewicz [2002]. In short, in each iteration a set of current solutions (CS) and the fitness values of these solutions are maintained. An external population (EP) is used to store non-dominated solutions. In MOGLS, two main parameters, $K$ and $S$, are used. $K$ determines the size of EP and $S$ is the size of the current solution. For each individual which is generated by genetic operations, a local search procedure is applied. When a pair of parent solutions is selected to generate new solutions, the fitness function (Weighted-sum or Tchebycheff) is utilized. A local search procedure is applied to new solutions to maximize its fitness value.

**MOEA/D** uses a decomposition method to decompose a multi-objective optimization problem into a number of single objective optimization problems. Then an EA is used to solve these sub-problems simultaneously. Each individual is assigned to a sub-problem. Based on the distance of sub-problem weight vectors, a neighborhood relationship among all sub-problems is defined. As two neighbouring sub-problems have a close optimal solution to optimize a sub-problem its neighborhood information is used. Since MOEA/D relies on the individuals' neighborhoods rather than the whole population to generate new offspring, it benefits from a lower computational cost compared to its counterparts such as MOGLS [Ishibuchi and Murata, 1998] and NSGA-II [Deb et al., 2002].

The general framework of MOEA/D is as follows:

An external population (EP) is used to store non-dominated solutions which are found during the search.

**Algorithm 2.3:** MOEA/D Algorithm

```
Initialize external Population to Null
Initialize Weight Vectors
while Exist a Weight Vector do
  Calculate T closet weight vectors to each weight vector
end while
Generate Initial Population
z_j = min_{1⩽i⩽N} f_j(x^i)
z = (z_1, ..., z_m)^T
while Termination Condition is Satisfied do
  Generate new offspring by applying genetic operators using the neighborhood
  information
  Update the neighborhood for each sub-problem
  Update EP
end while
```

During the time, MOEA/D has been studied and investigated from different aspects [Trivedi et al., 2017] such as: weight vector generalization, computational resource allocation, handing many-objective optimization, mating selection, replacement mechanism which we introduced them briefly in this section.

**Weight Vector Generalization**

1. UMOEA/D was proposed by Tan et al. [2013]. It uses good lattice point design (GLP) for weight vector initialization instead of simplex-lattice design. It has been shown that UMOEA/D can generate more uniform solutions than MOEA/D due to the use of GLP. Unlike simplex-lattice design, the use of GLP decouples the dependence between the number of objectives and the population size. As a result, UMOEA/D is scalable to a higher number of objectives.

2. More recently, Ma et al. [2014] proposed **MOEA/D-UDM** which replaces the initialization method of weight vector and the original Tchebycheff decomposition approach in MOEA/D. For weight vector initialization, MOEA/D-UDM combines the simplex-lattice design with a transformation method proposed in [Fang and Wang, 1993] and [Fang and Lin, 2003] to obtain uniformly distributed Pareto-optimal solutions over PF, then a uniform decomposition measurement [Ning et al., 2011] is used to select a uniform set of weight vectors. A Modified Tchybecheff ($M - TCH$) [Jain and Deb, 2013] which is defined as $M - TCH(x, w, z^*) = max_{i=1}^{M} \left| {}_{f_i(x) - z_i^*} \right| / w_i$ is adopted in

MOEA/D-UDM. It has been shown that MOEAD/UDM outperforms MOEA/D and UMOEA/D in terms of both diversity and convergence.

3. MACE-gD [Giagkiozis et al., 2014] is based on generalized decomposition (gD) [Giagkiozis et al., 2013a] and the Cross Entropy method (CE) [Rubinstein, 1999]. Generalized decomposition (gD) is used to select weight vectors to satisfy the distribution of solutions on Pareto-optimal solutions along PF. CE is used as the main optimization algorithm. In MACE-gD the geometry of Pareto front should be made available before the search process and a method to generate distribution along geometry based on the DM requirement should be available as well.

**Computational Resource Allocation**

1. Bi-criterion Evolution(BCE) [Li et al., 2015] is another decomposition-based EMO that attempts to maintain the diversity of solutions. BCE uses two populations, namely PC (Pareto Criterion) and NPC (Non-Pareto Criterion), where NPC guides the search towards the optimal front while PC mainly focuses on maintaining the diversity of solutions by exploring undeveloped or not well-developed regions in the objective space. These two populations communicate with each other and use the suitable individuals generated by either of them.

2. MOEA/D with the adaptive weight vector adjustment (**MOEA/D-AWA**) was proposed by Qi et al. [2014]. It uses a new weight vector initialization method with adaptive weight vector adjustment. MOEA/D-AWA initializes the weight vectors based on the geometric relationship between weight vectors and solutions under the Tchebycheff decomposition. To update the weight vectors, MOEA/D-AWA uses an elite population which introduces new sub-problems into the sparse regions of the search space. It has been shown that MOEA/D-AWA can outperform MOEA/D [Qi et al., 2014].

3. Pareto-adaptive weight vectors ($pa\lambda$)($pa\lambda$-MOEA/D) [Jain and Deb, 2014] was inspired by the idea of e-dominance to divide the objective space into different hyper boxes based on the geometry information of Pareto front. $pa\lambda$ has two features; first it is based on the Mixture Uniform Design (MUD) and can generate an arbitrary number of weight vectors for any number of objectives. Secondly, since $pa\lambda$ uses the Hypervolume metric, it is able to maintain diversity and convergence better than NSGA-II and MOEA/D.

**Handing Many-objective optimization**

1. DBEA which is proposed by Asafuddoula et al. [2015] uses reference directions to guide the search process. Sampling points are used to generate reference directions, similar to that of NSGA-III [Deb and Jain, 2014]. To maintain the diversity and convergence of solutions, two distance measures have been used. One distance is measured along the reference direction to control the convergence. The second measures the perpendicular distance from the solution to the reference direction and is used to maintain the diversity of solutions. To handle constraint optimization problems, adaptive $\epsilon$ level-based schema are adopted. In the proposed approach, the number of the reference directions is the same as the population size. Where a problem with a complex Pareto-optimal front is needed to maintain convergence and diversity properly, a large number of reference directions is required and consequently a large number of individuals is needed. This may not be cost-effective or practical.

2. RVEA [Cheng et al., 2016] is another decomposition-based EMO algorithm that uses reference vectors inspired by ideas from MOEA/D-M2M [Liu et al., 2014] to balance between convergence and diversity. The main idea behind RVEA is to use the reference vector to divide the objective space to some sub-spaces. In each sub-space, Angle-Penalized Distance (APD) is used to measure the closeness of solutions to the ideal point and reference vectors can be used to measure diversity or satisfaction of preferences. To maintain the uniformity of solutions in the objective space, reference vectors have been adapted based on the distribution of candidate solutions in the objective space.

3. [Deb and Jain, 2014] proposed a reference-point-based many-objective evolutionary algorithm NSGA-III which is based on the NSGA-II framework but with the significant changes in its selection operator. Diversity and uniformity in NSGA-III are maintained by providing and adapting well distributed reference points. NSGA-III applied to many-objective problems up to 15 objectives and its performance compared with two versions of MOEA/D.

**Mating Selection and Replacement Mechanism**

1. MOEA/D-STM was proposed by Li et al. [2014]. It uses a stable matching model (STM) to coordinate the selection process in MOEA/D. In MOEA/D-STM, sub-problems and solutions are expressed as two sets of agents. Each sub-problem agent ranks all solutions and prefers the solution to have a better aggregation function value. Each solution

agent ranks sub-problems based on its distance to the direction vector of sub-problems and prefers sub-problems with the lowest distance. This assignment can balance the diversity and convergence of a search.

2. MOEA/D-STM2L which is the extended version of MOEA/D-STM is proposed by Wu et al. [2015]. The proposed algorithm added another level to improve diversity of solutions. In other words, the first level is used of match a solution to one of its most preferred subproblems and the second level is used to match the solutions to the remaining subproblems. Experimental results show that MOEA/D-STM2L outperforms other state-of-the-art variants of MOEA/D as well as MOEA/D-STM.

## 2.5 Integrating Preferences in EMO Algorithms

There are three ways of involving a decision maker (DM) in an optimization process [Van Veldhuizen and Lamont, 2000]: The specification of preference can be done before the optimization process (*a priori*), during the optimization process (*interactive*) and after optimization process (*a posteriori*). These preference mechanisms were originally introduced in multi-Criteria decision making [Gandibleux, 2006]. Most EMOs can be referred to as a posteriori approach as they try to find a well distributed set of solutions on the Pareto-optimal front before allowing a decision maker to look at solutions and choose the most preferable solutions. In this research, we consider (*a priori*) approach. In other words, a decision maker can provide his/her preference(s) before the optimization process as a reference point and a reference point can be one of the existing solutions. It might not always be practical for the user to specify his/her exact preference(s). However, we assume that the user has an approximate idea about the objective space. Integrating the DM to the search process saves considerable computational resources by focusing on more desirable regions of DM's interest.

### 2.5.1 *a priori* Methods

In this section, we describe some of the EMO algorithms which incorporate the preference information prior to the optimization process. Since *a priori* methods are used in this thesis to propose new approaches, our literature has mainly focused on these methods. In this thesis, we have categorized *a priori* algorithms to three main categories based on the techniques that they used to incorporate the preference information: 1) Goal attainment 2) Reference Point Based 3) Light Beam Based.

Some popular approaches in **Goal attainment** are described below. Goal Programming, which is proposed by Deb [1998], is one of the first attempts to apply EMO approaches to classical goal programming [Ignizio, 1976]. The ability of an EA to find multiple solutions makes it possible to simultaneously minimize the deviations from individual goals, which eliminates the need for a user-defined weight vector. The effectiveness of this evolutionary approach to goal programming has been verified empirically using several test problems as well as a real-world engineering design problem [Deb, 1998]. In this approach, the emphasis was mainly on goal satisfaction and the algorithm does not try to find Pareto-optimal solutions close to the supplied goal. Since a DM needs to supply his/her goals before the optimization process, this approach is categorized as a *a priori* method.

The guided multi-objective evolutionary algorithm (**G-MOEA**), proposed by Branke et al. [2001], is another *a priori* method user needs to specify the linear trade-off between objectives before the search begin. For example, in a two-objective problem the decision maker has to specify how many units of the first objectives he/she is willing to trade for one unit of the second objective. G-MOEA then uses this trade-off information to guide the search towards the more desired regions of the Pareto-optimal front. Although G-MOEA is more flexible and intuitive than other approaches, it is not always an easy task for the decision maker to specify the trade-off between objectives, especially for many-objective problems.

Biased-Sharing, which was proposed by Deb [2003], applies the biased sharing technique to NSGA [Srinivas and Deb, 1994] where the biased Pareto-optimal solutions are generated on a desired region. To achieve this the user needs to assign weights to objectives before the optimization process. An objective with a higher priority takes a higher weight value. The main disadvantage of this technique is that it cannot find solutions on a compromise region where all objectives are of similar importance to the decision maker. Branke and Deb [2005] improved the idea of biased sharing and compared its performance with G-MOEA. They proposed a biased crowding distance in NSGA-II which has more flexibility than biased sharing in terms of finding solutions within the region of interest.

Some popular approaches in **Reference Point Based Algorithms** are described below.

Deb et al. [2006] proposed a method that integrated use-preference information with NSGA-II [Deb et al., 2002]. The new method which was called **R-NSGA-II** requires the decision maker to provide one or more reference points at the beginning of the search process. In R-NSGA-II, a modified version of the crowding distance operator [Deb et al., 2002] *preference distance* was used to favour the solutions which are closer to the reference point(s). To compute the preference distance, the Euclidean distances of all solutions to the reference point(s) are calculated and the solutions with lower distances are ranked higher. To maintain the diversity of solutions in the desired regions, an extra parameter $\epsilon$ was introduced. In this thesis, the performance of the proposed approach has been compared with R-NSGA-II. In short, the following changes have been applied to NSGA-II.

**Step 1:** The solution closest to the reference point should be found and assigned the rank one. To achieve this, the Euclidean distance between each solution and the reference point(s) are calculated and sorted.

**Step 2:** Solutions with the smaller crowding distance should be preferred. To achieve this, for each reference point the lowest rank calculated in the previous step is assigned as the crowding distance to a solution. This means that solutions closest to the reference point(s) receive lowest crowding distance values.

**Step 3:** To maintain diversity of solutions a $\epsilon$ parameter is used. To do this, the sum of normalized differences in objective values for all solutions is calculated. Those which have the value of $\epsilon$ or less have been grouped. From each group a random solution will be picked up and the rest of the members will be assigned a large number to remain in the race.

Wickramasinghe and Li [2009] integrated reference points and light beams with particle swarm optimization (PSO). The new approach (**Preference-based NSPSO**), which is based on a distance metric, changes the position of particles based on the user supplied information to find the preferred regions. This distance-metric based method was compared with another user-preference based EMO (Dominance-based reference point NSPSO) [Wickramasinghe and Li, 2008] which uses dominance-based comparison. It was shown that the distance metric approach performed better than NSPSO.

The proposed approach provides different options for DM in terms of directing solutions to desired regions. For example, if the user wants to choose several non-dominated solutions in step 4, the proposed approach can display solutions which have a next best achievement function value or use clustering the current populations. The main drawback of this approach is that visualizing the Pareto-optimal front for DM beyond three objectives is difficult.

**r-MOEA/D-STM** [Li et al., 2014] is the user preference version of MOEA/D-STM [Li et al., 2014]. In this approach a decision maker provides his/her preferences as reference points ($r$). It is widely accepted that using preferences in MOEAs potentially reduces the computational cost and drives the search direction to particular areas. In this study, reference points are used in both feasible and infeasible regions, it has been shown that r-MOEA/D-STM is able to provide solutions close to the reference points where there are two or three objectives. It should be noted that the performance of r-MOEA/D-STM has not been measured on complicated Pareto-optimal shapes and many-objective problems.

**Light Beam Based Algorithms** are another type of *a priori* user-preference methods which are very similar to reference point based algorithms. However, a light beam is passed to the optimizer rather than a reference point. One of the popular light beam algorithms is (**LBS-NSGA-II**) which is proposed by Deb and Kumar [2007a]. They applied light beam search to NSGA-II. The decision maker provides aspiration, reservation and a preference threshold for each objective and the procedure can be continued until a single preferred solution is obtained. To control the density of solutions, the parameter $\epsilon$ is used. A decision maker can choose more than one light beam so more than one set of preferred regions can be found simultaneously. It has been shown that the proposed approach can find solutions on the Pareto-optimal front up to three-objective problems, and solutions can converge satisfactorily if there are more objectives. Another light beam approach is **Distance Metric** which is proposed by Wickramasinghe and Li [2009]. They integrated light beams with particle swarm optimization (PSO). The new approach changes the position of particles based on the user-supplied information to find the preferred region. Distance metric was compared with another user-preference based EMO (NSPSO) [Wickramasinghe and Li, 2008] which uses a dominance-based comparison and it was shown that the distance metric approach performed better than NSPSO.

### 2.5.2 *a posteriori* Methods

In this section we describe some of the EMO algorithms which incorporate the preference information after the optimization process.

Preference-inspired coevolutionary algorithm **PICEA** [Wang et al., 2013] is based on *a posterior* decision making process which provides both proximal and diverse representation of the entire front to the decision maker. It coevolves a population of solutions with preference concepts. Solution can be awarded good fitness values if they perform well against preferences.

In **Weighted Stress Function Method (WFSM)** [Ferreira et al., 2007], firstly a Pareto-optimal set is generated, then a methodology is used to consider the preference of a decision maker by selecting a single solution or a region of the Pareto frontier. In other words, the search and decision processes are sequential. The main idea behind this method is that the best solution that satisfies the preference of the user must have the following properties: 1) the solution must be a non-dominated solution and belong to the Pareto front; 2) the ideal objective vector should be considered by the search process. For instance, where the aim is to maximize the criteria, the optimization of each criteria corresponds to the maximum value for that criterion. However, in the case of a multi-objective optimization problem, the importance of each criterion will induce a stress (pressure) for searching for solutions that maximize each of the criteria. For example, if there are two criteria $f_1$ and $f_2$, $\gamma_{w_1}$ and $\gamma_{w_2}$ are two stresses which belong to a solution and are associated to each corresponding criterion $f_1$ and $f_2$. $w_1$ and $w_2$ are weights associated with each criterion. It should be noted that the proposed method works well on convex and non-convex Pareto fronts but it is not suitable for problems with discontinuous Pareto fronts.

### 2.5.3 *Interactive* Methods

In this section we describe some of the EMO algorithms which incorporate preference information during the optimization process.

PBEA [Thiele et al., 2009] is developed based on an indicator-based evolutionary algorithm (IBEA) [Zitzler and Künzli, 2004]. The idea of interactive user-preference has been combined with PBEA and the new approach is called **Preference-Based Evolutionary Algorithm (PBEA)**. The main steps are described below:

**Step1** *Initialization:* By using PBEA, the approximation of a Pareto-optimal set is generated. A small set of solutions is displayed to a DM for evaluation.

**Step2:** *Reference Point:* DM specifies a reference point which indicates a desirable value for the objective function.

**Step3:** *Local Approximation:* Here, the reference point information is used in the PBEA algorithm to generate the local approximation of the Pareto-optimal set.

**Step4:** *Reference Point Projection:* Non-dominated solutions generated in the previous step received the smallest value for the achievement function.

**Step5:** *Termination:* If DM found most preferable solutions as a good estimate, he/she can stop the search. Otherwise, DM can continue to search by starting from step 1.

**RD-NSGA-II**, which is proposed by Deb and Kumar [2007b], applies reference direction to NSGA-II. In this approach, one or more reference directions can be obtained by the user in the objective space in each iteration. A number of representative points along the reference direction will be generated. A set of solutions corresponding to these representative points are obtained by using EMO strategies. A single solution which is obtained by utility function is used for further analysis. This procedure will continue until there is no further improvement.

In $UTA^{GMS}$ [Greco et al., 2008], the decision maker provides his/her preferences by using a set of pairwise comparisons based on the reference alternative which is a subset of $A^R \subseteq A$. A preference model is built based on the additive value functions. This model is used to define two rankings in the set $A$: 1) a necessary ranking for any two members of set $A$ such as $a$ and $b$ if, and only if, $a$ is preferred to $b$ for all compatible value functions; 2) possible ranking holds for this set if, and only if, $a$ is preferred to $b$ for at least one compatible value function. $UTA^{GMS}$ is an interactive approach as it used progressive pairwise comparisons to increase the subset $A^R$. Where no preference information is provided, a necessarily weak preference creates a weak dominance relation and a possibly weak relation is a complete relation.

The preference information can be specified directly or indirectly. If it is specified directly, values of some parameters are used in the preference model. This direct preference information is used in the traditional aggregation paradigm based on which aggregation model is constructed. This preference is then applied to set $A$ to rank alternatives. Indirect preference information is used in a regression paradigm based on the historic preference on the subset of alternatives. Then a consistent aggregation model which comes from this indirect information is applied to set $A$ to rank alternatives. This indirect preference information provided by DM is a good reason for involving DM in the loop of optimization and to incorporate his/her responses.

In the **Interactive Hybrid Approach** [Klamroth and Miettinen, 2008], a rough approximation of a non-dominate set is first generated. To provide an efficient overview of non-dominated solutions for users, a piecewise-linear approximation tool is used. The main involvement of a decision maker is to check the accuracy of this approximation in various ways: 1) bounding number of solutions are generated or any error in the approximation; 2) refining the approximation where DM can learn about the problem by studying the approximation. This gives an opportunity to DM to direct the search to the best non-dominated solution without comparing a lot of solutions at a time; 3) locating the most satisfactory region by specifying the least acceptable values in the form of a reference point. Since the number of optimization problems to be solved during the whole solution process can stay relatively small, this approach is not computationally expensive.

It should be noted that the there are some shortcoming with mentioned decomposition and user-preference based evolutionary algorithms. Those which are proposed in couple of years ago did not take into account the uniformity of solutions and they mainly focused on the convergence of solutions. More recent algorithms try to produce uniform solutions in the objective space by generating uniform weight vectors. However, this cannot be generalized to all algorithms as there is not always a direct relationship between weight vectors and solutions in the objective space. Recently, some approaches try to produce a uniform set of solutions in the objective space by adopting the weight vectors. However, the effect of weight vector adoption has not been measured during the optimization process. In this thesis, we proposed a user-preference based decomposition method which produces uniform solutions in the objective space by adopting the weight vectors and the effect of adoption is measured and take into account during the optimization process. Since the user-preference information has been adopted simultaneously with weight vectors, it can solve many-objective optimization problems efficiently.

## 2.6 Performance Metrics in EMO

This section gives an overview of some widely used metrics for evaluating multi-objective evolutionary algorithms.

The performance of EMO algorithms is typically measured by the following two aspects: 1) closeness of the solutions to the Pareto-optimal front (convergence); 2) the diversity and the spread of the solutions. A property which is sometimes overlooked is that the metric should measure the performance of a set of algorithms without relying on knowledge of the Pareto-optimal front. This problem becomes more serious when the Pareto-optimal front is unknown, which is most often the case in many real-world problems. In the remainder of this section, we review some existing performance metrics for EMO algorithms. The major classifications of metrics presented in this section are adopted from [Okabe et al., 2003].

### 2.6.1 Cardinality-Based Metrics

These metrics measure the performance of various algorithms by counting the total number of non-dominated solutions found by each algorithm [Veldhuizen, 1999a; Veldhuizen and Lamont, 2000]. However, producing a large number of non-dominated solutions does not necessarily make an algorithm better than another. For example, an algorithm may have only one solution that dominates all the solutions of another algorithm. In order to alleviate this problem, many cardinality-based approaches rely on a *reference set* and measure the contribution from each algorithm with respect to that reference set [Veldhuizen, 1999a; van Veldhuizen and Lamont, 1999; Zitzler, 1999; Czyzak and Jaszkiewicz, 1998; Hansen and Jaszkiewicz, 1998].

There are many different ways of constructing a reference set. For example, a reference set may be formed by aggregating all known solutions to a problem by various means, or by merging the solution sets that are generated by a set of algorithms that are to be compared [Mohan and Mehrotra, 2011]. These reference sets may contain all possible solutions, or just the non-dominated solutions. It should be noted that the ranking of a set of algorithms may change depending on the choice of the reference set [Mohan and Mehrotra, 2011]. The reference set used in this chapter is formed by taking the non-dominated solutions from the merged solution sets of several algorithms.

#### 2.6.1.1 Set Convergence Metric

This metric is used to measure the relative convergence of two solution sets with respect to each other [Zitzler and Thiele, 1999]. Let $A$, and $B$ be the solution sets of two different algorithms. $\mathcal{C}(A, B)$ is calculated as follows:

$$\mathcal{C}(A, B) = \frac{|\{b \in B | \exists a \in A : a \preceq b\}|}{|B|}, \tag{2.7}$$

If $\mathcal{C}(A, B) = 1$ then $A$ dominates all members of $B$, and if $\mathcal{C}(A, B) = 0$, none of the solutions from $B$ are dominated by $A$. The result of $\mathcal{C}$ metric is not always reliable. For instance, there are cases where the surfaces covered by two fronts are equal, but one front is closer to the Pareto-optimal front than the other.

#### 2.6.1.2 Convergence Difference of Two Sets

This metric, which is called $\mathcal{D}$ metric [Zitzler, 1999], is an improved version of the $\mathcal{C}$ metric. Let $A$ and $B$ be the solution sets of two different algorithms. Then $\mathcal{D}(A, B)$ is the size of the region which is *only* dominated by solutions in $A$, and $\mathcal{D}(B, A)$ is the size of the region which is *only* dominated by solutions in $B$. For a maximization problem if $\mathcal{D}(A, B) < \mathcal{D}(B, A)$, then it is concluded that $B$ dominates $A$.

Neither $\mathcal{C}$ and $\mathcal{D}$ metrics measure the diversity and spread of the solutions. Additionally, these two metrics are not efficient when comparing more than two algorithms. Another major drawback of these two metrics is that they become increasingly inaccurate as the number of objectives increases. The values for $\mathcal{C}(A, B)$ converge to $\mathcal{C}(\mathcal{B}, \mathcal{A})$ as the number of objectives increases. This is because most of the solutions in many-objective problems are non-dominated to each other. Therefore, the areas covered by the two sets of solutions become equal.

### 2.6.2 Distance-Based Metrics

*Generational Distance(GD) [Veldhuizen, 1999a]* is a metric widely used to measure the convergence of EMO algorithms by calculating the average closest distances of obtained solutions to the Pareto-optimal front. More precisely, the GD value is calculated in the following way. Let $Q$ be the obtained solution set and $P^*$ a set of non-dominated solutions on the Pareto-optimal front. Then,

$$GD(Q, P^*) = \frac{\sum_{v \in Q} d(v, P^*)}{|Q|} \tag{2.8}$$

where $|Q|$ is the number of solutions in $Q$, and $d(v, P^*)$ is the closest Euclidean distance from each point $v$ to a point in $P^*$. Since GD is calculated based on the Pareto-optimal front, it gives an accurate measure for the convergence of an algorithm. However, GD does not measure the diversity and spread of the solutions on the Pareto-optimal front. Another disadvantage of GD is that it becomes difficult to calculate when dealing with many-objective problems. For a reasonable sampling of the Pareto-optimal front, a large number of points are required, which makes the calculation of GD computationally expensive. It should be noted that GD cannot be applied without the existence of a reference front such as the Pareto-optimal front.

*Inverted Generational Distance (IGD) [Zitzler et al., 2003]* is an improved version of GD that takes both diversity and convergence of solutions into account. However unlike GD, which calculates the average closest distance of solutions to the Pareto-optimal front, it calculates the average closest distance of sample points on the Pareto-optimal front to the obtained solutions. Therefore,

$$IGD(P^*, Q) = \frac{\sum_{v \in P^*} d(v, Q)}{|P^*|}. \tag{2.9}$$

A major advantage of IGD is that it can measure both convergence and diversity of the solutions simultaneously. Similar to GD, IGD becomes exponentially expensive as the number of objectives increases.

### 2.6.3 Volume-Based Metrics

*Hypervolume Metric (HV) [Veldhuizen, 1999a; Zitzler and Thiele, 1998]* is a metric which measures the volume between all solutions in an obtained non-dominated set and a nadir point. A nadir point is a vector of the worst objective function values obtained by the solution set. To calculate the HV value, a set of hypercubes ($c_i$) is constructed by taking a solution $i$ and the nadir point as its diagonal corners. Finally, the HV value is the total volume of the hypercubes.

$$HV = \text{volume}\left(\bigcup_{i=1}^{|Q|} c_i\right), \tag{2.10}$$

where $Q$ is the solution set. Higher HV values indicate a better convergence and diversity of solutions on the Pareto-optimal front. A major advantage of HV is that it does not depend on knowledge of the Pareto-optimal front and its disadvantage is that HV value depends on the estimation of the nadir point.

### 2.6.4 User-Preference Performance Metrics in EMO

All of the metrics discussed so far were designed to measure the performance of EMO algorithms that approximate the entire Pareto-optimal front. There has been limited work on developing metrics for comparing user-preference based EMO algorithms.

To the best of our knowledge, Wickramasinghe *et al.* [Wickramasinghe et al., 2010] were the first to propose a metric for comparing user-preference based EMO algorithms. This metric works by combining the solution sets of all algorithms that need to be compared. Then, the closest solution to the ideal point is used as the center of a hypercube that defines a preferred region. Figure 2.5 shows how a preferred region is defined for two different reference points. The size of the preferred region is determined by a parameter, $\delta$, which is half the edge length of the hypercube. Finally, for each of the algorithms, HV is calculated with respect to a nadir point for all the solutions that fall within the preferred region. To calculate the nadir point, this metric uses the solutions from all algorithms inside the preferred region. The choice of the ideal point is the origin of the coordinate system for minimization problems.

An advantage of this metric is that it does not require knowledge of the Pareto-optimal

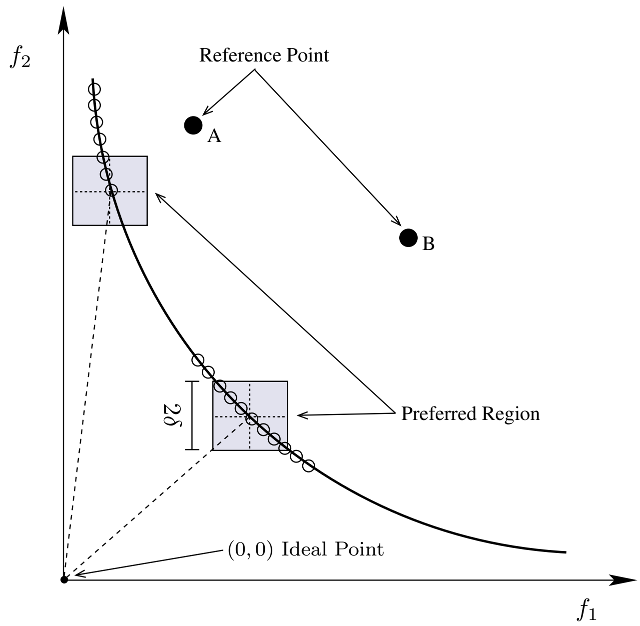



*Figure 2.5: An example to depict the deficiency of the metric proposed in [Wickramasinghe et al., 2010].*

front. However, the major drawback of this metric is its dependency on the location of the ideal point. This may cause misleading results when the reference point is biased towards one objective more than others. This misleading result is shown in Figure 2.5. As it be seen from the figure solutions, which are generated for the reference point A, converge on the Pareto-optimal front with a minimum distance to the reference point. In other words, they are fairly good solutions and the algorithm performs reasonably well. However, the proposed metric does not indicate this results due to the a bad choice of the ideal point which causes many high quality solutions to fall outside the preferred region. This shows that the results of this metric can be sensitive depending on the location of the reference point.

In chapter 7 of this thesis, we have developed a metric to measure the performance of user-preference based approaches without being dependent in any ideal/nadir point. The proposed metric is independent of the knowledge of the Pareto-optimal front, measure both convergence and diversity of solutions in the objective space and scale well as the number of objectives increases.

## 2.7 Summary

In this chapter, multi-objective optimization, dominance relation and many-objective optimization have been described. Since adopting EAs is one of the promising approaches to solve multi-objective optimization problems, basic functionality of EAs and the process of using them have been presented. Two main types of EA approaches to solve EMOs are described which are dominance-based and decomposition-based. It has been observed that dominance-based approaches are not very effective to solve many-objective optimization problems and decomposition-based EMO approaches are more promising in this regard. It has been also seen in the literature that user-preference approaches can solve many-objective optimization problems more effectively than non-user preference based approaches. As a result, there are great advantages for combining these two approaches. In the next chapter, we have developed a method that combines decomposition and user-preference methods. To discover whether this combination is effective in solving many-objective optimization problems, in chapter 4 a user-preference based decomposition approach has been proposed and its performance has been evaluated. Some of the existing works which attempted to maintain the uniformity of solutions in the objective space by adjusting the weight vectors during the course of optimization are reviewed. It has been found that there is a lack of a systematic way to adapt weight vectors which leads us to propose a new strategy for adapting weight vectors in chapter 6. Definitions of some widely used multi-objective metrics are presented. However, there is a lack of an effective metric to evaluate the performance of user-preference based approaches. Therefore, in chapter 7 a new metric for this purpose has been proposed.

# Chapter 3

# Applying User Preferences on Decomposition Methods

## 3.1 Introduction

In chapter 1 we raised the idea of combining decomposition methods and user-preference approaches. Decomposition methods alleviate selection pressure problems of dominance-based evolutionary algorithms. User-preference approaches reduce the computational cost by focusing the search on the preferred region of Pareto-optimal front. In this chapter, we propose a multi-objective optimization algorithm that integrates the user-preference with decomposition based EMO algorithms in order to provide a mechanism for tackling many-objective problems more effectively. In particular, the proposed algorithm has four major advantages: 1) Less susceptible to the selection pressure problem resulting from the use of dominance comparisons, by using a decomposition based method; 2) More computationally efficient by searching the regions which are preferred by the decision maker; 3) Faster convergence to the Pareto-optimal front; 4) Better scalability to higher objective spaces.

This chapter is organized as follows. The proposed algorithm is described in Section 3.2. The experimental results are presented and analyzed in Section 3.3 and finally, Section 3.5 concludes the chapter.

In this section we describe the details of a reference point based evolutionary multi-objective optimization problem through decomposition, which we refer to as the R-MEAD algorithm. As mentioned in Section 3.1, we seek to design an algorithm that scales better when the number of objectives increases. Therefore, instead of tackling a multi-objective

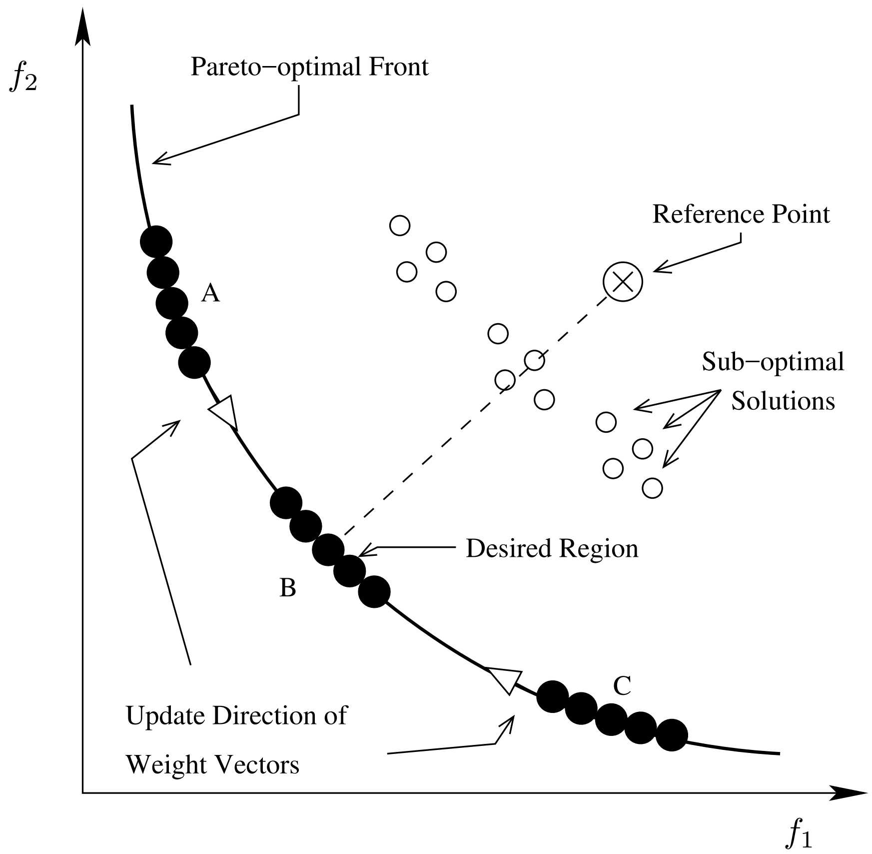



*Figure 3.1: This figure shows the effect of updating the weight vectors in moving the solutions closer to the desired region.*

problem directly, we first decompose it into a set of single-objective problems using a decomposition strategy such as weighted-sum or Tchebycheff. However, unlike decomposition methods such as MOGLS [Ishibuchi and Murata, 1998] and MOEA/D [Zhang and Li, 2007] where a set of weight vectors is generated to cover the *entire* Pareto-optimal front, we generate a smaller set of weight vectors just to find a number of solutions close to the user-supplied reference. Theoretically, if we manage to find a suitable weight vector, it is possible to find a solution close to the reference point. However, finding an appropriate weight vector that resulting in a solution close to the reference point is not trivial. To solve this problem, we initially select a sub-optimum weight vector and adapt it in the course of evolution until it results in a solution as close as possible to the user-provided reference point as well as to the Pareto-optimal front.

Figure 3.1 shows how updating the weight vector might result in solutions closer to the user's desired region. Region B in Figure 3.1 shows the desired region close to the reference point that we wish to find. Regions A and C show several solutions which are obtained on the Pareto-optimal front far from B (the desired region). The white solid arrows show the effect of updating the weight vectors. Note that the optimization of the population and the weight vectors happen simultaneously and as a result, the sub-optimal solutions which are shown in a white circle in Figure 3.1 might move directly towards the desired region on the Pareto-optimal front. However, if the individuals converge to any other location on the Pareto-optimal curve, the adaptation of the weight vector will move them closer to the desired region.

## 3.2 R-MEAD

Algorithm 3.1 shows the details of this process. The major stages of the algorithm are outlined as follow:

### Step 1 - Initialization

An initial population is randomly initialized within the lower and upper boundaries and evaluated for a limited number of iterations of an evolutionary algorithm with a predetermined decomposition method. Once the population is evolved, all of the objectives are evaluated and the results are stored in $\mathcal{PF}$ matrix (Algorithm 3.1 lines 1-4). It should be noted that each individual in the population is associated with only one weight vector which is later used to decompose the problem. The size of this initial population is determined by $size1$.

### Step 2 - Finding the base weight vectors

At this stage, for each of the reference points provided by the user, the closest point in the objective space ($\mathcal{PF}$) is found by using Euclidean distance. The corresponding weight vector of the closest point to each reference point is chosen as the base weight vector around which a set of new weight vectors is generated. The number and the spread of weight vectors are determined by the $size2$ and $radius$ variables respectively (Algorithm 3.1 lines 6-10). Since the number of individuals and the weight vectors are equal, the variable $size2$ is practically the population size.

**Algorithm 3.1:** R-MEAD

**Inputs:**

$size1$ : the initial population size<br>
$size2$ : number of solutions to be generated close to the reference point<br>
$radius$: determines the size of the solution region close to the reference point<br>
$m$ : number of objectives<br>
$n$ : number of dimensions<br>
$nref$ : number of provided reference points<br>
$\mathcal{RF}$ : A $nref \times m$ dimensional matrix of reference points with each row representing a single reference point.<br>
$dm$ : the decomposition method (weighted-sum or Tchebycheff)<br>
$F(\mathbf{x})$ : the objective function

**Variables:**

$\mathcal{IW}$ : A $size1 \times m$ dimensional matrix of the initial weight vectors<br>
$\mathcal{W}^i$ : A $size2 \times m$ dimensional matrix of weight vectors for the $i^{th}$ reference point.<br>
$\mathcal{BW}$ : A $nref \times m$ dimensional matrix of best weight vectors for reference points

1. $\mathcal{P} \leftarrow$ `rand`$(lbounds,\ ubounds,\ size1,\ n)$
2. $\mathcal{IW} \leftarrow$ `init_weight`$(size1)$
3. `evolve`$(\mathcal{P},\ F(\mathbf{x}),\ \mathcal{IW},\ dm)$
4. $\mathcal{PF} \leftarrow$ `evaluate`$(\mathcal{P},\ F(\mathbf{x}))$
5. $step \leftarrow radius/size2$
6. **for** $i \leftarrow 1$ **to** $nref$ **do**
7.     $ind \leftarrow$ `min_ind(euclid_dist`$(\mathcal{PF},\ \mathcal{RF}_{[i,:]})$`)`
8.     $\mathcal{BW}_{[i,:]} \leftarrow \mathcal{IW}_{[ind,:]}$
9.     $\mathcal{W}^i \leftarrow$ `init_weight`$(size2,\ raduis,\ \mathcal{BW}_{[i,:]})$
10. **end for**
11. $\mathcal{P} \leftarrow$ `rand`$(lbounds,\ ubounds,\ size2 \times nref\ ,\ n)$
12. **while** stop criteria not met **do**
13.     `evolve`$(\mathcal{P},\ F(\mathbf{x}))$
14.     $\mathcal{PF} \leftarrow$ `evaluate`$(\mathcal{P},\ \mathcal{W},\ dm)$
15.     **for** $j \leftarrow 1$ **to** $nref$ **do**
16.         **dist** $\leftarrow$ `euclid_dist`$(\mathcal{PF},\ \mathcal{RF}_{[i,:]})$
17.         $best\_index \leftarrow$ `min_ind(`**dist**`)`
18.         $worst\_index \leftarrow$ `max_ind(`**dist**`)`
19.         **direction** $\leftarrow \mathcal{W}^j_{[best\_index,:]} - \mathcal{W}^j_{[worst\_index,:]}$
20.         `update_weight`$(\mathcal{W}^j$, **direction**, $step)$
21.     **end for**
22. **end while**

### Step 3 - Evolving the population

Here the population is evolved and the weight vectors associated with each reference point are updated in a round-robin fashion until a termination criteria is met (Algorithm 3.1 lines 12-22).

### Step 3.1 - Updating weights

At this stage, the Euclidean distance of each reference point to its corresponding solutions in the population is calculated. In order to find the direction in which the weights should be updated, the best and the worst weights that correspond to the closest and the farthest points to the reference points are found. Next, an updated direction is calculated based on the best and the worst weight vectors (Algorithm 3.1 line 19). Once the updated direction is determined the weights are updated by moving them in the gradient direction with the amount determined by the *step* variable (Algorithm 3.1 line 20). This process is repeated for all weight vectors associated with each reference point.

It should be noted that unlike MOEA/D, our algorithm does not form a neighborhood for each individual. Instead, the evolutionary optimizer relies on the entire population to create new offspring. This eliminates the need to calculate the Euclidean distances between all weight vectors to form the neighborhoods. Because we only need to find a limited set of solutions near the user-supplied reference point rather than forming the entire frontier, the population size that we use is smaller than MOEA/D, which makes R-MEAD more computationally effective.

## 3.3 Experimental Settings

In this section we report the performance of our algorithm on a set of benchmark problems including ZDT1-ZDT4 and ZDT6 from the ZDT test suite [Zitzler et al., 2000] for two-objective problems and DTLZ1-DTLZ2 functions from DTLZ test suite [Deb et al., 2001] for three-objective problems using the Tchebycheff and the weighted-sum approaches.

### 3.3.1 Parameter Settings

The initial population size ($size1$) is set to 100 and 250 for the two-objective and three-objective problems respectively and the algorithm is executed for 10 iterations in order to generate the base weight vectors (See Algorithm 3.1). In the second stage of the optimization, the population size ($size2$) is set to 30 and 60 for two and three-objective problems respectively. We consistently used 450 iterations for the two-objective problems and 600 for the three-objective problems. The radius is set to 0.02 for the weighted-sum and 0.05 for the Tchebycheff approaches. The performance of the algorithm is not sensitive to the radius value and its sole purpose is to allow the users to adjust the size of the regions they wish to cover near the reference points. Since the way weight vectors are used in the Tchebycheff and the weighted-sum approaches is different, we chose the radius value such that both approaches cover a region with relatively equal sizes. The optimizer we used in our framework is a simple implementation of Differential Evolution [Storn and Price, 1995].

## 3.4 Analysis of Results

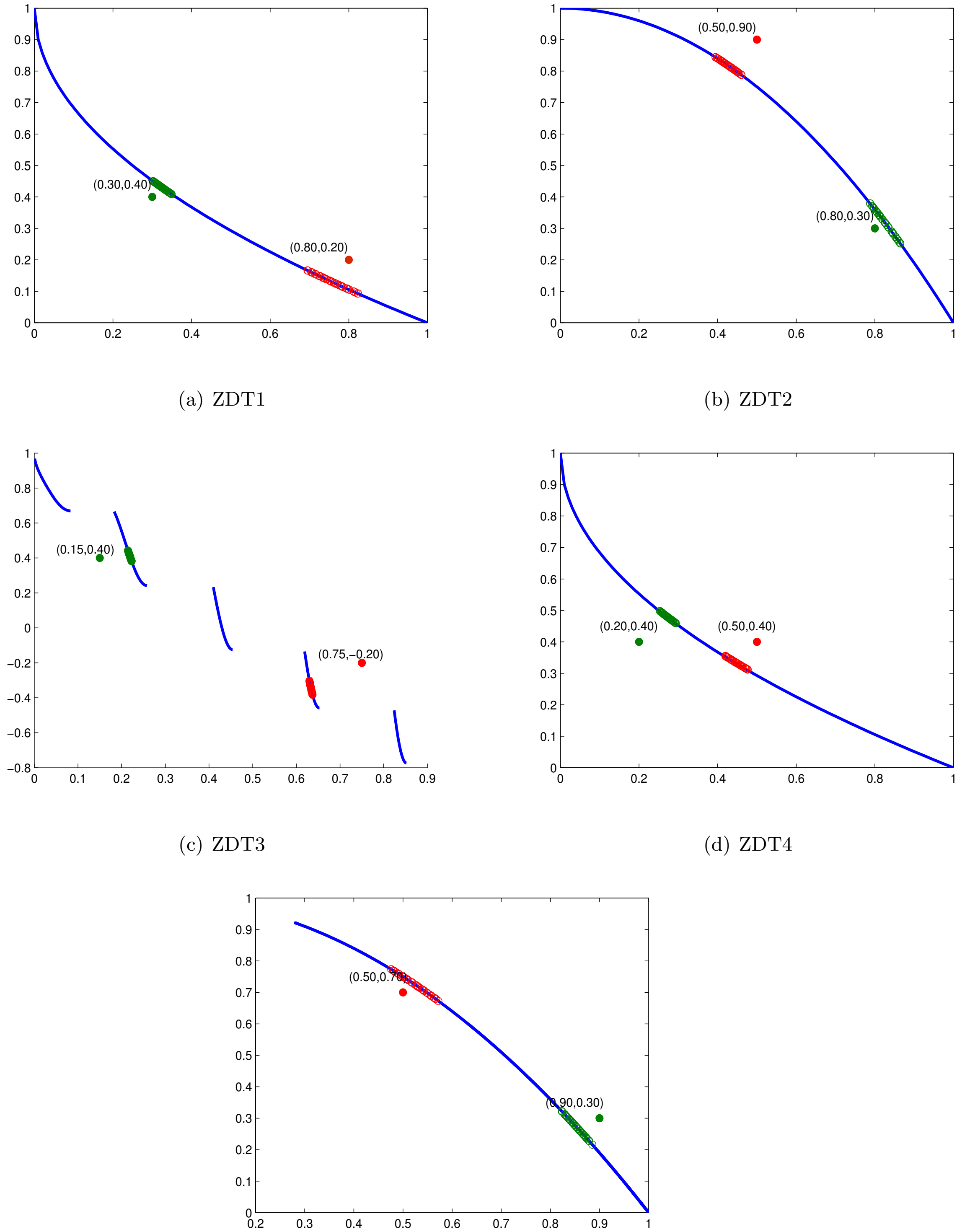
### 3.4.1 Benchmark Results

For all the benchmark problems we used two reference points in the infeasible and the feasible regions where one is closer to the Pareto-optimal front and the other one is farther away from it. On ZDT1, the reference points $(0.3, 0.4)$ and $(0.8, 0.2)$ were used. Figure 3.2(a) shows the promising result of our algorithm on this function. On ZDT2 the reference points $(0.8, 0.3)$ and $(0.5, 0.9)$ were used. Figure 3.2(b) shows that our algorithm managed to find a reasonable region on the Pareto-optimal front and close to the reference points. Our reference points for ZDT3 were $(0.15, 0.40)$ and $(0.75, -0.20)$. Figure 3.2(c) illustrates our results on this function. On ZDT4 $(0.2, 0.4)$ and $(0.5, 0.4)$ were our reference points. Figure 3.2(d) shows the results on ZDT4 where a set of solutions were found on the Pareto-optimal front close to the reference points. We chose $(0.9, 0.3)$ and $(0.5, 0.7)$ as reference points for ZDT6. Figure 3.2(e) shows that we get excellent results for this function.

On DTLZ1 $(0.2, 0.4, 0.9)$ and $(0.2, 0.2, 0.5)$ are our reference points. We have chosen $(0.2, 0.5, 0.6)$ and $(0.7, 0.8, 0.5)$ as reference points for DTLZ2. These points were chosen in feasible and infeasible regions with various distances from the Pareto-optimal front. It can be seen from Figures 3.4(a) and 3.4(b) that the R-MEAD algorithm also managed to find suitable regions for the three-objective problems. Overall, it can be seen from Figures 3.2 and 3.4 that the R-MEAD algorithm managed to find the desired regions as close as possible to the reference points on the Pareto-optimal front. The algorithm works consistently for the reference points in both feasible and infeasible regions.

### 3.4.2 Comparison between Weighted-Sum and Tchebycheff

In this section, instead of Tchebycheff we used weighted-sum as the decomposition method and ran the algorithm with the same reference points. In all test functions we stopped the algorithm after 450 generations. The results of the algorithm on test functions illustrate the weighted-sum approach does not work properly for non-convex Pareto-optimal fronts. Figure 3.3 shows the performance of the algorithm using the weighted-sum decomposition method on two-objective problems. For ZDT2, ZDT3 and ZDT6 with non-convex Pareto-optimal fronts, the weighted-sum method could not find nicely distributed solutions close to the reference points on the Pareto-optimal front, as shown in Figures 3.3(b), 3.3(c) and 3.3(e). In contrast, the functions with a convex Pareto-optimal front such as ZDT1 and




(a) ZDT1

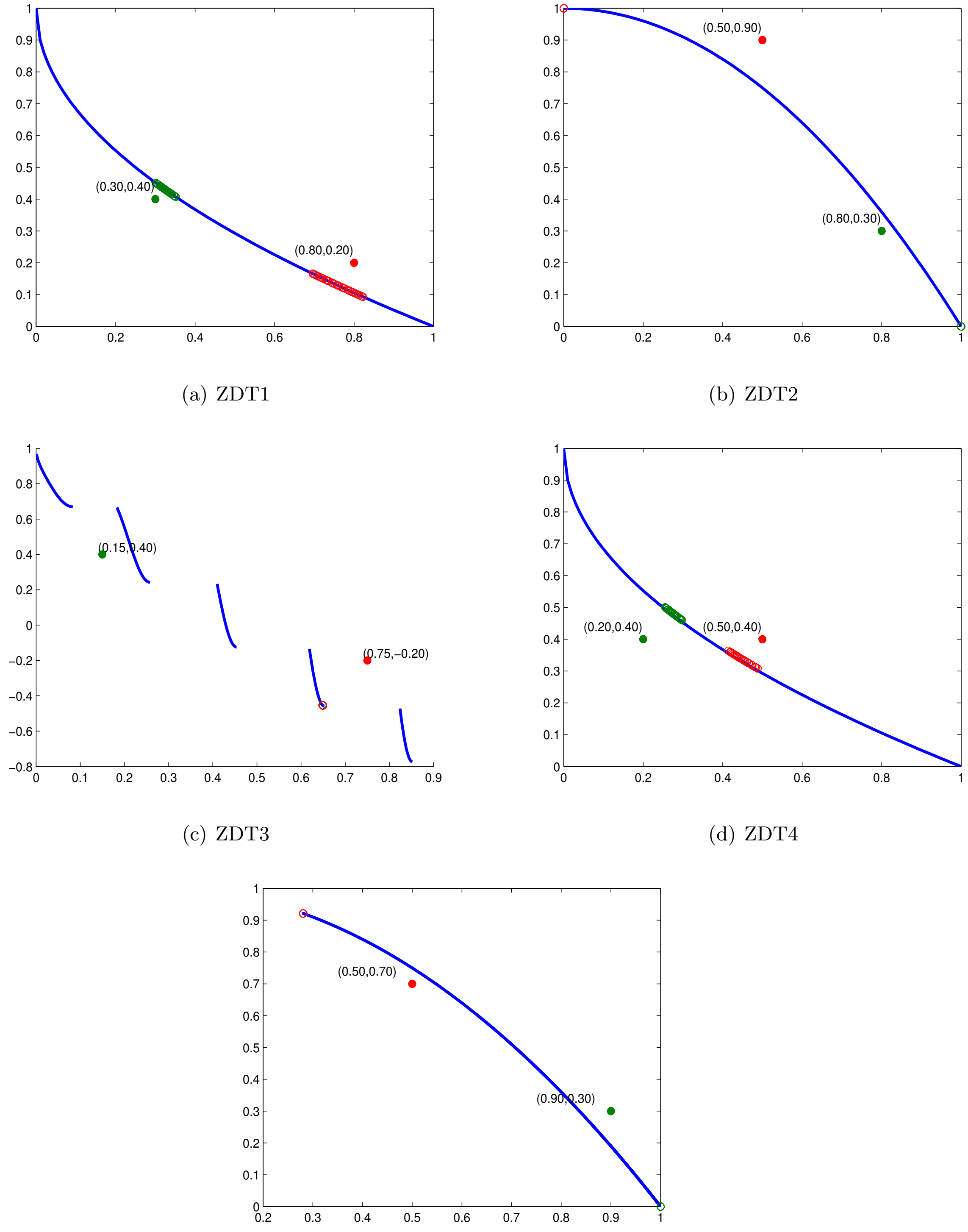
(b) ZDT2

(c) ZDT3

(d) ZDT4

(e) ZDT6

*Figure 3.2: Experimental results on ZDT1-ZDT4, ZDT6 benchmark functions using Tchebycheff decomposition.*




(a) ZDT1

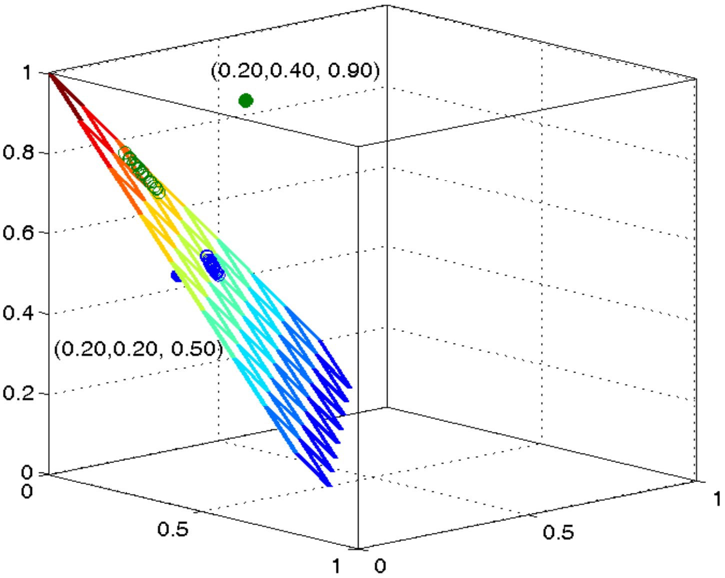
(b) ZDT2

(c) ZDT3

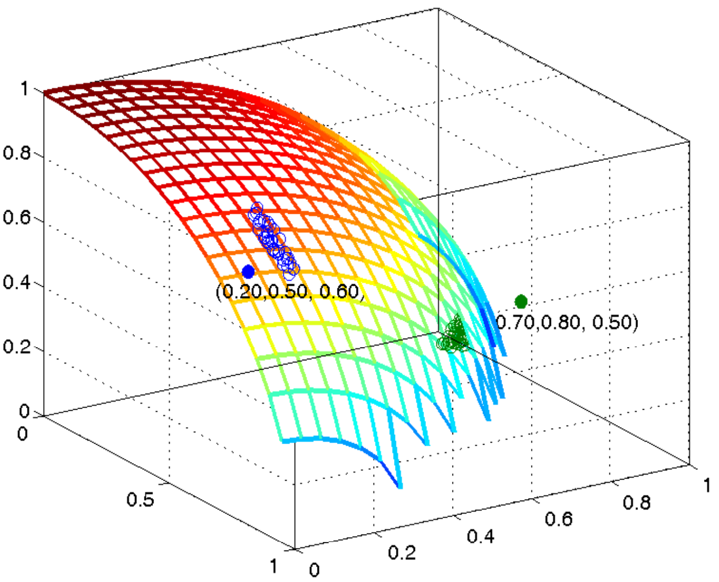
(d) ZDT4

(e) ZDT6

*Figure 3.3: Experimental results of R-MEAD on ZDT1-ZDT4 and ZDT6 benchmark functions using weighted-sum decomposition.*




(a) DTLZ1




(b) DTLZ2

*Figure 3.4: Experimental results of R-MEAD on DTLZ1 and DTLZ2 benchmark functions using Tchebycheff decomposition.*

ZDT4, gave similar results to those of the Tchebycheff method, as shown in Figures 3.3(a) and 3.3(d).

### 3.4.3 Faster Convergence to the Pareto-optimal front

In Section 3.1, we mentioned that one of the difficulties faced by EMO algorithms in solving many-objective problems is the exponential increase in the number of solutions required to approximate the Pareto-optimal front as the number of objectives increases. It has been suggested that a reference point based approach allows a more focused search and eliminates the need to approximate the entire Pareto-front [Wickramasinghe and Li, 2008]. This approach can result in significant computational savings, especially in high dimensional objective spaces. In this section, we attempt to show this potential saving more quantitatively through experimentation.

We used Generational Distance (GD) [Veldhuizen, 1999b] to measure the closeness of a solution front ($PF_{\text{sol}}$) to the Pareto-optimal front ($PF_{\text{true}}$). For GD formula please refer to equation 2.8 in chapter 2. We used a $p$ value of 2 in our experiments.

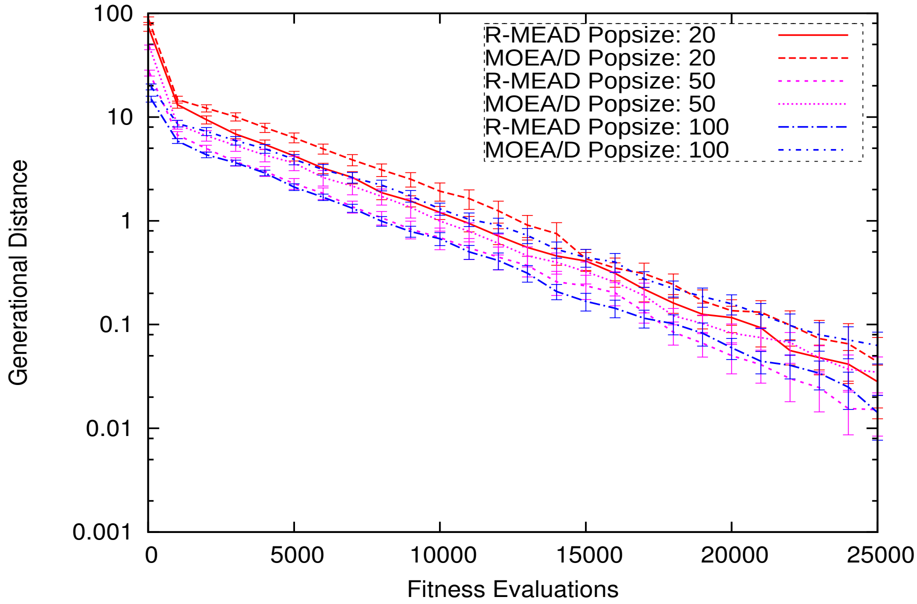
Since we use Tchebycheff decomposition method in both R-MEAD and MOEA/D, they result in a similar spread of solutions in the regions they cover. This allows us to exclude the spread and focus solely on the convergence in our comparison. Additionally, in order to have a fair comparison, we consistently used the same value for the variable $n$ in both R-MEAD and MOEA/D. Unlike dominance-based approaches where the first front is usually used to calculate the $GD$ value, we set $n$ to the population size in order to capture the convergence behavior of the entire population.

For our experiments we used a two-objective and a three-objective version of DTLZ1 function. In order to analyze the sensitivity of the algorithms to the population size, we used 20, 50 and 100 population sizes for two-objective. For three-objective we used 60 and 250 as the population sizes. The total number of fitness evaluations was set to $2.5 \times 10^4$ and $7.5 \times 10^4$ for the two-objective and the three-objective DTLZ1 respectively. Figure 3.5 shows how fast the GD value decreases through the course of evolution. Each point on the convergence plot is the average of 25 independent runs. It can be seen from Figure 3.5 that when an equal population size is used, R-MEAD consistently has a lower GD value than MOEA/D on both two-objective and three-objective DTLZ1. The fact that the error bars for the same population sizes do not overlap shows that R-MEAD algorithm converges significantly faster than MOEA/D. Figure 3.5(a), shows that the best results are achieved by R-MEAD with population sizes of 50 and 100 respectively.

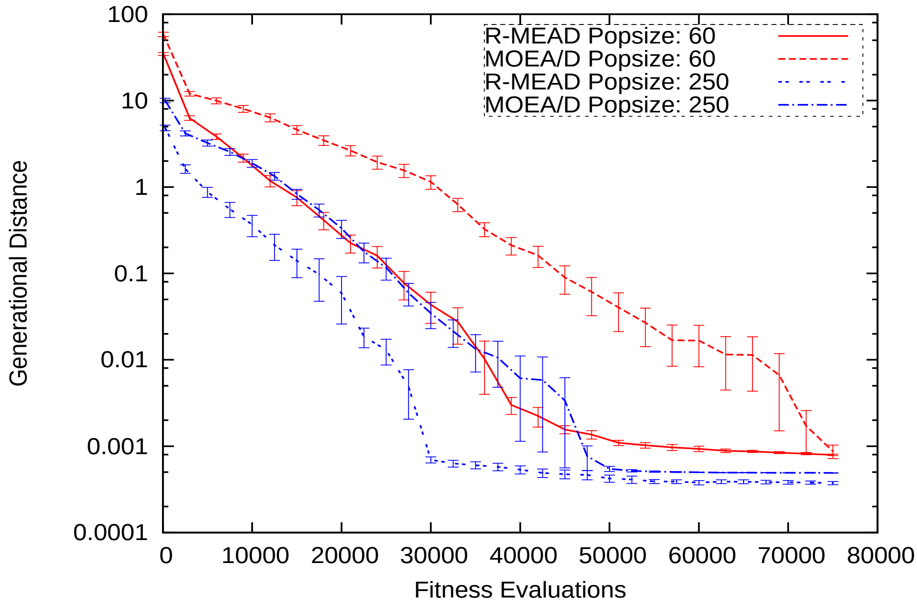
On the 3-objective DTLZ1, Figure 3.5(b) shows that there is a more significant difference between the convergence speed of MOEA/D and R-MEAD compared to that of the two-objective DTLZ1. R-MEAD managed to obtain a similar convergence accuracy compared to MOEA/D with significantly fewer fitness evaluations from around $2 \times 10^4$ to $3.5 \times 10^4$ depending on the population size. The increased 'gap' in the convergence speed from a two-objective problem to a 3-objective problem suggests that a considerable amount of computational resources can be saved using a reference point based approach, especially when dealing with many-objective problems. It should be noted that although in the case of R-MEAD we have not taken the closeness to the reference point into account, the experiments that we conducted in Section 3.4.1 have shown that a region close to the reference point can be formed using around $3.6 \times 10^4$ fitness evaluations which are far fewer than $7.5 \times 10^4$ that we used in this section. By looking at Figure 3.5(b) we can see that R-MEAD has a much lower GD value, around $3.6 \times 10^4$ which shows that not only that it converged faster than MOEA/D but also managed to find a suitable region close to the reference point.

### 3.5 Chapter Summary

In this chapter, we propose a user-preference based evolutionary algorithm that relies on decomposition strategies to convert a multi-objective problem into a set of single-objective problems. The use of a reference point allows the algorithm to focus the search on more preferred regions which can potentially save considerable computational resources. The algorithm that is proposed dynamically adapts the weight vectors and is able to converge close to the preferred regions. Combining decomposition strategies with reference point approaches paves the way for more effective optimization of many-objective problems. The use of a decomposition method alleviates the selection pressure problem associated with dominance-based approaches while a reference point allows a more focused search. The experimental results show that the proposed algorithm is capable of finding solutions close to the reference points specified by a decision maker. Moreover, our results show that high quality solutions can be obtained using less computational effort compared to a state-of-the-art decomposition based evolutionary multi-objective algorithm. In this chapter, we have presented a very simple way of updating the weight vectors. In the next chapter, we investigate the effectiveness of adapting the weight vectors when dealing with many-objective problems with complex non-convex Pareto-optimal fronts.




(a) Two-objective DTLZ1




(b) Three-objective DTLZ1

*Figure 3.5: Convergence of Generational Distance measure for R-MEAD and MOEA/D on two-objective and three-objective DTLZ1.*

# Chapter 4

# A User-preference Based Method for Solving Many Objective Problems

## 4.1 Introduction

In chapter 3, we showed that combining user-preference and decomposition methods is a promising strategy to tackle the selection pressure issue of dominance approaches, especially in the case of many-objective optimization problems. However, the proposed approach (R-MEAD) only applies to two and three-objectives, and not more than that. In this chapter, we want to extend R-MEAD to allow it to work for many-objective problems. The main difficulty to achieve this lies in initializing a set of weight vectors. As weight vector initialization in R-MEAD inherited the simplex-lattice design method from MOEA/D, the number of sample points is governed by the dimensionality of the problem. In this chapter, we aim to develop an algorithm called R-MEAD2 which resolves this issue of R-MEAD. In short, R-MEAD2 extends R-MEAD in the following aspects: 1) A new method for initializing the weight vector is used which makes R-MEAD2 scalable and applicable to many-objective optimization problems. This method decouples the population size from the number of objectives. 2) A uniform random number generator (ie., RNG) is used to simplify the update mechanism of weight vector and make it more efficient. 3) The performance of R-MEAD2 is compared with the-state-of-the-art R-NSGA-II [Deb et al., 2006]. Recently, an algorithm called UMOEA/D [Tan et al., 2013] has been proposed which detaches the population size from the number of objectives. It replaces the simplex-lattice design with the good lattice point (GLP) [Fang and Wang, 1993]. In Section 4.3.1, we will show that when the number of objectives exceeds eight, a simple uniform random number generator(RNG), which is a commonly used technique in various programming languages, produces more uniformly distributed solutions than GLP according to a measure called centered $L_2$-discrepancy [Fang and Lin, 2003]. R-MEAD2 replaces the simplex-lattice design with a uniform random number generator (RNG). Another improvement of R-MEAD2 over R-MEAD relates to updating weight vectors. In R-MEAD2, a simple hill climber is used to generate mutants based on a uniform distribution to maintain the uniformity of the weight vectors. In each iteration, a set of new weight vectors are generated based on a uniform distribution within a certain region around the best weight vector that is found so far and the process is repeated until the weights converge to a solution. Finally, R-MEAD2 with two well-known decomposition methods, namely PBI and Tchebycheff are compared with R-NSGA-II [Deb et al., 2006] to show the advantage of decomposition-based approaches over dominance based approaches.

The remainder of this chapter is organized as follows. The proposed algorithm is described in section 4.2. Section 4.3 contains the experimental results and comparison between R-NSGA-II and R-MEAD2. The chapter is concluded in Section 4.4.

## 4.2 Proposed approach (R-MEAD2)

This section describes the R-MEAD2 algorithm which is a reference point based evolutionary algorithm that uses decomposition methods for solving many-objective optimization problems.

The population size of algorithms such as MOEA/D and R-MEAD that rely on a simplex-lattice design grows dramatically as the number of objectives increases. More precisely, the weight values $w_1, w_2, \ldots, w_m$ are chosen from the set $\{\frac{0}{H}, \frac{1}{H}, \ldots, \frac{H}{H}\}$ where $H$ is a parameter chosen by the user. Therefore, the number of these vectors and consequently the population size is calculated by the following formula: $\binom{H+m-1}{m-1}$. Table 4.1 shows how the population size grows with number of objectives when the simplex-lattice design is used to generate the

*Table 4.1: R-MEAD pop-size for different objectives ($H = 10$).*

| # Objs ($m$) | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| Pop-size | 286 | 1001 | 3003 | 8008 | 19448 | 43758 | 92378 |

weight vectors. As can be seen, most of these population sizes are not commonly used in evolutionary algorithms.

To remove the coupling between the population size and the number of objectives, UMOEA/D [Tan et al., 2013] replaced the simplex-lattice design with the good lattice point (GLP) in MOEA/D [Zhang and Li, 2007] for generating the weight vectors. However, as shown in Section 4.3.1, GLP does not necessarily have a better uniformity as compared to a uniform random number generator (RNG) when the number of objectives exceeds eight. Additionally, using RNG for generating the weight vectors also removes the coupling between the population size and the number of objectives. This allows the user to pick any suitable population size irrespective of the number of objectives.

### 4.2.1 The R-MEAD2 Algorithm

In the proposed decomposition-based user-preference approach in order to guide the solutions towards the desired region, the weight vectors should be dynamically updated so that the solutions can converge in the direction of the reference point, and ideally on the Pareto-optimal front. To achieve this effect each solution is assigned to a weight vector which is updated in the course of optimization.

Figure 4.1 shows how the weights are updated during the optimization. On the left the black squares denote the solutions at some iteration $t$. The gray squares represent the solutions after running an iteration of the evolutionary optimizer. The arrows show the updated direction which is determined by the weight vectors associated with each individual. At this stage the weight vector associated with the closest solution to the reference point is marked as the best weight vector ($\mathbf{w}_b$). Once the **best weight vector** is identified, a set of new weight vectors is generated using RNG within a region centered around the best weight vector. This forms a hypercube in an $m$ dimensional space and the size of the region is determined by parameter $r$ which is the edge size of hypercube. The weight vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are generated with a uniform distribution around $\mathbf{w}_b$ as shown in the box at the center of Figure 4.1. Next, these weights are assigned back to the solutions that were obtained in the last iteration. Finally, the solutions are optimized with the newly assigned weights. This process is shown on the right-hand side of Figure 4.1. As can be seen, the new solutions marked by '$\star$' gradually converge within a confined region in the direction of the reference point.

Algorithm 4.1 shows the details of the proposed method. The main steps are outlined below:

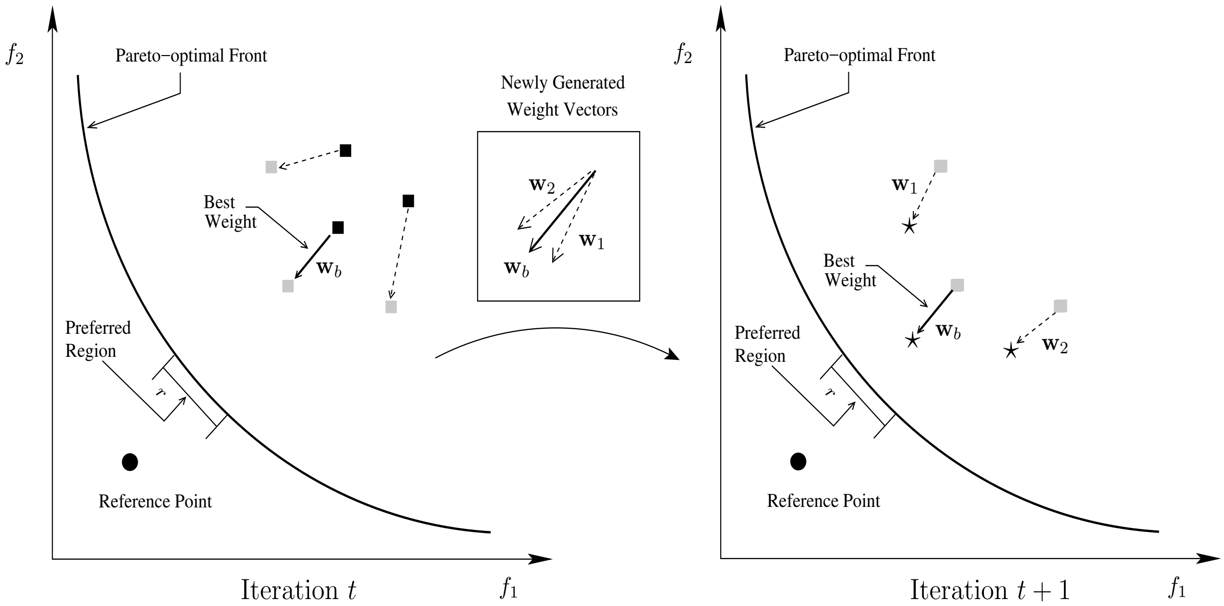



*Figure 4.1: Illustration how weight vectors are updated in RMEAD-2*

### Step 1 - Initialization

The initial population is created, and the weight vectors are initialized using `rng` and `init_weights` functions respectively (lines 1-2). The initial weight vectors are generated using RNG over the entire space of weight vectors in order to increase the probability of finding a good initial weight vector. Also the weights are normalized so that the components of a weight vector add up to one.

### Step 2 - Main Evolutionary Cycle

The population is evolved and the weight vectors are updated until a stopping criterion is met (lines 3-10).

### Step 2.1 - Evolving the Population

The population is evolved to find better solutions. The `evolve` function first uses a decomposition method as specified by the $dm$ variable to convert the multi-objective problem $\mathbf{F}(\mathbf{x})$ to a set of single-objective problems similar to MOEA/D [Zhang and Li, 2007]. Then it applies several genetic operations in order to evolve the population (lines 4). Finally, the new

---

**Algorithm 4.1:** R-MEAD2

---

1 **Inputs:**

*popsize*: the initial population size.
$r$ : determines the size of the preferred region.
$m$ : number of objectives.
$n$ : number of decision variables.
$dm$ : the decomposition method (Tchebycheff or PBI).
$\mathbf{R}$ : the reference point.
$\mathbf{F}(\mathbf{x})$ : the objective function.

2 ———

3 **Variables:**

$\mathbf{P}$ : population of individuals ($popsize \times n$).
$\mathbf{W}$ : matrix of weight vectors ($popsize \times m$).
$\mathbf{PF}$ : solution set ($popsize \times m$).

4 ———

1. $\mathbf{P} \leftarrow$ `rng(`*lbounds, ubounds, popsize , n*`)`
2. $\mathbf{W} \leftarrow$ `init_weights(`*popsize, m*`)`
3. **while** stop criteria not met **do**
4. $\quad \mathbf{P} \leftarrow$ `evolve(`$\mathbf{P}$, $\mathbf{F}(\mathbf{x})$, $\mathbf{W}$, $dm$`)`
5. $\quad \mathbf{PF} \leftarrow$ `evaluate(`$\mathbf{P}$, $\mathbf{F}(\mathbf{x})$`)`
6. $\quad \mathbf{d} \leftarrow$ `euclid_dist(`$\mathbf{PF}$, $\mathbf{R}$`)`
7. $\quad best \leftarrow$ `min_ind(`$\mathbf{d}$`)`
8. $\quad \mathbf{w}_b \leftarrow \mathbf{W}[best, :]$
9. $\quad \mathbf{W} \leftarrow$ `uniform(`*popsize, r,* $\mathbf{w}_b$`)`
10. **end while**

---

population is evaluated using the original objective function $\mathbf{F}(\mathbf{x})$ to find a set of solutions ($\mathbf{PF}$) in the objective space (line 5).

### Step 2.2 - Updating weights

In order to update the weight vectors, the best weight vector which is associated to the solution with the shortest Euclidean distance to the reference point, has to be found. The `euclid_dist` function calculates the distance of all solutions to the reference point $\mathbf{R}$. Then the `min_ind` function finds the index of the closest solution to the reference point which matches the index of the best weight vector. The closest solution helps the solutions to converge in the direction of reference points. Finally, the `uniform` function uses RNG to generate a set of uniform weight vectors around the best weight vector $\mathbf{w}_b$ (lines 7-9). The process repeats from Step 2.1.

*Table 4.2: The average $CD_2$ value of 25 independent runs for two initialization methods: A uniform random number generator(RNG), and Good lattice point (GLP) for different objectives. Those leading to statistically significantly lower $CD_2$ values (at a significant level of 5% according to Mann-Whitney-Wilcoxon (MWW) test) over others are highlighted in bold*

| # Objectives | Sample Sizes | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 50 | | 100 | | 250 | | 500 | | 1000 | | 5000 | |
| | RNG | GLP | RNG | GLP | RNG | GLP | RNG | GLP | RNG | GLP | RNG | GLP |
| 4 | 2.74 | **2.71** | 2.73 | 2.73 | 2.75 | 2.74 | 2.75 | 2.75 | 2.75 | 2.75 | 2.75 | 2.75 |
| 5 | 2.99 | **2.94** | 2.97 | **2.96** | 2.98 | **2.97** | 2.98 | 2.98 | 2.98 | 2.98 | 2.98 | 2.98 |
| 6 | 3.21 | 3.19 | 3.23 | 3.22 | 3.24 | **3.22** | 3.23 | 3.23 | **3.22** | 3.23 | 3.23 | **3.22** |
| 7 | 3.48 | 3.47 | 3.51 | 3.50 | 3.50 | **3.49** | 3.49 | 3.50 | 3.50 | 3.50 | 3.50 | 3.50 |
| 8 | 3.82 | **3.76** | **3.78** | 3.81 | **3.78** | 3.81 | **3.78** | 3.81 | 3.80 | **3.79** | 3.79 | 3.79 |
| 9 | 4.12 | 4.14 | **4.12** | 4.15 | **4.11** | 4.14 | **4.11** | 4.14 | **4.10** | 4.14 | 4.11 | 4.11 |
| 10 | 4.48 | 4.48 | **4.47** | 4.54 | **4.46** | 4.50 | **4.46** | 4.50 | **4.45** | 4.50 | 4.45 | 4.45 |
| 11 | **4.91** | 4.92 | **4.86** | 4.94 | **4.83** | 4.89 | **4.82** | 4.90 | **4.83** | 4.89 | **4.82** | 4.87 |
| 12 | **5.33** | 5.43 | **5.27** | 5.41 | **5.24** | 5.32 | **5.23** | 5.32 | **5.24** | 5.31 | **5.22** | 5.30 |
| 13 | **5.83** | 6.01 | **5.73** | 5.94 | **5.68** | 5.69 | **5.69** | 5.79 | **5.67** | 5.78 | **5.66** | 5.75 |
| 14 | **6.46** | 6.70 | **6.30** | 6.58 | **6.16** | 6.17 | **3.16** | 6.23 | **6.15** | 6.29 | **6.14** | 6.23 |
| 15 | **6.97** | 7.50 | **6.80** | 7.33 | **6.73** | 6.86 | **6.69** | 6.71 | **6.66** | 6.76 | **6.64** | 6.76 |

### 4.3 Experimental Results and Analysis

This section includes three main experimental settings. Firstly, the uniformity of GLP and RNG is compared to measure which method is more suitable for many-objective problems. Secondly, the convergence behavior of weight vectors on different reference points are analyzed. Finally, based on the parameter setting in section 4.3.2 the performance of R-MEAD2 using Tchebycheff and PBI on DTLZ1-DTLZ6 benchmark problems is compared with R-NSGA-II. All algorithms are tested on problems with 4 to 10 objectives.

#### 4.3.1 RNG vs GLP

In order to compare the performance RNG and GLP, we use a discrepancy measure called centered $L_2$-discrepancy ($CD_2$) [Fang and Lin, 2003]. Since we are interested in the uniformity of sample points, a lower value of $CD_2$ implies better uniformity. Let $\mathbf{P}$ be a $n \times m$ matrix of sample points where the sample point $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})$ is the $i$th row vector in matrix $\mathbf{P}$, and $m$ is the dimensionality of the sample points that, in this context, matches the number of objectives. Centered $L_2$-discrepancy [Fang and Lin, 2003] is defined as follows:

$$CD_2(\mathbf{P}) = \left(\frac{13}{12}\right)^m -$$
$$\frac{2}{n}\sum_{k=1}^{n}\prod_{j=1}^{m}\left(1+\frac{1}{2}|x_{kj}-\frac{1}{2}|-\frac{1}{2}|x_{kj}-\frac{1}{2}|^2\right)+\frac{1}{n^2}$$
$$\sum_{k=1}^{n}\sum_{j=1}^{n}\prod_{i=1}^{m}\left[1+\frac{1}{2}|x_{ki}-\frac{1}{2}|+\frac{1}{2}|x_{ji}-\frac{1}{2}|+\frac{1}{2}|x_{ki}-x_{ji}|\right],$$

Table 4.2 contains $CD_2$ values for RNG and GLP using different sample sizes and dimensions (number of objectives $m$). As can be seen, for all population sizes (50-5000) when the number of objectives grows beyond eight, RNG is consistently better than GLP. For typical population sizes such as 100, 250, and 500 which are commonly used for solving many-objective problems, RNG is better when the number of objectives is more than seven. Table 4.2 shows that in the case of fewer objectives (less than eight) when the population size is relatively small, then GLP can be beneficial. GLP can be useful when solving problems with expensive objective function evaluation where a small population size is more practical. It is easy to see that when a small number of sample points are allowed, then a more systematic approach such as GLP might be advantageous. However, when the number of sample points grow, GLP and RNG may perform similarly. This behavior can be observed in table 4.2 when the sample size is 1000 and 5000. It is clear that in most cases, both RNG and GLP have similar $CD_2$ values.

### 4.3.2 Parameter Settings and Performance Metrics

We used the population size of 200 for 4-7 objective problems, 300 for 8-objectives and 350 for 9 and 10-objective problems. A single reference point is used for all test problems ($f_i = 0.25, i \in \{1, \ldots, m\}$). In R-MEAD2-Te and R-MEAD2-PBI the parameter $r$ which determines the size of the preferred region is set to 2, and in R-NSGA-II the parameter $\epsilon$ is set to 0.002. It should be noted that $r$ parameter can be varied based on the preference of the user.

To compare the performances of R-MEAD2-PBI, R-MEAD2-Te and R-NSGA-II we have adopted inverted generational distance (IGD) [Zitzler et al., 2003] that measures both convergence and diversity of solutions. The calculation of IGD is based on the average closest distances between sample points on the Pareto-optimal front and the obtained solutions. We used $10^m$ sample points for four, five, six and seven objective problems and $5^m$ for eight, nine and ten objective problems, where $m$ is the number of objectives. It should be noted that because of memory limitations, fewer samples points were generated to approximate the Pareto-optimal front for problems having more than seven objectives. For IGD formula, please refer to equation 2.9 in Chapter 2. Since all algorithms used in this study are user-preference based, we only consider the solutions that fall within a desired region. The desired region is a hypersphere with radius $\rho$ around the sample point on the Pareto-optimal front which is closest to the reference point. For all experiments in this study the parameter $\rho$ is set to 2.

### 4.3.3 Weight Vector Convergence

Figure 4.2 shows the convergence behavior of weight vectors for both PBI and Tchebycheff on DTLZ1 using two different reference points. As it can be seen the weight values fluctuate at the beginning of the search, but they gradually stabilize and converge to a fixed value towards the end of a run. Depending on the position of a reference point which is shown as $\mathbf{R}$ for the purpose of this experiment, weight vectors converge to different values. For example, when $\mathbf{R} = (0.25, 0.25)$ which is near the center of the Pareto-optimal front, both weight values converge to a value close to 0.5. However, when the reference point is biased towards a particular objective (e.g. $\mathbf{R} = (0.2, 0.5)$) the weight values differ as it can be seen in Figures 4.2(c) and 4.2(d).

### 4.3.4 Numerical Results

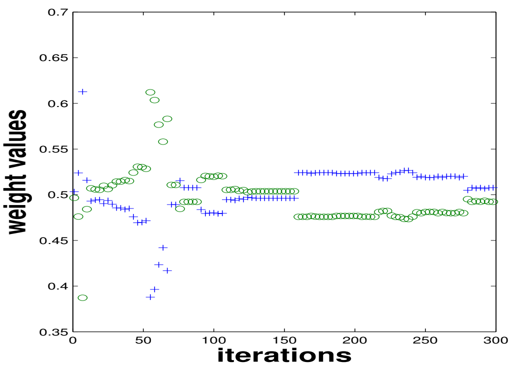
In this section we compare the performance of R-MEAD2 based on two decomposition methods Tchebycheff and PBI (abbreviated as R-MEAD2-Te and R-MEAD2-PBI respectively) with R-NSGA-II which is a dominance-based approach.

To test the significance of the obtained results we used the non-parametric Kruskal-Wallis one-way ANOVA [Sheskin, 2003] to detect if there is any significant difference between the performance of the three algorithms. The null and alternative hypotheses for Kruskal-Wallis are as follow:

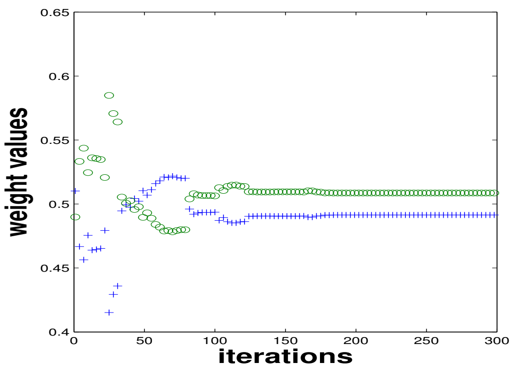
$$H_0 : \text{all samples come from the same distribution.}$$

$$H_{\text{a}} : \text{at least one sample comes from a different distribution.}$$

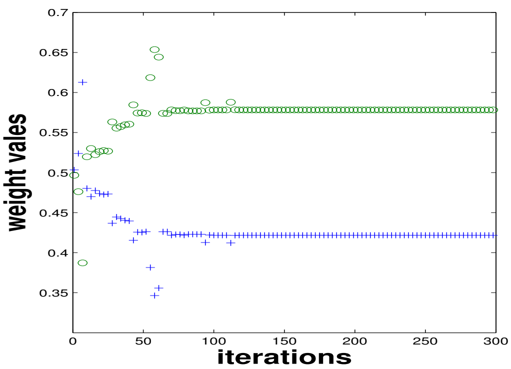
In order to rank the algorithms, we used Mann-Whitney-Wilcoxon (MWW) test with Bonferroni correction only when the null hypothesis of Kruskal-Wallis was rejected under 95%



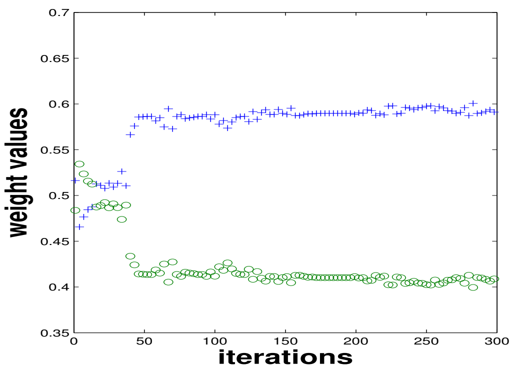

(a) DTLZ1 R-MEAD2-PBI $\mathbf{R} = (0.25, 0.25)$




(b) DTLZ1 R-MEAD2-Te $\mathbf{R} = (0.25, 0.25)$




(c) DTLZ1 R-MEAD2-PBI $\mathbf{R} = (0.2, 0.5)$




(d) DTLZ1 R-MEAD2-Te $\mathbf{R} = (0.2, 0.5)$

*Figure 4.2: Weight vector convergence behavior on 2-obj DTLZ1 using PBI and Tchebycheff. Objective-1 is shown by '+' and objective-2 by 'O'.*

confidence interval. Bonferroni correction is a simple technique for controlling the family-wise error rate [Sheskin, 2003]. Family-wise error rate is the accumulation of type I errors when more than one pair-wise comparison is used to rank a set of results. Under Bonferroni correction, in order to achieve an overall significance level of $\alpha$, the pair-wise tests should be performed with a significance level of $\alpha' = \frac{\alpha}{h}$, where $h$ is the number of pair-wise comparisons which is three in this study. For our experiments, the significance level of Kruskal-Wallis was set to 5%, and for pair-wise MWW tests a significance level of 1.67% was used, which results in an overall significant level of approximately 5%. Table 4.3 contains the median for 25 independent runs of the three algorithms. The last three columns show the $p$-value for three MWW pair-wise tests and the column labeled 'K-W' contains the $p$-value for the Kruskal-Wallis ANOVA test.

Table 4.3 shows IGD results for all three algorithms. It can be observed that the difference between algorithms is significant for all functions and over all numbers of objectives. So in all cases MWW test is used and the best performing algorithm is shown in bold. If the performance of two algorithms are statistically similar, both entries are shown in bold. The table shows that in general a decomposition approach is superior to the dominance-based approach, R-NSGA-II. For the sake of clarity we have summarized the comparison between R-NSGA-II and both versions of R-MEAD2 in table 4.4. From a total of 42 experiments, R-MEAD2-Te outperforms R-NSGA-II on 37 functions and ties on 2, and R-MEAD2-PBI outperforms R-NSGA-II on 35 functions and ties on 2. By looking back at table 4.3 we can see that R-NSGA-II outperforms R-MEAD-PBI on DTLZ1 when the number of objectives is less than 9. However, R-NSGA-II on DTLZ1 with 9 and 10 objectives, is outperformed by R-MEAD-PBI. Although R-NSGA-II in some objectives of DTLZ1 and DTLZ6 performs better than R-MEAD-PBI and R-MEAD-TE, in other test problems (DTLZ2-DTLZ5) R-NSGA-II is outperformed by both versions of R-MEAD. It should be noted that IGD takes both convergence and diversity of solutions into account. Therefore, we can conclude that decomposition methods generally perform better than R-NSGA-II in terms of both convergence and diversity on most many-objective problems. Finally, by comparing the results of R-MEAD2-PBI and R-MEAD2-Te we can see that both decomposition methods have very similar performance, but the PBI version performs slightly better than Tchebycheff on 24 functions but less well on 18 functions.

## 4.4 Chapter Summary

In this chapter, a user-preference based evolutionary multi-objective algorithm is proposed that uses decomposition methods for solving many-objective problems. Decomposition techniques that are widely used in multi-objective evolutionary optimization require a set of evenly distributed weight vectors to generate a diverse set of solutions on the Pareto-optimal front. The newly proposed algorithm, R-MEAD2, improves the scalability of its previous version, R-MEAD, which uses a simplex-lattice design method for generating weight vectors. That makes the population size dependent on the dimension size of the objective space. R-MEAD2 uses a uniform random number generator to remove the coupling between dimension and the population size. In this chapter, we show that a uniform random number generator is simple and able to generate evenly distributed points in a high dimensional space. Our comparative study shows that R-MEAD2 outperforms the dominance-based method

R-NSGA-II on many-objective problems. Since our experimental results suggest that both PBI and Tchebycheff preform similarly, in the next chapter we will investigate further the effect of penalty parameter ($\theta$) in PBI and for different types of problems which ($\theta$) is more suitable.

*Table 4.3: Columns 3-5 show IGD values on DTLZ1-DTLZ6 test problems. Columns 7-9 show the p-value for three MWW pair-wise tests.*

| Func. | # Obj | R-MEAD2-PBI | R-MEAD2-Te | R-NSGA-II | K-W | R-MEAD2-PBI vs R-NSGA-II | R-MEAD2-PBI vs R-NSGA-II | R-MEAD2-Te vs R-NSGA-II |
|---|---|---|---|---|---|---|---|---|
| | 4 | 2.7179e-02 | **2.0047e-02** | 2.6913e-02 | 2.37e-12 | 2.77e-12 | 4.43e-01 | 9.73e-11 |
| | 5 | 1.5969e-02 | **1.3061e-02** | **1.3489e-02** | 1.89e-12 | 2.77e-12 | 9.73e-11 | 1.99e-01 |
| | 6 | 1.9221e-03 | **1.4998e-03** | **1.5202e-03** | 8.51e-13 | 2.77e-12 | 9.73e-11 | 2.02e-02 |
| DTLZ1 | 7 | 5.7668e-03 | **4.9279e-03** | 5.1213e-03 | 4.82e-13 | 2.77e-12 | 9.73e-11 | 4.50e-03 |
| | 8 | 1.0997e-02 | **8.7635e-03** | 1.0280e-02 | 1.10e-13 | 2.77e-12 | 1.05e-04 | 9.73e-11 |
| | 9 | 8.2376e-03 | **6.0180e-03** | 8.7701e-03 | 4.43e-14 | 2.77e-12 | 1.10e-05 | 9.73e-11 |
| | 10 | 1.3500e-02 | **1.0895e-02** | 1.5834e-02 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 4 | **8.0403e-03** | 8.6697e-03 | 2.2126e-02 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 5 | **4.4710e-03** | 4.9080e-03 | 9.1054e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 6 | **4.8012e-04** | 5.0287e-04 | 7.2498e-04 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| DTLZ2 | 7 | **1.1177e-03** | 1.4606e-03 | 1.6713e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 8 | **1.7828e-03** | 2.1404e-03 | 2.7116e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 9 | **9.8180e-04** | 1.0688e-03 | 1.4740e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 10 | **1.8515e-03** | 2.0637e-03 | 2.5305e-03 | 3.74e-16 | 2.77e-12 | 9.72e-11 | 9.72e-11 |
| | 4 | **7.1956e-03** | 9.1197e-03 | 2.2122e-02 | 1.04e-15 | 4.27e-11 | 7.54e-10 | 9.73e-11 |
| | 5 | **4.2700e-03** | 5.8662e-03 | 9.1130e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 6 | 1.1976e-03 | **4.6246e-04** | 7.2545e-04 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| DTLZ3 | 7 | **1.2094e-03** | 1.3654e-03 | 1.6704e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 8 | **1.7592e-03** | 2.5105e-03 | 2.7037e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 9 | **9.9098e-04** | 1.1612e-03 | 1.4536e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 10 | **1.8656e-03** | 2.4228e-03 | 2.5326e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 4 | **8.2896e-03** | 1.0301e-02 | 2.2212e-02 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 5 | **4.2842e-03** | 8.5091e-03 | 9.1201e-03 | 1.46e-15 | 2.77e-12 | 9.73e-11 | 2.64e-09 |
| | 6 | **4.1579e-04** | 7.0932e-04 | 8.3514e-04 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| DTLZ4 | 7 | **1.0711e-03** | 1.6623e-03 | 2.0305e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 8 | **1.6788e-03** | 2.6767e-03 | 2.9353e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 9 | **1.1728e-03** | 1.3986e-03 | 1.5749e-03 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 10 | **1.7495e-03** | 2.4007e-03 | 2.6921e-03 | 4.40e-16 | 4.43e-12 | 1.38e-10 | 9.73e-11 |
| | 4 | 1.6671e-02 | **1.6204e-02** | 1.7121e-02 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 5 | 5.8838e-03 | **5.7738e-03** | 5.9900e-03 | 9.53e-13 | 2.77e-12 | 5.51e-08 | 5.51e-08 |
| | 6 | 1.8945e-03 | **1.7910e-03** | 1.9025e-03 | 1.46e-15 | 2.77e-12 | 2.64e-09 | 9.73e-11 |
| DTLZ5 | 7 | 6.4055e-04 | **6.2722e-04** | 6.4189e-04 | 5.11e-15 | 2.77e-12 | 5.51e-08 | 9.73e-11 |
| | 8 | 9.6134e-04 | **9.1614e-04** | 9.6588e-04 | 3.74e-16 | 2.77e-12 | 9.73e-11 | 9.73e-11 |
| | 9 | 8.1450e-04 | **7.9596e-04** | 8.1874e-04 | 1.46e-15 | 2.77e-12 | 2.64e-09 | 9.73e-11 |
| | 10 | 7.2022e-04 | **6.7489e-04** | 7.2718e-04 | 1.46e-15 | 2.77e-12 | 2.64e-09 | 9.73e-11 |
| | 4 | **1.6829e-02** | 1.6831e-02 | 1.7129e-02 | 2.89e-11 | 2.77e-12 | 8.85e-07 | 8.85e-07 |
| | 5 | 5.8428e-03 | **5.7819e-03** | 5.9870e-03 | 3.42e-09 | 2.77e-12 | 4.50e-03 | 1.10e-05 |
| | 6 | **1.8867e-03** | 1.9213e-03 | 1.9077e-03 | 1.59e-14 | 2.77e-12 | 8.85e-07 | 9.73e-11 |
| DTLZ6 | 7 | **6.3685e-04** | 6.6313e-04 | 6.4248e-04 | 4.43e-14 | 2.77e-12 | 1.10e-05 | 9.73e-11 |
| | 8 | 9.5918e-04 | **9.2020e-04** | 9.6915e-04 | 1.46e-15 | 2.77e-12 | 2.64e-09 | 9.73e-11 |
| | 9 | **8.1404e-04** | 8.2422e-04 | 8.2033e-04 | 5.11e-15 | 2.77e-12 | 5.51e-08 | 9.73e-11 |
| | 10 | 6.6082e-04 | **6.2611e-04** | 6.7947e-04 | 1.33e-12 | 4.43e-12 | 5.07e-02 | 9.73e-11 |

*Table 4.4: R-NSGA-II's number of wins, loses and ties against R-MEAD2-Te and R-MEAD2-PBI.*

| | R-MEAD2-Te | | | R-MEAD2-PBI | | |
|---|---|---|---|---|---|---|
| | Wins | Loses | Ties | Wins | Loses | Ties |
| R-NSGA-II | 3 | 37 | 2 | 5 | 35 | 2 |

# Chapter 5

# Sensitivity Analysis of the Penalty Parameter in PBI

## 5.1 Introduction

In the previous chapter, we concluded that there is a need for testing PBI with a wide variety of penalty parameters to be able to compare PBI and Tchebycheff accurately. There have been very few studies analyzing the systematic sensitivity of PBI [Ishibuchi et al., 2013b] and in particular on the performance of MOEA/D which is basis of our proposed approaches in this thesis. Ishibuchi et al. [2013a] studied the effect of various aggregation functions on MOEA/D based on a range of Knapsack problems. In another study, Sato [2014] proposed Inverted Penalty-based Boundary Intersection (IPBI) which is an extension of PBI. Similar to PBI, IPBI retains the penalty parameter, but it uses the nadir point instead of the ideal point. The drawback of the both these studies is that they are mostly based on a specific problem type, Knapsack problem and they did not study the effect of the penalty parameter of PBI on convergence and uniformity separately.

Although the penalty parameter of PBI can have a significant effect on the convergence and the uniformity of the solution set, this has not been studied in a systematic way particularly for user-preference based algorithms such as R-MEAD2 [Mohammadi et al., 2014] which is an aggregation-based method and can be used with PBI. The aim of this chapter is to study the effect of the penalty parameter of PBI ($\theta$) from the following aspects: 1) The influence of $\theta$ on problems with and without having user-preference information; 2) The impact of $\theta$ on convergence and uniformity of solutions separately and simultaneously; and 3) The relationship between $\theta$ and the number of objectives. In particular, we use MOEA/D [Zhang and Li, 2007] and R-MEAD2 [Mohammadi et al., 2014] to study the effect of the penalty parameter of PBI ($\theta$) on uniformity and convergence. We also investigate the relationship between the number of objectives and the optimal value of $\theta$. We have adopted Generational Distance (GD) [Veldhuizen, 1999a] as a measure of convergence, $L_2$-discrepancy [Fang and Lin, 2003] as a measure of uniformity, and Inverted Generational Distance (IGD) [Zitzler et al., 2003] as a measure that simultaneously captures both convergence and uniformity. We have also used the DTLZ benchmark suite [Deb et al., 2001] on functions with two to ten objectives for this study.

The rest of this chapter is organized as follows: In section 5.2 gives a brief description of parameter settings, and the performance metrics used in this chapter. section 5.2 contains the analysis and discussion of the results and section 5.4 concludes this chapter.

## 5.2 Experimental Design

This section contains the parameter settings of the algorithms, and the benchmark problems that we used for our experiments.

### 5.2.1 Parameter Settings

To analyze the effect of PBI's penalty parameter ($\theta$), we used R-MEAD2 [Mohammadi et al., 2014] as a user-preference aggregation-based EMO, and MOEA/D [Zhang and Li, 2007] as a decomposition-based Evolutionary multi-objective Optimization (EMO) which does not use any user-preference information. For both algorithms, the population size was set to 200, and the maximum number of iterations was limited to 100. The performance of both algorithms was tested on DTLZ1-DTLZ6 benchmark problems from the DTLZ benchmark suite [Deb et al., 2001] using two, four, six, eight, and ten objectives. Each problem instance (a function with a particular number of objectives) was solved with both R-MEAD2 and MOEA/D using the following $\theta$ values: $\{0, 0.1, 1, 5, 10, 50, 500\}$. The reference point used for R-MEAD2 is $x_i = 0.25, i \in \{1, \ldots, m\}$, and the radius parameter of R-MEAD2, which determines the size of preferred regions was set to 1.

### 5.2.2 Performance Metrics

In order to study the effect of $\theta$ on convergence behavior, and solution uniformity of MOEA/D and R-MEAD2 the following metrics are used:

1. **Generational Distance (GD)** [Veldhuizen, 1999a]
   is a metric which can be used to measure the convergence of EMO algorithms. It calculates the average closest distances of obtained solutions to the Pareto-optimal front. For GD formula, please refer to equation 2.8 in chapter 2.

2. **$L_2$-discrepancy** [Fang and Lin, 2003]
   is one of the most popular metrics used to measure the uniformity of solutions. Since we are interested in the uniformity of solutions, lower values of $L_2$ imply better uniformity.

3. **Inverted Generational Distance (IGD)** [Zitzler et al., 2003]
   is a metric that takes both diversity and convergence of solutions into account. It calculates the average closest distance of sample points on the Pareto-optimal front to the obtained solutions. For IGD formula, please refer to equation 2.9 in chapter 2.

## 5.3 Analysis and Discussion

In this section, the penalty parameter of PBI ($\theta$) is analyzed from the following points of view:

- The effect of $\theta$ on convergence as measured by GD;
- The effect of $\theta$ on uniformity as measured by $L_2$-discrepancy;
- The effect of $\theta$ on both convergence and uniformity as measured by IGD;
- The relative performances of MOEA/D and R/MEAD2 with respect to different values of $\theta$;
- The relationship between $\theta$ and the number of objectives.

Tables B.1-B.6[1] in Appendix B contains the experimental results of MOEA/D and R-MEAD2 on DTLZ1-DTLZ6 functions for various number of objectives and $\theta$ values. For $m = 6$ the values are omitted from the table for the sake of space. The performance of each algorithm is measured using the three metrics explained in Section 5.2.2. For each function with a given number of objectives, the best performing $\theta$ is highlighted. The size of tables B.1-B.6 makes analysis of the results prohibitive. For this reason, the information in this table is summarized in tables II-V. Most of the analysis in this chapter uses statistical

[1]tables have been placed at the appendix B for smoother follow of the content.

ranking of the results in order to summarize the detailed information in tables B.1-B.6 into a series of ordinal numbers that highlight the major trends in the data. Although no formal statistical tests are used, statistical ranking is a reliable technique that forms the basis of all non-parametric statistical methods [Sheskin, 2003].

One relatively obvious trend that can be seen in tables B.1-B.6 is that, in general, larger $\theta$ values will result in lower $L_2$-discrepancy values, which implies better uniformity of solutions. Theoretically, as $\theta$ increases, PBI more closely resembles NBI which has better uniformity. Unlike $L_2$-discrepancy, the relationship between IGD, GD and $\theta$ cannot be clearly seen. For a better understanding about the effect of $\theta$ on uniformity and convergence, we calculated the Spearman correlation coefficient ($\rho$) [Sheskin, 2003] between $\theta$ and the results based on each metric separately. Spearman correlation is a nonparametric method that can measure nonlinear correlation between two variables. It is formulated as

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)} \tag{5.1}$$

where $d$ is the pairwise distances of the ranks of the variables $xi$ and $yi$. $n$ is the number of samples.

The experimental results on 6 functions (DTLZ1-DTLZ6) with a different number of objectives ($\{2, 4, 6, 8, 10\}$) resulted in 30 different correlation coefficients. Basic descriptive statistics for these 30 values of $\rho$ are reported in table II. We can see that the results of table II is consistent with our observation on the relationship between $\theta$ and $L_2$-discrepancy. Large negative values indicate a strong negative correlation between $\theta$ and the respective metrics i.e., a larger $\theta$ value will result in lower values for the metrics (lower values means better performance). We can see that the mean and median $\rho$ for GD and IGD are relatively close to zero, especially for GD. We know that both uniformity and convergence affect the IGD values; therefore, we expect that the $\rho$ values for IGD will be influenced by large negative values of $\rho$ on $L_2$-discrepancy. We can see that in most cases the reported value for IGD is between the values reported for GD and $L_2$-discrepancy.

In order to get a better insight into the relationship between convergence and $\theta$, we counted the number of times that each $\theta$ value outperformed other $\theta$ values. This frequency information is reported in table III. The trend that we observed earlier for $L_2$ is also reflected here, with larger $\theta$ values having a higher success rate than others. With respect to convergence, we can see that for MOEA/D the success rate drops as $\theta$ increases. For R-MEAD2 the trend is not very clear but we can see that R-MEAD2 has a tendency towards larger

*Table II: Spearman correlation coefficients ($\rho$) of $\theta$ with respect to different metrics (descriptive statistics of 30 functions (DTLZ1-DTLZ6, number of objectives $\{2, 4, 6, 8, 10\}$).*

| | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|
| Stats | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| Min | -0.96 | -0.96 | -1.00 | -0.98 | -0.90 | -1.00 |
| Median | -0.32 | -0.08 | -0.89 | -0.63 | -0.14 | -0.96 |
| Max | +0.84 | +0.91 | -0.39 | +0.70 | +0.96 | +0.13 |
| Mean | -0.26 | +0.02 | -0.85 | -0.45 | -0.06 | -0.90 |
| Std | +0.51 | +0.60 | +0.17 | +0.53 | +0.62 | +0.23 |

*Table III: Success frequency of various $\theta$ values. Using IGD,GD and $L_2$ metrics on MOEA/D and R-MEAD2*

| | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|
| $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| 0 | 3 | 13 | 1 | 2 | 6 | 0 |
| 0.1 | 8 | 7 | 2 | 7 | 9 | 0 |
| 1.0 | 6 | 5 | 0 | 3 | 4 | 0 |
| 5.0 | 3 | 3 | 2 | 5 | 7 | 1 |
| 10 | 7 | 2 | 3 | 1 | 0 | 2 |
| 50 | 1 | 0 | 8 | 1 | 1 | 3 |
| 500 | 2 | 1 | 14 | 10 | 2 | 25 |

$\theta$ values relative to MOEA/D. We hypothesize that, in general, R-MEAD2 has a tendency to higher $\theta$ values compared with MOEA/D on the same problem with the same number of objectives. On one hand, Ishibuchi et al. [2013a] suggested higher values of $\theta$ can have a detrimental effect on convergence but on the other study by Mohammadi et al. [2012] suggested that inclusion of user-preference information may help the convergence rate by confining the search to the regions that are of interest to a decision maker. Considering these two observations together suggests that R-MEAD2 which uses preference information has a relatively faster convergence than MOEA/D. Therefore, a slight increase in the $\theta$ value can improve its uniformity without dramatically affecting the convergence behavior. We speculate that this is the reason behind R-MEAD2's tendency towards larger $\theta$ values relative to MOEA/D (Table III). However, the results in table III are not sufficient to reach such a conclusion. Therefore, we calculated the difference in the index of the best performing $\theta$

between R-MEAD2 and MOEA/D. More specifically, the $\theta$ values are assigned an index in the range $\{1, \ldots, 7\}$ in ascending order of $\theta$ values. Then, the difference between the index of the entry with the lowest value of a given metric is calculated. For instance, in the case of two-objective DTLZ2, R-MEAD2 attains its best GD value when $\theta$ equals 0.1 (index=2). Similarly, MOEA/D attains its best GD value when $\theta$ equals 0 (index=1). The difference between these two indices is presented in Table IV. With respect to tables B.1-B.6 this difference, which is denoted by $\Delta$, tells us how much the $\theta$ value of a highlighted entry of R-MEAD2 is larger or smaller than the $\theta$ of MOEA/D. Positive numbers indicate that the best performing $\theta$ of R-MEAD2 is larger than the best performing $\theta$ of MOEA/D. Negative numbers show the opposite, and zero means that the best performing $\theta$ of R-MEAD2 and MOEA/D are equal.

Table IV contains the sum of $\Delta$ values for different functions and number of objectives reported for IGD, GD, and $L_2$-discrepancy. It shows that the values are predominantly positive, which suggests that R-MEAD2 tends towards larger $\theta$ values compared to MOEA/D tested on the same function. The total sum of all $\Delta$ values on all functions and number of objectives ($m$) are 20, 18, and 26 for IGD, GD, and $L_2$-discrepancy, which confirms that R-MEAD2's bias towards larger $\theta$ values. Figure 5.1(a) shows a bar chart for sums of $\Delta$ values for different numbers of objectives based on the GD metric. We can see that the difference between the best performing $\theta$ of R-MEAD2 and MOEA/D is larger for fewer objectives. However, as the number of objectives increases, so the difference between the best performing $\theta$ of the two algorithms vanishes. This suggests that as the number of objectives increases, R-MEAD2 and MOEA/D exhibit similar convergent behavior.

*Table IV: Differences between indices of best performing $\theta$ values of MOEA/D and R-MEAD2.*

| $m$ | Function | IGD | GD | L2 |
|---|---|---|---|---|
| 2 | DTLZ1 | 0 | 0 | -1 |
| | DTLZ2 | 2 | 1 | 1 |
| | DTLZ3 | 0 | 0 | 0 |
| | DTLZ4 | 1 | 0 | 5 |
| | DTLZ5 | 0 | 0 | 0 |
| | DTLZ6 | 0 | 5 | 5 |
| Sum | – | 3 | 6 | 10 |
| 4 | DTLZ1 | 0 | 0 | 5 |
| | DTLZ2 | 0 | 1 | 1 |
| | DTLZ3 | 1 | -1 | 0 |
| | DTLZ4 | 1 | 3 | 0 |
| | DTLZ5 | 1 | 2 | -2 |
| | DTLZ6 | 4 | -1 | 0 |
| Sum | – | 7 | 4 | 4 |
| 6 | DTLZ1 | 1 | 1 | 0 |
| | DTLZ2 | 2 | 2 | 0 |
| | DTLZ3 | 1 | 1 | 1 |
| | DTLZ4 | -3 | 3 | -3 |
| | DTLZ5 | 0 | 0 | 3 |
| | DTLZ6 | 0 | -3 | 0 |
| Sum | – | 1 | 4 | 1 |
| 8 | DTLZ1 | 1 | 1 | 2 |
| | DTLZ2 | 2 | 0 | 3 |
| | DTLZ3 | -3 | -2 | 2 |
| | DTLZ4 | 5 | 3 | 0 |
| | DTLZ5 | 1 | 0 | 1 |
| | DTLZ6 | 3 | 1 | 2 |
| Sum | – | 9 | 3 | 10 |
| 10 | DTLZ1 | -1 | -1 | 0 |
| | DTLZ2 | 2 | 1 | 0 |
| | DTLZ3 | 2 | 2 | 0 |
| | DTLZ4 | -1 | 0 | 0 |
| | DTLZ5 | -1 | 0 | 0 |
| | DTLZ6 | -1 | -1 | 1 |
| Sum | – | 0 | 1 | 1 |
| Total | – | 20 | 18 | 26 |

*Table V: Differences between indices of the best performing $\theta$ values of IGD and GD, as well as IGD and $L_2$-discrepancy.*

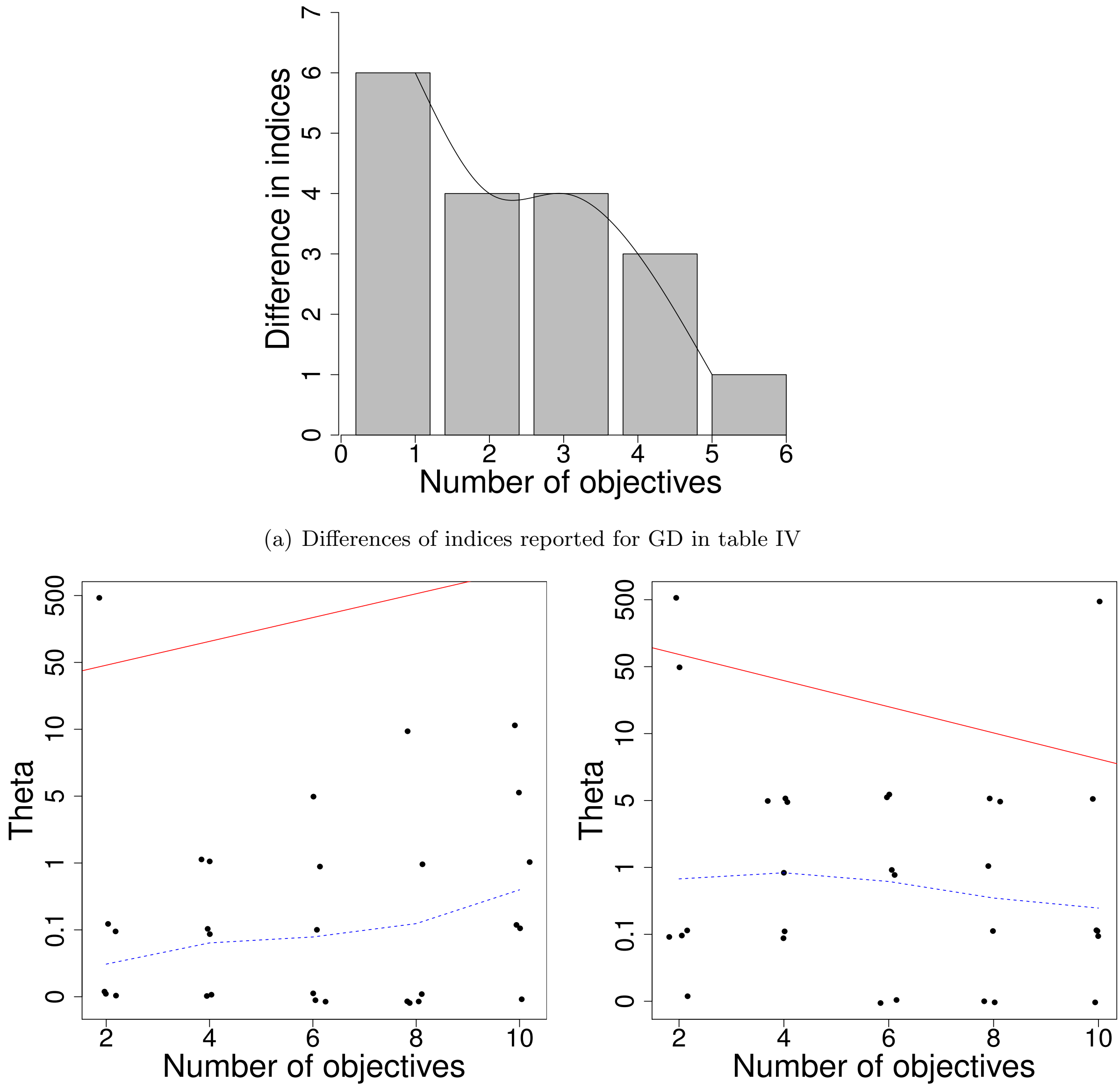
| | | MOEA/D | | R-MEAD2 | |
|---|---|---|---|---|---|
| Function | # Objs | $\Gamma_{\mathrm{GD}}$ | $\Gamma_{\mathrm{L_2}}$ | $\Gamma_{\mathrm{GD}}$ | $\Gamma_{\mathrm{L_2}}$ |
| DTLZ1 | 2 | 0 | 4 | 0 | 3 |
| | 4 | 0 | 0 | 0 | 5 |
| | 6 | 0 | 5 | 0 | 4 |
| | 8 | 0 | 4 | 0 | 5 |
| | 10 | 0 | 5 | 0 | 6 |
| Sum | – | 0 | 18 | 0 | 23 |
| DTLZ2 | 2 | 4 | 1 | 5 | 0 |
| | 4 | 4 | -1 | 3 | 0 |
| | 6 | 4 | 1 | 4 | 0 |
| | 8 | 4 | -1 | 6 | 0 |
| | 10 | 3 | 2 | 4 | 0 |
| Sum | – | 19 | 2 | 22 | 0 |
| DTLZ3 | 2 | 0 | 0 | 0 | 0 |
| | 4 | 0 | 3 | 2 | 3 |
| | 6 | 0 | 3 | 0 | 3 |
| | 8 | 1 | -1 | 0 | 4 |
| | 10 | 0 | 2 | 0 | 0 |
| Sum | – | 1 | 7 | 2 | 10 |
| DTLZ4 | 2 | 3 | -3 | 4 | 1 |
| | 4 | 2 | 3 | 0 | 2 |
| | 6 | 6 | 0 | 0 | 0 |
| | 8 | 1 | 5 | 3 | 0 |
| | 10 | 1 | 2 | 0 | 3 |
| Sum | – | 13 | 7 | 7 | 6 |
| DTLZ5 | 2 | 0 | 0 | 0 | 0 |
| | 4 | 1 | 5 | 0 | 3 |
| | 6 | 0 | 3 | 0 | 6 |
| | 8 | 0 | 5 | 1 | 5 |
| | 10 | 1 | 4 | 0 | 5 |
| Sum | – | 2 | 17 | 1 | 19 |
| DTLZ6 | 2 | 3 | -3 | -2 | 2 |
| | 4 | 1 | 3 | 6 | -1 |
| | 6 | 1 | 2 | 4 | 2 |
| | 8 | 1 | 1 | 3 | 0 |
| | 10 | 0 | 3 | 0 | 5 |
| Sum | – | 6 | 6 | 11 | 8 |

So far, we have seen that $\theta$ can significantly affect the convergence and the uniformity of solutions. Here, we use the technique of index differences that we used previously to study the relationship between convergence and uniformity in different functions. The analysis for this part revolves around the difference in the index of best performing $\theta$ values as measured by IGD and GD, as well as by IGD and $L_2$-discrepancy. This difference that we call $\Gamma$ measures the degree to which IGD is affected by either the convergence or uniformity measures. Table V contains the sum of $\Gamma$ values on various objectives and functions. The results are recorded separately for MOEA/D and R-MEAD2. The quantity $\Gamma_{\mathrm{GD}}$ measures the difference between the index of the best performing $\theta$ on IGD and GD. Positive values mean that the best performing $\theta$ for IGD is *larger* than the best performing $\theta$ for GD. Similarly, the quantity $\Gamma_{\mathrm{L}_2}$ measures the difference between the index of the best performing $\theta$ on $L_2$ and IGD. Positive values mean that the best performing $\theta$ for IGD is *smaller* than the best performing $\theta$ for $L_2$. Generally speaking, we expect to see an inverse effect between uniformity and convergence as $\theta$ increases. Therefore, observing a positive value for both $\Gamma_{\mathrm{GD}}$ and $\Gamma_{\mathrm{L}_2}$ means that the best performing $\theta$ for IGD lies between those of GD and $L_2$ which validates our expectation about the inverse relationship between uniformity and convergence. We can see from table V that the values are predominantly positive. Spearman correlation coefficient between $\Gamma_{\mathrm{L}_2}$ and $\Gamma_{\mathrm{GD}}$ based on the reported sums are -0.81 and -0.94 for MOEA/D and R-MEAD2 respectively. This suggests a clear inverse relationship between convergence and uniformity. The information in table V enables us to identify the dominant factor affecting the overall performance of MOEA/D and R-MEAD2. Observing a small value (close to zero) for $\Gamma_{\mathrm{GD}}$ suggests that the dominant factor is convergence, while a small value for $\Gamma_{\mathrm{L}_2}$ suggests that the dominant factor is uniformity. It is notable that the number of objectives does not affect the dominance level of either measures. Table V clearly shows that the balance between convergence and uniformity is different for each function. On functions DTLZ1, DTLZ3, and DTLZ5 convergence is the dominant factor. However, On DTLZ2 and to some extent on DTLZ4, uniformity is the dominant factor. On DTLZ6, convergence and uniformity are of equal importance. It is interesting to note that on multi-modal functions (DTLZ1 and DTLZ3), convergence is the dominant factor. This is intuitive, because the existence of local optima makes convergence more difficult. As mentioned in Section 5.2, a small $\theta$ value allows the solutions to move around the search space more freely by being able to deviate from their respective reference lines. This strategy may allow the solutions to escape local optima of multi-modal problems.

Finally, we briefly investigate the relationship between $\theta$ values and the number of objectives. To do this a set of scatter plots are constructed that show the trend between $\theta$ and $m$ for MOEA/D and R-MEAD2. Each point on the plots represent the best performing $\theta$ for each function. To avoid the points from overlapping, a random Gaussian noise has been added (Jittering technique [J. M. Chambers, W. S. Cleveland, B. Kleiner, and P. A. Tukey, 1983]). The solid line is the line of best fit based on linear regression, and the dashed line represents a locally-weighted polynomial regression (LOWESS) [Cleveland, 1979]. The fan-out effect visible in Figure 5.1(b) and the presence of outliers in Figure 5.1(c) suggest a lack of homoscedasticity of variances, which makes the resultant regression lines unreliable. Therefore, we conclude that there is no major relationship between $\theta$ and the number of objectives based on our experimental results.

### 5.4 Chapter Summary

This chapter is dedicated to a comprehensive analysis of PBI penalty parameter ($\theta$), and its effect on a user-preference algorithm (R-MEAD2) and a non-user-preference algorithm (MOEA/D). Unlike previous studies that only rely on Hypervolume as their performance measure, we study the effect of $\theta$ on convergence, uniformity, and the combination of convergence and uniformity independently. The experimental results suggest that user-preference algorithms consistently perform better with a relatively larger $\theta$ value compared to their non-user-preference counterparts. The results also suggest that with some problems, such as multi-modal functions, convergence is the dominant factor the overall performance, where a smaller $\theta$ is preferable. Conversely, in some other problems, a larger $\theta$ is suggested where uniformity is the dominant factor. Finally, we briefly investigate the relationship between $\theta$ and the number of objectives. The findings of this sensitivity analysis study have been used to develop feedback mechanism which is the topic of our next chapter. In other words, after this study we can estimate which penalty parameter ($\theta$) is more suitable for each type of problem.




(a) Differences of indices reported for GD in table IV



(b) $m$ vs $\theta$ (MOEA/D)



(c) $m$ vs $\theta$ (R-MEAD2)

*Figure 5.1: Relationship between $\theta$ and number of objectives. In (b) and (c) the solid line is the line of best fit and the dashed line is (LOWESS)*

# Chapter 6

# Feedback Mechanism for Decomposition-Based Evolutionary Many-Objective Optimization

## 6.1 Introduction

In this chapter, we use two existing metrics called Overlapping Hypervolume (OHV) [Van Moffaert et al., 2014] and Potential Energy (PE) [Ong et al., 2012]. Since the computational complexity of these two metrics is relatively low, the calculation of potential energy is not affected by the shape of boundary and it agrees with human perception of global uniformity, they turn out to be a suitable choice for this work. A new uniformity metric is also proposed called Potential Energy with direction vector (PEV). It should be noted that the proposed feedback mechanism approach can be applied to both user-preference and non-user preference decomposition algorithms. Since the focus of this chapter is on many-objective problems, the proposed feedback mechanism will be only applied here to show that a user-preference approach can be used to reduce the computational cost. As a result, the proposed approach called UR-MEAD2, should satisfy the preference of a user as well as uniformity of solutions.

The main contributions of this work can be summarized as follows: 1) Two existing measures are used as a feedback mechanism: Overlapping Hypervolume (OHV) [Van Moffaert et al., 2014] and Potential Energy (PE) [Ong et al., 2012]. These two metrics have been modified in a way that ranks each individual in terms of its contribution towards improving overall uniformity. 2) A new uniformity metric is also proposed, called Potential Energy with direction vector (PEV). PEV can measure the contribution that each solution makes to improve or degrade uniformity. This feature of PEV provides some gradient information to the algorithm in order to update weight vectors in a meaningful way. 3) Unlike existing studies which only use the modified version of Tchebycheff as a decomposition method to somehow remedy the non-linearity relationship between weight vectors and solutions, our proposed approach (UR-MEAD2) can work with any decomposition methods such as Weighted-Sum [Miettinen, 1999], Boundary Intersection [Das and Dennis, 1998], Penalty-based Boundary Intersection (PBI) [Zhang and Li, 2007]. 4) The performance of decomposition methods (Tchebycheff and PBI with different penalty factors) under each feedback mechanism method is compared. The experimental results show that Tchebycheff which uses our proposed uniformity metric (PEV) as a feedback mechanism has the best performance. 5) The performance of UR-MEAD2 is compared with a state-of-the-art (R-NSGA-II [Deb et al., 2006]).

The rest of this chapter is organized as follows. The proposed feedback mechanism algorithm is described in section 6.2. Section 6.3 discusses the experimental results. Finally Section 6.4 summarizes the research findings.

## 6.2 Proposed Approach UR-MEAD2

This section describes UR-MEAD2 algorithms which use three different feedback mechanisms to find a uniform set of solutions in the preferred region of the Pareto-front in the objective space.

A wide range of decomposition algorithms has been proposed with an attempt to improve the uniformity of solutions in the objective space [Tan et al., 2013; Qi et al., 2014; Ma et al., 2014]. These methods are based on the hypothesis that a uniform set of weight vectors will result in a uniform set of solutions in the objective space. However, depending on the decomposition method and the complexity of the Pareto optimal front, this assumption is not always true. Indeed, even for very simple Pareto fronts, we have shown in chapter 4 in table 4.2 a uniform random number generator can perform better than sophisticated methods such as good lattice point, especially on many-objective problems. As it can be observed from the table a range population size from small to large for all dimensions have been used. It is therefore imperative to devise algorithms that are first able to ascertain the uniformity of the solutions during the course of optimization, and then adopt more appropriate weight vectors to maximize the uniformity of the solutions.

Decomposition-based EMO algorithms are appealing for our purpose in the sense that the dynamic adjustment of weights gives us the flexibility to direct the solutions in order to maximize uniformity. However, without knowing the effect of each weight on the solution uniformity in the objective space, it is difficult to adjust the weights in a meaningful way. The problem with existing metrics is that they mainly measure the uniformity of solutions globally but they cannot rank each individual based on its contribution with respect to the entire population uniformity. It is clear that for an efficient optimization of weights to maximize the overall uniformity, we need some form of *gradient* information that tells us the direction in which the solutions should move in order to improve the uniformity. Unfortunately, a fully-fledged optimization of the weights is very costly due to the need for repeated evaluation of the solution set. It is clear that even a random walk requires the evaluation of multiple *sets* of weight vectors in order to find a set that improves the uniformity. To evaluate each set of weight vector, at least one iteration of optimizing the multi-objective problem is needed, which makes the process computationally expensive. To avoid this problem, we propose to use Overlapping Hypervolume (OHV) [Van Moffaert et al., 2014] and Potential Energy (PE) [Ong et al., 2012] metrics to detect the sparse areas of the objective space and bias the search in order to fill those areas (Sections 6.2.1 and 6.2.2). In other words, the weight associated with each solution is influenced by the weight associated with the solution in the most sparse area. The overall effect of this behavior is to force the solutions to move from dense areas to sparser areas of the objective space. Finally, we propose an improved version of the PE metric which gives us the exact direction in which a solution should move in order to improve uniformity (Section 6.2.3).

### 6.2.1 Overlapping Hypervolume

Overlapping Hypervolume can be used to measure the sparsity around a solution and use it as feedback to update the weights associated with the solutions, in order to move them from denser areas to sparser areas. OHV measures percentage of overlap and the unique hypervolume of two solutions in the objective space. This is formulated as follows:

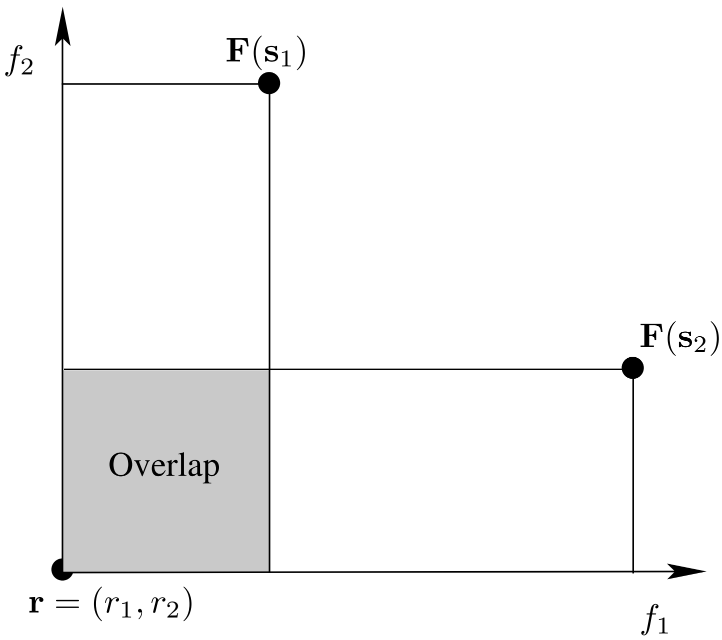



*Figure 6.1: Illustration of the overlapping Hypervolume method.*

$$\mathrm{OHV}(\mathbf{s}_i, \mathbf{s}_j) = \frac{V_{\text{overlap}}}{V_{\text{total}}}, \tag{6.1}$$

$$V_{\text{overlap}} = \prod_{k=1}^{m} |r_k - \min\{s_{ik}, s_{jk}\}|, \tag{6.2}$$

$$V_i = \prod_{k=1}^{m} |r_k - s_{ik}|, \tag{6.3}$$

$$V_j = \prod_{k=1}^{m} |r_k - s_{jk}|, \tag{6.4}$$

$$V_{\text{total}} = V_i + V_j - V_{\text{overlap}}, \tag{6.5}$$

where $\mathbf{r} = (r_1, \cdots, r_m)$ is a reference point in objective space, and $m$ is the number of objectives. Figure 6.1 shows how OHV measures the percentage of overlap between two solutions $\mathbf{s}_1 = (s_{11}, s_{12})$ and $\mathbf{s}_2 = (s_{21}, s_{22})$ in the objective space. The calculations for this particular example are as follows:

$$\begin{aligned} V_{\text{overlap}} &= |r_1 - \min\{s_{11}, s_{21}\}| \times \\ &\quad |r_2 - \min\{s_{12}, s_{22}\}| \qquad (6.6) \\ V_1 &= |(s_{11} - r_1) \cdot (s_{12} - r_2)| \qquad (6.7) \\ V_2 &= |(s_{21} - r_1) \cdot (s_{22} - r_2)| \qquad (6.8) \end{aligned}$$

It is clear that a smaller OHV value indicates better uniformity. If OHV equals one, it means two solutions are situated on top of each other. The overall Overlapping Hypervolume for a set of points is calculated as follows:

$$U_i^{\text{OHV}} = \sum_{j=1, j\neq i}^{\mu} \text{OHV}(\mathbf{s}_i, \mathbf{s}_j), \qquad (6.9)$$

where $\mathbf{s}_i$ and $\mathbf{s}_j$) are two solutions and $\mu$ is total number of points.

### 6.2.2 Potential Energy

Potential Energy (PE) was first introduced by Saff and Kuijlaars [1997] and was improved in [Ong et al., 2012]. It is defined as the energy stored in a body because of its relative position in a repulsive field with respect to its neighbors. In our proposed approach, we use this metric as feedback function. The average value of PE is assigned to its lowest when the points are placed as far as possible from each other. The potential energy between $i$th and $j$th point is formulated as follows:

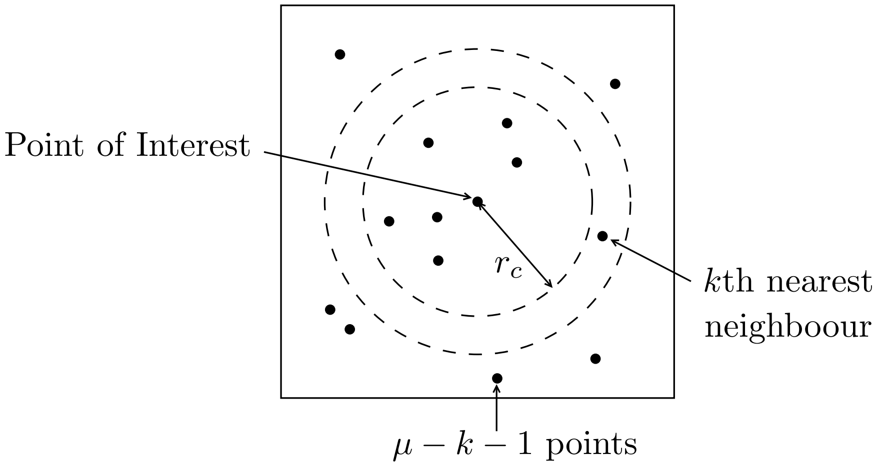
$$\text{PE}(\mathbf{s}_i, \mathbf{s}_j) = \frac{\theta}{\theta + \|\mathbf{s}_i - \mathbf{s}_j\|}, \qquad (6.10)$$

where $\|\mathbf{s}_i - \mathbf{s}_j\|$ is the Euclidean distance between $i$th and $j$th solutions, and $\theta$ is an arbitrary constant to prevent division by zero in the formula when $\|\mathbf{s}_i - \mathbf{s}_j\| = 0$.

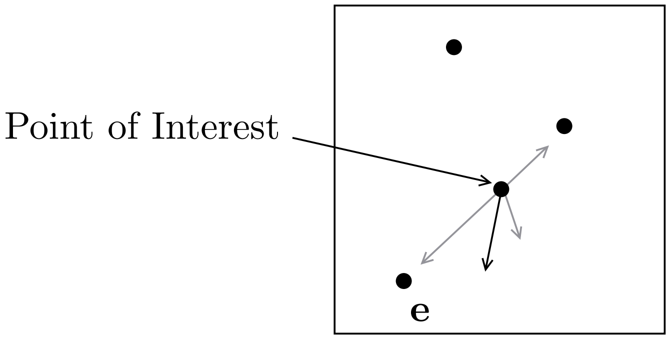
As the uniformity of points improves, the PE value decreases. The overall average potential energy for a set of points is calculated as follows:

$$\frac{\sum_{i=1}^{\mu} \sum_{j\in\mathcal{N}_i} \text{PE}(\mathbf{s}_i, \mathbf{s}_j)}{\mu|\mathcal{N}_i|}, \qquad (6.11)$$

where $\mu$ is the total number of points, $\mathcal{N}_i$ is a set that defines the neighborhood of the $i$th solution, and $|\mathcal{N}_i|$ is the cardinality of the set. Assuming that $k = |\mathcal{N}_i|$, it is clear that the




(a) Neighborhood.




(b) PE with Direction Vector.

*Figure 6.2: Illustration of the Potential Energy (PE) method.*

PE of the $i$th point is calculated based on its $k$-nearest neighbors. The neighborhood of the $i$th point is defined to be the points which are within $r_c + \delta_r$ distance from $\mathbf{s}_i$. The distance $r_c$ between $\mu$ points in a region with the area of $a$ is calculated as $\sqrt{\frac{a}{\mu}}$. The details about why this is the case can be found in Saff and Kuijlaars [1997]. $\theta$ is set to $3r_c^2$ such that PE always lies between 0 and 1. It should be noted that $\delta_r$ is not a control parameter and is not set by the user. Figure 6.2(a) shows the point of interest for PE calculation, including its $k$th nearest neighbor and the rest of the points in the field.

The main motivation to use PE is to identify sparse areas in the solution space and update the weight vectors in order to move the solutions from the denser areas to more sparse areas. The problem with Equation (6.11) is that it gives an overall measure of the uniformity but is not suitable of our purpose. To alleviate this problem, for a solution $i$, the sparsity around it is estimated by aggregating its potential energy with respect to all other solutions in its neighborhood:

$$U_i^{\mathrm{PE}} = \sum_{j \in \mathcal{N}_i} \mathrm{PE}(\mathbf{s}_i, \mathbf{s}_j), \tag{6.12}$$

where $\mathcal{N}_i$ is a set that defines the neighborhood of the $i$th solution. The solution with the smallest $U_i^{\mathrm{PE}}$ is considered to be in a sparse area relative to other solutions.

### 6.2.3 Potential Energy with Direction Vector

Potential Energy with Direction Vector (PEV) is an improved version of potential energy. Physically speaking, two charged particles exert forces of equal magnitude on each other, the direction of which is determined by the nature of the charged particles. Opposite charges produce an attractive force while similar charges produce a repulsive force along the straight line connecting the two charges. The force on each charge is calculated by Coulomb's law which is defined as $F = \frac{k_{\mathrm{e}} q_1 q_2}{r^2}$, where $k_{\mathrm{e}}$ is the Coulomb's constant, $q_1$ is the quantity of the first charge, $q_2$ is the quantity of the second charge, and $r$ is the distance between the two. If the net force on a particle is non-zero, it will accelerate in the direction of the net force. A set of charged particles is said to be in static equilibrium if the vectorial sum of all the forces on each particle is zero. In a system this is equilibrium, all objects stay as far as possible from each other.

Before a system enters equilibrium, all particles of the system move along a trajectory which is determined by the direction of the net force on it over time. In this chapter, we are inspired by this observation on the equilibrium state. Here, we assume that the solutions have unit charges of the same type. Therefore, the force that two particles exert on each other is only a function of their relative distance. The net force (vectorial sum of all the forces) for each solution is denoted by $\mathbf{e}$. This vector defines the direction that each solution should move in the objective space until all solutions are as far as possible from each other. It is clear that, the solutions are bounded in the feasible region, and once they converge on the Pareto-optimal front, their movement is constrained by the shape of the Pareto-optimal surface. PEV is formulated as follows:

$$\mathrm{PEV}(\mathbf{s}_i, \mathbf{s}_j) = \frac{\theta}{\theta + \|\mathbf{s}_i - \mathbf{s}_j\|} \hat{\mathbf{e}}_{i,j}, \tag{6.13}$$

$$\mathbf{U}_i^{\mathrm{PEV}} = \sum_{j=i, j\neq i}^{\mu} \mathrm{PEV}(\mathbf{s}_i, \mathbf{s}_j), \tag{6.14}$$

where $\hat{\mathbf{e}}_{ij}$ is a unit vector that is based on $\mathbf{s}_j$ and points in the direction of straight line connecting $\mathbf{s}_i$ and $\mathbf{s}_j$ point away from $\mathbf{s}_i$. The parameter $\theta$ is defined in the same way as PE was defined. Figure 6.2(b) shows the forces exerted on a point of interest and the net force exerted on it. An advantage of PEV is that its direction vector can be added to the weight vector that guides the solutions close to the reference point; this combines the improvement on uniformity and closeness to the reference point into a single step.

### 6.2.4 The UR-MEAD2 Algorithm

As previously mentioned, to generate a uniform set of solutions in the preferred region, the weight vectors should be updated to guide the solution towards the region of interest and to maximize the uniformity of the solutions. To guide solutions into the region of interest, the weight vector of the closest solution to the reference point is used to control the convergence direction of the solutions. To maintain the uniformity, a feedback mechanism based on OHV, PE or PEV is used to measure individual contributions to improving uniformity. Algorithm 6.1 shows the high-level structure of the UR-MEAD2 algorithm. First, the population of individuals ($\mathbf{P}$) is uniformly initialized within the upper ($\overline{\mathbf{x}}$) and the lower bounds ($\underline{\mathbf{x}}$), and the weight vectors are also initialized using the `init_weight` function (lines 6-10). The initial population is then evaluated using the objective functions ($\mathbf{F}$) to get the initial solution set ($\mathbf{S}$). Lines 12-52 form the main evolutionary cycle, in which the population of solution is optimized, and the weight vectors are adapted to maintain uniformity and to guide the solution towards the region of interest.

In the main evolutionary cycle, first the weight vector of the solution with the minimum Euclidean distance to the reference point ($\mathbf{r}$) is identified. On line 14 the distance of each solution from the reference point is calculated and returned by the `euclid_dist` function. Then the index of the closest solution is found using the `min_ind` function which is then used to extract the $i$th row of the weight matrix ($\mathbf{W}$). The extracted vector ($\mathbf{w}_{\mathrm{r}}^{\star}$)[1] is then used to guide the solutions towards the reference point. On lines 20-48, before the optimization process the feedback mechanism that is selected by the user is checked, and the weight vectors

[1]The subscript 'r' denotes the weight vector(s) that are generated to guide the solutions towards the *reference* point. In Algorithms 6.2-6.4, the subscript 'u' denotes the weight vector(s) used to improve the uniformity of the solutions.

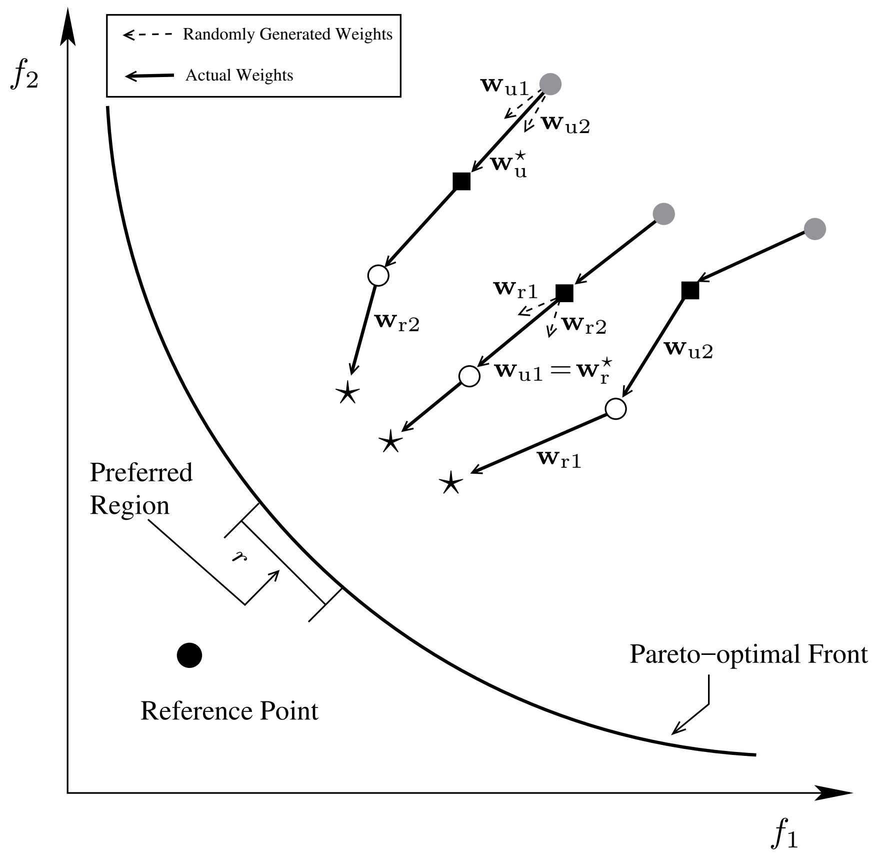



*Figure 6.3: Illustration of how the weight vectors are updated when PE and OHV are used in the proposed approach (UR-MEAD2). The updating of the weights alternates between the uniformity measure and the closeness to the reference point.* $\mathbf{w}_{u}^{\star}$ *refers to weight vector of the solution that has the best PE or OHV value, and* $\mathbf{w}_{r}^{\star}$ *is the weight vector of the solution closest to the reference point.*

are updated in order to maximize the uniformity of the solutions in the objective space. This is done by a series of feedback functions (`FDBK_OHV`, `FDBK_PE`, or `FDBK_PEV`). It should be noted that when OHV or PE are active, first a set of weight vectors ($\mathbf{W}_r$) are generated around $\mathbf{w}_r^{\star}$ and an iteration of optimization is executed to guide the solutions towards the region of interest. The weight vectors are then updated using the corresponding feedback mechanism, and the population is optimized with respect to improving the uniformity. It should be noted that the frequency of updating the weight vectors is controlled by two parameters, $\alpha$ and $\beta$. Updating the weight vectors for improving uniformity occurs every $\alpha$ iterations, and updating the weight vectors to move the solutions closer to the preferred region occurs exactly $\beta$ iterations after the updating of the weights to improve the uniformity (lines 22-30 for OHV and lines 34-42 for PE). This requires that $0 < \beta < \alpha$. This alternating process for OHV and PE is depicted in Figure 6.3. The squares, stars and circles denote solutions at successive iteration steps, and the arrows show the update direction which is based on the weight vectors associated with the individuals. It can be seen that a set of weight vectors is randomly first generated around the weight vector of the solution in the most sparse area ($\mathbf{w}^\star_\mathrm{u}$). The new weights are generated with either `FDBK_OHV` or `FDBK_PE` to move the solutions from the denser areas to sparser areas. Then, the solution with the closest distance to the reference point is identified and a new set of weight vectors is generated around its weight vector ($\mathbf{w}^\star_\mathrm{r}$). This step guides the solutions towards the reference point.

Unlike OHV and PE, the PEV method can update the weights to simultaneously improve the uniformity of the solutions and guide them towards the region of interest. For this reason, the `FDBK_PEV` takes $\mathbf{w}^\star_\mathrm{r}$ as input rather than a set of uniformly generated weights around it ($\mathbf{W}_\mathrm{r}$). For this reason the only frequency parameter needed by PEV is $\alpha$. Next, the feedback functions based on OHV, PE and PEV are explained, which are shown in Algorithms 6.2-6.4 respectively.

#### 6.2.4.1 OHV

In Algorithm 6.2, the matrix $\mathbf{M}$ contains the individual OHV values for every pair of solutions, i.e., $\mathbf{M}_{ij} = \mathtt{OHV}(\mathbf{s}_i, \mathbf{s}_j)$. Next, the values of $\mathbf{M}$ should be aggregated to associate a scalar value to each solution as an indication of its contribution to the overall uniformity. To do this, the OHV values of a solution $\mathbf{s}_i$ with every other solution $\mathbf{s}_j$ is added up and assigned to $\mathbf{s}_i$ (Equation (6.9). In Algorithm 6.2, this is done on line 14 using a vectorized implementation. The product $\mathbf{1}_\mu^\top \mathbf{M}$ gives the column sum of the matrix $\mathbf{M}$. Finally, on lines 14-20, the weight vector of the solution with the smallest aggregated OHV value ($\mathbf{w}^\star_\mathrm{u}$) is used to generate a new set of weight vectors ($\mathbf{W}$) for the next optimization cycle.

#### 6.2.4.2 PE

The feedback mechanism of PE is similar to that of OHV except that it forms a neighborhood for each solution. This is done by using the `NBHD` function that returns a binary matrix $\mathbf{\Theta}$. The $i$th column of $\mathbf{\Theta}$ contains the neighborhood information of the $i$th solution. If $\mathbf{\Theta}_{ij} = 1$, then the $j$th solution is inside the neighborhood of $i$. Therefore, the Hadamard product (entry-wise product) of M and $\mathbf{\Theta}$ excludes the values that do not belong to the neighborhood of a solution. Similar to OHV, the product $\mathbf{1}_\mu^\top(\mathbf{M} \circ \mathbf{\Theta})$ aggregates the PE values for each

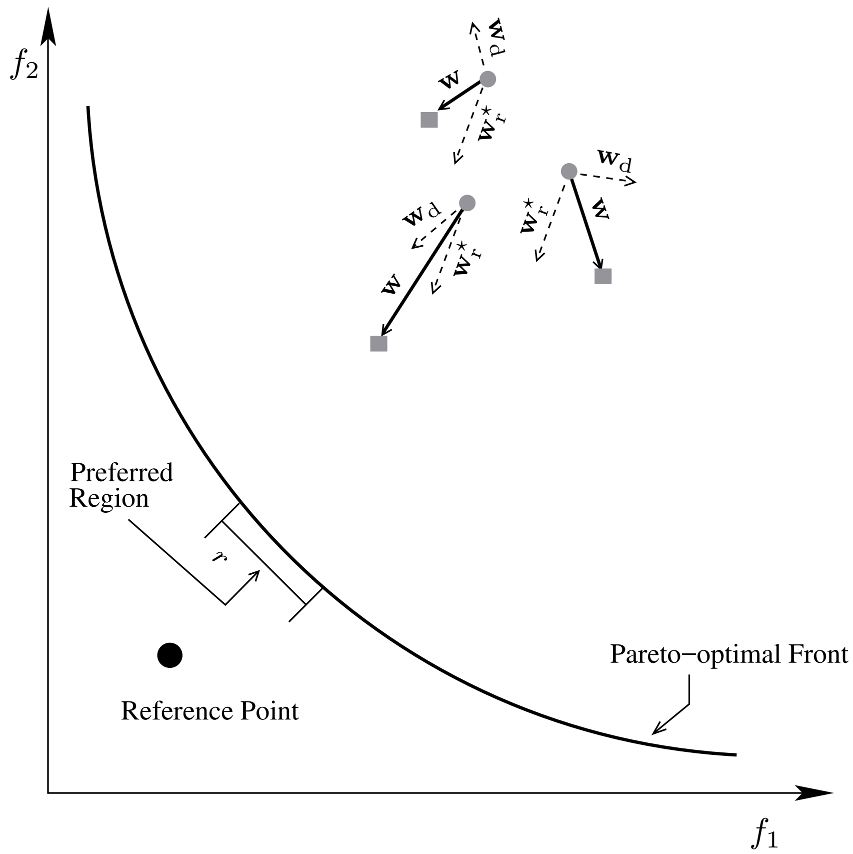



*Figure 6.4: Illustration of updating weight vectors based on direction vector and closeness to the reference point in UR-MEAD2 (PEV).* $\mathbf{w}_\mathrm{d}$ *refers to direction vector (analogous to the net virtual force exerted on each solution by all other solutions),* $\mathbf{w}_\mathrm{r}^\star$ *refers to the weight vector of the closest solution to the reference point, and w is the vectorial sum of* $\mathbf{w}_\mathrm{d}$ *and* $\mathbf{w}_\mathrm{r}^\star$*.*

solution (Equation (6.12)). Finally, on lines 16-22, the weight vector of the solution with smallest aggregated PE value ($\mathbf{w}_\mathrm{u}^\star$) is used to generate a new set of weight vectors ($\mathbf{W}$) for the next optimization cycle.

It was mentioned in Section 6.2.3 that in electrostatics, charged particles on a conductive surface flow until an equilibrium is reached. We take the net force vector on each particle (solution) to be the direction in which each particle should move in order to reach an electrostatic equilibrium. In Algorithm 6.4, each row of $\mathbf{M}$ represents the net force on each solution. In other words, the $i$th row ($\mathbf{M}_{i\bullet}$) of $\mathbf{M}$ is the net force on that all other particles (solutions) exert on the $i$th particle (solution)[2]. This is calculated using the `PEV` based on

[2]The symbol '•' references all the rows or columns in a matrix, i.e., $\mathbf{M}_{i\bullet}$ refers to the $i$th row of $\mathbf{M}$, and

Equation (6.13). Finally, the new weight vector of the $i$th solution is taken to be the vectorial sum of the net force exerted on it ($\mathbf{M}_{i\bullet}$) and the weight of the solution closest to the reference point ($\mathbf{w}_{\mathrm{r}}^{\star}$). This has the effect of accounting for both uniformity and convergence of the solutions in the region of interest. The product on line 14 results in a $\mu \times m$ matrix in which each row is a copy of $\mathbf{w}_{\mathrm{r}}^{\star}$. This is because $\mathbf{1}_{\mu}$ is a $\mu$-dimensional column vector and $\mathbf{w}_{\mathrm{r}}^{\star}$ is an $m$-dimensional row vector.

#### 6.2.4.3 PEV

Figure 6.4 shows the update process for UR-MEAD2 based on PEV. The figure shows the current weight vector of each solution and the net force exerted on each solution. The weight vector of the closest solution to the reference point is marked as $\mathbf{w}_{\mathrm{r}}^{\star}$. The vectorial sum of the net force and $\mathbf{w}_{\mathrm{r}}^{\star}$ for each solution is marked as $\mathbf{W}$ which is then used in the next optimization cycle. It should be noted that in PEV, the solutions are simultaneously guided towards the sparse area and the preferred regions. In PE and OHV when some weights are generated around the best weight vector (whether in terms of uniformity or closeness to the reference point), they will be randomly assigned to solutions. However, in PEV the weight vector of each solution is systematically updated to account for both uniformity and closeness to the reference point directions.

## 6.3 Experimental Results and Analysis

In this section, we first study the effect of using a feedback mechanism to maintain uniformity of solutions in the preferred regions and compare its performance with two existing user-preference algorithms, namely R-NSGA-II [Deb et al., 2006] and R-MEAD2 [Mohammadi et al., 2014]. The performance of different feedback mechanisms is then compared. Finally, the efficacy of our proposed feedback mechanism under different decomposition methods, i.e., Tchebycheff and PBI, is investigated.

### 6.3.1 Parameter Settings and Performance Metrics

Our experimental results are based on two widely used benchmark suites: DTLZ [Deb et al., 2001] and WFG [Huband et al., 2005]. These benchmarks are chosen to test the proposed algorithm on problems with different Pareto-optimal shapes and in particular, the WFG suite contains functions with convex, disconnected, and mixed Pareto-optimal sets.

$\mathbf{M}_{\bullet j}$ refers to the $j$th column of $\mathbf{M}$.

A single reference point ($f_i = 0.25, i \in \{1, \ldots, m\}$) is used for all test problems . All the algorithms are tested on problems with 4 to 10 objectives. In UR-MEAD2 and R-MEAD2, the $r$ parameter which determines the size of preferred region is set to 2 and in R-NSGA-II the $\epsilon$ parameter is set to 0.002 which is suggested by the authors of R-NSGA-II [Deb et al., 2006] algorithm. It should be noted that $r$ parameter in the proposed approach can be assigned to any number.The population size of all algorithms is set to 200. In order to compare the performance of R-MEAD2, UR-MEAD2 and R-NSGA-II, Hypervolume (HV) [Van Veldhuizen, 1999; Van Veldhuizen and Lamont, 1998] and Inverted Generational Distance (IGD) [Zitzler et al., 2003] are adopted in this chapter. For an accurate calculation of IGD, $20^m$, $10^m$, and $7^m$ points are sampled on the Pareto-optimal front for 4-6 objective, 7-8 objective, and 9-10 objective problems respectively, where $m$ is number of objectives. Since all algorithms used in this study are user-preference based, only the solutions inside the desired region are considered. This desired region is a hypersphere with radius $\rho$ around a so-called mid-point on the Pareto-optimal front. The mid-point is defined as the point closest to the reference point on the Pareto-optimal front. We used Tchebycheff and PBI as decomposition methods to run the proposed algorithms. To adjust the penalty parameter in PBI, we used the findings of chapter 5. The parameter $\rho$ is set to 0.2 in order to form a preferred region of a reasonable size close to the reference point. Both IGD and HV values are calculated. Summarized ttable results are presented in this chapter and complete table results are presented in Appendix C. Since results of both HV and IGD present a similar trend, in this section for the purpose of discussing the different types of experiments, HV is chosen.

### 6.3.2 The Effect of Using a Feedback Mechanism

Tables C.1-C.4 in Appendix C contain the HV values obtained by the proposed UR-MEAD2 algorithm, R-NSGA-II, and R-MEAD2 on the DTLZ and WFG benchmark problems. To establish the efficacy of UR-MEAD2, two separate statistical tests are conducted. First, all variants of UR-MEAD2 are compared with R-MEAD2 (baseline) using a series of pair-wise Wilcoxon rank-sum tests with a 95% confidence interval. Since the difference between R-MEAD2 and UR-MEAD2 is the updated mechanism of the weight vectors, this test can tell us whether the feedback mechanism of UR-MEAD2 is effective. The symbols '$\uparrow$' and '$\downarrow$' indicate that UR-MEAD2 performs significantly better and worse than R-MEAD2 respectively. The symbol '$\approx$' indicates that both algorithms are statistically similar.

In the second statistical test, all algorithms including a state-of-the-art are compared to find the best performing algorithm across all benchmark problems. To do so, the Kruskal-Wallis one-way ANOVA [Sheskin, 2003] is used to test for any significant difference between the algorithms. If such a difference is detected, a series of pair-wise Wilcoxon rank-sum tests with Holm $p$-value correction [Sheskin, 2003] is used to find the best performing algorithm[3]. Finally, all algorithms which are not significantly outperformed by any other algorithm are marked in bold. The rejection of the null hypothesis for all statistical tests is based on $\alpha = 0.05$.

Tables C.1-C.4 in Appendix C clearly show that all UR-MEAD2 variants outperform R-MEAD2 which does not have any feedback mechanism to update the weight vectors and improve uniformity. For better clarity, the win/tie/loss counts are included at the bottom of each table that summarizes the results. The reported counts clearly show the dominance of UR-MEAD2 (with OHV, PE, and PEV) over R-MEAD2 on both the DTLZ and WFG benchmark suites. It is notable, that the same trend continues for both the Tchebycheff and PBI decomposition methods. The comparison with R-NSGA-II also indicates the superiority of a feedback mechanism for adjusting the weight vectors.

Table 6.1 contains the total number of highlights for all algorithms across all benchmark suites. It clearly shows that UR-MEAD2 with PEV is the overall best performer. We can also see that the good performance of UR-MEAD2-PEV is consistent on the WFG problems which have complex Pareto-optimal fronts. The better performance of PEV compared with other feedback mechanisms such as OHV and PE is due to an explicit use of a direction vector that allows us to update the weight vectors in a meaningful way. Unlike PEV, OHV and PE randomly generate a set of solutions around the weight vector of the solution residing in the most sparse area of the objective space, whereas the PEV method uses the net force on each solution to obtain relevant gradient information for updating each individual weight. The effectiveness of such a feedback mechanism is reflected in the presented experimental results.

### 6.3.3 The Effect of Decomposition Methods on PEV

In this section, we investigate the effect of Tchebycheff and PBI with different $\theta$ values on the performance of PEV feedback mechanism that was proposed in Section 6.2.4.3.

The summarized results in table 6.1 suggest that PEV with Tchebycheff generally per-

[3] Holm correction is a simple technique for controlling the family-wise error rate. When more than one pair-wise comparisons are used to rank a set of results. Family-wise error rate is an accumulation of type I errors.

forms better than PBI. We can also see that PBI with $\theta = 0$ performs better than PBI with $\theta = 5$. However, the comparison of decomposition methods based on table 6.1 is not reliable due to effect that other algorithms such as R-NSGA-II or R-MEAD2 may have on the overall rankings. For this reason, we conducted an independent comparison between the performance of various decomposition methods on PEV, the HV results of which are reported in tables C.5 and C.6 in Appendix C for DTLZ and WFG respectively. The statistical test is based on a series of Wilcoxon rank-sum tests with Holm correction similar to the multiple comparison in the previous section. The total number of highlights for each algorithm is reported at the bottom of each table. These tables are summarized in table 6.3.

The result summary clearly indicates the superiority of Tchebycheff over PBI when it is used with the PEV feedback mechanism. This is contrary to the conventional view that the solutions obtained by PBI should be more uniformly distributed than those of Tchebycheff [Zhang and Li, 2007; Das and Dennis, 1998]. This is because PBI has a mechanism to force the convergence of solutions along a set of reference lines defined by the weight vectors, whereas Tchebycheff lacks such a mechanism. In this chapter, we have demonstrated that this shortcoming of Tchebycheff can be alleviated by using PEV as the guiding principle for improving the uniformity of solutions. This explains the better performance of Tchebycheff over that of PBI. With respect to PBI, we know that a larger $\theta$ value penalizes $d_2$ and forces the solutions to move towards the optimal front along the provided reference lines. It has been shown that larger $\theta$ values result in better uniformity of solutions [Mohammadi et al., 2015]. However, a strict constraint (i.e., a larger $\theta$) to force the solutions to converge along the reference lines may conflict with PEV's updated mechanism of the weight vectors. In other words, in the presence of a feedback mechanism such as PEV which accounts for improving the uniformity of solutions, enforcing yet another uniformity constraint may deteriorate the overall performance. This is why PBI with $\theta = 0$ performs better than PBI with $\theta = 5$ when PEV is used. Since the choice of an optimal $\theta$ value is problem dependent and may require extensive experimentation, Tchebycheff with a feedback mechanism to account for the uniformity of solutions is recommended.

**Algorithm 6.1:** $(\mathbf{S},\mathbf{P}) \leftarrow$ `UR-MEAD2`$(\mathbf{F}(\cdot), \mathbf{r}, \underline{\mathbf{x}}, \overline{\mathbf{x}}, n, m, \mu, r, v, dm)$

**Inputs:**

$\mu$ **:** the population size.
$r$ **:** determines the size of the preferred region.
$m$ **:** number of objectives.
$n$ **:** number of decision variables.
$dm$ **:** the decomposition method (Tchebycheff or PBI).
$v$ **:** version of the algorithm (`OHV|PE|PEV`).
$\alpha$ **:** update frequency of weights for uniformity
$\beta$ **:** update frequency of weights for user-preference
$\mathbf{r}$ **:** the reference point.
$\overline{\mathbf{x}}_{1\times n}$ **:** upper-bounds of the decision variables.
$\underline{\mathbf{x}}_{1\times n}$ **:** lower-bounds of the decision variables.
$\mathbf{F}(\mathbf{x})$ **:** the objective function.

---

**Variables:**

$\mathbf{P}_{\mu\times n}$ **:** population of individuals.
$\mathbf{W}_{\mu\times m}$**:** matrix of weight vectors.
$\mathbf{S}_{\mu\times m}$ **:** solution set.

---

$\mathbf{P} \leftarrow$ `rand`$(\mu, n, \underline{\mathbf{x}}, \overline{\mathbf{x}})$; $c = 1$;
$\mathbf{W} \leftarrow$ `init_weights`$(\mu, m)$;
$\mathbf{S} \leftarrow \mathbf{F}(\mathbf{P})$;
**while** *stop criteria not met* **do**
  $\mathbf{d} \leftarrow$ `euclid_dist`$(\mathbf{S}, \mathbf{r})$;
  $i \leftarrow$ `min_ind`$(\mathbf{d})$;
  $\mathbf{w}_{\mathrm{r}}^{\star} \leftarrow \mathbf{W}_{i\bullet}$;
  **if** $v =$ *'OHV'* **then**
    **if** $(c \mod \alpha) = \beta \vee c = 1$ **then**
      $\mathbf{W}_{\mathrm{r}} \leftarrow$ `uniform`$(\mu, r, \mathbf{w}_{\mathrm{r}}^{\star})$;
    $\mathbf{P} \leftarrow$ `optimize`$(\mathbf{P}, \mathbf{F}(\mathbf{x}), \mathbf{W}_{\mathrm{r}}, dm)$;
    **if** $(c \mod \alpha) = 0$ **then**
      $\mathbf{W} \leftarrow$ `FDBK_OHV`$(\mathbf{S}, \mathbf{W})$;
  **else if** $v =$ *'PE'* **then**
    **if** $(c \mod \alpha) = \beta \vee c = 1$ **then**
      $\mathbf{W}_{\mathrm{r}} \leftarrow$ `uniform`$(\mu, r, \mathbf{w}_{\mathrm{r}}^{\star})$;
    $\mathbf{P} \leftarrow$ `optimize`$(\mathbf{P}, \mathbf{F}(\mathbf{x}), \mathbf{W}_{\mathrm{r}}, dm)$;
    **if** $(c \mod \alpha) = 0$ **then**
      $\mathbf{W} \leftarrow$ `FDBK_PE`$(\mathbf{S}, \mathbf{W})$
  **else if** $v =$ *'PEV'* **then**
    **if** $(c \mod \alpha) = 0$ **then**
      $\mathbf{W} \leftarrow$ `FDBK_PEV`$(\mathbf{S}, \mathbf{w}_{\mathrm{r}}^{\star})$;
  $\mathbf{P} \leftarrow$ `optimize`$(\mathbf{P}, \mathbf{W}, \mathbf{F}(\cdot), dm)$;
  $\mathbf{S} \leftarrow \mathbf{F}(\mathbf{P})$; $c = c + 1$;
**return** $(\mathbf{S}, \mathbf{P})$;

**Algorithm 6.2:** $\mathbf{W} \leftarrow$ `FDBK_OHV`$(\mathbf{S}, \mathbf{W})$

**Inputs:**
$\mathbf{S}_{\mu \times m}$ : solution set.
$\mathbf{W}_{\mu \times m}$ : matrix of weight vectors.

---

**Variables:**
$\mathbf{M}_{\mu \times \mu}$ : matrix of OHV values for all pairs of solutions.
$\mathbf{1}_{\mu \times 1}$ : a column vector of ones with size $\mu$.

---

$\mathbf{M} \leftarrow \mathbf{0}_{\mu \times \mu}$;
**for** $i \leftarrow 1 \rightarrow \mu$ **do**
    **for** $j \leftarrow i + 1 \rightarrow \mu$ **do**
        $\mathbf{M}_{i,j} \leftarrow \mathbf{M}_{j,i} \leftarrow \texttt{OHV}(\mathbf{S}_{i\bullet}, \mathbf{S}_{j\bullet})$;
$\mathbf{m} \leftarrow \mathbf{1}_{\mu}^{\top} \mathbf{M}$;
$i \leftarrow \texttt{min\_ind}(\mathbf{m})$;
$\mathbf{w}_{\mathrm{u}}^{\star} \leftarrow \mathbf{W}_{i\bullet}$;
$\mathbf{W} \leftarrow \texttt{uniform}(\mu, r, \mathbf{w}_{\mathrm{u}}^{\star})$;
**return** $\mathbf{W}$;

**Algorithm 6.3:** $\mathbf{W} \leftarrow$ `FDBK_PE`$(\mathbf{S}, \mathbf{W})$

**Inputs:**
$\mathbf{S}_{\mu \times m}$ : solution set.
$\mathbf{W}_{\mu \times m}$ : matrix of weight vectors.

---

**Variables:**
$\mathbf{M}_{\mu \times \mu}$ : matrix of PE values for all pairs of solutions.
$\mathbf{1}_{\mu \times 1}$ : a column vector of ones with size $\mu$.
$\mathbf{\Theta}_{\mu \times \mu}$ : binary matrix of neighborhood calculations.

---

$\mathbf{M} \leftarrow \mathbf{0}_{\mu \times \mu}$;
**for** $i \leftarrow 1 \rightarrow \mu$ **do**
    **for** $j \leftarrow i \rightarrow \mu$ **do**
        $\mathbf{M}_{i,j} \leftarrow \mathbf{M}_{j,i} \leftarrow \texttt{PE}(\mathbf{S}_{i\bullet}, \mathbf{S}_{j\bullet})$;
$\mathbf{\Theta} \leftarrow \texttt{NBHD}(\mathbf{S})$;
$\mathbf{m} \leftarrow \mathbf{1}_{\mu}^{\top} (\mathbf{M} \circ \mathbf{\Theta})$;
$i \leftarrow \texttt{min\_ind}(\mathbf{m})$;
$\mathbf{w}_{\mathrm{u}}^{\star} \leftarrow \mathbf{W}_{i\bullet}$;
$\mathbf{W} \leftarrow \texttt{uniform}(\mu, r, \mathbf{w}_{\mathrm{u}}^{\star})$;
**return** $\mathbf{W}$;

**Algorithm 6.4:** $\mathbf{W} \leftarrow$ FDBK_PEV($\mathbf{S}, \mathbf{w}_{\mathrm{r}}^{\star}$)

```
Inputs:
S_{μ×m} : solution set.
w_r^⋆   : m-dimensional weight vector of the solution closest to r.
────────────────────────────
Variables:
M_{μ×m}: matrix of PEV vectors.
W_{μ×m}: matrix of weight vectors.
1_{μ×1}  : a column vector of ones with size μ.
────────────────────────────
M ← 0_{μ×m};
for i ← 1 → μ do
    for j ← i → μ do
        M_{i•} ← M_{i•} + PEV(S_{i•}, S_{j•});
W_u ← 1_μ w_r^⋆;
W ← M + W_u;
return W;
```

*Table 6.1: Number of times each algorithm has the best performance as compared to other algorithms(HV results).*

| Func. | Algorithm | Tchebycheff | PBI $\theta = 5$ | PBI $\theta = 0$ |
|---|---|---|---|---|
| DTLZ | UR-MEAD2-OHV | 3 | 2 | – |
| | UR-MEAD2-PE | 6 | 1 | – |
| | UR-MEAD2-PEV | 30 | 14 | 22 |
| | R-MEAD2 | 3 | 3 | – |
| | R-NSGA-II | 0 | 1 | – |
| WFG | UR-MEAD2-OHV | 1 | 2 | – |
| | UR-MEAD2-PE | 14 | 1 | – |
| | UR-MEAD2-PEV | 38 | 19 | 34 |
| | R-MEAD2 | 4 | 0 | – |
| | R-NSGA-II | 7 | 8 | – |

*Table 6.2: Number of times each algorithm has the best performance as compared to other algorithms.(IGD results)*

| Func. | Algorithm | Tchebycheff | PBI $\theta = 5$ | PBI $\theta = 0$ |
|---|---|---|---|---|
| DTLZ | UR-MEAD2-OHV | 3 | 0 | – |
| | UR-MEAD2-PE | 3 | 3 | – |
| | UR-MEAD2-PEV | 36 | 17 | 22 |
| | R-MEAD2 | 0 | 0 | – |
| | R-NSGA-II | 0 | 0 | – |
| WFG | UR-MEAD2-OHV | 5 | 3 | – |
| | UR-MEAD2-PE | 8 | 5 | – |
| | UR-MEAD2-PEV | 49 | 9 | 47 |
| | R-MEAD2 | 2 | 0 | – |
| | R-NSGA-II | 10 | 11 | – |

*Table 6.3: Number of times UR-MEAD2-PEV has the best performance in different decomposition methods(HV Values)*

| | | | PBI | |
|---|---|---|---|---|
| Func. | Algorithm | Tchebycheff | $\theta = 5$ | $\theta = 0$ |
| DTLZ | UR-MEAD2-PEV | 27 | 5 | 10 |
| WFG | UR-MEAD2-PEV | 34 | 8 | 21 |

*Table 6.4: Number of times UR-MEAD2-PEV has the best performance in different decomposition methods(IGD Values)*

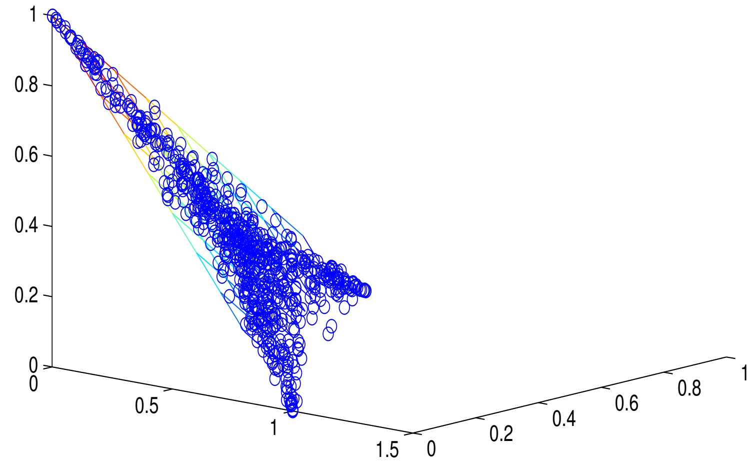
| | | | PBI | |
|---|---|---|---|---|
| Func. | Algorithm | Tchebycheff | $\theta = 5$ | $\theta = 0$ |
| DTLZ | UR-MEAD2-PEV | 20 | 10 | 14 |
| WFG | UR-MEAD2-PEV | 41 | 2 | 23 |

### 6.3.4 Behavior of UR-MEAD2 without reference point

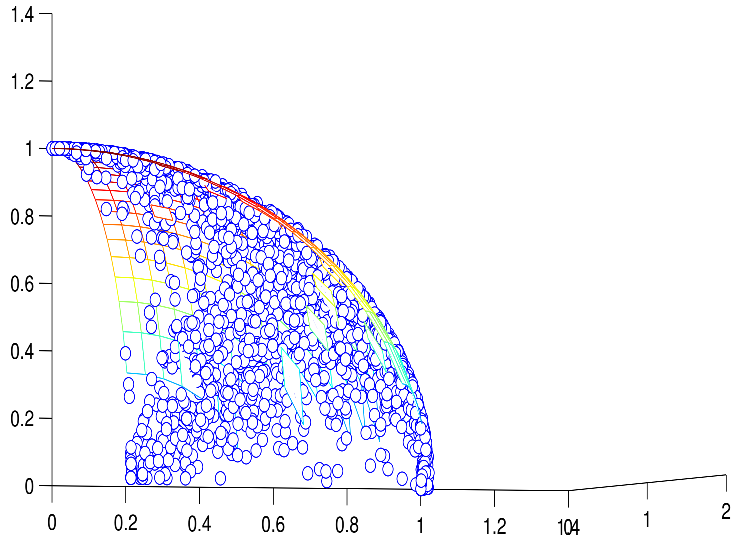
In this section, the performance of UR-MEAD2 without using any preference information has been investigated. To do this, UR-MEAD2 with PEV has been chosen. In order to ignore the effect of reference point in UR-MEAD2 with PEV, $\mathbf{w}_{\mathrm{r}}^{\star}$ vector has been removed and $\mathbf{w}_{\mathrm{d}}$ is the only vector uses during the optimization process. Figures 6.5(a) and 6.5(b) show the convergence behavior of UR-MEAD2 (with PEV) on DTLZ1 and DTLZ2 respectively. In this experiment, Tchebycheff is used as decomposition method. As can be observed, most solutions have been converged relatively diversely on the entire Pareto front. Figures 6.6(a) and 6.6(b) show the objective value plots of UR-MEAD2 (with PEV) on DTLZ2 and WFG7 for 10 objectives. In this experiment, PBI is used as decomposition method. It can be observed that the objective values varied between zero and one which indicates the reasonable coverage of Pareto front by solutions.

### 6.3.5 Sensitivity analysis of UR-MEAD2 to weight vector update frequency

Giagkiozis et al. [2013b] suggested that the adaptation of the weight vectors is ill-advised and can affect the convergence rate of decomposition-based EMO algorithms. In their study, they used only Generational Distance (GD) as their measure of performance, which does not take the uniformity of the solutions into account. It is conceivable that the adaptation of the weights may have some detrimental effect on the convergence rate; however, the improvement

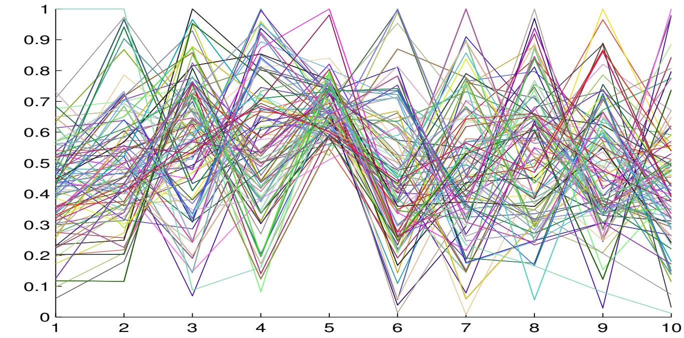



(a) 3-objective DTLZ1

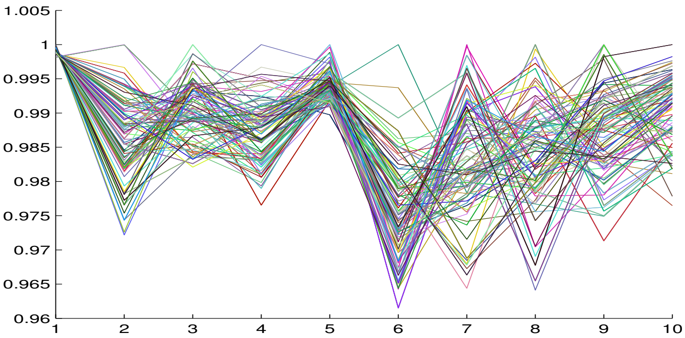



(b) 3-objective DTLZ2

*Figure 6.5: Convergence of solutions on 3-objective DTLZ1 and DTLZ2 using UR-MEAD2 with Tchebycheff.*




(a) 10-objective DTLZ2




(b) 10-objective WFG7

*Figure 6.6: Objective value plots on 10-objective DTLZ2 and WFG7 using UR-MEAD2 with PBI.*

one can gain in terms of diversity may result in a better overall performance as measured by IGD and/or HV (see the following discussion and table 6.5). It should also be noted that the incorporation of user-preference can improve the convergence behavior of the algorithm by focusing the search on a smaller region rather than the entire Pareto front [Mohammadi et al., 2012]. This can compensate for the potentially slower convergence due to adaptation of the weight vectors.

In this section, we study the sensitivity of UR-MEAD2 to the frequency of updating the weight vectors. A set of parameters is added to the UR-MEAD2 algorithm to control the frequency of updating the weight vectors. For PE and OHV there are two control parameters $\alpha$ and $\beta$, which control the frequency of updating the weights for closeness to reference point and uniformity of the solutions respectively. For PEV only one control parameter ($\alpha$) is needed because both uniformity and closeness to the reference point are handled in a single step. Table 6.5 contains the experimental results of UR-MEAD2 (with Tchebycheff) using various values for $\alpha$ and $\beta$ on two selected functions. The values of $\alpha$ and $\beta$ are chosen to ensure an equal spacing between adaptation of the weights. It should be noted that lower values of the frequency parameters result in a higher adaptation frequency. The experiments are based on three frequency levels to resemble low, medium, and high frequencies.

To investigate the effect of different frequency levels, the low and medium frequency levels are compared with the high level which is the baseline, where the adaptation occurs at every iteration. To ensure statistical significance, a series of pair-wise Wilcoxon rank-sum tests with a 95% confidence interval is used. The symbols '↑' and '↓' indicate that the baseline performs significantly better and worse than the other cases respectively. The symbol '≈' indicates that both cases are statistically similar. Table 6.5 clearly shows that by decreasing the adaptation frequency of the weights the overall performance of the algorithm drops. This suggests that the degradation in performance as suggested by Giagkiozis et al. [2013b] may not be very severe and the incorporation of preference information can easily compensate for it, while the adaptation mechanism improves the uniformity of the solutions.

### 6.4 Chapter Summary

In this chapter, we have proposed several mechanisms that allow us to assess the contribution of each solution towards improving the overall uniformity of the solution set, and use this information to update the weight vectors dynamically such that the overall uniformity is maintained. The general idea is to identify the sparse areas of the objective space, and move the solutions from denser areas to more sparse areas. The proposed approach UR-MEAD2 uses three uniformity metrics to measure the uniformity of solutions in objective space. These metrics are: Overlapping Hypervolume (OHV), Potential Energy (PE), and Potential Energy with Direction Vector (PEV). OHV and PE are two existing metrics which have been modified to fit the purpose of this chapter and PEV is proposed in this chapter. PEV uses the idea of electrostatic equilibrium to calculate the direction in which each solution should move in order to improve the overall uniformity. The experimental results suggest that the proposed algorithm outperforms the state of-the-art algorithm on a wide range of benchmark problems with complex Pareto-optimal fronts.

*Table 6.5: HV values of UR-MEAD2 with TE using different frequency parameters of updating weight vectors.*

| Alg. | Func | # Obj | Frequencies<br>High $\alpha = 1$ | <br>Medium $\alpha = 5$ | <br>Low $\alpha = 30$ |
|---|---|---|---|---|---|
| PEV | DTLZ1 | 4 | 10.33e-02 | 10.24e-02$^{\approx}$ | 5.21e-02$^{\downarrow}$ |
| | | 5 | 9.98e-02 | 7.91e-02$^{\downarrow}$ | 10.22e-03$^{\downarrow}$ |
| | | 6 | 1.29e-02 | 0.52e-02$^{\downarrow}$ | 4.21e-03$^{\downarrow}$ |
| | | 7 | 0.80e-02 | 0.32e-02$^{\downarrow}$ | 12.88e-03$^{\downarrow}$ |
| | | 8 | 2.76e-01 | 2.01e-01$^{\approx}$ | 8.01e-03$^{\downarrow}$ |
| | | 9 | 3.25e-01 | 2.15e-01$^{\downarrow}$ | 12.09e-03$^{\downarrow}$ |
| | | 10 | 2.36e-02 | 1.00e-02$^{\downarrow}$ | 10.04e-03$^{\downarrow}$ |
| PEV | DTLZ5 | 4 | 16.80e-02 | 14.39e-02$^{\downarrow}$ | 10.09e-03$^{\downarrow}$ |
| | | 5 | 12.95e-02 | 10.98e-02$^{\downarrow}$ | 11.30e-03$^{\downarrow}$ |
| | | 6 | 8.00e-02 | 6.78e-02$^{\downarrow}$ | 10.31e-03$^{\downarrow}$ |
| | | 7 | 4.60e-02 | 3.26e-02$^{\downarrow}$ | 6.03e-03$^{\downarrow}$ |
| | | 8 | 14.25e-02 | 13.81e-02$^{\downarrow}$ | 9.17e-03$^{\downarrow}$ |
| | | 9 | 14.38e-03 | 12.18e-02$^{\downarrow}$ | 12.10e-03$^{\downarrow}$ |
| | | 10 | 15.10e-02 | 14.01e-02$^{\downarrow}$ | 11.29e-02$^{\downarrow}$ |
| win/tie/loss counts | | | | 0/2/12 | 0/0/14 |
| Alg. | Func | # Obj | $\alpha = 2, \beta = 1$ | $\alpha = 10, \beta = 5$ | $\alpha = 60, \beta = 30$ |
| PE | DTLZ1 | 4 | 8.12e-02 | 7.98e-02$^{\approx}$ | 7.38e-03$^{\downarrow}$ |
| | | 5 | 9.92e-02 | 8.90e-02$^{\downarrow}$ | 14.00e-03$^{\downarrow}$ |
| | | 6 | 1.01e-02 | 0.45e-02$^{\downarrow}$ | 9.89e-03$^{\downarrow}$ |
| | | 7 | 1.26e-02 | 0.95e-02$^{\downarrow}$ | 16.90e-03$^{\downarrow}$ |
| | | 8 | 1.31e-02 | 0.89e-02$^{\downarrow}$ | 4.56e-03$^{\downarrow}$ |
| | | 9 | 1.33e-02 | 0.95e-02$^{\downarrow}$ | 11.03e-03$^{\downarrow}$ |
| | | 10 | 1.59e-02 | 0.51e-02$^{\downarrow}$ | 9.20e-03$^{\downarrow}$ |
| PE | DTLZ5 | 4 | 11.11e-02 | 11.26e-02$^{\approx}$ | 9.42e-03$^{\downarrow}$ |
| | | 5 | 8.36e-02 | 6.91e-02$^{\downarrow}$ | 7.22e-03$^{\downarrow}$ |
| | | 6 | 7.37e-02 | 7.41e-02$^{\uparrow}$ | 11.98e-03$^{\downarrow}$ |
| | | 7 | 8.68e-02 | 7.31e-02$^{\downarrow}$ | 6.30e-03$^{\downarrow}$ |
| | | 8 | 13.87e-03 | 11.29e-03$^{\downarrow}$ | 10.41e-04$^{\downarrow}$ |
| | | 9 | 6.62e-02 | 5.22e-02$^{\downarrow}$ | 13.60e-03$^{\downarrow}$ |
| | | 10 | 16.53e-02 | 14.91e-02$^{\downarrow}$ | 11.99e-03$^{\downarrow}$ |
| win/tie/loss counts | | | | 1/2/11 | 0/0/14 |
| OHV | DTLZ1 | 4 | 7.65e-02 | 7.41e-02$^{\approx}$ | 4.39e-03$^{\downarrow}$ |
| | | 5 | 9.74e-02 | 8.78e-02$^{\downarrow}$ | 22.71e-03$^{\downarrow}$ |
| | | 6 | 1.02e-02 | 0.50e-02$^{\downarrow}$ | 10.13e-03$^{\downarrow}$ |
| | | 7 | 1.18e-02 | 0.91e-02$^{\downarrow}$ | 14.04e-03$^{\downarrow}$ |
| | | 8 | 1.31e-02 | 1.00e-02$^{\downarrow}$ | 7.02e-03$^{\downarrow}$ |
| | | 9 | 1.42e-02 | 1.05e-02$^{\downarrow}$ | 10.99e-03$^{\downarrow}$ |
| | | 10 | 1.58e-02 | 1.12e-02$^{\downarrow}$ | 8.01e-03$^{\downarrow}$ |
| OHV | DTLZ5 | 4 | 10.43e-02 | 10.03e-02$^{\approx}$ | 8.61e-03$^{\downarrow}$ |
| | | 5 | 7.67e-02 | 5.58e-02$^{\downarrow}$ | 9.61e-03$^{\downarrow}$ |
| | | 6 | 6.81e-02 | 5.88e-02$^{\downarrow}$ | 8.40e-03$^{\downarrow}$ |
| | | 7 | 9.26e-02 | 8.02e-02$^{\downarrow}$ | 7.21e-03$^{\downarrow}$ |
| | | 8 | 18.47e-02 | 16.36e-02$^{\downarrow}$ | 10.33e-03$^{\downarrow}$ |
| | | 9 | 4.57e-02 | 3.07e-02$^{\downarrow}$ | 8.78e-03$^{\downarrow}$ |
| | | 10 | 7.61e-02 | 6.09e-02$^{\downarrow}$ | 9.18e-03$^{\downarrow}$ |
| win/tie/loss counts | | | | 0/2/12 | 0/0/14 |

# Chapter 7

# A Novel Performance Metric for User-preference based Algorithms

## 7.1 Introduction

The previous chapters proposed several user-preference based evolutionary multi-objective and many-objective algorithms (R-MEAD, R-MEAD2 and UR-MEAD2) and their performances have been compared with other existing works. Despite the growing interest in developing user-preference based algorithms, very few performance measures have been developed to facilitate a fair comparison of such algorithms. As a result, in this chapter we develop a metric to evaluate the performance of use-preference based algorithms.

A metric which has been recently developed by Wickramasinghe *et al.* [Wickramasinghe et al., 2010] is specifically designed for comparing user-preference based EMO algorithms. However, a major drawback of this metric is that its results can be misleading, depending on the choice of the reference point (ch.2 section 2.6.4). An ideal metric for user-preference based algorithms should have the following properties:

1. Form a preferred region closest to the reference point provided by the user;
2. Measure both convergence and diversity of the solutions with respect to the preferred region;
3. Be independent of knowledge of Pareto-optimal front for its calculation and;
4. Scale as well as the number of objectives increases.

Although many performance metrics have been proposed for comparing EMO algorithms, none of them is suitable to evaluate user-preference based EMO algorithms by satisfying all four properties.

The proposed metric in this chapter borrows the idea of a *reference set* [Mohan and Mehrotra, 2011] from cardinality-based metrics to form a *composite front* that acts as a replacement for the Pareto-optimal front. This composite front is then used to define a preferred region based on the location of a user-supplied reference point. Once the preferred region is defined, existing EMO metrics can be applied to the preferred region.

The chapter is organized in the following way. Section 7.2 describes the details of the proposed metric. Experimental results and their analysis are presented in Section 7.3, and Section 7.4 concludes the chapter.

## 7.2 Proposed Metric (UPCF)

In this section, we propose a metric to evaluate the performance of user-preference based multi-objective evolutionary algorithms.

In a nutshell, the proposed metric which hereafter is called user-preference metric based on a composite front (UPCF), merges the solution set of all algorithms and uses the non-dominated solutions of the merged solution sets as a replacement for the Pareto-optimal front. This so-called *composite front* is a type of reference set commonly used in several cardinality-based metrics. The composite front is then used to form a preferred region based on the position of a reference point provided by the decision maker. Finally, the performance of each algorithm is measured by calculating IGD or HV for solutions of each algorithm which are within the preferred region. UPCF can be coupled with either IGD or HV. In this chapter both these popular techniques are used for the sake of comparison. Measuring both convergence and diversity of the solution set makes both IGD and HV desirable candidates for this new metric. The following details the procedure for applying UPCF:

### Step 1 - Generating a Composite Front

The solution set of all the algorithms to be compared are merged, and all non-dominated solutions from this merged set are placed in another set called the *composite front*. In Figure 7.1, squares and circles show the solution sets for two different user-preference based algorithms. The solutions shown as black squares form the composite front, and the solutions shown as gray circles are those dominated by at least one solution in the composite front.

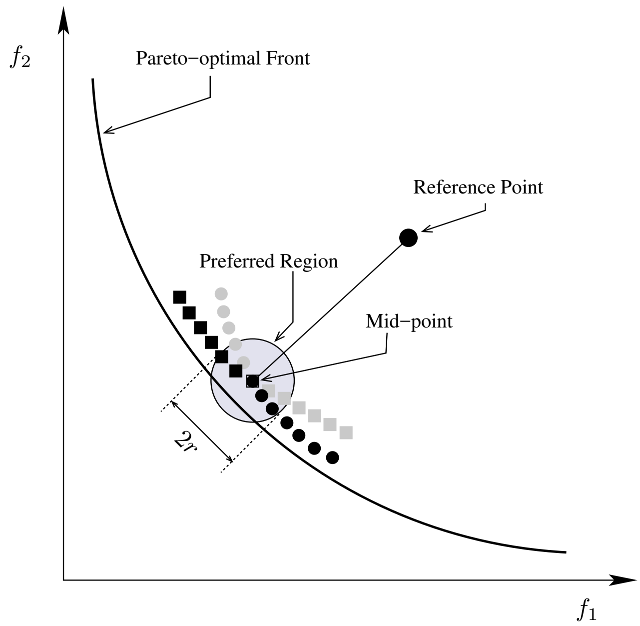



*Figure 7.1: An example of a composite front which is used to define a preferred region.*

### Step 2 - Generating a Preferred Region For Each Reference Point

To define the preferred region, the Euclidean distances between all the solutions in the composite front and a reference point is calculated. The solution with the least distance to the reference point is then identified. This point is called *mid-point* as shown in Figure 7.1. Finally, the solutions within $r$ distance of the *mid-point* are considered to be in the preferred region. The parameter $r$ is specified by the user, which determines the size of the preferred region. In real-world applications where objectives do not have the same units, objectives should be normalized otherwise the parameter $r$ will not be meaningful.

### Step 3 - Calculating IGD and HV

IGD and HV are calculated based on the solutions inside the preferred region. To calculate the IGD values, instead of using sample points on the Pareto-optimal front, the solutions in the composite front are used. The IGD based on the composite front is abbreviated as IGD-CF while HV based on the composite front is abbreviated as HV-CF.

A major advantage of UPCF is that it can be applied to situations where the Pareto-optimal front is unknown. This property has significant implications for the scalability and computational cost of the metric. For example, many distance-based approaches, such as GD

*Table 7.1: Nadir points for two and three-objective test problems*

| Test problem | Nadir Point |
|---|---|
| ZDT1 | (0.87, 0.30) |
| ZDT2 | (1.00, 0.60) |
| ZDT3 | (0.85, 1.00) |
| ZDT4 | (1.00, 28.59) |
| ZDT6 | (1.00, 2.77) |
| DTLZ1 | (2.00, 1.03, 2.00) |
| DTLZ2 | (0.37, 0.37, 1.00) |
| DTLZ3 | (0.85, 1.00, 1.00) |
| DTLZ4 | (1.00, 1.00, 0.96) |
| DTLZ5 | (0.48, 0.48, 1.00) |
| DTLZ6 | (0.43, 0.43, 1.00) |

and IGD, require a set of sample points on the Pareto-optimal front. For small problems with only two or three-objectives, it is easy to generate a set of points on the Pareto-optimal front. However, as the number of objectives increases, the cost of this process grows exponentially. In addition to the computational cost of sampling the Pareto-optimal front, for many real-world problems the Pareto-optimal front is either very difficult to generate or is completely unknown.

### 7.3 Simulation Results

In order to evaluate the effectiveness of the proposed metric, we compared it with IGD metric based on the true Pareto-optimal front which is liked a based-line for our comparison and we called it IGD-OF. Since IGD-OF uses Pareto-optimal front information, it can be pretty reliable to be used as based-line. R-NSGA-II [Deb et al., 2006], R-MEAD-Te [Mohammadi et al., 2012], and R-MEAD-Ws [Mohammadi et al., 2012] are used to run the proposed metrics on them. R-NSGA-II is a modified version of the popular NSGA-II [Deb et al., 2002] algorithm that can handle multiple reference points. R-MEAD-Te and R-MEAD-Ws are two user-preference based algorithms which are based on the MOEA/D algorithm [Zhang and Li, 2007]. R-MEAD-Te and R-MEAD-Ws rely on the Tchebycheff [Miettinen, 1999] and Weighted-Sum [Miettinen, 1999] decomposition methods respectively, to convert a multi-objective optimization problem into a single-objective problem. To evaluate the performance of UPCF on many-objective problems, it has been tested on R-MEAD2-TE, UR-MEAD2-PEV(TE) and R-NSGA-II.

*Table 7.2: Nadir points for 3,5 and 7 objective test problems*

| Test problem | #obj | Nadir Point |
|---|---|---|
| DTLZ1 | 5 | (4.75, 5.47, 5.71, 7.25, 0.52) |
| | 7 | (15.41, 19.48, 16.74, 16.99, 12.54, 16.02, 16.81) |
| | 10 | (9.25, 10.29, 12.32, 7.03, 9.10, 10.39, 9.83, 9.08, 9.96, 0.63) |
| DTLZ2 | 5 | (1.58, 1.30, 1.12, 1.20, 1.27) |
| | 7 | (0.63, 0.68, 1.19, 1.12, 1.33, 1.17, 1.38) |
| | 10 | (0.53, 0.83, 0.51, 1.17, 1.06, 1.35, 1.15, 1.32, 1.09, 1.08) |
| DTLZ3 | 5 | (1.03, 0.50, 1.32, 1.06, 1.19) |
| | 7 | (5.46, 1.70, 9.84, 8.83, 1.90, 9.58, 5.15) |
| | 10 | (4.61, 7.00, 5.52, 4.11, 6.22, 3.80, 2.78, 8.91, 9.60, 11.97) |
| DTLZ4 | 5 | (1.00, 1.00, 1.00, 1.00, 1.01) |
| | 7 | (1.02, 1.05, 1.04, 1.05, 1.03, 1.01, 1.03) |
| | 10 | (1.07, 1.01, 1.04, 1.02, 1.03, 1.06, 1.03, 1.06, 1.00, 1.04) |
| DTLZ5 | 5 | (0.26, 0.29, 0.36, 0.65, 1.33) |
| | 7 | (0.23, 0.29, 0.40, 0.33, 0.51, 0.65, 1.37) |
| | 10 | (0.24, 0.20, 0.21, 0.25, 0.37, 0.29, 0.32, 0.59, 0.74, 1.82) |
| DTLZ6 | 5 | (0.55, 0.54, 0.70, 1.14, 2.44) |
| | 7 | (0.54, 0.44, 0.43, 0.68, 0.75, 1.16, 3.72) |
| | 10 | (0.41, 0.28, 0.37, 0.26, 0.52, 0.57, 0.83, 1.00, 1.12, 3.14) |

To understand whether UPCF coupled with IGD-CF and HV-CF is an accurate metric to measure the performance of a user-preference based algorithm, their results have been compared with our based-line IGD-OF. In addition to IGD-CF and HV-CF, the average number of solutions that each algorithm contributes to the composite front (NS-CF) is also reported for further analysis.

The benchmark problems used in this chapter are ZDT1-ZDT4 and ZDT6 functions for two-objective problems, and DTLZ1-DTLZ6 functions for three and many-objective problems. We used $(0.7, 0.2)$ and $(0.2, 0.2, 0.6)$ as reference points for two-objective and three-objective test problems respectively. A different reference point $(0.2, 0.4, 0.9)$ was used for DTLZ1 since the point $(0.2, 0.2, 0.6)$ is located on its Pareto-optimal front. A single reference point ($f_i = 0.25 \in 1, ..., m$) is used for 5,7 and 10 objectives. The population size for two-objective test problems was set to 50. The number of iterations in each run is 150 for ZDT1 and ZDT2, 300 for ZDT3, and 500 for ZDT4 and ZDT6.

The population size is set to 200 for three, five and seven-objective problems. For ten objective problems the population size is set to 350. The number of iterations in each run is 200 for DTLZ1, DTLZ2, DTLZ5 and DTLZ6, and 400 for DTLZ3 and DTLZ4.

To specify the size of the preferred region on the composite front and the Pareto-optimal front, parameter $r$ (see Section 7.2) is set to 0.1 for all test problems. The $\epsilon$ parameter of R-NSGA-II is set to 0.001 for two-objective problems and 0.002 for three and many-objective problems. The radius parameter of R-MEAD-Te and R-MEAD-Ws, which can be used to control the size of the preferred region, is set to 0.05 and 0.02 for Tchebycheff and Weighted-Sum approaches respectively. The initial population size is set to 100 and 250 for two and three-objective problems respectively. The nadir point used by HV-CF is calculated by taking the worst objective value for each of the objective functions from all solutions generated by all three algorithms in 25 independent runs. Table 7.1 shows the nadir points calculated for two and three-objective problems and table 7.2 shows nadir points for many-objective problems.

Tables 7.3 and 7.4 show the mean and the standard deviation for 25 independent runs of R-NSGA-II, R-MEAD-Te and R-MEAD-Ws using four different performance measures. As mentioned previously, IGD-OF is not part of the proposed metric and is solely used as a baseline against which other algorithms are compared. The last three columns are the results of t-test (p-values) using a 95% confidence interval.

*Table 7.3: Results on the two-objective test problems. The mean, p-value and standard deviation of 25 independent runs are reported. The statistical significance results are based on the t-test using a 95% confidence interval.*

| Func. | Metric | R-MEAD-Te | R-MEAD-Ws | R-NSGA-II | R-MEAD-Te vs R-NSGA-II | R-MEAD-Te vs R-MEAD-Ws | R-MEAD-Ws vs R-NSGA-II |
|---|---|---|---|---|---|---|---|
| ZDT1 | HV-CF | 2.67e-02 (6.10e-03) | **3.66e-02 (8.56e-04)** | 2.70e-02 (2.20e-03) | 8.30e-01 | 4.79e-08 | 1.84e-16 |
| | IGD-CF | 5.70e-03 (1.43e-02) | **4.31e-04 (2.70e-04)** | 2.50e-03 (1.00e-03) | 2.74e-01 | 2.57e-09 | 1.24e-11 |
| | NS-CF | 3.96e+01 (1.21e+01) | 3.68e+01 (3.60e+00) | **4.95e+01 (9.18e-01)** | 4.44e-04 | 3.24e-01 | 2.32e-15 |
| | IGD-OF | 5.60e-03 (1.02e-02) | **7.44e-04 (3.28e-04)** | 3.50e-03 (7.06e-04) | 3.18e-01 | 2.81e-02 | 2.99e-14 |
| ZDT2 | HV-CF | **5.81e-02 (1.20e-03)** | 0.00e+00 (0.00e+00) | 5.16e-02 (1.13e-02) | 9.20e-03 | 1.88e-42 | 8.45e-18 |
| | IGD-CF | **3.60e-03 (1.30e-03)** | 1.04e-01 (1.23e-02) | 1.05e-02 (2.42e-02) | 6.60e-03 | 6.61e-23 | 4.42e-17 |
| | NS-CF | **4.75e+01 (3.51e+00)** | 0.00e+00 (0.00e+00) | 2.77e+01 (1.61e+01) | 3.93e-06 | 6.61e-29 | 8.90e-09 |
| | IGD-OF | **1.00e-03 (1.01e-04)** | 2.80e-02 (7.08e-18) | 3.60e-03 (5.10e-03) | 1.60e-09 | 5.40e-60 | 3.18e-18 |
| ZDT3 | HV-CF | **1.12e-01 (1.33e-01)** | 7.77e-02 (1.12e-01) | 3.16e-02 (6.64e-02) | 2.83e-02 | 1.39e-01 | 1.34e-01 |
| | IGD-CF | 3.12e-01 (7.46e-01) | 3.12e-01 (7.47e-01) | 3.11e-01 (7.29e-01) | 8.49e-01 | 6.67e-01 | 9.13e-01 |
| | NS-CF | **2.15e+01 (2.40e+01)** | **2.20e+01(2.53e+01)** | 4.80e+00 (9.46e+00) | 9.50e-03 | 9.40e-01 | 1.02e-02 |
| | IGD-OF | 1.74e-01 (4.84e-02) | 1.80e-01 (4.42e-02) | 1.78e-01 (6.04e-02) | 8.65e-01 | 9.77e-01 | 8.80e-01 |
| ZDT4 | HV-CF | **9.63e+00 (1.17e-01)** | 0.00e+00 (0.00e+00) | **9.48e+00 (6.57e-01)** | 3.40e-01 | 1.06e-47 | 1.43e-29 |
| | IGD-CF | **7.20e-04 (6.49e-04)** | 8.96e-02 (1.23e-02) | **1.00e-03 (7.42e-04)** | 1.56e-01 | 1.38e-22 | 1.30e-22 |
| | NS-CF | **3.32e+01 (1.34e+01)** | 0.00e+00 (0.00e+00) | **3.26e+01 (1.96e+01)** | 9.32e-01 | 6.94e-12 | 1.62e-08 |
| | IGD-OF | **2.60e-03 (1.34e-04)** | 5.42e-02 (2.83e-17) | **2.60e-03 (1.00e-03)** | 8.67e-01 | 9.70e-64 | 8.66e-43 |
| ZDT6 | HV-CF | **4.22e-01 (1.95e-02)** | 0.00e+00 (0.00e+00) | 2.94e-01 (2.32e-01) | 1.01e-02 | 9.06e-34 | 1.54e-06 |
| | IGD-CF | **8.50e-04 (1.10e-03)** | 1.23e-01 (4.50e-03) | 4.81e-02 (5.97e-02) | 6.46e-04 | 4.51e-34 | 9.71e-07 |
| | NS-CF | **5.00e+01 (0.00e+00)** | 0.00e+00 (0.00e+00) | 2.68e+00 (3.92e+00) | 1.02e-27 | 0.00e+00 | 2.30e-03 |
| | IGD-OF | **4.40e-03 (1.80e-03)** | 9.86e-02 (5.67e-17) | 3.94e-02 (4.54e-02) | 7.86e-04 | 3.40e-43 | 9.78e-07 |

*Table 7.4: Results on the three-objective test problems. The mean, p-value and standard deviation of 25 independent runs are reported. The statistical significance results are based on the t-test using a 95% confidence interval.*

| Func. | Metric | R-MEAD-Te | R-MEAD-Ws | R-NSGA-II | R-MEAD-Te vs R-NSGA-II | R-MEAD-Te vs R-MEAD-Ws | R-MEAD-Ws vs R-NSGA-II |
|---|---|---|---|---|---|---|---|
| DTLZ1 | HV-CF | 7.33e-01 (9.98e-01) | 0.00e+00 (0.00e+00) | **1.93e+00 (7.18e-02)** | 4.00e-06 | 1.20e-03 | 5.12e-36 |
| | IGD-CF | 4.69e-02 (5.52e-02) | 6.46e-02 (4.57e-02) | **1.20e-03 (2.70e-03)** | 2.90e-04 | 2.00e-03 | 1.63e-07 |
| | NS-CF | 3.94e+01 (6.77e+01) | 0.00e+00 (0.00e+00) | **1.70e+02 (5.98e+01)** | 4.27e-07 | 7.70e-03 | 3.28e-13 |
| | IGD-OF | 4.12e-02 (2.73e-02) | 6.12e-02 (1.42e-17) | **4.20e-03 (1.20e-03)** | 6.54e-07 | 1.20e-03 | 9.25e-42 |
| DTLZ2 | HV-CF | **5.06e-04 (1.01e-04)** | 0.00e+00 (0.00e+00) | 4.28e-04 (1.07e-05) | 7.45e-04 | 1.06e-18 | 3.46e-04 |
| | IGD-CF | **1.46e-04 (1.65e-05)** | 5.12e-02 (3.60e-04) | 8.83e-04 (3.73e-05) | 7.97e-32 | 1.85e-53 | 2.36e-53 |
| | NS-CF | 1.82e+02 (5.34e+00) | 0.00e+00 (0.00e+00) | **2.00e+02 (0.00e+00)** | 1.04e-14 | 1.61e-38 | 0.00e+00 |
| | IGD-OF | **2.10e-03 (4.42e-05)** | 5.94e-02 (0.00e+00) | 3.00e-03 (3.96e-05) | 7.11e-03 | 1.95e-76 | 2.01e-77 |
| DTLZ3 | HV-CF | 2.11e-02 (1.94e-02) | 0.00e+00 (0.00e+00) | **3.74e-02 (3.28e-04)** | 3.08e-04 | 1.42e-05 | 4.44e-51 |
| | IGD-CF | 3.22e-02 (3.49e-02) | 6.12e-02 (1.00e-02) | **7.64e-04 (8.05e-04)** | 1.79e-04 | 1.25e-05 | 5.22e-02 |
| | NS-CF | 8.94e+01 (9.51e+01) | 0.00e+00 (0.00e+00) | **2.00e+02 (0.00e+00)** | 5.39e-06 | 8.82e-05 | 0.00e+00 |
| | IGD-OF | 2.85e-02 (2.80e-02) | 5.94e-02 (0.00e+00) | **3.10e-03 (2.48e-04)** | 1.46e-04 | 1.11e-05 | 2.70e-58 |
| DTLZ4 | HV-CF | 2.02e-02 (1.57e-02) | 0.00e+00 (0.00e+00) | **2.97e-02 (1.80e-03)** | 5.70e-03 | 1.10e-06 | 4.24e-31 |
| | IGD-CF | 2.69e-02 (3.36e-02) | 6.09e-02 (9.00e-03) | **8.95e-04 (8.83e-04)** | 8.82e-04 | 1.07e-06 | 4.32e-21 |
| | NS-CF | 8.74e+01 (8.45e+01) | 0.00e+00 (0.00e+00) | **2.00e+02 (6.24e-01)** | 7.24e-07 | 2.67e-05 | 7.43e-62 |
| | IGD-OF | 2.42e-02 (2.70e-02) | 5.94e-02 (0.00e+00) | **3.00e-03 (1.16e-04)** | 6.31e-04 | 2.90e-66 | 9.50e-07 |
| DTLZ5 | HV-CF | 2.90e-03 (4.86e-04) | 0.00e+00 (0.00e+00) | **4.00e-03 (3.60e-05)** | 4.00e-11 | 1.86e-02 | 8.82e-51 |
| | IGD-CF | 2.03e-04 (3.31e-05) | 5.79e-02 (1.90e-03) | **4.92e-05 (5.45e-06)** | 1.22e-18 | 2.32e-37 | 2.58e-37 |
| | NS-CF | 1.33e+02 (1.96e+01) | 0.00e+00 (0.00e+00) | **1.66e+02 (2.33e+00)** | 1.25e-08 | 8.61e-22 | 3.19e-46 |
| | IGD-OF | 2.16e-04 (3.70e-05) | 2.76e-02 (7.08e-18) | **3.12e-05 (5.39e-06)** | 9.09e-19 | 1.39e-07 | 9.75e-91 |
| DTLZ6 | HV-CF | 1.50e-03 (2.63e-05) | 0.00e+00 (0.00e+00) | **2.00e-03 (8.13e-06)** | 4.24e-32 | 1.13e-43 | 4.49e-59 |
| | IGD-CF | 1.77e-04 (2.86e-05) | 5.02e-02 (1.35e-04) | **3.92e-05 (4.46e-06)** | 8.28e-19 | 1.63e-63 | 1.99e-63 |
| | NS-CF | **2.00e+02 (0.00e+00)** | 0.00e+00 (0.00e+00) | 1.98e+02 (2.14e+00) | 9.06e-06 | 0.00e+00 | 6.75e-49 |
| | IGD-OF | 2.77e-04 (2.19e-05) | 2.76e-02 (7.08e-18) | **3.42e-05 (9.85e-06)** | 7.20e-26 | 5.06e-76 | 1.84e-84 |

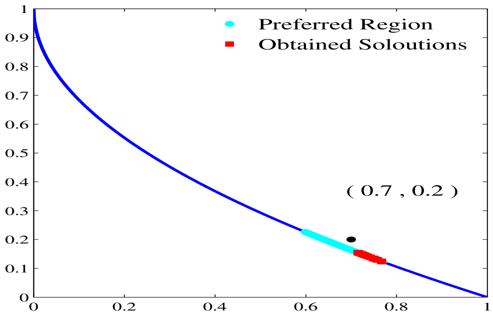
### 7.3.1 Two-Objective Test Problems

The ZDT1 test problem has a convex Pareto-optimal front. Table 7.3 shows the result of R-NSGA-II, R-MEAD-Te and R-MEAD-Ws on this test problem. The results of the statistical test show that R-NSGA-II and R-MEAD-Te are not significantly different using the HV-CF, IGD-OF and IGD-CF measures. However, we can significantly distinguish the performance of R-MEAD-Ws from R-NSGA-II and R-MEAD-Te. Results of IGD-CF and HV-CF are consistent with IGD-OF, which suggests that R-MEAD-Ws performs significantly better than the other two algorithms. However, this conclusion is different when NS-CF is used. Figures 7.2(a), 7.2(b), and 7.2(c) also show this behavior.

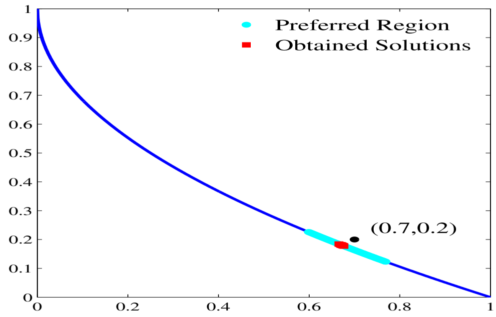
The next test problem is ZDT2, which has a non-convex Pareto-optimal front. According to p-values in table 7.3, all three methods are significantly different. The results of all three measures are consistent with IGD-OF, which suggests that R-MEAD-Te outperforms the other two algorithms. Figures 7.2(d), 7.2(f) and 7.2(e) visually confirm the results generated by the metrics. On ZDT3, the only measure which is consistent with IGD-OF is IGD-CF. The test results of IGD-CF and IGD-OF show that no algorithm performs significantly better than any other. However, the conclusion is different when HV-CF and NS-CF measures are used. On the ZDT4 function, the results of all three measures are consistent with IGD-OF. The t-test shows that R-MEAD-Te and R-NSGA-II exhibit similar performances, and both algorithms outperform R-MEAD-Ws. ZDT6 test problem has a concave Pareto-optimal front. As with ZDT4, all three measures are consistent with IGD-OF which suggests that R-MEAD-Te performs significantly better than the other algorithms. Figures 7.2(j), 7.2(k), and 7.2(l) also confirm those numerical results.

### 7.3.2 Three-Objective Test Problems

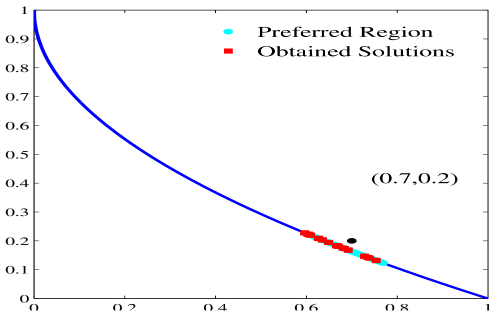
According to the t-test results shown in table 7.4, all algorithms are statistically distinguishable. Except for NS-CF on DTLZ2 and DTLZ6, all other measures are consistent with IGD-OF on all functions. It can be seen from table 7.4 that R-NSGA-II outperforms other algorithms on almost all functions. It is interesting to note that R-MEAD-Ws fails to converge on all functions. Consequently the HV-CF and NS-CF for R-MEAD-Ws are consistently zero for all functions. Figure 7.3 shows the performance of all three algorithms on the two selected functions.



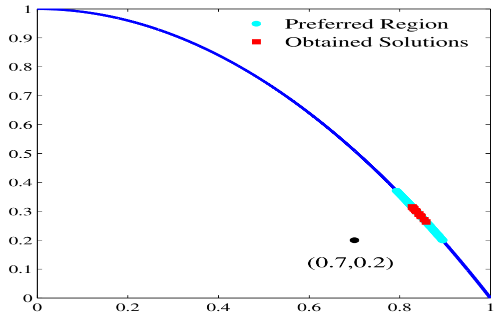

(a) ZDT1-R-MEAD-Te



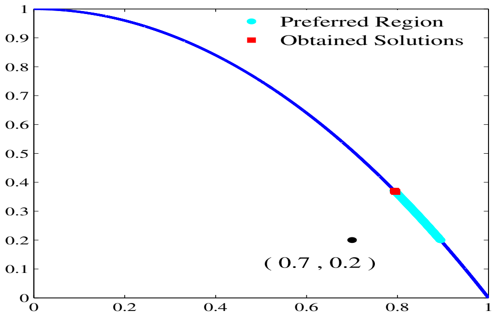

(b) ZDT1-RNSGA-II



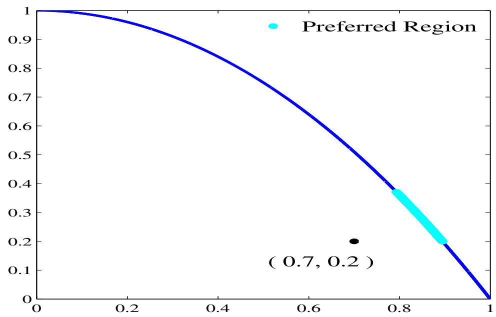

(c) ZDT1-R-MEAD-Ws



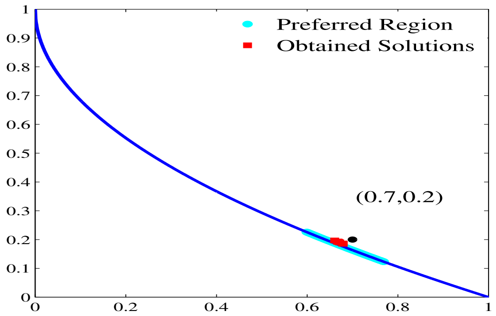

(d) ZDT2-R-MEAD-Te



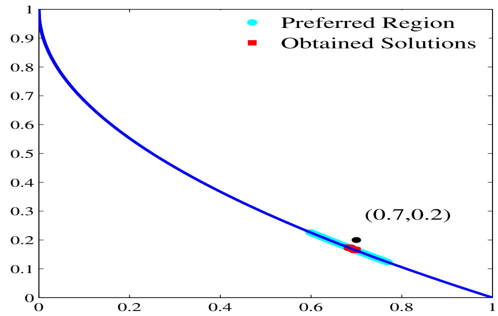

(e) ZDT2-RNSGA-II



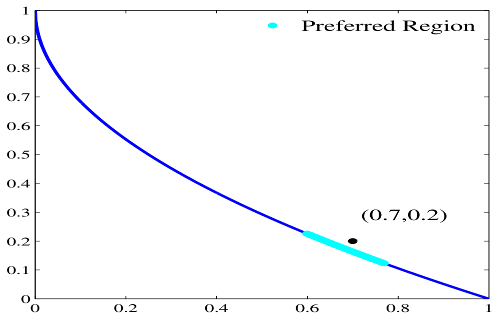

(f) ZDT2-R-MEAD-Ws



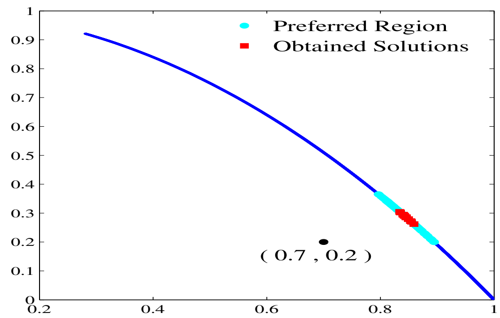

(g) ZDT4-R-MEAD-Te



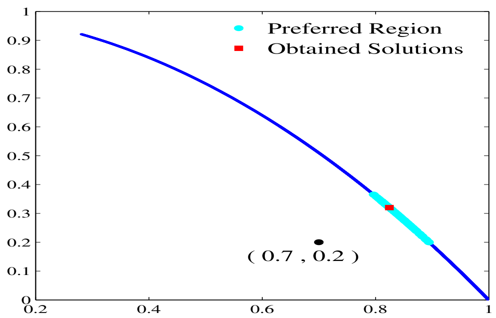

(h) ZDT4-RNSGA-II



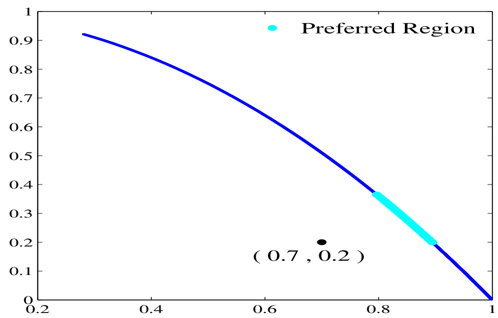

(i) ZDT4-R-MEAD-Ws




(j) ZDT6-R-MEAD-Te

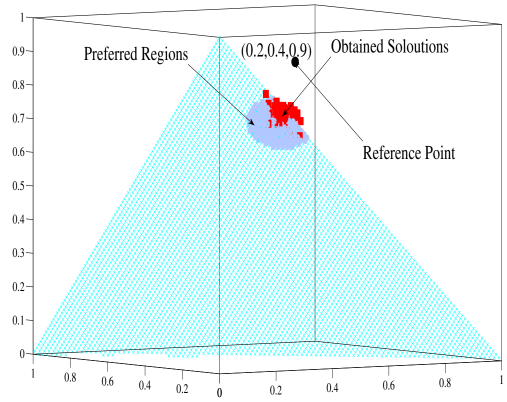



(k) ZDT6-RNSGA-II

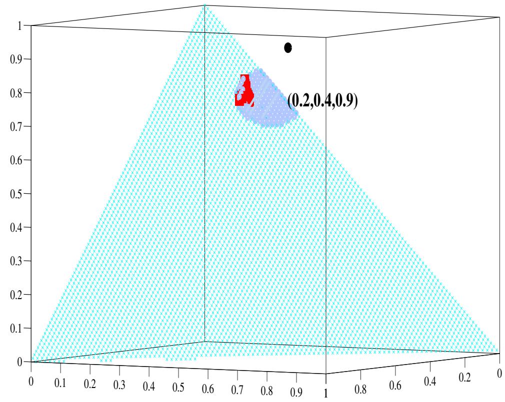



(l) ZDT6-R-MEAD-Ws

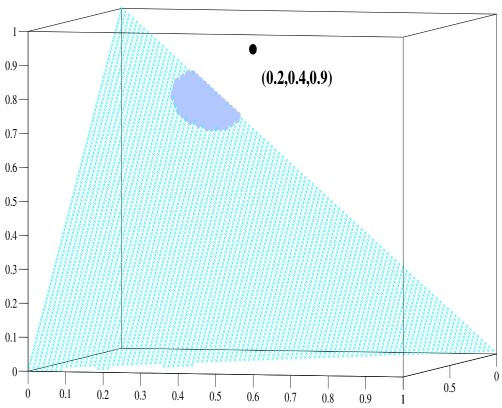
*Figure 7.2: Results on ZDT1, ZDT2, ZDT4 and ZDT6 functions using R-NSGA-II, R-MEAD-Te and R-MEAD-Ws. Preferred region on Pareto-optimal front is shown in light blue color and solutions found by each algorithm in the region are shown in red.*



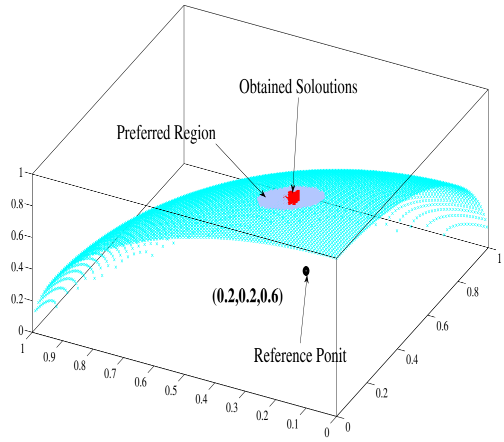

(a) DTLZ1-R-NSGA-II



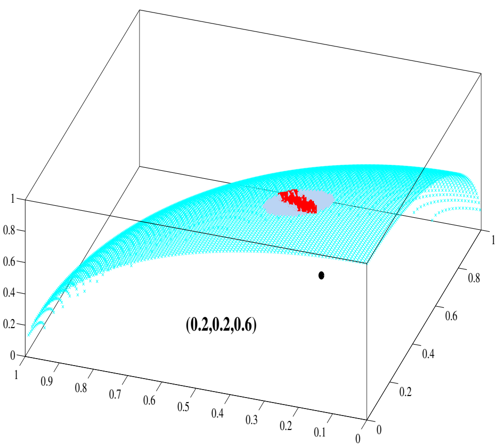

(b) DTLZ1-R-MEAD-Te



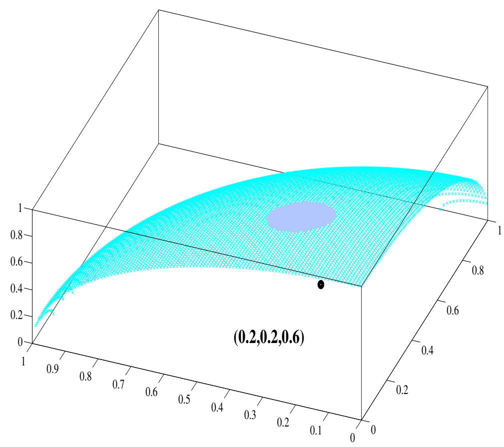

(c) DTLZ1-RMEAD-Ws




(d) DTLZ2-R-NSGA-II




(e) DTLZ2-R-MEAD-Te




(f) DTLZ2-RMEAD-Ws

*Figure 7.3: Results on DTLZ1 and DTLZ2 functions using R-MEAD-Te, R-MEAD-Ws and R-NSGA-II. Preferred region on Pareto-optimal front is shown in light blue color and solutions found by each algorithm in the region are shown in red.*

### 7.3.3 Many-objective Test Problems

Tables 7.5, 7.6 and 7.7 show the median of R-MEAD2-TE, UR-MEAD2-PEV(TE) and R-NSGA-II for five, seven and ten objective problems. To test the significance of the obtained results we used the non-parametric Kruskal-Wallis one-way ANOVA [Sheskin, 2003]. In order to rank the algorithms, Mann-Whitney-Wilcoxon (MWW) test with Bonferroni correction is used only when the null hypothesis of Kruskal-Wallis was rejected under 95% confidence interval. Description and usage of Bonferroni correction and Kruskal-Wallis can be found in chapter 4 According to Kruskal-Wall and pair-wise MWW tests in tables 7.5, 7.6 and 7.7,all three algorithms are significantly different. To be efficient in terms of space, the p-values are not reported.

### 7.3.4 Result Analysis

Results of tables 7.3 and 7.4 are summarized in table 7.8. As can be observed IGD-CF is consistent with IGD-OF in most cases. HV-CF and NS-CF are in the second and third positions respectively. However a different trend can be seen from table 7.9 which is the summarized of tables 7.5 , 7.6 and 7.7. Based on table 7.9 HV-CF is consistent with IGD-OF in most cases. IGD-CF and NS-CF are second and third respectively. To explain this trend, it should be noted that where there are only two or three-objectives the composite front is fairly close to the actual Pareto-optimal front, so applying IGD on the composite front or Pareto-optimal front will be very similar. As a result, IGD-CF is very similar to IGD-OF. However, in the case of a large number of objectives (5, 7 and 10) where the Pareto-optimal front is more complex, it is likely that composite front is not very accurate. Therefore, HV-CF offers a more reliable performance measure where there is a higher number of objectives.

It can be seen from tables 7.8 and 7.9 that a cardinality-based approach is less reliable than HV-CF and IGD-CF. The results tend to be worse when dealing with many-objective problems. As the number of objectives increases, a greater portion of the solutions becomes non-dominated, causing different algorithms to make a very close or even identical contribution to the composite front. This makes cardinality-based approaches less accurate as the number of objectives increases and may lead to an incorrect conclusion.

In this chapter the maximum number of objectives which are used to run the experiments was 10 due to memory limitation for calculating IGD-OF. It should be noted that calculating IGD-OF for larger numbers of objectives needs a huge memory which might not be accessible in a normal PC and it is very slow process. However, as mentioned earlier the proposed metric

*Table 7.5: Results on the 5-objective test problems. The median of 25 independent runs are reported.*

| Func. | Metric | R-MEAD2-Te | UR-MEAD2-PEV (TE) | R-NSGA-II |
|---|---|---|---|---|
| DTLZ1 | HV-CF | 4.70e+02 | **4.47e+02** | 3.10e+02 |
| | IGD-CF | **3.04e-03** | 3.16e-03 | 1.11e-02 |
| | NS-CF | **2.00e+02** | **2.00e+02** | 0.00e+00 |
| | IGD-OF | 1.31e-02 | **1.06e-02** | 1.34e-02 |
| DTLZ2 | HV-CF | 9.73e-01 | **10.93e-1** | 4.17e-01 |
| | IGD-CF | **1.04e-02** | 1.67e-02 | 2.40e-02 |
| | NS-CF | 1.29e+02 | 1.88e+02 | **2.00e+02** |
| | IGD-OF | 4.91e-03 | **3.85e-03** | 9.10e-03 |
| DTLZ3 | HV-CF | **1.27e-01** | 5.38e-02 | 1.49e-02 |
| | IGD-CF | 9.29e-03 | **1.84e-02** | 2.27e-02 |
| | NS-CF | 1.61e+02 | 9.60e+01 | **2.00e+02** |
| | IGD-OF | 5.87e-03 | **2.89e-03** | 9.11e-03 |
| DTLZ4 | HV-CF | 5.97e-02 | **6.44e-03** | 5.48e-02 |
| | IGD-CF | 2.49e-02 | **1.51e-02** | 2.87e-02 |
| | NS-CF | 1.31e+02 | 1.66e+02 | **2.00e+02** |
| | IGD-OF | 8.51e-03 | **6.98e-03** | 9.12e-03 |
| DTLZ5 | HV-CF | 9.23e-07 | **2.18e-06** | 8.70e-09 |
| | IGD-CF | **4.87e-04** | 1.46e-03 | 7.40e-03 |
| | NS-CF | 1.96e+02 | **2.00e+02** | 1.99e+02 |
| | IGD-OF | 5.77e-03 | **4.78e-03** | 5.99e-03 |
| DTLZ6 | HV-CF | 2.13e-02 | **4.40e-02** | 3.54e-02 |
| | IGD-CF | 4.93e-03 | 4.35e-03 | **1.09e-03** |
| | NS-CF | 1.88e+02 | 1.74e+02 | **1.98e+02** |
| | IGD-OF | 5.78e-03 | **4.73e-03** | 5.98e-03 |

*Table 7.6: Results on the 7-objective test problems. The median of 25 independent runs are reported.*

| Func. | Metric | R-MEAD2-Te | UR-MEAD2-PEV (TE) | R-NSGA-II |
|---|---|---|---|---|
| DTLZ1 | HV-CF | **2.80e+08** | **2.80e+08** | 2.63e+08 |
| | IGD-CF | **1.97e-03** | 3.92e-03 | 1.67e-02 |
| | NS-CF | **2.00e+02** | **2.00e+02** | 2.60e+00 |
| | IGD-OF | 4.93e-03 | **4.06e-03** | 5.12e-03 |
| DTLZ2 | HV-CF | **1.05e-01** | 1.67e-02 | 3.68e-02 |
| | IGD-CF | 2.05e-02 | **1.19e-02** | 2.53e-02 |
| | NS-CF | 1.07e+02 | 1.39e+02 | **2.00e+02** |
| | IGD-OF | 1.46e-03 | **1.18e-03** | 1.67e-03 |
| DTLZ3 | HV-CF | **5.53e+04** | 7.37e+00 | 3.64e+04 |
| | IGD-CF | **1.74e-02** | 4.47e-01 | 2.73e-02 |
| | NS-CF | 1.31e+02 | 0.00e+00 | **2.00e+02** |
| | IGD-OF | **1.37e-03** | 5.20e-03 | 1.67e-03 |
| DTLZ4 | HV-CF | 5.39e-05 | **6.72e-02** | 2.71e-02 |
| | IGD-CF | 3.26e-02 | **3.00e-02** | 3.14e-02 |
| | NS-CF | 5.55e+01 | 9.43e+01 | **2.00e+02** |
| | IGD-OF | 1.66e-03 | **1.22e-03** | 2.03e-03 |
| DTLZ5 | HV-CF | 3.98e-06 | **6.49e-06** | 5.59e-06 |
| | IGD-CF | **2.37e-03** | 1.65e-03 | 8.57e-03 |
| | NS-CF | **2.00e+02** | 1.90e+02 | 1.96e+02 |
| | IGD-OF | 6.27e-04 | **5.05e-04** | 6.41e-04 |
| DTLZ6 | HV-CF | 4.47e-03 | **9.24e-02** | 1.20e-02 |
| | IGD-CF | 6.18e-03 | **5.73e-03** | 1.14e-02 |
| | NS-CF | 1.76e+02 | 1.76e+02 | **1.99e+02** |
| | IGD-OF | 6.63e-04 | **6.09e-04** | 6.42e-04 |

*Table 7.7: Results on the 10-objective test problems. The median of 25 independent runs are reported.*

| Func. | Metric | R-MEAD2-Te | UR-MEAD2-PEV (TE) | R-NSGA-II |
|---|---|---|---|---|
| DTLZ1 | HV-CF | 4.05e+08 | **5.85e+08** | 2.24e+08 |
| | IGD-CF | **2.40e-03** | 5.47e-03 | 2.71e-02 |
| | NS-CF | **2.00e+02** | 1.90e+02 | 5.60e-01 |
| | IGD-OF | 1.09e-02 | **1.06e-02** | 1.58e-02 |
| DTLZ2 | HV-CF | **2.11e-02** | 4.49e-03 | 7.62e-03 |
| | IGD-CF | 2.06e-02 | **1.13e-02** | 2.27e-02 |
| | NS-CF | 8.60e+01 | 1.93e+02 | **2.00e+02** |
| | IGD-OF | 2.06e-03 | **6.37e-04** | 2.53e-03 |
| DTLZ3 | HV-CF | 2.45e+07 | **5.08e+07** | 2.70e+07 |
| | IGD-CF | 2.62e-02 | **6.54e-01** | 2.72e-02 |
| | NS-CF | 9.00e+01 | 0.00e+00 | **2.00e+02** |
| | IGD-OF | 2.42e-03 | **1.33e-03** | 2.53e-03 |
| DTLZ4 | HV-CF | 4.38e-02 | **8.57e-02** | 4.51e-02 |
| | IGD-CF | 4.92e-02 | 4.52e-02 | **2.77e-02** |
| | NS-CF | 1.20e+01 | 5.42e+01 | **2.00e+02** |
| | IGD-OF | 2.40e-03 | **2.02e-03** | 2.69e-03 |
| DTLZ5 | HV-CF | **2.34e-07** | 1.70e-07 | 1.23e-07 |
| | IGD-CF | **1.53e-03** | 3.55e-03 | 9.60e-03 |
| | NS-CF | 1.93e+02 | **2.00e+02** | **2.00e+02** |
| | IGD-OF | 6.75e-04 | **6.53e-04** | 7.27e-04 |
| DTLZ6 | HV-CF | 7.85e-05 | 1.94e-04 | **4.45e-04** |
| | IGD-CF | 7.61e-03 | **6.47e-03** | 1.54e-02 |
| | NS-CF | 1.73e+02 | 1.65e+02 | **2.00e+02** |
| | IGD-OF | 6.26e-04 | **5.60e-04** | 6.79e-04 |

does not have these limitations and it can be easily run for more than 10 objectives such as 15 and 20. Based on the experimental results, HV-CF is more reliable performance measure than the other two metrics, especially for higher numbers of objectives. Therefore, HV-CF can be a reasonable replacements for IGD metric in case of higher number of objectives since HV-CF uses less computational cost and has a faster process.

*Table 7.8: Consistency of each measure with IGD-OF (IGD based on Pareto-optimal front) for two and three-objective problems is shown in below table. (✓: consistent, ×: inconsistent).*

| Function | IGD-CF | HV-CF | NS-CF |
|---|---|---|---|
| ZDT1 | ✓ | ✓ | × |
| ZDT2 | ✓ | ✓ | ✓ |
| ZDT3 | ✓ | × | × |
| ZDT4 | ✓ | ✓ | ✓ |
| ZDT6 | ✓ | ✓ | ✓ |
| DTLZ1 | ✓ | ✓ | ✓ |
| DTLZ2 | ✓ | ✓ | × |
| DTLZ3 | ✓ | ✓ | ✓ |
| DTLZ4 | ✓ | ✓ | ✓ |
| DTLZ5 | ✓ | ✓ | ✓ |
| DTLZ6 | ✓ | ✓ | × |
| Total | 11 | 10 | 7 |

## 7.4 Chapter Summary

In this chapter, we proposed a metric for evaluating the performance of user-preference based evolutionary multi-objective algorithms by defining a preferred region based on the location of a user-supplied reference point. This metric uses a *composite front* which is a type of reference set and is used as a replacement for the Pareto-optimal front. This composite front is constructed by extracting the non-dominated solutions from the merged solution sets of all algorithms that are to be compared. A preferred region is then defined on the *composite front* based on the location of a reference point. Once the preferred region is defined, existing evolutionary multi-objective performance metrics can be applied with respect to the preferred region. In this chapter the performance of a cardinality-based metric, a distance-based metric, and a volume-based metric were compared against a baseline which relies on knowledge of the Pareto-optimal front. The experimental results show that the distance-based and the volume-

*Table 7.9: Consistency of each measure with IGD-OF (IGD based on Pareto-optimal front) for 5,7 and 10-objective problems is shown in below table. (✓: consistent, ×: inconsistent).*

| # Obj | Function | IGD-CF | HV-CF | NS-CF |
|---|---|---|---|---|
| 5 obj | DTLZ1 | × | ✓ | ✓ |
| | DTLZ2 | × | ✓ | × |
| | DTLZ3 | ✓ | × | × |
| | DTLZ4 | ✓ | ✓ | × |
| | DTLZ5 | × | ✓ | ✓ |
| | DTLZ6 | × | ✓ | × |
| 7 obj | DTLZ1 | × | ✓ | ✓ |
| | DTLZ2 | ✓ | × | × |
| | DTLZ3 | ✓ | ✓ | × |
| | DTLZ4 | ✓ | ✓ | × |
| | DTLZ5 | × | ✓ | × |
| | DTLZ6 | ✓ | ✓ | × |
| 10 obj | DTLZ1 | × | ✓ | × |
| | DTLZ2 | ✓ | × | × |
| | DTLZ3 | ✓ | ✓ | × |
| | DTLZ4 | × | ✓ | × |
| | DTLZ5 | × | × | ✓ |
| | DTLZ6 | ✓ | × | × |
| | Total | 9 | 13 | 4 |

based metrics are consistent with the baseline, showing meaningful comparisons. However, the cardinality-based approach shows some inconsistencies and is not suitable for comparing the algorithms.

# Chapter 8

# Conclusion

This thesis has been dedicated to examining and developing methods to solve many-objective optimization problems. This was achieved by combing two methods, i.e., decomposition and user preference-based methods. 1) Firstly, user preference which focuses the search space on the desired regions provided by the decision makers. This allows a considerable saving in computational resources; 2) Secondly, decomposition which increases the selection pressure to move the population towards the Pareto-optimal front and alleviate the selection pressure issue of dominance-based approaches. As an initial investigation, we developed a user-preference decomposition algorithm which was effective in solving problems with two and three-objectives. The promising results motivated us to extend the algorithm to solve many-objective problems. The major challenge was to generate and initialize weight vectors for a larger number of objectives with low computational costs. This challenge was resolved by removing the dependence between population size and the number of objectives. The results clearly indicate that the proposed methods can solve many-objective optimization problems effectively. Further investigation showed that the proposed algorithm did not work well on many-objective problems with non-linear or with complex Pareto-optimal fronts. To overcome this weakness, weight vectors were adapted dynamically during the course of optimization based on the position of solutions in the objective space. It has been shown empirically that the proposed dynamic adaptation outperforms R-NSGA-II which is one of popular dominance-based approaches on many-objective optimization problems. Finally, since there was not an accurate and reliable user-preference EMO metric to measure the performance of our proposed algorithms without relying on the knowledge of Pareto-optima, we developed a user-preference performance metric.

The reminder of this chapter is organized in two sections: 1) in Section 8.1, the research objectives which were outlined in chapter 1 are revisited along with our findings; 2) in Section 8.2, several future works are described and discussed.

## 8.1 Research Objectives Revisited

1. **To develop a novel method by combining the decomposition and user preference methods for better handling many-objective optimization problems.**

   In Chapter 3, we proposed a preference-based evolutionary multi-objective optimization through decomposition. The proposed approach, R-MEAD, has the following benefits:

   (a) Using decomposition we can convert a multi-objective problem into a single-objective which is not susceptible to the selection pressure problem common to dominance-based approaches.

   (b) Using the reference point allows a search to focus on the desired regions which can potentially save a considerable amount of computational resources.

   R-MEAD has been evaluated using two decomposition approaches, Tchebycheff and Weighted-sum. Experimental results showed that the Tchebycheff approach performs better than the Weighted-sum especially when dealing with non-convex problems. It has also been shown that R-MEAD results in a faster convergence compared to MOEA/D, especially when the number of objectives increases. It might sounds unfair to compare a user-preference algorithm with non-user preference algorithm. However, the reason for comparison was to show the effect of adopting preference information in saving computational resources.

   In chapter 4, we improved R-MEAD (developed in chapter 3) in the following ways:

   (a) Unlike R-MEAD, in R-MEAD2 the population size and the number of objectives are decoupled and the increased number of objectives does not cause a growth in population size. Consequently, we can conclude that R-MEAD2 is better suited for solving many-objective problems.

   (b) R-MEAD2 uses a simple random number generator (RNG) to initialize the weight vectors. The results of centered $L_2$-discrepancy showed that RNG can generate more uniform weights than good lattice point (GLP) when the number of objectives grows beyond seven with a typical sample size of 100, 250 and 500. For other sample sizes (50, 1000 and 5000) RNG can produce points with lower discrepancy than GLP when the number of objectives goes beyond eight.

(c) R-MEAD2 is less susceptible to the selection pressure issue compared to dominance-based approaches such as R-NSGA-II. The experimental results showed that both versions of R-MEAD2 (Tchebycheff and PBI) outperform R-NSGA-II over a range of many-objective problems.

From the experimental results, it was observed that the performance of PBI and Tchebycheff appeared to be very similar. However, a stronger conclusion requires further investigation on penalty parameter in PBI as we run our experiments with only one theta value ($\theta$=5). This was investigated in chapter 5.

Another observation from experimental results is that a uniform set of weight vectors does not necessarily map to a uniform set of solutions in the objective space, especially on highly non-linear and complex Pareto-optimal fronts. This requires a feedback mechanism to adjust the weights in order to obtain a set of uniform solutions, which was investigated in chapter 6.

2. **To evaluate the effect of penalty parameters in PBI on the performance of user-preference and non-user preference EMO algorithms.**

   In chapter 5, the effect of the penalty parameter of PBI was analyzed from various aspects, and the following findings were achieved:

   (a) A larger $\theta$ generally improves the uniformity, but it often has a detrimental effect on convergence, especially with MOEA/D.

   (b) User-preference based approaches such as R-MEAD2 perform better in terms of both uniformity and convergence with higher $\theta$ values compared to EMO approaches that do not use any user-preference information, such as MOEA/D.

   (c) On some problems, such as multi-modal functions, convergence is the dominant factor where smaller $\theta$ values are suggested.

   (d) On most uni-modal problems, uniformity is the dominant factor, therefore a larger $\theta$ is suggested.

   (e) There is no strong relationship between the number of objectives and $\theta$.

The overall observation based on the findings of this chapter is that there is no unique $\theta$ value that works well on different types of problems with different number of objectives. Although R-MEAD2 showed noticeable tendency towards larger $\theta$ values than MOEA/D, the exact value of $\theta$ that works well on different problems varies from case to case. An implicit assumption of this chapter was that the effect of $\theta$ on uniformity and convergence does not change over the course of optimization.

3. **To improve the uniformity of solutions in the preferred region particularly when the shape of the Pareto-optimal front is complex or highly non-linear. To investigate a mechanism which can find a diverse set of solutions without relying on the information of a Pareto-optimal front.**

   In chapter 6, we proposed a method to improve the uniformity of solutions generated by many-objective decomposition-based EMO algorithms. The proposed method uses a mechanism to update the weight vectors of decomposition-based algorithms according to a feedback that quantifies the uniformity of the solutions in the objective space. In addition, the weights are updated in parallel to move the solutions towards a region of interest which is specified by a decision maker through a reference point.

   The feedback mechanism adapted to quantify the uniformity/discrepancy of solutions in the objective space are: Overlapping Hypervolume (OHV), Potential Energy (PE) and Potential Energy with Direction Vector (PEV). Among these, OHV and PE are two existing metrics which have been modified to fit the purpose of this research. In particular, the main idea revolves around the identification of sparse regions in the objective space and updating the weight vectors to move the solutions from denser areas to more sparse areas. These methods are stochastic in nature and do not estimate the gradient base on which the weight vectors should be updated. Unlike OHV and PE, PEV is based on a vectorized implementation of PE in which the solution set is treated as a set of charged particles which strive to reach static equilibrium. In PEV, the gradient information to update each weight vector is calculated based on the virtual net force that is exerted on each solution (particle) from all other solutions.

   The experimental results showed that UR-MEAD2 (using OHV, PE, and PEV) can perform better than their counterparts with no feedback mechanism (R-MEAD2 and R-NSGA-II). One of our findings was that PEV had the best performance among three feedback mechanisms, which could be attributed to the use of accurate gradient information for updating the weight vectors. The performance of PEV with respect to various decomposition methods was analyzed and studied. The analysis showed that PEV is most compatible with Tchebycheff. We realized that Tchebycheff's lack of control on uniformity is effectively compensated by PEV, making it superior to PBI. With respect to PBI, we have also realized that PEV generally works better when its penalty value is smaller. Overall, the proposed framework allows effective integration of user preference while maintaining uniformity on complex Pareto fronts. After investigating the effect of the new weight vector adaption method on UR-MEAD2 without a reference point, we found that the proposed feedback mechanisms are capable of generating the uniform set of solutions for non-user preference approaches as well.

4. **To define a metric to evaluate the performance of different user-preference based algorithms.**

   In chapter 7, a metric (UPCF) is proposed for measuring the performance of user-preference based EMO algorithms. UPCF works by combining the solution sets of the algorithms that are to be compared and extracting the non-dominated solutions into a *composite front* which is then used to define a preferred region based on the location of a user-supplied reference point. Once the preferred region is defined, we used IGD (IGD-CF), HV (HV-CF) and the cardinality-based approach (NS-CF) to measure the convergence and diversity of the solution set of each algorithm. *Composite front* can be used as a replacement of Pareto-optimal front. To ensure the effectiveness of our proposed metric, we compared it with a baseline metric (IGD-OF) that used the Pareto-optimal front information. We found that IGD-CF and HV-CF are consistent with IGD-OF, but NS-CF shows some inconsistency. These inconsistencies tend to be magnified when dealing with problems with a higher number of objectives.

   Another important lesson we learned was that HV-CF has the best consistency with IGD-OF in case of the large number of objectives. We speculate that this behavior is due to easy convergence of algorithms on the Pareto-optimal front due to the small number of objectives. In other words, the composite front happens to closely resemble a desired portion of the Pareto-optimal front. However, this is not the case when the algorithm cannot fully converge, particularly in higher numbers of objectives. Therefore, HV-CF becomes the more reliable performance measure.

## 8.2 Future Research

This research is mainly dedicated to solving many-objective problems by integrating user-preference and decomposition methods. Several algorithms have been developed for this purpose. However, there are more aspects and areas which can be investigated in the future.

- In chapter 5, a comprehensive analysis of PBI penalty parameter ($\theta$), and its effect on a user-preference algorithm (R-MEAD2) and a non-user-preference algorithm (MOEA/D) has been conducted. However, we did not investigate the effect of adapting $\theta$ during the course of optimization. Adaptation of $\theta$ is appealing from two different viewpoints. Firstly, the optimum $\theta$ is problem dependent. Secondly, the effect of $\theta$ on convergence and uniformity may vary during the course of optimization.

- In chapter 6, a feedback mechanism is proposed to maintain the uniformity of solutions in the objective space. Three different metrics are used as a feedback mechanism. This chapter was mainly dedicated to adjusting the weight vectors to improve the uniformity of solutions in the objective space. It is worthwhile investigating the effect of other factors to the uniformity of solutions in the objective space such as distribution of solutions in the decision space, speed of convergence and optimization algorithm parameters.

- In chapter 7, a metric (UPCF) is proposed to measure the performance of user-preference based EMO algorithms. UPCF uses a *composite front* as the replacement of Pareto-optimal front. *Composite front* is a combination of non-dominated solutions from algorithms which need to be compared. The current *composite front* might not be accurate enough where there are problems with complex and non-linear Pareto-optimal front shapes. As a result, there is a need to investigate the development of more accurate ways of generating *composite fronts* which can express the features of the Pareto-optimal front during the course of optimization.

- In this thesis, we are mainly focused on *a priori* and *a posteriori* user-preference approaches, so there is a great potential to apply *interactive* approaches to developed frameworks in the particular algorithms developed in chapter 6. Since the weight vectors are updated dynamically, incorporation of interactive preference information into the optimization loop will boost the performance of the algorithm dramatically. Additionally, decision makers can have a much better understanding of solutions in the objective space and this helps them to choose their preferred regions with greater accuracy.

- Applying our developed methods to real-world applications is another area for future investigation, particularly problems that use interactive optimization process. Stewart et al. [2008] introduced several interactive many-objective real-world applications. If a many-objective user-preference algorithm uses a suitable visualization schema, it can benefit significantly as an interactive optimization process for real world problems.

# Appendices

# Appendix A

# Multi-objective Optimization Test Problems

In this section, we present a brief overview of three widely used multi-objective test functions namely, ZDT [Zitzler et al., 2000], DTLZ Deb et al. [2001] and WFG Huband et al. [2005]. These test suites provide a common platform for researchers to evaluate their algorithms on problems with a range of pre-defined characteristics.

1. **ZDT Test Suite**

   ZDT functions were firstly introduced by Zitzler et al. [2000]. These functions are designed based on the features of Pareto-optimal fronts which may cause difficulties for an EMO to find diverse Pareto-optimal solutions. These features include convexity or non-convexity, discreteness and non-uniformity. For each of these features a corresponding test function is constructed. ZDT test functions are restricted to two-objective. Table A.1 shows ZDT suites with their Pareto front shapes as well as the number of decision variables ($N$). $x_i \in [0, 1]$ in case of all ZDT functions except ZDT4 where $x_i \in [-5, 5]$ and $x_1 \in [0, 1]$. ZDT1 to ZDT3 are simple two-objective problems with one global optimal front. However, ZDT4 is multi-modal with many local Pareto-fronts.

2. **DTLZ Test Suite**

   DTLZ functions were proposed by Deb et al. [2001]. The main difference between DTLZ and ZDT functions is that DTLZ functions are scalable. Three approaches are used to design DTLZ test problems. The first approach uses a mostly-translated single-objective function. In the second approach, the procedure is constructed with the

*Table A.1: Features of ZDT Test Problems*

| Functions | Features | N |
|---|---|---|
| ZDT1 | convex | 30 |
| ZDT2 | concave | 30 |
| ZDT3 | convex and disconnected | 30 |
| ZDT4 | convex | 10 |
| ZDT6 | concave | 10 |

*Table A.2: Features of DTLZ Test Problems*

| Functions | Features | N |
|---|---|---|
| DTLZ1 | linear,multi-modal | M+4 |
| DTLZ2 | concave,uni-modal | M+9 |
| DTLZ3 | concave,multi-modal | M+9 |
| DTLZ4 | concave,uni-modal | M+9 |
| DTLZ5 | concave,uni-modal | M+9 |
| DTLZ6 | concave,uni-modal | M+9 |

assumption of the mathematical formula of a Pareto-optimal front. The third approach is constructed with the assumption that the shape of the search space is a rectangular hype-box. In DTLZ test problems, all decision variables are $x_i \in [0, 1]$. Table A.2 illustrates the properties of these functions.

3. **WFG Test Suite**

WFG problems were proposed by Huband et al. [2005]. They cover a wide variety of Pareto-optimal geometries such as convex, concave, mixed convex/concave, linear, degenerated and disconnected. WFG functions use vector of $x$ which is associated with an underlying problem to define fitness space. $x$ is driven from the vector of working parameters $z$. Unlike the previous test suites, WFG Toolkits allow the designer to control the complexity. In other words, designers are able to build scalable problems which are both non-separable and multi-modal.

To build a test problem, designers can determine the geometry of fitness shape by selecting several shape functions and they can facilitate the creation of transition vectors by employing a number of transformation functions which can be summarized in three main steps: 1) specify value for underlying formalism; 2)specify the shape function, and 3) specify transition vectors. Shape function specifies the nature of the Pareto-optimal front and map parameter with domain [0,1] to [0,1]. Different types of Pareto-optimal fronts are included: linear, convex, concave, mixed convex/concave and disconnected. Transformation function map the input parameters with domain [0,1] to [0,1]. Transformation functions are included Bias:Polynomial, Bias:Flat Region, Bias:Parameter Dependent, Shift:Linear Shift:Deceptive, Shift:Multi-modal, Reduction: Weighted-Sum, Reduction: non-separable. Properties of WFG1-WFG9 functions are described in table A.3.

*Table A.3: Features of WFG Test Problems*

| Functions | Features |
|---|---|
| WFG1 | convex, mixed, uni-modal |
| WFG2 | convex, disconnected, uni-modal |
| WFG3 | linear, degenerate, multi-modal |
| WFG4 | concave, multi-modal |
| WFG5 | concave, deceptive |
| WFG6 | concave, uni-modal |
| WFG7 | concave, uni-modal |
| WFG8 | concave, uni-modal |
| WFG9 | concave, multi-modal, deceptive |

Appendix B

# IGD,GD and $L_2$ results on DTLZ1-DTLZ6 test problems using PBI decomposition with different penalty parameters

Table B.1: IGD,GD and $L_2$ values on DTLZ1 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ1, $m=2$ | 0 | 4.07e-2(1.42e-17) | 2.97e-2(1.42e-17) | 4.85e-27(1.43e-45) | 1.74e-2(3.76e-03) | 3.39e-2(2.01e-02) | 1.50e-01(1.25e-01) |
| | 0.1 | **3.82e-2(2.12e-17)** | **2.96e-2(7.08e-18)** | 2.86e-27(1.43e-45) | **1.71e-2(4.17e-03)** | **2.54e-2(1.81e-03)** | 1.50e-01(1.25e-01) |
| | 1.0 | 1.06e+1(8.80e+00) | 7.50e+0(6.21e+00) | 4.80e-28(1.43e-45) | 1.80e-2(1.04e-03) | 4.12e-2(9.07e-03) | 4.14e-02(3.45e-08) |
| | 5.0 | 1.04e+1(8.67e+00) | 7.38e+0(6.13e+00) | 1.66e-28(1.43e-45) | 1.73e-2(1.18e-03) | 4.03e-2(5.26e-03) | 2.82e-04(2.35e-11) |
| | 10 | 3.88e-2(1.42e-17) | 3.07e-2(1.06e-17) | 5.50e-30(1.46e-42) | 1.78e-2(1.69e-03) | 3.79e-2(6.33e-03) | **1.75e-06(1.46e-06)** |
| | 50 | 3.86e-2(2.12e-17) | 3.01e-2(1.77e-17) | **5.06e-30(1.46e-42)** | 1.80e-2(1.50e-03) | 3.97e-2(8.93e-03) | 2.91e-06(2.42e-06) |
| | 500 | 3.88e-2(1.42e-17) | 3.07e-2(1.42e-17) | 1.27e-29(2.93e-42) | 1.78e-2(1.76e-03) | 3.76e-2(6.32e-03) | 3.16e-06(2.64e-06) |
| DTLZ1, $m=4$ | 0 | 2.25e-2(3.54e-18) | 2.42e-2(3.54e-18) | 1.16e-10(0.00e+00) | 3.69e-2(9.98e-03) | 3.87e-2(9.76e-03) | 3.38e-01(2.81e-01) |
| | 0.1 | **1.93e-2(3.54e-18)** | **2.06e-2(7.08e-18)** | **3.41e-15(8.05e-31)** | **3.14e-2(9.33e-03)** | **3.27e-2(9.44e-03)** | 1.89e-02(1.58e-02) |
| | 1.0 | 3.30e-2(7.08e-18) | 3.44e-2(1.42e-17) | 5.41e-11(1.98e-26) | 3.35e-2(5.47e-05) | 3.49e-2(1.07e-05) | 2.71e-10(2.74e-10) |
| | 5.0 | 2.95e-2(1.42e-17) | 3.08e-2(7.08e-18) | 1.96e-11(6.60e-27) | 3.33e-2(2.64e-03) | 3.36e-2(2.85e-03) | 3.81e-11(3.17e-11) |
| | 10 | 2.93e-2(1.42e-17) | 3.08e-2(7.08e-18) | 3.66e-11(1.32e-26) | 3.25e-2(1.15e-03) | 3.39e-2(1.22e-03) | 4.51e-11(2.42e-11) |
| | 50 | 2.95e-2(7.08e-18) | 3.07e-2(1.42e-17) | 5.21e-12(0.00e+00) | 3.24e-2(1.13e-03) | 3.38e-2(1.23e-03) | 1.19e-11(3.28e-12) |
| | 500 | 2.95e-2(1.42e-17) | 3.08e-2(7.08e-18) | 1.13e-11(0.00e+00) | 3.22e-2(1.40e-03) | 3.37e-2(1.42e-03) | **8.49e-12(5.49e-12)** |
| DTLZ1, $m=6$ | 0 | 1.54e-3(4.43e-19) | 3.14e-2(3.14e-2) | 8.09e-04(2.21e-19) | 2.42e-3(6.05e-04) | 3.84e-2(5.92e-3) | 4.29e-01(3.32e-01) |
| | 0.1 | **1.39e-3(0.00e+00)** | **2.73e-2(1.06e-17)** | 2.74e-06(0.00e+00) | 2.18e-3(6.27e-04) | 3.84e-2(5.23e-3) | 2.67e-01(2.22e-01) |
| | 1.0 | 2.05e-3(8.85e-19) | 4.33e-2(7.08e-18) | 3.89e-08(2.70e-23) | **2.07e-3(8.64e-06)** | **3.33e-2(1.90e-4)** | 2.77e-08(4.82e-09) |
| | 5.0 | 1.97e-3(4.43e-19) | 6.08e-2(2.53e-9) | 1.39e-08(6.75e-24) | 2.14e-3(4.96e-05) | 4.90e-2(1.03e-3) | 2.03e-08(5.13e-09) |
| | 10 | 1.96e-3(4.43e-19) | 6.08e-2(3.74e-8) | 9.77e-09(3.38e-24) | 2.14e-3(5.70e-05) | 4.26e-2(1.10e-3) | 1.95e-08(6.04e-09) |
| | 50 | 1.95e-3(1.11e-18) | 4.12e-2(0.00e+0) | 9.44e-09(0.00e+00) | 2.15e-3(5.68e-05) | 4.26e-2(7.89e-4) | 2.45e-09(1.94e-10) |
| | 500 | 1.95e-3(4.43e-19) | 6.08e-2(2.19e-10) | **9.17e-09(1.69e-24)** | 2.15e-3(6.59e-05) | 4.25e-2(9.48e-4) | **2.39e-09(2.14e-10)** |
| DTLZ1, $m=8$ | 0 | **8.78e-3(1.77e-18)** | **2.36e-2(1.06e-17)** | 2.30e-02(3.54e-18) | 1.39e-2(3.04e-03) | | 5.20e-01(3.07e-01) |
| | 0.1 | 8.97e-3(4.43e-18) | 2.47e-2(7.08e-18) | 9.09e-05(6.33e-25) | **1.06e-2(2.83e-03)** | **3.08e-2(1.14e-03)** | 1.50e-01(1.25e-01) |
| | 1.0 | 1.13e-2(0.00e+00) | 3.86e-2(7.08e-18) | 1.89e-06(2.16e-22) | 1.14e-2(6.24e-05) | 3.93e-2(3.68e-04) | 2.97e-06(7.42e-08) |
| | 5.0 | 1.11e-2(0.00e+00) | 3.80e-2(2.12e-17) | 1.29e-06(0.00e+00) | 1.14e-2(1.31e-04) | 3.88e-2(4.63e-04) | 2.74e-06(2.25e-07) |
| | 10 | 1.09e-2((0.00e+0) | 3.78e-2(2.12e-17) | **5.77e-07(3.24e-22)** | 1.15e-2(1.44e-04) | 3.88e-2(4.84e-04) | 2.71e-06(2.51e-07) |
| | 50 | 1.09e-2(1.77e-18) | 3.75e-2(2.12e-17) | 7.13e-07(0.00e+00) | 1.15e-2(1.31e-04) | 3.82e-2(6.90e-05) | 1.17e-06(1.56e-06) |
| | 500 | 1.10e-2(1.77e-18) | 3.78e-2(0.00e+00) | 1.16e-06(6.48e-22) | 1.15e-2(2.20e-04) | 3.82e-2(2.40e-04) | **1.08e-06(2.08e-06)** |
| DTLZ1, $m=10$ | 0 | 1.16e-2(1.77e-18) | 3.54e-2(1.42e-17) | 1.37e-01(5.67e-17) | **1.58e-2(1.87e-03)** | **1.42e-1(4.93e-02)** | 3.13e-01(8.10e-02) |
| | 0.1 | **1.10e-2(7.08e-18)** | **3.20e-2(1.42e-17)** | 1.82e-05(6.92e-21) | 1.58e-2(1.02e-03) | 1.42e-1(4.93e-02) | 2.98e-05(1.17e-05) |
| | 1.0 | 1.28e-2(1.77e-18) | 3.64e-2(0.00e+00) | 4.61e-05(1.06e-25) | 1.58e-2(1.02e-03) | 1.42e-1(4.93e-02) | 2.84e-05(3.57e-06) |
| | 5.0 | 1.36e-2(5.31e-18) | 3.82e-2(2.12e-17) | 2.61e-05(1.04e-20) | 1.58e-2(2.12e-03) | 1.42e-1(4.93e-02) | 2.75e-05(2.05e-06) |
| | 10 | 1.24e-2(7.08e-18) | 3.56e-2(1.42e-17) | 3.26e-10(1.58e-25) | 1.58e-2(1.02e-03) | 1.42e-1(4.93e-02) | 2.48e-05(2.85e-06) |
| | 50 | 1.25e-2(0.00e+00) | 3.55e-2(2.12e-17) | 2.90e-10(0.00e+00) | 1.58e-2(1.02e-03) | 1.42e-1(4.93e-02) | 2.32e-05(2.37e-06) |
| | 500 | 1.24e-2(1.77e-18) | 3.53e-2(2.12e-17) | **2.74e-10(1.58e-25)** | 1.58e-2(1.01e-03) | 1.42e-1(4.93e-02) | **2.21e-05(1.27e-06)** |

Table B.2: IGD,GD and $L_2$ values on DTLZ2 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ2, $m=2$ | 0 | 4.63e-2(7.31e-18) | **8.16e-4(0.00e+00)** | 3.76e-86(2.55e-111) | 1.00e-3(1.57e-11) | 6.49e-2(1.49e-09) | 2.50e-01(0.00e+00) |
| | 0.1 | 4.63e-2(1.42e-17) | 8.16e-4(1.77e-18) | 3.76e-86(2.55e-111) | 8.62e-4(1.73e-04) | **1.02e-2(4.34e-03)** | 1.50e-01(1.25e-01) |
| | 1.0 | 1.48e-3(0.00e+00) | 8.31e-4(3.32e-19) | 3.39e-90(6.79e-111) | 2.89e-4(1.27e-04) | 2.25e-2(2.17e-03) | 1.22e-04(1.02e-04) |
| | 5.0 | 1.51e-3(2.21e-19) | 8.51e-4(4.43e-19) | 9.15e-94(2.17e-109) | 3.05e-4(1.58e-04) | 3.61e-2(1.53e-03) | 7.16e-08(2.70e-09) |
| | 10 | **1.29e-3(0.00e+00)** | 5.46e-3(3.32e-19) | 2.74e-94(1.30e-108) | 2.98e-4(1.20e-04) | 2.28e-2(8.07e-04) | 6.33e-08(3.61e-09) |
| | 50 | 1.75e-3(6.64e-19) | 1.76e-3(1.76e-03) | **1.31e-96(1.82e-102)** | 2.87e-4(1.35e-04) | 2.13e-2(2.56e-03) | 6.09e-08(5.07e-09) |
| | 500 | 1.40e-3(6.64e-19) | 8.61e-3(2.21e-19) | 9.15e-96(2.17e-109) | **2.86e-4(1.36e-04)** | 2.11e-2(2.77e-03) | **6.01e-08(7.01e-09)** |
| DTLZ2, $m=4$ | 0 | 2.63e-2(1.06e-17) | 8.16e-2(1.77e-18) | 3.76e-55(2.55e-111) | 3.89e-2(5.08e-03) | 5.12e-2(3.25e-03) | 4.37e-01(1.56e-01) |
| | 0.1 | 2.82e-2(1.06e-17) | 7.07e-2(2.83e-17) | 4.17e-56(7.08e-18) | 3.93e-2(3.08e-04) | 7.30e-2(5.45e-03) | 3.62e-01(2.50e-01) |
| | 1.0 | 2.65e-2(7.08e-18) | **1.07e-2(2.83e-17)** | 1.18e-58(0.00e+00) | 3.42e-2(7.42e-04) | 6.01e-2(7.85e-03) | 3.63e-01(2.49e-01) |
| | 5.0 | 7.02e-3(0.00e+00) | 1.57e-2(7.08e-18) | 2.55e-59(8.46e-76) | 1.73e-2(5.88e-03) | **2.78e-2(1.12e-03)** | 7.67e-05(5.60e-09) |
| | 10 | 7.39e-3(8.85e-19) | 1.46e-2(7.08e-18) | 2.38e-59(1.32e-77) | 1.70e-2(5.96e-03) | 3.75e-2(1.19e-03) | 6.55e-09(5.46e-10) |
| | 50 | 7.53e-3(0.00e+00) | 1.61e-2(7.08e-18) | **9.51e-60(2.26e-75)** | 1.71e-2(5.80e-03) | 3.81e-2(1.12e-03) | 5.46e-09(4.55e-10) |
| | 500 | **6.87e-3(8.85e-19)** | 1.42e-2(8.85e-18) | 2.90e-59(1.35e-74) | **1.68e-2(6.24e-03)** | 3.66e-2(1.31e-03) | **5.31e-09(4.42e-10)** |
| DTLZ2, $m=6$ | 0 | 1.17e-3(7.36e-05) | **4.80e-2(1.19e-9)** | 5.17e-01(2.22e-01) | 1.23e-3(2.29e-19) | 4.80e-2(3.17e-10) | 5.17e-01(2.22e-01) |
| | 0.1 | 1.19e-3(4.75e-05) | 4.80e-2(6.34e-10) | 5.87e-01(1.35e-01) | 1.23e-3(2.29e-19) | 4.80e-2(2.74e-9) | 6.94e-01(1.68e-16) |
| | 1.0 | 1.20e-3(4.16e-05) | 4.80e-2(1.03e-10) | 5.94e-01(1.25e-01) | 1.23e-3(0.00e+00) | **4.79e-2(2.36e-3)** | 5.17e-01(2.22e-01) |
| | 5.0 | 8.09e-4(3.36e-04) | 6.20e-2(1.94e-2) | 1.92e-03(1.60e-03) | 4.41e-4(0.00e+00) | 5.72e-2(7.89e-3) | 1.25e-02(1.57e-04) |
| | 10 | **7.97e-4(3.51e-04)** | 6.09e-2(2.08e-2) | 1.88e-03(1.57e-03) | 4.44e-4(0.00e+00) | 5.57e-2(9.86e-3) | 2.67e-02(2.99e-04) |
| | 50 | 8.08e-4(3.37e-04) | 6.26e-2(1.86e-2) | 1.85e-03(1.54e-03) | 4.29e-4(0.00e+00) | 6.44e-2(1.64e-2) | 1.82e-03(1.52e-05) |
| | 500 | 8.01e-4(3.46e-04) | 6.07e-2(2.10e-2) | **1.85e-03(1.54e-03)** | **4.25e-4(1.14e-19)** | 6.19e-2(1.95e-2) | **1.81e-03(1.53e-05)** |
| DTLZ2, $m=8$ | 0 | 4.57e-3(4.78e-04) | **5.41e-2(1.98e-03)** | 6.84e-01(1.02e-01) | 4.66e-3(3.68e-04) | **5.41e-2(1.98e-03)** | 6.30e-01(2.58e-01) |
| | 0.1 | 4.50e-3(5.62e-04) | 5.41e-2(1.98e-03) | 6.16e-01(1.88e-01) | 4.66e-3(3.68e-04) | 5.41e-2(1.98e-03) | 6.16e-01(1.87e-01) |
| | 1.0 | 4.57e-3(4.78e-04) | 5.41e-2(1.98e-03) | 6.84e-01(1.02e-01) | 4.66e-3(3.68e-04) | 5.41e-2(1.98e-03) | 6.84e-01(1.02e-01) |
| | 5.0 | 2.65e-3(6.68e-04) | 8.21e-2(5.30e-03) | **1.88e-03(1.57e-03)** | 2.85e-3(3.32e-04) | 8.00e-2(5.21e-03) | 8.11e-03(6.39e-33) |
| | 10 | **2.62e-3(7.00e-04)** | 8.25e-2(4.75e-03) | 1.89e-03(1.57e-03) | 2.82e-3(3.71e-04) | 7.73e-2(8.58e-03) | 7.20e-03(8.99e-36) |
| | 50 | 2.64e-3(6.79e-04) | 8.13e-2(6.27e-03) | 1.88e-03(1.56e-03) | 2.70e-3(6.06e-04) | 7.49e-2(1.80e-03) | 1.89e-03(1.57e-03) |
| | 500 | 2.69e-3(6.12e-04) | 8.35e-2(3.55e-03) | 1.97e-03(1.64e-03) | **2.68e-3(6.30e-04)** | 7.27e-2(4.50e-03) | **1.09e-03(1.58e-03)** |
| DTLZ2, $m=10$ | 0 | 1.11e-3(0.00e+00) | **5.73e-2(4.78e-03)** | 4.86e-01(4.05e-01) | 1.40e-3(4.96e-11) | 6.78e-2(2.61e-09) | 6.82e-01(1.60e-01) |
| | 0.1 | 1.25e-3(0.00e+00) | 5.73e-2(4.78e-03) | 4.82e-01(3.13e-01) | 1.39e-3(6.64e-06) | **5.44e-2(1.39e-03)** | 6.82e-01(1.60e-01) |
| | 1.0 | 1.20e-3(0.00e+00) | 5.73e-2(4.78e-03) | 4.80e-01(2.15e-01) | 1.39e-3(6.64e-06) | 6.44e-2(1.39e-03) | 7.42e-02(8.50e-02) |
| | 5.0 | **6.12e-4(1.14e-19)** | 5.04e-2(4.20e-03) | 3.84e-01(3.20e-01) | 7.38e-4(9.19e-05) | 6.14e-2(6.21e-03) | 2.50e-03(3.02e-04) |
| | 10 | 6.20e-4(1.14e-19) | 6.84e-2(4.03e-03) | 3.80e-01(3.20e-01) | 7.36e-4(9.14e-05) | 6.25e-2(4.91e-03) | 7.30e-04(8.68e-05) |
| | 50 | 6.28e-4(1.14e-19) | 6.55e-2(3.79e-03) | 1.06e-05(8.37e-06) | 6.22e-4(1.67e-04) | 6.63e-2(1.19e-03) | 1.05e-05(8.71e-06) |
| | 500 | 6.24e-4(1.14e-19) | 6.55e-2(3.79e-03) | **1.06e-05(8.87e-06)** | **6.22e-4(1.66e-04)** | 6.61e-2(1.21e-03) | **1.05e-05(8.71e-06)** |

Table B.3: IGD,GD and $L_2$ values on DTLZ3 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ3, $m = 2$ | 0 | 3.30e-2(1.55e-03) | 9.49e-2(9.21e-03) | 4.33e-01(2.28e-01) | 9.38e-4(2.26e-04) | 6.67e-2(4.83e-03) | 2.50e-01(1.42e-16) |
| | 0.1 | 5.54e-2(2.33e-02) | 1.51e-1(5.32e-02) | 2.50e-01(1.18e-01) | 1.10e-3(6.97e-05) | 1.04e-1(5.97e-03) | 2.50e-01(2.78e-15) |
| | 1.0 | 4.55e-2(1.82e-02) | 1.05e-1(7.29e-02) | 1.21e-01(1.08e-01) | 1.69e-3(1.86e-04) | 1.17e-1(3.95e-02) | 2.10e-01(1.78e-15) |
| | 5.0 | 6.61e-2(2.41e-02) | 1.34e-1(6.59e-02) | 6.51e-02(1.16e-02) | 8.56e-4(1.78e-04) | 5.99e-2(4.42e-03) | 1.99e-03(7.15e-05) |
| | 10 | 4.27e-2(8.92e-03) | 8.16e-2(8.16e-03) | 4.78e-02(2.01e-02) | 1.71e-3(9.18e-04) | 9.38e-2(7.41e-03) | 1.78e-03(6.76e-05) |
| | 50 | 5.53e-2(6.83e-03) | 1.10e-1(2.75e-02) | 2.22e-02(1.06e-02) | 8.93e-4(2.96e-04) | 3.56e-2(3.46e-03) | 1.61e-03(5.88e-05) |
| | 500 | **2.63e-2(2.12e-02)** | **4.50e-2(3.57e-03)** | **5.21e-03(1.05e-03)** | **6.89e-4(1.21e-04)** | **1.36e-2(9.64e-03)** | **1.38e-03(1.15e-01)** |
| DTLZ3, $m = 4$ | 0 | 9.29e-2(6.87e-02) | 1.65e-1(1.32e-02) | 5.51e+01(1.88e-01) | 3.86e-2(7.84e-04) | 2.35e-1(2.20e-02) | 5.62e-01(3.59e-02) |
| | 0.1 | 6.03e-2(2.79e-02) | 1.01e-1(5.25e-02) | 1.13e+01(1.14e-01) | 3.86e-2(7.84e-04) | **6.53e-2(2.04e-03)** | 5.62e-01(4.87e-02) |
| | 1.0 | **2.24e-2(1.88e-03)** | **3.23e-2(3.54e-03)** | 4.46e-01(1.46e-01) | 4.05e-2(6.11e-03) | 7.39e-2(2.36e-03) | 5.62e-01(5.10e-02) |
| | 5.0 | 2.70e-2(1.50e-02) | 4.48e-2(2.00e-03) | 1.54e-01(1.29e-01) | **3.55e-2(4.64e-03)** | 7.48e-2(2.07e-03) | 3.28e-01(9.55e-02) |
| | 10 | 2.72e-2(1.55e-02) | 4.34e-2(2.31e-03) | 1.25e-01(1.05e-01) | 3.92e-2(4.85e-03) | 7.48e-2(2.18e-03) | 1.91e-01(5.78e-02) |
| | 50 | 3.96e-2(3.33e-03) | 6.61e-2(1.10e-03) | 8.84e-02(7.36e-02) | 4.10e-2(4.41e-03) | 7.48e-2(2.06e-03) | 1.11e-01(3.93e-02) |
| | 500 | 3.65e-2(6.50e-04) | 6.08e-2(4.35e-03) | **8.84e-02(7.36e-02)** | 4.69e-2(2.22e-03) | 1.05e-1(6.30e-02) | **2.31e-02(1.85e-03)** |
| DTLZ3, $m = 6$ | 0 | 1.86e-3(3.65e-04) | 8.22e-2(2.11e-2) | 6.94e-01(2.77e-02) | 1.77e-3(6.48e-04) | 8.91e-2(5.06e-2) | 6.94e-01(2.19e-02) |
| | 0.1 | 2.08e-3(1.03e-03) | 1.08e-1(7.44e-2) | 6.94e-01(3.85e-01) | 1.78e-3(6.50e-04) | 9.68e-2(5.94e-2) | 6.94e-01(1.18e-02) |
| | 1.0 | **1.22e-3(2.67e-05)** | **4.86e-2(2.29e-4)** | 6.60e-01(4.32e-02) | 1.61e-3(4.50e-04) | 6.91e-2(2.52e-2) | 4.07e-01(1.59e-02) |
| | 5.0 | 1.24e-3(7.14e-06) | 4.86e-2(2.28e-4) | 5.33e-01(2.01e-01) | **9.45e-4(3.83e-04)** | **4.67e-2(2.67e-3)** | 6.15e-02(3.39e-03) |
| | 10 | 1.34e-3(1.14e-04) | 5.33e-2(5.72e-3) | 5.20e-01(1.55e-01) | 1.88e-3(6.88e-04) | 1.21e-1(3.12e-2) | 8.84e-03(7.36e-04) |
| | 50 | 1.30e-3(5.56e-05) | 7.72e-2(2.43e-2) | **1.56e-01(1.58e-01)** | 1.28e-3(5.48e-04) | 7.68e-2(2.37e-2) | 7.54e-03(3.01e-04) |
| | 500 | 1.34e-3(1.32e-06) | 7.90e-2(2.17e-2) | 2.90e-01(3.33e-01) | 9.91e-4(4.16e-04) | 7.71e-2(2.44e-2) | **7.26e-03(6.05e-04)** |
| DTLZ3, $m = 8$ | 0 | 1.35e-2(7.86e-03) | 2.43e-1(1.51e-02) | 5.62e-01(3.13e-03) | 1.74e-2(1.55e-02) | 3.19e-1(2.73e-02) | 7.65e-01(2.19e-02) |
| | 0.1 | 1.37e-2(7.99e-03) | 2.46e-1(1.53e-02) | 4.78e-01(2.37e-02) | 6.98e-3(1.64e-03) | 1.37e-1(3.03e-02) | 7.62e-01(4.01e-02) |
| | 1.0 | 1.01e-2(2.30e-03) | 1.82e-1(5.65e-02) | 3.78e-01(4.63e-03) | **1.13e-3(5.83e-03)** | **1.02e-1(1.18e-02)** | 7.50e-01(1.91e-02) |
| | 5.0 | 9.23e-3(1.46e-03) | 1.49e-1(2.77e-02) | 2.94e-01(3.43e-02) | 9.56e-3(9.38e-04) | 1.65e-1(2.68e-02) | 5.62e-01(1.57e-02) |
| | 10 | 6.02e-3(1.49e-03) | **9.23e-2(2.63e-03)** | **1.33e-01(1.12e-02)** | 7.80e-3(2.94e-03) | 1.36e-1(6.24e-02) | 3.72e-01(2.00e-02) |
| | 50 | **6.01e-3(1.42e-03)** | 9.24e-2(2.56e-03) | 1.52e-01(9.93e-03) | 8.23e-2(2.18e-03) | 2.05e-1(4.19e-02) | 1.22e-01(6.39e-03) |
| | 500 | 7.35e-3(2.70e-04) | 1.15e-1(3.28e-03) | 1.50e-01(8.76e-03) | 8.95e-2(2.90e-03) | 2.28e-1(5.30e-02) | **7.64e-02(7.15e-03)** |
| DTLZ3, $m = 10$ | 0 | 2.99e-2(6.50e-03) | 5.97e-1(1.46e-01) | 4.14e+01(3.20e+01) | 8.16e-3(3.26e-03) | 1.45e-1(6.42e-02) | 9.35e-01(8.03e-02) |
| | 0.1 | 1.25e-2(6.21e-03) | 2.51e-1(1.40e-01) | 8.10e-01(1.20e-01) | 8.06e-3(3.15e-03) | 1.52e-1(4.96e-02) | 8.19e-01(9.13e-02) |
| | 1.0 | 7.43e-3(1.26e-04) | 1.49e-1(1.87e-02) | 7.99e-01(5.12e-02) | 7.23e-3(4.22e-05) | 1.44e-1(1.70e-02) | 7.59e-01(9.10e-02) |
| | 5.0 | 6.18e-3(1.33e-03) | 1.24e-1(4.23e-02) | 7.65e-01(9.10e-02) | 6.11e-3(1.44e-03) | 1.23e-1(4.40e-02) | 7.39e-01(7.54e-02) |
| | 10 | **6.06e-3(1.48e-03)** | **1.22e-1(4.49e-02)** | 7.55e-01(7.20e-02) | 6.06e-3(1.50e-03) | 1.22e-1(4.53e-02) | 7.25e-01(5.17e-02) |
| | 50 | 6.06e-3(1.48e-03) | 1.22e-1(4.49e-02) | 6.34e-01(3.20e-02) | 6.19e-3(1.36e-03) | 1.24e-1(4.27e-02) | 8.09e-02(2.34e-02) |
| | 500 | 6.06e-3(1.48e-03) | 1.24e-1(7.67e-04) | **6.14e-01(3.10e-02)** | **6.04e-3(1.40e-03)** | **1.02e-1(2.71e-02)** | **6.33e-02(4.11e-03)** |

Table B.4: IGD,GD and $L_2$ values on DTLZ4 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ4, $m=2$ | 0 | 1.00e-1(2.93e-17) | 9.95e-2(1.03e-09) | 2.50e-01(0.00e+00) | 1.00e-3(3.27e-13) | 9.95e-2(3.26e-11) | 2.50e-01(0.00e+00) |
| | 0.1 | 4.63e-2(0.00e+00) | **5.98e-2(4.95e-03)** | **1.50e-01(1.25e-01)** | 7.85e-4(2.68e-04) | **5.99e-2(4.95e-03)** | 1.50e-01(1.25e-01) |
| | 1.0 | 8.61e-4(2.29e-19) | 5.99e-2(4.96e-03) | 1.50e-01(1.25e-01) | 6.51e-4(4.36e-04) | 6.95e-2(3.75e-03) | 1.50e-01(1.25e-01) |
| | 5.0 | 1.28e-3(2.29e-19) | 5.99e-2(4.94e-03) | 1.50e-01(1.25e-01) | 6.49e-4(4.39e-04) | 6.92e-2(3.79e-03) | 1.50e-01(1.25e-01) |
| | 10 | **8.51e-4(2.29e-19)** | 5.99e-2(4.96e-03) | 1.50e-01(1.25e-01) | 6.48e-4(4.40e-04) | 6.91e-2(3.80e-03) | 1.50e-01(1.25e-01) |
| | 50 | 1.02e-3(2.29e-19) | 5.99e-2(4.95e-03) | 1.50e-01(1.25e-01) | **6.46e-4(4.42e-04)** | 6.87e-2(3.85e-03) | 1.49e-01(1.12e-01) |
| | 500 | 1.01e-3(2.29e-19) | 5.99e-2(4.96e-03) | 1.50e-01(1.25e-01) | 6.55e-4(4.32e-04) | 6.99e-2(3.70e-03) | **1.45e-01(2.45e-01)** |
| DTLZ4, $m=4$ | 0 | 3.75e-2(9.32e-03) | **3.29e-2(1.81e-02)** | 3.38e-01(2.81e-01) | 4.36e-2(7.06e-03) | 7.52e-2(1.09e-03) | 5.40e-01(1.12e-01) |
| | 0.1 | 4.50e-2(2.87e-11) | 7.74e-2(4.94e-11) | 5.63e-01(0.00e+00) | 4.36e-2(7.06e-03) | 7.54e-2(1.13e-03) | 5.40e-01(1.12e-01) |
| | 1.0 | **1.50e-2(8.29e-10)** | 1.03e-1(3.19e-03) | 5.62e-01(2.56e-15) | 4.43e-2(3.26e-03) | 6.30e-2(1.80e-03) | 3.62e-01(2.50e-01) |
| | 5.0 | 1.57e-2(7.93e-03) | 3.32e-2(1.83e-03) | 2.98e-04(1.73e-04) | **1.57e-2(9.51e-03)** | **3.08e-2(1.06e-03)** | 3.75e-02(3.13e-02) |
| | 10 | 1.57e-2(7.92e-03) | 3.39e-2(1.74e-03) | 2.94e-04(2.45e-04) | 1.74e-2(6.45e-03) | 3.81e-2(1.25e-03) | 2.43e-05(3.03e-05) |
| | 50 | 1.57e-2(8.02e-03) | 3.39e-2(1.80e-03) | **2.76e-04(2.30e-04)** | 1.71e-2(6.18e-03) | 3.72e-2(1.34e-03) | **1.89e-05(1.57e-05)** |
| | 500 | 1.58e-2(7.82e-03) | 3.39e-2(1.75e-03) | 2.76e-04(2.30e-04) | 1.69e-2(6.42e-03) | 3.70e-2(1.35e-03) | 3.43e-05(2.86e-05) |
| DTLZ4, $m=6$ | 0 | 1.35e-3(2.52e-04) | **5.48e-2(3.15e-2)** | 4.17e-01(3.47e-01) | 1.45e-3(5.36e-12) | 7.36e-2(2.72e-10) | 6.94e-01(0.00e+00) |
| | 0.1 | 1.30e-3(1.87e-04) | 8.81e-2(1.49e-2) | 4.28e-01(3.33e-01) | 1.45e-3(3.76e-12) | 7.36e-2(1.91e-10) | 6.94e-01(0.00e+00) |
| | 1.0 | 1.45e-3(2.26e-12) | 8.15e-2(9.83e-3) | 3.94e-01(2.47e-14) | 1.45e-3(3.10e-14) | 7.85e-2(6.06e-3) | 6.94e-01(1.14e-14) |
| | 5.0 | 8.42e-3(3.04e-04) | 5.72e-2(3.07e-2) | 3.17e-02(3.25e-03) | **6.61e-4(2.27e-04)** | **1.03e-2(3.80e-2)** | **1.67e-02(1.39e-02)** |
| | 10 | 8.52e-3(4.16e-04) | 5.84e-2(2.93e-2) | 3.17e-02(1.71e-02) | 7.46e-4(1.21e-04) | 6.89e-2(9.25e-3) | 2.78e-02(1.06e-17) |
| | 50 | 8.52e-3(2.04e-04) | 7.78e-2(5.01e-3) | 3.17e-02(3.25e-03) | 8.71e-4(3.93e-04) | 6.34e-2(2.30e-2) | 2.06e-02(1.72e-02) |
| | 500 | **8.53e-4(4.15e-04)** | 6.28e-2(2.38e-2) | **2.06e-02(1.71e-02)** | 8.78e-4(3.83e-04) | 5.80e-2(2.10e-4) | 2.06e-02(1.71e-02) |
| DTLZ4, $m=8$ | 0 | 4.32e-3(7.86e-04) | **4.88e-2(2.65e-03)** | 5.66e-01(3.75e-01) | 4.96e-3(3.28e-11) | 8.03e-2(5.31e-10) | 7.66e-01(0.00e+00) |
| | 0.1 | **3.95e-3(5.14e-11)** | 6.14e-1(6.67e-02) | 5.49e-01(2.71e-01) | 4.96e-3(1.95e-11) | 8.03e-2(3.17e-10) | 7.66e-01(0.00e+00) |
| | 1.0 | 4.95e-3(2.34e-11) | 1.16e-1(4.52e-02) | 4.66e-01(1.02e-13) | 4.96e-3(1.22e-11) | 1.16e-1(4.44e-02) | 7.66e-01(5.05e-14) |
| | 5.0 | 4.81e-3(1.43e-03) | 6.45e-1(1.97e-02) | 4.66e-01(3.75e-01) | 3.12e-3(5.83e-04) | **5.06e-2(1.47e-03)** | 9.25e-02(0.00e+00) |
| | 10 | 4.64e-3(1.64e-03) | 6.43e-1(2.00e-02) | 4.59e-01(3.83e-01) | 3.23e-3(4.93e-05) | 5.53e-2(8.21e-03) | 8.37e-02(3.91e-02) |
| | 50 | 4.29e-3(1.09e-03) | 6.66e-1(1.38e-02) | 4.45e-02(2.74e-02) | 3.17e-3(3.07e-04) | 5.53e-2(2.53e-04) | 7.50e-02(1.90e-16) |
| | 500 | 4.76e-3(9.14e-04) | 5.29e-1(1.27e-02) | **4.39e-02(3.66e-02)** | **3.03e-3(7.92e-04)** | 6.23e-2(1.17e-03) | **6.73e-02(3.99e-03)** |
| DTLZ4, $m=10$ | 0 | 4.45e-3(6.30e-04) | 7.98e-2(1.94e-03) | 6.30e-01(2.25e-01) | 4.25e-3(1.46e-11) | 7.64e-2(2.74e-10) | 8.10e-01(0.00e+00) |
| | 0.1 | 4.95e-3(8.43e-11) | 7.64e-2(1.50e-09) | 8.10e-01(0.00e+00) | 4.25e-3(6.71e-11) | 7.72e-2(1.08e-03) | 8.10e-01(0.00e+00) |
| | 1.0 | 4.95e-3(1.43e-12) | 1.70e-1(1.17e-02) | 8.10e-01(2.19e-04) | 4.25e-3(3.11e-11) | 7.94e-2(3.73e-03) | 8.10e-01(3.36e-14) |
| | 5.0 | 3.78e-3(1.47e-03) | **6.69e-2(9.82e-03)** | 5.22e-01(3.60e-01) | **2.56e-3(8.71e-04)** | **6.58e-2(1.32e-03)** | 5.22e-01(3.60e-01) |
| | 10 | **2.19e-3(4.83e-04)** | 6.85e-2(2.58e-03) | 7.00e-02(2.50e-02) | 2.89e-3(1.69e-04) | 7.32e-2(5.72e-03) | 1.48e-01(3.50e-02) |
| | 50 | 2.87e-3(7.74e-04) | 7.60e-2(1.20e-03) | 7.00e-02(2.50e-02) | 2.72e-3(5.34e-04) | 6.90e-2(2.62e-03) | 1.32e-01(3.50e-02) |
| | 500 | 2.23e-3(4.57e-19) | 7.03e-2(2.16e-03) | **5.80e-02(4.00e-02)** | 3.07e-3(3.20e-04) | 6.96e-2(1.65e-03) | **8.80e-02(6.00e-02)** |

Table B.5: IGD,GD and $L_2$ values on DTLZ5 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ5, $m=2$ | 0 | 4.27e-2(7.15e-12) | **5.35e-2(6.00e-12)** | 3.76e-09(2.55e-11) | 4.27e-4(7.24e-14) | **5.35e-2(6.08e-12)** | 3.76e-09(2.55e-11) |
| | 0.1 | **4.27e-2(8.43e-12)** | 5.35e-2(6.73e-12) | 3.76e-09(2.55e-11) | **4.27e-4(1.75e-13)** | 5.35e-2(1.19e-11) | 3.76e-09(2.55e-11) |
| | 1.0 | 4.27e-2(7.88e-12) | 5.35e-2(8.45e-12) | 3.76e-09(2.55e-11) | 4.27e-4(2.89e-14) | 5.35e-2(3.10e-12) | 3.76e-09(2.55e-11) |
| | 5.0 | 4.27e-2(1.94e-12) | 5.35e-2(3.19e-12) | 3.76e-09(2.55e-11) | 4.27e-4(3.62e-14) | 5.35e-2(5.50e-12) | 3.76e-09(2.55e-11) |
| | 10 | 4.27e-2(9.66e-12) | 5.35e-2(5.20e-12) | 3.76e-09(2.55e-11) | 4.27e-4(3.17e-13) | 5.35e-2(6.54e-12) | 3.76e-09(2.55e-11) |
| | 50 | 4.27e-2(1.65e-11) | 5.35e-2(8.71e-12) | 3.76e-09(2.55e-11) | 4.27e-4(2.79e-14) | 5.35e-2(7.08e-12) | 3.76e-09(2.55e-11) |
| | 500 | 4.27e-2(3.50e-12) | 5.35e-2(1.62e-12) | **3.76e-09(2.55e-11)** | 4.27e-4(6.47e-14) | 5.35e-2(1.17e-11) | **3.76e-09(2.55e-11)** |
| DTLZ5, $m=4$ | 0 | 1.72e-2(2.12e-04) | **3.25e-2(4.05e-04)** | 3.38e-01(2.81e-01) | 1.73e-2(3.60e-12) | 3.32e-2(6.64e-12) | 3.20e-07(1.73e-08) |
| | 0.1 | **1.71e-2(3.08e-04)** | 3.25e-2(5.89e-04) | 5.63e-01(0.00e+00) | 1.73e-2(4.12e-06) | 3.32e-2(7.87e-06) | 4.20e-08(1.17e-09) |
| | 1.0 | 1.73e-2(9.78e-05) | 3.31e-2(1.87e-04) | 5.62e-01(2.56e-15) | **1.71e-2(9.75e-05)** | **3.31e-2(1.86e-04)** | 3.34e-08(9.93e-09) |
| | 5.0 | 1.71e-2(1.79e-04) | 3.26e-2(3.41e-04) | 2.08e-04(1.73e-04) | 1.71e-2(1.40e-04) | 3.31e-2(2.66e-04) | 3.15e-08(7.25e-09) |
| | 10 | 1.71e-2(2.62e-04) | 3.26e-2(4.98e-04) | 2.94e-04(2.45e-04) | 1.71e-2(1.52e-04) | 3.31e-2(2.88e-04) | **2.79e-08(9.41e-09)** |
| | 50 | 1.71e-2(2.66e-04) | 3.26e-2(5.06e-04) | 2.76e-04(2.30e-04) | 1.71e-2(1.74e-04) | 3.31e-2(3.29e-04) | 2.80e-08(2.85e-09) |
| | 500 | 1.71e-2(2.64e-04) | 3.26e-2(5.02e-04) | **2.76e-04(2.30e-04)** | 1.71e-2(1.74e-04) | 3.31e-2(3.28e-04) | 1.07e-07(7.42e-08) |
| DTLZ5, $m=6$ | 0 | **1.57e-3(2.25e-03)** | **2.74e-2(4.30e-2)** | 1.00e-06(4.63e-07) | **1.06e-3(2.74e-03)** | **2.75e-2(4.29e-2)** | 1.04e-06(8.70e-07) |
| | 0.1 | 2.51e-3(5.35e-04) | 4.80e-2(1.02e-2) | 1.05e-06(8.73e-07) | 1.68e-3(7.36e-04) | 4.84e-2(9.78e-3) | 1.04e-06(8.70e-07) |
| | 1.0 | 1.92e-3(9.85e-06) | 3.67e-2(1.88e-4) | 2.20e-18(5.08e-19) | 1.81e-3(1.76e-05) | 3.67e-2(1.72e-4) | 2.16e-08(4.92e-09) |
| | 5.0 | 1.91e-3(1.49e-05) | 3.65e-2(2.86e-4) | **2.12e-18(4.69e-19)** | 1.80e-3(1.24e-05) | 3.66e-2(2.19e-4) | 1.92e-08(2.15e-09) |
| | 10 | 1.91e-3(1.83e-05) | 3.64e-2(3.51e-4) | 2.12e-18(5.87e-19) | 1.80e-3(1.10e-05) | 3.66e-2(2.23e-4) | 1.90e-08(3.14e-09) |
| | 50 | 1.91e-3(1.64e-05) | 3.65e-2(3.13e-4) | 2.12e-18(6.45e-19) | 1.80e-3(1.49e-05) | 3.65e-2(2.77e-4) | 1.04e-08(4.16e-09) |
| | 500 | 1.90e-3(1.95e-05) | 3.64e-2(3.64e-2) | 2.35e-18(9.34e-19) | 1.80e-3(1.16e-05) | 3.65e-2(2.49e-4) | **1.94e-09(4.49e-10)** |
| DTLZ5, $m=8$ | 0 | **9.63e-4(1.29e-05)** | **4.05e-2(5.43e-04)** | 1.39e-08(1.66e-08) | 9.63e-4(1.27e-05) | **4.05e-2(5.35e-04)** | 1.16e-08(1.38e-08) |
| | 0.1 | 9.63e-4(1.26e-05) | 4.05e-2(5.32e-04) | 1.89e-09(1.52e-09) | **9.53e-4(3.75e-06)** | 4.05e-2(5.04e-04) | 8.29e-08(3.20e-10) |
| | 1.0 | 9.72e-4(4.67e-06) | 4.09e-2(1.95e-04) | 1.88e-09(1.37e-12) | 9.75e-4(2.18e-06) | 4.09e-2(1.56e-04) | 9.69e-08(6.53e-11) |
| | 5.0 | 9.73e-4(9.02e-06) | 4.09e-2(3.76e-04) | 1.84e-09(5.10e-10) | 9.74e-4(8.50e-06) | 4.09e-2(3.53e-04) | 1.86e-09(6.91e-10) |
| | 10 | 9.71e-4(1.23e-05) | 4.08e-2(5.15e-04) | 1.86e-09(7.50e-10) | 9.73e-4(8.81e-06) | 4.09e-2(3.67e-04) | 1.73e-09(7.03e-10) |
| | 50 | 9.70e-4(9.41e-06) | 4.08e-2(3.93e-04) | **1.21e-09(2.25e-10)** | 9.72e-4(9.25e-06) | 4.09e-2(3.85e-04) | 1.35e-09(5.02e-10) |
| | 500 | 9.69e-4(1.11e-05) | 4.07e-2(4.62e-04) | 1.30e-09(1.71e-10) | 9.70e-4(9.28e-06) | 4.08e-2(3.87e-04) | **1.18e-09(2.53e-10)** |
| DTLZ5, $m=10$ | 0 | 1.27e-3(5.33e-04) | 8.80e-2(3.24e-02) | 1.65e-04(5.70e-04) | 1.32e-3(5.50e-04) | 8.61e-2(3.08e-03) | 9.84e-04(1.89e-06) |
| | 0.1 | 6.85e-3(4.59e-05) | **5.06e-2(3.19e-03)** | 8.31e-04(6.80e-04) | **7.25e-4(5.04e-05)** | **5.05e-2(3.21e-03)** | 8.40e-04(6.88e-04) |
| | 1.0 | **6.88e-4(4.06e-06)** | 5.06e-2(4.82e-04) | 1.66e-05(4.20e-06) | 7.32e-4(4.80e-06) | 5.07e-2(4.83e-03) | 1.63e-05(4.63e-06) |
| | 5.0 | 6.88e-4(7.32e-06) | 5.06e-2(3.16e-04) | 8.69e-06(3.40e-06) | 7.29e-4(4.85e-06) | 5.07e-2(3.46e-03) | 8.89e-06(3.14e-06) |
| | 10 | 6.88e-4(5.24e-06) | 5.06e-2(2.61e-04) | 8.04e-06(2.15e-06) | 7.28e-4(2.25e-06) | 5.07e-2(3.13e-03) | 8.15e-06(2.03e-06) |
| | 50 | 6.88e-4(6.31e-06) | 5.06e-2(3.09e-04) | 5.86e-06(6.65e-07) | 7.28e-4(5.54e-06) | 5.07e-2(3.14e-03) | 5.93e-06(5.86e-07) |
| | 500 | 6.88e-4(9.34e-06) | 5.06e-2(3.87e-04) | **5.77e-06(6.76e-07)** | 7.18e-4(4.16e-06) | 5.07e-2(2.71e-03) | **4.07e-06(3.01e-07)** |

Table B.6: IGD,GD and $L_2$ values on DTLZ6 test problems. The mean and STD of 25 independent runs are reported.

| | | MOEA/D | | | R-MEAD2 | | |
|---|---|---|---|---|---|---|---|
| Function, $m$ | $\theta$ | IGD | GD | $L_2$ | IGD | GD | $L_2$ |
| DTLZ6, $m=2$ | 0 | 4.24e-2(4.29e-04) | **5.32e-2(2.49e-04)** | **3.76e-09(1.10e-10)** | 4.24e-4(3.69e-06) | 5.63e-2(2.13e-04) | 3.76e-06(1.10e-10) |
| | 0.1 | 4.26e-2(1.45e-04) | 5.32e-2(8.24e-05) | 3.76e-09(1.08e-10) | 4.30e-4(1.82e-05) | 5.54e-2(1.97e-03) | 3.76e-06(3.54e-10) |
| | 1.0 | 4.27e-2(7.08e-18) | 5.32e-2(7.08e-18) | 3.76e-09(2.55e-11) | 4.40e-4(4.35e-06) | 5.39e-2(5.28e-04) | 3.76e-06(1.10e-10) |
| | 5.0 | **4.23e-2(4.08e-04)** | 5.32e-2(1.58e-04) | 3.76e-09(3.25e-11) | **4.26e-4(1.79e-07)** | 5.34e-2(8.43e-05) | 3.76e-06(1.10e-10) |
| | 10 | 4.24e-2(3.33e-04) | 5.32e-2(1.16e-04) | 3.76e-09(3.41e-11) | 4.26e-4(1.66e-19) | 5.35e-2(7.08e-18) | 3.76e-06(2.55e-11) |
| | 50 | 4.27e-2(1.41e-05) | 5.32e-2(1.98e-04) | 3.76e-09(3.96e-10) | 4.26e-4(3.26e-07) | **5.33e-2(1.06e-04)** | **3.75e-06(1.09e-10)** |
| | 500 | 4.27e-2(2.45e-06) | 5.32e-2(2.06e-04) | 3.76e-09(5.38e-10) | 4.27e-4(1.13e-06) | 5.34e-2(2.69e-04) | 3.77e-06(2.92e-10) |
| DTLZ6, $m=4$ | 0 | 1.73e-2(1.06e-17) | 3.35e-2(1.19e-17) | 9.02e-03(7.28e-39) | 1.77e-2(6.95e-05) | **3.29e-2(1.29e-04)** | 4.29e-03(1.03e-03) |
| | 0.1 | 1.73e-2(5.19e-05) | **3.33e-2(6.15e-04)** | 7.04e-03(8.48e-37) | 1.77e-2(7.22e-05) | 3.29e-2(1.35e-04) | 3.66e-03(1.19e-03) |
| | 1.0 | **1.72e-2(4.60e-05)** | 3.35e-2(8.54e-05) | 1.72e-03(1.09e-38) | 1.77e-2(1.62e-05) | 3.30e-2(3.34e-05) | 3.57e-03(1.64e-03) |
| | 5.0 | 1.74e-2(4.55e-04) | 3.35e-2(8.54e-04) | 1.92e-03(1.60e-32) | 1.77e-2(4.62e-04) | 3.34e-2(8.66e-04) | 3.51e-03(3.01e-03) |
| | 10 | 1.75e-2(5.18e-04) | 3.35e-2(9.74e-04) | 7.67e-03(6.39e-32) | 1.77e-2(5.44e-04) | 3.34e-2(1.02e-03) | 2.54e-03(1.29e-03) |
| | 50 | 1.75e-2(5.89e-04) | 3.35e-2(1.11e-03) | **1.71e-03(1.42e-31)** | 1.76e-2(6.07e-04) | 3.35e-2(1.14e-03) | **1.73e-03(1.45e-03)** |
| | 500 | 1.75e-2(6.96e-04) | 3.35e-2(1.31e-03) | 2.57e-03(2.14e-31) | **1.75e-2(6.52e-04)** | 3.35e-2(1.22e-03) | 2.40e-03(2.00e-03) |
| DTLZ6, $m=6$ | 0 | 2.02e-2(6.90e-03) | 1.95e-1(1.32e-1) | 3.22e-06(1.56e-06) | 1.14e-2(7.98e-03) | **2.85e-2(1.32e-1)** | 6.59e-08(5.76e-09) |
| | 0.1 | 2.72e-3(6.65e-04) | 5.20e-2(1.27e-2) | 3.12e-06(6.76e-06) | 2.95e-3(9.54e-04) | 5.26e-2(1.32e-2) | 4.99e-08(1.66e-09) |
| | 1.0 | 1.96e-3(3.36e-05) | 3.83e-2(3.17e-4) | 4.38e-05(3.52e-05) | 1.92e-3(5.15e-05) | 3.87e-2(7.31e-4) | 3.64e-08(2.72e-09) |
| | 5.0 | 1.91e-3(3.65e-06) | **3.60e-2(6.99e-5)** | 1.02e-08(6.64e-09) | 1.82e-3(2.31e-05) | 3.66e-2(7.31e-5) | 4.36e-09(1.20e-10) |
| | 10 | **1.90e-3(1.25e-05)** | 3.64e-2(2.39e-4) | 7.99e-09(3.92e-09) | **1.80e-3(8.20e-06)** | 3.64e-2(2.53e-4) | 4.31e-09(1.26e-10) |
| | 50 | 1.90e-3(2.29e-19) | 3.63e-2(2.70e-4) | 3.82e-09(1.30e-09) | 1.80e-3(2.13e-06) | 3.64e-2(1.79e-4) | 4.28e-09(3.42e-10) |
| | 500 | 1.90e-3(9.03e-06) | 3.64e-2(1.73e-4) | **3.43e-09(1.79e-09)** | 1.80e-3(1.03e-06) | 3.64e-2(1.89e-4) | **3.28e-09(1.97e-10)** |
| DTLZ6, $m=8$ | 0 | 4.20e-3(2.69e-03) | 1.77e-1(1.13e-01) | 1.73e-08(1.10e-09) | 6.40e-3(4.52e-03) | 2.71e-1(1.88e-01) | 1.24e-007(8.67e-8) |
| | 0.1 | 9.76e-4(2.16e-06) | 4.10e-2(8.78e-05) | 1.78e-09(1.00e-09) | 9.80e-4(5.30e-06) | 4.15e-2(1.69e-04) | 6.87e-08(4.61e-09) |
| | 1.0 | 9.85e-4(2.83e-06) | **4.08e-2(1.16e-04)** | 1.18e-08(4.44e-09) | 9.84e-4(7.93e-06) | 4.14e-2(3.25e-04) | 1.27e-08(1.30e-08) |
| | 5.0 | **9.70e-4(1.17e-05)** | 4.08e-2(4.87e-04) | 1.65e-09(1.21e-09) | 9.76e-4(1.08e-05) | **4.10e-2(4.45e-04)** | 1.23e-09(8.37e-10) |
| | 10 | 9.70e-4(8.36e-06) | 4.08e-2(3.52e-04) | **3.37e-10(2.50e-10)** | 9.76e-4(1.37e-05) | 4.10e-2(5.72e-04) | 5.41e-10(4.00e-10) |
| | 50 | 9.71e-4(1.19e-05) | 4.08e-2(4.95e-04) | 3.37e-10(1.97e-10) | 9.76e-4(1.42e-05) | 4.10e-2(5.94e-04) | 3.85e-10(3.00e-10) |
| | 500 | 9.71e-4(1.29e-05) | 4.08e-2(5.38e-04) | 3.37e-10(1.84e-10) | **9.75e-4(1.39e-05)** | 4.10e-2(5.80e-04) | **3.60e-10(2.87e-10)** |
| DTLZ6, $m=10$ | 0 | 1.37e-3(6.12e-04) | 9.49e-2(3.57e-03) | 1.39e-04(1.16e-04) | 1.57e-3(7.23e-04) | 1.02e-1(3.82e-02) | 9.77e-04(8.12e-04) |
| | 0.1 | 6.98e-4(5.26e-05) | 5.40e-2(3.97e-03) | 5.24e-04(4.16e-04) | **6.23e-4(7.10e-05)** | **5.08e-2(4.68e-03)** | 8.22e-04(6.72e-04) |
| | 1.0 | **6.97e-4(3.73e-06)** | **5.17e-2(8.49e-4)** | 1.22e-04(2.50e-05) | 7.29e-4(1.11e-05) | 5.27e-2(1.35e-03) | 1.04e-04(6.67e-05) |
| | 5.0 | 7.84e-4(4.08e-06) | 5.17e-2(1.04e-03) | 3.83e-06(9.24e-07) | 7.09e-4(3.80e-06) | 5.29e-2(2.51e-04) | 2.95e-06(2.02e-06) |
| | 10 | 7.84e-4(2.70e-07) | 5.18e-2(8.12e-04) | 2.88e-06(1.21e-06) | 7.01e-4(1.28e-05) | 5.11e-2(1.59e-03) | 2.44e-06(1.74e-06) |
| | 50 | 7.85e-4(1.65e-06) | 5.18e-2(4.41e-04) | **1.54e-06(7.73e-07)** | 7.10e-4(3.08e-06) | 5.15e-2(4.59e-04) | 1.70e-06(7.17e-07) |
| | 500 | 7.84e-4(2.40e-06) | 5.17e-2(6.48e-04) | 1.66e-06(3.14e-07) | 7.15e-4(1.39e-06) | 5.13e-2(6.68e-04) | **1.32e-06(8.24e-07)** |

## Appendix C

# Hypervolume and IGD results for UR-MEAD2 Method

*Table C.1: HV values for PBI decomposition method on DTLZ1-DTLZ6 test problems. The parameter $\theta = 5$ unless specified. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | | |
|---|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV | PEV ($\theta$=0) |
| DTLZ1 | 4 | 8.90e-03 | 1.16e-02 | 8.03e-02$^{\uparrow}$ | 9.26e-02$^{\uparrow}$ | 10.16e-02$^{\uparrow}$ | **10.28e-02**$^{\uparrow}$ |
| | 5 | 1.00e-02 | 1.19e-02 | **9.64e-02**$^{\uparrow}$ | 9.52e-02$^{\uparrow}$ | 0.17e-02$^{\downarrow}$ | 0.34e-02$^{\downarrow}$ |
| | 6 | 8.91e-02 | 1.31e-02 | 2.09e-02$^{\uparrow}$ | 1.31e-02$^{\approx}$ | 4.11e-02$^{\uparrow}$ | 9.82e-02$^{\uparrow}$ |
| | 7 | 5.40e-03 | 0.40e-02 | 1.21e-02$^{\uparrow}$ | 1.21e-02$^{\uparrow}$ | **2.23e-02**$^{\uparrow}$ | 0.16e-02$^{\uparrow}$ |
| | 8 | 3.19e-03 | 1.51e-02 | **2.33e-02**$^{\uparrow}$ | **2.33e-02**$^{\uparrow}$ | 2.22e-02$^{\uparrow}$ | 2.21e-02$^{\uparrow}$ |
| | 9 | 7.53e-03 | 1.66e-02 | 2.45e-02$^{\uparrow}$ | 2.46e-02$^{\uparrow}$ | **3.15e-02**$^{\uparrow}$ | 3.13e-02$^{\uparrow}$ |
| | 10 | 1.75e-02 | 0.98e-02 | 1.60e-02$^{\uparrow}$ | 1.60e-02$^{\uparrow}$ | 1.98e-02$^{\uparrow}$ | **2.21e-02**$^{\uparrow}$ |
| DTLZ2 | 4 | 80.90e-04 | 71.11e-03 | 15.35e-02$^{\uparrow}$ | 11.83e-02$^{\uparrow}$ | **33.33e-02**$^{\uparrow}$ | 33.31e-02$^{\uparrow}$ |
| | 5 | 12.41e-02 | 9.38e-02 | 14.15e-02$^{\uparrow}$ | 13.52e-02$^{\uparrow}$ | 10.0e-02$^{\uparrow}$ | **14.64e-02**$^{\uparrow}$ |
| | 6 | 11.56e-02 | 6.64e-02 | 7.38e-02$^{\uparrow}$ | 13.52e-02$^{\uparrow}$ | 15.0e-02$^{\uparrow}$ | **16.10e-02**$^{\uparrow}$ |
| | 7 | 10.71e-03 | 71.53e-03 | 12.82e-02$^{\uparrow}$ | 13.93e-02$^{\uparrow}$ | 1.51e-02$^{\downarrow}$ | **17.71e-02**$^{\uparrow}$ |
| | 8 | 10.04e-03 | 79.70e-03 | 12.02e-02$^{\uparrow}$ | 19.48e-02$^{\uparrow}$ | **20.21e-02**$^{\uparrow}$ | 19.48e-02$^{\uparrow}$ |
| | 9 | 92.17e-03 | 10.03e-02 | 12.27e-02$^{\uparrow}$ | 13.92e-02$^{\uparrow}$ | 29.31e-02$^{\uparrow}$ | **40.92e-02**$^{\uparrow}$ |
| | 10 | 80.38e-3 | 28.41e-02 | 11.34e-02$^{\downarrow}$ | 23.57e-02$^{\downarrow}$ | 28.41e-02$^{\approx}$ | **45.01e-02**$^{\uparrow}$ |
| DTLZ3 | 4 | 80.90e-03 | 10.01e-02 | 12.02e-02$^{\uparrow}$ | 12.30e-02$^{\uparrow}$ | **37.34e-02**$^{\uparrow}$ | 36.44e-02$^{\uparrow}$ |
| | 5 | 12.39e-03 | 14.56e-03 | 9.44e-02$^{\uparrow}$ | 9.63e-02$^{\uparrow}$ | 10.32e-02$^{\uparrow}$ | **10.49e-02**$^{\uparrow}$ |
| | 6 | 11.37e-03 | 5.37e-03 | 12.02e-02$^{\uparrow}$ | 12.30e-02$^{\uparrow}$ | **14.62e-02**$^{\uparrow}$ | 14.33e-02$^{\uparrow}$ |
| | 7 | 11.41e-03 | 3.73e02 | 7.27e-02$^{\uparrow}$ | 8.81e-02$^{\uparrow}$ | 9.36e-02$^{\uparrow}$ | **10.22e-02**$^{\uparrow}$ |
| | 8 | 10.70e-03 | 1.23e-02 | 5.37e-02$^{\uparrow}$ | 6.33e-02$^{\uparrow}$ | **9.21e-02**$^{\uparrow}$ | 8.33e-02$^{\uparrow}$ |
| | 9 | 5.09e-03 | 7.69e-03 | 7.43e-02$^{\downarrow}$ | 9.75e-02$^{\uparrow}$ | 9.83e-02$^{\uparrow}$ | **10.03e-02**$^{\uparrow}$ |
| | 10 | 7.88e-03 | 6.01e-03 | 4.24e-02$^{\uparrow}$ | 5.23e-02$^{\uparrow}$ | 5.38e-02$^{\uparrow}$ | **6.66e-02**$^{\uparrow}$ |
| DTLZ4 | 4 | 13.58e-03 | 10.36e-02 | 11.75e-02$^{\uparrow}$ | 12.37e-02$^{\uparrow}$ | **37.38e-02**$^{\uparrow}$ | 37.31e-02$^{\uparrow}$ |
| | 5 | 15.39e-03 | 10.44e-02 | 14.64e-02$^{\uparrow}$ | 17.25e-02$^{\uparrow}$ | 18.06e-02$^{\uparrow}$ | **24.64e-02**$^{\uparrow}$ |
| | 6 | 11.62e-03 | 14.01e-03 | 16.10e-02$^{\uparrow}$ | 15.66e-02$^{\uparrow}$ | **16.13e-02**$^{\uparrow}$ | 44.05e-03$^{\uparrow}$ |
| | 7 | 15.27e-03 | 10.33e-02 | 11.71e-02$^{\uparrow}$ | 10.33e-02$^{\approx}$ | 15.31e-02$^{\uparrow}$ | **17.82e-02**$^{\uparrow}$ |
| | 8 | 11.79e-03 | 10.99e-02 | 19.48e-02$^{\uparrow}$ | 19.48e-02$^{\uparrow}$ | 20.27e-02$^{\uparrow}$ | **37.20e-02**$^{\uparrow}$ |
| | 9 | 11.17e-03 | 6.03e-03 | 21.43e-02$^{\uparrow}$ | 6.03e-02$^{\approx}$ | 25.30e-02$^{\uparrow}$ | **30.92e-02**$^{\uparrow}$ |
| | 10 | 10.00e-03 | 5.93e-03 | 23.57e-02$^{\uparrow}$ | 12.48e-02$^{\uparrow}$ | 14.46e-02$^{\uparrow}$ | **64.49e-02**$^{\uparrow}$ |
| DTLZ5 | 4 | 13.37e-03 | 29.43e-03 | 11.90e-02$^{\uparrow}$ | 12.17e-02$^{\uparrow}$ | **16.75e-02**$^{\uparrow}$ | 15.88e-02$^{\uparrow}$ |
| | 5 | 12.90e-03 | 37.99e-03 | 11.15e-02$^{\uparrow}$ | 11.23e-02$^{\uparrow}$ | 12.27e-02$^{\uparrow}$ | **12.73e-02**$^{\uparrow}$ |
| | 6 | 12.95e-03 | **73.92e-02** | 11.06e-02$^{\downarrow}$ | 10.98e-02$^{\downarrow}$ | 70.14e-02$^{\downarrow}$ | 70.26e-02$^{\downarrow}$ |
| | 7 | 13.35e-03 | 9.57e-02 | 11.87e-02$^{\uparrow}$ | 12.32e-02$^{\uparrow}$ | **13.72e-02**$^{\uparrow}$ | 13.71e-02$^{\uparrow}$ |
| | 8 | 14.01e-03 | 3.46e-02 | 12.60e-02$^{\uparrow}$ | 12.84e-02$^{\uparrow}$ | **13.77e-02**$^{\uparrow}$ | 13.25e-02$^{\uparrow}$ |
| | 9 | 14.87e-03 | 12.56e-02 | 13.94e-02$^{\uparrow}$ | 13.88e-02$^{\uparrow}$ | **13.95e-02**$^{\uparrow}$ | 10.76e-02$^{\downarrow}$ |
| | 10 | 15.98e-03 | 1.36e-02 | 13.87e-02$^{\uparrow}$ | 13.89e-02$^{\uparrow}$ | 14.08e-02$^{\uparrow}$ | **14.10e-02**$^{\uparrow}$ |
| DTLZ6 | 4 | 5.83e-02 | 2.05e-02 | 6.85e-02$^{\uparrow}$ | 7.06e-02$^{\uparrow}$ | 8.11e-02$^{\uparrow}$ | **9.17e-02**$^{\uparrow}$ |
| | 5 | 10.07e-03 | 2.71e-02 | 5.70e-02$^{\uparrow}$ | 6.54e-02$^{\uparrow}$ | **7.19e-02**$^{\uparrow}$ | 7.16e-02$^{\uparrow}$ |
| | 6 | 12.93e-03 | 6.38e-03 | 9.83e-02$^{\uparrow}$ | 8.90e-02$^{\uparrow}$ | 10.15e-02$^{\uparrow}$ | **10.17e-02**$^{\uparrow}$ |
| | 7 | 13.29e-03 | **12.19e-02** | 8.44e-02$^{\downarrow}$ | 10.06e-03$^{\downarrow}$ | 10.26e-02$^{\downarrow}$ | 10.22e-08$^{\downarrow}$ |
| | 8 | 7.62e-03 | **19.31e-02** | 11.71e-02$^{\downarrow}$ | 12.69e-02$^{\downarrow}$ | 13.20e-02$^{\downarrow}$ | 13.86e-02$^{\downarrow}$ |
| | 9 | **14.01e-02** | 3.29e-02 | 11.44e-02$^{\uparrow}$ | 10.98e-02$^{\uparrow}$ | 12.28e-02$^{\uparrow}$ | 13.44e-02$^{\uparrow}$ |
| | 10 | 7.58e-02 | 5.68e-03 | 8.10e-02$^{\uparrow}$ | 7.19e-02$^{\uparrow}$ | 9.01e-02$^{\uparrow}$ | **10.05e-02**$^{\uparrow}$ |
| win/tie/loss counts | | | | 37/0/5 | 35/3/4 | 36/1/5 | 37/0/5 |

*Table C.2: HV values for Tchebycheff decomposition method on DTLZ1-DTLZ6 test problems. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | |
|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV |
| DTLZ1 | 4 | 80.90e-03 | 1.41e-02 | 7.65e-02$^{\uparrow}$ | 8.12e-02$^{\uparrow}$ | **10.33e-02**$^{\uparrow}$ |
| | 5 | 1.00e-02 | 1.54e-02 | 9.74e-02$^{\uparrow}$ | 9.92e-02$^{\uparrow}$ | **9.98e-02**$^{\uparrow}$ |
| | 6 | 8.91e-03 | **1.73e-02** | 1.02e-02$^{\downarrow}$ | 1.01e-02$^{\downarrow}$ | 1.29e-02$^{\downarrow}$ |
| | 7 | 5.40e-03 | 1.09e-02 | 1.18e-02$^{\uparrow}$ | **1.26e-02**$^{\uparrow}$ | 0.80e-02$^{\downarrow}$ |
| | 8 | 3.97e-03 | 2.05e-02 | 1.31e-02$^{\downarrow}$ | 1.31e-02$^{\downarrow}$ | **2.76e-01**$^{\uparrow}$ |
| | 9 | 5.53e-03 | 1.33e-02 | 1.42e-02$^{\uparrow}$ | 1.33e-02$^{\approx}$ | **3.25e-01**$^{\uparrow}$ |
| | 10 | 8.97e-03 | 0.79e-02 | 1.58e-02$^{\uparrow}$ | 1.59e-02$^{\uparrow}$ | **2.36e-02**$^{\uparrow}$ |
| DTLZ2 | 4 | 8.90e-03 | 5.58e-03 | 12.13e-02$^{\uparrow}$ | 19.13e-02$^{\uparrow}$ | **34.17e-02**$^{\uparrow}$ |
| | 5 | 2.41e-03 | 5.18e-03 | **12.60e-02**$^{\uparrow}$ | 12.49e-02$^{\uparrow}$ | 5.00e-02$^{\uparrow}$ |
| | 6 | 11.56e-03 | 26.92e-03 | 13.41e-02$^{\uparrow}$ | 13.74e-02$^{\uparrow}$ | **17.05e-02**$^{\uparrow}$ |
| | 7 | 10.71e-04 | 67.09e-04 | 13.87e-02$^{\uparrow}$ | 15.09e-02$^{\uparrow}$ | **18.56e-02**$^{\uparrow}$ |
| | 8 | 10.04e-04 | 18.11e-03 | 15.21e-02$^{\uparrow}$ | 18.04e-02$^{\uparrow}$ | **18.64e-02**$^{\uparrow}$ |
| | 9 | 92.17e-03 | 26.03e-03 | 14.01e-02$^{\uparrow}$ | 18.20e-02$^{\uparrow}$ | **20.19e-02**$^{\uparrow}$ |
| | 10 | 25.94e-04 | 36.31e-03 | 17.90e-02$^{\uparrow}$ | 15.62e-02$^{\uparrow}$ | **25.94e-02**$^{\uparrow}$ |
| DTLZ3 | 4 | 80.90e-04 | 5.58e-03 | 5.87e-02$^{\uparrow}$ | **11.53e-02**$^{\uparrow}$ | 10.96e-02$^{\uparrow}$ |
| | 5 | 12.39e-04 | 51.18e-03 | 0.87e-02$^{\uparrow}$ | 7.62e-02$^{\uparrow}$ | **10.61e-02**$^{\uparrow}$ |
| | 6 | 11.37e-04 | 26.92e-03 | 2.34e-02$^{\uparrow}$ | 8.48e-02$^{\uparrow}$ | **15.84e-02**$^{\uparrow}$ |
| | 7 | 11.41e-04 | 6.09e-03 | 7.23e-03$^{\downarrow}$ | 10.23e-03$^{\downarrow}$ | **11.34e-02**$^{\uparrow}$ |
| | 8 | 10.70e-04 | 18.11e-03 | 10.87e-02$^{\uparrow}$ | 18.58e-02$^{\uparrow}$ | **24.80e-02**$^{\uparrow}$ |
| | 9 | 95.09e-04 | 26.03e-03 | 6.72e-02$^{\uparrow}$ | 1.52e-02$^{\uparrow}$ | **14.85e-02**$^{\uparrow}$ |
| | 10 | 63.56e-03 | 36.31e-03 | 4.45e-02$^{\uparrow}$ | **7.71e-02**$^{\uparrow}$ | 6.76e-02$^{\uparrow}$ |
| DTLZ4 | 4 | 13.58e-03 | 88.45e-03 | 2.46e-02$^{\uparrow}$ | 2.66e-02$^{\uparrow}$ | **3.11e-02**$^{\uparrow}$ |
| | 5 | 15.39e-03 | 95.40e-03 | 4.04e-02$^{\uparrow}$ | 6.48e-02$^{\uparrow}$ | **8.38e-02**$^{\uparrow}$ |
| | 6 | 11.62e-03 | 1.20e-02 | 14.94e-02$^{\uparrow}$ | 17.64e-02$^{\uparrow}$ | **34.35e-02**$^{\uparrow}$ |
| | 7 | 15.27e-03 | 1.33e-03 | 14.74e-02$^{\uparrow}$ | **19.03e-02**$^{\uparrow}$ | 18.14e-02$^{\uparrow}$ |
| | 8 | 11.79e-03 | 1.65e-02 | 1.70e-02$^{\uparrow}$ | 1.65e-02$^{\approx}$ | **6.90e-02**$^{\uparrow}$ |
| | 9 | 11.17e-03 | 4.25e-02 | 9.75e-02$^{\uparrow}$ | 7.18e-02$^{\uparrow}$ | **10.66e-02**$^{\uparrow}$ |
| | 10 | 19.67e-03 | 7.87e02 | 20.49e-02$^{\uparrow}$ | 22.23e-02$^{\uparrow}$ | **0.25e-01**$^{\uparrow}$ |
| DTLZ5 | 4 | 13.37e-03 | 17.02e-03 | 10.43e-02$^{\uparrow}$ | 11.11e-02$^{\uparrow}$ | **16.80e-02**$^{\uparrow}$ |
| | 5 | 12.90e-03 | 3.44e-02 | 7.67e-02$^{\uparrow}$ | 8.36e-02$^{\uparrow}$ | **12.95e-02**$^{\uparrow}$ |
| | 6 | 12.95e-03 | 1.05e-02 | 6.81e-02$^{\uparrow}$ | 7.37e-02$^{\uparrow}$ | **8.00e-02**$^{\uparrow}$ |
| | 7 | 13.35e-03 | **15.47e-02** | 9.26e-02$^{\downarrow}$ | 8.68e-02$^{\downarrow}$ | 4.60e-02$^{\downarrow}$ |
| | 8 | 14.01e-03 | 1.00e-02 | **18.47e-02**$^{\uparrow}$ | 13.87e-03$^{\downarrow}$ | 14.25e-02$^{\uparrow}$ |
| | 9 | 14.87e-03 | 3.39e-02 | 4.57e-02$^{\uparrow}$ | 6.62e-02$^{\uparrow}$ | **14.38e-02**$^{\uparrow}$ |
| | 10 | 15.98e-03 | 6.07e-02 | 7.61e-02$^{\uparrow}$ | **16.53e-02**$^{\uparrow}$ | 15.10e-02$^{\uparrow}$ |
| DTLZ6 | 4 | 5.83e-03 | 2.71e-02 | 3.65e-03$^{\downarrow}$ | 9.29e-02$^{\uparrow}$ | **10.10e-02**$^{\uparrow}$ |
| | 5 | 10.07e-03 | **11.95e-02** | 10.65e-02$^{\downarrow}$ | 10.82e-03$^{\downarrow}$ | 11.54e-02$^{\downarrow}$ |
| | 6 | 12.93e-03 | 8.75e-02 | 8.98e-02$^{\uparrow}$ | 9.27e-02$^{\uparrow}$ | **10.35e-02**$^{\uparrow}$ |
| | 7 | 13.29e-03 | 6.74e-02 | 5.35e-02$^{\downarrow}$ | 7.06e-02$^{\uparrow}$ | **11.14e-02**$^{\uparrow}$ |
| | 8 | 2.62e-02 | 6.40e-02 | **36.69e-02**$^{\uparrow}$ | 6.40e-02$^{\approx}$ | 11.21e-02$^{\uparrow}$ |
| | 9 | 14.01e-03 | 3.37e-02 | 9.01e-02$^{\uparrow}$ | **27.15e-02**$^{\uparrow}$ | 10.95e-02$^{\uparrow}$ |
| | 10 | 27.58e-03 | 15.18e-02 | 15.08e-02$^{\uparrow}$ | 20.40e-02$^{\uparrow}$ | **20.48e-02**$^{\uparrow}$ |
| win/tie/loss counts | | | | 35/0/7 | 33/3/6 | 38/0/4 |

*Table C.3: HV values for PBI decomposition method on WFG1-WFG9 test problems. The parameter $\theta = 5$ unless specified. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | | |
|---|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV | PEV ($\theta$=0) |
| WFG1 | 4 | 17.14e-03 | 13.29e-03 | 13.52e-02$^{\uparrow}$ | 13.61e-02$^{\uparrow}$ | 15.40e-02$^{\uparrow}$ | **16.36e-02**$^{\uparrow}$ |
| | 5 | 20.44e-03 | 20.30e-03 | **22.97e-02**$^{\uparrow}$ | 18.30e-02$^{\downarrow}$ | 10.57e-02$^{\downarrow}$ | 21.03e-02$^{\downarrow}$ |
| | 6 | 26.05e-03 | 24.70e-03 | 25.07e-02$^{\uparrow}$ | 23.72e-02$^{\downarrow}$ | **25.55e-02**$^{\uparrow}$ | 25.35e-02$^{\uparrow}$ |
| | 7 | 37.02e-03 | 27.59e-03 | 27.92e-02$^{\uparrow}$ | 27.16e-02$^{\downarrow}$ | **28.62e-02**$^{\uparrow}$ | 28.35e-02$^{\uparrow}$ |
| | 8 | 31.90e-03 | 30.21e-03 | 10.30e-02$^{\uparrow}$ | 10.25e-02$^{\uparrow}$ | 10.36e-02$^{\uparrow}$ | **11.17e-02**$^{\uparrow}$ |
| | 9 | 34.49e-03 | 31.89e-03 | 32.01e-02$^{\uparrow}$ | 32.06e-02$^{\uparrow}$ | **32.94e-02**$^{\uparrow}$ | 29.05e-02$^{\downarrow}$ |
| | 10 | 31.23e-03 | 33.13e-03 | 33.21e-02$^{\uparrow}$ | 33.15e-02$^{\uparrow}$ | 34.05e-02$^{\uparrow}$ | **35.05e-02**$^{\uparrow}$ |
| WFG2 | 4 | 58.19e-03 | 3.59e-03 | 0.73e-02$^{\uparrow}$ | 2.65e-02$^{\uparrow}$ | **2.92e-02**$^{\uparrow}$ | 2.84e-02$^{\uparrow}$ |
| | 5 | 7.00e-03 | 3.83e-03 | 6.12e-02$^{\uparrow}$ | 2.39e-02$^{\uparrow}$ | **6.34e-02**$^{\uparrow}$ | 5.53e-02$^{\uparrow}$ |
| | 6 | 10.30e-02 | 13.64e-03 | 13.72e-02$^{\uparrow}$ | 11.38e-02$^{\uparrow}$ | 10.84e-02$^{\uparrow}$ | **1.03e-01**$^{\uparrow}$ |
| | 7 | **18.71e-02** | 9.15e-03 | 7.15e-03$^{\downarrow}$ | 4.97e-03$^{\downarrow}$ | 0.16e-03$^{\downarrow}$ | 2.47e-02$^{\uparrow}$ |
| | 8 | 4.95e-03 | 5.52e-03 | 0.55e-02$^{\uparrow}$ | **9.13e-02**$^{\uparrow}$ | 0.38e-02$^{\uparrow}$ | 3.97e-02$^{\uparrow}$ |
| | 9 | 21.24e-03 | 5.38e-03 | 1.06e-02$^{\uparrow}$ | 1.51e-02$^{\uparrow}$ | 2.32e-02$^{\uparrow}$ | **7.91e-02**$^{\uparrow}$ |
| | 10 | **15.37e-02** | 7.54e-02 | 6.51e-02$^{\downarrow}$ | 7.46e-02$^{\downarrow}$ | 7.41e-02$^{\downarrow}$ | 8.01e-02$^{\uparrow}$ |
| WFG3 | 4 | 1.16e-02 | 5.74e-02 | 6.01e-02$^{\uparrow}$ | 12.80e-02$^{\uparrow}$ | 31.85e-02$^{\uparrow}$ | **9.62e-01**$^{\uparrow}$ |
| | 5 | 11.59e-03 | 7.80e-02 | 4.28e-02$^{\downarrow}$ | 6.09e-02$^{\downarrow}$ | **10.16e-02**$^{\downarrow}$ | 9.05e-02$^{\uparrow}$ |
| | 6 | 13.82e-02 | 12.64e-03 | 11.21e-02$^{\uparrow}$ | 12.16e-02$^{\uparrow}$ | 15.54e-02$^{\uparrow}$ | **15.76e-02**$^{\uparrow}$ |
| | 7 | 8.84e-03 | 5.05e-03 | 1.20e-02$^{\uparrow}$ | 4.25e-02$^{\uparrow}$ | 5.12e-02$^{\uparrow}$ | **6.05e-02**$^{\uparrow}$ |
| | 8 | 1.25e-03 | 3.52e-03 | 8.90e-03$^{\uparrow}$ | 7.95e-02$^{\uparrow}$ | 6.93e-02$^{\uparrow}$ | **10.13e-02**$^{\uparrow}$ |
| | 9 | 9.66e-02 | 11.57e-03 | 8.61e-02$^{\uparrow}$ | 12.97e-02$^{\uparrow}$ | **17.05e-02**$^{\uparrow}$ | 10.17e-02$^{\uparrow}$ |
| | 10 | 30.55e-02 | 9.60e-02 | 10.01e-02$^{\uparrow}$ | 9.60e-02$^{\approx}$ | 12.10e-02$^{\uparrow}$ | **3.94e-01**$^{\uparrow}$ |
| WFG4 | 4 | 4.39e-02 | 3.60e-02 | 3.90e-02$^{\uparrow}$ | 4.21e-02$^{\uparrow}$ | **9.83e-02**$^{\uparrow}$ | 9.02e-02$^{\uparrow}$ |
| | 5 | 10.92e-03 | 1.00e-03 | 1.81e-02$^{\uparrow}$ | 8.59e-02$^{\uparrow}$ | **9.29e-02**$^{\uparrow}$ | 8.34e-02$^{\uparrow}$ |
| | 6 | 23.37e-03 | 4.19e-03 | 0.79e-02$^{\uparrow}$ | 4.18e-02$^{\uparrow}$ | **6.46e-02**$^{\uparrow}$ | 6.01e-02$^{\uparrow}$ |
| | 7 | 32.78e-03 | 17.60e-03 | 1.08e-02$^{\uparrow}$ | 6.03e-02$^{\uparrow}$ | 7.90e-02 $^{\uparrow}$ | **8.81e-02**$^{\uparrow}$ |
| | 8 | 2.85e-03 | 7.69e-03 | 2.36e-02$^{\uparrow}$ | 7.69e-02$^{\uparrow}$ | 8.14e-02$^{\uparrow}$ | **9.01e-02**$^{\uparrow}$ |
| | 9 | 1.66e-03 | 3.76e-03 | **5.01e-02**$^{\uparrow}$ | 3.76e-02$^{\uparrow}$ | 4.25e-02$^{\uparrow}$ | 4.21e-02$^{\uparrow}$ |
| | 10 | **4.40e-02** | 1.26e-03 | 4.02e-02$^{\uparrow}$ | 1.23e-02$^{\uparrow}$ | 3.33e-02$^{\uparrow}$ | 4.01e-02$^{\uparrow}$ |
| WFG5 | 4 | **12.60e-02** | 5.77e-02 | 3.39e-02$^{\downarrow}$ | 5.20e-02$^{\downarrow}$ | 7.22e-02$^{\downarrow}$ | 6.35e-02$^{\uparrow}$ |
| | 5 | 2.57e-03 | 2.32e-02 | 5.41e-02$^{\uparrow}$ | 8.16e-02$^{\uparrow}$ | **11.25e-02**$^{\uparrow}$ | 10.36e-02$^{\uparrow}$ |
| | 6 | 4.44e-02 | 4.58e-02 | 5.21e-02$^{\uparrow}$ | 2.90e-02$^{\downarrow}$ | 9.11e-02$^{\uparrow}$ | **1.86e-01**$^{\uparrow}$ |
| | 7 | 4.58e-01 | 3.96e-02 | 6.16e-02$^{\uparrow}$ | 8.85e-02$^{\uparrow}$ | 0.16e-01$^{\uparrow}$ | **9.12e-01**$^{\uparrow}$ |
| | 8 | 4.14e-012 | 2.60e-02 | 5.47e-02$^{\uparrow}$ | 2.65e-02$^{\uparrow}$ | 1.11e-02$^{\downarrow}$ | **12.14e-02**$^{\uparrow}$ |
| | 9 | 1.46e-02 | 6.0e-03 | 7.01e-02$^{\uparrow}$ | 6.18e-02$^{\uparrow}$ | 8.59e-02$^{\uparrow}$ | **8.39e-02**$^{\uparrow}$ |
| | 10 | 1.41e-02 | 2.64e-02 | 2.90e-02$^{\uparrow}$ | 3.21e-02$^{\uparrow}$ | **4.33e-02**$^{\uparrow}$ | 4.01e-02$^{\uparrow}$ |
| WFG6 | 4 | 5.52e-03 | 6.39e-03 | 7.40e-02$^{\uparrow}$ | 8.20e-02$^{\uparrow}$ | 10.16e-02$^{\uparrow}$ | **11.32e-02**$^{\uparrow}$ |
| | 5 | 6.14e-03 | 2.64e-03 | 2.90e-02$^{\uparrow}$ | 3.42e-02$^{\uparrow}$ | **2.22e-02**$^{\uparrow}$ | 2.20e-02$^{\uparrow}$ |
| | 6 | **1.90e-02** | 0.06e-03 | 0.26e-02$^{\uparrow}$ | 0.28e-02$^{\uparrow}$ | 0.40e-02$^{\uparrow}$ | 0.36e-02$^{\uparrow}$ |
| | 7 | **4.80e-02** | 1.41e-03 | 1.01e-02$^{\uparrow}$ | 2.18e-02$^{\uparrow}$ | **4.24e-02**$^{\uparrow}$ | 4.00e-02$^{\uparrow}$ |
| | 8 | 12.56e-02 | 2.83e-03 | 3.02e-02$^{\uparrow}$ | 4.07e-02$^{\uparrow}$ | 9.15e-02$^{\uparrow}$ | **9.20e-02**$^{\uparrow}$ |
| | 9 | 5.22e-02 | 5.00e-02 | 8.33e-02$^{\uparrow}$ | 5.00e-02$^{\approx}$ | 6.63e-02$^{\uparrow}$ | **6.90e-02**$^{\uparrow}$ |
| | 10 | 4.35e-02 | 7.22e-02 | 8.09e-02$^{\uparrow}$ | 15.43e-02$^{\uparrow}$ | **15.50e-02**$^{\uparrow}$ | 10.00e-02$^{\uparrow}$ |
| WFG7 | 4 | 11.06e-02 | 6.39e-02 | 9.41e-02$^{\uparrow}$ | 10.81e-02$^{\uparrow}$ | 10.54e-02$^{\uparrow}$ | **15.91e-02**$^{\uparrow}$ |
| | 5 | 23.64e-03 | 4.64e-02 | 5.24e-02$^{\uparrow}$ | 5.13e-02$^{\uparrow}$ | 10.80e-02$^{\uparrow}$ | **16.65e-02**$^{\uparrow}$ |

*Table C.3: HV values for PBI decomposition method on WFG1-WFG9 test problems. The parameter $\theta = 5$ unless specified. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | | |
|---|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV | PEV ($\theta$=0) |
| | 6 | 5.72e-02 | 2.30e-02 | 2.49e-02$^\uparrow$ | 3.98e-02$^\uparrow$ | **6.80e-02**$^\uparrow$ | 6.31e-02$^\uparrow$ |
| | 7 | **22.16e-01** | 9.41e-02 | 10.24e-02$^\uparrow$ | 25.01e-02$^\uparrow$ | 10.15e-02$^\uparrow$ | 40.75e-02$^\uparrow$ |
| | 8 | 1.57e-02 | 2.83e-02 | 3.23e-02$^\uparrow$ | 3.81e-02$^\uparrow$ | 4.68e-02$^\uparrow$ | **5.02e-02**$^\uparrow$ |
| | 9 | 3.94e-02 | 5.00e-02 | 10.01e-02$^\uparrow$ | 12.59e-02$^\uparrow$ | 10.20e-02$^\uparrow$ | **17.45e-02**$^\uparrow$ |
| | 10 | 3.59e-02 | 1.22e-02 | 1.40e-02$^\uparrow$ | 2.93e-02$^\uparrow$ | 3.26e-02$^\uparrow$ | **4.76e-02**$^\uparrow$ |
| | 4 | **14.11e-02** | 6.39e-02 | 9.21e-02$^\uparrow$ | 10.63e-02$^\uparrow$ | 11.05e-02$^\uparrow$ | 12.01e-02$^\uparrow$ |
| | 5 | 32.40e-03 | 4.01e-02 | 4.34e-02$^\uparrow$ | 5.13e-02$^\uparrow$ | 6.99e-02$^\uparrow$ | **7.08e-02**$^\uparrow$ |
| WFG8 | 6 | 2.60e-03 | 1.41e-02 | 2.01e-02$^\uparrow$ | 1.51e-02$^\uparrow$ | 2.13e-02$^\uparrow$ | **3.01e-02**$^\uparrow$ |
| | 7 | 14.07e-03 | 1.26e-02 | 2.31e-02$^\uparrow$ | 5.27e-02$^\uparrow$ | **7.21e-02**$^\uparrow$ | 7.02e-02$^\uparrow$ |
| | 8 | 6.90e-02 | 2.41e-02 | 3.36e-02$^\uparrow$ | 8.39e-02$^\uparrow$ | 10.18e-02$^\uparrow$ | **12.39e-02**$^\uparrow$ |
| | 9 | 7.59e-02 | 3.33e-02 | 8.00e-02$^\uparrow$ | 3.323e-02$^\downarrow$ | 8.82e-02$^\uparrow$ | **9.01e-02**$^\uparrow$ |
| | 10 | 3.67e-03 | 3.36e-02 | 2.01e02$^\downarrow$ | 2.33e-02$^\downarrow$ | 2.49e-02$^\downarrow$ | **3.01e-02**$^\uparrow$ |
| | 4 | 4.41e-03 | 6.05e-03 | 1.09e-02$^\uparrow$ | 1.94e-02$^\uparrow$ | 2.45e-02$^\uparrow$ | **3.53e-02**$^\uparrow$ |
| | 5 | 16.59e-03 | 1.18e-03 | 1.39e-02$^\uparrow$ | 3.02e-02$^\uparrow$ | 6.14e-02$^\uparrow$ | **6.26e-02**$^\uparrow$ |
| WFG9 | 6 | 1.14e-03 | 5.24e-03 | 1.71e-02$^\uparrow$ | 2.28e-02$^\uparrow$ | 2.33e-02$^\uparrow$ | **3.60e-02**$^\uparrow$ |
| | 7 | 6.40e-03 | 3.89e-03 | 2.12e-02$^\uparrow$ | 2.52e-02$^\uparrow$ | **10.92e-02**$^\uparrow$ | 3.91e-02$^\uparrow$ |
| | 8 | 15.24e-03 | 10.44e-03 | 8.88e-02$^\uparrow$ | 9.43e-02$^\uparrow$ | 8.40e-02$^\uparrow$ | **9.32e-02**$^\uparrow$ |
| | 9 | 5.33e-03 | 7.29e-03 | 6.39e-02$^\uparrow$ | 7.21e-02$^\uparrow$ | **8.48e-02**$^\uparrow$ | 8.58e-02$^\uparrow$ |
| | 10 | 3.06e-03 | 2.10e-03 | 4.21e-02$^\uparrow$ | 5.02e-02$^\uparrow$ | 8.07e-02$^\uparrow$ | **8.09e-02**$^\uparrow$ |
| win/tie/loss counts | | | | 59/0/4 | 51/2/10 | 56/0/7 | 60/0/3 |

*Table C.4: HV values for Tchebycheff decomposition method on WFG1-WFG9 test problems. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | |
|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV |
| WFG1 | 4 | 17.14e-03 | 16.46e-02 | 10.29e-02$^{\downarrow}$ | 19.29e-02$^{\uparrow}$ | **20.88e-02**$^{\uparrow}$ |
| | 5 | 20.44e-03 | 66.92e-02 | 33.87e-02$^{\downarrow}$ | 34.69e-02$^{\downarrow}$ | **77.18e-02**$^{\uparrow}$ |
| | 6 | 26.05e-03 | 2.76e-02 | 5.19e-02$^{\uparrow}$ | 2.76e-02$^{\approx}$ | **6.55e-02**$^{\uparrow}$ |
| | 7 | 37.02e-03 | 11.83e-02 | 12.45e-02$^{\uparrow}$ | 14.36e-02$^{\uparrow}$ | **15.81e-02**$^{\uparrow}$ |
| | 8 | 31.90e-03 | 44.16e-02 | 40.14e-02$^{\downarrow}$ | **66.85e-02**$^{\uparrow}$ | 61.32e-02$^{\uparrow}$ |
| | 9 | 34.49e-03 | 9.27e-02 | 13.91e-02$^{\uparrow}$ | 14.39e-02$^{\uparrow}$ | **14.45e-02**$^{\uparrow}$ |
| | 10 | 50.63e-03 | 38.46e-02 | 39.57e-02$^{\uparrow}$ | **47.24e-02**$^{\uparrow}$ | 47.11e-02$^{\uparrow}$ |
| WFG2 | 4 | 58.19e-03 | 14.23e-02 | 19.29e-02$^{\uparrow}$ | 20.41e-02$^{\uparrow}$ | **21.41e-02**$^{\uparrow}$ |
| | 5 | 7.00e-03 | 16.37e-02 | 20.27e-02$^{\uparrow}$ | 29.94e-02$^{\uparrow}$ | **30.97e-02**$^{\uparrow}$ |
| | 6 | **80.30e-02** | 65.87e-02 | 54.28e-02$^{\downarrow}$ | 14.16e-02$^{\downarrow}$ | 20.84e-02$^{\downarrow}$ |
| | 7 | 8.71e-02 | 8.48e-02 | 9.31e-02$^{\uparrow}$ | 7.31e-02$^{\downarrow}$ | **10.57e-02**$^{\uparrow}$ |
| | 8 | 4.95e-02 | 5.76e-02 | 7.69e-02$^{\uparrow}$ | 5.08e-02$^{\downarrow}$ | **9.20e-03**$^{\uparrow}$ |
| | 9 | 2.24e-02 | 2.89e-02 | 3.36e-02$^{\uparrow}$ | 3.59e-02$^{\uparrow}$ | **8.32e-02**$^{\uparrow}$ |
| | 10 | 15.37e-02 | 17.65e-02 | 28.49e-02$^{\uparrow}$ | 29.04e-02$^{\uparrow}$ | **30.84e-02**$^{\uparrow}$ |
| WFG3 | 4 | 1.166e-02 | **37.38e-03** | 18.29e-02$^{\downarrow}$ | 20.65e-02$^{\downarrow}$ | 22.50e-03$^{\downarrow}$ |
| | 5 | 11.59e-03 | 16.36e-03 | 16.36e-02$^{\approx}$ | **28.79e-02**$^{\uparrow}$ | 15.30e-02$^{\uparrow}$ |
| | 6 | **13.82e-02** | 7.39e-02 | 7.39e-02$^{\approx}$ | 3.71e-02$^{\downarrow}$ | 3.86e-02$^{\downarrow}$ |
| | 7 | 3.84e-02 | 6.89e-02 | 5.45e-02$^{\downarrow}$ | 4.26e-02$^{\downarrow}$ | **7.48e-02**$^{\uparrow}$ |
| | 8 | 1.25e-03 | 2.70e-03 | 3.76e-02$^{\uparrow}$ | 4.10e-03$^{\uparrow}$ | **11.27e-03**$^{\uparrow}$ |
| | 9 | 9.66e-02 | 3.50e-02 | 4.59e-02$^{\uparrow}$ | 3.51e-02$^{\uparrow}$ | **10.22e-02**$^{\uparrow}$ |
| | 10 | 30.55e-02 | 31.69e-02 | 33.19e-02$^{\uparrow}$ | **57.36e-02**$^{\uparrow}$ | 50.11e-02$^{\uparrow}$ |
| WFG4 | 4 | 4.39e-02 | 1.17e-02 | 4.81e-02$^{\uparrow}$ | **19.11e-02**$^{\uparrow}$ | 10.48e-02$^{\uparrow}$ |
| | 5 | 2.92e-02 | 3.20e-02 | 3.54e-02$^{\uparrow}$ | 3.66e-02$^{\uparrow}$ | **9.65e-02**$^{\uparrow}$ |
| | 6 | 3.37e-02 | 4.07e-02 | 4.17e-02$^{\uparrow}$ | 5.75e-02$^{\uparrow}$ | **8.83e-02**$^{\uparrow}$ |
| | 7 | 3.78e-02 | 2.17e-02 | 2.17e-02$^{\approx}$ | **8.35e-02**$^{\uparrow}$ | 8.34e-02$^{\uparrow}$ |
| | 8 | 2.85e-02 | 3.56e-02 | 3.73e-02$^{\uparrow}$ | **5.80e-02**$^{\uparrow}$ | 5.10e-02$^{\uparrow}$ |
| | 9 | 1.66e-02 | 2.34e-02 | 2.98e-02$^{\uparrow}$ | 2.70e-02$^{\uparrow}$ | **5.15e-02**$^{\uparrow}$ |
| | 10 | 7.40e-02 | 10.46e-02 | 11.01e-02$^{\uparrow}$ | 11.03e-02$^{\uparrow}$ | **12.56e-02**$^{\uparrow}$ |
| WFG5 | 4 | 10.60e-02 | 11.19e-02 | 12.18e-02$^{\uparrow}$ | 12.73e-02$^{\uparrow}$ | **13.56e-02**$^{\uparrow}$ |
| | 5 | 2.57e-02 | 3.59e-02 | 5.61e-02$^{\uparrow}$ | 8.37e-02$^{\uparrow}$ | **10.46e-02**$^{\uparrow}$ |
| | 6 | 4.44e-02 | 5.95e-02 | 6.01e-02$^{\uparrow}$ | 3.38e-02$^{\downarrow}$ | **6.19e-02**$^{\uparrow}$ |
| | 7 | 4.58e-02 | 5.80e-02 | 6.54e-02$^{\uparrow}$ | 5.80e-02$^{\approx}$ | **8.06e-02**$^{\uparrow}$ |
| | 8 | 14.14e-02 | 10.70e-02 | **15.98e-02**$^{\uparrow}$ | 12.26e-02$^{\uparrow}$ | 13.22e-02$^{\uparrow}$ |
| | 9 | 11.46e-03 | **17.62e-03** | 12.85e-03$^{\downarrow}$ | 12.26e-03$^{\downarrow}$ | 13.55e-02$^{\downarrow}$ |
| | 10 | 1.41e-02 | 3.83e-02 | 4.01e-02$^{\uparrow}$ | **10.77e-02**$^{\uparrow}$ | 9.17e-02$^{\uparrow}$ |
| WFG6 | 4 | 5.52e-02 | 9.61e-02 | 10.00-e02$^{\uparrow}$ | 2.10e-02$^{\uparrow}$ | **11.91e-02**$^{\uparrow}$ |
| | 5 | 6.14e-02 | 9.20e-02 | 10.45e-02$^{\uparrow}$ | 9.57e-02$^{\uparrow}$ | **10.39e-02**$^{\uparrow}$ |
| | 6 | 11.90e-02 | 1.61e-02 | 8.49e-02$^{\uparrow}$ | 2.06e-02$^{\uparrow}$ | **9.95e-02**$^{\uparrow}$ |
| | 7 | 14.80e-02 | 11.31e-02 | 10.49e-02$^{\downarrow}$ | 6.36e-02$^{\downarrow}$ | **15.19e-02**$^{\uparrow}$ |
| | 8 | **12.56e-02** | 10.02e-02 | 8.99e-02$^{\downarrow}$ | 7.33e-02$^{\downarrow}$ | 10.19e-02$^{\uparrow}$ |
| | 9 | 9.22e-02 | 9.82e-02 | 9.88e-02$^{\uparrow}$ | **37.34e-02**$^{\uparrow}$ | 36.76e-02$^{\uparrow}$ |
| | 10 | **4.35e-02** | 5.66e-02 | 5.66e-02$^{\approx}$ | **14.08e-02**$^{\uparrow}$ | 10.39e-02$^{\uparrow}$ |
| | 4 | 11.06e-02 | 8.33e-02 | 10.38e-02$^{\uparrow}$ | 10.39e-02$^{\uparrow}$ | **10.61e-02**$^{\uparrow}$ |
| | 5 | 3.64e-02 | 2.98e-02 | 7.19e-02$^{\uparrow}$ | 9.21e-02$^{\uparrow}$ | **9.22e-02**$^{\uparrow}$ |
| WFG7 | | | | | | |

*Table C.4: HV values for Tchebycheff decomposition method on WFG1-WFG9 test problems. The median of 25 independent runs is reported.*

| Func. | # Obj | R-NSGA-II | R-MEAD2 | UR-MEAD2 OHV | UR-MEAD2 PE | UR-MEAD2 PEV |
|---|---|---|---|---|---|---|
| | 6 | 5.72e-02 | 6.88e-02 | 7.01e-02↑ | **9.15e-02**↑ | 8.80e-02↑ |
| | 7 | 2.16e-02 | 3.05e-02 | 4.29e-02↑ | 3.38e-02↑ | **4.52e-02**↑ |
| | 8 | **9.57e-02** | 1.62e-02 | 4.71e-02↑ | 6.89e-02↑ | 3.94e-02↑ |
| | 9 | 3.94e-02 | 2.38e-02 | 3.15e-02↑ | 2.91e-02↑ | **7.72e-02**↑ |
| | 10 | 3.59e-02 | 4.45e-02 | 4.98e-02↑ | 4.46e-02↑ | **5.01e-02**↑ |
| WFG8 | 4 | **14.11e-02** | 9.98e-02 | 10.39e-02↑ | 13.62e-02↑ | 10.12e-02↑ |
| | 5 | 3.04e-02 | 2.03e-02 | 2.03e-02≈ | 3.18e-02↑ | **6.59e-02**↑ |
| | 6 | 2.60e-02 | 2.04e-02 | 2.10e-02↑ | 2.49e-02↑ | **3.36e-02**↑ |
| | 7 | 4.07e-02 | **5.68e-02** | 4.02e-02↓ | 3.38e-02↓ | 4.39e-02↓ |
| | 8 | 6.90e-02 | 7.03e-02 | 8.19e0-2↑ | **17.59e-02**↑ | 10.34e-02↑ |
| | 9 | 7.59e-02 | 5.89e-02 | 7.36e-02↑ | 6.06e-02↑ | **7.77e-02**↑ |
| | 10 | 3.67e-02 | 4.19e-02 | 10.87e-02↑ | 7.00e-02↑ | **11.82e-02**↑ |
| WFG9 | 4 | 4.41e-02 | **6.58e-02** | 3.44e-02↓ | 2.99e-02↓ | 5.45e-02↓ |
| | 5 | **6.59e-02** | 1.27e-02 | 3.12e-02↑ | 4.21e-02↑ | 5.65e-02↑ |
| | 6 | 1.14e-02 | 2.04e-02 | 2.88e-02↑ | 3.20e-02↑ | **8.62e-02**↑ |
| | 7 | 6.40e-02 | 10.12e-02 | 8.99e-02↓ | 6.64e-02↓ | **10.45e-02**↑ |
| | 8 | 5.24e-02 | 7.69e-02 | 7.69e-02≈ | 7.35e-02↓ | **10.35e-02**↑ |
| | 9 | 5.33e-02 | 6.05e-02 | 6.15e-02↑ | **7.40e-02**↑ | 6.31e-02↑ |
| | 10 | 3.06e-02 | 4.14e-02 | 7.87e-02↑ | **8.67e-02**↑ | 8.65e-02↑ |
| win/tie/loss counts | | | | 45/6/12 | 46/2/15 | 57/0/6 |

*Table C.5: HV values for Tchebycheff and PBI (PEV variant) on DTLZ.he median of 25 independent runs is reported*

| | | UR-MEAD2 | | |
|---|---|---|---|---|
| Func. | # Obj | PBI ($\theta$=5) | PBI ($\theta$=0) | TE |
| DTLZ1 | 4 | 10.16e-02 | 10.28e-02 | **10.33e-02** |
| | 5 | 0.17e-02 | 0.34e-02 | **9.98e-02** |
| | 6 | 4.11e-02 | **9.82e-02** | 1.29e-02 |
| | 7 | **2.23e-02** | 0.16e-02 | 0.80e-02 |
| | 8 | 2.22e-02 | 2.21e-02 | **2.76e-01** |
| | 9 | 3.15e-02 | 3.13e-02 | **3.25e-01** |
| | 10 | 1.98e-02 | 2.21e-02 | **2.36e-02** |
| DTLZ2 | 4 | 33.33e-02 | 33.31e-02 | **34.17e-02** |
| | 5 | 10.0e-02 | **14.64e-02** | 5.00e-02 |
| | 6 | 15.0e-02 | 16.10e-02 | **17.05e-02** |
| | 7 | 1.51e-02 | 17.71e-02 | **18.56e-02** |
| | 8 | **20.21e-02** | 19.48e-02 | 18.64e-02 |
| | 9 | 29.31e-02 | **40.92e-02** | 20.19e-02 |
| | 10 | 28.41e-02 | **45.01e-02** | 25.94e-02 |
| DTLZ3 | 4 | **37.34e-02** | 36.44e-02 | 10.96e-02 |
| | 5 | 10.32e-02 | 10.49e-02 | **10.61e-02** |
| | 6 | 14.62e-02 | 14.33e-02 | **15.84e-02** |
| | 7 | 9.36e-02 | 10.22e-02 | **11.34e-02** |
| | 8 | 9.21e-02 | 8.33e-02 | **24.80e-02** |
| | 9 | 9.83e-02 | 10.03e-02 | **14.85e-02** |
| | 10 | 5.38e-02 | 6.66e-02 | **6.76e-02** |
| DTLZ4 | 4 | **37.38e-02** | 37.31e-02 | 3.11e-02 |
| | 5 | 18.06e-02 | **24.64e-02** | 8.38e-02 |
| | 6 | 16.13e-02 | 44.05e-03 | **34.35e-02** |
| | 7 | 15.31e-02 | 17.82e-02 | **18.14e-02** |
| | 8 | 20.27e-02 | **37.20e-02** | 6.90e-02 |
| | 9 | 25.30e-02 | **30.92e-02** | 10.66e-02 |
| | 10 | 14.46e-02 | 64.49e-02 | **0.25e-01** |
| DTLZ5 | 4 | 16.75e-02 | 15.88e-02 | **16.80e-02** |
| | 5 | 12.27e-02 | 12.73e-02 | **12.95e-02** |
| | 6 | 70.14e-02 | **70.26e-02** | 8.00e-02 |
| | 7 | **13.72e-02** | 13.71e-02 | 4.60e-02 |
| | 8 | 13.77e-02 | 13.25e-02 | **14.25e-02** |
| | 9 | 13.95e-02 | 10.76e-02 | **14.38e-02** |
| | 10 | 14.08e-02 | 14.10e-02 | **15.10e-02** |
| DTLZ6 | 4 | 8.11e-02 | 9.17e-02 | **10.10e-02** |
| | 5 | 7.19e-02 | 7.16e-02 | **11.54e-02** |
| | 6 | 10.15e-02 | 10.17e-02 | **10.35e-02** |
| | 7 | 10.26e-02 | 10.22e-08 | **11.14e-02** |
| | 8 | 13.20e-02 | **13.86e-02** | 11.21e-02 |
| | 9 | 12.28e-02 | **13.44e-02** | 10.95e-02 |
| | 10 | 9.01e-02 | 10.05e-02 | **20.48e-02**$^{\uparrow}$ |
| Total highlights | | 5 | 10 | 27 |

*Table C.6: HV values for Tchebycheff and PBI (PEV variant) on WFG test problems. The median of 25 independent runs is reported.*

| | | UR-MEAD2 | | |
|---|---|---|---|---|
| Func. | # Obj | PBI ($\theta$=5) | PBI ($\theta$=0) | TE |
| WFG1 | 4 | 15.40e-02 | 16.36e-02 | **20.88e-02** |
| | 5 | 10.57e-02 | 21.03e-02 | **77.18e-02** |
| | 6 | **25.55e-02** | 25.35e-02 | 6.55e-02 |
| | 7 | **28.62e-02** | 28.35e-02 | 15.81e-02 |
| | 8 | 10.36e-02 | 11.17e-02 | **61.32e-02** |
| | 9 | **32.94e-02** | 29.05e-02 | 14.45e-02 |
| | 10 | 34.05e-02 | 35.05e-02 | **47.11e-02** |
| WFG2 | 4 | 2.92e-02 | 2.84e-02 | **21.41e-02** |
| | 5 | 6.34e-02 | 5.53e-02 | **30.97e-02** |
| | 6 | 10.84e-02 | **1.03e-01** | 20.84e-02 |
| | 7 | 0.16e-03 | 2.47e-02 | **10.57e-02** |
| | 8 | 0.38e-02 | **3.97e-02** | 9.20e-03 |
| | 9 | 2.32e-02 | 7.91e-02 | **8.32e-02** |
| | 10 | 7.41e-02 | 8.01e-02 | **30.84e-02** |
| WFG3 | 4 | 31.85e-02 | **9.62e-01** | 22.50e-03 |
| | 5 | 10.16e-02 | 9.05e-02 | **15.30e-02** |
| | 6 | 15.54e-02 | **15.76e-02** | 3.86e-02 |
| | 7 | 5.12e-02 | 6.05e-02 | **7.48e-02** |
| | 8 | 6.93e-02 | **10.13e-02** | 11.27e-03 |
| | 9 | **17.05e-02** | 10.17e-02 | 10.22e-02 |
| | 10 | 12.10e-02 | **3.94e-01** | 50.11e-02 |
| WFG4 | 4 | 9.83e-02 | 9.02e-02 | **10.48e-02** |
| | 5 | 9.29e-02 | 8.34e-02 | **9.65e-02** |
| | 6 | 6.46e-02 | 6.01e-02 | **8.83e-02** |
| | 7 | 7.90e-02 | **8.81e-02** | 8.34e-02 |
| | 8 | 8.14e-02 | **9.01e-02** | 5.10e-02 |
| | 9 | 4.25e-02 | 4.21e-02 | **5.15e-02** |
| | 10 | 3.33e-02 | 4.01e-02 | **12.56e-02** |
| WFG5 | 4 | 7.22e-02 | 6.35e-02 | **13.56e-02** |
| | 5 | **11.25e-02** | 10.36e-02 | 10.46e-02 |
| | 6 | 9.11e-02 | **1.86e-01** | 6.19e-02 |
| | 7 | 0.16e-01 | **9.12e-01** | 8.06e-02 |
| | 8 | 1.11e-02 | 12.14e-02 | **13.22e-02** |
| | 9 | 8.59e-02 | 8.39e-02 | **13.55e-02** |
| | 10 | 4.33e-02 | 4.01e-02 | **9.17e-02** |
| WFG6 | 4 | 10.16e-02 | 11.32e-02 | **11.91e-02** |
| | 5 | 2.22e-02 | 2.20e-02 | **10.39e-02** |
| | 6 | 0.40e-02 | 0.36e-02 | **9.95e-02** |
| | 7 | 4.24e-02 | 4.00e-02 | **15.19e-02** |
| | 8 | 9.15e-02 | 9.20e-02 | **10.19e-02** |
| | 9 | 6.63e-02 | 6.90e-02 | **36.76e-02** |
| | 10 | **15.50e-02** | 10.00e-02 | 10.39e-02 |
| WFG7 | 4 | 10.54e-02 | **15.91e-02** | 10.61e-02 |
| | 5 | 10.80e-02 | **16.65e-02** | 9.22e-02 |

*Table C.6: HV values for Tchebycheff and PBI (PEV variant) on WFG test problems. The median of 25 independent runs is reported.*

| | | UR-MEAD2 | | |
|---|---|---|---|---|
| Func. | # Obj | PBI ($\theta$=5) | PBI ($\theta$=0) | TE |
| | 6 | 6.80e-02 | 6.31e-02 | **8.80e-02** |
| | 7 | 10.15e-02 | **40.75e-02** | 4.52e-02 |
| | 8 | 4.68e-02 | **5.02e-02** | 3.94e-02 |
| | 9 | 10.20e-02 | **17.45e-02** | 7.72e-02 |
| | 10 | 3.26e-02 | 4.76e-02 | **5.01e-02** |
| WFG8 | 4 | 11.05e-02 | **12.01e-02** | 10.12e-02 |
| | 5 | 6.99e-02 | **7.08e-02** | 6.59e-02 |
| | 6 | 2.13e-02 | 3.01e-02 | **3.36e-02** |
| | 7 | **7.21e-02** | 7.02e-02 | 4.39e-02 |
| | 8 | 10.18e-02 | **12.39e-02** | 10.34e-02 |
| | 9 | 8.82e-02 | **9.01e-02** | 7.77e-02 |
| | 10 | 2.49e-02 | 3.01e-02 | **11.82e-02** |
| WFG9 | 4 | 2.45e-02 | 3.53e-02 | **5.45e-02** |
| | 5 | 6.14e-02 | **6.26e-02** | 5.65e-02 |
| | 6 | 2.33e-02 | 3.60e-02 | **8.62e-02** |
| | 7 | **10.92e-02** | 3.91e-02 | 10.45e-02 |
| | 8 | 8.40e-02 | 9.32e-02 | **10.35e-02** |
| | 9 | 8.48e-02 | **8.58e-02** | 6.31e-02 |
| | 10 | 8.07e-02 | 8.09e-02 | **8.65e-02** |
| Total highlights | | 8 | 21 | 34 |

*Table C.7: IGD values for PBI decomposition method on DTLZ1-DTLZ6 test problems. The parameter $\theta = 5$ unless specified. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | | |
|---|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV | PEV ($\theta$=0) |
| DTLZ1 | 4 | 2.69e-02 | 2.72e-02 | 2.38e-02$^\uparrow$ | 2.46e-02$^\uparrow$ | **1.66e-02**$^\uparrow$ | 2.07e-02$^\uparrow$ |
| | 5 | 1.38e-02 | 1.63e-02 | 1.55e-02$^\uparrow$ | 1.60e-02$^\uparrow$ | **1.36e-02**$^\uparrow$ | 1.45e-02$^\uparrow$ |
| | 6 | 1.58e-03 | 1.99e-03 | 1.63e-03$^\uparrow$ | 1.66e-03$^\uparrow$ | **1.19e-03**$^\uparrow$ | 1.58e-03$^\uparrow$ |
| | 7 | 5.12e-03 | 5.77e-03 | 5.03e-03$^\uparrow$ | 5.08e-03$^\uparrow$ | 5.23e-03$^\uparrow$ | **4.06e-03**$^\uparrow$ |
| | 8 | 1.02e-02 | 1.10e-02 | 1.11e-02$^\downarrow$ | 1.12e-02$^\downarrow$ | **1.00e-02**$^\uparrow$ | 1.08e-02$^\uparrow$ |
| | 9 | 8.77e-03 | 8.24e-03 | 6.49e-03$^\uparrow$ | 6.51e-03$^\uparrow$ | **3.30e-03**$^\uparrow$ | 6.35e-03$^\uparrow$ |
| | 10 | 1.58e-02 | 1.35e-02 | 1.18e-02$^\uparrow$ | 1.18e-02$^\uparrow$ | 1.23e-02$^\uparrow$ | **1.16e-02**$^\uparrow$ |
| DTLZ2 | 4 | 2.21e-02 | 8.05e-03 | 4.63e-03$^\uparrow$ | 6.03e-03$^\uparrow$ | 5.48e-03$^\uparrow$ | **4.27e-03**$^\uparrow$ |
| | 5 | 9.10e-03 | 4.47e-03 | 3.22e-03$^\uparrow$ | 3.08e-03$^\uparrow$ | 3.46e-03$^\uparrow$ | **1.85e-03**$^\uparrow$ |
| | 6 | 7.24e-04 | 4.80e-04 | 7.89e-04$^\downarrow$ | 6.46e-04$^\downarrow$ | 3.23e-04$^\uparrow$ | **1.68e-04**$^\uparrow$ |
| | 7 | 1.67e-03 | 1.12e-03 | 1.01e-03$^\uparrow$ | 1.36e-03$^\downarrow$ | 1.71e-03$^\downarrow$ | **8.58e-04**$^\uparrow$ |
| | 8 | 2.71e-03 | 1.79e-03 | 1.22e-03$^\uparrow$ | 2.22e-03$^\downarrow$ | 2.22e-03$^\downarrow$ | **1.08e-03**$^\uparrow$ |
| | 9 | 1.47e-03 | 9.82e-04 | 3.72e-04$^\uparrow$ | 3.72e-04$^\uparrow$ | 4.54e-04$^\uparrow$ | **1.00e-04**$^\uparrow$ |
| | 10 | 2.53e-03 | 1.86e-03 | 9.21e-04$^\uparrow$ | 1.39e-03$^\uparrow$ | 1.40e-03$^\uparrow$ | **5.97e-04**$^\uparrow$ |
| DTLZ3 | 4 | 2.25e-02 | 7.26e-03 | 6.13e-03$^\uparrow$ | 6.09e-03$^\uparrow$ | 9.41e-03$^\downarrow$ | **3.38e-03**$^\uparrow$ |
| | 5 | 9.11e-03 | 4.27e-03 | 3.48e-03$^\uparrow$ | 3.48e-03$^\uparrow$ | **2.47e-03**$^\uparrow$ | 3.54e-03$^\uparrow$ |
| | 6 | 7.25e-04 | 1.20e-03 | 1.05e-03$^\uparrow$ | **1.04e-03**$^\uparrow$ | 1.30e-03$^\downarrow$ | 1.13e-03$^\uparrow$ |
| | 7 | 1.67e-03 | 1.21e-03 | 1.10e-03$^\uparrow$ | 1.29e-03$^\downarrow$ | 4.24e-03$^\downarrow$ | **1.07e-03**$^\uparrow$ |
| | 8 | 2.70e-03 | 1.76e-03 | 1.45e-03$^\uparrow$ | 1.28e-03$^\uparrow$ | **1.09e-03**$^\uparrow$ | 1.27e-03$^\uparrow$ |
| | 9 | 1.49e-03 | 9.97e-04 | 6.29e-04$^\uparrow$ | 5.76e-04$^\uparrow$ | **3.13e-04**$^\uparrow$ | 6.69e-04$^\uparrow$ |
| | 10 | 2.53e-03 | 1.87e-03 | 1.26e-03$^\uparrow$ | 1.26e-03$^\uparrow$ | **1.05e-03**$^\uparrow$ | 1.23e-03$^\uparrow$ |
| DTLZ4 | 4 | 2.22e-02 | 8.29e-03 | 6.50e-03$^\uparrow$ | **5.78e-03**$^\uparrow$ | 6.50e-03$^\uparrow$ | 6.50e-03$^\uparrow$ |
| | 5 | 9.12e-03 | 4.28e-03 | 3.79e-03$^\uparrow$ | **3.29e-03**$^\uparrow$ | 3.79e-03$^\uparrow$ | 3.79e-03$^\uparrow$ |
| | 6 | 8.35e-04 | 4.16e-04 | 3.45e-04$^\uparrow$ | 4.08e-04$^\uparrow$ | **3.40e-04**$^\uparrow$ | 3.55e-04$^\uparrow$ |
| | 7 | 2.03e-03 | 1.07e-03 | 1.21e-03$^\downarrow$ | 1.94e-03$^\downarrow$ | 1.21e-03$^\downarrow$ | **1.05e-03**$^\uparrow$ |
| | 8 | 2.93e-03 | 1.68e-03 | 1.56e-03$^\uparrow$ | 1.56e-03$^\uparrow$ | 1.56e-03$^\uparrow$ | **1.07e-03**$^\uparrow$ |
| | 9 | 1.57e-03 | 1.17e-03 | 1.13e-03$^\uparrow$ | 1.13e-03$^\uparrow$ | 1.13e-03$^\uparrow$ | **1.06e-03**$^\uparrow$ |
| | 10 | 2.69e-03 | 1.75e-03 | 1.25e-03$^\uparrow$ | 1.19e-03$^\uparrow$ | 1.25e-03$^\uparrow$ | **1.10e-03**$^\uparrow$ |
| DTLZ5 | 4 | 1.71e-02 | 1.67e-02 | 1.52e-02$^\uparrow$ | 1.52e-02$^\uparrow$ | 1.53e-02$^\uparrow$ | **1.50e-02**$^\uparrow$ |
| | 5 | 5.99e-03 | 5.88e-03 | 4.99e-03$^\uparrow$ | 4.98e-03$^\uparrow$ | 4.94e-03$^\uparrow$ | **4.90e-03**$^\uparrow$ |
| | 6 | 1.90e-03 | 1.89e-03 | 1.65e-03$^\uparrow$ | 1.66e-03$^\uparrow$ | **1.57e-03**$^\uparrow$ | 1.59e-03$^\uparrow$ |
| | 7 | 6.41e-04 | 6.41e-04 | 6.46e-04$^\downarrow$ | 6.32e-04$^\uparrow$ | **5.09e-04**$^\uparrow$ | 5.24e-04$^\uparrow$ |
| | 8 | 9.65e-04 | 9.61e-04 | 9.58e-04$^\uparrow$ | 9.57e-04$^\uparrow$ | **9.28e-04**$^\uparrow$ | 9.44e-04$^\uparrow$ |
| | 9 | 8.18e-04 | 8.14e-04 | 7.27e-04$^\uparrow$ | 7.27e-04$^\uparrow$ | **7.03e-04**$^\uparrow$ | 7.22e-04$^\uparrow$ |
| | 10 | 7.27e-04 | 7.20e-04 | 7.12e-04$^\uparrow$ | 7.12e-04$^\uparrow$ | **6.60e-04**$^\uparrow$ | 7.03e-04$^\uparrow$ |
| DTLZ6 | 4 | 1.71e-02 | 1.68e-02 | 1.59e-02$^\uparrow$ | 1.56e-02$^\uparrow$ | 1.53e-02$^\uparrow$ | **1.48e-02**$^\uparrow$ |
| | 5 | 6.18e-03 | 6.12e-03 | 5.39e-03$^\uparrow$ | 5.38e-03$^\uparrow$ | 5.21e-03$^\uparrow$ | **4.96e-03**$^\uparrow$ |
| | 6 | 1.90e-03 | 1.89e-03 | 1.89e-03$^\approx$ | 1.99e-03$^\downarrow$ | **1.80e-03**$^\uparrow$ | 1.85e-03$^\uparrow$ |
| | 7 | 6.42e-04 | 6.37e-04 | 6.32e-04$^\uparrow$ | 5.89e-04$^\uparrow$ | **5.29e-04**$^\uparrow$ | 5.85e-04$^\uparrow$ |
| | 8 | 9.69e-04 | 9.59e-04 | 9.57e-04$^\uparrow$ | 9.49e-04$^\uparrow$ | 9.54e-04$^\uparrow$ | **9.39e-04**$^\uparrow$ |
| | 9 | 8.20e-04 | 8.14e-04 | 8.19e-04$^\downarrow$ | 8.08e-04$^\uparrow$ | 8.30e-04$^\downarrow$ | **8.02e-04**$^\uparrow$ |
| | 10 | 6.81e-04 | 6.79e-04 | 6.32e-04$^\uparrow$ | 6.25e-04$^\uparrow$ | 6.27e-04$^\uparrow$ | **6.12e-04**$^\uparrow$ |
| win/tie/loss counts | | | | 36/1/5 | 35/0/7 | 35/0/7 | 42/0/0 |

*Table C.8: IGD values for Tchebycheff decomposition method on DTLZ1-DTLZ6 test problems. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | |
|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV |
| DTLZ1 | 4 | 2.69e-02 | 2.00e-02 | 1.29e-02$^{\uparrow}$ | 1.86e-02$^{\uparrow}$ | **1.56e-02**$^{\uparrow}$ |
| | 5 | 1.34e-02 | 1.31e-02 | 1.27e-02$^{\uparrow}$ | 1.16e-02$^{\uparrow}$ | **1.06e-02**$^{\uparrow}$ |
| | 6 | 1.52e-03 | 1.50e-03 | 2.14e-03$^{\downarrow}$ | 2.19e-03$^{\downarrow}$ | **1.26e-03**$^{\downarrow}$ |
| | 7 | 5.12e-03 | 4.93e-03 | 3.50e-03$^{\uparrow}$ | 4.81e-03$^{\uparrow}$ | **4.06e-03**$^{\uparrow}$ |
| | 8 | 1.02e-02 | 8.76e-03 | 1.17e-03$^{\uparrow}$ | 1.18e-03$^{\uparrow}$ | **1.03e-03**$^{\uparrow}$ |
| | 9 | 8.77e-03 | 6.02e-03 | 5.81e-03$^{\uparrow}$ | 5.89e-03$^{\uparrow}$ | **5.44e-03**$^{\uparrow}$ |
| | 10 | 1.58e-02 | 1.09e-02 | 1.30e-02$^{\downarrow}$ | 1.03e-02$^{\uparrow}$ | **1.06e-02**$^{\uparrow}$ |
| DTLZ2 | 4 | 2.21e-02 | 8.67e-03 | 6.31e-03$^{\uparrow}$ | 5.08e-03$^{\uparrow}$ | **7.38e-03**$^{\uparrow}$ |
| | 5 | 9.10e-03 | 4.91e-03 | 3.67e-03$^{\uparrow}$ | **3.54e-03**$^{\uparrow}$ | 3.85e-03$^{\uparrow}$ |
| | 6 | 7.24e-04 | 5.03e-04 | 3.32e-04$^{\uparrow}$ | 3.36e-04$^{\uparrow}$ | **3.23e-04**$^{\uparrow}$ |
| | 7 | 1.67e-03 | 1.46e-03 | 1.53e-03$^{\downarrow}$ | 1.55e-03$^{\downarrow}$ | **1.18e-03**$^{\uparrow}$ |
| | 8 | 2.71e-03 | 2.15e-03 | 1.52e-03$^{\uparrow}$ | 1.19e-03$^{\uparrow}$ | **1.13e-03**$^{\uparrow}$ |
| | 9 | 1.47e-03 | 1.07e-03 | 1.08e-03$^{\downarrow}$ | 1.05e-03$^{\uparrow}$ | **1.02e-03**$^{\uparrow}$ |
| | 10 | 2.53e-03 | 2.06e-03 | 1.08e-03$^{\uparrow}$ | 1.16e-03$^{\uparrow}$ | **6.37e-04**$^{\uparrow}$ |
| DTLZ3 | 4 | 2.21e-02 | 9.12e-03 | 5.96e-03$^{\uparrow}$ | **5.84e-03**$^{\uparrow}$ | 5.96e-03$^{\uparrow}$ |
| | 5 | 9.11e-03 | 5.87e-03 | 3.54e-03$^{\uparrow}$ | 3.54e-03$^{\uparrow}$ | **2.89e-03**$^{\uparrow}$ |
| | 6 | 7.25e-04 | 4.62e-04 | 5.70e-04$^{\downarrow}$ | 4.56e-04$^{\uparrow}$ | **3.03e-04**$^{\uparrow}$ |
| | 7 | 1.67e-03 | 1.37e-03 | **1.15e-03**$^{\uparrow}$ | 2.74e-03$^{\downarrow}$ | 5.20e-03$^{\uparrow}$ |
| | 8 | 2.70e-03 | 2.51e-03 | **2.15e-03**$^{\uparrow}$ | 3.19e-03$^{\downarrow}$ | 3.45e-03$^{\downarrow}$ |
| | 9 | 1.45e-03 | 1.16e-03 | 1.12e-03$^{\uparrow}$ | 1.16e-03$^{\approx}$ | **1.06e-03**$^{\uparrow}$ |
| | 10 | 2.53e-03 | 2.42e-03 | 1.72e-03$^{\uparrow}$ | **1.24e-03**$^{\uparrow}$ | 1.33e-03$^{\uparrow}$ |
| DTLZ4 | 4 | 2.22e-02 | 1.03e-02 | 1.03e-02$^{\approx}$ | 1.01e-02$^{\uparrow}$ | **8.92e-03**$^{\uparrow}$ |
| | 5 | 9.12e-03 | 8.51e-03 | 7.77e-03$^{\uparrow}$ | 7.54e-03$^{\uparrow}$ | **6.98e-03**$^{\uparrow}$ |
| | 6 | 8.35e-04 | 7.09e-04 | 3.41e-04$^{\uparrow}$ | 3.20e-04$^{\uparrow}$ | **3.08e-04**$^{\uparrow}$ |
| | 7 | 2.03e-03 | 1.66e-03 | 2.06e-03$^{\downarrow}$ | 1.96e-03$^{\downarrow}$ | **1.22e-03**$^{\uparrow}$ |
| | 8 | 2.93e-03 | 2.68e-03 | 1.72e-03$^{\uparrow}$ | 1.11e-03$^{\uparrow}$ | **1.03e-03**$^{\uparrow}$ |
| | 9 | 1.57e-03 | 1.40e-03 | 1.27e-03$^{\uparrow}$ | 1.22e-03$^{\uparrow}$ | **1.10e-03**$^{\uparrow}$ |
| | 10 | 2.69e-03 | 2.40e-03 | 2.11e-03$^{\uparrow}$ | 2.12e-03$^{\uparrow}$ | **2.02e-03**$^{\uparrow}$ |
| DTLZ5 | 4 | 1.71e-02 | 1.62e-02 | 1.53e-02$^{\uparrow}$ | 1.61e-02$^{\uparrow}$ | **1.45e-02**$^{\uparrow}$ |
| | 5 | 5.99e-03 | 5.77e-03 | 4.89e-03$^{\uparrow}$ | 4.95e-03$^{\uparrow}$ | **4.78e-03**$^{\uparrow}$ |
| | 6 | 1.90e-03 | 1.79e-03 | 1.75e-03$^{\uparrow}$ | 1.70e-03$^{\uparrow}$ | **1.52e-03**$^{\uparrow}$ |
| | 7 | 6.41e-04 | 6.27e-04 | 6.13e-04$^{\uparrow}$ | 3.09e-04$^{\uparrow}$ | **5.05e-04**$^{\uparrow}$ |
| | 8 | 9.65e-04 | 9.16e-04 | 7.78e-04$^{\uparrow}$ | 9.13e-04$^{\uparrow}$ | **7.14e-04**$^{\uparrow}$ |
| | 9 | 8.18e-04 | 7.96e-04 | **7.24e-04**$^{\uparrow}$ | 8.12e-04$^{\downarrow}$ | 7.45e-04$^{\uparrow}$ |
| | 10 | 7.27e-04 | 6.75e-04 | 6.65e-04$^{\uparrow}$ | 6.58e-04$^{\uparrow}$ | **6.53e-04**$^{\uparrow}$ |
| DTLZ6 | 4 | 1.71e-02 | 1.68e-02 | 1.62e-02$^{\uparrow}$ | 3.41e-02$^{\downarrow}$ | **1.37e-02**$^{\uparrow}$ |
| | 5 | 5.98e-03 | 5.78e-03 | 5.02e-03$^{\uparrow}$ | 5.72e-03$^{\uparrow}$ | **4.73e-03**$^{\uparrow}$ |
| | 6 | 1.90e-03 | 1.92e-03 | 1.88e-03$^{\uparrow}$ | 1.89e-03$^{\uparrow}$ | **1.08e-03**$^{\uparrow}$ |
| | 7 | 6.42e-04 | 6.63e-04 | 7.62e-04$^{\downarrow}$ | 6.25e-04$^{\uparrow}$ | **6.09e-04**$^{\uparrow}$ |
| | 8 | 9.69e-04 | 9.20e-04 | 9.14e-04$^{\uparrow}$ | 9.06e-04$^{\uparrow}$ | **9.02e-04**$^{\uparrow}$ |
| | 9 | 8.20e-04 | 8.24e-04 | 8.12e-04$^{\uparrow}$ | 8.14e-04$^{\uparrow}$ | **7.98e-04**$^{\uparrow}$ |
| | 10 | 6.79e-04 | 6.26e-04 | 6.34e-04$^{\downarrow}$ | 6.15e-04$^{\uparrow}$ | **5.60e-04**$^{\uparrow}$ |
| win/tie/loss counts | | | | 34/1/7 | 34/1/7 | 40/0/2 |

*Table C.9: IGD values for PBI decomposition method on WFG1-WFG9 test problems. The parameter $\theta = 5$ unless specified. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | | |
|---|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV | PEV ($\theta$=0) |
| WFG1 | 4 | 2.76e-02 | 2.75e-02 | 2.74e-02$^{\uparrow}$ | 2.73e-02$^{\uparrow}$ | 2.73e-02$^{\uparrow}$ | **2.72e-02**$^{\uparrow}$ |
| | 5 | 6.48e-03 | 6.47e-03 | 6.46e-03$^{\uparrow}$ | 6.45e-03$^{\uparrow}$ | 6.44e-03$^{\uparrow}$ | **6.42e-03**$^{\uparrow}$ |
| | 6 | 1.53e-03 | 1.52e-03 | 1.51e-03$^{\uparrow}$ | 1.51e-03$^{\uparrow}$ | 1.50e-03$^{\uparrow}$ | **1.49e-03**$^{\uparrow}$ |
| | 7 | 2.92e-03 | 2.80e-03 | 2.80e-03$^{\approx}$ | 2.80e-03$^{\approx}$ | 2.79e-03$^{\uparrow}$ | **2.78e-03**$^{\uparrow}$ |
| | 8 | 5.34e-03 | 5.34e-03 | 5.33e-03$^{\uparrow}$ | 5.32e-03$^{\uparrow}$ | 5.31e-03$^{\uparrow}$ | **5.30e-03**$^{\uparrow}$ |
| | 9 | 2.28e-03 | 2.39e-03 | 2.28e-03$^{\uparrow}$ | 2.28e-03$^{\uparrow}$ | 2.28e-03$^{\uparrow}$ | **2.27e-03**$^{\uparrow}$ |
| | 10 | 2.17e-03 | 2.28e-03 | 2.17e-03$^{\uparrow}$ | 2.17e-03$^{\uparrow}$ | 2.17e-03$^{\uparrow}$ | **2.16e-03**$^{\uparrow}$ |
| WFG2 | 4 | **1.37e-01** | 1.41e-01 | 1.41e-01$^{\approx}$ | 1.39e-01$^{\uparrow}$ | 1.39e-01$^{\uparrow}$ | **1.37e-01**$^{\uparrow}$ |
| | 5 | **1.56e-02** | 1.60e-02 | 1.60e-02$^{\approx}$ | 1.60e-02$^{\approx}$ | 1.59e-02$^{\uparrow}$ | **1.56e-02**$^{\uparrow}$ |
| | 6 | 2.30e-02 | 2.36e-02 | 2.29e-02$^{\uparrow}$ | 2.29e-02$^{\uparrow}$ | 2.28e-02$^{\uparrow}$ | **2.27e-02**$^{\uparrow}$ |
| | 7 | 8.46e-03 | 8.68e-03 | 8.45e-03$^{\uparrow}$ | 8.44e-03$^{\uparrow}$ | 8.43e-03$^{\uparrow}$ | **8.42e-03**$^{\uparrow}$ |
| | 8 | **3.05e-03** | 3.13e-03 | 3.13e-03$^{\approx}$ | 3.12e-03$^{\uparrow}$ | 3.11e-03$^{\uparrow}$ | **3.05e-03**$^{\uparrow}$ |
| | 9 | **8.02e-03** | 8.23e-03 | **8.02e-03**$^{\uparrow}$ | **8.02e-03**$^{\uparrow}$ | **8.02e-03**$^{\uparrow}$ | 8.09e-03$^{\uparrow}$ |
| | 10 | 8.12e-03 | 8.33e-03 | 8.11e-03$^{\uparrow}$ | 8.11e-03$^{\uparrow}$ | 8.10e-03$^{\uparrow}$ | 8.09e-03$^{\uparrow}$ |
| WFG3 | 4 | 1.20e-01 | 1.34e-01 | 1.20e-01$^{\uparrow}$ | 1.20e-01$^{\uparrow}$ | 1.19e-01$^{\uparrow}$ | **1.18e-01**$^{\uparrow}$ |
| | 5 | 6.23e-02 | 6.83e-02 | 6.22e-02$^{\uparrow}$ | 6.21e-02$^{\uparrow}$ | 6.20e-02$^{\uparrow}$ | **6.19e-02**$^{\uparrow}$ |
| | 6 | 3.45e-02 | 3.81e-02 | 3.44e-02$^{\uparrow}$ | 3.45e-02$^{\uparrow}$ | 3.45e-02$^{\uparrow}$ | **3.43e-02**$^{\uparrow}$ |
| | 7 | 1.11e-01 | 1.23e-01 | 1.10e-01$^{\uparrow}$ | 1.09e-01$^{\uparrow}$ | 1.08e-01$^{\uparrow}$ | **1.07e-01**$^{\uparrow}$ |
| | 8 | 1.03e-01 | 9.18e-02 | 9.17e-02$^{\uparrow}$ | 9.18e-02$^{\uparrow}$ | 9.11e-02$^{\uparrow}$ | **8.27e-02**$^{\uparrow}$ |
| | 9 | 1.97e-01 | 2.19e-01 | 1.96e-01$^{\uparrow}$ | 1.95e-01$^{\uparrow}$ | 1.94e-01$^{\uparrow}$ | **1.93e-01**$^{\uparrow}$ |
| | 10 | 1.69e-01 | 1.89e-01 | 1.68e-01$^{\uparrow}$ | 1.67e-01$^{\uparrow}$ | 1.66e-01$^{\uparrow}$ | **1.65e-01**$^{\uparrow}$ |
| WFG4 | 4 | 3.15e-02 | 3.18e-02 | 3.12e-02$^{\uparrow}$ | **3.08e-02**$^{\uparrow}$ | 3.11e-02$^{\uparrow}$ | 3.10e-02$^{\uparrow}$ |
| | 5 | **9.40e-03** | 9.43e-03 | 9.42e-03$^{\uparrow}$ | 9.41e-03$^{\uparrow}$ | 9.41e-03$^{\uparrow}$ | **9.40e-03**$^{\uparrow}$ |
| | 6 | 1.50e-02 | 1.50e-02 | 1.49e-02$^{\uparrow}$ | 1.48e-02$^{\uparrow}$ | 1.47e-02$^{\uparrow}$ | **1.45e-02**$^{\uparrow}$ |
| | 7 | 6.64e-03 | 6.63e-03 | 6.63e-03$^{\approx}$ | **6.62e-03**$^{\uparrow}$ | **6.62e-03**$^{\uparrow}$ | 6.64e-03$^{\downarrow}$ |
| | 8 | 3.03e-03 | 3.05e-03 | **3.01e-03**$^{\uparrow}$ | **3.01e-03**$^{\uparrow}$ | 3.02e-03$^{\uparrow}$ | **3.01e-03**$^{\uparrow}$ |
| | 9 | 5.96e-03 | 5.87e-03 | 5.87e-03$^{\approx}$ | 5.86e-03$^{\uparrow}$ | 5.85e-03$^{\uparrow}$ | **5.83e-03**$^{\uparrow}$ |
| | 10 | 3.37e-03 | 3.32e-03 | 3.32e-03$^{\approx}$ | 3.31e-03$^{\uparrow}$ | 3.30e-03$^{\uparrow}$ | **3.29e-03**$^{\uparrow}$ |
| WFG5 | 4 | 4.84e-02 | 4.85e-02 | 4.88e-02$^{\downarrow}$ | 5.60e-02$^{\downarrow}$ | **4.81e-02**$^{\uparrow}$ | 5.91e-02$^{\downarrow}$ |
| | 5 | 2.10e-02 | 2.10e-02 | 2.10e-02$^{\approx}$ | 2.08e-02$^{\uparrow}$ | 2.09e-02$^{\uparrow}$ | **1.95e-03**$^{\uparrow}$ |
| | 6 | 4.38e-02 | 4.38e-02 | 4.37e-02$^{\uparrow}$ | 4.36e-02$^{\uparrow}$ | 4.36e-02$^{\uparrow}$ | **4.34e-03**$^{\uparrow}$ |
| | 7 | 2.38e-02 | 2.38e-02 | 2.37e-02$^{\uparrow}$ | 2.36e-02$^{\uparrow}$ | 2.35e-02$^{\uparrow}$ | **2.11e-02**$^{\uparrow}$ |
| | 8 | 1.27e-02 | 1.60e-02 | 1.26e-02$^{\uparrow}$ | 1.26e-02$^{\uparrow}$ | 1.25e-02$^{\uparrow}$ | **1.24e-02**$^{\uparrow}$ |
| | 9 | 2.77e-02 | 2.77e-02 | 2.76e-02$^{\uparrow}$ | 2.75e-02$^{\uparrow}$ | **2.74e-02**$^{\uparrow}$ | **2.74e-02**$^{\uparrow}$ |
| | 10 | 1.74e-02 | 1.74e-02 | 1.73e-02$^{\uparrow}$ | 1.73e-02$^{\uparrow}$ | 1.72e-02$^{\uparrow}$ | **1.71e-02**$^{\uparrow}$ |
| WFG6 | 4 | **9.95e-02** | 1.33e-01 | 1.23e-01$^{\uparrow}$ | 1.33e-01$^{\approx}$ | 1.27e-01$^{\uparrow}$ | 1.23e-01$^{\uparrow}$ |
| | 5 | 3.64e-02 | 4.06e-02 | 3.05e-02$^{\uparrow}$ | 3.06e-02$^{\uparrow}$ | **3.03e-02**$^{\uparrow}$ | 3.06e-02$^{\uparrow}$ |
| | 6 | 6.52e-02 | 7.42e-02 | 6.42e-02$^{\uparrow}$ | 6.41e-02$^{\uparrow}$ | 6.39e-02$^{\uparrow}$ | **6.37e-02**$^{\uparrow}$ |
| | 7 | 3.42e-02 | 3.69e-02 | 3.29e-02$^{\uparrow}$ | 3.29e-02$^{\uparrow}$ | **3.28e-02**$^{\uparrow}$ | 3.29e-02$^{\uparrow}$ |
| | 8 | **1.76e-02** | 2.15e-02 | 1.93e-02$^{\uparrow}$ | 1.97e-02$^{\uparrow}$ | 1.90e-02$^{\uparrow}$ | 1.87e-02$^{\uparrow}$ |
| | 9 | **3.66e-02** | 3.89e-02 | 3.89e-02$^{\approx}$ | 3.89e-02$^{\approx}$ | 3.88e-02$^{\uparrow}$ | 3.87e-02$^{\uparrow}$ |
| | 10 | 2.22e-02 | 2.34e-02 | 2.16e-02$^{\uparrow}$ | 2.15e-02$^{\uparrow}$ | 2.14e-02$^{\uparrow}$ | **2.13e-02**$^{\uparrow}$ |
| | 4 | 4.75e-02 | 4.76e-02 | 4.75e-02$^{\uparrow}$ | 4.75e-02$^{\uparrow}$ | 4.75e-02$^{\uparrow}$ | **4.71e-02**$^{\uparrow}$ |
| | 5 | 1.55e-02 | 1.56e-02 | 1.55e-02$^{\uparrow}$ | 1.55e-02$^{\uparrow}$ | **1.53e-02**$^{\uparrow}$ | 1.55e-02$^{\uparrow}$ |
| WFG7 | | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 6 | 2.73e-02 | 2.73e-02 | 2.72e-02↑ | 2.72e-02↑ | 2.71e-02↑ | **2.70e-02**↑ |
| | 7 | 1.33e-02 | 1.33e-02 | 1.33e-02≈ | 1.33e-02↑ | 1.32e-02↑ | **1.31e-02**↑ |
| | 8 | 6.65e-03 | 6.67e-03 | 6.64e-03↑ | 6.62e-03↑ | 6.61e-03↑ | **6.60e-03**↑ |
| | 9 | **1.41e-02** | 1.42e-02 | 1.42e-02≈ | 1.42e-02≈ | **1.41e-02**↑ | 1.42e-02≈ |
| | 10 | **8.63e-03** | 8.67e-03 | 8.67e-03≈ | 8.67e-03≈ | 8.68e-03↓ | 8.66e-03↑ |
| | 4 | 9.17e-02 | 1.07e-01 | 1.08e-01↓ | **1.01e-01**↑ | 1.07e-01≈ | 1.09e-01↓ |
| | 5 | 3.44e-02 | 3.91e-02 | 3.42e-02↑ | 3.41e-02↑ | 3.41e-02↑ | **3.40e-02**↑ |
| WFG8 | 6 | 6.46e-02 | 7.22e-02 | **6.32e-02**↑ | 7.31e-02↓ | 7.33e-02↓ | 7.30e-02↓ |
| | 7 | **3.27e-02** | 3.60e-02 | 3.58e-02↑ | 3.60e-02≈ | 3.57e-02↑ | 3.56e-02↑ |
| | 8 | 1.66e-02 | 1.81e-02 | 1.61e-02↑ | 1.61e-02↑ | 1.60e-02↑ | **1.59e-02**↑ |
| | 9 | 3.54e-02 | 3.81e-02 | 3.41e-02↑ | 3.41e-02↑ | 3.40e-02↑ | **3.39e-02**↑ |
| | 10 | 2.16e-02 | 2.30e-02 | 2.10e-02↑ | 2.09e-02↑ | 2.08e-02↑ | **2.07e-02**↑ |
| | 4 | 3.01e-02 | 3.05e-02 | 3.05e-02≈ | 3.32e-02↓ | **3.00e-02**↑ | **3.00e-02**↑ |
| | 5 | 1.19e-02 | 8.81e-03 | 8.90e-03↓ | 8.95e-03↓ | 8.73e-03↑ | **8.70e-03**↑ |
| WFG9 | 6 | 2.26e-02 | 1.38e-02 | 1.37e-02↑ | 1.37e-02↑ | 1.36e-02↑ | **1.27e-02**↑ |
| | 7 | 1.40e-02 | 5.95e-03 | 5.96e-03↓ | 6.14e-03↓ | 5.94e-03↑ | **5.15e-03**↑ |
| | 8 | 8.49e-03 | 2.67e-03 | 1.67e-03↑ | 1.67e-03↑ | 1.66e-03↑ | **1.63e-03**↑ |
| | 9 | 2.00e-02 | 5.06e-03 | 5.05e-03↑ | 5.06e-03≈ | 5.04e-03↑ | **5.01e-03**↑ |
| | 10 | 1.33e-02 | 2.81e-03 | 2.73e-03↑ | 2.72e-03↑ | 2.70e-03↑ | **2.12e-03**↑ |
| win/tie/loss counts | | | | 46/13/4 | 50/8/5 | 60/1/2 | 58/1/4 |

*Table C.10: IGD values for Tchebycheff decomposition method on WFG1-WFG9 test problems. The median of 25 independent runs is reported.*

| | | | | UR-MEAD2 | | |
|---|---|---|---|---|---|---|
| Func. | # Obj | R-NSGA-II | R-MEAD2 | OHV | PE | PEV |
| WFG1 | 4 | 2.76e-02 | 3.04e-02 | 2.71e-02$^{\uparrow}$ | **2.73e-03**$^{\uparrow}$ | **2.70e-02**$^{\uparrow}$ |
| | 5 | 6.48e-03 | 7.20e-03 | 6.46e-03$^{\uparrow}$ | 6.47e-02$^{\uparrow}$ | **6.43e-03**$^{\uparrow}$ |
| | 6 | 1.53e-03 | 1.67e-03 | 1.51e-03$^{\uparrow}$ | 1.52e-03$^{\uparrow}$ | **1.50e-03**$^{\uparrow}$ |
| | 7 | 2.92e-03 | **2.80e-03** | 2.87e-03$^{\downarrow}$ | 2.84e-03$^{\downarrow}$ | **2.80e-03**$^{\approx}$ |
| | 8 | 5.34e-03 | 5.59e-03 | 5.28e-03$^{\uparrow}$ | 5.30e-03$^{\uparrow}$ | **5.25e-03**$^{\uparrow}$ |
| | 9 | **2.28e-03** | 2.36e-03 | 2.39e-03$^{\downarrow}$ | 2.54e-03$^{\downarrow}$ | 2.29e-03$^{\uparrow}$ |
| | 10 | 2.17e-03 | 2.23e-03 | 2.16e-03$^{\uparrow}$ | 2.14e-03$^{\uparrow}$ | **2.10e-03**$^{\uparrow}$ |
| WFG2 | 4 | 1.37e-01 | 1.45e-01 | 1.33e-01$^{\uparrow}$ | 1.32e-01$^{\uparrow}$ | **1.31e-01**$^{\uparrow}$ |
| | 5 | 1.56e-02 | 1.85e-02 | 1.50e-02$^{\uparrow}$ | 1.45e-02$^{\uparrow}$ | **1.39e-02**$^{\uparrow}$ |
| | 6 | 2.30e-02 | 2.67e-02 | 2.25e-02$^{\uparrow}$ | 2.24e-02$^{\uparrow}$ | **2.23e-02**$^{\uparrow}$ |
| | 7 | 8.46e-03 | 9.14e-03 | 8.40e-03$^{\uparrow}$ | 8.39e-02$^{\uparrow}$ | **8.36e-03**$^{\uparrow}$ |
| | 8 | 3.05e-03 | 3.61e-03 | **3.02e-03**$^{\uparrow}$ | **3.02e-03**$^{\uparrow}$ | 3.03e-03$^{\uparrow}$ |
| | 9 | 8.02e-03 | 9.09e-03 | 9.08e-03$^{\uparrow}$ | 9.26e-02$^{\downarrow}$ | **8.01e-03**$^{\downarrow}$ |
| | 10 | 8.12e-03 | 8.95e-03 | 8.11e-03$^{\uparrow}$ | 8.10e-03$^{\uparrow}$ | **8.09e-03**$^{\uparrow}$ |
| WFG3 | 4 | 1.18e-01 | 1.30e-01 | 1.18e-01$^{\uparrow}$ | 1.15e-01$^{\uparrow}$ | **1.11e-01**$^{\uparrow}$ |
| | 5 | 6.23e-02 | 7.85e-02 | 6.21e-02$^{\uparrow}$ | 6.20e-02$^{\uparrow}$ | **6.13e-02**$^{\uparrow}$ |
| | 6 | 3.43e-02 | 4.19e-02 | 3.42e-02$^{\uparrow}$ | 3.40e-02$^{\uparrow}$ | **3.39e-02**$^{\uparrow}$ |
| | 7 | 1.11e-01 | 1.25e-01 | 1.10e-01$^{\uparrow}$ | 1.09e-01$^{\uparrow}$ | **1.04e-01**$^{\uparrow}$ |
| | 8 | 1.03e-01 | 8.27e-02 | 8.24e-02$^{\uparrow}$ | 8.22e-02$^{\uparrow}$ | **8.12e-02**$^{\uparrow}$ |
| | 9 | 1.97e-01 | 2.33e-01 | 1.92e-01$^{\uparrow}$ | 1.91e-02$^{\uparrow}$ | **1.90e-01**$^{\uparrow}$ |
| | 10 | 1.69e-01 | 1.95e-01 | 1.65e-01$^{\uparrow}$ | 1.64e-01$^{\uparrow}$ | **1.62e-01**$^{\uparrow}$ |
| WFG4 | 4 | 3.15e-02 | 3.22e-02 | 3.10e-02$^{\uparrow}$ | **3.09e-02**$^{\uparrow}$ | 3.25e-02$^{\downarrow}$ |
| | 5 | **9.40e-03** | 9.65e-03 | **9.40e-03**$^{\uparrow}$ | **9.40e-03**$^{\uparrow}$ | **9.40e-03**$^{\uparrow}$ |
| | 6 | 1.50e-02 | 1.56e-02 | 1.49e-02$^{\uparrow}$ | 1.47e-02$^{\uparrow}$ | **1.41e-02**$^{\uparrow}$ |
| | 7 | 6.64e-03 | 7.02e-03 | 6.60e-03$^{\uparrow}$ | 6.55e-03$^{\uparrow}$ | **6.53e-03**$^{\uparrow}$ |
| | 8 | 3.03e-03 | 3.23e-03 | 3.03e-03$^{\uparrow}$ | 3.02e-03$^{\uparrow}$ | **2.95e-03**$^{\uparrow}$ |
| | 9 | 5.96e-03 | 1.93e-02 | 5.95e-03$^{\uparrow}$ | 5.92e-03$^{\uparrow}$ | **5.90e-03**$^{\uparrow}$ |
| | 10 | 3.37e-03 | 1.73e-02 | **3.31e-03**$^{\uparrow}$ | **3.31e-03**$^{\uparrow}$ | **3.31e-02**$^{\uparrow}$ |
| WFG5 | 4 | 4.84e-02 | 6.20e-02 | 4.80e-02$^{\uparrow}$ | **4.47e-01**$^{\uparrow}$ | 4.49e-02$^{\uparrow}$ |
| | 5 | 2.10e-02 | 2.42e-02 | 2.04e-02$^{\uparrow}$ | 2.04e-02$^{\uparrow}$ | **2.01e-02**$^{\uparrow}$ |
| | 6 | 4.38e-02 | 6.27e-03 | **4.27e-03**$^{\uparrow}$ | **4.27e-03**$^{\uparrow}$ | 4.28e-02$^{\uparrow}$ |
| | 7 | 2.38e-02 | 2.71e-02 | 2.37e-02$^{\uparrow}$ | 2.36e-02$^{\uparrow}$ | **2.35e-02**$^{\uparrow}$ |
| | 8 | **1.27e-02** | 1.37e-02 | 1.31e-02$^{\uparrow}$ | 1.29e-02$^{\uparrow}$ | 1.29e-02$^{\uparrow}$ |
| | 9 | 2.77e-02 | 3.12e-02 | 2.75e-02$^{\uparrow}$ | 2.75e-02$^{\uparrow}$ | **2.70e-02**$^{\uparrow}$ |
| | 10 | 1.74e-02 | 1.86e-02 | 1.71e-02$^{\uparrow}$ | 2.70e-02$^{\downarrow}$ | **1.66e-02**$^{\uparrow}$ |
| WFG6 | 4 | **9.85e-02** | 1.07e-01 | 1.05e-01$^{\uparrow}$ | 3.47e-01$^{\downarrow}$ | 9.88e-02$^{\uparrow}$ |
| | 5 | **3.64e-02** | 4.04e-02 | 4.15e-03$^{\downarrow}$ | 4.15e-03$^{\downarrow}$ | 3.68e-03$^{\uparrow}$ |
| | 6 | **6.52e-02** | 7.36e-02 | 7.27e-03$^{\uparrow}$ | 6.92e-03$^{\uparrow}$ | 6.86e-03$^{\uparrow}$ |
| | 7 | **3.42e-02** | 3.60e-02 | 3.60e-02$^{\approx}$ | 3.59e-02$^{\uparrow}$ | 3.44e-02$^{\uparrow}$ |
| | 8 | 1.73e-02 | 1.84e-02 | 1.64e-02$^{\uparrow}$ | 1.62e-02$^{\uparrow}$ | **1.55e-02**$^{\uparrow}$ |
| | 9 | 3.66e-02 | 3.85e-02 | 3.59e-02$^{\uparrow}$ | 3.58e-02$^{\uparrow}$ | **3.08e-02**$^{\uparrow}$ |
| | 10 | 2.22e-02 | 2.32e-02 | 2.21e-02$^{\uparrow}$ | 2.18e-02$^{\uparrow}$ | **2.13e-02**$^{\uparrow}$ |
| WFG7 | 4 | 4.75e-02 | 4.82e-02 | **4.10e-02**$^{\uparrow}$ | 4.17e-02$^{\uparrow}$ | **4.10e-02**$^{\uparrow}$ |
| | 5 | 1.55e-02 | 1.60e-02 | 1.50e-02$^{\uparrow}$ | 1.51e-02$^{\uparrow}$ | **1.48e-02**$^{\uparrow}$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 6 | 2.73e-02 | 2.90e-02 | 2.70e-02$^{\uparrow}$ | 2.67e-03$^{\uparrow}$ | **2.60e-02**$^{\uparrow}$ |
| | 7 | 1.33e-02 | 1.41e-02 | 1.32e-02$^{\uparrow}$ | 3.33e-02$^{\uparrow}$ | **1.30e-02**$^{\uparrow}$ |
| | 8 | 6.65e-03 | 7.26e-03 | 6.38e-03$^{\uparrow}$ | 6.34e-02$^{\uparrow}$ | **6.29e-03**$^{\uparrow}$ |
| | 9 | 1.41e-02 | 1.55e-02 | 1.40e-02$^{\uparrow}$ | **1.39e-02**$^{\uparrow}$ | **1.39e-02**$^{\uparrow}$ |
| | 10 | 8.63e-03 | 9.76e-03 | 8.62e-03$^{\uparrow}$ | 8.61e-03$^{\uparrow}$ | **8.60e-03**$^{\uparrow}$ |
| WFG8 | 4 | 9.17e-02 | 9.99e-02 | 9.90e-02$^{\uparrow}$ | 9.47e-02$^{\uparrow}$ | **9.52e-02**$^{\uparrow}$ |
| | 5 | 3.44e-02 | 3.87e-02 | 3.43e-02$^{\uparrow}$ | 3.40e-02$^{\uparrow}$ | **3.39e-02**$^{\uparrow}$ |
| | 6 | 6.46e-02 | **6.27e-03** | 6.37e-03$^{\downarrow}$ | 6.34e-02$^{\downarrow}$ | 6.32e-02$^{\downarrow}$ |
| | 7 | 3.27e-02 | 3.47e-02 | 3.27e-02$^{\uparrow}$ | 3.26e-02$^{\uparrow}$ | **3.25e-02**$^{\uparrow}$ |
| | 8 | **1.66e-02** | 1.78e-02 | 1.79e-02$^{\downarrow}$ | 1.76e-02$^{\uparrow}$ | 1.69e-02$^{\uparrow}$ |
| | 9 | **3.54e-02** | 3.75e-02 | 3.81e-02$^{\downarrow}$ | 3.75e-02$^{\approx}$ | 3.60e-02$^{\uparrow}$ |
| | 10 | 2.16e-02 | 2.26e-02 | 2.26e-02$^{\approx}$ | 2.11e-02$^{\uparrow}$ | **2.08e-02**$^{\uparrow}$ |
| WFG9 | 4 | **3.01e-02** | 5.94e-02 | 3.06e-02$^{\uparrow}$ | 3.07e-02$^{\uparrow}$ | 3.06e-02$^{\uparrow}$ |
| | 5 | 1.19e-02 | 2.34e-02 | 2.84e-03$^{\uparrow}$ | 9.53e-03$^{\uparrow}$ | **9.13e-03**$^{\uparrow}$ |
| | 6 | 2.26e-02 | 6.27e-02 | 1.79e-02$^{\uparrow}$ | 1.58e-02$^{\uparrow}$ | **1.50e-02**$^{\uparrow}$ |
| | 7 | 1.40e-02 | 2.90e-02 | 7.44e-03$^{\uparrow}$ | 7.41e-03$^{\uparrow}$ | **7.17e-03**$^{\uparrow}$ |
| | 8 | 8.49e-03 | 1.46e-02 | 8.46e-03$^{\uparrow}$ | 7.77e-03$^{\uparrow}$ | **3.11e-03**$^{\uparrow}$ |
| | 9 | 2.00e-02 | 3.49e-02 | 1.55e-02$^{\uparrow}$ | 1.70e-02$^{\uparrow}$ | **1.40e-02**$^{\uparrow}$ |
| | 10 | 1.33e-02 | 1.96e-02 | 1.02e-02$^{\uparrow}$ | 1.03e-02$^{\uparrow}$ | **5.47e-03**$^{\uparrow}$ |
| win/tie/loss counts | | | | 55/2/6 | 55/1/7 | 59/1/3 |

*Table C.11: IGD values for Tchebycheff and PBI (PEV variant) on WFG test problems. The median of 25 independent runs is reported.*

| | | UR-MEAD2 | | |
|---|---|---|---|---|
| Func. | # Obj | PBI ($\theta$=5) | PBI ($\theta$=0) | TE |
| WFG1 | 4 | 2.73e-02 | 2.72e-02 | **2.70e-02** |
| | 5 | 6.44e-03 | 6.42e-03 | **6.43e-03** |
| | 6 | 1.50e-03 | **1.49e-03** | 1.50e-03 |
| | 7 | 2.79e-03 | **2.78e-03** | 2.80e-03 |
| | 8 | 5.31e-03 | 5.30e-03 | **5.25e-03** |
| | 9 | 2.28e-03 | **2.27e-03** | 2.29e-03 |
| | 10 | 2.17e-03 | 2.16e-03 | **2.10e-03** |
| WFG2 | 4 | 1.39e-01 | 1.37e-01 | **1.31e-01** |
| | 5 | 1.59e-02 | 1.56e-02 | **1.39e-02** |
| | 6 | 2.28e-02 | 2.27e-02 | **2.23e-02** |
| | 7 | 8.43e-03 | 8.42e-03 | **8.36e-03** |
| | 8 | 3.11e-03 | 3.05e-03 | **3.03e-03** |
| | 9 | 8.02e-03 | 8.09e-03 | **8.01e-03** |
| | 10 | 8.10e-03 | **8.09e-03** | **8.09e-03** |
| WFG3 | 4 | 1.19e-01 | 1.18e-01 | **1.11e-01** |
| | 5 | 6.20e-02 | 6.19e-02 | **6.13e-02** |
| | 6 | 3.45e-02 | 3.43e-02 | **3.39e-02** |
| | 7 | 1.08e-01 | 1.07e-01 | **1.04e-01** |
| | 8 | 9.11e-02 | 8.27e-02 | **8.12e-02** |
| | 9 | 1.94e-01 | 1.93e-01 | **1.90e-01** |
| | 10 | 1.66e-01 | 1.65e-01 | **1.62e-01** |
| WFG4 | 4 | 3.11e-02 | **3.10e-02** | 3.25e-02 |
| | 5 | 9.41e-03 | **9.40e-03** | **9.40e-03** |
| | 6 | 1.47e-02 | 1.45e-02 | **1.41e-02** |
| | 7 | 6.62e-03 | 6.64e-03 | **6.53e-03** |
| | 8 | 3.02e-03 | 3.01e-03 | **2.95e-03** |
| | 9 | 5.85e-03 | **5.83e-03** | 5.90e-03 |
| | 10 | 3.30e-03 | **3.29e-03** | 3.31e-02 |
| WFG5 | 4 | 4.81e-02 | 5.91e-02 | **4.49e-02** |
| | 5 | 2.09e-02 | **1.95e-03** | 2.01e-02 |
| | 6 | 4.36e-02 | 4.34e-03 | **4.28e-02** |
| | 7 | 2.35e-02 | **2.11e-02** | 2.35e-02 |
| | 8 | 1.25e-02 | **1.24e-02** | 1.29e-02 |
| | 9 | 2.74e-02 | 2.74e-02 | **2.70e-02** |
| | 10 | 1.72e-02 | 1.71e-02 | **1.66e-02** |
| WFG6 | 4 | 1.27e-01 | 1.23e-01 | **9.88e-02** |
| | 5 | **3.03e-02** | 3.06e-02 | 3.68e-03 |
| | 6 | 6.39e-02 | **6.37e-02** | 6.86e-03 |
| | 7 | 3.28e-02 | 3.29e-02 | 3.44e-02 |
| | 8 | 1.90e-02 | 1.87e-02 | **1.55e-02** |
| | 9 | 3.88e-02 | 3.87e-02 | **3.08e-02** |
| | 10 | 2.14e-02 | **2.13e-02** | **2.13e-02** |
| WFG7 | 4 | 4.75e-02 | 4.71e-02 | **4.10e-02** |
| | 5 | 1.53e-02 | 1.55e-02 | **1.48e-02** |

| | | | | |
|---|---|---|---|---|
| | 6 | 2.71e-02 | 2.70e-02 | **2.60e-02** |
| | 7 | 1.32e-02 | 1.31e-02 | **1.30e-02** |
| | 8 | 6.61e-03 | 6.60e-03 | **6.29e-03** |
| | 9 | 1.41e-02 | 1.42e-02 | **1.39e-02** |
| | 10 | 8.68e-03 | 8.66e-03 | **8.60e-03** |
| WFG8 | 4 | 1.07e-01 | 1.09e-01 | **9.52e-02** |
| | 5 | 3.41e-02 | 3.40e-02 | **3.39e-02** |
| | 6 | 7.33e-02 | 7.30e-02 | **6.32e-02** |
| | 7 | 3.57e-02 | 3.56e-02 | **3.25e-02** |
| | 8 | 1.60e-02 | **1.59e-02** | 1.69e-02 |
| | 9 | 3.40e-02 | **3.39e-02** | 3.60e-02 |
| | 10 | 2.08e-02 | **2.07e-02** | 2.08e-02 |
| WFG9 | 4 | **3.00e-02** | **3.00e-02** | 3.06e-02 |
| | 5 | 8.73e-03 | **8.70e-03** | 9.13e-03 |
| | 6 | 1.36e-02 | **1.27e-02** | 1.50e-02 |
| | 7 | 5.94e-03 | **5.15e-03** | 7.17e-03 |
| | 8 | 1.66e-03 | **1.63e-03** | 3.11e-03 |
| | 9 | 5.04e-03 | **5.01e-03** | 1.40e-02 |
| | 10 | 2.70e-03 | **2.12e-03** | 5.47e-03 |
| Total highlights | | 2 | 23 | 41 |

*Table C.12: IGD values for Tchebycheff and PBI (PEV variant) on DTLZ test problems. The median of 25 independent runs is reported.*

| | | UR-MEAD2 | | |
|---|---|---|---|---|
| Func. | # Obj | PBI ($\theta$=5) | PBI ($\theta$=0) | TE |
| DTLZ1 | 4 | 1.66e-02 | 2.07e-02 | **1.56e-02** |
| | 5 | 1.36e-02 | 1.45e-02 | **1.06e-02** |
| | 6 | **1.09e-03** | 1.48e-03 | 1.26e-03 |
| | 7 | 5.23e-03 | **4.06e-03** | **4.06e-03** |
| | 8 | **1.00e-02** | 1.08e-02 | 1.03e-03 |
| | 9 | **3.30e-03** | 6.35e-03 | 5.44e-03 |
| | 10 | 1.23e-02 | 1.16e-02 | **1.06e-2** |
| DTLZ2 | 4 | 5.48e-03 | **4.27e-03** | 7.38e-03 |
| | 5 | 3.46e-03 | **1.85e-03** | 3.85e-03 |
| | 6 | 3.23e-04 | **1.68e-04** | 3.23e-04 |
| | 7 | 1.71e-03 | **8.58e-04** | 1.18e-03 |
| | 8 | 2.22e-03 | **1.08e-03** | 1.13e-03 |
| | 9 | 4.54e-04 | **1.00e-04** | 1.03e-03 |
| | 10 | 1.40e-03 | **5.97e-04** | 6.37e-04 |
| DTLZ3 | 4 | 9.41e-03 | **3.38e-03** | 5.96e-03 |
| | 5 | **2.47e-03** | 3.54e-03 | 2.89e-03 |
| | 6 | 1.30e-03 | 1.13e-03 | **3.03e-04** |
| | 7 | 4.24e-03 | **1.07e-03** | 5.20e-03 |
| | 8 | **1.09e-03** | 1.27e-03 | 3.45e-03 |
| | 9 | **3.13e-04** | 6.69e-04 | 1.06e-03 |
| | 10 | **1.05e-03** | 1.23e-03 | 1.33e-03 |
| DTLZ4 | 4 | 6.50e-03 | 6.50e-03 | **8.92e-03** |
| | 5 | **3.79e-03** | **3.79e-03** | 6.98e-03 |
| | 6 | 3.40e-04 | 3.55e-04 | **3.08e-04** |
| | 7 | 1.21e-03 | **1.05e-03** | 1.22e-03 |
| | 8 | 1.56e-03 | 1.07e-03 | **1.03e-03** |
| | 9 | 1.13e-03 | **1.06e-03** | 1.10e-03 |
| | 10 | 1.25e-03 | **1.10e-03** | 2.02e-03 |
| DTLZ5 | 4 | 1.53e-02 | 1.50e-2 | **1.45e-02** |
| | 5 | 4.94e-03 | 4.90e-03 | **4.78e-03** |
| | 6 | 1.57e-03 | 1.59e-03 | **1.52e-03** |
| | 7 | 5.09e-04 | 5.24e-04 | **5.05e-04** |
| | 8 | 9.28e-04 | 9.44e-04 | **7.14e-04** |
| | 9 | **7.03e-04** | 7.22e-04 | 7.45e-04 |
| | 10 | 6.60e-04 | 7.03e-04 | **6.53e-04** |
| DTLZ6 | 4 | 1.53e-02 | 1.48e-02 | **1.37e-02** |
| | 5 | 5.21e-03 | 4.96e-03 | **4.73e-03** |
| | 6 | 1.80e-03 | 1.85e-03 | **1.08e-03** |
| | 7 | **5.29e-04** | 5.85e-04 | 6.09e-04 |
| | 8 | 9.54e-04 | 9.39e-04 | **9.02e-04** |
| | 9 | 8.30e-04 | 8.02e-04 | **7.98e-04** |
| | 10 | 6.27e-04 | 6.12e-04 | **5.60e-04** |
| Total highlights | | 10 | 14 | 20 |

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