

# Interaction of Ocean Waves with Wave Generated by Surfing Ship

- D. Anderson (George Mason University)
- R. Buckmire (Occidental College)
- P. Cook (University of Delaware)
- E. Cumberbatch (Claremont Graduate University)
- A. Fitt (University of Southampton)
- P. Howell (University of Nottingham)
- P. Kramer (Rensselaer Polytechnic Institute)
- H. Ockendon (Oxford Centre for Industrial and Applied Mathematics)
- E. Phifer (University of Delaware)
- R. Platte (University of Delaware)
- C. Please (University of Southampton)
- S. Rethorst (Vehicle Research Corporation)
- L. Rossi (University of Massachusetts–Lowell)
- D. Schwendeman (Rensselaer Polytechnic Institute)
- B. Tilley (New Jersey Institute of Technology)
- E. Varley (Lehigh University)

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# 1 Introduction

There is significant interest in understanding the fluid motions within the ocean in the applications of improved navigation techniques, climate modelling and ecological responses to pollutants, and their eventual removal. A fundamental question in ocean motions is how small scale gravity-capillary waves interact with larger scale fluid structures, such as tsunamis, wave motions generated by ships and the like. These interactions are important in ship design, since a significant contribution to the drag is found at the bow of the ship. Called a splash, the fluid appears to run up the bow and overturn onto the fluid in front of the ship.

In using potential flow to model ocean flows, the splash is a singularity in the flow analogous to that found in airfoils at the leading edge. Cumberbatch [10] considered a two-dimensional linear analysis of the fluid motion local to a parabolic interface in a frame moving with a ship, and found the forced wave generated by a pressure distribution under its hull. The splash singularity was removed by the choice of the hull shape that satisfies the Kutta condition. This was implemented in the limit of large Froude number  $F_s = g\ell/U^2$ , where  $g$  is the gravitational constant,  $\ell$  is the half-length of the ship, and  $U$  is the ship's velocity. In Figure 1, we show Cumberbatch's interfacial profile. Note that the interfacial shape is monotonically increasing toward the bow of the ship, and the long-wavelength gravity wave that emanates in the wake. This wake differs significantly from the work of Vandenberg and Keller [32], in which the motion of a flat surfboard propelled by a solitary wave was considered numerically in a finite-depth fluid layer. Their work focused on the analytic solutions to the potential flow system.

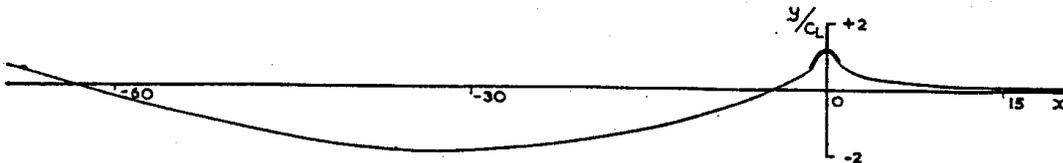


Figure 1: Wave generated by surfing ship, which is moving from the left to the right with  $F_s = \sqrt{20}$ . Reproduced from Figure 3 of [10].

It is of technological interest to take advantage of the work [10] in order to reduce the drag of conventional ship designs. If the theoretical solution can be achieved in practice, this novel-hull ship could run at speeds approximating 200 knots (350 km/h) by “surfing” using a dynamic lift mechanism with the same drag forces that are seen in conventional ships moving at 20 knots (35 km/h) [7]. One point of concern is that the theory for the surfing ship has been designed under the assumption that the surface of the ocean is smooth, whereas in reality it is choppy due to the ubiquitous presence of waves (mostly wind-driven). These waves may

well induce significant sources of drag on the ship [26, Sec. 6] . It was hypothesized, however, in [7] that the ocean waves would be damped out before they encountered the ship. Several physical mechanisms were proposed by which this might happen:

- At high speeds (100 – 200 knots), the surfing ship will ride on a large wave of height 5 – 15 m, and the ocean waves, which have much smaller height (1 – 2 m) will not have the energy to climb up the ocean wave to encounter the surfing ship.
- The fluid motion created by the surfing ship will generate an underwater pressure profile which damps out the waves in front of the ship.
- The ocean waves will break as they encounter the large wave on which the surfing ship rides.

The primary focus of this report will be to explore whether the ocean waves can indeed be expected to damp out before they encounter the surfing ship through the use of a mathematical formulation which incorporates all of the above possible mechanisms.

## 1.1 Outline of Report

We discuss the methods we use in Section 2, and summarize the results and conclusions in Section 3. These first three sections provide a brief overview of the work in this report.

The remainder of the report presents details of the calculations. The setup of the mathematical problem, including a more extensive discussion of governing equations, assumptions, and the data, are presented in Section 4. Our first calculation, based on a linearization of the ocean waves about a base state given by the ship wave, is described in Section 5. A more extensive nonlinear calculation is next explained in Section 6 and then executed in Section 7. In Appendix A, we describe and motivate the conformal mapping formalism which underlies our nonlinear asymptotic calculation.

## 2 Summary of Method of Analysis

We adopt a precise mathematical framework for describing the dynamics of waves on an ocean surface, making the following main simplifications:

- The problem is reduced to two dimensions: one along the direction of ship motion and one in the vertical direction.
- Viscosity is neglected.
- The generation of vorticity at the surface is neglected.
- The dynamics of both the ship and ocean waves is dominated by gravity (so they can be called “gravity waves.”) Surface tension, etc. is neglected. Of course, the shape of the ship wave is largely determined by the need to balance the pressure imposed on it by the ship.

These are all fairly reasonable approximations to make to get a good first look at the situation, and will be discussed further in Subsection 4.1. The main mathematical question is how the amplitude of a packet or train of ocean waves will evolve as they encounter the massive “ship wave” which the surfing ship generates.

The equations which govern the evolution of the waves are a coupled system of partial differential equations with a free boundary. We do not know how to obtain exact solutions to such a complicated system, and even numerical simulations would be quite expensive. We can however exploit the presence of the following small quantities to pursue a systematic *asymptotic analysis*:

- The aspect ratio of the ship wave, which we express as  $F^{-2}$  where  $F$  is a large quantity called the *Froude number*, which is different from that referred to in Section 1.
- The ratio  $\delta$  of the wavelength of the ocean waves to the length scale of the ship wave.
- The ratio  $\varepsilon$  of the typical height of the ocean waves to that of the ship wave.

More details on the sizes of these parameters based on the data are presented in Subsections 4.4 and 4.5.

Under the assumption that the above three nondimensional parameters are small, one can employ various asymptotic approaches to obtain an approximate answer. A variety of such calculations appear in the literature over the last four decades, but strangely, we have not found any which we could apply with confidence to the problem at hand with the given data for the ocean and ship waves. The issue is that the behavior of the solution to a mathematical problem with several small

parameters (such as  $\delta$ ,  $\varepsilon$ , and  $F^{-1}$ ), depends quite sensitively on the ordering of the small parameters. Any proper asymptotic calculation must assume some ordering relation between the small parameters in order to arrive at a definite answer. It just so happens that all the published research we have found is applicable only to orderings of  $\delta$ ,  $\varepsilon$ , and  $F^{-1}$  which are not relevant to our problem with our given parameter values. Consequently, we had to devise our own techniques.

We will first mention in Subsection 2.1 some of the methods for calculating wave interactions which have appeared in the literature and explain why we could not or did not apply them. Then we discuss in Subsection 2.2 the approaches we took.

## 2.1 Inapplicability of Previous Methods

A direct multiple-scale asymptotic calculation for the change in the wavelength and height of a short gravity wave as it rides over a much longer gravity wave was presented in [25]. This is exactly the kind of problem we are facing, but unfortunately, their analysis uses a Taylor expansion of the surface elevation which is only valid when the wavelength of both the short and long waves are much larger than the height of the long wave. In nondimensional parameters, this translates to  $F^{-2} \ll \delta \ll 1$ , which is not satisfied by the data for the surfing ship.

A weakly nonlinear theory [8, Ch. 3] is certainly appropriate to our problem, since  $F^{-2}$  is small, and much work (reviewed in [12]) has been done to show that weakly nonlinear gravity waves can be described by a “nonlinear Schrödinger (NLS) equation” (or some variation) which is a great simplification over the exact surface wave equations (Subsection 4.2). The derivation by Dysthe [13], however also Taylor expands the boundary condition, which we already said is not appropriate for our problem. The derivations of Hasimoto and Ono [16] and Davey and Stewartson [11] do not make this assumption, but instead assume that the depth of the water is much smaller than the horizontal scale of the wave packets. In our present case, this is not satisfied because the length scale of the ship wave (up to 2 km ) is smaller or comparable than the typical ocean depth (several km).

Another attempt suggested by the wide disparity of the length scale between the ocean waves and the ship wave ( $\delta \ll 1$ ) is the averaged Lagrangian theory initiated by Whitham [9, 17, 33]. While this theory is quite elegant and been used successfully in various applications similar to our problem, it has some difficulties which discouraged us from attempting it ([8, Sec. 11.2],[11]). First of all, it works best for producing a prediction for changes to wave amplitudes which are comparable to their original size (“order unity”). It is much more difficult to implement when one wants to compute higher order approximations ([3],[17, Sec. 8.2]), as will be necessary in our problem because we will see there is no “order unity” change to the ocean waves as they climb the ship wave. Moreover, unlike the direct multiple-scales approaches described in the previous paragraphs, it is difficult to obtain an

estimate of the error made in the approximation, especially when there are several small parameters [6] (but see [3, 33]). It appears that the theory is best designed for problems in which  $\delta$  is by far the smallest parameter, but this does not hold for our present problem. Finally, the method involves a number of subtle points which are difficult to anticipate when applied to the evolution of waves in a fluid medium which is itself moving [4] (as is the case when the ocean waves encounter the ship wave). In particular, we found three apparently different suggestions for how the averaged Lagrangian equations should appear in a moving medium [4, 9, 33]. We did not take the time to reconcile these various equations or sort out which was right because of other limitations previously mentioned. One might think to simply read off the result cited in [9] for how gravity waves interact with a current, but this analysis does not take into account an important “vertical acceleration” term which is needed to emend the gravity wave-current interaction result to apply to short gravity wave–long gravity wave interaction problem ([9, 25],[27, Sec. 3.7]).

An older technique ([8, Ch. 11],[27, Ch. 3],[31, 34]) which describes interaction between short waves and long waves through a “radiation stress” term was awkward to apply for similar reasons. While the theory is elegant and physically meaningful, it is hard to estimate errors and conduct higher order approximations, and it appears that  $\delta$  should be the smallest parameter in the problem. Fortunately, though, a published prediction for the interaction between a long and short gravity wave based on the radiation stress theory is available [25], so we simply report it in Section 3 even though we can’t vouch for its applicability to our problem.

One last interesting technique of which we are aware for studying weakly nonlinear gravity wave interactions is the use of a hodograph transformation which maps the free-surface fluid domain to a fixed half-plane (parameterized by the velocity potential and stream function of the flow) [26, 28]. This removes the complication of the free surface at the cost of some extra complications in the governing equations. This technique has been used, to our knowledge, only to describe the nonlinear evolution and stability of a single wave [23, 24, 29]. It may well work for our problem with interacting waves, but we have not explored it because the calculations for a single wave already seem quite complicated. We opted instead for some more direct and transparent approaches.

## 2.2 Calculation Approaches Used in This Report

We first present in Section 5 a WKB asymptotic analysis based on a *linearization* of the problem about the basic ship wave state. This analysis effectively assumes that  $\varepsilon$  is the smallest parameter in the problem, which is not really true, but this assumption allows us to ascertain the effects of some of the physical interaction between the ship wave and the ocean waves. In particular, this analysis will include the direct effects of the flow and pressure of the ship wave on the ocean waves.

This distinguishes it from a naive fully linear analysis, which would miss the ocean wave–ship wave interactions. In particular, we handle the distortion of the ocean surface due to the ship wave in a precise manner. Rather than Taylor expanding the equations about the equilibrium ocean surface level (which we are not allowed to do since the ocean waves vary on length scales small or comparable to the height of the ship wave), we apply a “flattening” coordinate transformation to convert the curvy fluid domain described by the ship wave to the lower half plane. The effects of the ship wave distortion then appear as extra terms in the governing equations, which we handle in a precise perturbative manner.

We then present in Sections 6 and 7 a more precise nonlinear analysis, which includes not only the direct coupling between the ocean waves and the ship wave, but “self-nonlinear” interaction of the ocean waves. Our reasons for pursuing this analysis were twofold:

- A scaling estimate based on the data indicates that the “self-nonlinear” effects can modify the results computed by the linearized analysis.
- The consideration of “self-nonlinear” effects will allow us to study whether the ocean waves will undergo a significant change in shape which would cause them to break as they ride up the ship wave.

As discussed in Appendix A, straightforward adaptations of traditional nonlinear multiple-scales analysis runs into difficulties when applied to our problem. We therefore devised a (to our knowledge) novel approach which is a simpler variation of the hodograph transformation approach used in [23, 24, 29]. Briefly, we define a time-dependent conformal transformation which maps the fluid domain, including both ship wave and ocean wave distortions, to a simple half-plane. Unlike the hodograph transformation, our conformal transformation is a near-identity mapping. Consequently, we can express it perturbatively. Unlike the simpler flattening transformation used in Section 5, the conformal transformation does not introduce extra terms in the governing equations (though one must be very careful to properly interpret partial derivatives evaluated at the surface). Our technique is otherwise a standard multiple scales, weakly nonlinear perturbation analysis. We order the small parameters  $\varepsilon$  and  $\delta$  and  $F^{-1}$  as suggested by the data (which is different for the surfing ship at 100 knot and 200 knot speeds).

### 3 Summary of Results

Nowhere in any of our mathematical calculations, nor from our research into the literature of related interacting wave problems, have we found any evidence that the ocean waves will be damped out before they encounter the ship hull. A linearized calculation (Section 5) shows that the ocean waves for which there was most hope for significant damping by the ship wave ( $\delta \sim F^{-4}$ ) in fact had their amplitude only change by  $O(F^{-2})$  during their ascent of the ship wave. More precisely, the ratio of the amplitude of the ocean waves as they climb the ship wave to their original amplitude is (56):

$$r(x) = \exp \left[ \frac{1}{4} F^{-2} (3\eta^{(s)}(x) - \eta_{xx}^{(s)}(x)) + O(F^{-4}) \right], \quad (1)$$

where  $\eta^{(s)}(x)$  is the shape of the ship wave (in a comoving frame of reference), and  $\eta_{xx}^{(s)}(x)$  is its second derivative. More precisely,  $\eta^{(s)}(x)$  is the elevation of the ocean surface induced by the surfing ship in the absence of any other ocean wave disturbances. It suffices to use the linearized solution computed in [10]. It appears from inspection of the graph of this solution (Figure 1) that the ocean waves will actually amplify by an  $O(F^{-2})$  amount as they climb the ship wave, but a more detailed calculation could be undertaken.

The linearized calculation discussed above accounts for the nonlinear interaction between the ocean wave and ship wave, but not “self-nonlinear” effects of the ocean waves as they interact with the ship wave. Upon inclusion of all the nonlinearities, which includes any possible mechanism for wave breaking, we found no effect on the ocean wave amplitude through  $O(F^{-1})$  in the asymptotic analysis for either the 100 knot or 200 knot ship. Based on the linearized analysis discussed above, we do expect the ocean wave amplitude to change at  $O(F^{-2})$ , but the formula (1) may need to be modified due to self-nonlinear effects of the ocean waves. To compute these corrections would require a great deal more labor than the calculations conducted in this report.

Though the previously published results for the interaction between short waves and long waves rely on calculations which do not appear to be valid for the parameter values relevant to the surfing ship, we mention that they all predict that the shorter waves will *increase* in amplitude (and contract in wavelength) as they climb up the longer wave (by an amount proportional to  $F^{-2}$ ) ([25],[26, Sec.2.F],[27, Ch. 3]). These analyses range from multiple scales analyses using a Taylor expansion of the surface distortion ([25],[26, Sec. 2.F]) to the radiation stress theory ([25],[27, Ch. 3]). None of these analyses include self-nonlinear interactions of the ocean waves, which could be relevant and modify the predictions for our problem.

The main conclusion is that the ocean wave amplitudes should not change by more than few percent as they encounter a 200 knot ship or a few tenths as they

encounter a 100 knot ship. Quantifying “a few” and determining whether the ocean amplitudes grow or decay as they ascend the ship wave would require more extensive calculation. Based on the results of the direct interaction calculation of Section 5 and other calculations which are not truly valid for the parameter values of interest ([25],[26, Sec. 2.F],[27, Sec. 3.7]), it seems likely that the ocean waves will actually amplify a bit as they climb the ship wave, unless their self-nonlinear interaction somehow opposes this tendency.

We finally remark on the physical mechanisms which have been proposed for why the ocean waves should be damped or smoothed out as they ascend the ship wave:

- The idea that the waves do not have the gravitational energy to ascend the ship wave is physically flawed. Waves can climb an arbitrary fluid surface without requiring significant energy. The correct way to compute the gravitational energy of a wave packet is to consider the *distortion* of the fluid surface from its wave-free state [27, Sec. 3.6]. This distortion could be the same whether the waves are on the crest of a big ship wave or simply on the flat ocean surface.
- We find no evidence for the hypothesis that the pressure induced by the ship will smooth out the ocean waves. Indeed, we have been able to compute the leading order effect of the fluid flow and pressure induced by the ship on the ocean waves, neglecting wave-breaking phenomena, and arrived at the formula (1). This formula indicates that the ocean waves will grow for at least part, if not all, of their ride up the ship wave.
- We find no evidence that the ocean waves will tend to break significantly as they climb the ship wave. We conducted a nonlinear analysis in Section 7 which indicated that the ship wave did not cause any steepening of the shape of the ocean wave to the order  $O(F^{-1})$  considered. By considering the magnitude of higher order terms neglected in our analysis, we estimate that the ocean waves could change shape by at most a few percent as they encounter the wave generated by a 200 knot surfing ship, and at most by a few tenths as they encounter the wave generated by a 100 knot surfing ship. To determine whether the change of shape is actually this large and conducive to breaking would require explicit calculation beyond that performed in the report.

### 3.1 Possible Directions for Future Work

To obtain further insight into how the ocean waves will behave as they encounter the ship wave, one could carry out the nonlinear analysis to one higher order. We stress that an accurate result to this order requires special caution as noted in

Appendix A.3. While the steps to execute this calculation are described in Sections 6 and 7 and Appendix A, the work would be considerably more tedious than that reported here. It is probably not worth the effort, given the pessimistic prognosis based on the results obtained so far.

Direct numerical simulations of the governing equations could be contemplated, but they would be quite expensive because of the large range of scales involved between the ship and the ocean waves.

One different avenue of possible exploration is three-dimensional effects. Sometimes waves can be unstable in the direction transverse to their motion [12, Sec. 5], and such effects are not included in the two-dimensional analysis conducted here.

## 4 Mathematical Problem Formulation

We now set up the mathematical formulation of the problem concerning the interaction of the ocean waves with the wave generated by the surfing ship. We lay down the fundamental modeling assumptions in Subsection 4.1 and explain why we make them. The governing equations are presented in Subsection 4.2 and then nondimensionalized in Subsection 4.3. In the nondimensionalized formulation, there are three important nondimensional groups of parameters which influence the physical interaction between the waves. We report on the values of these parameters for both 100 knot and 200 knot speed operating conditions of the ship in Subsection 4.4, and use these to motivate asymptotic analyses with certain distinguished limits.

### 4.1 Modeling Assumptions

The dynamics of both the ship wave and the ocean waves of interest can be assumed to be entirely governed by gravity and the conservation and kinematic laws of hydrodynamics. In particular, surface tension effects can be neglected because all waves of interest have wavelengths far in excess of a centimeter [1, p. 73]. Similarly, because the waves of interest are no more than about a kilometer long, we can neglect the influence of the Coriolis force [15, Sec. 7.5].

We are interested in the development of the ocean waves as they meet the ship wave in the open waters, so we assume there are no boundaries confining the fluid and model the ocean as having infinite depth. This latter assumption is justified provided the wavelengths of the waves are much less than the depth of the ocean, which is certainly true for the ordinary ocean waves but marginally true for the ship wave [15, Sec. 5.5]. The effects of finite ocean depth could be incorporated [1, Sec. 3.5], but it does not seem that finiteness of the ocean depth should bear significantly on the central question we are investigating. As we are focusing on surface phenomena, we neglect the density stratification of the ocean, assuming it to be instead given by a constant density (equal to the density near the surface). This assumption filters out “baroclinic modes” involving internal waves, which have only a very small influence on the surface motion [15, Sec. 6.2].

Because the Reynolds number is at least of order  $10^3$ , and no large rigid boundaries need to be modelled [1, Sec. 2.2], we can neglect viscosity. The viscous boundary layer at the free surface is only about a centimeter thick ([8, Sec. 10.1],[27, Sec. 3.4]). Of course the bottom of the ship presents a rigid boundary for which viscosity should be considered in computing the drag [26, Sec. 6], but since we are only interested in how the ocean waves evolve *before* they meet the ship, we can neglect this rigid boundary in our analysis. We naturally do assume that the boat generates a pressure on the ocean which is responsible for driving the ship wave underneath the surfing ship.

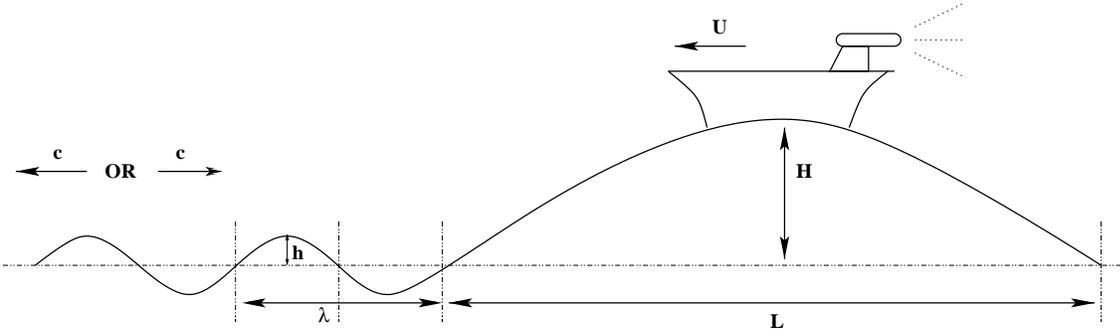


Figure 2: Ship generates a wave with length scale  $L$  and height  $H$  moving with speed  $U$  to the left. The packet of ocean waves consists of waves with wavelength  $\lambda$  and height  $h$  and phase speed  $\pm c$ .

The only apparent sources of vorticity generation are wind stress and the ship hull, and we will neglect these in our model. Even if substantial vorticity were generated by the surfing ship, it would appear downstream and have no bearing on the ocean waves as they climb up the ship wave. Wind stress can indeed induce significant vorticity on the ocean surface ([2],[15, Ch. 9]), but such vorticity is not essential to the existence and sustenance of the waves, and we therefore leave vorticity out of our model. (Otherwise we would have to include a model for the wind stress, which is not generic.) The irrotational assumption is moreover quantitatively legitimate when the wave steepness (height divided by wavelength) is small [20], and we see from the data in Table 1 that both the ship wave and incoming waves have moderately small steepness. See ([25, Sec. 2],[27, Sec. 3.1]) for further discussion concerning the validity of the irrotational assumption.

## 4.2 Governing Equations in Dimensional Form

Based on the discussion above in Subsection 4.1, we shall base our analysis on the equations of motion for an incompressible, inviscid fluid undergoing potential flow (since the effects of vorticity are negligible). We envision the ship moving *from right to left* at a steady operating speed  $-U$ , and then formulate our coordinates in a frame of reference moving with the ship, so that the ship wave appears stationary (Figure 2). (Note that this is the opposite direction of that used in [10] and displayed in Figure 1.) To keep the analysis as focused as possible on the main question of how the ship wave and ocean waves interact, we take a two-dimensional model where  $x^\bullet$  and  $y^\bullet$  are the horizontal and vertical spatial coordinates, and  $t^\bullet$  denotes time. We adorn these variables with solid circles to denote they are dimensional; we will define nondimensionalized coordinates in Subsection 4.3.

We choose  $y^\bullet = 0$  to denote the undisturbed level of the ocean surface, and  $x^\bullet = 0$  to line up with the center of the ship. As we mentioned above, we take a deep ocean limit so that fluid extends all the way down to  $y^\bullet \rightarrow -\infty$ . We neglect the effects of variations in the second horizontal direction (transverse to the ship motion). Of course the ship wave does have significant variation across the ship, and three-dimensional effects can be expected to be relevant and change the quantitative details of the mathematical predictions. Nonetheless, most of the essential physics of the wave interactions should be captured in the two-dimensional model. In particular, the physical hypotheses concerning mechanisms proposed in the introduction by which ocean waves could be attenuated as they climb the ship wave do not seem to rely on three-dimensional effects. Their validity can be explored well with a two dimensional model.

The functions which completely describe the waves are

- $\eta^\bullet(x^\bullet, t^\bullet)$ , which is the elevation of the surface above its normal resting height.
- $\phi^\bullet(x^\bullet, y^\bullet, t^\bullet)$ , which is the potential function for the fluid velocity relative to the constant stream speed  $U$ . Specifically, the velocity of the fluid in the frame comoving with the ship is

$$\mathbf{v}^\bullet(x^\bullet, y^\bullet, t^\bullet) = \begin{bmatrix} U + \phi_{x^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet) \\ \phi_{y^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet) \end{bmatrix}.$$

Partial derivatives are here, as throughout the report, usually denoted by subscripts on functions.

The functions describing the waves are determined as the solutions to the following coupled set of partial differential equations with a free boundary [1, Sec. 3.2]:

#### 4.2.1 Laplace's Equation

$$\nabla^2 \phi^\bullet(x^\bullet, y^\bullet, t^\bullet) = 0 \text{ for } y^\bullet < \eta^\bullet(x^\bullet, t^\bullet) \quad (2)$$

This equation just expresses the incompressibility of the flow.

#### 4.2.2 Bernoulli Equation

$$\begin{aligned} \phi_{t^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet) + \frac{1}{2} [(U + \phi_{x^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet))^2 + (\phi_{y^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet))^2] + g\eta^\bullet(x^\bullet, t^\bullet) + \frac{\Pi^\bullet(x^\bullet)}{\rho} \\ = \frac{1}{2}U^2 \text{ for } y^\bullet = \eta^\bullet(x^\bullet, t^\bullet) \end{aligned} \quad (3)$$

where  $\Pi^\bullet(x^\bullet)$  is the excess pressure (above atmospheric pressure) which the ship induces on the ocean surface. It is nonzero only underneath the ship, and is responsible for the generation of the ship wave. The Bernoulli equation is an integrated form of the Euler equations for fluid motion, made possible due to the assumption that the flow can be described by a potential  $\phi^\bullet$ .

### 4.2.3 Kinematic Condition on Surface

$$\phi_{y^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet) = \eta_{t^\bullet}^\bullet(x^\bullet, t^\bullet) + (U + \phi_{x^\bullet}^\bullet(x^\bullet, y^\bullet, t^\bullet))\eta_{x^\bullet}^\bullet(x^\bullet, t^\bullet) \text{ for } y^\bullet = \eta^\bullet(x^\bullet, t^\bullet) \quad (4)$$

This equation just expresses a self-consistency condition: the surface elevation evolves at a rate given by the velocity of the fluid at the surface.

### 4.2.4 Initial and Boundary Conditions

Clearly surface disturbances should decay deep in the ocean, so we impose

$$\phi^\bullet(x^\bullet, y^\bullet, t^\bullet) \rightarrow 0 \text{ as } y^\bullet \searrow -\infty.$$

This also fixes an arbitrary constant in the definition of the velocity potential.

The ship wave and ocean waves will arise due to initial and/or boundary conditions at the ocean surface.

**Ship Wave** Let  $\eta^{(s)\bullet}(x^\bullet)$  denote the steady surface distortion and  $\phi^{(s)\bullet}(x^\bullet, y^\bullet)$  denote the steady velocity potential induced by the ship in its comoving frame of reference, in the absence of any ocean waves. This steady wave is driven in a self-consistent manner by the pressure  $\Pi^\bullet(x)$  imposed at the surface by the ship, which we take as given. The complete system of equations determining the functions  $\eta^{(s)\bullet}$  and  $\phi^{(s)\bullet}$  are:

$$\begin{aligned} 0 &= \nabla^2 \phi^{(s)\bullet}(x^\bullet, y^\bullet) \text{ for } y^\bullet < \eta^{(s)\bullet}(x^\bullet), \\ 0 &= U \phi_{x^\bullet}^{(s)\bullet}(x^\bullet, y^\bullet) + \frac{1}{2} \left[ (\phi_{x^\bullet}^{(s)\bullet}(x^\bullet, y^\bullet))^2 + (\phi_{y^\bullet}^{(s)\bullet}(x^\bullet, y^\bullet))^2 \right] + g \eta^{(s)\bullet}(x^\bullet) \\ &\quad + \frac{\Pi^\bullet(x^\bullet)}{\rho} \text{ for } y^\bullet = \eta^{(s)\bullet}(x^\bullet), \\ 0 &= (U + \phi_{x^\bullet}^{(s)\bullet}(x^\bullet, y^\bullet)) \eta_{x^\bullet}^{(s)\bullet}(x^\bullet) - \phi_{y^\bullet}^{(s)\bullet}(x^\bullet, y^\bullet) \text{ for } y^\bullet = \eta^{(s)\bullet}(x^\bullet), \end{aligned}$$

$$\phi^{(s)\bullet}(x^\bullet, y^\bullet) \rightarrow 0 \text{ as } y^\bullet \searrow -\infty.$$

We will characterize the ship wave by its speed  $U$  and height  $H$ , for which we have clean data (Subsection 4.4). Using the general relation between the length scale and

speed of gravity waves, we associate a horizontal length scale

$$L = \frac{U^2}{g} \quad (5)$$

to the ship wave, where  $g$  is the gravitational constant

$$g = 9.8\text{m/s}^2.$$

For mathematical purposes, we will never need to actually use the precise value of the length of the ship wave; rather it is the combination  $U^2/g$  which is crucial. Consequently, we don't bother with worrying about what sort of numerical prefactor to introduce so that  $L$  really describes the length of the ship wave in a precise (rather than order-of-magnitude) sense.

**Ocean Waves** We also need to supply an initial or additional boundary condition to complete the determination of the ocean waves. We consider two possibilities, one of which we will use for the linearized analysis and one for the nonlinear analysis

- **Wave Train Conditions (Linearized Analysis):** We imagine a train of ocean waves with constant wavelength  $\lambda$  extending infinitely far in front of the ship, which have constant amplitude  $h$  until they encounter the ship wave:

$$\lim_{x^\bullet \rightarrow -\infty} \eta^\bullet(x^\bullet, t^\bullet) \sim h \exp(ik^\bullet(x^\bullet - (U \pm c)t^\bullet)). \quad (6)$$

We are assuming that the incoming waves all have the same wavenumber

$$k^\bullet = \frac{2\pi}{\lambda} \quad (7)$$

and therefore phase speed [21]:

$$c = \sqrt{g/k^\bullet} = \frac{g\lambda}{2\pi}.$$

Note that the phase speed of the ocean waves is  $U \pm c$  in a frame comoving with the ship. We consider ocean waves both moving toward and away from the ship because the ship will overtake either kind of wave as it moves.

The  $x^\bullet \rightarrow -\infty$  behavior of the velocity potential  $\phi^\bullet(x^\bullet, y^\bullet, t^\bullet)$  is similarly prescribed with

$$\lim_{x^\bullet \rightarrow -\infty} \phi^\bullet(x^\bullet, y^\bullet, t^\bullet) \sim \mp ich \exp(ik^\bullet(x^\bullet - (U \pm c)t^\bullet)) \exp(k^\bullet(y^\bullet - \eta^{(s)\bullet}(x^\bullet))),$$

which is appropriate to describe a train of linear Stokes waves [1, Sec. 3.2] moving either to the left or to the right (depending on the sign of  $\pm$ .)

We will only use these wave train conditions for our *linearized* analysis of the problem in Section 5, because constant amplitude wave trains are unstable due to nonlinear effects (Benjamin-Feir instability [17, Sec. 5.3.2]).

- **Wave Packet Conditions (Nonlinear Analysis):** We can also study a slightly less idealized problem in which the ocean waves are initialized at  $t^\bullet = 0$  to be in the form of a packet of Stokes gravity waves extending over a finite or infinite region. We write the initial surface elevation therefore in the form of the ship wave plus a modulated disturbance at a given wavenumber  $k^\bullet$ :

$$\eta^\bullet(x^\bullet, t^\bullet = 0) = \eta^{(s)\bullet}(x^\bullet) + [A^{(0)\bullet}(x^\bullet) \exp(ik^\bullet x^\bullet) + \text{c.c.}],$$

where c.c. denotes the complex conjugate of the other terms grouped with it. Because we will be using the wave packet conditions for our nonlinear analysis, it is important for us to keep track of the complex conjugate terms which arise from the reality of the surface elevation function  $\eta^\bullet$ ; we did not need to do this for the wave train conditions we are using with the linearized analysis. The wavenumber  $k^\bullet$  is related to the wavelength  $\lambda$  of the ocean waves as in Eq. (7), and the function  $A^{(0)\bullet}$  has a maximum height  $h$ .

The velocity potential is, roughly speaking, initialized in the same way as the surface elevation, but it is tricky to write down in a quantitative manner because the velocity potential of the ship wave is defined on the region  $y^\bullet < \eta^{(s)\bullet}(x^\bullet)$  whereas the velocity potential of the superposition of the ship and ocean waves is defined on a different domain  $y^\bullet < \eta^\bullet(x^\bullet, t^\bullet)$ . This point is relevant for a nonlinear analysis, but can be ignored for the linear analysis because the whole problem can be formulated on the fixed base domain  $y^\bullet < \eta^{(s)\bullet}(x^\bullet)$ ; see Eq. 16 in Section 5.

To simplify our nonlinear analysis, we will implicitly define the initialized velocity potential  $B^{(0)\bullet}(x^\bullet, y^\bullet) = \phi^\bullet(x^\bullet, y^\bullet, t^\bullet = 0)$  as the one which corresponds to the ship wave superposed with an ocean wave packet which moves only in one direction (either left or right relative to the fixed ocean). This is sensible if we think of the ocean wave packet as being driven by some moderately coherent wind stress. If we were to initialize  $\phi^\bullet$  in some arbitrary way, the ocean wave packet would generally immediately break up into two wave packets moving at group velocities  $\pm u = \pm \frac{1}{2} \sqrt{g/k^\bullet}$  relative to the fixed ocean or  $U \pm u$  relative to the ship. This problem could be analyzed with the same techniques developed here, but the equations would be complicated by terms which had nothing to do with the primary feature of interest here: the interaction between the ship wave and the ocean waves. There are exactly two special choices of  $\phi^\bullet(x^\bullet, y^\bullet, t^\bullet = 0)$  which will be consistent with a unidirectional packet of ocean waves encountering the ship wave (one for each direction relative to the fixed ocean), and we implicitly choose one of these two initializations.

### 4.3 Nondimensionalization of Governing Equations

We nondimensionalize the variables with respect to length and time scales characterizing the ship wave:

$$\begin{aligned}x^\bullet &= \frac{U^2}{g}x = x/L, & y^\bullet &= \frac{U^2}{g}y = y/L, \\t^\bullet &= \frac{U}{g}t.\end{aligned}$$

Since we will be pursuing an asymptotic analysis, it is important that we nondimensionalize the functions with respect to an appropriate combination of governing parameters which characterize their actual amplitude, otherwise the subsequent perturbative analysis would be nonsensical [22]. Clearly, the amplitude for the surface displacements of the ocean and ship waves are  $\eta^{(s)\bullet} \sim H$  and  $A^{(0)\bullet} \sim h$ . As the aspect ratios  $H/L$  and  $h/\lambda$  of both ship and ocean waves are small, we can estimate the magnitude of the fluid velocity fluctuations using linear theory [1, Sec. 3.2] as  $Hg/U$  and  $hg/u$  respectively. Note that the magnitude of the fluid velocity fluctuations is *not* generally on the same order of magnitude as the speed of the wave. Indeed, the fluid velocity in gravity waves with small aspect ratio is small compared to the speed of the wave. This was a point of confusion in Appendix F of [7]. Proceeding, then, the magnitude of the velocity potential scales as the product of the velocity fluctuation magnitude and the length scale ( $L \sim U^2/g$  or  $\lambda \sim u^2/g$ ), so  $\phi^{(s)\bullet} \sim UH$ , and  $B^{(0)\bullet} \sim uh$ .

Finally, the magnitude of the pressure imposed on the surface by the ship can be estimated by noticing that the only negative term in the Bernoulli equation (3) that can balance the pressure  $\Pi^\bullet$  imposed by the ship is the counterflow term  $\rho U \phi_{x^\bullet}^{(s)\bullet} \sim \rho g H$ . With these considerations, we then nondimensionalize the functions describing the waves as follows:

$$\begin{aligned}\eta^\bullet(x^\bullet, t^\bullet) &= H\eta(x, t), \\ \eta^{(s)\bullet}(x^\bullet) &= H\eta^{(s)}(x), \\ A^{(0)\bullet}(x^\bullet) &= hA^{(0)}(x), \\ \phi^\bullet(x^\bullet, y^\bullet, t^\bullet) &= UH\phi(x, y, t), \\ \phi^{(s)\bullet}(x^\bullet, y^\bullet) &= UH\phi^{(s)}(x, y), \\ B^{(0)\bullet}(x^\bullet, y^\bullet) &= uhB^{(0)}(x, y), \\ \Pi^\bullet(x^\bullet) &= \rho g H\Pi(x).\end{aligned}$$

We must take note, however, that the components of the fluid functions  $\eta^\bullet(x^\bullet, t^\bullet)$  and  $\phi^\bullet(x^\bullet, y^\bullet, t^\bullet)$  which describe the perturbations due to the ocean wave will vary on

both the ship wave length scale  $L \sim U^2/g$  and the ocean wave length scale  $\lambda$ , and will vary on both the time scale of the ship  $\mathcal{T} \sim U/g$  and the time scale of the ocean waves  $\tau \sim u/g \sim \sqrt{\lambda/g}$ . Consequently, we need to recognize in the asymptotic analysis that the nondimensionalized ocean wave functions  $\eta^\bullet$  and  $\phi^\bullet$  will have multiple-scale dependence on space and time. In terms of the nondimensionalized coordinates  $x$ ,  $y$ , and  $t$  which vary over scales of order unity relative to the ship wave, the ocean waves will depend on both order unity scales (corresponding to the change in their environment as they cross the ship wave) and faster scales (corresponding to their natural wavelength and frequency nondimensionalized with respect to the ship wave properties).

In rewriting the governing fluid equations in terms of these nondimensionalized equations and variables, we will find it useful to introduce the following nondimensional parameters:

- The ratio between the wavelength of the ocean waves and the length scale of the ship wave:

$$\delta = \lambda/L = \frac{g\lambda}{U^2}.$$

Note that the last expression involves quantities which we can define precisely from the data.

- The ratio between the height of the ocean waves and the ship wave:

$$\varepsilon = H/h.$$

- *Froude number*

$$F = \frac{U}{\sqrt{gH}}. \tag{8}$$

Note that it is related to the aspect ratio of the ship wave as follows, using Eq. (5):

$$\frac{H}{L} = F^{-2}$$

We emphasize that Eq. (8) defines a *different Froude number* than the one based on the ship length which was used in [10]. In this manuscript we shall specifically intend the definition (8) whenever we refer to a Froude number  $F$ , unless specifically mentioned otherwise.

We can now nondimensionalize the equations from Subsection 4.3 according to the above prescriptions:

$$\nabla^2 \phi(x, y, t) = 0 \text{ for } y < F^{-2}\eta(x, t), \quad (9a)$$

$$\phi_y(x, y, t) = \eta_t(x, t) + \eta_x(x, t) + F^{-2}\phi_x(x, y, t)\eta_x(x, t) \text{ for } y = F^{-2}\eta(x, t), \quad (9b)$$

$$\phi_t(x, y, t) + \phi_x(x, y, t) + \frac{F^{-2}}{2} (\phi_x^2(x, y, t) + \phi_y^2(x, y, t)) \quad (9c)$$

$$+ \eta(x, t) + \Pi(x) = 0 \text{ for } y = F^{-2}\eta(x, t),$$

$$\phi(x, y, t) \rightarrow 0 \text{ for } y \rightarrow -\infty, \quad (9d)$$

In addition, we supply one of the following auxiliary conditions to introduce ocean waves:

- **Wave Train Conditions (Linearized Analysis)**

$$\lim_{x \rightarrow -\infty} \eta(x, t) \sim \varepsilon \exp(i\delta^{-1}(x - t) \mp i\delta^{-1/2}t), \quad (9e)$$

$$\lim_{x \rightarrow -\infty} \phi(x, y, t) \sim \mp i\varepsilon\delta^{1/2} \exp(i\delta^{-1}(x - t) \mp i\delta^{-1/2}t) e^{\delta^{-1}y} \text{ for } y < F^{-2}\eta(x, t)$$

- **Wave Packet Conditions (Nonlinear Analysis)**

$$\eta(x, t = 0) = \eta^{(s)}(x) + \varepsilon \left[ A^{(0)}(x)e^{i\delta^{-1}x} + \text{c.c.} \right], \quad (9f)$$

with  $\phi(x, y, t = 0)$  chosen so that the ocean wave packet propagates in only one direction.

The nondimensionalized problem for the ship wave is:

$$0 = \nabla^2 \phi^{(s)}(x, y) \text{ for } y < F^{-2}\eta^{(s)}(x), \quad (10a)$$

$$0 = \phi_x^{(s)}(x, y) + \frac{1}{2}F^{-2} [(\phi_x^{(s)}(x, y))^2 + (\phi_y^{(s)}(x, y))^2] + \eta^{(s)}(x) \\ + \Pi(x) \text{ for } y = F^{-2}\eta^{(s)}(x), \quad (10b)$$

$$0 = \eta_x^{(s)}(x) + F^{-2}\phi_x^{(s)}(x, y)\eta_x^{(s)}(x) - \phi_y^{(s)}(x, y) \text{ for } y = \eta^{(s)}(x), \quad (10c)$$

$$\phi^{(s)}(x, y) \rightarrow 0 \text{ as } y \searrow -\infty. \quad (10d)$$

#### 4.4 Parameter Values for Waves

Based on the data presented in the ‘‘Surfing Wave Heights’’ figure in [7] for a surfing ship with a 200 foot beam and 30 foot chord, we estimate the parameters of the waves in Table 1. Some entries for the ship wave give a range of values, since the properties of the ship wave depend on the speed and loading of the ship. When such

ranges are given, the left value corresponds to the surfing ship moving at 100 knots, which is one speed value emphasized in the report [7]. The right value corresponds to the optimal cruising speed of 200 knots. Note that sometimes the left number in a range of values will be greater than the right number, since we wish to maintain the correspondence between the left values and the surfing ship moving at 100 knots, and the right values and the surfing ship moving at 200 knots. Note that the height of the ship wave depends not only on the ship speed but on the loading of the ship. We have assumed a loading of 900 tons per foot of span, which yields the reported wave heights from the data tabulated in the “Surfing Wave Heights” figure of [7]. In estimating the group velocity of the ocean wave and the wavelength of the ship wave, we have used the relationship

$$u = \frac{1}{2} \sqrt{\frac{g\lambda}{2\pi}}.$$

All data in the table is in metric units because of its greater familiarity to the university scientists, notwithstanding their origins from the United States and the United Kingdom!

Table 1: Approximate values of physical parameters. These values are only used as a guide to motivate the development of asymptotic procedures.

	Ship Wave	Ocean Waves
Wavelength	$L = 500 - 2000$ m	$\lambda = 10m,$
Group Velocity	$U = 50 - 100$ m/s	$u = 1$ m/s,
Waveheight	$H = 15 - 5$ m	$h = 1m$

## 4.5 Small Parameters and Distinguished Limits

The nondimensional parameters appearing in the governing nondimensional equations in Subsection 4.3 have the approximate values given in Table 2, computed using the data from Table 1. We note that  $F^{-1}$ ,  $\delta$ , and  $\varepsilon$  can all reasonably be taken as small parameters, and therefore a perturbative analysis is suggested. Perturbation theories with multiple small parameters, however, generally require some assumption about the relative magnitude of the small parameters.

For a 100 knot speed of the surfing ship, we see that roughly

$$\varepsilon \sim F^{-2} \quad \delta \sim F^{-3} \tag{11}$$

and for a 200 knot speed of the surfing ship,

$$\varepsilon \sim F^{-1} \quad \delta \sim F^{-2}. \tag{12}$$

Table 2: Approximate values of nondimensional parameters.

	100 knot ship speed	200 knot ship speed
F	4	15
$\delta$	0.02	0.005
$\varepsilon$	0.07	0.2

We describe now how these balances between the small parameters motivate our analysis

We explore first in Section 5 whether the interaction between the ship wave and ocean waves could possibly cause an *order unity* change in the ocean wave amplitude, by which we mean a change comparable to their original size. A scaling consideration of the equations suggests the distinguished limit  $\delta = k^{-1}F^{-4}$ , with  $k$  a fixed constant, as the one most likely to produce such a sizable change. The data suggests that  $\delta$  should be chosen larger than this for our application, but consideration of this distinguished limit actually encompasses the wider regime  $F^{-6} \ll \delta \ll F^{-2}$  by general principles from the theory of asymptotic expansions [19, Sec. 1.4]. If we were to find a significant effect on ocean waves which happened to satisfy  $\delta \sim F^{-4}$ , then we would be able to quickly evaluate the changes to the ocean waves encountering the 100 knot surfing ship (with  $\delta \sim F^{-3}$ ), and then we could conduct a separate analysis for the 200 knot surfing ship (with  $\delta \sim F^{-2}$ ). For this first analysis, it is helpful to include only the terms in the equations which are first order in  $\varepsilon$ , which means that we are assuming that  $\varepsilon$  is infinitesimally small. This is again not strictly consistent with the data, since  $\varepsilon$  is small but not very small compared to the other nondimensional parameters, but this assumption makes the calculations much simpler. Indeed, the equations then become linear in the ocean wave amplitudes. As we mentioned in Section 2, however, these equations will include nonlinear interaction between the ocean wave and the ship wave. What is neglected are some terms which are nonlinear in the ocean wave amplitude. We say that the calculation in Section 5 is a *linearized* analysis because it is linear in the amplitude of the perturbation (the ocean waves) about the base state (the ship wave), but it is *not* a completely linear problem because the ocean wave and ship wave interact rather than simply superpose.

As discussed in Section 3, we find that even for this most favorable choice of scaling  $\delta \sim F^{-4}$ , the amplitude of the ocean waves would not suffer an order unity change as they encountered the ship wave. Rather they would change by some amount proportional to some power of the small parameter  $F^{-1}$ . While this change would be negligible in the  $F^{-1} \rightarrow 0$  limit, we are of course working with a problem where the parameters  $\varepsilon$ ,  $\delta$ , and  $F^{-1}$  are small but nonzero. Therefore, a change in the ocean wave amplitude proportional to  $F^{-n}$  for some  $n > 0$  might well be significant, especially if the coefficient of  $F^{-n}$  was large. Our goal, then, becomes to identify

and quantify the *leading order* change in the amplitude of the ocean waves. In other words, we want to compute the most important term in the perturbation expansion for the ocean wave amplitude which describes its change upon encountering the ship wave. The calculation in Section 5 suggests that the leading order change occurs at  $O(F^{-2})$  according to the linearized theory. (The notation  $O(F^{-n})$  indicates terms comparable to or smaller than  $F^{-n}$ ). However, one readily checks by scaling considerations that while the inclusion of the nonlinear terms will not affect the evolution of the ocean wave amplitudes at  $O(1)$ , they can affect it at  $O(F^{-1})$ .

Therefore consideration of the nonlinear terms could change the prediction of the leading order change of the ocean wave amplitudes (but it will not change the prediction that nothing happens to the ocean wave amplitudes at  $O(1)$ ). We therefore pursue a calculation which accounts for the nonlinear terms. This approach includes the possibility of wave-breaking effects.

Since the linearized analysis indicated that there would be no  $O(1)$  change of the ocean wave amplitudes even for the most hopeful distinguished limit  $\delta = kF^{-4}$ , we opt in the nonlinear case to choose instead the distinguished limit  $\delta = kF^{-2}$  which is closer to the data. For the nonlinear analysis, we must also choose a distinguished limit for the value of  $\varepsilon$ . The data for the 100 knot and 200 knot surfing ship suggest different distinguished limits.

For the 100 knot surfing ship, we take

$$\delta = k^{-1}F^{-2}, \quad \varepsilon = aF^{-2},$$

where  $k$  and  $a$  are constants which remain fixed as  $F \rightarrow \infty$ . This distinguished limit will cover the regime

$$F^{-4} \ll \delta \ll 1 \quad F^{-3} \ll \varepsilon \ll F^{-1}.$$

For the 200 knot surfing ship, we take

$$\delta = kF^{-2}, \quad \varepsilon = aF^{-1},$$

where  $k$  and  $a$  are constants which remain fixed as  $F \rightarrow \infty$ . This distinguished limit will cover the regime

$$F^{-4} \ll \delta \ll 1 \quad F^{-2} \ll \varepsilon \ll 1.$$

The nonlinearities are more important for the case of the 200 knot surfing ship, and we will show the calculations for this case explicitly in Section 7. The results for the 100 knot case can be obtained from the same kind of analysis by simply reducing the importance of the nonlinearity, and we do not provide the details.

## 5 Asymptotic Analysis Linearized About Ship Wave

The dimensionless problem to be solved for the velocity potential  $\phi(x, y, t)$  and free-surface displacement  $\eta(x, t)$  is

$$\nabla^2 \phi = 0 \quad y < F^{-2} \eta, \quad (13a)$$

$$\phi_y = \eta_t + \eta_x + F^{-2} \phi_x \eta_x y = F^{-2} \eta, \quad (13b)$$

$$\phi_t + \phi_x + \frac{F^{-2}}{2} (\phi_x^2 + \phi_y^2) + \eta = 0 \quad y = F^{-2} \eta, \quad (13c)$$

$$\phi \rightarrow 0 \quad y \rightarrow -\infty, \quad (13d)$$

where  $F$  is the Froude number, which is assumed to be large. We take as our base state a steady solution of (13), denoted by  $\{\phi_0(x, y), \eta_0(x)\}$ , which therefore satisfy

$$\nabla^2 \phi_0 = 0 \quad y < F^{-2} \eta_0, \quad (14a)$$

$$\phi_{0y} = \eta_0' + F^{-2} \phi_{0x} \eta_0' y = F^{-2} \eta_0, \quad (14b)$$

$$\phi_{0x} + \frac{F^{-2}}{2} (\phi_{0x}^2 + \phi_{0y}^2) + \eta_0 = 0 \quad y = F^{-2} \eta_0, \quad (14c)$$

$$\phi_0 \rightarrow 0 \quad y \rightarrow -\infty. \quad (14d)$$

Note that for this section, we modify notation slightly from that discussed in Section 4 so that the ship wave elevation and potential is denoted here by  $\eta_0$  and  $\phi_0$  rather than  $\eta^{(s)}$  and  $\phi^{(s)}$ .

Now we perturb this base solution with an infinitesimal unsteady disturbance:

$$\eta = \eta_0 + \epsilon \eta_1, \quad \phi = \phi_0 + \epsilon F^{-2} \phi_1, \quad (15)$$

where  $\epsilon \ll 1$ . The perturbed equations are obtained by substituting (15) into (13) and linearising with respect to  $\epsilon$ , which is assumed to be sufficiently small that the linearisation is uniformly valid. A more specific bound on the size of  $\epsilon$  for which our analysis applies will be determined *a posteriori*. Thus we obtain the following linear problem for  $\phi_1$  and  $\eta_1$ :

$$\nabla^2 \phi_1 = 0 \quad y < F^{-2} \eta_0, \quad (16a)$$

$$F^{-2} \phi_{1y} + F^{-2} \phi_{0yy} \eta_1 = \eta_{1t} + \eta_{1x} + F^{-2} \phi_{0x} \eta_{1x} + F^{-4} \eta_0' (\phi_{1x} + \phi_{0xy} \eta_1) y = F^{-2} \eta_0, \quad (16b)$$

$$F^{-2} (\phi_{1t} + \phi_{1x} + \phi_{0xy} \eta_1) + F^{-4} (\phi_{0x} \phi_{1x} + \phi_{0y} \phi_{1y}) + F^{-4} (\phi_{0x} \phi_{0xy} + \phi_{0y} \phi_{0yy}) \eta_1 + \eta_1 = 0 \quad y = F^{-2} \eta_0, \quad (16c)$$

$$\phi_1 \rightarrow 0 \quad y \rightarrow -\infty. \quad (16d)$$

It is convenient to perform the following ‘‘flattening’’ coordinate transformation which shifts the base free surface onto the  $X$ -axis:

$$X = x, \quad Y = y - F^{-2}\eta_0(x). \quad (17)$$

Derivatives in the two systems are related by the chain rules

$$\partial_x = \partial_X - F^{-2}\eta'_0(X)\partial_Y, \quad \partial_y = \partial_Y; \quad (18)$$

since  $t$ -derivatives are unaffected by the transformation there is no need to re-label the time coordinate. In terms of the new variables, the base problem (14) may be rewritten as

$$\phi_{0XX} + (1 + F^{-4}(\eta'_0)^2) \phi_{0YY} = F^{-2} (2\eta'_0\phi_{0XY} + \eta''_0\phi_{0Y})Y < 0, \quad (19a)$$

$$(1 + F^{-4}(\eta'_0)^2) \phi_{0Y} = \eta'_0 + F^{-2}\eta'_0\phi_{0X}Y = 0, \quad (19b)$$

$$\phi_{0X} + \frac{F^{-2}}{2} [\phi_{0X}^2 - (1 + F^{-4}(\eta'_0)^2) \phi_{0Y}^2] + \eta_0 = 0Y = 0, \quad (19c)$$

$$\phi_0 \rightarrow 0Y \rightarrow -\infty, \quad (19d)$$

while the perturbed problem (16) becomes

$$\phi_{1XX} + (1 + F^{-4}(\eta'_0)^2) \phi_{1YY} = F^{-2} (2\eta'_0\phi_{1XY} + \eta''_0\phi_{1Y})Y < 0, \quad (20a)$$

$$\begin{aligned} \eta_{1t} + (1 + F^{-2}\phi_{0X} - F^{-4}\eta'_0\phi_{0Y}) \eta_{1X} + \\ F^{-4}\eta'_0 (\phi_{1X} + \phi_{0XY}\eta_1) = \\ F^{-2} (1 + F^{-4}(\eta'_0)^2) (\phi_{1Y} + \phi_{0YY}\eta_1)Y = 0, \end{aligned} \quad (20b)$$

$$\begin{aligned} F^{-2} (1 + F^{-2}\phi_{0X} - F^{-4}\eta'_0\phi_{0Y}) (\phi_{1X} + \phi_{0XY}\eta_1) + \\ \eta_1 + F^{-2}\phi_{1t} = 0Y = 0, \end{aligned} \quad (20c)$$

$$\phi_1 \rightarrow 0Y \rightarrow -\infty. \quad (20d)$$

Next we consider the conditions to be imposed as  $X \rightarrow -\infty$ . Here the base solution is assumed to have a flat free surface and uniform velocity:

$$\eta_0 \rightarrow 0, \quad \phi_0 \rightarrow 0 \quad \text{as } X \rightarrow -\infty. \quad (21)$$

In this limit, (20) reduces to the classical problem of linear Stokes gravity waves on a uniform stream. We impose an incoming wave train with dimensionless wavelength of order  $F^{-4}$  and hence

$$\left. \begin{aligned} \eta_1 &\sim \exp\left(iF^4k(X - t) \mp iF^2\sqrt{k}t\right) \\ \phi_1 &\sim \mp \frac{i}{\sqrt{k}} \exp\left(iF^4k(X - t) + F^4kY \mp iF^2\sqrt{k}t\right) \end{aligned} \right\} \text{as } X \rightarrow -\infty, \quad (22)$$

where  $k > 0$ . (It is this Stokes waves calculation which suggests the factor of  $F^{-2}$  absorbed into  $\phi_1$  in (15).) Notice that  $\epsilon$  may be chosen such that the coefficient multiplying the exponential in  $\eta_1$  is unity. If the  $\mp$  in (22) is set to  $-$ , the waves travel in the same direction as the free stream; otherwise they propagate in the opposite direction. Both possibilities should be considered: the ship travels faster than the phase speed and hence may catch up with waves travelling away from it.

Motivated by (22), we apply the WKBJ ansatz,

$$\phi_1 = \exp(F^4 u(X, Y, t)), \quad \eta_1 = \exp(F^4 v(X, t)), \quad (23)$$

which transforms (20) to

$$\begin{aligned} & u_X^2 + u_Y^2 - 2F^{-2}\eta'_0 u_X u_Y + \\ & F^{-4} [u_{XX} + u_{YY} + (\eta'_0)^2 u_Y^2] - \\ & F^{-6} [2\eta'_0 u_{XY} + \eta''_0 u_Y] + F^{-8} (\eta'_0)^2 u_{YY} = 0Y < 0, \end{aligned} \quad (24a)$$

$$\begin{aligned} & R(v_t + v_X) + F^{-2} [\phi_{0X} R v_X - u_Y] + \\ & F^{-4} \eta'_0 (u_X - \phi_{0Y} R v_X) - F^{-6} [\phi_{0YY} R + (\eta'_0)^2 u_Y] + \\ & F^{-8} \eta'_0 \phi_{0XY} R - F^{-10} (\eta'_0)^2 \phi_{0YY} R = 0Y = 0, \end{aligned} \quad (24b)$$

$$\begin{aligned} & u_t + u_X + F^{-2} (\phi_{0X} u_X + R) + \\ & F^{-4} (\phi_{0XY} R - \eta'_0 \phi_{0Y} u_X) + \\ & F^{-6} \phi_{0X} \phi_{0XY} R - F^{-8} \eta'_0 \phi_{0Y} \phi_{0XY} R = 0Y = 0, \end{aligned} \quad (24c)$$

$$u \rightarrow -\infty Y \rightarrow -\infty, \quad (24d)$$

where

$$R = \exp[F^4 (v(X, t) - u(X, 0, t))]. \quad (25)$$

Note that we require

$$v(X, t) = u(X, 0, t) + O(F^{-4});$$

otherwise the WKBJ expressions for  $\phi_1$  and  $\eta_1$  cannot be consistent.

Now we are ready to write the dependent variables as asymptotic expansions in the small parameter  $F^{-2}$ :

$$u \sim u_0 + F^{-2} u_1 + F^{-4} u_2 + \dots, \quad v \sim v_0 + F^{-2} v_1 + F^{-4} v_2 + \dots \quad (26)$$

Similarly we may expand the base solution:

$$\phi_0 \sim \phi_{00} + F^{-2} \phi_{01} + \dots, \quad \eta_0 \sim \eta_{00} + F^{-2} \eta_{01} + \dots \quad (27)$$

By substituting these into (19) we obtain the following boundary conditions satisfied by the base solution, which will be useful later:

$$\begin{aligned}\phi_{00Y} &= \eta'_{00}, & \phi_{01Y} &= \eta'_{01} - \eta_{00}\eta'_{00}, \\ \phi_{00X} &= -\eta_{00}, & \phi_{01X} &= -\eta_{01} - \frac{1}{2}\eta_{00}^2 + \frac{1}{2}(\eta'_{00})^2,\end{aligned}\tag{28}$$

all on  $Y = 0$ .

Now we simply have to substitute the expansions (26) into (24) and equate like powers of  $F$ . Note that

$$v_0(X, t) = u_0(X, 0, t), \quad v_1(X, t) = u_1(X, 0, t),\tag{29}$$

and, if

$$R \sim R_0 + F^{-2}R_1 + \dots,\tag{30}$$

then

$$R_0 = \exp(v_2(X, t) - u_2(X, 0, t)).\tag{31}$$

The leading-order solution consistent with the imposed incoming wave train (22) is

$$u_0 = ik(X - t) + kY, \quad v_0 = ik(X - t).\tag{32}$$

Then at order  $F^{-2}$  we have

$$iu_{1X} + u_{1Y} = ik\eta'_{00}Y < 0,\tag{33a}$$

$$R_0(v_{1t} + v_{1X}) = k - ikR_0\phi_{00X}Y = 0,\tag{33b}$$

$$u_{1t} + u_{1X} = -R_0 - ik\phi_{00X}Y = 0.\tag{33c}$$

Now, (29) implies that  $u_{1t} + u_{1X} = v_{1t} + v_{1X}$  on  $Y = 0$  and hence

$$R_0^2 = -k \quad \Rightarrow \quad R_0 = \pm i\sqrt{k}.\tag{34}$$

Notice that the  $\pm$  here corresponds to the  $\mp$  in (22). Then we may solve for  $v_1$ :

$$v_1 = -R_0t - ik\phi_{00}(X, 0);\tag{35}$$

the arbitrary function of  $X - t$  is set to zero by comparing with the behaviour (22) as  $X \rightarrow -\infty$ . Finally, the solution for  $u_1$  which equals  $v_1$  on  $Y = 0$  is

$$u_1 = k\eta_{00}(X) - k\eta_{00}(X - iY) - ik\phi_{00}(X - iY, 0) - R_0t.\tag{36}$$

Moving on to order  $F^{-4}$  we find the following problem for  $u_2$  and  $\eta_2$  (which correspond to the amplitude modulation of  $\phi_1$  and  $\eta_1$ ):

$$iu_{2X} + u_{2Y} = ik\eta'_{01}(X) - k\eta'_{00}(X) [\eta'_{00}(X - iY) - i\eta_{00}(X - iY)]Y < 0, \quad (37a)$$

$$v_{2t} + v_{2X} = R_1 - \eta_{00}R_0 + ik \left[ \frac{3}{2}\eta_{00}^2 + \frac{1}{2}(\eta'_{00})^2 + \eta_{01} \right] Y = 0, \quad (37b)$$

$$u_{2t} + u_{2X} = -R_1 - \eta''_{00}R_0 + ik \left[ \frac{3}{2}\eta_{00}^2 + \frac{1}{2}(\eta'_{00})^2 + \eta_{01} \right] Y = 0. \quad (37c)$$

Now, from (31) we have

$$v_2 = u_2 + \log R_0 \quad \Rightarrow \quad v_{2t} + v_{2X} = u_{2t} + u_{2X} \quad \text{on } Y = 0, \quad (38)$$

and hence

$$R_1 = \frac{R_0}{2} (\eta_{00} - \eta''_{00}). \quad (39)$$

Then we may solve for  $v_2$  in the form

$$v_2 = v_2(X) = \frac{R_0}{2} (\phi_{00}(X, 0) - \eta'_{00}(X)) - ik\phi_{01}(X, 0) + ik \int_{-\infty}^X \eta_{00}(x)^2 + \eta'_{00}(x)^2 dx, \quad (40)$$

using the behaviour (22) as  $X \rightarrow -\infty$  to eliminate the arbitrary function of  $X - t$ . Thus the solution for  $u_2$  is

$$u_2 = k\eta_{01}(X) + ik\eta_{00}(X) [\eta'_{00}(X - iY) - i\eta_{00}(X - iY)] + f_2(X - iY), \quad (41)$$

where the function of integration is given by

$$f_2(X) = v_2(X) - k\eta_{01}(X) - k\eta_{00}(X) [i\eta'_{00}(X) + \eta_{00}(X)] - \log R_0. \quad (42)$$

The upshot of all this is the following. Since  $R_0 = \pm i\sqrt{k}$ , both  $v_1$  and  $v_2$  are pure imaginary. This means that the terms considered so far in the analysis act only upon the phase of the surface waves. Thus the analysis predicts that the small waves ride up the big wave without any reduction in amplitude, but with varying phase speed and wavelength. This appears to be bad news for the surfing ship! It may be that  $v_3$  contains real terms which give higher-order damping (or amplification) of the little waves. However, these would correspond to amplitude variations of order  $F^{-2}$ .

In order to obtain  $v_3(X, t)$  we need only examine the kinematic and Bernoulli equations. At  $\mathcal{O}(F^{-6})$  these are

$$\begin{aligned} R_3(v_{0t} + v_{0X}) + R_2(v_{1t} + v_{1X}) + R_1(v_{2t} + v_{2X}) + R_0(v_{3t} + v_{3X}) = \\ u_{2Y} - \phi_{00X}(v_{0X}R_2 + v_{1X}R_1 + v_{2X}R_0) - \phi_{01X}(v_{0X}R_1 + v_{1X}R_0) \\ - \phi_{02X}v_{0X}R_0 + \eta'_{01}\phi_{00Y}v_{0X}R_0 + \eta'_{00}\phi_{01Y}v_{0X}R_0 + \eta'_{00}\phi_{00Y}v_{1X}R_0 + \\ \eta'_{00}\phi_{00Y}v_{0X}R_1 - \eta'_{01}u_{0X} - \eta'_{00}u_{1X} + \phi_{00YY}R_0 + (\eta'_{00})^2u_{0Y} \end{aligned} \quad (43a)$$

$$\begin{aligned} -(u_{3t} + u_{3X}) = R_2 + \phi_{00X}u_{2X} + \phi_{01X}u_{1X} + \phi_{02X}u_{0X} + \phi_{01XY}R_0 \\ + \phi_{00XY}R_1 - \eta'_{01}\phi_{00Y}u_{0X} - \eta'_{00}\phi_{01Y}u_{0X} - \eta'_{00}\phi_{00Y}u_{1X} + \phi_{00X}\phi_{00XY}R_0 \end{aligned} \quad (43b)$$

where these equations are evaluated at  $Y = 0$ .

In order to simplify and solve for  $v_3$  we shall use the following results from the above analysis

$$R_0^2 = -k \quad (44)$$

$$R_1 = \frac{R_0}{2}(\eta_{00} - \eta''_{00}) \quad (45)$$

We also note that  $R_1 = R_0[v_3(X, t) - u_3(X, 0, t)]$  which leads to the result that

$$u_{3t} + u_{3X} = v_{3t} + v_{3X} - \frac{1}{2}(\eta'_{00} - \eta'''_{00}) \quad (46)$$

Additionally, we have

$$v_{0t} + v_{0X} = 0 \quad (47a)$$

$$v_{1t} + v_{1X} = \frac{k}{R_0} - ik\phi_{00X} \quad (47b)$$

$$v_{2t} + v_{2X} = R_1 - \eta_{00}R_0 + ik\left[\frac{3}{2}\eta_{00}^2 + \frac{1}{2}(\eta'_{00})^2 + \eta_{01}\right] \quad (47c)$$

and

$$u_{0t} + u_{0X} = 0 \quad (48a)$$

$$u_{1t} + u_{1X} = -R_0 - ik\phi_{00X} \quad (48b)$$

$$u_{2t} + u_{2X} = -R_1 - \eta''_{00}R_0 + ik\left[\frac{3}{2}\eta_{00}^2 + \frac{1}{2}(\eta'_{00})^2 + \eta_{01}\right] \quad (48c)$$

all of which are evaluated on  $Y = 0$ . We also note that

$$u_{1X}\Big|_{Y=0} = v_{1X}\Big|_{Y=0} = -ik\phi_{00X} \quad (49)$$

$$u_{2X}\Big|_{Y=0} = v_{2X}\Big|_{Y=0} = \frac{R_0}{2} [\phi_{00X} - \eta''_{00}] - ik\phi_{01X} + ik [\eta_{00}^2 + (\eta'_{00})^2] \quad (50)$$

$$u_{2Y}\Big|_{Y=0} = -iv_{2X} + ik\eta'_{01} + k\eta'_{00} [i\eta_{00} - \eta'_{00}] \quad (51)$$

After some manipulations we find that

$$\begin{aligned} v_{3t} + v_{3X} &= R_2 + \phi_{00YY} + \frac{R_1}{2} [\eta_{00} + \eta''_{00}] + \left[ \eta_{00} + \frac{iR_0}{k} \right] v_{2X} \\ &\quad + ik \left[ \eta_{00}\eta_{01} - \frac{1}{2}(\eta'_{00})^2\eta_{00} + \frac{1}{2}\eta_{00}^3 + 2\eta'_{00}\eta'_{01} - \phi_{02X} \right] \end{aligned} \quad (52a)$$

$$\begin{aligned} u_{3t} + u_{3X} &= -R_2 - R_1\eta''_{00} - R_0 [\eta''_{01} - (\eta'_{00})^2 - 2\eta_{00}\eta''_{00}] + \eta_{00}v_{2X} \\ &\quad + ik \left[ \eta_{00}\eta_{01} - \frac{1}{2}(\eta'_{00})^2\eta_{00} + \frac{1}{2}\eta_{00}^3 + 2\eta'_{00}\eta'_{01} - \phi_{02X} \right] \end{aligned} \quad (52b)$$

We can combine these to get an equation for  $v_3$  of the form

$$v_{3t} + v_{3X} = \frac{3}{4}\eta'_{00} - \frac{1}{4}\eta'''_{00} + v_{3COMPLEX}, \quad (53)$$

where

$$\begin{aligned} v_{3COMPLEX} &= \frac{R_1}{4} [\eta_{00} - \eta''_{00}] + \left[ \eta_{00} + \frac{iR_0}{2k} \right] v_{2X} \\ &\quad - \frac{R_0}{2} [\eta''_{01} - (\eta'_{00})^2 - 2\eta_{00}\eta''_{00}] \\ &\quad + ik \left[ \eta_{00}\eta_{01} - \frac{1}{2}(\eta'_{00})^2\eta_{00} + \frac{1}{2}\eta_{00}^3 + 2\eta'_{00}\eta'_{01} - \phi_{02X} \right] \end{aligned} \quad (54)$$

This suggests that  $v_3$  has the form

$$v_3(X, t) = \frac{3}{4}\eta_{00} - \frac{1}{4}\eta''_{00} + i(\dots) \quad (55)$$

It follows that the sea surface perturbation has the form

$$\eta_1 = \exp \left\{ \left[ \frac{1}{4}F^{-2} (3\eta_{00} - \eta''_{00}) + \mathcal{O}(F^{-4}) \right] + iv_C(X, t) \right\} \quad (56)$$

where  $v_C(X, t)$  is a real function given by

$$\begin{aligned} v_C(X, t) &= F^4 k(X - t) - F^2 \left( \pm\sqrt{k}t + k\phi_{00} \right) \pm \frac{\sqrt{k}}{2} (\phi_{00} - \eta'_{00}) - k\phi_{01} \\ &\quad + k \int_{-\infty}^X [\eta_{00}^2 + (\eta'_{00})^2] dx + \mathcal{O}(F^{-2}) \end{aligned} \quad (57)$$

Note that the  $\mathcal{O}(F^{-2})$  term in  $v_C(X, t)$  can be obtained from equations (53) and (54).

There is growth (decay) of the perturbations if the quantity  $3\eta_{00} - \eta''_{00}$  is positive (negative). Based on figure 3 in Cumberbatch (1958) the upstream wave (which I believe in his calculation corresponds to  $x \rightarrow +\infty$ ) appears to have  $\eta_{00} > 0$  and  $\eta''_{00} < 0$  which would suggest growth of the disturbance. The growth does not seem to depend on the wavenumber  $k$  of the disturbance and, as noted above, is small,  $\mathcal{O}(F^{-2})$ . Also note that if we are in the ship's frame of reference the growth is in space rather than in time. That is, if we focus our attention on a fixed point in front of the ship the amplitude of those waves are not growing in time.

Recall that we assumed in this analysis that the ratio of the wavelengths of the ocean waves to the ship waves is related to the Froude number by  $\delta = kF^{-4}$ , but as discussed in Subsection 4.5, the analysis here is appropriate for the wider range  $F^{-6} \ll \delta \ll F^{-2}$  by just allowing  $k$  to range between  $F^{-2} \ll k \ll F^2$ . For this entire range, the change in the amplitude remains  $\mathcal{O}(F^{-2})$ . Consulting the data in Table 2 in Subsection 4.5, we see that this suggests a change on the order of ten percent of the original wave amplitude. A more precise value can be computed by evaluating (56) using the solution for the ship wave from [10].

## 6 Preparation for Nonlinear Asymptotic Analysis

The work in the previous section was predicated on a linearized theory for the evolution of the ocean waves, which was motivated by the fact that the relevant parameters indicate that the nonlinearity in the governing equations is small. However, we saw from that analysis that the amplitude of the ocean waves suffered only a small change as they encountered the ship wave. We derived formulas for this small change, but there are two ways we can improve the accuracy of these results:

- Include the effects of the nonlinearity. Even though the nonlinearity is formally weak, the effects of the linear terms on the ocean waves was found to be formally weak, so the effects of the nonlinear terms might be comparable or greater than those of the linear terms. That is, while we can conclusively argue that the nonlinear terms will not give rise to an order unity change in the ocean wave amplitude, the weak nonlinear terms may provide a relevant or even dominant contribution to the formula for the leading order effect of the ship wave on the ocean waves. Though the changes in the ocean wave amplitude will be weak in a mathematically formal sense, they may still be significant because the small parameters have finite values and may be multiplied by a large coefficient.
- The formulas derived in Section 5 assumed that a complex-valued function on the real axis could be analytically continued into the entire lower half of the complex plane (see for example Eq. (36)). This is not generally true [18, Ch. 8], and we don't know for certain whether this operation was really legitimate for the function  $\phi_{00}$  in Eq. (36). We will avoid this problem in the weakly nonlinear derivations by mapping the ocean region onto a half-plane with a *conformal* mapping rather than with the direct “flattening” mapping (17). See Appendix A for a discussion of why the standard multiple-scales analysis, of the type pursued in Section 5, runs into difficulties and why the conformal mapping approach ameliorates it. From this more precise analysis, we will see that the formal assumption that an analytical continuation exists which we used in Section 5 creates some error, and we will be able to estimate and to avoid this error. We are able to show, though, that the error does not change the main prediction (56) (but maybe the “self-nonlinear” terms explicitly excluded from that analysis will).

### 6.1 Overview of Solution Strategy

We summarize here the general approach we will use to obtain a systematic approximation for the evolution of the ocean wave packet perturbation, allowing for the effects of nonlinearity. Details will be provided in subsequent subsections.

1. We reformulate the free surface fluid equations (9) using a conformal mapping formalism described in detail in Appendix A. The reason we do this is to avoid the problem we encountered in Section 5 in which the equations generated by a direct perturbation expansion on the standard equations can only be solved under an assumption of analytic continuability of certain surface functions, which cannot generally be expected. The source of this problem and the reason why conformal mapping avoids it is discussed in Appendix A. The reformulated equations, along with their decomposition into ship wave and ocean packet perturbation contributions, are developed in Subsection 6.2.
2. Following standard practice in asymptotic analysis, we identify different space and time scales on which the nondimensionalized ocean wave packet will vary in Subsection 6.3.
3. We next write down an asymptotic series expansion for the ocean wave packet perturbation where the dependence on the fast and intermediate scales of variation is explicitly identified (Subsection 6.4).
4. We are left then only to solve for the “amplitude coefficients” in this perturbation expansion, which depend on the “slow” space and time variables. We write down the general procedure for solving these amplitude coefficients in Subsection 6.5.

Explicit calculations are presented in Section 7 after we discuss the above steps in a general way in this section.

## **6.2 Reformulation of Equations Using Conformal Mapping**

In this subsection, we will set up the asymptotics for the weakly nonlinear analysis of the fluid equations using the conformal mapping formalism, which is explained in

more detail in Appendix A. The governing equations which we use are therefore:

$$0 = \eta_t(x, t) + \eta_x(x, t) - \phi_y + F^{-2} \bar{\phi}_x(x, t) \eta_x(x, t), \quad (58a)$$

$$0 = \bar{\phi}_t(x, t) + \bar{\phi}_x(x, t) + \frac{F^{-2}}{2} \left( \bar{\phi}_x^2(x, t) + \bar{\phi}_y^2(x, t) \right) + \eta(x, t) + \Pi(x), \quad (58b)$$

$$\phi(x, y, t) = \int_{-\infty}^{\infty} e^{i\xi \mathcal{X}(x, y, t)} e^{|\xi| \mathcal{Y}(x, y, t)} P(\xi, t) d\xi, \quad (58c)$$

$$\bar{\phi}(x, t) = \phi(x, y = F^{-2} \eta(x, t), t), \quad (58d)$$

$$\bar{\phi}_x(x, t) = \left. \frac{\partial \phi(x, y, t)}{\partial x} \right|_{y=F^{-2} \eta(x, t)}, \quad (58e)$$

$$\bar{\phi}_y(x, t) = \left. \frac{\partial \phi(x, y, t)}{\partial y} \right|_{y=F^{-2} \eta(x, t)}, \quad (58f)$$

$$\bar{\phi}_t(x, t) = \left. \frac{\partial \phi(x, y, t)}{\partial t} \right|_{y=F^{-2} \eta(x, t)}, \quad (58g)$$

with the conformal mapping functions  $\mathcal{X}$  and  $\mathcal{Y}$  determined by solving the following equations:

$$\frac{\partial \tilde{\mathcal{X}}^\#}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}^\#}{\partial Y} - F^{-2} \eta_X \frac{\partial \tilde{\mathcal{X}}^\#}{\partial Y} = 0, \quad (58h)$$

$$\frac{\partial \tilde{\mathcal{Y}}^\#}{\partial X} + \frac{\partial \tilde{\mathcal{X}}^\#}{\partial Y} - F^{-2} \eta_X \frac{\partial \tilde{\mathcal{Y}}^\#}{\partial Y} = 0 \text{ for } Y < 0, \quad (58i)$$

$$\tilde{\mathcal{Y}}^\#(X, Y = 0, T) = 0, \quad (58j)$$

$$\lim_{Y \rightarrow -\infty} \left( \tilde{\mathcal{Y}}^\#(X, Y, T) - (Y + F^{-2} \eta(X, T)) \right) = 0. \quad (58k)$$

which are expressed in terms of flattened variables:

$$\begin{aligned} X &= x, \\ Y &= y - F^{-2} \eta(x, t), \\ T &= t, \end{aligned} \quad (58l)$$

$$\tilde{\mathcal{X}}^\#(X, Y, T) = \mathcal{X}(x, y, t),$$

$$\tilde{\mathcal{Y}}^\#(X, Y, T) = \mathcal{Y}(x, y, t).$$

For our nonlinear analysis, we impose wave packet initial conditions [17, Sec. 5.3.2]:

$$\eta(x, t = 0) = \eta^{(s)}(x) + \varepsilon \left[ A^{(0)}(x) e^{i\delta^{-1}x} + \text{c.c.} \right], \quad (58m)$$

with  $\phi(x, y, t = 0)$  chosen so that the system corresponds to a ship wave superposed with a ocean wave packet moving in one direction relative to the fixed ocean.

This formulation is mathematically equivalent to the standard set of surface wave equations (9). One nice feature of this “conformally mapped” representation is that we can express the fluid equations purely in terms of the surface elevation  $\eta$  and the surface velocities  $\bar{v}^{(x)} = \overline{\phi_x}$  and  $\bar{v}^{(y)} = \overline{\phi_y}$ . The Laplace equation for the velocity potential  $\phi$  in the interior of the fluid has been replaced by its exact solution (58c) represented in terms of the conformal mapping functions  $\mathcal{X}(x, y, t)$  and  $\mathcal{Y}(x, y, t)$ . This allows us a way to decompose the fluid functions into a ship wave component and a perturbation due to the ocean wave.

### 6.2.1 Reformulation of Pure Ship Wave Problem in Terms of Surface Quantities

To see this, let us write down the system of equations determining the ship wave functions in isolation of the ocean wave packet:

$$0 = \eta_x^{(s)}(x) + F^{-2} \overline{\phi_x}^{(s)}(x) \eta_x^{(s)}(x) - \overline{\phi_y}^{(s)}(x), \quad (59a)$$

$$0 = \overline{\phi_x}^{(s)}(x) + \frac{1}{2} F^{-2} \left[ (\overline{\phi_x}^{(s)}(x))^2 + (\overline{\phi_y}^{(s)}(x))^2 \right] + \eta^{(s)}(x) + \Pi(x), \quad (59b)$$

$$\phi^{(s)}(x, y) = \int_{-\infty}^{\infty} e^{i\xi \mathcal{X}^{(s)}(x, y)} e^{|\xi| \bar{\mathcal{Y}}^{(s)}(x, y)} P(\xi) d\xi, \quad (59c)$$

$$\bar{\phi}^{(s)}(x) = \phi^{(s)}(x, y = F^{-2} \eta^{(s)}(x, t)), \quad (59d)$$

$$\overline{\phi_x}^{(s)}(x, t) = \left. \frac{\partial \phi^{(s)}(x, y)}{\partial x} \right|_{y=F^{-2} \eta^{(s)}(x)}, \quad (59e)$$

$$\overline{\phi_y}^{(s)}(x, t) = \left. \frac{\partial \phi^{(s)}(x, y)}{\partial y} \right|_{y=F^{-2} \eta^{(s)}(x)}, \quad (59f)$$

The conformal mapping equations for the ship wave are:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{X}}^{\#(s)}}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}^{\#(s)}}{\partial Y} - F^{-2} \eta_X^{(s)} \frac{\partial \tilde{\mathcal{X}}^{\#(s)}}{\partial Y} &= 0, \\ \frac{\partial \tilde{\mathcal{Y}}^{\#(s)}}{\partial X} + \frac{\partial \tilde{\mathcal{X}}^{\#(s)}}{\partial Y} - F^{-2} \eta_X^{(s)} \frac{\partial \tilde{\mathcal{Y}}^{\#(s)}}{\partial Y} &= 0 \text{ for } Y < 0, \end{aligned} \quad (59g)$$

$$\tilde{\mathcal{Y}}^{\#(s)}(X, Y = 0, T) = 0,$$

$$\lim_{Y \rightarrow -\infty} \left( \tilde{\mathcal{Y}}^{\#(s)}(X, Y, T) - (Y + F^{-2} \eta^{(s)}(X, T)) \right) = 0.$$

which are expressed in terms of flattened variables:

$$\begin{aligned}
X &= x, \\
Y &= y - F^{-2}\eta^{(s)}(x, t), \\
T &= t, \\
\tilde{\mathcal{X}}^{\sharp(s)}(X, Y, T) &= \mathcal{X}(x, y, t), \\
\tilde{\mathcal{Y}}^{\sharp(s)}(X, Y, T) &= \mathcal{Y}(x, y, t), \\
\eta^{(s)}(X, T) &= \eta^{(s)}(x, t).
\end{aligned} \tag{59h}$$

### 6.2.2 Reformulation of Ship Wave Interacting with Ocean Wave Packet Problem in Terms of Surface Quantities

Now we can write the surface elevation and surface values of the velocity potential for the full problem (58) with the interacting ocean wave packet and ship wave in terms of a ship wave component and a perturbation due to the ocean wave packet:

$$\begin{aligned}
\eta(x, t) &= \eta^{(s)}(x) + \varepsilon\eta^{(p)}(x, t), \\
\bar{\phi}(x, t) &= \bar{\phi}^{(s)}(x) + \varepsilon\delta^{1/2}\bar{\phi}_x^{(p)}(x, t), \\
\bar{\phi}_x(x, t) &= \bar{\phi}_x^{(s)}(x) + \varepsilon\delta^{1/2}\bar{\phi}_x^{(p)}(x, t), \\
\bar{\phi}_y(x, t) &= \bar{\phi}_y^{(s)}(x) + \varepsilon\delta^{1/2}\bar{\phi}_y^{(p)}(x, t).
\end{aligned} \tag{60}$$

We have observed from the initial data (58m) that the perturbation to the surface elevation is  $O(\varepsilon)$ ; the perturbations to the velocity potential are scaled by using the fact that both ship and ocean waves have small aspect ratio, so that the wave height and potential are related to leading order by the linear Stokes relations [1, Sec. 3.2]. (Note that  $\bar{\phi}^{(p)}(x)$  is scaled to be  $O(1)$ ; its first derivatives  $\bar{\phi}_x^{(p)}$  and  $\bar{\phi}_y^{(p)}$  will be  $O(\delta^{-1})$  because of the fast variation of the ocean waves relative to the ship.) Substituting the expansions (60) into the dynamical equations (58a) and (58b), using the corresponding ship wave equations (59) to remove the  $O(1)$  terms, and then dividing the kinematic equation (58a) by  $\varepsilon$  and the Bernoulli equation (58b) by  $\varepsilon\delta^{1/2}$  gives us the following equations for the perturbation represented by

the ocean wave packet:

$$\begin{aligned}
0 &= \eta_t^{(p)}(x, t) + \eta_x^{(p)}(x, t) - \delta^{1/2} \overline{\phi_y^{(p)}}(x, t) + \delta^{1/2} \varepsilon \mathbb{F}^{-2} \overline{\phi_x^{(p)}}(x, t) \eta_x^{(p)}(x, t) \\
&\quad + \mathbb{F}^{-2} \left[ \overline{\phi_x^{(s)}}(x) \eta_x^{(p)}(x, t) + \delta^{1/2} \overline{\phi_x^{(p)}}(x, t) \eta_x^{(s)}(x) \right], \\
0 &= \overline{\phi_t^{(p)}}(x, t) + \overline{\phi_x^{(p)}}(x, t) + \delta^{-1/2} \eta^{(p)}(x, t) \\
&\quad + \mathbb{F}^{-2} \left( \overline{\phi_x^{(s)}}(x) \overline{\phi_x^{(p)}}(x, t) + \overline{\phi_y^{(s)}}(x) \overline{\phi_y^{(p)}}(x, t) \right) \\
&\quad + \frac{\varepsilon \delta^{-1} \mathbb{F}^{-2}}{2} \left( (\overline{\phi_x^{(p)}}(x, t))^2 + (\overline{\phi_y^{(p)}}(x, t))^2 \right).
\end{aligned} \tag{61}$$

These equations are initialized at  $t = 0$  by a wave packet:

$$\eta^{(p)}(x, t = 0) = \left[ A^{(0)}(x) e^{i\delta^{-1}x} + \text{c.c.} \right], \tag{62}$$

with  $\phi^{(p)}(x, t = 0)$  defined in a corresponding way so that the wave packet moves in a single direction relative to the fixed ocean.

We cannot however so simply separate out purely perturbative pieces of the full velocity potential  $\phi(x, y, t)$  and of the conformal mapping functions  $(\mathcal{X}, \mathcal{Y})$  because these functions are defined on the interior of the fluid domain of the perturbed problem, which is different from the interior of the fluid domain for the ship wave in isolation. We therefore proceed more carefully, still with the aim of separating out quantities associated to the ocean wave perturbations from the full fluid functions whenever possible. The role of the rest of the mathematical problem which is not included in Eq. (61) above is to relate the derivatives of the potential at the surface  $\overline{\phi_x^{(p)}}$  and  $\overline{\phi_y^{(p)}}$  to the surface elevation  $\eta^{(p)}$  and surface value of the potential  $\overline{\phi^{(p)}}$ , thereby closing the system (61). We will massage these remaining equations to facilitate the derivation of this relationship in our detailed calculations. The final form of the equations will be presented in Subsubsection 6.2.3.

**Relation of Surface Derivatives of Potential to Surface Value of Velocity Potential** Note first of all that while we cannot divide  $\phi(x, y, t)$  into a ship wave plus ocean wave piece, we can do this for the function  $P(\xi, t)$  appearing in (58c) because this function does not depend on  $y$ :

$$P(\xi, t) = P^{(s)}(\xi) + \varepsilon \delta^{1/2} P^{(p)}(\xi, t).$$

Similarly, we can subdivide the *surface* values of the conformal mapping function  $\mathcal{X}(x, y, t)$  and their derivatives into a ship wave plus ocean wave perturbation. For this development, it will be cleaner to simply write down expressions for complete

partial derivatives with respect to  $x$ ,  $y$ , and  $t$  coordinates and not separate them into contributions from fast and slow variation. We will connect back to the fast and slow derivatives when we present the final general formulas for the derivatives of the potential at the surface. Proceeding, then, we have:

$$\bar{\mathcal{X}}(x, t) \equiv \mathcal{X}(x, y = F^{-2}\eta(x, t), t) = \bar{\mathcal{X}}^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{X}}^{(p)}(x, t), \quad (63a)$$

$$\bar{\mathcal{X}}_x(x, t) \equiv \left. \frac{\partial \mathcal{X}(x, y, t)}{\partial x} \right|_{y=F^{-2}\eta(x, t)} = \bar{\mathcal{X}}_x^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{X}}_x^{(p)}(x, t), \quad (63b)$$

$$\bar{\mathcal{X}}_y(x, t) \equiv \left. \frac{\partial \mathcal{X}(x, y, t)}{\partial y} \right|_{y=F^{-2}\eta(x, t)} = \bar{\mathcal{X}}_y^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{X}}_y^{(p)}(x, t), \quad (63c)$$

$$\bar{\mathcal{X}}_t(x, t) \equiv \left. \frac{\partial \mathcal{X}(x, y, t)}{\partial t} \right|_{y=F^{-2}\eta(x, t)} = \varepsilon F^{-2} \bar{\mathcal{X}}_t^{(p)}(x, t), \quad (63d)$$

$$\bar{\mathcal{Y}}(x, t) \equiv \mathcal{Y}(x, y = F^{-2}\eta(x, t), t) = 0, \quad (63e)$$

$$\bar{\mathcal{Y}}_x(x, t) \equiv \left. \frac{\partial \mathcal{Y}(x, y, t)}{\partial x} \right|_{y=F^{-2}\eta(x, t)} = \bar{\mathcal{Y}}_x^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{Y}}_x^{(p)}(x, t), \quad (63f)$$

$$\bar{\mathcal{Y}}_y(x, t) \equiv \left. \frac{\partial \mathcal{Y}(x, y, t)}{\partial y} \right|_{y=F^{-2}\eta(x, t)} = \bar{\mathcal{Y}}_y^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{Y}}_y^{(p)}(x, t), \quad (63g)$$

$$\bar{\mathcal{Y}}_t(x, t) \equiv \left. \frac{\partial \mathcal{Y}(x, y, t)}{\partial t} \right|_{y=F^{-2}\eta(x, t)} = \varepsilon F^{-2} \bar{\mathcal{Y}}_t^{(p)}(x, t). \quad (63h)$$

The ship wave contributions (with superscript  $(s)$ ) to these surface evaluations of the conformal mapping functions are just the surface evaluations of the conformal mapping functions  $\mathcal{X}^{(s)}$  and  $\bar{\mathcal{Y}}^{(s)}$  which appear in the pure ship wave problem (59). The perturbation quantities  $\bar{\mathcal{X}}^{(p)}(x, t)$  and  $\bar{\mathcal{Y}}^{(p)}(x, t)$  are well scaled (so that their leading terms are order unity), but since they have fast variations on the scale  $\delta^{-1}$ , the surface derivatives ( $\bar{\mathcal{X}}_x^{(p)}$ , etc.) are actually  $O(\delta^{-1})$ .

Now we can relate the ocean wave packet components of the surface potential and its derivatives to the ocean wave perturbation in the surface values of the conformal mapping functions and their derivatives by evaluating (58c) and its derivatives at the surface, and using the decompositions in Eq. (60) and Eq. (63):

$$\begin{aligned} \bar{\phi}(x, t) &= \bar{\phi}^{(s)}(x) + \varepsilon \delta^{1/2} \bar{\phi}^{(p)}(x, t) \\ &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x) + \varepsilon F^{-2} \bar{\mathcal{X}}^{(p)}(x, t))} (P^{(s)}(\xi, t) + \varepsilon \delta^{1/2} P^{(p)}(\xi, t)) d\xi \\ &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} P^{(s)}(\xi, t) d\xi + \varepsilon F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} i\xi \bar{\mathcal{X}}^{(p)}(x, t) P^{(s)}(\xi) d\xi \\ &\quad + \varepsilon \delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x, t))} P^{(p)}(\xi, t) d\xi + O(\varepsilon^2 F^{-4}), \end{aligned}$$

$$\begin{aligned}
\overline{\phi}_x(x, t) &= \overline{\phi}_x^{(s)}(x, t) + \varepsilon\delta^{1/2}\overline{\phi}_x^{(p)}(x, t) \\
&= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x,t) + \varepsilon F^{-2}\bar{\mathcal{X}}^{(p)}(x,t))} (P^{(s)}(\xi) + \varepsilon\delta^{1/2}P^{(p)}(\xi, t)) \\
&\quad \times \left( i\xi(\bar{\mathcal{X}}_x^{(s)}(x) + \varepsilon F^{-2}\bar{\mathcal{X}}_x^{(p)}(x, t)) + |\xi|(\bar{\mathcal{Y}}_x^{(s)}(x) + \varepsilon F^{-2}\bar{\mathcal{Y}}_x^{(p)}(x, t)) \right) d\xi, \\
&= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} (i\xi(\bar{\mathcal{X}}_x^{(s)}(x) + |\xi|\bar{\mathcal{Y}}_x^{(s)}(x))P^{(s)}(\xi, t) d\xi \\
&\quad + \varepsilon F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} (i\xi(\bar{\mathcal{X}}_x^{(p)}(x, t) + |\xi|\bar{\mathcal{Y}}_x^{(p)}(x, t))P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon\delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x,t))} (i\xi(\bar{\mathcal{X}}_x(x, t) + |\xi|\bar{\mathcal{Y}}_x(x, t))P^{(p)}(\xi, t) d\xi \\
&\quad + O(\varepsilon F^{-2}) + O(\delta^{-1}\varepsilon^2 F^{-4}),
\end{aligned}$$

$$\begin{aligned}
\overline{\phi}_y(x, t) &= \overline{\phi}_y^{(s)}(x, t) + \varepsilon\delta^{1/2}\overline{\phi}_y^{(p)}(x, t) \\
&= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x) + \varepsilon F^{-2}\bar{\mathcal{X}}^{(p)}(x,t))} (P^{(s)}(\xi) + \varepsilon\delta^{1/2}P^{(p)}(\xi, t)) \\
&\quad \times \left( i\xi(\bar{\mathcal{X}}_y^{(s)}(x) + \varepsilon F^{-2}\bar{\mathcal{X}}_y^{(p)}(x, t)) \right. \\
&\quad \quad \left. + |\xi|(\bar{\mathcal{Y}}_y^{(s)}(x) + \varepsilon F^{-2}\bar{\mathcal{Y}}_y^{(p)}(x, t)) \right) d\xi, \\
&= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} (i\xi\bar{\mathcal{X}}_y^{(s)}(x) + |\xi|\bar{\mathcal{Y}}_y^{(s)}(x))P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} (i\xi\bar{\mathcal{X}}_y^{(p)}(x, t) + |\xi|\bar{\mathcal{Y}}_y^{(p)}(x, t))P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon\delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x,t))} (i\xi\bar{\mathcal{X}}_y(x, t) + |\xi|\bar{\mathcal{Y}}_y(x, t))P^{(p)}(\xi, t) d\xi \\
&\quad + O(\varepsilon F^{-2}) + O(\delta^{-1}\varepsilon^2 F^{-4}),
\end{aligned}$$

$$\begin{aligned}
\overline{\phi}_t(x, t) + \overline{\phi}_x(x, t) &= \overline{\phi}_x^{(s)}(x) + \varepsilon\delta^{1/2}(\overline{\phi}_t^{(p)}(x, t) + \overline{\phi}_x^{(p)}(x, t)) \\
&= \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}^{(s)}(x) + \varepsilon F^{-2}\overline{\mathcal{X}}^{(p)}(x, t))} (P^{(s)}(\xi) + \varepsilon\delta^{1/2}P^{(p)}(\xi, t)) \\
&\quad \left( i\xi(\overline{\mathcal{X}}_x^{(s)}(x) + \varepsilon F^{-2}(\overline{\mathcal{X}}_t^{(p)}(x, t) + \overline{\mathcal{X}}_x^{(p)}(x, t))) \right. \\
&\quad \left. + |\xi|(\overline{\mathcal{Y}}^{(s)}(x) + \varepsilon F^{-2}(\overline{\mathcal{Y}}_x^{(p)}(x, t) + \overline{\mathcal{Y}}_t^{(p)}(x, t))) \right) d\xi \\
&\quad + \varepsilon\delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}^{(s)}(x) + \varepsilon F^{-2}\overline{\mathcal{X}}^{(p)}(x, t))} \frac{\partial P^{(p)}(\xi, t)}{\partial t} d\xi, \\
&= \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}^{(s)}(x))} (i\xi(\overline{\mathcal{X}}_x^{(s)}(x) + |\xi|\overline{\mathcal{Y}}_x^{(s)}(x))P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}^{(s)}(x))} i\xi\overline{\mathcal{X}}^{(p)}(x, t) (i\xi(\overline{\mathcal{X}}_x^{(s)}(x) + |\xi|\overline{\mathcal{Y}}_x^{(s)}(x))P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}^{(s)}(x))} \left( i\xi(\overline{\mathcal{X}}_t^{(p)}(x, t) + \overline{\mathcal{X}}_x^{(p)}(x, t)) \right. \\
&\quad \left. + |\xi|(\overline{\mathcal{Y}}_t^{(p)}(x, t) + \overline{\mathcal{Y}}_x^{(p)}(x, t)) \right) P^{(s)}(\xi) d\xi \\
&\quad + \varepsilon\delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}(x, t))} (i\xi(\overline{\mathcal{X}}_t(x, t) + \overline{\mathcal{X}}_x(x, t)) \\
&\quad \left. + |\xi|(\overline{\mathcal{Y}}_t(x, t) + \overline{\mathcal{Y}}_x(x, t))) P^{(p)}(\xi, t) d\xi \\
&\quad + \varepsilon\delta^{1/2} \int_{-\infty}^{\infty} e^{i\xi(\overline{\mathcal{X}}(x, t))} \frac{\partial P^{(p)}(\xi, t)}{\partial t} d\xi + O(\delta^{-1/2}\varepsilon^2 F^{-4}).
\end{aligned}$$

Since we only need the expansions of  $\overline{\phi}$  through  $O(\varepsilon F^{-2})$ , its spatial derivatives through  $O(\varepsilon)$ , and  $\overline{\phi}_t(x, t) + \overline{\phi}_x(x, t)$  through  $O(\varepsilon\delta^{1/2})$  to our order of approximation, and the domain of validity of our distinguished limit encompasses  $F^{-4} \ll \delta \ll 1$ , we have not explicitly written out terms which are automatically of higher order. We have also not at this point done any expansion of the term involving  $P^{(p)}$ . Some caution would need to be used here since  $P^{(p)}$  will be peaked at values  $\xi \sim nk\delta^{-1}$ , and it seems easier to do this asymptotic analysis later with explicit expressions.

Now, in each equation, the first term in the last expression is the pure ship wave component:

$$\begin{aligned}
\bar{\phi}^{(s)}(x) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} P^{(s)}(\xi) d\xi, \\
\bar{\phi}_x^{(s)}(x) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} (i\xi\bar{\mathcal{X}}_x^{(s)}(x) + |\xi|\bar{\mathcal{Y}}_x^{(s)}(x)) P^{(s)}(\xi) d\xi, \\
\bar{\phi}_y^{(s)}(x) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} (i\xi\bar{\mathcal{X}}_y^{(s)}(x) + |\xi|\bar{\mathcal{Y}}_y^{(s)}(x)) P^{(s)}(\xi) d\xi.
\end{aligned} \tag{64}$$

Subtracting these pure ship wave terms out, we obtain the following expressions for the ocean wave perturbations to the value and derivatives of the fluid potential at the surface:

$$\begin{aligned}
\bar{\phi}^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}(x, t)} P^{(p)}(\xi, t) d\xi \\
&\quad + \delta^{-1/2}F^{-2} \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} i\xi\bar{\mathcal{X}}^{(p)}(x, t) P^{(s)}(\xi) d\xi + O(\varepsilon\delta^{-1/2}F^{-4}), \\
\bar{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}(x)} (i\xi\bar{\mathcal{X}}_x(x) + |\xi|\bar{\mathcal{Y}}_x(x)) P^{(p)}(\xi, t) d\xi \\
&\quad + \delta^{-1/2}F^{-2} \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} (i\xi\bar{\mathcal{X}}_x^{(p)}(x, t) + |\xi|\bar{\mathcal{Y}}_x^{(p)}(x, t)) P^{(s)}(\xi) d\xi \\
&\quad + O(\delta^{-1/2}F^{-2}) + O(\delta^{-3/2}\varepsilon F^{-4}), \\
\bar{\phi}_y^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}(x)} (i\xi\bar{\mathcal{X}}_y(x) + |\xi|\bar{\mathcal{Y}}_y(x)) P^{(p)}(\xi, t) d\xi \\
&\quad + \delta^{-1/2}F^{-2} \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}^{(s)}(x)} (i\xi\bar{\mathcal{X}}_y^{(p)}(x, t) + |\xi|\bar{\mathcal{Y}}_y^{(p)}(x, t)) P^{(s)}(\xi) d\xi \\
&\quad + O(\delta^{-1/2}F^{-2}) + O(\delta^{-3/2}\varepsilon F^{-4}),
\end{aligned} \tag{65}$$

$$\begin{aligned}
\overline{\phi}_t^{(p)}(x, t) + \overline{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi\bar{\mathcal{X}}(x,t)} \frac{\partial P^{(p)}(\xi, t)}{\partial t} d\xi \\
&+ \delta^{-1/2} F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} \left( i\xi(\bar{\mathcal{X}}_t(x, t) + \bar{\mathcal{X}}_x(x, t)) \right. \\
&\quad \left. + |\xi|(\bar{\mathcal{Y}}_t(x, t) + \bar{\mathcal{Y}}_x(x, t)) \right) P^{(p)}(\xi) d\xi \\
&+ \delta^{-1/2} F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} \left( i\xi(\bar{\mathcal{X}}_t^{(p)}(x, t) + \bar{\mathcal{X}}_x^{(p)}(x, t)) \right. \\
&\quad \left. + |\xi|(\bar{\mathcal{Y}}_t^{(p)}(x, t) + \bar{\mathcal{Y}}_x^{(p)}(x, t)) \right) P^{(s)}(\xi) d\xi \\
&+ \delta^{-1/2} F^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} i\xi \bar{\mathcal{X}}^{(p)}(x, t) (i\xi(\bar{\mathcal{X}}_x^{(s)}(x) + |\xi| \bar{\mathcal{Y}}_x^{(s)}(x)) P^{(s)}(\xi) d\xi \\
&+ O(\delta^{-1} \varepsilon F^{-4})
\end{aligned}$$

It can be checked using the computation technique for the conformal mapping functions discussed in Subsubsection 6.5.2 that

$$\begin{aligned}
\bar{\mathcal{X}}_x^{(s)}(x, t) &= 1 + O(F^{-2}), & \bar{\mathcal{Y}}_x^{(s)}(x, t) &= O(F^{-2}), \\
\bar{\mathcal{X}}_y^{(s)}(x, t) &= O(F^{-2}), & \bar{\mathcal{Y}}_y^{(s)}(x, t) &= 1 + O(F^{-2}).
\end{aligned}$$

Using these facts along with the expressions (65) for the surface potential derivatives associated to the ship wave (and analogous ones for the second partial derivatives),

we have

$$\begin{aligned} \bar{\phi}^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} P^{(p)}(\xi, t) d\xi \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} i\xi \bar{\mathcal{X}}^{(p)}(x, t) P^{(s)}(\xi) d\xi \\ &\quad + O(\varepsilon \mathbf{F}^{-2}), \end{aligned} \tag{66a}$$

$$\begin{aligned} \bar{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} (i\xi \bar{\mathcal{X}}_x(x) + |\xi| \bar{\mathcal{Y}}_x(x)) P^{(p)}(\xi, t) d\xi \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}_x^{(p)}(x, t) \bar{\phi}_x^{(s)}(x) + \bar{\mathcal{Y}}_x^{(p)}(x, t) \bar{\phi}_y^{(s)}(x) \right) \\ &\quad + O(\delta^{-1/2} \mathbf{F}^{-2}) + O(\delta^{-3/2} \mathbf{F}^{-4}), \end{aligned}$$

$$\begin{aligned} \bar{\phi}_y^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} (i\xi \bar{\mathcal{X}}_y(x) + |\xi| \bar{\mathcal{Y}}_y(x)) P^{(p)}(\xi, t) d\xi \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}_y^{(p)}(x, t) \bar{\phi}_x^{(s)}(x) + \bar{\mathcal{Y}}_y^{(p)}(x, t) \bar{\phi}_y^{(s)}(x) \right) \\ &\quad + O(\delta^{-1/2} \mathbf{F}^{-2}) + O(\delta^{-3/2} \mathbf{F}^{-4}), \end{aligned} \tag{66b}$$

$$\begin{aligned} \bar{\phi}_t^{(p)}(x, t) + \bar{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi \bar{\mathcal{X}}(x, t)} \frac{\partial P^{(p)}(\xi, t)}{\partial t} d\xi \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} \left( i\xi (\bar{\mathcal{X}}_t(x, t) + \bar{\mathcal{X}}_x(x, t)) \right. \\ &\quad \quad \left. + |\xi| (\bar{\mathcal{Y}}_t(x, t) + \bar{\mathcal{Y}}_x(x, t)) \right) P^{(p)}(\xi) d\xi \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \left( (\bar{\mathcal{X}}_t^{(p)}(x, t) + \bar{\mathcal{X}}_x^{(p)}(x, t)) \bar{\phi}_x^{(s)}(x) \right) \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \left( (\bar{\mathcal{Y}}_t^{(p)}(x, t) + \bar{\mathcal{Y}}_x^{(p)}(x, t)) \bar{\phi}_y^{(s)}(x) \right) \\ &\quad + \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}^{(p)}(x, t) \bar{\phi}_{xx}^{(s)}(x) \right) + O(\delta^{-1} \mathbf{F}^{-4}). \end{aligned} \tag{66c}$$

This is a good point to leave these expressions in general form; we can process them further for particular examples once we have explicit expansions for the conformal mapping functions  $\mathcal{X}$  and  $\mathcal{Y}$  and the surface potential  $\bar{\phi}^{(p)}(x, t)$ . We can then use the evaluation of Eq. (58c) on the surface to write down an expansion for  $P(\xi, t)$  in terms of the expansion coefficients of these functions, and then use Eqs. (66) to obtain an asymptotic expansion for the surface derivatives of the potential purely in terms of surface quantities.

### 6.2.3 Summary of Governing Equations of Ocean Wave Packet Perturbation

The evolution equations for the surface functions are:

$$0 = \eta_t^{(p)}(x, t) + \eta_x^{(p)}(x, t) - \delta^{1/2} \overline{\phi_y^{(p)}}(x, t) + \delta^{1/2} \varepsilon F^{-2} \overline{\phi_x^{(p)}}(x, t) \eta_x^{(p)}(x, t) \quad (67a)$$

$$+ F^{-2} \left[ \overline{\phi_x^{(s)}}(x) \eta_x^{(p)}(x, t) + \delta^{1/2} \overline{\phi_x^{(p)}}(x, t) \eta_x^{(s)}(x) \right],$$

$$0 = \overline{\phi_t^{(p)}}(x, t) + \overline{\phi_x^{(p)}}(x, t) + \delta^{-1/2} \eta^{(p)}(x, t) \quad (67b)$$

$$+ F^{-2} \left( \overline{\phi_x^{(s)}}(x) \overline{\phi_x^{(p)}}(x, t) + \overline{\phi_y^{(s)}}(x) \overline{\phi_y^{(p)}}(x, t) \right)$$

$$+ \frac{\varepsilon \delta^{-1} F^{-2}}{2} \left( (\overline{\phi_x^{(p)}}(x, t))^2 + (\overline{\phi_y^{(p)}}(x, t))^2 \right).$$

These equations are initialized at  $t = 0$  by a wave packet:

$$\eta^{(p)}(x, t = 0) = \left[ A^{(0)}(x) e^{i\delta^{-1}x} + \text{c.c.} \right], \quad (67c)$$

This system of equations is not closed because the derivatives  $\overline{\phi_x^{(p)}}$ ,  $\overline{\phi_t^{(p)}}$ , and  $\overline{\phi_y^{(p)}}$  of the potential on the surface cannot be expressed simply in terms of the surface value of the potential, both because the surface is moving and because the surface is not perfectly aligned along the  $x$  or  $y$  direction.

To obtain these surface derivatives, we use a conformal map between the fluid region and the lower half plane, which requires the solution of the equations expressed in flattened variables:

$$\frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y} - F^{-2} \eta_X \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y} = 0, \quad (67d)$$

$$\frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial X} + \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y} - F^{-2} \eta_X \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y} = 0 \text{ for } Y < 0, \quad (67e)$$

$$\tilde{\mathcal{Y}}^\sharp(X, Y = 0, T) = 0, \quad (67f)$$

$$\lim_{Y \rightarrow -\infty} \left( \tilde{\mathcal{Y}}^\sharp(X, Y, T) - (Y + F^{-2} \eta(X, T)) \right) = 0, \quad (67g)$$

which we then re-express in terms of original physical variables:

$$X = x,$$

$$Y = y - F^{-2} \eta(x, t),$$

$$T = t, \quad (67h)$$

$$\mathcal{X}(x, y, t) = \tilde{\mathcal{X}}^\sharp(X, Y, T),$$

$$\mathcal{Y}(x, y, t) = \tilde{\mathcal{Y}}^\sharp(X, Y, T).$$

We then evaluate these conformal mapping functions at the surface, decomposing them into ship wave and ocean wave perturbation components as in Eq. (63).

Then we compute the function  $P^{(p)}(\xi, t)$  from the equation

$$\begin{aligned} \bar{\phi}^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} P^{(p)}(\xi, t) d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x))} i\xi \bar{\mathcal{X}}^{(p)}(x, t) P^{(s)}(\xi) d\xi + O(\varepsilon \mathbf{F}^{-2}), \end{aligned} \quad (67i)$$

and then compute the derivatives of the surface potential:

$$\begin{aligned} \bar{\phi}^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} P^{(p)}(\xi, t) d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} i\xi \bar{\mathcal{X}}^{(p)}(x, t) P^{(s)}(\xi) d\xi \\ &+ O(\varepsilon \mathbf{F}^{-2}), \end{aligned} \quad (67j)$$

$$\begin{aligned} \bar{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} (i\xi \bar{\mathcal{X}}_x(x) + |\xi| \bar{\mathcal{Y}}_x(x)) P^{(p)}(\xi, t) d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}_x^{(p)}(x, t) \bar{\phi}_x^{(s)}(x) + \bar{\mathcal{Y}}_x^{(p)}(x, t) \bar{\phi}_y^{(s)}(x) \right) \\ &+ O(\delta^{-1/2} \mathbf{F}^{-2}) + O(\delta^{-3/2} \mathbf{F}^{-4}), \end{aligned}$$

$$\begin{aligned} \bar{\phi}_y^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} (i\xi \bar{\mathcal{X}}_y(x) + |\xi| \bar{\mathcal{Y}}_y(x)) P^{(p)}(\xi, t) d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}_y^{(p)}(x, t) \bar{\phi}_x^{(s)}(x) + \bar{\mathcal{Y}}_y^{(p)}(x, t) \bar{\phi}_y^{(s)}(x) \right) \\ &+ O(\delta^{-1/2} \mathbf{F}^{-2}) + O(\delta^{-3/2} \mathbf{F}^{-4}), \end{aligned} \quad (67k)$$

$$\begin{aligned} \bar{\phi}_t^{(p)}(x, t) + \bar{\phi}_x^{(p)}(x, t) &= \int_{-\infty}^{\infty} e^{i\xi \bar{\mathcal{X}}(x, t)} \frac{\partial P^{(p)}(\xi, t)}{\partial t} d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}(x))} \left( i\xi (\bar{\mathcal{X}}_t(x, t) + \bar{\mathcal{X}}_x(x, t)) \right. \\ &\quad \left. + |\xi| (\bar{\mathcal{Y}}_t(x, t) + \bar{\mathcal{Y}}_x(x, t)) \right) P^{(p)}(\xi) d\xi \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \left( (\bar{\mathcal{X}}_t^{(p)}(x, t) + \bar{\mathcal{X}}_x^{(p)}(x, t)) \bar{\phi}_x^{(s)}(x) \right) \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \left( (\bar{\mathcal{Y}}_t^{(p)}(x, t) + \bar{\mathcal{Y}}_x^{(p)}(x, t)) \bar{\phi}_y^{(s)}(x) \right) \\ &+ \delta^{-1/2} \mathbf{F}^{-2} \left( \bar{\mathcal{X}}^{(p)}(x, t) \bar{\phi}_{xx}^{(s)}(x) \right) + O(\delta^{-1} \mathbf{F}^{-4}). \end{aligned} \quad (67l)$$

This closes the surface evolution equations Eqs. (67a) and (67b).

### 6.3 Fast and Intermediate Rescaled Variables

As usual in mathematical problems with multiple small parameters, we must separately study distinguished limits in which the orders of magnitude of the small parameters are related in some way. Using the distinguished limits relating  $\delta$ ,  $F$ , and  $\varepsilon$  discussed in Subsection 4.5, we base our asymptotic analysis on the single small parameter  $F^{-1}$ . It will be useful to introduce rescaled space and time variables [19, Sec. 1.3] to track the variation of the ocean wave packet perturbation which is small-scale (or “fast”) compared to the time scale of the ship:

$$\begin{aligned}\tilde{x} &= F^\beta x, & \tilde{X} &= F^\beta X, \\ \tilde{y} &= F^\beta y, & \tilde{Y} &= F^\beta Y, \\ \tilde{t} &= F^\beta t, & \tilde{T} &= F^\beta T, \\ t^* &= F^{\beta/2} t, & T^* &= F^{\beta/2} T,\end{aligned}$$

when  $\delta$  and  $F$  are linked by  $\delta = k^{-1}F^{-\beta}$ . The fast space variables ( $\tilde{x}$ ,  $\tilde{X}$ ,  $\tilde{y}$ , and  $\tilde{Y}$ ) are generated directly by the initial data (67c), while the fast time ( $\tilde{t}$ ,  $\tilde{T}$ ) and intermediately fast time ( $t^*$ ,  $T^*$ ) variables will be generated because an initial packet of waves  $e^{i\delta^{-1}x}$  will evolve to first approximation (when the weak nonlinearity is completely neglected) as a wavepacket proportional to  $e^{i\delta^{-1}(\tilde{x}-\tilde{t})\mp i\delta^{-1/2}t^*}$  [1, Sec. 3.2]. The fast time variables ( $\tilde{t}$ ,  $\tilde{T}$ ) correspond to the time scale of motion for the ocean wave packet in the frame of the ship, while the intermediate time variables ( $t^*$ ,  $T^*$ ) correspond to the time scale over which the ocean wave packet propagates relative to the fixed ocean. It is readily checked that no other fast variables need to be introduced into the analysis.

### 6.4 General Dependence of Solution on Fast and Intermediate Time and Space Variables

Because we are allowing nonlinear effects in the subsequent analysis, we must allow not only the amplitude but the shape of the ocean waves to change as they encounter the ship wave. These will generate additional terms in the expansions for the surface displacement and velocity potential of the ocean wave which are not proportional to the incoming sinusoidal variation  $e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*}$ . We could just proceed by positing general asymptotic expansions of all functions appearing in (67) in powers of  $F^{-1}$ , substituting them into the governing equations, then solving them order by order. This rapidly becomes tedious. We elect instead to first write down the general form the solution will take, and then simply solve for the coefficients (depending only on slow variables) in this general expansion. We hope the presentation will be clearer because we will be able to present a general algorithm for obtaining the coefficients from the outset. The algebraic details will then hopefully be less distracting.

Two features of our mathematical problem make it possible for us to write down immediately how the ocean wave functions  $\eta^{(p)}(x, t)$  and  $\phi^{(p)}(x, y, t)$  depend on the fast and intermediate space-time variables:

- All nonlinearities in the fluid equations are polynomials (in particular quadratic polynomials),
- The environment experienced by the ocean waves as they encounter the ship wave depends only on the slow variables ( $x$ ,  $y$ , and  $t$ ), but not on the intermediate  $t^*$  or fast variables  $\tilde{x}$ ,  $\tilde{y}$ , and  $\tilde{t}$ .

Some thought about what kind of terms can be generated at arbitrary order by a systematic expansion, given these properties of the equations, leads to the conclusion

that the general solution for ocean wave perturbation must have the form:

$$\eta^{(p)} = \sum_{\ell=0}^2 \sum_{n=-N(\ell)}^{N(\ell)} \sum_{\mu \in \mathcal{M}} F^{-\ell} \eta_{n\mu\ell}(x, t) e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*} + O(F^{-3}), \quad (68a)$$

$$\bar{\phi}^{(p)} = \sum_{\ell=0}^2 \sum_{n=-N(\ell)}^{N(\ell)} \sum_{\mu \in \mathcal{M}} F^{-\ell} \bar{\phi}_{n\mu\ell}(x, t) e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*} + O(F^{-3}), \quad (68b)$$

$$\varepsilon^{-1} \delta^{-1/2} \phi = \varepsilon^{-1} \delta^{-1/2} \int_{-\infty}^{\infty} e^{i(\xi)\mathcal{X}} e^{|\xi|\mathcal{Y}} \phi^{(s)}(\xi, t) d\xi \quad (68c)$$

$$+ \sum_{\ell=0}^2 \sum_{n=-N(\ell)}^{N(\ell)} \sum_{\mu \in \mathcal{M}} F^{-\ell} \int_{-\infty}^{\infty} e^{i(n\delta^{-1}k+\xi)\mathcal{X}} e^{-ink\tilde{t} \mp i\mu k^{1/2} t^*} e^{|n\delta^{-1}k+\xi|\mathcal{Y}} P_{n\mu\ell}(\xi, t) d\xi + O(F^{-3}),$$

$$\mathcal{X} = x + \sum_{\ell=0}^4 \sum_{n=-N(\ell)}^{N(\ell)} \sum_{m=0}^{M(\ell)} \sum_{\mu \in \mathcal{M}} F^{-\ell} \mathcal{X}_{nm\mu\ell}(x, y, t) e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*} e^{mk(\tilde{y}-\eta)} + O(F^{-5}), \quad (68d)$$

$$\mathcal{Y} = y + \sum_{\ell=0}^4 \sum_{n=-\tilde{N}(\ell)}^{\tilde{N}(\ell)} \sum_{m=0}^{\tilde{M}(\ell)} \sum_{\mu \in \tilde{\mathcal{M}}} F^{-\ell} \mathcal{Y}_{nm\mu\ell}(x, y, t) e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*} e^{mk(\tilde{y}-\eta)} + O(F^{-5}), \quad (68e)$$

$$\tilde{\mathcal{X}}^{\sharp} = X + \sum_{\ell=0}^4 \sum_{n=-\tilde{N}(\ell)}^{\tilde{N}(\ell)} \sum_{m=0}^{\tilde{M}(\ell)} \sum_{\mu \in \tilde{\mathcal{M}}} F^{-\ell} \tilde{\mathcal{X}}_{nm\mu\ell}^{\sharp}(X, Y, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{mk\tilde{Y}} + O(F^{-5}), \quad (68f)$$

$$\tilde{\mathcal{Y}}^{\sharp} = Y + \sum_{\ell=0}^4 \sum_{n=-\tilde{N}(\ell)}^{\tilde{N}(\ell)} \sum_{m=0}^{\tilde{M}(\ell)} \sum_{\mu \in \tilde{\mathcal{M}}} F^{-\ell} \tilde{\mathcal{Y}}_{nm\mu\ell}^{\sharp}(X, Y, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{mk\tilde{Y}} + O(F^{-5}), \quad (68g)$$

where:

- $N(\ell)$  and  $M(\ell)$  are nonnegative integer functions of  $\ell$  which generally increase with  $\ell$ . They denote how many superharmonics of the wave are generated at each order. The  $\ell < 0$  components arise only because of contributions to  $\phi$  from the pure ship wave, and consequently  $N(\ell) = M(\ell) = 0$  for  $\ell < 0$ .
- $\mathcal{M}$  is, for each  $\ell \geq 0$ , a finite set of real numbers.
- We will refer to the coefficients  $\{\eta_{n\mu\ell}\}_{n,\mu,\ell}$  as *amplitude coefficients*, noting that the physical amplitude describing the height of the ocean waves is really

determined by the *magnitudes*  $\{|\eta_{m\mu\ell}|\}_{n,\mu,\ell}$  of the amplitude coefficients for the surface elevation. The amplitude coefficients obey the following complex conjugacy relations due to the reality of the functions  $\eta^{(p)}(x, t)$  and  $\phi^{(p)}(x, y, t)$ :

$$\begin{aligned}
\eta_{m\mu\ell}(x, t) &= \eta_{-n-\mu\ell}^*(x, t), \\
\bar{\phi}_{n\mu\ell}(x, t) &= \bar{\phi}_{-n-\mu\ell}^*(x, t), \\
P_{n\mu\ell}(\xi, t) &= P_{-n-\mu\ell}^*(-\xi, t), \\
\mathcal{X}_{nm\mu\ell}(x, y, t) &= \mathcal{X}_{-nm-\mu\ell}^*(x, y, t), \\
\mathcal{Y}_{nm\mu\ell}(x, y, t) &= \mathcal{Y}_{-nm-\mu\ell}^*(x, y, t), \\
\tilde{\mathcal{X}}_{nm\mu\ell}^\#(x, y, t) &= \tilde{\mathcal{X}}_{-nm-\mu\ell}^{\#*}(x, y, t), \\
\tilde{\mathcal{Y}}_{nm\mu\ell}^\#(x, y, t) &= \tilde{\mathcal{Y}}_{-nm-\mu\ell}^{\#*}(x, y, t).
\end{aligned} \tag{69}$$

- The amplitude coefficients for the conformal mapping functions in physical and flattened variables are related directly by the definition of these variables (67h):

$$\begin{aligned}
\tilde{\mathcal{X}}_{nm\mu\ell}^\# &= \tilde{\mathcal{X}}_{nm\mu\ell}^\#, \\
\tilde{\mathcal{Y}}_{nm\mu\ell}^\# &= \begin{cases} \mathcal{Y}_{nm\mu\ell} & \text{for } m \neq 0, \\ \mathcal{Y}_{nm\mu\ell} + \eta_{n\mu\ell-\alpha} & \text{for } m = 0, (\ell, n, \mu) \neq (\alpha, 0, 0), \\ \mathcal{Y}_{0002} + \eta^{(s)} & \text{for } (\ell, n, m, \mu) = (\alpha, 0, 0, 0), \end{cases}
\end{aligned}$$

when  $\varepsilon$  is linked to  $F$  by  $\varepsilon = aF^{-\alpha}$ .

- We have indicated the terms retained for the 200 knot ship speed calculations ( $\delta = F^{-2}$ ,  $\varepsilon = F^{-1}$ ). Similar expansions apply for the 100 knot ship speed calculation, except the conformal mapping functions must be expanded through fifth order with  $O(F^{-6})$  error. (The calculations however are of similar or lesser complexity because the difficult terms emerge one order later than in the 200 knot calculation).

We record here some important remarks regarding these expansions:

- Note that we do not expand the ship wave functions  $\eta^{(s)}$  or  $\bar{\phi}^{(s)}$  in power series of  $F^{-1}$ , because this would create extra clutter in our expansions. We will instead simply express our equations for the ocean wave packet in terms of the *exact* ship wave functions  $\eta^{(s)}$  and  $\bar{\phi}^{(s)}$ . Since the resulting equations will only have accuracy to  $O(F^{-1})$ , we can without loss *at that point* substitute in an asymptotic approximation for the ship waves which is  $O(F^{-1})$  accurate.

Since the nonlinearities in Eq. (59) appear only at  $O(F^{-2})$ , and there is no need for multiple scales analysis on this pure ship wave problem, it suffices to take the ship wave solution for the *linearized* version of that system, which was computed in [10].

- The expansion for  $P$  has been renormalized by  $\varepsilon^{-1}\delta^{-1/2}$  so that the  $O(F^{-\ell})$  terms in this expansion will correspond to the  $O(F^{-\ell})$  terms in the ocean wave packet surface functions  $\eta^{(p)}$  and  $\bar{\phi}^{(p)}$ .
- By comparing the asymptotic expansion (68c) for the potential function  $P$  in the bulk of the ocean with the expression (67i), we note that the integration variable  $\xi$  is shifted so that

$$P^{(p)}(\xi, t) = \sum_{\ell=0}^2 \sum_{n=-N(\ell)}^{N(\ell)} \sum_{\mu \in \mathcal{M}} F^{-\ell} e^{-ink\tilde{T} \mp i\mu k^{1/2} T^*} P_{n\mu\ell}(\xi - n\delta^{-1}k, t) + O(F^{-3}).$$

The functions  $P_{n\mu\ell}(\xi, t)$  are determined by the slow variation of the wave packet and so are centered on order unity values of  $\xi$ ; we see then that  $P^{(p)}(\xi, t)$  is expressed as a superposition of contributions centered at  $\xi = n\delta^{-1}k$ , which corresponds to the fast variation of the waves in the ocean wave packet (and their superharmonics generated by nonlinear interaction).

- At first blush, it may appear that  $\mu$  should only take integer values, but other irrational frequencies will arise due to the need to match with the initial wave packet data (67c). So for example, the equations (67a) and (67b) may directly generate a term in the solution proportional to  $F^{-\ell} e^{ink(\tilde{X}-\tilde{T}) \mp ink^{1/2} T^*}$ , but this does not match the spatial profile of the initial data (67c) for  $n \neq 1$ . Consequently, one must add in this case a complementary homogenous solution to the equations (67a) and (67b) to match to the initial data. These homogenous solutions are just free Stokes waves, with fast space-time variation given by  $e^{ink(\tilde{X}-\tilde{T}) \mp in^{1/2} k^{1/2} T^*}$ , corresponding to  $\mu = n^{1/2}$ . The nonlinear interaction between homogenous and forced inhomogenous solutions in the asymptotic hierarchy will generate ever more values of  $\mu$ , though the number is finite at any order  $\ell$ . Fortunately, a careful check will demonstrate that the  $\mu \neq n$  terms will not be relevant for our analysis to our desired accuracy.
- We have truncated the expansions at the orders we will need them. Further complications will emerge in higher order terms of the conformal mapping functions. This is because we will find that that the terms in the expansion of the conformal mapping functions (such as  $\mathcal{X}$ ) cannot always be factorized into a slowly varying amplitude coefficient  $\mathcal{X}_{nm\mu\ell}(x, y, t)$  and a rapidly varying factor  $e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{mk\tilde{Y}}$ . The fast and slow dependence is intermingled

for the reasons discussed in Appendix A.2, and this intermingling occurs even in some of the relevant (and retained) terms in our expansion. However, as explained in Appendix A.3, we can *approximately* factorize the more precise expression for the terms in the conformal mapping expansion *for the purposes of evaluating the terms and their derivatives at the surface* with a relative error of  $O(F^{-3})$ . We will see that these intermingled terms will only occur at  $\ell \geq 3$ , so that the error in factorization will be  $O(F^{-6})$ , and therefore negligible.

- For the same reason as discussed in the previous point, in Eq. (68c) we can and will replace the factor  $e^{n\delta^{-1}k+\xi|\mathcal{Y}}$  in terms with  $n \neq 0$  by  $e^{(n\delta^{-1}k+\xi)\mathcal{Y}}$ , which permits the terms in the expansion of  $\phi$  to be written in as products of factors depending only on slow and on fast/intermediate space-time variables. (The  $n = 0$  terms don't involve any fast variables so are already trivially in such factorized form.) This replacement is permissible for the purpose of evaluating surface terms, at the cost of a relative error  $O(F^{-3})$ . Since the dependence on fast variables only begins at order  $\ell \geq 0$ , this factorization error is negligibly small compared to the retained terms. If we were to pursue a more accurate approximation, we would be forced to work with the more precise expressions!

## 6.5 Solution of the Amplitude Coefficients

We describe now how to solve for the amplitude coefficients in the expansions (68). The general approach is naturally to substitute these expansions into the governing equations (67), reorganize both sides into clean expansions in powers of  $F^{-1}$  and dependence on the fast and intermediate space-time variables, and then equate coefficients of these terms on both sides of the equations. We describe now what kind of equations for the amplitude coefficients this procedure will produce. In the following, we will take  $\delta = k^{-1}F^{-2}$ , which is assumed in both distinguished limits for 100 knot and 200 knot ship speeds.

### 6.5.1 Solution of Amplitude Coefficients in Surface Equations

By substituting the expansions (68) into the surface wave equations, one finds that each amplitude coefficient  $\eta_{n\mu\ell}(x, t)$  and  $\bar{\phi}_{n\mu\ell}(x, t)$  appears to leading order as

$$F^{-\ell+1} \left( \mp i\mu\sqrt{k}\eta_{n\mu\ell}(x, t) - nk\bar{\phi}_{n\mu\ell} \right) e^{ink(\tilde{x}-\tilde{t})\mp i\mu k^{1/2}t^*} + O(F^{-\ell})$$

in the kinematic equation (67a) and as

$$F^{-\ell+1} \left( \mp i\mu\sqrt{k}\bar{\phi}_{n\mu\ell}(x, t) + \eta_{n\mu\ell}(x, t) \right) e^{ink(\tilde{x}-\tilde{t})\mp i\mu k^{1/2}t^*} + O(F^{-\ell})$$

in the Bernoulli equation (67b). Suppose that other terms of the form

$$K_{n\mu\ell-1}^{(1)}(x, t)F^{-\ell+1}e^{ink(\tilde{x}-\tilde{t})\mp i\mu k^{1/2}t^*}, \quad K_{n\mu\ell-1}^{(2)}(x, t)F^{-\ell+1}e^{ink(\tilde{x}-\tilde{t})\mp i\mu k^{1/2}t^*},$$

appear respectively in these equations; the functions  $K_{n\mu\ell-1}^{(1)}$  and  $K_{n\mu\ell-1}^{(2)}$  only involve amplitude coefficients which already appeared at an earlier order (smaller  $\ell$ ). Then we can match terms by solving the linear system:

$$\mp i\mu\sqrt{k}\eta_{n\mu\ell} - nk\bar{\phi}_{n\mu\ell} = K_{n\mu\ell-1}^{(1)}, \quad (70a)$$

$$\eta_{n\mu\ell} + \mp i\mu\sqrt{k}\bar{\phi}_{n\mu\ell} = K_{n\mu\ell-1}^{(2)}. \quad (70b)$$

This system is solvable provided the determinant is nonzero:

$$n \neq \mu^2$$

If  $n = \mu^2$  then this system cannot be solved in general. Rather we had better arrange that the lower order terms (smaller  $\ell$ ) are chosen so that the terms  $K_{n\mu\ell-1}^{(1)}$  and  $K_{n\mu\ell-1}^{(2)}$  fall in the solvable linear subspace of Eq. (70). This will in practice involve dynamical equations for  $\eta_{n\mu\ell-1}$  and  $\bar{\phi}_{n\mu\ell-1}$ , as we shall see below. Assuming this can be done, then Eq. (70) sets one linear relationship between  $\eta_{n\mu\ell}$  and  $\bar{\phi}_{n\mu\ell}$ . The other equation which determines these functions will occur by enforcing that the analogous system (70) for  $\ell \rightarrow \ell+1$  is solvable. This amounts to balancing terms in the equations which are proportional to  $F^{-\ell} e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*}$ . This second equation for  $\eta_{n\mu\ell}$  and  $\bar{\phi}_{n\mu\ell}$  will be dynamical, involving their space and time derivatives. It is these equations which will allow us to satisfy the initial conditions. Indeed, if there are modes with  $n \neq \mu^2$  generated by inhomogeneities in the equations, their amplitude coefficients are directly prescribed, without regard to the initial data; any mismatch so created must be compensated by a mode with the same value of  $n$  and  $\mu = \sqrt{n}$ . All the above statements work fine except for the case  $n = \mu = 0$ , which corresponds to a ‘‘mean flow’’ mode (since there is no fast dependence on  $x$ ).

For these mean flow modes ( $n = \mu = 0$ ), we must arrange that  $K_{00\ell-1}^{(1)} = 0$  and then Eq. (70) would appear to prescribe a value for  $\eta_{00\ell}$ , but this would spoil the flexibility needed to arrange that  $K_{00\ell}^{(1)} = 0$  at the next order. Indeed,  $K_{00\ell-1}^{(1)}$  is the inhomogeneity in the kinematic boundary condition for which solvability conditions result in a dynamical equation for  $\eta_{00\ell-1}$ . Therefore,  $\eta_{00\ell}$  is really determined by the need to zero out the inhomogeneity  $K_{00\ell}^{(1)}$  at the next order. How can we then satisfy Eq. (70b)? The answer is that Eq. (70) at the previous order  $O(F^{\ell-2})$  put no constraint on  $\bar{\phi}_{00\ell-1}$ , so that *it* can be used to balance the inhomogeneity  $K_{00\ell-1}^{(2)}$  and whatever term  $\eta_{00\ell}$  is needed to kill off  $K_{00\ell}^{(1)}$ . Like the case  $n = \mu^2 \neq 0$ , both  $\eta_{00\ell}$  and  $\bar{\phi}_{00\ell}$  will satisfy dynamical equations involving their (slow) space and time derivatives. But the twist is that the equation for  $\bar{\phi}_{00\ell}$  will involve  $\eta_{00\ell+1}$ , which will be solved one order later.

The above discussion is useful in indicating which modes (parametrized by  $n$  and  $\mu$ ) will be generated at each order  $\ell$ . We can summarize the rule as follows. An inhomogeneity of the form  $F^{-\ell} e^{ink(\tilde{x}-\tilde{t}) \mp i\mu k^{1/2} t^*}$  appearing in either the kinematic (67a) or Bernoulli (67b) equations will generate the following amplitude coefficients:

- $\bar{\phi}_{n\mu\ell+1}$  and  $\eta_{n\mu\ell+1}$  if  $n \neq \mu^2$ ,
- $\bar{\phi}_{n\mu\ell}$  and  $\eta_{n\mu\ell}$  if  $n = \mu^2 \neq 0$ ,
- $\bar{\phi}_{00\ell-1}$  and  $\eta_{00\ell}$  if  $n = \mu = 0$  and the inhomogeneity appears in the Eq. (67a),
- $\bar{\phi}_{00\ell}$  and  $\eta_{00\ell+1}$  if  $n = \mu = 0$  and the inhomogeneity appears in the Eq. (67b)

Also, the generation of any term  $\bar{\phi}_{n\mu\ell}$  or  $\eta_{n\mu\ell}$  with  $n \neq \mu^2$  will generally induce terms with the same values of  $\ell$  and  $n$ , but with  $\mu = \pm\sqrt{n}$ , in order to satisfy the initial conditions.

### 6.5.2 Solution of Coefficients in Conformal Mapping Functions

Turning now to the conformal mapping equations (67d)–(67g), we find upon substitution of the asymptotic expansions (68f) and (68g) that the coefficients of the conformal mapping expansion appear to leading order as

$$F^{-\ell+2} \left[ ink\tilde{\mathcal{X}}_{nm\mu\ell}^\#(X, Y, T) - mk\tilde{\mathcal{Y}}_{nm\mu\ell}^\#(X, Y, T) \right] e^{ink(\tilde{X}-\tilde{T})\mp i\mu k^{1/2}T^*} + O(F^{1-\ell})$$

and

$$F^{-\ell+2} \left[ ink\tilde{\mathcal{Y}}_{nm\mu\ell}^\#(X, Y, T) + mk\tilde{\mathcal{X}}_{nm\mu\ell}^\#(X, Y, T) \right] e^{ink(\tilde{X}-\tilde{T})\mp i\mu k^{1/2}T^*} + O(F^{1-\ell}).$$

Therefore if other terms of the form

$$K_{nm\mu\ell-2}^{(1)}(X, Y, T)F^{-\ell+2}e^{ink(\tilde{X}-\tilde{T})\mp i\mu k^{1/2}T^*}, \quad K_{nm\mu\ell-2}^{(2)}(X, Y, T)F^{-\ell+2}e^{ink(\tilde{X}-\tilde{T})\mp i\mu k^{1/2}T^*}$$

appear, with  $K_{nm\mu\ell-2}^{(1)}$  and  $K_{nm\mu\ell-2}^{(2)}$  involving only amplitude coefficients with smaller values of  $\ell$ , then we can balance these terms by choosing  $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  by solving a linear system of the form:

$$ink\tilde{\mathcal{X}}_{nm\mu\ell}^\# - mk\tilde{\mathcal{Y}}_{nm\mu\ell}^\# = K_{nm\mu\ell-2}^{(1)}, \quad (71a)$$

$$mk\tilde{\mathcal{X}}_{nm\mu\ell}^\# + ink\tilde{\mathcal{Y}}_{nm\mu\ell}^\# = K_{nm\mu\ell-2}^{(2)} \quad (71b)$$

The solution of this system is possible provided that  $n \neq m$  so that the associated determinant is nonzero. If  $n = m$ , then this system is only solvable if  $K_{nm\mu\ell-2}^{(1)} - iK_{nm\mu\ell-2}^{(2)} = 0$ , in which case only one linear relation is enforced between  $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$ . The right hand side of the system (71b) can be made to satisfy  $K_{nm\mu\ell-2}^{(1)} - iK_{nm\mu\ell-2}^{(2)} = 0$  by noting that  $\tilde{\mathcal{X}}_{nm\mu\ell-2}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell-2}^\#$  have only one linear relation imposed on them from  $O(F^{-\ell+4})$  and using the remaining flexibility in their choice.  $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  will similarly be uniquely determined by a solvability condition at

$O(F^{-\ell})$ . The equations resulting from the solvability conditions for  $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  will be partial differential equations which can be used to satisfy the boundary condition (67f). The resulting solution for  $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  will not however satisfy the decay condition (67g)! For our present problem, we can still use these solutions to obtain approximations which are accurate through  $O(F^{-1})$ , which is as far as we compute in this report. The reason is explained in Appendix A.3.

Summarizing, then, an inhomogeneity proportional to  $F^{-\ell} e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*}$  in the conformal mapping equations (67d)–(67g) generates the following coefficients in the conformal mapping asymptotic expansions (68f) and (68g):

- $\tilde{\mathcal{X}}_{nm\mu\ell-2}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell-2}^\#$  if  $m \neq n$ ,
- $\tilde{\mathcal{X}}_{nm\mu\ell}^\#$  and  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  if  $m = n$ .

Also, the generation of any nonzero term  $\tilde{\mathcal{Y}}_{nm\mu\ell}^\#$  with  $m \neq n$  will in general require the inclusion of coefficients in the conformal mapping expansion with the same value of  $(\ell, n, \mu)$  but with  $m = n$ .

## 7 Nonlinear Asymptotic Analysis for 200 Knot Surfing Ship

The distinguished limit we impose for this singular perturbation theory is

$$\delta = k^{-1}F^{-2}, \quad \varepsilon = aF^{-1},$$

where  $a$  and  $k$  are constants held fixed in the asymptotic limit  $F \rightarrow \infty$ . The strength of the nonlinearity in the equation for the ocean wave perturbations is measured by  $\varepsilon$ . By setting it to scale as  $\varepsilon \sim F^{-1}$  in this analysis, we retain the effects of the nonlinear terms. Because they are small, however, we will be able to treat them perturbatively, using the methods of weakly nonlinear wave theory ([8, Sec. 8,11,19.1],[17, Ch. 5]).

The asymptotic expansion of the surface equations reads:

$$\begin{aligned} 0 = & F^2 \left( \eta_t^{(p)} + \eta_{\tilde{x}}^{(p)} \right) + F \left( \eta_{t^*}^{(p)} - \phi_y^{(p)} \right) + \left( \eta_t^{(p)} + \eta_x^{(p)} + \overline{\phi_x}^{(s)} \eta_x^{(s)} + a \phi_{\tilde{x}}^{(p)} \eta_{\tilde{x}}^{(p)} \right) \\ & + F^{-1} \left( -\phi_y^{(p)} + \eta_x^{(s)} \phi_{\tilde{x}}^{(p)} \right) + O(F^{-2}). \end{aligned} \tag{72a}$$

$$\begin{aligned} 0 = & F^2 \left( \phi_t^{(p)} + \phi_{\tilde{x}}^{(p)} \right) + F \left( \phi_{t^*}^{(p)} + \eta^{(p)} \right) \\ & + \left( \phi_t^{(p)} + \phi_x^{(p)} + \phi_x^{(s)} \phi_{\tilde{x}}^{(p)} + \phi_y^{(s)} \phi_y^{(p)} + a(\phi_{\tilde{x}}^{(p)})^2 + a(\phi_y^{(p)})^2 \right) + O(F^{-2}). \end{aligned}$$

The asymptotic expansion of the conformal mapping equations are:

$$\begin{aligned} 0 = & F^2 \left( \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial \tilde{X}} - \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial \tilde{Y}} \right) - Fa \eta_{\tilde{X}}^{(p)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial \tilde{Y}} + \left( \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y} - \eta_X^{(s)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial \tilde{Y}} \right) \\ & - F^{-1} a \left( \eta_{\tilde{X}}^{(p)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y} + \eta_X^{(p)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial \tilde{Y}} \right) \\ & - F^{-2} \eta_X^{(s)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y} - F^{-3} a \eta_X^{(p)} \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y}, \end{aligned} \tag{72b}$$

$$\begin{aligned} 0 = & F^2 \left( \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial \tilde{X}} + \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial \tilde{Y}} \right) - Fa \eta_{\tilde{X}}^{(p)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial \tilde{Y}} + \left( \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial X} + \frac{\partial \tilde{\mathcal{X}}^\sharp}{\partial Y} - \eta_X^{(s)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial \tilde{Y}} \right) \\ & - F^{-1} a \left( \eta_{\tilde{X}}^{(p)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y} + \eta_X^{(p)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial \tilde{Y}} \right) - F^{-2} \eta_X^{(s)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y} - F^{-3} a \eta_X^{(p)} \frac{\partial \tilde{\mathcal{Y}}^\sharp}{\partial Y}, \end{aligned}$$

We have retained only as many orders as we shall use in our subsequent analysis, in which the goal is to describe the changes in the amplitude of the wavepacket

of ocean waves through  $O(F^{-1})$  as it passes the ship wave. This wavepacket is prescribed through the initial condition (67c):

$$\eta^{(p)}(x, t = 0) = [A^{(0)}(x)e^{ikx} + \text{c.c.}],$$

with the potential  $\bar{\phi}^{(p)}$  chosen implicitly so that the ocean waves are all moving in the same direction.

## 7.1 Determination of Relevant Terms Through $O(F^{-1})$

By simple iteration, we can ascertain the terms in the expansion (68) which are generated through  $O(F^{-1})$ , and list these in Table 3. A formal consideration of the kinematic equation (67a) suggests at first that an order unity mean flow term  $\eta_{00}(x, t)$  and corresponding term  $\bar{\phi}_{0,-1}(x, t)$  might be generated by the order unity  $a\phi_{\bar{x}}^{(p)}\eta_{\bar{x}}^{(p)}$  term. But by taking a closer look at the inhomogeneity which appears to generate the  $\eta_{00}(x, t)$  term:

$$\begin{aligned} a\phi_{\bar{x}}^{(p)}\eta_{\bar{x}}^{(p)} &= a \left( ik\bar{\phi}_{10}(x, t)e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} + \text{c.c.} + O(F^{-1}) \right) \\ &\quad \times \left( ik\eta_{10}(x, t)e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} + \text{c.c.} + O(F^{-1}) \right) \\ &= k^2(\bar{\phi}_{10}(x, t)\eta_{10}^*(x, t) + \bar{\phi}_{10}^*(x, t)\eta_{10}(x, t)) \\ &\quad - \left[ k^2\bar{\phi}_{10}(x, t)\eta_{10}(x, t)e^{2ik(\bar{x}-\bar{t})\mp 2ik^{1/2}t^*} + \text{c.c.} \right] + O(F^{-1}), \end{aligned}$$

we note that  $\bar{\phi}_{10}$  and  $\eta_{10}$  are related by (see Eq. (84) below)

$$\bar{\phi}_{10}(x, t) = \frac{\mp i}{\sqrt{k}}\eta_{10}(x, t)$$

so that

$$k^2(\bar{\phi}_{10}(x, t)\eta_{10}^*(x, t) + \bar{\phi}_{10}^*(x, t)\eta_{10}(x, t)) = 0$$

and there is no order unity forcing of a mean flow, so  $\eta_{00}(x, t) = 0$ . (The other order unity term depends on the fast variables and will generate a term in the solution for  $\eta^{(p)}$  which also depends on fast variables, following the prescription discussed in Subsubsection 6.5.1.) The  $\bar{\phi}_{00}(x, t)$  term may be nonzero though, so we say that a  $\ell = n = \mu = 0$  mode is present when we list modes in Table 3.

The subset of the generated terms which are relevant to the evolution of the amplitude of the main ocean wave packet ( $n = \mu = 1$ ) through order  $O(F^{-1})$  are also listed in Table 3. (No  $O(F^{-2})$  terms affect the  $O(F^{-1})$  evolution of the main ocean wave packet).

Table 3: Accounting of wave modes  $\eta_{n\mu\ell}$ ,  $\bar{\phi}_{n\mu\ell}$ , and  $P_{n\mu\ell}$  generated and relevant at order  $\ell = 0, 1$ . We only list modes with  $n \geq 0$ ; to every mode  $(n, \mu)$  there corresponds a mode  $(-n, -\mu)$  due to the reality conditions (69).

$\ell$	Generated $(n, \mu)$	Relevant $(n, \mu)$
0	(0, 0), (1, 1)	(1, 1)
1	(0, 0), (1, 1), (2, 2), (2, $\sqrt{2}$ )	(1, 1), (2, 2)

Table 4: Accounting of modes  $\mathcal{X}_{nm\mu\ell}$  and  $\mathcal{Y}_{nm\mu\ell}$  generated and relevant in conformal map at orders  $0 \leq \ell \leq 4$ . We only list modes with  $n \geq 0$ ; to every mode  $(n, m, \mu)$  there corresponds a mode  $(-n, m, -\mu)$  due to the reality conditions (69).

$\ell$	Generated $(n, m, \mu)$	Relevant $(n, m, \mu)$
0	none	none
1	none	none
2	(0, 0, 0)	(0, 0, 0)
3	(1, 0, 1), (1, 1, 1)	(1, 0, 1), (1, 1, 1)
4	(0, 0, 0), (1, 0, 1), (1, 1, 1), (2, 0, 2), (2, 1, 2), (2, 2, 2), (2, 0, $\sqrt{2}$ ), (2, 1, $\sqrt{2}$ ), (2, 2, $\sqrt{2}$ )	(0, 0, 0), (1, 0, 1), (1, 1, 1), (2, 0, 2), (2, 1, 2), (2, 2, 2)

Note that all the relevant terms have  $\mu = n$  so we will drop the  $\mu$  label in the following. From the above mode analysis, we need only consider the following

terms:

$$\eta^{(p)} = (\eta_{10}(x, t) + F^{-1}\eta_{11}(x, t) + F^{-2}\eta_{12}(x, t) + O(F^{-3})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \quad (73a)$$

$$+ (F^{-1}\eta_{21}(x, t) + O(F^{-2})) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots ,$$

$$\bar{\phi}^{(p)} = (\bar{\phi}_{10}(x, t) + F^{-1}\bar{\phi}_{10}(x, t) + F^{-2}\bar{\phi}_{12}(x, t) + O(F^{-3})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \quad (73b)$$

$$+ (F^{-1}\bar{\phi}_{21}(x, t) + O(F^{-2})) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots ,$$

$$F^3\phi = F^3 \int_{-\infty}^{\infty} e^{i(\xi)\mathcal{X}} e^{|\xi|\mathcal{Y}} \phi^{(s)}(\xi, t) d\xi \quad (73c)$$

$$+ \left[ e^{ikF^2\mathcal{X}-ik\tilde{t}\mp ik^{1/2}t^*} e^{kF^{-2}\mathcal{Y}} \int_{-\infty}^{\infty} e^{i\xi\mathcal{X}} e^{\xi\mathcal{Y}} (P_{10}(x, t) + F^{-1}P_{11}(x, t) + F^{-2}P_{12}(x, t) \right.$$

$$\left. + O(F^{-3})) \right.$$

$$\left. + e^{2ikF^{-2}\mathcal{X}-2ik\tilde{t}\mp 2ik^{1/2}t^*} e^{2kF^{-2}\mathcal{Y}} \int_{-\infty}^{\infty} e^{i\xi\mathcal{X}} e^{\xi\mathcal{Y}} (F^{-1}P_{21}(x, t) + O(F^{-2})) \right. \quad (73d)$$

$$\left. + \text{c.c.} \right] + \dots , \quad (73e)$$

$$\mathcal{X} = x + (F^{-2}\mathcal{X}_{002}(x, y, t) + F^{-4}\mathcal{X}_{004}(x, y, t) + O(F^{-5})) \quad (73f)$$

$$+ \left[ (F^{-3}\mathcal{X}_{103}(x, y, t) + F^{-4}\mathcal{X}_{104}(x, y, t) + O(F^{-5})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \right.$$

$$+ (F^{-3}\mathcal{X}_{113}(x, y, t) + F^{-4}\mathcal{X}_{114}(x, y, t) + O(F^{-5})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} e^{k\tilde{Y}}$$

$$+ F^{-4}\mathcal{X}_{204}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + F^{-4}\mathcal{X}_{214}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} e^{k\tilde{Y}}$$

$$\left. + F^{-4}\mathcal{X}_{224}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} e^{2k\tilde{Y}} + \text{c.c.} \right] + \dots ,$$

$$\mathcal{Y} = y + (F^{-2}\mathcal{Y}_{002}(x, y, t) + F^{-4}\mathcal{Y}_{004}(x, y, t) + O(F^{-5})) \quad (73g)$$

$$+ \left[ (F^{-3}\mathcal{Y}_{103}(x, y, t) + F^{-4}\mathcal{Y}_{104}(x, y, t) + O(F^{-5})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \right.$$

$$+ (F^{-3}\mathcal{Y}_{113}(x, y, t) + F^{-4}\mathcal{Y}_{114}(x, y, t) + O(F^{-5})) e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} e^{k\tilde{Y}}$$

$$+ F^{-4}\mathcal{Y}_{204}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + F^{-4}\mathcal{Y}_{214}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} e^{k\tilde{Y}}$$

$$\left. + F^{-4}\mathcal{Y}_{224}(x, y, t) e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} e^{2k\tilde{Y}} + \text{c.c.} \right] + \dots ,$$

We have used the notation c.c. to denote the complex conjugate of the terms already listed, and  $\dots$  denote other terms which are not relevant to the determination of the amplitude of the main ocean wave mode through  $O(F^{-1})$ .

## 7.2 Expression of Conformal Map in Terms of Surface Disturbance

As described in Appendix A.1, it is convenient to solve the conformal mapping equations perturbatively in terms of flattened variables (95), so we write

$$\begin{aligned}
\tilde{\mathcal{X}}^\sharp &= X + \left( F^{-2} \tilde{\mathcal{X}}_{002}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{X}}_{004}^\sharp(x, y, t) + O(F^{-5}) \right) \\
&+ \left[ (F^{-3} \tilde{\mathcal{X}}_{103}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{X}}_{104}^\sharp(x, y, t) + O(F^{-5})) e^{ik(\tilde{X}-\tilde{T}) \mp ik^{1/2} T^*} \right. \\
&\quad + (F^{-3} \tilde{\mathcal{X}}_{113}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{X}}_{114}^\sharp(x, y, t) + O(F^{-5})) e^{ik(\tilde{X}-\tilde{T}) \mp ik^{1/2} T^*} e^{k\tilde{Y}} \\
&\quad + F^{-4} \tilde{\mathcal{X}}_{204}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} \\
&\quad + F^{-4} \tilde{\mathcal{X}}_{214}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} e^{k\tilde{Y}} \\
&\quad \left. + F^{-4} \tilde{\mathcal{X}}_{224}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} e^{2k\tilde{Y}} + \text{c.c.} \right] + \dots, \\
\tilde{\mathcal{Y}}^\sharp &= y + \left( F^{-2} \tilde{\mathcal{Y}}_{002}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{Y}}_{004}^\sharp(x, y, t) + O(F^{-5}) \right) \\
&+ \left[ (F^{-3} \tilde{\mathcal{Y}}_{103}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{Y}}_{104}^\sharp(x, y, t) + O(F^{-5})) e^{ik(\tilde{X}-\tilde{T}) \mp ik^{1/2} T^*} \right. \\
&\quad + (F^{-3} \tilde{\mathcal{Y}}_{113}^\sharp(x, y, t) + F^{-4} \tilde{\mathcal{Y}}_{114}^\sharp(x, y, t) + O(F^{-5})) e^{ik(\tilde{X}-\tilde{T}) \mp ik^{1/2} T^*} e^{k\tilde{Y}} \\
&\quad + F^{-4} \tilde{\mathcal{Y}}_{204}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} \\
&\quad + F^{-4} \tilde{\mathcal{Y}}_{214}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} e^{F^2 k\tilde{Y}} \\
&\quad \left. + F^{-4} \tilde{\mathcal{Y}}_{224}^\sharp(x, y, t) e^{2ik(\tilde{X}-\tilde{T}) \mp 2ik^{1/2} T^*} e^{2k\tilde{Y}} + \text{c.c.} \right] + \dots,
\end{aligned}$$

where the functions of flattened variables are related to the corresponding functions of physical variables as:

$$\begin{aligned}
\tilde{\mathcal{X}}^\sharp(X, Y, T) &= \mathcal{X}(x, y, t), \\
\tilde{\mathcal{X}}_{nm\ell}^\sharp(X, Y, T) &= \mathcal{X}_{nm\ell}(x, y, t), \\
\tilde{\mathcal{Y}}^\sharp(X, Y, T) &= \mathcal{Y}(x, y, t), \\
\tilde{\mathcal{Y}}_{nm\ell}^\sharp(X, Y, T) &= \mathcal{Y}_{nm\ell}(x, y, t) \text{ for } m \neq 0, \\
\tilde{\mathcal{Y}}_{002}^\sharp(X, Y, T) &= \mathcal{Y}_{002}(x, y, t) + \eta^{(s)}(x, t), \\
\tilde{\mathcal{Y}}_{004}^\sharp(X, Y, T) &= \mathcal{Y}_{004}(x, y, t), \\
\tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T) &= \mathcal{Y}_{103}(x, y, t) + \eta_{10}(x, y, t), \\
\tilde{\mathcal{Y}}_{104}^\sharp(X, Y, T) &= \mathcal{Y}_{104}(x, y, t) + \eta_{11}(x, y, t), \\
\tilde{\mathcal{Y}}_{204}^\sharp(X, Y, T) &= \mathcal{Y}_{204}(x, y, t) + \eta_{21}(x, y, t).
\end{aligned}$$

Note that the  $\tilde{\mathcal{Y}}_{nm\ell}^\sharp(X, Y, T)$  has a correction term for  $m = 0$  because  $\tilde{\mathcal{Y}}^\sharp$  is expressed as a perturbation series with base  $Y = y - F^{-2}\eta(x, t)$ , while  $\mathcal{Y}$  is expressed as a perturbation series with base  $y$ .

We proceed now to solve for those components which are in the end really needed. (Some appeared to be relevant based on formal mode-coupling considerations, but will be seen in the detailed analysis to have no influence on the quantities of interest.)

### 7.2.1 Solution for $\tilde{\mathcal{X}}_{002}^\sharp(X, Y, T)$ and $\tilde{\mathcal{Y}}_{002}^\sharp(X, Y, T)$

We can solve the system

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{X}}_{002}^\sharp}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}_{002}^\sharp}{\partial Y} &= 0, \\
\frac{\partial \tilde{\mathcal{Y}}_{002}^\sharp}{\partial X} + \frac{\partial \tilde{\mathcal{X}}_{002}^\sharp}{\partial Y} &= \eta_X^{(s)}, \\
\tilde{\mathcal{Y}}_{002}^\sharp(X, Y = 0, T) &= 0, \\
\lim_{Y \rightarrow -\infty} (\tilde{\mathcal{Y}}_{002}^\sharp(X, Y, T) - \eta^{(s)}(X, T)) &= 0.
\end{aligned}$$

by first looking for a particular solution which takes care of the inhomogeneity  $\eta^{(s)}$ , then adding to it a homogenous solution which allows the total solution to satisfy

the required boundary conditions:

$$\begin{aligned}\tilde{\mathcal{X}}_{002}^\sharp(X, Y, T) &= \int_{-\infty}^{\infty} e^{i\xi X} e^{|\xi|Y} (i \operatorname{sgn} \xi) \hat{\eta}^{(s)}(\xi, T) d\xi, \\ \tilde{\mathcal{Y}}_{002}^\sharp(X, Y, T) &= \eta^{(s)}(X) - \int_{-\infty}^{\infty} e^{i\xi X} e^{|\xi|Y} \hat{\eta}^{(s)}(\xi, T) d\xi,\end{aligned}$$

where

$$\hat{\eta}^{(s)}(\xi, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi X} \eta^{(s)}(X, T) dX$$

is the Fourier transform of  $\eta^{(s)}$  and

$$\operatorname{sgn} \xi = \begin{cases} 1 & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \\ -1 & \text{if } \xi < 0 \end{cases}$$

is the signum function.

### 7.2.2 Solution for $\tilde{\mathcal{X}}_{103}^\sharp(X, Y, T)$ and $\tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T)$

The system

$$\begin{aligned}ik\tilde{\mathcal{X}}_{103}^\sharp(X, Y, T) &= 0, \\ ik\tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T) - ik a \eta_{10}(X, T) &= 0\end{aligned}$$

has the solution

$$\tilde{\mathcal{X}}_{103}^\sharp(X, Y, T) = 0, \quad \tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T) = a \eta_{10}(X, T)$$

### 7.2.3 Solution for $\tilde{\mathcal{X}}_{113}^\sharp(X, Y, T)$ and $\tilde{\mathcal{Y}}_{113}^\sharp(X, Y, T)$

As discussed in Subsubsection 6.5.1, we obtain the relation

$$\tilde{\mathcal{Y}}_{113}^\sharp = i\tilde{\mathcal{X}}_{113}^\sharp \tag{74}$$

at  $O(F^{-1})$  and the system

$$\begin{aligned}\frac{\partial \tilde{\mathcal{X}}_{113}^\sharp}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}_{113}^\sharp}{\partial Y} &= 0, \\ \frac{\partial \tilde{\mathcal{Y}}_{113}^\sharp}{\partial X} + \frac{\partial \tilde{\mathcal{X}}_{113}^\sharp}{\partial Y} &= 0, \\ \tilde{\mathcal{Y}}_{103}^\sharp(X, Y=0, T) + \tilde{\mathcal{Y}}_{113}^\sharp(X, Y=0, T) &= 0, \\ \lim_{Y \rightarrow -\infty} (\tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T) + \tilde{\mathcal{Y}}_{113}^\sharp(X, Y, T) - a \eta_{10}(X, T)) &= 0.\end{aligned} \tag{75}$$

at  $O(F^{-3})$ . Substitution of the relation (74) and the solution (74) into Eq. (75) yields a Cauchy-Riemann equation for  $\tilde{\mathcal{Y}}_{113}^\sharp$ , with the formal solution

$$\begin{aligned}\tilde{\mathcal{Y}}_{113}^\sharp(X, Y, T) &= -a \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} \hat{\eta}_{10}(\xi, T) d\xi, \\ \tilde{\mathcal{X}}_{113}^\sharp(X, Y, T) &= ia \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} \hat{\eta}_{10}(\xi, T) d\xi,\end{aligned}\tag{76}$$

where

$$\hat{\eta}_{m\ell}(\xi, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi X} \eta_{m\ell}(X, T) dX.$$

This solution would only be a true solution if  $\hat{\eta}_{10}(\xi, T)$  decays very rapidly for  $\xi < 0$  which is not true. However, as explained in Appendix A.3, this formal solution will suffice for making calculations involving surface quantities with relative error  $O(F^{-3})$ .

#### 7.2.4 Solution for $\tilde{\mathcal{X}}_{104}^\sharp(X, Y, T)$ and $\tilde{\mathcal{Y}}_{103}^\sharp(X, Y, T)$

The system

$$\begin{aligned}ik\tilde{\mathcal{X}}_{104}^\sharp(X, Y, T) &= 0, \\ ik\tilde{\mathcal{Y}}_{104}^\sharp(X, Y, T) - ika\eta_{11}(X, T) &= 0\end{aligned}$$

has the solution

$$\tilde{\mathcal{X}}_{104}^\sharp(X, Y, T) = 0, \quad \tilde{\mathcal{Y}}_{104}^\sharp(X, Y, T) = a\eta_{11}(X, T)$$

#### 7.2.5 Solution for $\tilde{\mathcal{X}}_{204}^\sharp(X, Y, T)$ and $\tilde{\mathcal{Y}}_{204}^\sharp(X, Y, T)$

The system

$$\begin{aligned}2ik\tilde{\mathcal{X}}_{204}^\sharp(X, Y, T) &= 0, \\ 2ik\tilde{\mathcal{Y}}_{204}^\sharp(X, Y, T) - 2ika\eta_{21}(X, T) &= 0\end{aligned}$$

has the solution

$$\tilde{\mathcal{X}}_{204}^\sharp(X, Y, T) = 0, \quad \tilde{\mathcal{Y}}_{204}^\sharp(X, Y, T) = a\eta_{21}(X, T)\tag{77}$$

### 7.2.6 Solution for $\tilde{\mathcal{X}}_{214}^\#(X, Y, T)$ and $\tilde{\mathcal{Y}}_{214}^\#(X, Y, T)$

The system

$$\begin{aligned} 2ik\tilde{\mathcal{X}}_{214}^\#(X, Y, T) - k\tilde{\mathcal{Y}}_{214}^\#(X, Y, T) - (ika\eta_{10}(X, T))k\tilde{\mathcal{X}}_{113}^\#(X, Y, T) &= 0, \\ 2ik\tilde{\mathcal{Y}}_{214}^\#(X, Y, T) + k\tilde{\mathcal{X}}_{214}^\#(X, Y, T) - (ika\eta_{10}(X, T))k\tilde{\mathcal{Y}}_{113}^\#(X, Y, T) &= 0 \end{aligned}$$

has the solution

$$\begin{aligned} \tilde{\mathcal{X}}_{214}^\#(X, Y, T) &= \frac{ak}{3}\eta_{10}(X, T)(2\tilde{\mathcal{X}}_{113}^\#(X, Y, T) - i\tilde{\mathcal{Y}}_{113}^\#(X, Y, T)), \\ \tilde{\mathcal{Y}}_{204}^\#(X, Y, T) &= \frac{ak}{3}\eta_{10}(X, T)(i\tilde{\mathcal{X}}_{113}^\#(X, Y, T) + 2\tilde{\mathcal{Y}}_{113}^\#(X, Y, T)) \end{aligned}$$

Substituting in the previously found solutions (76) for  $\tilde{\mathcal{X}}_{113}^\#(X, Y, T)$  and  $\tilde{\mathcal{Y}}_{113}^\#(X, Y, T)$  yields

$$\begin{aligned} \tilde{\mathcal{X}}_{214}^\#(X, Y, T) &= ika^2\eta_{10}(X, T) \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} \hat{\eta}_{10}(\xi, T) d\xi, \\ \tilde{\mathcal{Y}}_{214}^\#(X, Y, T) &= -ka^2\eta_{10}(X, T) \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} \hat{\eta}_{10}(\xi, T) d\xi. \end{aligned} \tag{78}$$

### 7.2.7 Solution for $\tilde{\mathcal{X}}_{224}^\#(X, Y, T)$ and $\tilde{\mathcal{Y}}_{113}^\#(X, Y, T)$

As discussed in Subsubsection 6.5.1, we obtain the relation

$$\tilde{\mathcal{Y}}_{224}^\# = i\tilde{\mathcal{X}}_{224}^\# \tag{79}$$

at  $O(F^{-2})$  and the system

$$\begin{aligned} \frac{\partial \tilde{\mathcal{X}}_{224}^\#}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}_{224}^\#}{\partial Y} &= 0, \\ \frac{\partial \tilde{\mathcal{Y}}_{224}^\#}{\partial X} + \frac{\partial \tilde{\mathcal{X}}_{224}^\#}{\partial Y} &= 0, \end{aligned} \tag{80}$$

$$\tilde{\mathcal{Y}}_{204}^\#(X, Y=0, T) + \tilde{\mathcal{Y}}_{214}^\#(X, Y=0, T) + \tilde{\mathcal{Y}}_{224}^\#(X, Y=0, T) = 0,$$

$$\lim_{Y \rightarrow -\infty} (\tilde{\mathcal{Y}}_{204}^\#(X, Y, T) + \tilde{\mathcal{Y}}_{214}^\#(X, Y, T) + \tilde{\mathcal{Y}}_{224}^\#(X, Y, T) - a\eta_{21}(X, T)) = 0.$$

at  $O(F^{-4})$ . Substitution of the relation (79) and the solutions (77) and (78) into Eq. (80) yields a Cauchy-Riemann equation for  $\tilde{\mathcal{Y}}_{224}^\#$ , with the formal solution

$$\begin{aligned} \tilde{\mathcal{Y}}_{224}^\#(X, Y, T) &= \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} (-a\hat{\eta}_{21}(\xi, T) + ka^2\hat{\eta}_{10}^2(\xi, T)) d\xi, \\ \tilde{\mathcal{X}}_{224}^\#(X, Y, T) &= i \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi y} (a\hat{\eta}_{21}(\xi, T) - ka^2\hat{\eta}_{10}^2(\xi, T)) d\xi, \end{aligned}$$

where

$$\widehat{\eta}_{10}^2(\xi, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi X} \eta_{10}^2(X, T) dX.$$

Again, this is only a formal solution, but will give us accurate surface values and derivatives with relative error  $O(F^{-3})$ , as described in Appendix A.3.

### 7.2.8 Summary of Conformal Map Solution in Physical Coordinates

Expressing the above results in terms of the original physical variables  $(x, y, t)$ , we obtain:

$$\mathcal{X}_{002}(x, y, t) = \int_{-\infty}^{\infty} e^{i\xi x} e^{|\xi|(y-F^{-2}\eta(x,t))} (i \operatorname{sgn} \xi) \widehat{\eta}^{(s)}(\xi, t) d\xi,$$

$$\mathcal{Y}_{002}(x, y, t) = - \int_{-\infty}^{\infty} e^{i\xi x} e^{|\xi|(y-F^{-2}\eta(x,t))} \widehat{\eta}^{(s)}(\xi, t) d\xi,$$

$$\mathcal{X}_{103}(x, y, t) = 0,$$

$$\mathcal{Y}_{103}(x, y, t) = 0,$$

$$\mathcal{X}_{113}(x, y, t) = ia \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} \widehat{\eta}_{10}(\xi, t) d\xi,$$

$$\mathcal{Y}_{113}(x, y, t) = -a \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} \widehat{\eta}_{10}(\xi, t) d\xi,$$

$$\mathcal{X}_{104}(x, y, t) = 0,$$

$$\mathcal{Y}_{104}(x, y, t) = 0,$$

$$\mathcal{X}_{204}(x, y, t) = 0,$$

$$\mathcal{Y}_{204}(x, y, t) = 0,$$

$$\mathcal{X}_{214}(x, y, t) = ika^2 \eta_{10}(x, t) \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} \widehat{\eta}_{10}(\xi, t) d\xi,$$

$$\mathcal{Y}_{214}(x, y, t) = -ka^2 \eta_{10}(x, t) \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} \widehat{\eta}_{10}(\xi, t) d\xi,$$

$$\mathcal{X}_{224}(x, y, t) = i \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} (a\widehat{\eta}_{21}(\xi, t) - ka^2 \widehat{\eta}_{10}^2(\xi, t)) d\xi,$$

$$\mathcal{Y}_{224}(x, y, t) = \int_{-\infty}^{\infty} e^{i\xi x} e^{\xi(y-F^{-2}\eta(x,t))} (-a\widehat{\eta}_{21}(\xi, t) + ka^2 \widehat{\eta}_{10}^2(\xi, t)) d\xi.$$

The solutions for  $\mathcal{X}_{004}$ ,  $\mathcal{Y}_{004}$ ,  $\mathcal{X}_{114}$ ,  $\mathcal{Y}_{114}$ , are actually not needed explicitly, as we shall see.

Note that the terms in our expansion of the conformal map actually themselves depend on the expansion parameter  $F$ , but only through the exponential dependence on  $y - F^{-2}\eta(x, t)$ . We can think of our arrangement as a useful renormalization of the usual asymptotic series.

The behavior of the conformal mapping functions and their first derivatives at the ocean surface will be of key importance in the ensuing calculations. Following the development in Subsection 6.2, we find it useful to decompose these surface expressions into ship wave and perturbation components, with the notation of Eq. (63). The ship wave contributions to the conformal mapping functions at the surface are:

$$\begin{aligned}
\bar{\mathcal{X}}^{(s)}(x, t) &= x + F^{-2}\bar{\mathcal{X}}_{002}(x, t) + F^{-4}\bar{\mathcal{X}}_{004}(x, t) + O(F^{-5}), \\
\bar{\mathcal{Y}}^{(s)}(x, t) &= 0, \\
\bar{\mathcal{X}}_x^{(s)}(x, t) &= 1 + F^{-2}(\bar{\mathcal{X}}_x)_{002}(x, t) + O(F^{-4}), \\
\bar{\mathcal{Y}}_x^{(s)}(x, t) &= F^{-2}(\bar{\mathcal{Y}}_x)_{002}(x, t) + O(F^{-4}), \\
\bar{\mathcal{X}}_y^{(s)}(x, t) &= F^{-2}(\bar{\mathcal{X}}_y)_{002}(x, t) + O(F^{-4}), \\
\bar{\mathcal{Y}}_y^{(s)}(x, t) &= 1 + F^{-2}(\bar{\mathcal{Y}}_y)_{002}(x, t) + O(F^{-4}),
\end{aligned} \tag{81a}$$

while the perturbative surface contributions to the conformal mapping functions due to the ocean wave packet are

$$\begin{aligned}
aF^{-3}\bar{\mathcal{X}}^{(p)}(x, t) &= (F^{-3}\bar{\mathcal{X}}_{113}(x, t) + F^{-4}\bar{\mathcal{X}}_{114}(x, t) + O(F^{-5}))e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \\
&\quad + F^{-4}(\bar{\mathcal{X}}_{214}(x, t) + \bar{\mathcal{X}}_{224}(x, t))e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots, \\
aF^{-3}\bar{\mathcal{Y}}^{(p)}(x, t) &= 0, \\
aF^{-3}\bar{\mathcal{X}}_x^{(p)}(x, t) &= ik(F^{-1}\bar{\mathcal{X}}_{113}(x, t) + F^{-2}\bar{\mathcal{X}}_{114}(x, t) + O(F^{-3}))e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \\
&\quad + 2ikF^{-2}(\bar{\mathcal{X}}_{214}(x, t) + \bar{\mathcal{X}}_{224}(x, t))e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots, \\
aF^{-3}\bar{\mathcal{Y}}_x^{(p)}(x, t) &= ik(F^{-1}\bar{\mathcal{Y}}_{113}(x, t) + F^{-2}\bar{\mathcal{Y}}_{114}(x, t) + O(F^{-3}))e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \\
&\quad + 2ikF^{-2}(\bar{\mathcal{Y}}_{214}(x, t) + \bar{\mathcal{Y}}_{224}(x, t))e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots, \\
aF^{-3}\bar{\mathcal{X}}_y^{(p)}(x, t) &= k(F^{-1}\bar{\mathcal{X}}_{113}(x, t) + F^{-2}\bar{\mathcal{X}}_{114}(x, t) + O(F^{-3}))e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \\
&\quad + kF^{-2}(\bar{\mathcal{X}}_{214}(x, t) + 2\bar{\mathcal{X}}_{224}(x, t))e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots, \\
aF^{-3}\bar{\mathcal{Y}}_y^{(p)}(x, t) &= k(F^{-1}\bar{\mathcal{Y}}_{113}(x, t) + F^{-2}\bar{\mathcal{Y}}_{114}(x, t) + O(F^{-3}))e^{ik(\tilde{x}-\tilde{t})\mp ik^{1/2}t^*} \\
&\quad + kF^{-2}(\bar{\mathcal{Y}}_{214}(x, t) + 2\bar{\mathcal{Y}}_{224}(x, t))e^{2ik(\tilde{x}-\tilde{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots.
\end{aligned} \tag{81b}$$

The coefficients appearing in these expansions are expressed in terms of the surface displacement as follows:

$$\begin{aligned}
\bar{\mathcal{X}}_{002}(x, t) &= -(\mathcal{H}\eta^{(s)})(x), \\
\bar{\mathcal{Y}}_{002}(x, t) &= -\eta^{(s)}(x), \\
\bar{\mathcal{X}}_{113}(x, t) &= i\eta_{10}(x, t), \\
\bar{\mathcal{Y}}_{113}(x, t) &= -\eta_{10}(x, t), \\
\bar{\mathcal{X}}_{214}(x, t) &= ika(\eta_{10}(x, t))^2, \\
\bar{\mathcal{Y}}_{214}(x, t) &= -ka(\eta_{10}(x, t))^2, \\
\bar{\mathcal{X}}_{224}(x, t) &= ia\eta_{21}(x, t) - ika^2(\eta_{10}(x, t))^2, \\
\bar{\mathcal{Y}}_{224}(x, t) &= -a\eta_{21}(x, t) + ka^2(\eta_{10}(x, t))^2, \\
(\bar{\mathcal{X}}_x)_{002}(x, t) &= -(\mathcal{H}\eta^{(s)})_x(x) + O(F^{-2}), \\
(\bar{\mathcal{Y}}_x)_{002}(x, t) &= -\eta_x^{(s)}(x) + O(F^{-2}), \\
(\bar{\mathcal{X}}_y)_{002}(x, t) &= \eta_x^{(s)}(x) + O(F^{-2}), \\
(\bar{\mathcal{Y}}_y)_{002}(x, t) &= -(\mathcal{H}\eta^{(s)})_x(x) + O(F^{-2}).
\end{aligned} \tag{81c}$$

We have only listed the surface derivatives which will be needed. The symbol  $\mathcal{H}$  denotes the Hilbert transform [30, p. 26], which is defined as:

$$\begin{aligned}
(\mathcal{H}f)(x, t) &= \int_{-\infty}^{\infty} e^{i\xi x} (-i \operatorname{sgn} \xi) \hat{f}(\xi, t) d\xi, \\
\hat{f}(\xi, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} f(x, t) dx.
\end{aligned}$$

### 7.3 Surface Derivatives of Potential

The evolution equations for the surface functions involves derivatives of the potential at the surface, which we would like to express in terms of surface quantities so that the system of equations (67a) and (67b) is closed. We do this following the procedure outlined in Subsubsection 6.2.2.

### 7.3.1 Expansion of $\hat{\phi}^{(p)}(\xi, t)$

Matching our expansion (73b) for the potential at the surface to the expansion of the potential in the interior of the fluid (73c), we have

$$\begin{aligned}
& (\bar{\phi}_{10}(x, t) + F^{-1}\bar{\phi}_{11}(x, t) + F^{-2}\bar{\phi}_{12}(x, t) + O(F^{-3})) e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} \\
& + (F^{-1}\bar{\phi}_{21}(x, t) + O(F^{-2})) e^{2ik(\bar{x}-\bar{t})\mp 2ik^{1/2}t^*} + \text{c.c.} + \dots \\
= & F^{-1} \int_{-\infty}^{\infty} e^{i\xi(\bar{\mathcal{X}}^{(s)}(x, t))} [i\xi\bar{\mathcal{X}}^{(p)}(x, t)P^{(s)}(\xi) + \\
& + (P_{10}(\xi, t) + F^{-1}P_{11}(\xi, t) + F^{-2}P_{12}(\xi, t) + O(F^{-3})) e^{ik(F^2\bar{\mathcal{X}}^{(s)}(x)-\bar{t})\mp ik^{1/2}t^*} \\
& + (F^{-1}P_{21}(\xi, t) + O(F^{-2})) e^{2ik(F^2\bar{\mathcal{X}}^{(s)}(x)-\bar{t})\mp 2ik^{1/2}t^*}] d\xi + \text{c.c.} + \dots
\end{aligned}$$

By substituting in the expansions (73f) and (73h) for the conformal mapping functions, reorganizing the right hand side into an asymptotic expansion in  $F^{-1}$ , and matching relevant terms from the left hand side, we obtain the following relationships between the expansion of the potential at the surface and the expansion of the potential in the bulk of the ocean:

$$\int_{-\infty}^{\infty} e^{i\xi x + k\bar{\mathcal{X}}_{002}(x, t)} P_{10}(\xi, t) d\xi = \bar{\phi}_{10}(x, t), \quad (82a)$$

$$\int_{-\infty}^{\infty} e^{i\xi x + k\bar{\mathcal{X}}_{002}(x, t)} P_{11}(\xi, t) d\xi = \bar{\phi}_{11}(x, t) - \overline{\phi_x}^{(s)} \bar{\mathcal{X}}_{113}(x, t), \quad (82b)$$

$$\int_{-\infty}^{\infty} e^{i\xi x + k\bar{\mathcal{X}}_{002}(x, t)} P_{12}(\xi, t) d\xi = \bar{\phi}_{11}(x, t) - \overline{\phi_x}^{(s)} \bar{\mathcal{X}}_{114}(x, t) \quad (82c)$$

$$\begin{aligned}
& - (ik\bar{\mathcal{X}}_{004}(x, t) + k^2|\bar{\mathcal{X}}_{113}(x, t)|^2 - ik\bar{\mathcal{X}}_{002}(x, t)(\bar{\mathcal{X}}_x)_{002}(x, t))\bar{\phi}_{10}(x, t) \\
& + \bar{\mathcal{X}}_{002}(x, t)\bar{\phi}_{10,x}(x, t) + 2ik\bar{\mathcal{X}}_{113}^*(x, t)\bar{\phi}_{21}(x, t) \\
& - ik(\bar{\mathcal{X}}_{214}(x, t) + \bar{\mathcal{X}}_{224}(x, t))\bar{\phi}_{10}^*(x, t),
\end{aligned}$$

$$\int_{-\infty}^{\infty} e^{i\xi x + k\bar{\mathcal{X}}_{002}(x, t)} P_{21}(\xi, t) d\xi = \bar{\phi}_{21}(x, t) - ik\bar{\mathcal{X}}_{113}(x, t)\bar{\phi}_{10}(x, t). \quad (82d)$$

Now we can obtain asymptotic expansions for the derivatives of the potential evaluated at the surface by substituting in the asymptotic expansion for  $P$  (73c) and the conformal mapping functions (81) into Eq. (66), and then using Eq. (82) to replace the bulk coefficients  $\{P_{n\ell}(\xi, t)\}_{\ell, n}$  for the potential by the surface potential

expansion coefficients. In this way, we obtain

$$F^2(\overline{\phi_{\bar{x}}}^{(p)} + \overline{\phi_{\bar{t}}}^{(p)}) + F\overline{\phi_{t^*}}^{(p)} + \overline{\phi_x}^{(p)} + \overline{\phi_t}^{(p)} = e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} [F(\mp ik^{1/2}\overline{\phi_{10}}) \quad (83a)$$

$$+ \left( \mp ik^{1/2}\overline{\phi_{11}} \mp ik^{1/2}\overline{\phi_y}^{(s)}\overline{\mathcal{Y}}_{113} + \overline{\phi_{10,x}} + \overline{\phi_{10,t}} + k(\overline{\mathcal{Y}}_x)_{002}\overline{\phi_{10}} \right) \\ \cdot + F^{-1} \left( \mp ik^{1/2}\overline{\phi_{12}} \mp ik^{1/2}\overline{\phi_y}^{(s)}\overline{\mathcal{Y}}_{114} + k(\overline{\mathcal{Y}}_x)_{002}\overline{\phi_{11}} + \overline{\phi_{11,x}} + \overline{\phi_{11,t}} \right. \\ \left. \pm 2k^{5/2}\overline{\mathcal{X}}_{113}\overline{\mathcal{Y}}_{113}^*\overline{\phi_{10}} \pm 2ik^{3/2}\overline{\mathcal{Y}}_{113}^*\overline{\phi_{21}} \mp 2ik^{3/2}(\overline{\mathcal{Y}}_{214} + \overline{\mathcal{Y}}_{224})\overline{\phi_{10}}^* \right. \\ \left. - \overline{\phi_x}^{(s)}(\overline{\mathcal{X}}_{113,x} + \overline{\mathcal{X}}_{113,t} + k(\overline{\mathcal{Y}}_x)_{002}\overline{\mathcal{X}}_{113}) \right) \\ + Fe^{2ik(\bar{x}-\bar{t})\mp 2ik^{1/2}t^*} (\mp 2ik^{1/2}\overline{\phi_{21}} \mp ik^{3/2}\overline{\mathcal{Y}}_{113}\overline{\phi_{10}}) + \text{c.c.} + \dots, \quad (83b)$$

$$F^2\overline{\phi_{\bar{x}}}^{(p)} + \overline{\phi_x}^{(p)} = e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} \left[ F^2ik\overline{\phi_{10}} + F \left( ik\overline{\phi_{11}} + ik\overline{\phi_y}^{(s)}\overline{\mathcal{Y}}_{113} \right) \right] \\ + Fe^{2ik(\bar{x}-\bar{t})\mp 2ik^{1/2}t^*} (2ik\overline{\phi_{21}} + ik^2\overline{\mathcal{Y}}_{113}\overline{\phi_{10}}) + \text{c.c.} + \dots, \quad (83c)$$

$$F^2\overline{\phi_{\bar{y}}}^{(p)} + \overline{\phi_y}^{(p)} = e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*} \left[ F^2k\overline{\phi_{10}} + F \left( k\overline{\phi_{11}} + k\overline{\phi_y}^{(s)}\overline{\mathcal{Y}}_{113} \right) \right. \\ \left. + \left( k\overline{\phi_{12}} + k\overline{\phi_y}^{(s)}\overline{\mathcal{Y}}_{114} + (ik(\overline{\mathcal{X}}_y)_{002} + k(\overline{\mathcal{Y}}_y)_{002}) \right. \right. \\ \left. \left. - k(\overline{\mathcal{X}}_x)_{002} - 2ik^3\overline{\mathcal{X}}_{113}\overline{\mathcal{Y}}_{113}^*\overline{\phi_{10}} - i\overline{\phi_{10,x}} \right. \right. \\ \left. \left. + 2k^2\overline{\mathcal{Y}}_{113}^*\overline{\phi_{21}} + k^2(-i\overline{\mathcal{X}}_{224} + \overline{\mathcal{Y}}_{214} + 2\overline{\mathcal{Y}}_{224})\overline{\phi_{10}}^* \right) \right] \\ + Fe^{2ik(\bar{x}-\bar{t})\mp 2ik^{1/2}t^*} (2k\overline{\phi_{21}} + k^2\overline{\mathcal{Y}}_{113}\overline{\phi_{10}}) + \text{c.c.} + \dots, \quad (83d)$$

We note that in actually computing these results, it is helpful to recognize that certain groups of terms appear in the same way in the expansion of  $\overline{\phi}^{(p)}$  and its derivatives. We also used the relation

$$\int_{-\infty}^{\infty} i\xi e^{i\xi x + k\overline{\mathcal{X}}_{002}(x,t)} P_{10}(\xi, t) d\xi = e^{ik\overline{\mathcal{X}}_{002}(x,t)} \frac{\partial}{\partial x} \left( e^{-ik\overline{\mathcal{X}}_{002}(x,t)} \overline{\phi_{10}}(x, t) \right),$$

which follows directly from Eq. (82a).

## 7.4 Solution of Surface Equations

By substituting into (67a) and (67b) the expressions (83) for the derivatives of the potential at the surface, we obtain a closed system of PDE's for the evolution of the amplitude coefficients of the expansion of the surface height  $\eta^{(p)}(x, t)$  (73a) and surface value of the potential  $\overline{\phi}^{(p)}(x, t)$  (73b), which we can solve order by order as described in Subsection 6.5.

### 7.4.1 Leading Order Equations ( $O(F^{-2})$ )

Trivially satisfied.

#### 7.4.2 Second Order Equations ( $O(F^{-1})$ )

$$\begin{aligned}\mp i\sqrt{k}\eta_{10} - kP_{10} &= 0, \\ \mp i\sqrt{k}P_{10} + \eta_{10} &= 0\end{aligned}$$

This system is satisfied provided

$$\bar{\phi}_{10}(x, t) = \frac{\mp i}{\sqrt{k}}\eta_{10}(x, t). \quad (84)$$

We have insisted that the initial data of the wavepacket satisfied this condition which some choice of  $\mp$ , because the waves were initialized so that they were all moving in the same direction (see Subsubsection 4.2.4). We can therefore proceed.

#### 7.4.3 Third Order Equations $O(1)$ , $e^{ik(\bar{x}-\bar{t})\mp ik^{1/2}t^*}$ component

$$\begin{aligned}0 &= \mp i\sqrt{k}\eta_{11} - (k\bar{\phi}_{11} + k\bar{\phi}_y^{(s)}\bar{\mathcal{Y}}_{113}) \\ &\quad + \frac{\partial\eta_{10}}{\partial t} + \frac{\partial\eta_{10}}{\partial x} + ik\bar{\phi}_x^{(s)}\eta_{10},\end{aligned} \quad (85a)$$

$$\begin{aligned}0 &= \mp i\sqrt{k}\bar{\phi}_{11} \mp i\sqrt{k}\bar{\phi}_y^{(s)}\bar{\mathcal{Y}}_{113} + k(\bar{\mathcal{Y}}_x)_{002}\bar{\phi}_{10} \\ &\quad + \frac{\partial\bar{\phi}_{10}}{\partial t} + \frac{\partial\bar{\phi}_{10}}{\partial x} + \eta_{11} + k(\bar{\mathcal{Y}}_x)_{002}\bar{\phi}_{10} \\ &\quad + ik\bar{\phi}_x^{(s)}\bar{\phi}_{10} + k\bar{\phi}_y^{(s)}\bar{\phi}_{10}.\end{aligned} \quad (85b)$$

After substituting Eq. (84) into Eq. (85c) and multiplying this equation by  $\pm i\sqrt{k}$ , and rearranging terms, we obtain:

$$\begin{aligned}0 &= \frac{\partial\eta_{10}}{\partial t} + \frac{\partial\eta_{10}}{\partial x} + ik\bar{\phi}_x^{(s)}\eta_{10} \mp i\sqrt{k}\eta_{11} \\ &\quad - k(\bar{\phi}_{11} + \bar{\phi}_y^{(s)}\bar{\mathcal{Y}}_{113}),\end{aligned} \quad (86a)$$

$$\begin{aligned}0 &= \frac{\partial\eta_{10}}{\partial t} + \frac{\partial\eta_{10}}{\partial x} + ik\bar{\phi}_x^{(s)}\eta_{10} \pm i\sqrt{k}\eta_{11} \\ &\quad + k(\bar{\phi}_{11} + \bar{\phi}_y^{(s)}\bar{\mathcal{Y}}_{113}) + k(\bar{\phi}_y^{(s)} + (\bar{\mathcal{Y}}_x)_{002})\eta_{10}.\end{aligned} \quad (86b)$$

We can simplify Eq. (86b) by noting from Eq. (81c) that  $(\bar{\mathcal{Y}}_x)_{002} = -\eta_x^{(s)} + O(F^{-2})$  and from (81c) that

$$\eta_x^{(s)} - \bar{\phi}_y^{(s)} = O(F^{-2}),$$

so we can drop the term  $k(\overline{\phi}_y^{(s)}(x) + (\overline{\mathcal{Y}}_x)_{002})\eta_{10}$ . (This manipulation necessitates the introduction of a correction term to  $O(F^{-2})$  terms in the expansion of the surface equation (67b), but we do not use these terms within the scope of our analysis.)

We can better process the system (86) by adding and subtracting the two equations, and multiplying each by a half. This gives us the equivalent system

$$\frac{\partial \eta_{10}(x, t)}{\partial t} + \frac{\partial \eta_{10}(x, t)}{\partial x} + ik\overline{\phi}_x^{(s)}(x)\eta_{10}(x, t) = 0, \quad (87a)$$

$$\mp i\sqrt{k}\eta_{11} - k\overline{\phi}_{11} - k\overline{\phi}_y^{(s)}\overline{\mathcal{Y}}_{113} = 0. \quad (87b)$$

Equation (87b) implies the following relation between the amplitude coefficients of the first order correction terms in the surface potential and surface height:

$$\overline{\phi}_{11}(x, t) = \frac{\mp i}{\sqrt{k}}\eta_{11}(x, t) + \overline{\phi}_y^{(s)}(x)\eta_{10}(x, t), \quad (88)$$

where we have used Eq. (81c). Eq. (87a) is the leading order equation for the ocean wave packet amplitude. It is to be supplemented with the initial wave packet condition (67c):

$$\eta_{10}(x, t = 0) = A^{(0)}(x)$$

The first order PDE (87a) is readily solved through the method of characteristics [35]. By introducing

$$I^{(x)}(x) = \int_{-\infty}^x \overline{\phi}_x^{(s)}(x') dx',$$

we can write the solution as

$$\eta_{10}(x, t) = e^{ik(I^{(x)}(x-t) - I^{(x)}(x))} A^{(0)}(x - t). \quad (89)$$

We see that there is no leading order change in the magnitude of the wavepacket amplitude coefficient; there is only a phase shift due to the flow induced by the ship wave which can be explained on kinematic groundsNSF00hjt:wpbgf.

#### 7.4.4 Third Order Equations $O(1)$ , $e^{2ik(\tilde{x}-\tilde{t}) \mp 2ik^{1/2}t^*}$ component

$$\begin{aligned} \mp 2i\sqrt{k}\eta_{21} - (2k\overline{\phi}_{21} - k^2\overline{\mathcal{Y}}_{113}\overline{\phi}_{10}) - ak^2\eta_{10}\overline{\phi}_{10} &= 0, \\ \mp 2i\sqrt{k}\overline{\phi}_{21} \mp ik^{3/2}\overline{\mathcal{Y}}_{113}\overline{\phi}_{10} + \eta_{21} - ak^2(\overline{\phi}_{10})^2 + ak^2(\overline{\phi}_{10})^2 &= 0. \end{aligned}$$

Simplifying and using the relations (84) and (81c) for  $\overline{\phi}_{10}(x, t)$  and  $\overline{\mathcal{Y}}_{113}(x, t)$  in terms of  $\eta_{10}(x, t)$ , we get

$$\begin{aligned} \mp 2i\sqrt{k}\eta_{21} - 2k\overline{\phi}_{21}(x, t) &= 0, \\ \mp 2i\sqrt{k}\overline{\phi}_{21}(x, t) + \eta_{21}(x, t) + ak(\eta_{10}(x, t))^2 &= 0. \end{aligned}$$

The solution of these equations is

$$\eta_{21}(x, t) = k a \eta_{10}^2(x, t), \quad \bar{\phi}_{21}(x, t) = \mp i k^{1/2} a \eta_{10}^2(x, t).$$

#### 7.4.5 Fourth Order Equations $O(F^{-1})$

We proceed to one higher order in an attempt to determine how the amplitude of the wavepacket is altered through  $O(F^{-1})$ . This is determined by the  $O(F^{-1})$  terms in the surface equations, of which only those terms proportional to  $e^{ik(\bar{x}-\bar{t}) \mp ik^{1/2}t^*}$  are relevant to determining the evolution of the amplitude of the dominant mode in the wave packet:

$$\begin{aligned} 0 &= \mp i \sqrt{k} \eta_{12} + \eta_{11,t} + \eta_{11,x} - k \bar{\phi}_{12} - k \overline{\phi_y^{(s)}} \bar{\mathcal{Y}}_{114} & (92a) \\ &\quad - (ik(\bar{\mathcal{X}}_y)_{002} - k(\bar{\mathcal{X}}_x)_{002} + k(\bar{\mathcal{Y}}_y)_{002} - 2ik^3 \bar{\mathcal{X}}_{113} \bar{\mathcal{Y}}_{113}^*) \bar{\phi}_{10} + i \bar{\phi}_{10,x} \\ &\quad - 2k^2 \bar{\mathcal{Y}}_{113}^* \bar{\phi}_{21} - k^2 (-i \bar{\mathcal{X}}_{224} + \bar{\mathcal{Y}}_{214} + 2\bar{\mathcal{Y}}_{224}) \bar{\phi}_{10}^* + ik \overline{\phi_x^{(s)}} \eta_{11} \\ &\quad + a(2ik \bar{\phi}_{21} + ik^2 \bar{\mathcal{Y}}_{113} \bar{\phi}_{10}) (-ik \eta_{10}^*) + a(-ik \bar{\phi}_{10}^*) (2ik \eta_{21}) + iak \eta_x^{(s)} \bar{\phi}_{10}, \\ 0 &= \mp ik^{1/2} \bar{\phi}_{12} \mp ik^{1/2} \overline{\phi_y^{(s)}} \bar{\mathcal{Y}}_{114} + k(\bar{\mathcal{Y}}_x)_{002} \bar{\phi}_{11} + \bar{\phi}_{11,x} + \bar{\phi}_{11,t} \pm 2k^{5/2} \bar{\mathcal{X}}_{113} \bar{\mathcal{Y}}_{113}^* \bar{\phi}_{10} \\ &\quad \pm 2ik^{3/2} \bar{\mathcal{Y}}_{113}^* \bar{\phi}_{21} \mp 2ik^{3/2} (\bar{\mathcal{Y}}_{214} + \bar{\mathcal{Y}}_{224}) \bar{\phi}_{10}^* \\ &\quad - \overline{\phi_x^{(s)}} (\bar{\mathcal{X}}_{113,x} + \bar{\mathcal{X}}_{113,t} + k(\bar{\mathcal{Y}}_x)_{002} \bar{\mathcal{X}}_{113}) + \eta_{21} + \overline{\phi_x^{(s)}} (ik \bar{\phi}_{11} + ik \overline{\phi_y^{(s)}} \mathcal{Y}_{113}) \\ &\quad + \overline{\phi_y^{(s)}} (k \bar{\phi}_{11} + k \overline{\phi_y^{(s)}} \mathcal{Y}_{113}) + a(2ik \bar{\phi}_{21} + ik^2 \bar{\mathcal{Y}}_{113} \bar{\phi}_{10}) (-ik \bar{\phi}_{10}^*) \\ &\quad + 2a(2k \bar{\phi}_{21} + k^2 \bar{\mathcal{Y}}_{113} \bar{\phi}_{10}) (k \bar{\phi}_{10}^*). \end{aligned}$$

Substituting in Eq. (88) and processing the resulting equations in the same way as we did Eq. (86), we obtain:

$$\begin{aligned} 0 &= \frac{\partial \eta_{11}}{\partial t} + \frac{\partial \eta_{11}}{\partial x} + ik \overline{\phi_x^{(s)}} \eta_{11} \pm \frac{1}{2\sqrt{k}} \eta_{10,x} \\ &\quad \pm \frac{i}{2} \left( k^{1/2} \overline{\phi_{xy}^{(s)}} - k^{3/2} ((\overline{\phi_x^{(s)}})^2 + (\overline{\phi_y^{(s)}})^2) \right) \eta_{10} & (93) \\ &\quad \mp 4ia^2 k^{5/2} |\eta_{10}|^2 \eta_{10}, \end{aligned}$$

and another equation involving  $\eta_{12}$  and  $\bar{\phi}_{12}$  which does not concern us. In obtaining this equation, we have also used relations like

$$\frac{\partial \overline{\phi_y^{(s)}}}{\partial x} = \overline{\phi_{xy}^{(s)}} + O(F^{-2}),$$

which follow from the considerations of Subsubsection 6.2.2. The notation on the right hand side of this equation, as usual, means

$$\overline{\phi_{xy}^{(s)}}(x) \equiv \left. \frac{\partial^2 \phi^{(s)}(x, y)}{\partial x \partial y} \right|_{y=F^{-2}\eta^{(s)}(x)}.$$

The source of the term  $\pm \frac{1}{2\sqrt{k}}\eta_{10,x}$  in (93) is readily associated to the group velocity of the wave packet relative to the ocean [1, Sec. 3.3]. The penultimate term describes an interaction between the ship wave and the ocean wave packet – but since the coefficient is imaginary, this will again only affect the phase and not the amplitude of the wave packet! The last term, which is nonlinear in the ocean wave packet amplitude but does not involve the ship wave, is the same as that found in equations describing the evolution of a weakly nonlinear wave packet (not interacting with another wave) [13, 16].

#### 7.4.6 Final Approximate Solution

Rather than solving (93) for  $\eta_{11}$  and adding it to the solution for  $\eta_{10}$  obtained above, it is useful to renormalize the asymptotic expansion by combining (87a) and (93) into a single equation for

$$\bar{\eta}(x, t) = \eta_{10}(x, t) + F^{-1}\eta_{11}(x, t),$$

which we then solve. Using the fact that  $|\eta_{10}(x, t)|^2 = |A^{(0)}(x - t)|^2$  (which follows from Eq. (89)), we can write

$$\begin{aligned} \frac{\partial \bar{\eta}(x, t)}{\partial t} + \left(1 \pm \frac{1}{2}F^{-1}k^{-1/2}\right) \frac{\partial \bar{\eta}(x, t)}{\partial x} \\ + i \left( k \overline{\phi_x^{(s)}}(x) \mp 4F^{-1}a^2k^{5/2}|A^{(0)}(x - t)|^2 \pm \frac{1}{2}F^{-1}k^{1/2}\overline{\phi_{xy}^{(s)}}(x) \right. \\ \left. \mp \frac{1}{2}F^{-1}k^{3/2}((\overline{\phi_x^{(s)}}(x))^2 + (\overline{\phi_y^{(s)}}(x))^2) \right) \bar{\eta}(x, t) + O(F^{-2}) = 0. \end{aligned}$$

with the initial wavepacket conditions:

$$\bar{\eta}(x, t = 0) = A^{(0)}(x).$$

Solution by the method of characteristics yields:

$$\begin{aligned} \bar{\eta}(x, t) = e^{\pm iF^{-1}a^2k^{5/2}|A^{(0)}(x-t)|^2t} \\ \times \exp \left[ i \int_{x - (1 \pm \frac{1}{2}F^{-1}k^{-1/2})t}^x \left( -k \overline{\phi_x^{(s)}}(x') \mp \frac{1}{2}F^{-1}k^{1/2}\overline{\phi_{xy}^{(s)}}(x') \right. \right. \\ \left. \left. \pm \frac{1}{2}F^{-1}k^{3/2}((\overline{\phi_x^{(s)}}(x'))^2 + (\overline{\phi_y^{(s)}}(x'))^2) \right) dx' \right] \\ \times A^{(0)} \left( x - \left(1 \pm \frac{1}{2}k^{-1/2}F^{-1}\right)t \right) + O(F^{-2}). \end{aligned}$$

This describes the evolution of the ocean wave packet amplitude through  $O(F^{-1})$ . (There are also superharmonic waves generated with twice the wavenumber.) We see that to this order, the ship wave only induces a phase shift, and neither amplifies nor attenuates the ocean wave packet.

## 7.5 Estimate of Magnitude of Change of Ocean Wave Packet Amplitude

In order to ascertain the leading order change to the ocean wave packet amplitude as it encountered the ship wave, we would have to proceed to the next order in the asymptotic expansion. However, as discussed in Appendix A.3, to proceed to consider  $O(F^{-2})$  changes to the ocean wave amplitude, we must make a more precise use of the conformal mapping formalism than what we have been able to do in this report which stopped consideration at  $O(F^{-1})$ . The analysis would be even more laborious.

We will instead attempt simply to estimate the maximum order to which the ocean wave amplitude can be expected to change upon encountering the ship wave. Naively, we would say that the next order corrections which we have not yet computed would be  $O(F^{-2})$ . However, we must remember the resonance contribution discussed in Subsubsection A.3.3, which induces a term of  $O(F^{-2})$ . Now we must remember that we assumed in our analysis that

$$\varepsilon = aF^{-1}, \quad \delta = k^{-1}F^{-2},$$

with fixed constants  $a$  and  $k$ , but that our analysis extends readily to the wider regime:

$$F^{-2} \ll \varepsilon \ll 1, \quad F^{-4} \ll \delta \ll 1.$$

For this wider regime, we should say instead that the next order corrections are at most  $O(F^{-2} + \varepsilon^2 + \delta)$ . Consulting the data in Table 2 in Subsection 4.5 for a 200 knot surfing ship, we see that the correction can be expected to be no more than a few percent of the original wave amplitude. Of course, to quantify “a few percent” and whether the ocean waves amplify or attenuate as they ride up the ship wave would require a detailed calculation.

## 7.6 Solution for 100 knot ship speed

As discussed in Subsection 4.5, the appropriate asymptotic limit to describe the surfing ship moving at 100 knots is  $\varepsilon = aF^{-2}$  and  $\delta = k^{-1}F^{-2}$ . This amounts to pushing the nonlinear terms involving the ocean wave packet to higher order, so the asymptotic analysis is simpler than that for the 200 knot case presented above. We

can immediately conclude, by dropping the nonlinear terms which were pushed to higher order, that the ocean wave packet perturbation for the 100 knot speed ship evolves as follows:

$$\begin{aligned} \eta^{(p)}(x, t) &= \bar{\eta}(x, t)e^{ik(\bar{x}-\bar{t}) \mp ik^{1/2}t^*} + \text{c.c.} + O(F^{-2}), \\ \bar{\eta}(x, t) &= \exp \left[ i \int_{x-(1 \pm \frac{1}{2}F^{-1}k^{-1/2})t}^x \left( -k\overline{\phi_x^{(s)}}(x') \mp \frac{1}{2}F^{-1}k^{1/2}\overline{\phi_{xy}^{(s)}}(x') \right. \right. \\ &\quad \left. \left. \pm \frac{1}{2}F^{-1}k^{3/2}((\overline{\phi_x^{(s)}})^2(x') + (\overline{\phi_y^{(s)}})^2(x')) \right) dx' \right] \\ &\quad \times A^{(0)} \left( x - (1 \pm \frac{1}{2}k^{-1/2}F^{-1})t \right). \end{aligned}$$

Again, the ocean wave packet amplitude is not changed in magnitude through  $O(F^{-1})$ ; only its phase is changed.

By the same considerations as those presented in Subsection 7.5, we would say that the ocean waves will only be changed by at most  $O(F^{-2} + \delta + \varepsilon)$ . Consulting Table 2 in Subsection 4.5 for the values of these nondimensional parameters for a 100 knot ship speed, we predict that the ocean wave amplitude will change by no more than a few tenths of their original height as they encounter the ship wave, though we can't say whether they increase or decrease. More precisely, the ocean waves can be expected to be affected more severely by the ship wave at 100 knots than at 200 knots. This may seem surprising since at 200 knots the wave equations are more nonlinear. The resolution is that the extra nonlinearities only induced a phase shift in the ocean waves to the order computed, and the small nondimensional parameters  $\delta$  and  $F^{-1}$  are smaller at 200 knots than at 100 knots.

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# A Perturbative Conformal Mapping Method for Solving Multiscale Potential Equations with Irregular Boundaries

One of the mathematically difficult aspects of the surface wave equations (9) is the need to solve the potential equation (9a) on the fluid domain  $y < F^{-2}\eta(x, t)$ , which will be irregular and time-dependent due to the distortion of the ocean surface by the waves. We can however exploit the small aspect ratio  $F^{-2}$  of the ocean surface (due to both ship and ocean waves) to take a perturbative approach. One method was described in Section 5, in which a perturbative change of variables (17) was performed which flattened the boundary and added some small terms to the potential equation. Since the leading order terms in the potential equation still give Laplace's equation, and we know how to solve Laplace's equation on a half-plane, we appear to have a good asymptotic setup, and indeed one can proceed formally as we did in Section 5. The problem, however, is that this approach will generate equations similar to the Cauchy-Riemann equation [18, Sec. 7] as amplitude coefficient equations describing the slow-scale variation of the potential function (see (36)). But the Cauchy-Riemann equation with prescribed data on the real axis does not in general have a solution which is well-defined on the lower-half plane [18, Ch. 8]. Indeed the existence of a solution to such a Cauchy-Riemann equation is equivalent to the possibility of extending the function defined on the real axis to an analytic complex function on the lower half plane. We implicitly assumed the existence of such a solution in Eq. (36), but we have not provided evidence that such a solution even exists. This seems to suggest that the standard assumptions of multiple-scale analysis do not apply to the solution of the potential equation with multi-scale variation in its boundary data.

We can provide an explanation for the underlying difficulty and develop an asymptotic procedure which avoids unsolvable equations by mapping the ocean surface to its undisturbed level through a *conformal mapping* rather than the flattening mapping (17). A conformal mapping is defined as one which preserves angles. For our purposes, the most important property of a conformal mapping is that it leaves the potential equation unchanged.

## A.1 Conformal Mapping Setup

In symbols, let  $(x, y) \rightarrow (\mathcal{X}, \mathcal{Y})$  be a conformal mapping which maps the region  $y < F^{-2}\eta(x, t)$  to  $\mathcal{Y} < 0$ . Note that this conformal mapping will depend on time since the free surface does, so  $\mathcal{X} = \mathcal{X}(x, y, t)$  and  $\mathcal{Y} = \mathcal{Y}(x, y, t)$ . We expect the conformal mapping to be a near-identity mapping, in the sense that  $\mathcal{X} = x + O(F^{-2})$

and  $\mathcal{Y} = y + O(F^{-2})$ , so the mapping is invertible for each  $t$  and the limit  $y \rightarrow -\infty$  will be equivalent to  $\mathcal{Y} \rightarrow -\infty$ .

Then, in terms of the new conformally mapped variables, the potential equation reads:

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial \mathcal{X}^2} + \frac{\partial^2 \Phi}{\partial \mathcal{Y}^2} &= 0, \\ \Phi(\mathcal{X}, \mathcal{Y} = 0, T) &= \bar{\Phi}(\mathcal{X}, T), \\ \lim_{\mathcal{Y} \rightarrow -\infty} \Phi(\mathcal{X}, \mathcal{Y}, T) &= 0,\end{aligned}$$

where  $\bar{\phi}(x, t) = \phi(x, y = F^{-2}\eta(x, t), t)$  is the surface value of the potential function, and the functions  $\Phi$  and  $\bar{\Phi}$  are just the functions  $\phi$  and  $\bar{\phi}$  in remapped variables

$$\phi(x, y, t) = \Phi(\mathcal{X}(x, y, t), \mathcal{Y}(x, y, t), t) \quad \bar{\phi}(x, t) = \bar{\Phi}(\mathcal{X}(x, y, t), t).$$

The potential equation in the conformally remapped variables is just Laplace's equation on the lower-half plane, which has the following exact solution in terms of the surface data:

$$\Phi(\mathcal{X}, \mathcal{Y}, t) = \int_{-\infty}^{\infty} e^{i\xi\mathcal{X}} e^{|\xi|\mathcal{Y}} \hat{\Phi}(\xi, t) d\xi,$$

where

$$\hat{\Phi}(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi\mathcal{X}} \bar{\Phi}(\mathcal{X}, t) d\mathcal{X}$$

is the Fourier transform of the surface value  $\bar{\phi}(x, t)$  of the potential function. Returning then to the original variables, we can relate the velocity potential throughout the ocean region  $y < F^{-2}\eta(x, t)$  to its surface values at  $y = F^{-2}\eta(x, t)$  through the following formula:

$$\phi(x, y, t) = \int_{-\infty}^{\infty} e^{i\xi\mathcal{X}(x, y, t)} e^{|\xi|\mathcal{Y}(x, y, t)} \hat{\Phi}(\xi, t) d\xi.$$

In other words, solving the potential equation (9a) is equivalent to finding a conformal mapping  $(x, y) \rightarrow (\mathcal{X}, \mathcal{Y})$  which maps the ocean surface  $y = F^{-2}\eta(x, t)$  to its undisturbed level  $\mathcal{Y} = 0$ .

The conditions which the mapping  $(x, y) \rightarrow (\mathcal{X}, \mathcal{Y})$  must satisfy to be conformal are [5]

$$\frac{\partial \mathcal{X}(x, y, t)}{\partial x} - \frac{\partial \mathcal{Y}(x, y, t)}{\partial y} = 0, \quad \frac{\partial \mathcal{Y}(x, y, t)}{\partial x} + \frac{\partial \mathcal{X}(x, y, t)}{\partial y} = 0. \quad (94a)$$

These equations are supplemented with the boundary conditions

$$\mathcal{Y}(x, y = F^{-2}\eta(x, t), t) = 0, \quad (94b)$$

$$\lim_{y \rightarrow -\infty} (\mathcal{Y}(x, y, t) - y) = 0. \quad (94c)$$

which just expresses the desideratum that the ocean surface  $y = F^{-2}\eta(x, t)$  gets mapped to  $\mathcal{Y} = 0$ , and that the conformal map distortion vanishes far away from the ocean surface. Actually it would have been enough to just demand that  $\mathcal{Y}(x, y, t) - y$  remain bounded over the fluid region  $y < F^{-2}\eta(x, t)$ . Because the ocean surface disturbance vanishes at large distances from the ship wave, however, we expect that the conformal map distortion decays for both large  $|x|$  and large negative  $y$ . Note that we do not prescribe any boundary condition for  $\mathcal{X}$ ; nonetheless the problem (94) has a unique solution because it is elliptic [14].

Now, we'd like to obtain a perturbative solution to the potential equation (9a) through a perturbative solution of the conformal mapping problem (94), exploiting the small parameter  $F^{-1}$ . While the potential equation and conformal mapping problem are equivalent when viewed as *exact problems*, the perturbative analysis of these two problems generates different sets of equations. As we will demonstrate with a simple example in Subsection A.2, the potential equation itself is not amenable to a standard singular perturbation approach, but we can develop a fairly standard singular perturbation theory for the conformal mapping problem. First we need to massage the conformal mapping problem a bit because in its present form, the boundary of the domain changes with each order of the perturbation hierarchy. We therefore apply the same “flattening mapping” to the conformal mapping problem (94) as we did for the potential problem in Section 5:

$$\begin{aligned} X &= x, \\ Y &= y - F^{-2}\eta(x, t), \\ T &= t. \end{aligned} \quad (95)$$

By expressing the functions in terms of these “flattened” variables:

$$\begin{aligned} \tilde{\mathcal{X}}^\sharp(X, Y, T) &= \mathcal{X}(x, y, t), \\ \tilde{\mathcal{Y}}^\sharp(X, Y, T) &= \mathcal{Y}(x, y, t) \end{aligned}$$

and applying the chain rule for derivatives:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial X} - F^{-2}\eta_x(x, t)\frac{\partial}{\partial Y} = \frac{\partial}{\partial X} - F^{-2}\eta_x(X, T)\frac{\partial}{\partial Y} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial T} - F^{-2}\eta_t(x, t)\frac{\partial}{\partial Y} = \frac{\partial}{\partial T} - F^{-2}\eta_t(X, T)\frac{\partial}{\partial Y},\end{aligned}\tag{96}$$

the conformal mapping problem (94) is now expressed on a fixed domain  $Y < 0$ :

$$\frac{\partial \tilde{\mathcal{X}}^\#}{\partial X} - \frac{\partial \tilde{\mathcal{Y}}^\#}{\partial Y} - F^{-2}\eta_x \frac{\partial \tilde{\mathcal{X}}^\#}{\partial Y} = 0,\tag{97a}$$

$$\frac{\partial \tilde{\mathcal{Y}}^\#}{\partial X} + \frac{\partial \tilde{\mathcal{X}}^\#}{\partial Y} - F^{-2}\eta_x \frac{\partial \tilde{\mathcal{Y}}^\#}{\partial Y} = 0,\tag{97b}$$

$$\tilde{\mathcal{Y}}^\#(X, Y = 0, T) = 0,\tag{97c}$$

$$\lim_{Y \rightarrow -\infty} \left( \tilde{\mathcal{Y}}^\#(X, Y, T) - (Y + F^{-2}\eta(X, T)) \right) = 0.\tag{97d}$$

This may be thought of as the equation which corrects the flattened variables  $(X, Y, T)$  so that they are *conformally* related to the original physical variables  $(x, y, t)$ .

If we now substitute in an appropriate modulated wave packet assumption for  $\eta(x, t)$ , we can perform a multiple-scales analysis on Eq. (97) and obtain a hierarchy of equations which can be solved in fairly explicit form. Details of this implementation on an explicit example are presented in Section 7.

## A.2 Conformal Mapping and Nonstandard Multiscale Form

The reader may well ask how the conformal mapping approach is able to avoid generating the unsolvable Cauchy-Riemann equations which plagued the asymptotic hierarchy when attempting to solve the potential equation perturbatively in terms of flattened nonconformal variables in Section 5. Indeed, we have even posed the conformal mapping problem itself in terms of these same nonconformal flattened variables (95).

To get some insight into what is happening, let us consider a very simple model singular perturbation problem:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ for } y < 0,\tag{98a}$$

$$\phi(x, y = 0) = \psi(x)e^{ix/\delta},\tag{98b}$$

$$\lim_{y \rightarrow -\infty} \phi(x, y) = 0.\tag{98c}$$

The parameter  $\delta$  is a small positive parameter. This is a simple, linear elliptic (Laplace) problem on a simple domain (the lower half plane) with prescribed boundary data. The only unusual feature about it is the form of the boundary data: a modulated packet of rapid oscillations (think of  $\psi(x)$  as a smooth, rapidly decaying function). The system (98c) can of course be exactly solved in a number of ways, such as by Fourier transform:

$$\phi(x, y) = \int_{-\infty}^{\infty} e^{i(\xi+\delta^{-1})x+|\xi+\delta^{-1}|y} \hat{\psi}(\xi) d\xi, \quad (99)$$

where

$$\hat{\psi}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x) e^{-i\xi x} dx.$$

We will now show, however, that a standard singular perturbation, multiple scales approach will not be able to give a good approximation to this answer! Indeed, following standard procedure, we would look for an expansion of  $\phi$  in the following form:

$$\phi(x, y) = \phi_0(x, y, \tilde{x}, \tilde{y}) + \delta\phi_1(x, y, \tilde{x}, \tilde{y}) + \delta^2\phi_2(x, y, \tilde{x}, \tilde{y}) + O(\delta^3) \quad (100)$$

for small  $\delta$ , where

$$\tilde{x} = x/\delta, \quad \tilde{y} = y/\delta$$

are the rapidly varying variables. Substitution of the assumed expansion (100) into the model system (98c), we would arrive at:

$$\begin{aligned} 0 = & \delta^{-2} \left( \frac{\partial^2 \phi_0}{\partial \tilde{x}^2} + \frac{\partial^2 \phi_0}{\partial \tilde{y}^2} \right) + \delta^{-1} \left( \frac{\partial^2 \phi_1}{\partial \tilde{x}^2} + \frac{\partial^2 \phi_1}{\partial \tilde{y}^2} + 2 \frac{\partial^2 \phi_0}{\partial \tilde{x} \partial x} + 2 \frac{\partial^2 \phi_0}{\partial \tilde{y} \partial y} \right) \\ & + \left( \frac{\partial^2 \phi_2}{\partial \tilde{x}^2} + \frac{\partial^2 \phi_2}{\partial \tilde{y}^2} + 2 \frac{\partial^2 \phi_1}{\partial \tilde{x} \partial x} + 2 \frac{\partial^2 \phi_1}{\partial \tilde{y} \partial y} + \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} \right) + O(\delta) = 0 \end{aligned}$$

with boundary conditions:

$$\begin{aligned} \phi_0(x, y = 0, \tilde{x}, \tilde{y} = 0) &= \psi(x) e^{i\tilde{x}}, \\ \phi_j(x, y = 0, \tilde{x}, \tilde{y} = 0) &= 0 \text{ for } j \geq 1, \\ \lim_{y \rightarrow -\infty} \phi_j(x, y, \tilde{x}, \tilde{y}) &= 0 \text{ for } j \geq 0. \end{aligned}$$

We now attempt to solve these equations order by order.

### A.2.1 First Equation in Asymptotic Hierarchy ( $O(\delta^{-2})$ )

$$\frac{\partial^2 \phi_0}{\partial \tilde{x}^2} + \frac{\partial^2 \phi_0}{\partial \tilde{y}^2} = 0$$

with boundary condition

$$\phi_0(x, y = 0, \tilde{x}, \tilde{y} = 0) = \psi(x)e^{i\tilde{x}}.$$

These equations fix the dependence of  $\phi_0$  on the fast variables:

$$\phi_0(x, y, \tilde{x}, \tilde{y}) = A_0(x, y)e^{i\tilde{x} + \tilde{y}}, \quad (101)$$

where  $A_0(x, y)$  is so far only constrained to satisfy

$$A_0(x, y = 0) = \psi(x).$$

### A.2.2 Second Equation in Asymptotic Hierarchy $O(\delta^{-1})$

$$\frac{\partial^2 \phi_1}{\partial \tilde{x}^2} + \frac{\partial^2 \phi_1}{\partial \tilde{y}^2} = -2 \left( i \frac{\partial A_0}{\partial x} + \frac{\partial A_0}{\partial y} \right) e^{i\tilde{x} + \tilde{y}} \quad (102)$$

with boundary condition

$$\phi_1(x, y = 0, \tilde{x}, \tilde{y} = 0) = 0.$$

If  $A_0$  is allowed to be arbitrary, then the solution to (102) would be

$$\phi_1(x, y, \tilde{x}, \tilde{y}) = - \left( i \frac{\partial A_0}{\partial x} + \frac{\partial A_0}{\partial y} \right) \tilde{y} e^{i\tilde{x} + \tilde{y}}.$$

This would be a secular term, though it is not as badly divergent as typical secularities in singular perturbation problems [19]. One way to look at such a term is to realize that it can be written in the form

$$\delta \phi_1 = \left( i \frac{\partial A_0}{\partial x} + \frac{\partial A_0}{\partial y} \right) y e^{i\tilde{x} + \tilde{y}}$$

which should be absorbable into the zeroth order term  $\phi_0$ . That means we should be able to choose  $A_0$  so that this contribution is absorbed in  $\phi_0$  rather than  $\phi_1$ , by simply making the right hand side of (102) vanish:

$$i \frac{\partial A_0}{\partial x} + \frac{\partial A_0}{\partial y} = 0. \quad (103)$$

While this equation for the amplitude coefficient is not forced upon us as insistently as in typical singular perturbation problems, balking will not lead to a more natural equation for  $A_0$  later. The choice not to enforce Eq. (103) merely complicates the asymptotic expansion but does not in the end avoid the problem which we now point out.

### A.2.3 Ill-Posed Equation for Amplitude Coefficient $A_0$

From the first two equations in the asymptotic hierarchy, we have deduced the following system for the amplitude coefficient  $A_0$ :

$$i\frac{\partial A_0}{\partial x} + \frac{\partial A_0}{\partial y} = 0 \text{ for } y < 0, \tag{104}$$

$$A_0(x, y = 0) = \psi(x).$$

This is a Cauchy-Riemann equation which is equivalent to finding an analytic continuation of the function  $\psi(x)$  from the real axis to the lower half complex plane [18, Sec. 7, Ch. 8]. This problem does not have a solution except for very special types of functions  $\psi(x)$  [18, Ch. 8]! Indeed, assuming  $\psi(x)$  is integrable (as it would be for a localized wave packet), we can write it as a Fourier integral

$$\psi(x) = \int_{-\infty}^{\infty} e^{i\xi x} \hat{\psi}(\xi) d\xi$$

and then attempt to write the following formal solution to (104):

$$A_0(x, y) = \int_{-\infty}^{\infty} e^{i\xi x + \xi y} \hat{\psi}(\xi) d\xi. \tag{105}$$

This would be a correct solution if  $\hat{\psi}(\xi)$  decayed faster than exponentially so that  $e^{\xi y} \hat{\psi}(\xi)$  would remain an integrable function of  $\xi$  for arbitrary  $y < 0$ . But most functions, even smooth functions, do not have a faster-than-exponentially decaying Fourier transform, and the integral representation (105) will then diverge once  $y$  becomes sufficiently negative. Even worse, the Fourier transform of most (even smooth) functions do not even decay exponentially so that the expression (105) will diverge for *all*  $y < 0$ . Therefore, we conclude that the amplitude coefficient equation (104) does not generally have a solution.

It may be surprising that the technical mathematical properties of  $\psi(x)$  should have such a strong influence on whether we can solve the amplitude coefficient equation emerging from the system (98c). We will come back in Subsubsection A.2.5 to discuss what these technical considerations imply in practical problems (such as the one studied in the main text of this report), but let us first discuss the mathematical reasons why the asymptotic expansion (100) produced an amplitude coefficient equation (104) which is not generally solvable.

### A.2.4 Mathematical Reason for Breakdown of Asymptotic Expansion

With knowledge of the exact solution, we can see what the trouble is; the exact solution is *not* of the standard multiple scales form assumed in (100). The culprit is

the factor  $e^{|\xi+\delta^{-1}|y}$  in the integrand. For values of  $\xi$  of order unity, this factor does behave as a function depending on separate scales  $\tilde{y} = y/\delta$  and  $y$ . But note that the Fourier integral extends over all values of  $\xi$ . Consider then the contribution from  $|\xi + \delta^{-1}| \leq 1$ :

$$\int_{-\delta^{-1}-1}^{-\delta^{-1}+1} e^{i(\xi+\delta^{-1})x+|\xi+\delta^{-1}|y} \hat{\psi}(\xi) d\xi = \int_{-1}^1 e^{i\xi'x+|\xi'|y} \hat{\psi}(-\delta^{-1} + \xi') d\xi'.$$

Note that this contribution does *not* have the fast exponential decay proportional to  $e^{\tilde{y}} = e^{y/\delta}$  which the contribution from order unity values of  $\xi$  do. Therefore, we see that the fast dependence (101) predicted the leading order term in the asymptotic expansion is not really correct. As  $\xi$  becomes large and comparable to  $\delta^{-1}$ , the contribution from the integral (99) does not have a clean separation of dependence on fast and slow scales. What is happening is that these Fourier coefficients of the modulation  $\psi(x)$  are in near resonance and therefore nearly cancel the fast oscillations  $e^{i\xi x}$  which gave rise to the rapid decay in  $y$ .

Observe, however, that the Fourier coefficients  $\hat{\psi}(\xi)$  will generally be small for  $\xi \sim O(\delta^{-1})$ . Then the exact solution (99) can roughly be thought of as a sum of two kinds of contributions:

- the *straightforward* contribution from  $|\xi| \sim O(1)$ , which is order unity near  $y = 0$  but decays very rapidly (like  $e^{y/\delta}$ ) for  $y < 0$ ,
- a *resonant* contribution from  $\xi \approx \delta^{-1}$  which has smaller amplitude but decays more slowly (like  $e^y$ ) for  $y < 0$ .

The resonant contribution does not fit into the framework assumed in the multiple scales asymptotic expansion (100). (The situation is not helped if one eschews the use of the amplitude coefficient equation (103). The practical question then, is, when can we safely ignore this troublesome contribution?

### A.2.5 Practical Implications for Asymptotic Expansions

The answer depends of course on what we are interested. We will want to make contact again shortly with the surface wave problem discussed in the main text of the report. There we solve the potential equation in order to obtain information about the derivatives of  $\phi$  at the surface. In our model problem, then, let us suppose we want an asymptotic approximation for the  $y$  derivative

$$\overline{\phi}_y(x) = \left. \frac{\partial \phi}{\partial y} \right|_y.$$

The exact expression, obtained from the exact solution (99) is

$$\overline{\phi}_y(x) = \int_{-\infty}^{\infty} e^{i(\xi+\delta^{-1})x} |\xi + \delta^{-1}| \hat{\psi}(\xi) d\xi. \quad (106)$$

Suppose we formally compute the surface  $y$  derivative of the formal approximate solution we obtained from the asymptotic expansion:

$$\phi^{(\text{asy})}(x) = \int_{-\infty}^{\infty} e^{i(\xi+\delta^{-1})x+(\delta^{-1}+\xi)y} \hat{\psi}(\xi) d\xi, \quad (107)$$

not worrying for the moment about the fact that this integral may well diverge for  $y < 0$ . We obtain:

$$\overline{\phi}_y^{\text{asy}}(x) = \int_{-\infty}^{\infty} e^{i(\xi+\delta^{-1})x} (\delta^{-1} + \xi) \hat{\psi}(\xi) d\xi.$$

Note that this result is well-defined and finite, but differs from the exact result (106). The difference is:

$$\overline{\phi}_y^{\text{asy}}(x) - \overline{\phi}_y(x) = 2 \int_{-\infty}^{-\delta^{-1}} e^{i(\xi+\delta^{-1})x} (\delta^{-1} + \xi) \hat{\psi}(\xi) d\xi,$$

which can be bounded by

$$|\overline{\phi}_y^{\text{asy}}(x) - \overline{\phi}_y(x)| \leq 2 \int_{-\infty}^{-\delta^{-1}} |\delta^{-1} + \xi| |\hat{\psi}(\xi)| d\xi.$$

If  $|\hat{\psi}(\xi)| \leq C|\xi|^{-\gamma}$  for some constants  $C$  and  $\gamma$ , then

$$|\overline{\phi}_y^{\text{asy}}(x) - \overline{\phi}_y(x)| \leq O(\delta^{\gamma-2}).$$

This estimate will generally be fairly sharp, meaning that this difference will usually have a nonzero  $\delta^{\gamma-2}$  contribution. Noting that  $\overline{\phi}_y \sim \text{ord}(\delta^{-1})$ , meaning strictly comparable to  $\delta^{-1}$ , we see that the *relative* size of the error made by neglecting the resonant contribution is  $O(\delta^{\gamma-1})$ . Therefore, if we can tolerate an error of this size in  $\overline{\phi}_y(x)$ , then we can use the naive, possibly divergent expression (107) to get a satisfactory approximation. To compute  $\overline{\phi}_y(x)$  with relative error  $O(\delta^{\gamma-1})$  or smaller requires us however to work with the precise expression (99).

With this, we conclude our discussion of the model problem (98c), and apply what we have learned to the asymptotic solution of the potential equation in the surface wave problem which we are studying in the main text of this report.

### A.3 Application to Interacting Surface Wave Problem

The same problems that plagued the simple model problem (98c) are naturally present in the solution of the potential equation in the interacting surface wave problem (9). There is the further complication that the surface itself is not flat, but has variations on fast and small scales. We will first discuss what the above considerations imply for the linearized analysis in Section 5, then describe how they motivate our solution strategy for the nonlinear asymptotic analysis in Sections 6 and 7.

### A.3.1 Application to Linearized Analysis

We mentioned above that the calculations in Section 5 assumed the analytical continuability of some functions from the real axis into the lower half plane (see Eq. 36), which generally cannot be assumed. This analytical continuation, however, is only needed to compute surface derivatives of the velocity potential and is analogous to writing down the naive, possibly divergent expression (107) as a solution to (98c). For the same reasons, this formal approach will produce an approximation which is mathematically correct, up to a *relative* error of  $O(\delta^{\gamma-1})$ , if  $\hat{\bar{\phi}}(\xi, t) \leq C|\xi|^{-\gamma}$  for some constants  $C$  and  $\gamma$ . The rate of decay of  $\hat{\bar{\phi}}(\xi, t)$  is determined by the smoothness of  $\bar{\phi}(x, t)$ . The most important source of nonsmoothness is the pressure imposed by the ship. From [10], the pressure imposed by the ship can be shown to behave like  $\Pi(x) \approx C_p|x - x_*|^{1/2}$ , where  $x_*$  is either position at which the ship-sea contact ends. (The constant  $C_p$  may be different at either end of the ship). From the Bernoulli equation (10b), we see that  $\bar{\phi}^{(s)}(x, t)$  should be one order smoother than  $\Pi(x)$ , so that  $\bar{\phi}^{(s)}(x, t)$  should behave like  $|x - x_*|^{3/2}$  near the edge of the ship, and this will be its least smooth component. Because of the interaction between the ship waves and ocean waves, we can generally expect that the amplitude coefficients of the ocean wave packet will have a similar degree of smoothness. Consequently, by the general relation between the smoothness of a function and the rate of decay of its Fourier transform, we have that the Fourier transform of the ship wave potential  $\bar{\phi}^{(s)}(x)$  and the ocean wave perturbation potential  $\bar{\phi}^{(p)}(x, t)$  decay fast enough so that [30, Sec. 8.5.1]

$$|\hat{\bar{\phi}}^{(s)}(\xi)|, |\hat{\bar{\phi}}^{(p)}(\xi, t)| \leq C|\xi|^{-5/2}, \quad (108)$$

for some positive constant  $C$ , but no faster. Consequently, the formal procedure, which neglects the resonant contribution, will produce expressions for surface derivatives of the potential which are correct up to errors of  $O(\delta^{3/2})$ , can be chosen as small as desired. For  $\delta = F^{-4}$  (as in the linearized calculation in Section 5), the relative error is therefore  $O(F^{-6})$ . This creates an absolute error in (56) of order  $O(F^{-4})$ , which is negligible to the order of accuracy reported.

### A.3.2 Comment on Applicability in Presence of Other Sources of Smoothing of Pressure

Before proceeding further, let us first emphasize that the above analysis is relevant even if there are other mechanisms not included in our framework (such as viscosity or some tapering of the pressure loading at the ship end) which actually smooth the pressure so that it vanishes more nicely than  $|x - x_*|^{1/2}$ . Any such smoothing will take place over a small length scale, and while it will imply that the Fourier transform of the velocity potentials *eventually* decay faster than indicated in Eq. (108), we should

still expect the decay law (108) to be sharp over a wide range of large values of  $\xi$ . In fact, if the physical length scale over which the smoothing occurs is  $\mathcal{L}$  and the ship length is  $L$ , then (108) will be sharp for  $1 \ll |\xi| \ll L/\mathcal{L}$ , but then the velocity potentials will decay more rapidly for  $|\xi| \gg L/\mathcal{L}$ . Since the resonant contribution comes from  $|\xi| \sim \delta^{-1}$ , our estimates of the error in neglecting it remain sharp even in the presence of smoothing of the pressure provided that  $\delta \lesssim \mathcal{L}/L$ , which is the same as saying  $\lambda \lesssim \mathcal{L}$  where  $\lambda$  is the wavelength of the ocean waves. If, however, we have instead that  $\lambda \gg \mathcal{L}$ , then the resonant contribution will be less important than the estimates we report here.

### A.3.3 Application to Nonlinear Analysis

We turn now to the implications for the nonlinear asymptotic analysis. It turns out that if we use the same flattening mapping (95) as in the linearized analysis of Section 5, and proceed formally, we eventually encounter equations of Cauchy-Riemann type for which we could not write down even a formal solution nor have confidence that one existed. The reason may be that the flattening mapping does not become close to the identity as  $y \rightarrow -\infty$ ; the flattening distortion is indeed independent of  $y$ . A *conformal* mapping which flattens the surface, however, allowed us to pursue the computations without such obstacles. Indeed, in terms of the conformally mapped coordinates, we can write down exact solutions for the potential  $\phi$  in terms of its surface value, as we discussed in Subsection A.1. No Cauchy-Riemann equation is involved here. Therefore, we can write down a clean asymptotic expansion of the form (68c) to describe the behavior of the potential in the bulk of the fluid in response to surface wave distortions. In the course of the calculations, however, we find it convenient to replace the factor

$$e^{n\delta^{-1}k+\xi|\mathcal{Y}} \longrightarrow e^{(n\delta^{-1}k+\xi)\mathcal{Y}}. \quad (109)$$

By the same reasoning as described above for the linearized analysis, this incurs a relative  $O(\delta^{3/2})$  error in the evaluation of derivatives of  $\phi$  at the surface. We took  $\delta = F^{-2}$  in our nonlinear analysis, so the relative error will be  $O(F^{-3})$ . Tracing its effects on the asymptotic hierarchy of equations, we find that this error will have at most an  $O(F^{-2})$  effect on the ocean wave functions. This error will be negligible so long as we only are explicitly computing changes through  $O(F^{-1})$  (as we are in this report), but if we wish to compute  $O(F^{-2})$  changes correctly, we will not be allowed to make the replacement (109).

We are also making a similar simplifying assumption in the nonlinear asymptotic analysis when we write down the asymptotic expansions (68d)–(68g) for the conformal mapping functions. Cauchy-Riemann equations then arise for the  $n = m$  terms, but these would be entirely avoided if we had written the more precise asymptotic

expansion

$$\begin{aligned} \tilde{\mathcal{X}}^\sharp = X + \sum_{\ell=0}^4 F^{-\ell} \sum_{n=-\tilde{N}(\ell)}^{\tilde{N}(\ell)} \left[ \sum_{\substack{m=-\tilde{M}(\ell) \\ m \neq n}}^{\tilde{M}(\ell)} \sum_{\mu \in \tilde{\mathcal{M}}} \tilde{\mathcal{X}}_{nm\mu\ell}^\sharp(X, Y, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{mk\tilde{Y}} \right. \\ \left. + \int_{-\infty}^{\infty} \Xi_{nm\mu\ell}(\xi, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{i\xi X} e^{-|nk\tilde{Y}+\xi Y|} \right] + O(F^{-5}), \quad (110a) \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{Y}}^\sharp = Y + \sum_{\ell=0}^4 F^{-\ell} \sum_{n=-\tilde{N}(\ell)}^{\tilde{N}(\ell)} \left[ \sum_{\substack{m=-\tilde{M}(\ell) \\ m \neq n}}^{\tilde{M}(\ell)} \sum_{\mu \in \tilde{\mathcal{M}}} \tilde{\mathcal{Y}}_{nm\mu\ell}^\sharp(X, Y, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{mk\tilde{Y}} \right. \\ \left. + \int_{-\infty}^{\infty} \Upsilon_{nm\mu\ell}(\xi, T) e^{ink(\tilde{X}-\tilde{T}) \mp i\mu k^{1/2} T^*} e^{i\xi X} e^{-|nk\tilde{Y}+\xi Y|} \right] + O(F^{-5}) \quad (110b) \end{aligned}$$

(with similar form for  $\mathcal{X}$  and  $\mathcal{Y}$ .) Making the replacement

$$e^{-|nk\tilde{Y}+\xi Y|} \rightarrow e^{nk\tilde{Y}+\xi Y} \quad (111)$$

would put this more precise expansion into the form (68d)–(68g), and also incurs a relative error of  $O(F^{-3})$ , once we observe via the governing equations that the surface elevation and conformal mapping functions should have the same smoothness as the potential. By following this relative error through the calculations, we find that the absolute error incurred by the simplification (111) is negligible to the order of accuracy we are pursuing. A more accurate computation may however require use of the more precise expansion (110).

## A.4 Summary of Motivation for Use of Conformal Mapping Solution Strategy

The reader may be puzzled at this point with why we have bothered with the above mathematical considerations when we have in the end indicated that a formal approach ignoring the mathematical perils does generate approximations which are good enough for our purposes. There are a few reasons:

- We have been able to quantify how accurate the formal solutions can be expected to be, and at what order of approximation the resonant contribution neglected by the formal calculations become practically relevant. It turns out that this resonant contribution occurs at the next order beyond that pursued in the present report, so an analysis which seeks to quantify  $O(F^{-2})$  changes to the ocean wave packet perturbation cannot make the formal simplifications (109) and (111).

- We have presented a framework for which asymptotic calculations can be carried out to arbitrarily high order, including resonant contributions when they are relevant.
- The formal perturbative approach based on the flattening mapping creates nasty equations which we do not know how to solve when nonlinearity is included, even if we use formal solutions like (105) for the Cauchy-Riemann equations. The conformal mapping framework, however, both allows a mathematically precise representation of the solution for the velocity potential and generates equations which can be solved without difficulty and with as much precision as desired.

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