Disregarding the 'Hole Argument'

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Abstract. Jim Weatherall has suggested that the hole argument of Earman and Norton (1987) is based on a misleading use of mathematics. I argue on the contrary that Weatherall demands an implausible restriction on how mathematics is used. The hole argument, on the other hand, is in no new danger at all.

1. Introduction

Jim Weatherall has argued that interpreters of general relativity may disregard Einstein's hole argument as presented by Earman and Norton (1987). He begins with the following innocuous observation: when we say two descriptions of the world are equivalent, we generally mean that they share some relevant structure. Moreover, the relevant structure for many mathematical descriptions is natural and obvious. For example, the category of sets is characterised by membership relations, which are preserved by functions[1]. So, functions provide a natural standard of set equivalence. The category of groups is characterised by binary operations, which are preserved by group homomorphisms[2]. So, group homomorphisms provide a natural notion of group equivalence. Weatherall quite correctly observes that when considering questions of equivalence, it is important not to conflate or otherwise mix up which structures we take to be relevant.

Fine. But Weatherall goes on to suggest that, for any legitimate[3] mathematical representation, isomorphic mathematical structures always represent the same physical situation. Here I beg to differ. When

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[1]That is, functions have the property that $a \in A$ only if $f(a) \in f(A)$.

[2]Namely, if $(R, \cdot)$ and $(S, \circ)$ are groups, then $\varphi : R \to S$ is a homomorphism iff $\varphi(a \cdot b) = \varphi(a) \circ \varphi(b)$.

[3]I use 'legitimate' to stand in for the many positive terms that Weatherall uses to describe representations satisfying his rule, including 'adequate,' 'attentive,' 'default,' 'mathematically natural,' 'philosophically satisfying,' 'reasonable,' and 'standard.'

one is concerned with facts about the real world, it is the world that ultimately adjudicates whether two mathematical descriptions are equivalent. This especially relevant for spacetime realism[4] about the spacetime manifold. Realism brings with it its own notion of equivalence, namely whether two unobservable descriptions represent the same real-world situations. One may balk at the realist concept of equivalence, claim that it is uninteresting, or otherwise to discuss it. But Weatherall's radical restriction on mathematical representation is no reason to ignore it, and it is no reason to disregard the hole argument.

2. Warming up

2.1. **Rotations of a vector.** Weatherall begins with a warm-up example that deals with abstract integers. Before reviewing it, let me suggest some further warming up that makes use of a more concrete structure, the rotations of the vector $v \in \mathbb{R}^2$ shown in Figure 1.

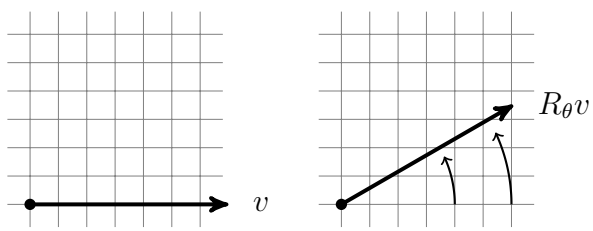


Figure 1.

We all understand what it means to rotate this figure. No mathematics is needed for that; we can just pick up the page and turn it. But suppose we also wish to describe these rotations in mathematical language. Writing v in Cartesian coordinates, we can define a group of rotations using the set of matrices $\{R_\theta : \theta \in [0, 2\pi)\}$ under the operation of matrix multiplication, where,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Each R_θ rotates the vector v counterclockwise through the angle θ, and satisfies $R_\theta R_{\theta'} = R_{\theta+\theta'}$. The identity is given by the matrix $R_0 = I$, since $IR_\theta = R_\theta I = R_\theta$ for all rotations R_θ. Thus we have a group $(R_\theta, \cdot)$ that allows us to model and describe the rotations of the concrete physical arrow above.

[4]By 'realism' I will mean the view that some unobservable proposition of a scientific theory is approximately true, approximately refers, or some similar variation; Kukla (1998) has helpfully coralled many species of realism.

However, there are many ways to instantiate a rotation group. Let me define a new binary operation '$*$' on the same matrices by the relation

$$R_\theta * R_{\theta'} := R_{\theta+\theta'-\pi}.$$

The identity element for the new group $(R_\theta, *)$ is not the identity matrix, but rather R_π, since $R_\pi * R_\theta = R_\theta * R_\pi = R_\theta$ for all rotations R_θ.

There is an isomorphism from $(R_\theta, *)$ to $(R_\theta, \cdot)$ given by $\rho(R_\theta) = R_{\theta-\pi}$. We thus have two isomorphic groups defined on the same underlying set of rotation matrices. Does this imply that both are equally correct ways to describe the physical rotations of the vector? Or that the matrices I and R_π are equally correct representatives of the identity rotation? Of course not: the first description (with the standard identity element I) is clearly correct, and the second description is not. We can say why this is without referring to any special mathematical objects: we began with a good non-mathematical understanding of what it means to rotate the vector v, and the second group fails to adequately capture that understanding.

Surprisingly, Weatherall's dictum prohibits us from making any such judgement in this language, because the two descriptions of the physical rotations are given by isomorphic mathematical models. According to Weatherall, 'isomorphic mathematical models in physics should be taken to have the same representational capacities' (Weatherall 2014, p.4). Thus one must seemingly conclude that both descriptions are equally good models of the physical rotations. This is absurd. When representing the world in terms of groups, as with many mathematical structures, there may be reasons external to the formalism that lead us to distinguish between isomorphic models. To prohibit any such distinction would be an implausible restriction on how mathematics is used.

One might be tempted to try to save Weatherall's dictum by adding more mathematical structure. For example, instead of describing the rotations of the arrow using the group $G = (R_\theta, \cdot)$, one could describe them using a *matrix representation*, which is a pair (G, ρ) with $\rho : G \to GL$ a homomorphism from G into the 'General Linear' group GL of 2x2 matrices over the real numbers. The first group $G = (R_\theta, \cdot)$ can be given a matrix representation using the identity mapping $\iota : R_\theta \to R_\theta$. The second group $G^* = (R_\theta, *)$ requires instead the mapping $\rho(R_\theta) = R_{\theta-\pi}$. We may now observe that as matrix representations of the abstract rotation group $SO(2)$, the structures (G, ι) and (G^*, ρ) are the same. They both take the group identity to the matrix identity I.

They both take the order-2 group element to R_π. And so on. On this more elaborate description of rotations, we have just one representation of the rotation group, and the previous difficulty does not arise.

I do not deny that a matrix representation theory provides one way to describe physical rotations. But I see no sense in which this helps Weatherall's restriction on representation hold water. At the risk of stating the obvious: the meaning of the mathematically-precise phrase 'matrix representation' should not be conflated with the common language use of 'representation' to mean a model or description of a physical situation. Weatherall requires a restriction on the latter in presuming that isomorphic models must represent the same situation. My response in this section is that, on the contrary, factors outside the formalism may determine that two isomorphic groups represent do not describe the same situation. This is an example of the general point I would like to argue for: when mathematical structures represent a physical situation, it is the physical situation itself that must ultimately determine what is equivalent and what is not.

2.2. **Weatherall's groups of integers.** A similar analysis is available in Weatherall's own warm-up exercise. Here we consider two more abstract structures, the groups of integers, $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$. The binary operation '$+$' of the first group is normal arithmetic addition, so that $3 + 5 = 8$, etc. The binary operation '$\tilde{+}$' of the second group is arithmetic addition followed by subtraction of 1, so that in general $n \tilde{+} m = n + m - 1$, and in particular $3 \tilde{+} 5 = 7$, etc.

Weatherall asks whether there is an ambiguity with regard to which number is the identity in the group of integers. The identity element of the first group $(\mathbb{Z}, +)$ is 0, since $0 + n = n + 0 = 0$ for all $n \in \mathbb{Z}$. The identity of the second group $(\mathbb{Z}, \tilde{+})$ is 1, since $1 \tilde{+} n = n \tilde{+} 1 = n$ for all $n \in \mathbb{Z}$. He concludes, quite correctly, that there is no ambiguity when 'the identity' is interpreted as either the group identity or the set element 0. If 'identity' means 'group identity,' then the term is only meaningful relative to a chosen group. So, 0 is the identity for $(\mathbb{Z}, +)$ and 1 is the identity for $(\mathbb{Z}, \tilde{+})$. On the other hand, if one means 'the set element n' where (say) $n = 0$, then again there is no ambiguity in specifying this object. Thus, each concept of identity conferred by the formalism provides an unambiguous way to represent the numbers.

All this is perfectly agreeable. But the set theory and group theory are not the only tools available for distinguishing numbers. One particularly relevant alternative arises when one presumes a certain kind of realism about numbers (more commonly known as a *platonism*). For the platonist, a number like (say) zero may have a mind-independent

existence, which provides the definitive criterion for whether or not a given object n can serve as its representative. The (platonist) realist standard of equivalence is reality, not group or set structure.

I do not wish to advocate mathematical platonism. The point I am making is that *if* one is a realist about some objects, then that realism provides an independent sense in which two such objects are or or not the same. For the realist about numbers there is a well-defined question of whether the identity of the group $(\mathbb{Z}, +)$ or the identity of the group $(\mathbb{Z}, \tilde{+})$ corresponds to the 'mind-independent' number 0. Some may not like the question, but the point is that it is a meaningful question, and it is the kind of question that comes up when one takes a formalism to be describing mind-independent reality. The fact that there is no ambiguity about the group identity of each is irrelevant.

Let me point out an alternative argument using Weatherall's example, which I think better sets the stage for the hole argument[5]. It is an argument against the plausibility of realism about the integers, which runs as follows.

(1) Suppose for reductio that the integers $\mathbb{Z}$ have a mind-independent existence, and as a matter of mind-independent fact form a group isomorphic to $(\mathbb{Z}, +)$, and with additive identity 0.
(2) Define the groups of integers $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$ as above, with additive identities 0 and 1, respectively.
(3) Observe that the group theoretic structure of these groups alone does not determine which (if either) of 0 or 1 is the true identity. Thus, realism about integers violates the principle (which someone might call 'Group Equivalence'[6]) that every group isomorphism relates equivalent metaphysical states of affairs.
(4) This failure may be too high a price to pay for a metaphysical view about numbers. Thus, realism about integers is implausible.

This argument is a much closer analogue of the hole argument. Of course, the final step here is questionable; I do not think the failure of 'Group Equivalence' is a convincing refutation of realism about numbers. But the analogous failure in the hole argument is much more dramatic, corresponding to a radical failure of a certain kind of Laplacian

[5]This argument is akin to a classic argument of Benacerraf (1965), and related ones discussed by (Kitcher 1984, Ch.6) and Shapiro (2000, Ch.10).

[6]As the name suggests, Group Equivalence is akin to Leibniz Equivalence in the discussion of the Hole Argument.

determinism. That is a somewhat more interesting argument against realism[7]. So, let me now turn to that argument.

3. The hole argument

What Weatherall calls the 'mathematical argument' appearing in the hole argument has the exact same character as the warm-up examples above. We begin with a spacetime (M, g_{ab}). Following Earman and Norton (1987) we construct a new spacetime $(M, \tilde{g}_{ab})$ that is related to the first by a non-trivial isometry. In particular, the new spacetime is constructed so as to be the identity function outside the open region $O \subset M$ (the 'hole'), and non-identical inside the region. And so, if one is a realist about the bare spacetime points (a view that Earman and Norton dub *manifold substantivalism*), then one might ask: isn't there an ambiguity as to which is the factual value of the metric at a given spacetime point, g or $\tilde{g}$?

If so, then this is a very serious ambiguity indeed. For it means that general relativity does not determine all the facts in the 'hole' region O on the basis of facts outside that region, and thus appears to allow for rampant indeterminim, which goes well beyond the usual difficulties with the Cauchy problem in General Relativity[8]. This, Norton and Earman famously argued, may be a high price to pay for a metaphysical view like substantivalism.

Weatherall answers that the problem lies not with manifold substantivalism, but with an illegitimate use of mathematics to represent the physical world. General relativity represents the world using Lorentzian manifolds. As Lorentzian manifolds, (M, g_{ab}) and $(M, \tilde{g}_{ab})$ are isomorphic. Weatherall's dictum now states that 'the fact that such an isometry exists provides the *only* sense in which the two spacetimes are empirically equivalent' (Weatherall 2014, p.11). Therefore, the two spacetimes should be considered equivalent from any interpretive perspective as well, manifold-substantivalism included. Since the formalism of general relativity suggests no further way to distinguish these models, Weatherall concludes that '[o]ne way or the other, the Hole Argument seems to be blocked' (Weatherall 2014, p.13).

However, many features external to the formalism of general relativity may still determine whether the physical situations represented

[7]Nethertheless, it may be eschewed by one or more of the many creative responses to the hole argument. For example, Butterfield (1989) exhibits a sense of determinism that can still be saved.

[8]John Earman charmingly calls the latter difficulties a 'dirty open secret' (Earman 1995, §3.8).

by (M, g_{ab}) and $(M, \tilde{g}_{ab})$ are the same. Most importantly, the physical world is what ultimately allows one to adjudicate physical equivalence, just as the physical rotations allowed us to adjudicate the equivalence of two rotation groups. To miss this possibility is to miss the whole point of the substantivalism debate.

If substantivalism is true, then spacetime points are real. So, manifold substantivalism implies that there is a matter of fact about what the metrical value of a spacetime point is, or whether a star is passing through that point, or any number of other properties that are left undetermined by the hole transformation. For the substantivalist, it is neither the set structure nor the metrical structure that determines the identity of spacetime points, but rather states of affairs in the real world.

Leibniz himself rejected substantivalism because it requires the introduction of vague structures that go beyond the standard formalism of physics. Weatherall is sympathetic with this point.

> [T]he would-be substantivalist, in order to reply effectively to the Hole Argument, needs to stipulate what the additional structure might be and why we should think it matters. And it is difficult to see how this could be done in a mathematically natural or philosophically satisfying way. (Weatherall 2014, p.21)

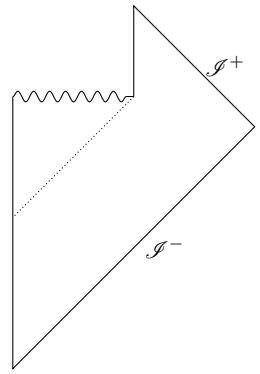
I agree that a good reason to reject manifold substantivalism is that it is poorly defined and physically unmotivated. But I do not agree that we should reject all representations which do not respect the standard of equivalence conferred by general relativity.

4. The danger of a priori physics

Let me now point out another sense in which, as a matter of good physical practice, empirical facts may bring us to distinguish between isometric spacetimes. Of course, Manifold substantivalism may not provide the sort of empirical facts that typically motivate such a distinction. But this is no reason to dismiss the physical practice in its entirety.

Mathematical models generally provide incomplete descriptions of the world. This includes models of spacetime that use the mathematical tools of classical general relativity: when we represent the world using one spacetime or another, we generally recognise that it is just an approximation of the truth. We should thus expect our understanding of the empirical world to occasionally force us to adjust the mathematical structures appearing in our mathematical representations.

Figure 2.



Weatherall's proposal asks us to do just the opposite, by taking 'legitimate' (or 'natural' or 'default') representations in a chosen formalism to force us to adjust our understanding of the empirical world:

> For the purposes of the present paper, the form of guidance I require is just this: the default sense of 'sameness' or 'equivalence' of mathematical models in physics should be the sense of equivalence given by the mathematics used in formulating those models. (Weatherall 2014, p.3)

His position is that he 'will not defend' such constraints, preferring to quietly presume them in order to analyse the hole argument.

This is a dangerous path, and one that can quickly lead to a priori physics. The real world includes all sorts of structures that are not captured by the language of Lorentzian manifolds, whether they be quantum fields, strings, loops, causal sets, or any number of other things. Given this incompleteness of representation, it is strange to say that an isometry must necessarily preserve all physical situations represented by a Lorentzian manifold. Consider the Lorentzian manifold describing an evaporating black hole, depicted in the conformal diagram of Figure 2. The evolution of a quantum field entering and leaving the black hole can be described in different ways. For example, some have taken it to be unitary, and some have taken it to be non-unitary evolution, even while adopting the same classical spacetime. But this should not dissuade anyone from the practice of representing spacetime using a Lorentzian manifold. One must simply recognise that isomorphic Lorentzian manifolds (in this case, one and the same Lorentzian manifold!) may still represent two completely different physical situations.

In short: the world generically contains distinct situations that our mathematical representations do not distinguish. One cannot simply prohibit this, on pain of doing a priori physics. I am quite sure that this is not Weatherall's intention, but it is an apparent consequence of his restriction on representation.

Let me emphasise that what I am defending in this section is the possibility of distinguishing isometric spacetimes for the purpose of making progress in physics. I am not interested in defending manifold substantivalism, which may or may not serve any such noble purpose. Semi-classical and quantum gravity may provide some interesting motivation for a distinction between two isometric spacetimes. So, it is fair to ask that manifold substantivalism find some similarly compelling motivation. It is also reasonable to demand that such distinctions find their ultimate motivation in empirical facts, or that they avoid the price of 'radical local indeterminism' identified by Earman and Norton (1987). But it is not reasonable to throw out a perfectly good mathematical representation just because the world may make distinctions that the mathematical formalism does not.

5. An alternative brand of quietism

There is gentler brand of quietism in the neighborhood of Weatherall's view that I think is worth clarifying. It is an attitude that I myself adopt from time to time, and provides some guidance on how to react to the Hole Argument. The main difference is that this view will be presented as a mere attitude, as opposed to a rule on how we are allowed to use mathematical representations. I know of no argument that establishes the present perspective. Some simply take comfort in the gentle, Buddhist-like perspective on realism that this attitude provides.

The attitude begins by stating propositions that we have good evidence to believe, in the normal language of science. For example, we may all agree that the region near the galactic centre has the structure of Kerr spacetime. But at this point, the attitude refuses all further interpretive claims. Questions like 'Is the manifold M is real?' are passed over silently. In their stead one adopts an attitude of quietism as far as the propositions of realism about unobservables are concerned.

I take this to capture a sense of what Arthur Fine has called the Natural Ontological Attitude (NOA), which he summarizes as the recommendation to 'try to take science on its own terms, and try not to read things into science' (Fine 1986, p.149). This perspective can be helpful, and indeed I often find myself joining its practitioners in the

monastery for a little peace of mind. However, there is no use pretending that this view is established by any rigorous argument or rule, as Fine is quick to point out:

> It does not comprise a doctrine, nor does it set a philosophical agenda. At most it orients us somewhat on how to pursue problems of interest, promoting some issues relative to others just because they more clearly connect with science itself. Such a redirection is exactly what we want and expect from an attitude, which is all that NOA advertises itself as being. (Fine 1986, p.10).

The NOA attitude toward manifold substantivalism, I take it, is an exercise in the discipline of silence.

However, the hole argument is not necessarily a case where this attitude is appropriate. The hole argument itself promotes a useful connection between the realism debate and philosophy of science, in establishing a link between manifold substantivalism and indeterminism. It has also promoted useful connections between the realism debate and modern physics in helping to motivate a relationist perspective on spacetime in quantum gravity[9]. Clearly, with too much quietism you may miss out on all the fun. But a healthy dose of it may still sometimes be helpful in the gentle form that I have described here.

6. Conclusion

Formal equivalence relations are only meaningful once a standard of equivalence has been identified. But it would be a mistake to suggest that the only equivalence relations are those provided by the standard formalism of general relativity. The ultimate standard of equivalence is the one conferred by the real world. And it is this standard that is at issue in the hole argument, and any other debate concerning scientific realism. The hole argument has not been 'blocked' by Weatherall's discussion, and previous commentators have not failed to 'recognize the mathematical significance of an isomorphism'(Weatherall 2014, p.13 fn.20). It simply concerns matters of realism that are precluded from Weatherall's discussion by fiat.

The danger in the radical restriction that Weatherall suggests is not only that it forgoes any question of realism. Along the way, it forgoes the practice of using mathematical representations that are incomplete. Earman and Norton (1987) argued that as a metaphysical doctrine, manifold substantivalism may come at too high a price. The price of

[9]For example, see Isham (1993), Belot and Earman (1999), and Rovelli (2004).

Weatherall's doctrine for applied mathematics, it appears, may be even higher.

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References

Belot, G. and Earman, J. (1999). From metaphysics to physics, *in* J. Butterfield and C. Pagonis (eds), *From Physics to Philosophy*, Cambridge: Cambridge University Press.

Benacerraf, P. (1965). What numbers could not be, *The Philosophical Review* **74**(1): 47–73.

Butterfield, J. (1989). The hole truth, *The British journal for the philosophy of science* **40**(1): 1–28.

Earman, J. (1995). *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetime*, New York: Oxford University Press.

Earman, J. and Norton, J. D. (1987). What Price Spacetime Substantivalism? The Hole Story, *British Journal for the Philosophy of Science* **38**: 515–525.

Fine, A. (1986). *The Shaky Game: Einstein, Realism and the Quantum Theory*, Chicago, London: The University of Chicago Press.

Isham, C. J. (1993). Canonical quantum gravity and the problem of time, *Integrable systems, quantum groups, and quantum field theories*, Springer, pp. 157–287. arXiv:gr-qc/9210011.

Kitcher, P. (1984). *The nature of mathematical knowledge*, Cambridge Univ Press.

Kukla, A. (1998). *Studies in Scientific Realism*, New York: Oxford University Press.

Rovelli, C. (2004). *Quantum Gravity*, Cambridge monographs on mathematical physics, Cambridge: Cambridge University Press.

Shapiro, S. (2000). *Thinking about mathematics*, Oxford: Oxford University Press.

Weatherall, J. O. (2014). Regarding the 'hole argument'. arXiv:1412.0303v1.