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3. The circular restricted three-body problem

3.1 Definition of the circular restricted three-body problem

[VK 5; MD 3; G09]

→ The general three-body problem can be stated as follows: known the positions and velocities of three gravitationally interacting bodies (i.e. point masses) at a given time, determine their positions and velocities at any other time.

→ The general three-body problem is extremely complex. An interesting and relevant simplified problem is the restricted problem, in which the mass of one of the three bodies is negligible. The motion of the two main bodies is an unperturbed 2-body orbit.

→ Let us further simplify the problem by assuming that the (two-body) orbit of the two more massive bodies is circular: this is the *circular restricted three-body problem*.

→ For convenience, let us call the bodies: primary (more massive of the primaries), secondary (less massive of the primaries) and test particle (negligible mass body).

3.2 Units and coordinates

→ Mass units. Total mass 1, secondary mass μ_2, primary mass $\mu_1 = 1 - \mu_2$. The mass of the test particle is m. Often used notation $\mu_2 = \mu$ and $\mu_1 = 1 - \mu$. We use μ to indicate μ_2 in this chapter (not to be confused with the gravitational mass μ used in chapter on the two-body problem).

→ Length units. (Constant) distance between primaries is $a = 1$. Primary and secondary are, respectively at distance μ and $1 - \mu$ from centre of mass.

→ Time units. It is assumed $G = 1$. From Kepler's third law we have

$$T^2 = \frac{4\pi^2}{G(\mu_1 + \mu_2)} a^3, \qquad \text{so} \qquad T = 2\pi,$$

because $a = 1$, $\mu_1 + \mu_2 = 1$. It follows $n = 2\pi/T = 1$. Even if $n = 1$ we keep n (which is an angular velocity) explicitly in the equations.

→ Take a sidereal (i.e. non-rotating, inertial) frame of reference ξ, η, ζ, centered in the centre of mass. The angle between (x, y) and (ξ, η) is the polar angular coordinate $\phi = nt$. The position of the primary as a function of time is

$$\xi_1 = -\mu \cos nt,\ \eta_1 = -\mu \sin nt,\ \zeta_1 = 0.$$

The position of secondary as a function of time is

$$\xi_2 = (1-\mu) \cos nt,\ \eta_2 = (1-\mu) \sin nt,\ \zeta_2 = 0.$$

→ Take now a synodic (i.e. rotating with the primaries) frame x, y, z, rotating with angular velocity $n = 1$. In this frame the primary has $x_1 = -\mu, y_1 = 0, z_1 = 0$ and the secondary has $x_2 = 1-\mu, y_2 = 0, z_2 = 0$, independent of time.

3.3 Equations of motion

[VK 5.2]

→ The Hamiltonian of the test particle in the sidereal (inertial) frame is

$$\mathcal{H} = \frac{1}{2m}\left({p_\xi}^2 + {p_\eta}^2 + {p_\zeta}^2\right) - \frac{(1-\mu)m}{r_1} - \frac{\mu m}{r_2},$$

where

$$r_1 = \sqrt{(\xi-\xi_1)^2 + (\eta-\eta_1)^2 + (\zeta-\zeta_1)^2} = \sqrt{(\xi+\mu\cos nt)^2 + (\eta+\mu\sin nt)^2 + \zeta^2},$$

$$r_2 = \sqrt{(\xi-\xi_2)^2 + (\eta-\eta_2)^2 + (\zeta-\zeta_2)^2} = \sqrt{[\xi-(1-\mu)\cos nt]^2 + [\eta-(1-\mu)\sin nt]^2 + \zeta^2}.$$

Note that $\mathcal{H}$ depends explicitly on time, because ξ_1 and ξ_2 are functions of t, so $\mathcal{H} = \mathcal{H}(\mathbf{p}, \mathbf{q}, t)$.

→ The coordinates x, y, z are related to ξ, η, ζ by

$$\xi = x\cos nt - y\sin nt, \qquad \eta = x\sin nt + y\cos nt, \qquad \zeta = z,$$

the inverse of which is

$$x = \xi\cos nt + \eta\sin nt,$$

$$y = -\xi\sin nt + \eta\cos nt,$$

$$z = \zeta.$$

See plot of x-y,ξ-η, FIG CM3.1 (Fig. 3.1 MD).

→ The above transformation (rotation) is a canonical transformation from $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$, where $\mathbf{q} = (\xi, \eta, \zeta)$, $\mathbf{p} = (p_\xi, p_\eta, p_\zeta)$, $\mathbf{Q} = (x, y, z)$, $\mathbf{P} = (p_x, p_y, p_z)$. The transformation is obtained by the following generating function (see G09) of the form $F = F(\mathbf{q}, \mathbf{P}, t)$:

$$F(\xi, \eta, \zeta, p_x, p_y, p_z, t) = (\xi\cos nt + \eta\sin nt)p_x + (-\xi\sin nt + \eta\cos nt)p_y + \zeta p_z,$$

because

$$x = \frac{\partial F}{\partial p_x} = \xi \cos nt + \eta \sin nt,$$

$$y = \frac{\partial F}{\partial p_y} = -\xi \sin nt + \eta \cos nt,$$

$$z = \frac{\partial F}{\partial p_z} = \zeta$$

The relations between the momenta are

$$p_\xi = \frac{\partial F}{\partial \xi} = p_x \cos nt - p_y \sin nt,$$

$$p_\eta = \frac{\partial F}{\partial \eta} = p_x \sin nt + p_y \cos nt,$$

$$p_\zeta = \frac{\partial F}{\partial \zeta} = p_z,$$

so ${p_x}^2 + {p_y}^2 + {p_z}^2 = {p_\xi}^2 + {p_\eta}^2 + {p_\zeta}^2$.

→ The Hamiltonian in the synodic frame is

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}.$$

We have

$$\frac{\partial F}{\partial t} = (-n\xi \sin nt + n\eta \cos nt)p_x + (-n\xi \cos nt - n\eta \sin nt)p_y = n(yp_x - xp_y)$$

so

$$\mathcal{H}' = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) + m\Phi(r_1, r_2) + n(yp_x - xp_y),$$

where

$$\Phi(r_1, r_2) \equiv -\frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

and, in the synodic coordinates,

$$r_1^2 = (x + \mu)^2 + y^2 + z^2, \qquad r_2^2 = [x - (1-\mu)]^2 + y^2 + z^2.$$

→ We can eliminate the test-particle mass m by performing a transformation $\tilde{p}_x = p_x/m$, $\tilde{p}_y = p_y/m$, $\tilde{p}_z = p_z/m$. The equations of motion keep the canonical form with the Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}'/m$ (see G09). This can be seen also by noting that

$$\dot{p}_x = -\frac{\partial \mathcal{H}'}{\partial x} \implies \frac{\dot{p}_x}{m} = -\frac{\partial(\mathcal{H}'/m)}{\partial x} \implies \dot{\tilde{p}}_x = -\frac{\partial \tilde{\mathcal{H}}}{\partial x}$$

and

$$\dot{x} = \frac{\partial \mathcal{H}'}{\partial p_x} = \frac{\partial(\mathcal{H}'/m)}{\partial p_x/m} \implies \dot{x} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_x}$$

So we get

$$\tilde{\mathcal{H}} = \frac{1}{2}(\tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2) + \Phi(r_1, r_2) + n(y\tilde{p}_x - x\tilde{p}_y).$$

→ The equations of motion are

$$\dot{x} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_x} = \tilde{p}_x + ny$$
$$\dot{y} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_y} = \tilde{p}_y - nx$$
$$\dot{z} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_z} = \tilde{p}_z$$
$$\dot{\tilde{p}}_x = -\frac{\partial \tilde{\mathcal{H}}}{\partial x} = n\tilde{p}_y - \frac{\partial \Phi}{\partial x}$$
$$\dot{\tilde{p}}_y = -\frac{\partial \tilde{\mathcal{H}}}{\partial y} = -n\tilde{p}_x - \frac{\partial \Phi}{\partial y}$$
$$\dot{\tilde{p}}_z = -\frac{\partial \tilde{\mathcal{H}}}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

→ The first three equations above can be written as $\tilde{p}_x = \dot{x} - ny$, $\tilde{p}_y = \dot{y} + nx$, $\tilde{p}_z = \dot{z}$. Differentiating these w.r.t. time we get

$$\dot{\tilde{p}}_x = \ddot{x} - n\dot{y}$$
$$\dot{\tilde{p}}_y = \ddot{y} + n\dot{x}$$
$$\dot{\tilde{p}}_z = \ddot{z},$$

which, combined with the last three give

$$\ddot{x} - n\dot{y} = n\dot{y} + n^2 x - \frac{\partial \Phi}{\partial x}$$
$$\ddot{y} + n\dot{x} = -n\dot{x} + n^2 y - \frac{\partial \Phi}{\partial y}$$
$$\ddot{z} = -\frac{\partial \Phi}{\partial z}$$

so

$$\ddot{x} = 2n\dot{y} + n^2 x - \frac{\partial \Phi}{\partial x}$$
$$\ddot{y} = -2n\dot{x} + n^2 y - \frac{\partial \Phi}{\partial y}$$
$$\ddot{z} = -\frac{\partial \Phi}{\partial z}$$

so

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y}$$
$$\ddot{z} = \frac{\partial U}{\partial z}$$

where

$$U = \frac{n^2}{2}(x^2 + y^2) - \Phi$$

is the (positive) effective potential. $2n\dot{y}$ and $-2n\dot{x}$ are the Coriolis terms, $n^2(x^2 + y^2)/2$ is the centrifugal potential.

3.4 Jacobi integral

[VK G09 MD]

→ The total energy is not conserved in the restricted three-body problem (because the gravitational effect of the test particle on the primaries is neglected). This can be seen also by noting that the Hamiltonian $\mathcal{H}$ depends explicitly on time. But there is another important integral of motion: the so-called *Jacobi integral* $C_{\rm J} \equiv -2\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is the mass-normalized Hamiltonian in the synodic frame (which does not depend explicitly on time).

→ Take the Hamiltonian

$$\tilde{\mathcal{H}} = \frac{1}{2}(\tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2) + ny\tilde{p}_x - nx\tilde{p}_y + \Phi,$$

substituting

$$\tilde{p}_x = \dot{x} - ny, \qquad \tilde{p}_y = \dot{y} + nx, \qquad \tilde{p}_z = \dot{z},$$

we get

$$\begin{aligned}\tilde{\mathcal{H}} &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{n^2}{2}(x^2 + y^2) + \Phi \\ &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U\end{aligned}$$

so

$$-2\tilde{\mathcal{H}} = 2U - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = const = C_{\rm J},$$

where $C_{\rm J} = 2U - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ is a constant known as the Jacobi integral.

→ $C_{\rm J}$ can be used to constrain regions allowed for the orbit, because $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - C_{\rm J} \geq 0$, so we must have $U \geq C_{\rm J}/2$. Note that U is positive by construction.

→ If we know at some time position and velocity of the test particle, we know the value of $C_{\rm J}$ at all times.

→ Taking $\dot{x} = \dot{y} = \dot{z} = 0$, for given $C_{\rm J}$, we can construct zero-velocity surfaces (*Hill surfaces*), which separate allowed and forbidden regions in the space x, y, z. Allowed regions are those for which $U \geq C_{\rm J}/2$

→ At fixed z we can consider zero-velocity curves, which separate allowed and forbidden regions in the space x, y. For instance, we can look at zero-velocity curves in the $z = 0$ plane. See plots: FIG CM3.2a (fig. 3.8 MD), FIG CM3.2b and FIG CM3.3 (fig. 5.2 VK).

→ Drawing plots of forbidden and allowed regions for decreasing $C_{\rm J}$, it is clear that for large values of $C_{\rm J}$ (i.e. large forbidden areas) the system is "Hill stable": one or two allowed regions around primary and secondary, not connected with the outer allowed region.

→ We note some particular points in which the zero-velocity curves cross: the collinear points L_1 between primary and secondary, L_2 (on the side of the secondary) and L_3 (on the side of the primary). Note that L_1, L_2, L_3 are saddle points. Other two particular points are the minima of U: the triangular points L_4 (leading) and L_5 (trailing), forming equilateral triangles with the positions of the primary and the secondary.

→ L_1, L_2, L_3, L_4, L_5 are known as the Lagrangian points or libration points. We will show below that these are equilibrium points and study their stability.

3.5 Tisserand relation

[MD 3.4]

→ Let us take the Jacobi integral C_J and write it in the inertial sidereal frame (ξ, η). It can be shown (see Problem 3.1) that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 + 2n(\eta\dot{\xi} - \xi\dot{\eta}) + n^2(\xi^2 + \eta^2).$$

→ Using the above relations into the Jacobi integral

$$C_J = 2U - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + n^2(x^2 + y^2) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2,$$

we get

$$C_J = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - \dot{\xi}^2 - \dot{\eta}^2 - \dot{\zeta}^2 + 2n(\xi\dot{\eta} - \eta\dot{\xi}),$$

because we recall that

$$x^2 + y^2 = \xi^2 + \eta^2,$$

as the transformation is just a rotation.

→ In several applications $\mu \ll 1$. For instance in the problem Sun-Jupiter-comet, $M_{\rm Jupiter} \sim 10^{-3} M_\odot$, so we can take the limit $1 - \mu \sim 1$. From know on we call r the Sun-comet distance: $r = r_1$.

→ When the comet is not close to Jupiter, we can also use $\mu/r_2 \ll 1/r$ and consider the approximation of the two-body motion comet-Sun. In this case we can use the following relations for the two-body problem:

$$\tilde{E} = \frac{v^2}{2} - \frac{G(M_\odot + m_{comet})}{r} = -\frac{G(M_\odot + m_{comet})}{2a}.$$

Here $G(M_\odot + m_{comet}) \approx GM_\odot \approx 1 - \mu \approx 1$, so

$$v^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = \frac{2}{r} - \frac{1}{a},$$

so

$$C_J = \frac{2}{r} - \frac{2}{r} + \frac{1}{a} + 2n(\xi\dot{\eta} - \eta\dot{\xi})$$

→ The angular momentum is

$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$$

so

$$m_{comet}(\xi\dot{\eta} - \eta\dot{\xi}) = L_\zeta = L_z = L\cos i,$$

where i is the inclination of the comet orbit, w.r.t. the Jupiter-Sun orbital plane. Using the two-body relation

$$\tilde{L} = L/m_{comet} = \sqrt{GM_\odot a(1-e^2)}$$

with $GM_\odot \simeq 1$, so

$$\frac{\tilde{L}^2}{=} a(1-e^2)$$

and

$$\xi\dot{\eta} - \eta\dot{\xi} = \sqrt{a(1-e^2)}\cos i$$

→ Altogether (in units such that the Sun-Jupiter mean motion is $n = 1$) we get

$$C_{\rm T} \approx \frac{C_{\rm J}}{2} = \frac{1}{2a} + \sqrt{a(1-e^2)}\cos i \approx const.$$

This is *Tisserand relation*, which can be used to verify, by measuring the orbital elements, whether a comet is new or is a new passage of a previous comet scattered by a close encounter with Jupiter.

→ See figs. 3.3 and 3.4 of MD (FIG CM3.4 and FIG CM3.5). Note that in deriving Tisserand relation we have assumed that the orbit of the primaries is circular (in fact $e_{\rm Jupiter} \simeq 0.05$).

→ Further discussion on the evolution of comet orbits can be found in VK 11.6.

→ Variation of orbital elements as a consequence of a close encounter is exploited in interplanetary missions. The close passages with planets are used to modify the orbital elements of artificial satellites (e.g. Voyager, Galileo, Cassini): in this case the mechanism is called “gravit assist” or “gravitational slingshot” (see Problem 3.2).

Problem 3.1

Write $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ in sidereal coordinates ξ, η and ζ (x, y and z are the synodic coordinates).

We have

$$x = \xi\cos nt + \eta\sin nt$$

$$y = -\xi\sin nt + \eta\cos nt$$

$$z = \zeta,$$

so

$$\dot{x} = \dot{\xi}\cos nt + \dot{\eta}\sin nt - n\xi\sin nt + n\eta\cos nt,$$

$$\dot{y} = -\dot{\xi}\sin nt + \dot{\eta}\cos nt - n\xi\cos nt - n\eta\sin nt,$$

$$\dot{z} = \dot{\zeta}$$

It follows:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 + 2n(\eta\dot{\xi} - \xi\dot{\eta}) + n^2(\xi^2 + \eta^2).$$

Problem 3.2

A spacescraft undergoes the following gravitational slingshot manoeuvre. It enters a planet gravitational field with velocity $\mathbf{v}_0$ *orthogonal to the planet velocity* $\mathbf{v}_p$ *(in the heliocentric reference system) and it exits from the planet gravitational field with a velocity vector* $\mathbf{v}_1$ *forming an angle* ϑ *with respect to* $\mathbf{v}_0$*. Compute the final speed* v_1 *in the heliocentric frame.*

In the planetcentric frame of reference the orbit is a hyperbola: the kinetic energy of the spacecraft is the same when it enters and when it exits the planet gravitational field:

$$\frac{1}{2}u_0^2 = \frac{1}{2}u_1^2,$$

where $\mathbf{u}_0 = \mathbf{v}_0 - \mathbf{v}_p$ and $\mathbf{u}_1 = \mathbf{v}_1 - \mathbf{v}_p$ are, respectively, the initial and final velocities of the spacecraft in the planetcentric frame. Taking x in the direction of the motion of the planet and y in the orthogonal direction, we have

$$v_p^2 + v_{0,y}^2 = v_{1,x}^2 + v_p^2 - 2v_p v_{1,x} + v_{1,y}^2,$$

which, given that $v_{0,y} = v_0$, $v_{1,x} = v_1 \sin\vartheta$ and $v_{1,y} = v_1 \cos\vartheta$, gives

$$v_0^2 = v_1^2 \sin^2\vartheta - 2v_p v_1 \sin\vartheta + v_1^2 \cos\vartheta,$$

which can be written as

$$v_1^2 - 2v_p \sin\vartheta v_1 - v_0^2 = 0.$$

The final speed in the heliocentric frame is thus

$$v_1 = v_p \sin\vartheta + \sqrt{v_p^2 \sin^2\vartheta + v_0^2}.$$

3.6 Location of the Lagrangian points

[VK]

→ We look now for equilibrium points in the synodic (rotating) frame: in these points the test particle is not at rest (in an inertial frame), but its orbit is such that its position with resopect to the two primaries is constant.

→ We recall that the equations of motion are

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

$$\ddot{z} = \frac{\partial U}{\partial z}$$

where

$$U = \frac{1}{2}(x^2 + y^2) - \Phi,$$

$$\Phi = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \qquad r_2^2 = [x-(1-\mu)]^2 + y^2 + z^2,$$

and we have used $n = 1$. Equilibrium points are such that $\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0$.

→ Consider first the equation for z and write it explicitly:

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} = -z\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right),$$

so we must have $z = 0$ for equilibrium. In other words all equilibrium points are in the orbital plane of the primaries. Therefore we restrict hereafter to the planar problem imposing $z = 0$.

→ So the relevant equations are

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

with

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = [x-(1-\mu)]^2 + y^2.$$

→ Equilibrium points are such that $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$, so

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$
$$y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} = 0$$

→ The equation for y can be written as

$$y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0,$$

so we have two families of solutions: $y = 0$ and $y \neq 0$

3.6.1 Collinear points

→ Let us first look at the case $y = 0$: in these cases the test particle is on the same straight line as the two primaries (collinear points). The equation for x is

$$x - \frac{(1-\mu)(x+\mu)}{[(x+\mu)^2]^{3/2}} - \frac{\mu(x-1+\mu)}{[(x-1+\mu)^2]^{3/2}} = 0,$$
$$x - \frac{1-\mu}{(x+\mu)^2}\frac{x+\mu}{|x+\mu|} - \frac{\mu}{(x-1+\mu)^2}\frac{x-1+\mu}{|x-1+\mu|} = 0$$

→ Let us consider three intervals $x < -\mu$ (to the left of both primaries), $-\mu < x < 1-\mu$ (between the primaries) and $x > 1-\mu$ (to the right of both primaries). The above equation becomes:

$$x + \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0, \qquad if \qquad x < -\mu \qquad (L_3)$$
$$x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0, \qquad if \qquad -\mu < x < 1-\mu \qquad (L_1)$$
$$x - \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0, \qquad if \qquad x > 1-\mu \qquad (L_2)$$

→ The above three equations are 5th-order polynomial equations, which in general cannot be solved analytically. The solutions for given μ can be found by solving numerically the equations. It turns out that each of the three has just one real solution in the interval where it is valid. These three solutions are the x coordinates of the collinear Lagrangian points L_1, L_2 and L_3. We call L_1 the point between the primaries, L_2 on the side of the secondary and L_3 the point on the side of the primary. This choice is standard, though in the literature there are also different choices for L_1, L_2, L_3.

→ Show plot in Fig. 5.3 of VK (FIG CM3.6). Note that for L_1 and L_2,

$$x > \frac{1}{2} - \mu$$

for all values of μ.

→ *Distance of L_1 and L_2 from the secondary in the limit $\mu \ll 1$.* Let us define $\delta = x - (1 - \mu)$, which is x coordinate, taking as origin the location of the secondary. It can be shown (see Problem 3.3) that when $\mu \ll 1$ the x coordinates of L_1 and L_2 scale as $\delta \propto \mu^{1/3}$.

3.6.2 Triangular points

→ Let us now consider the case $y \neq 0$ (triangular points). For equilibrium we must have (from the y equation):

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0,$$

which we multiply by $(x + \mu)$, so

$$x + \mu - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{(x+\mu)\mu}{r_2^3} = 0,$$

and subtract from the x equation

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu)}{r_2^3} + \frac{\mu}{r_2^3} = 0$$

to obtain

$$-\mu + \frac{\mu}{r_2^3} = 0, \qquad \Longrightarrow \qquad r_2 = 1$$

→ Take again

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0,$$

multiply by $(x - 1 + \mu)$, so

$$x - 1 + \mu - \frac{(1-\mu)(x-1+\mu)}{r_1^3} - \frac{(x-1+\mu)\mu}{r_2^3} = 0,$$

and subtract from the x equation

$$x - \frac{(1-\mu)(x+\mu-1)}{r_1^3} - \frac{1-\mu}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

to obtain

$$1 - \mu - \frac{1-\mu}{r_1^3} = 0 \implies r_1 = 1.$$

→ So $r_1 = r_2 = 1$ =distance between the primaries. These equilibrium points (L_4 and L_5) are the vertices of equilateral triangles having the primaries on the other vertices ($\Longrightarrow$ they are called triangular points).

→ Let us find the coordinates of L_3 and L_4:

$$(x - 1 + \mu)^2 + y^2 = 1 \qquad \text{i.e.} \quad r_2 = 1$$

$$(x + \mu)^2 + y^2 = 1 \qquad \text{i.e.} \quad r_1 = 1$$

The first can be written as

$$(x + \mu)^2 - 2(x + \mu) + y^2 = 0,$$

which, combined with the second gives:

$$2(x + \mu) = 1, \implies x = \frac{1}{2} - \mu,$$

so

$$y^2 = 1 - \frac{1}{4} \implies y = \pm\frac{\sqrt{3}}{2}$$

→ The solutions are easily found geometrically, considering the equilateral triangle, as we know that $r_1 = r_2 = 1$:

$$x = \frac{(1-\mu) + (-\mu)}{2} = \frac{1}{2} - \mu$$

$$y = \pm\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$$

<u>Problem 3.3</u>

Show that when $\mu \ll 1$ the x coordinates of L_1 and L_2 scale as $\delta \propto \mu^{1/3}$, where $\delta = x - (1 - \mu)$. [see 7.2.2 of G09]

Let us focus on L_2.

$$\delta = x - (1 - \mu) = x - 1 + \mu > 0.$$

So the equation for teh x coordinate of L_2 (see Section 3.6.1) becomes

$$\delta + 1 - \mu - \frac{1 - \mu}{(\delta + 1)^2} - \frac{\mu}{\delta^2} = 0$$

$$\delta + 1 - \frac{1}{(\delta + 1)^2} - \mu + \frac{\mu}{(\delta + 1)^2} - \frac{\mu}{\delta^2} = 0.$$

Multiplying by $\delta^2(\delta + 1)^2$ we get

$$\mu = \frac{\delta^2 - (\delta + 1)^3\delta^2}{\delta^2 - \delta^2(\delta + 1)^2 - (\delta + 1)^2} = ... =$$

$$= \frac{\delta^3[\delta^2 + 3\delta + 3]}{\delta^4 + 2\delta^3 + 3\delta^2 + 1}$$

We expand the above function $\mu = \mu(\delta)$ in the limit $\delta \ll 1$ (which is also the limit $\mu \ll 1$, because $\mu \to 0$ if $\delta \to 0$).

$$\mu = \mu(0) + \mu'(0)\delta + \frac{1}{2}\mu''(0)\delta^2 + \frac{1}{6}\mu'''(0)\delta^3 + ...$$

Let us write $\mu(\delta) = N/D$. We have $D(0) = 1$, $D'(0) = 2$, $D''(0) = 1$, $D'''(0) = 2$, and $N(0) = N'(0) = N''(0) = 0$, $N'''(0) = 18$. So $\mu(0) = \mu'(0) = \mu''(0) = 0$, $\mu'''(0) = 18$. For $\delta \ll 1$ we have

$$\mu(\delta) = \frac{1}{6}\mu'''(0)\delta^3 + ... = 3\delta^3 + \mathcal{O}(\delta^4),$$

so

$$\delta = \mathcal{O}(\mu^{1/3})$$

3.7 Stability of the Lagrangian points

3.7.1 Stability of equilibrium points and stability of orbits: some definitions

[S67 5.2]

→ It is useful to define the concept of stability of equilibrium solutions and stability of orbits.

→ The concept of stability applies in general to $\mathbf{w}(t)$, which is a solution of a system of differential equations $\dot{\mathbf{w}} = \mathbf{F}(\mathbf{w}, t)$. In the case of the motion of a particle $\mathbf{w}(t)$ is the orbit: $\mathbf{w} = (\mathbf{r}, \mathbf{v})$ are the phase-space coordinates (positions and velocities).

Stability of equilibrium points

→ *Stability of equilibrium points:* $\mathbf{w} = \mathbf{a}$, where $\mathbf{a} = const$, is a stable equilibrium point if, for given $\epsilon > 0$, there exists a $\delta > 0$ such that if at a reference (initial) time t_0

$$|\mathbf{w}(t_0) - \mathbf{a}| < \delta$$

then, for all $t > t_0$

$$|\mathbf{w}(t) - \mathbf{a}| < \epsilon.$$

→ *Linear stability:* an equilibrium point is linearly stable if it is stable against all *small* (i.e. linear) disturbances ($|\delta\mathbf{w}|/|\mathbf{w}| \ll 1$).

→ *Non-linear stability:* an equilibrium point is non-linearly stable if it is stable against all disturbances (not necessarily small).

→ In general linear stability does not imply non-linear stability.

Stability of orbits

→ The concept of stability of an orbit $\mathbf{w}(t)$ is based on the comparison of the orbit $\mathbf{w}(t)$ with other orbits (called perturbed orbits) that have initial conditions slightly different from the orbit $\mathbf{w}(t)$.

→ We have two different definitions of the stability of orbits: "Lyapunov stability" and "orbital stability".

→ *Definition (1): "Lyapunov stability".* The orbit $\mathbf{w}(t)$ is Lyapunov stable if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that any perturbed orbit $\mathbf{w}'(t)$ satisfying $|\mathbf{w}'(t_0) - \mathbf{w}(t_0)| < \delta$ satisfies $|\mathbf{w}'(t) - \mathbf{w}(t)| < \epsilon$ for $t > t_0$. Lyapunov stability is based on isochronous evaluation of the deviations.

→ *Definition (2): "orbital stability".* An orbit $\mathbf{w}(t)$ is orbitally stable if, given any $\epsilon > 0$ there exist a $\delta > 0$ such that for any perturbed orbit $\mathbf{w}'(t)$ satisfying $|\mathbf{w}'(t_0) - \mathbf{w}(t_0)| < \delta$ it is possible to find c such that $|\mathbf{w}'(t) - \mathbf{w}(t + c)| < \epsilon$ for $t > t_0$.

3.7.2 Lagrangian points: linearized equations

[MD, R05]

→ We study here the linear stability of the Lagrangian points.

→ Let us call x_0, y_0 and z_0 the coordinates of an equilibrium point (i.e. one of the Lagrangian points). We introduce the coordinates

$$X = x - x_0, \qquad Y = y - y_0, \qquad Z = z - z_0.$$

Note that $\dot{X} = \dot{x}$, $\dot{Y} = \dot{y}$, $\dot{Z} = \dot{z}$, because $\dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0$.

→ Let us assume that X, Y, Z are small displacements $\Longrightarrow$ linear perturbations $\Longrightarrow$ linear stability analysis.

→ We write equations for $X(t)$ and we study the solutions. If $X(t)$ oscillates or goes to zero the point is linearly stable. If $X(t)$ diverges the point is unstable. We do the same for all the other phase-space coordinates.

→ Consider a simple example: a 1-D mechanical system described by the equation $\ddot{x} = -\mathrm{d}\Phi/\mathrm{d}x$. Write the solution in the vicinity of the equilibrium point $x = x_0$. If $\Phi = \frac{1}{2}(x - x_0)^2$ the solution oscillates (x_0 is stable); if $\Phi = -\frac{1}{2}(x - x_0)^2$ the solution diverges exponentially (x_0 is unstable).

→ Let us consider the restricted three-body problem. We can expand in Taylor series the equations of motion

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x},$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y},$$

$$\ddot{z} = \frac{\partial U}{\partial z},$$

to obtain equations that describe the motion in the vicinity of the equilibrium point

→ Expanding the derivative of U we get:

$$\frac{\partial U}{\partial x} = \left(\frac{\partial U}{\partial x}\right)_0 + U_{xx}X + U_{xy}Y + U_{xz}Z + \ldots,$$

$$\frac{\partial U}{\partial y} = \left(\frac{\partial U}{\partial y}\right)_0 + U_{xy}X + U_{yy}Y + U_{yz}Z + \ldots,$$

$$\frac{\partial U}{\partial z} = \left(\frac{\partial U}{\partial z}\right)_0 + U_{xz}X + U_{yz}Y + U_{zz}Z + \ldots,$$

where

$$U_{xx} \equiv \left(\frac{\partial^2 U}{\partial x^2}\right)_0, \qquad U_{yy} \equiv \left(\frac{\partial^2 U}{\partial y^2}\right)_0, \qquad U_{zz} \equiv \left(\frac{\partial^2 U}{\partial z^2}\right)_0,$$

$$U_{xy} \equiv \left(\frac{\partial^2 U}{\partial x \partial y}\right)_0, \qquad U_{xz} \equiv \left(\frac{\partial^2 U}{\partial x \partial z}\right)_0, \qquad U_{yz} \equiv \left(\frac{\partial^2 U}{\partial y \partial z}\right)_0,$$

where subscript 0 means evaluated in x_0, y_0, z_0.

→ We recall that in the equilibrium points $\partial U/\partial x = \partial U/\partial y = \partial U/\partial z = 0$, so the linearized equations of motion read

$$\ddot{X} - 2\dot{Y} = U_{xx}X + U_{xy}Y + U_{xz}Z,$$

$$\ddot{Y} + 2\dot{X} = U_{xy}X + U_{yy}Y + U_{yz}Z,$$

$$\ddot{Z} = U_{xz}X + U_{yz}Y + U_{zz}Z.$$

3.7.3 Derivatives of U

$$r_1 = \sqrt{z^2 + y^2 + (x+\mu)^2}$$

$$r_2 = \sqrt{z^2 + y^2 + (x+\mu-1)^2}$$

$$\Phi = -(1-\mu)/r_1 - \mu/r_2 = \frac{\mu - 1}{\sqrt{z^2 + y^2 + (x+\mu)^2}} - \frac{\mu}{\sqrt{z^2 + y^2 + (x+\mu-1)^2}}$$

$$U = (x^2+y^2)/2 - \Phi = -\frac{\mu - 1}{\sqrt{z^2 + y^2 + (x+\mu)^2}} + \frac{\mu}{\sqrt{z^2 + y^2 + (x+\mu-1)^2}} + \frac{y^2 + x^2}{2}$$

$$U_x = \frac{\partial U}{\partial x} = \frac{(\mu - 1)\,(x+\mu)}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{\mu\,(x+\mu-1)}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}} + x$$

$$U_{xx} = \frac{\partial U_x}{\partial x} = \frac{\mu - 1}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{3\,(\mu - 1)\,(x+\mu)^2}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}} - \frac{\mu}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}} +$$

$$+ \frac{3\,\mu\,(x+\mu-1)^2}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}} + 1$$

$$U_y = \frac{\partial U}{\partial y} = \frac{(\mu - 1)\ y}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{\mu\, y}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}} + y$$

$$U_{yy} = \frac{\partial U_y}{\partial y} = \frac{\mu - 1}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{3\ (\mu - 1)\ y^2}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}} - \frac{\mu}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}} +$$
$$+ \frac{3\,\mu\, y^2}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}} + 1$$

$$U_{xy} = \frac{\partial U_x}{\partial y} = \frac{3\,\mu\ (x+\mu-1)\ y}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}} - \frac{3\ (\mu-1)\ (x+\mu)\ y}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}}$$

$$U_z = \frac{\partial U}{\partial z} = \frac{(\mu - 1)\ z}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{\mu\, z}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}}$$

$$U_{zz} = \frac{\partial U_z}{\partial z} = \frac{\mu - 1}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{3}{2}}} - \frac{3\ (\mu - 1)\ z^2}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}} - \frac{\mu}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{3}{2}}} +$$
$$+ \frac{3\,\mu\, z^2}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}}$$

$$U_{xz} = \frac{\partial U_x}{\partial z} = \frac{3\,\mu\ (x+\mu-1)\ z}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}} - \frac{3\ (\mu-1)\ (x+\mu)\ z}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}}$$

$$U_{yz} = \frac{\partial U_y}{\partial z} = \frac{3\,\mu\, y\, z}{\left(z^2 + y^2 + (x+\mu-1)^2\right)^{\frac{5}{2}}} - \frac{3\ (\mu-1)\ y\, z}{\left(z^2 + y^2 + (x+\mu)^2\right)^{\frac{5}{2}}}$$

→ We need to evaluate the above derivatives in the equilibrium points (x_0, y_0, z_0). It is then useful to introduce the following quantities:

$$\tilde{A} = \frac{\mu_1}{(r_1^3)_0} + \frac{\mu_2}{(r_2^3)_0}$$

$$\tilde{B} = 3\left[\frac{\mu_1}{(r_1^5)_0} + \frac{\mu_2}{(r_2^5)_0}\right]$$

$$\tilde{C} = 3\left[\frac{\mu_1(x_0 - x_1)}{(r_1^5)_0} + \frac{\mu_2(x_0 - x_2)}{(r_2^5)_0}\right]$$

$$\tilde{D} = 3\left[\frac{\mu_1(x_0 - x_1)^2}{(r_1^5)_0} + \frac{\mu_2(x_0 - x_2)^2}{(r_2^5)_0}\right]$$

where $\mu_1 = 1 - \mu$, $\mu_2 = \mu$, $x_1 = -\mu_2$, $x_2 = \mu_1$, and $(\cdots)_0$ means evaluated in the equilibrium point (x_0, y_0, z_0).

→ The derivatives of U, evaluated in x_0, y_0, z_0, read as follows

$$U_{xx} = 1 - \tilde{A} + \tilde{D},$$

$$U_{yy} = 1 - \tilde{A} + \tilde{B}y_0^2,$$

$$U_{xy} = \tilde{C}y_0,$$

$$U_{zz} = -\tilde{A} + \tilde{B}z_0^2,$$

$$U_{xz} = \tilde{C}z_0,$$

$$U_{yz} = \tilde{B}y_0z_0.$$

3.7.4 Linear stability analysis of Lagrangian points: method

[MD 3.7]

→ Let us first note that for each of the five Lagrangian points we have $U_{xz} = U_{yz} = 0$, because $z_0 = 0$, therefore the above equations become

$$\ddot{X} - 2\dot{Y} = U_{xx}X + U_{xy}Y,$$

$$\ddot{Y} + 2\dot{X} = U_{xy}X + U_{yy}Y,$$

$$\ddot{Z} = U_{zz}Z.$$

The Z equation is independent of the other two and it is just the equation of a harmonic oscillator, and can be treated separately. Our stability problem reduces to solve the Z equation and the system of coupled equations for X and Y.

→ Let us discuss the solution of the system

$$\ddot{X} - 2\dot{Y} = U_{xx}X + U_{xy}Y,$$

$$\ddot{Y} + 2\dot{X} = U_{xy}X + U_{yy}Y.$$

This is a system of second order ODEs. It can be reduced to a system of 4 first order ODEs for the 4-dimensional vector $\mathbf{w} = (w_1, w_2, w_3, w_4) = (X, Y, \dot{X}, \dot{Y})$, which can be written

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \dot{X}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \dot{Y}$$

$$\frac{\mathrm{d}\dot{X}}{\mathrm{d}t} = U_{xx}X + U_{xy}Y + 2\dot{Y},$$

$$\frac{\mathrm{d}\dot{Y}}{\mathrm{d}t} = U_{xy}X + U_{yy}Y - 2\dot{X},$$

or, in vectorial form,

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}.$$

→ *Eigenvalues, eigenvectors, characteristic polynomial.* Given a matrix $\mathbf{A}$, if $\mathbf{Ax} = \lambda\mathbf{x}$, $\mathbf{x}$ is an eigenvector and λ the corresponding eigenvalue. The system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ has non-trivial solution (i.e. $\mathbf{x} \neq 0$) if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. When $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is written explicitly, it is a polynomial in λ, known as the *characteristic polynomial* of the matrix.

→ The system $\dot{\mathbf{w}} = \mathbf{A}\mathbf{w}$ is coupled. We wish to transform it into an uncoupled system. To do so we perform the transformation $\mathbf{w}' = \mathbf{B}\mathbf{w}$, where $\mathbf{B}$ is a constant matrix to be specified. Therefore $\mathbf{w} = \mathbf{B}^{-1}\mathbf{w}'$ and $\dot{\mathbf{w}} = \mathbf{B}^{-1}\dot{\mathbf{w}}'$. So the system becomes

$$\mathbf{B}^{-1}\dot{\mathbf{w}}' = \mathbf{A}\mathbf{B}^{-1}\mathbf{w}' \implies \dot{\mathbf{w}}' = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}\mathbf{w}'.$$

If $\mathbf{C} \equiv \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$ is diagonal, then our system in $\mathbf{w}'$ is uncoupled. We can construct $\mathbf{B}^{-1}$ using the (column) eigenvectors so that

$$\mathbf{B}\mathbf{A}\mathbf{B}^{-1} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

where λ_i are the eigenvalues (see Problem 3.4).

→ Our linear system in $\mathbf{w}'$ has become just

$$\dot{\mathbf{w}}' = \mathbf{\Lambda}\mathbf{w}', \implies \dot{w}_i' = \lambda_i w_i',$$

the solutions of which are

$$w_i' = c_i e^{\lambda_i t},$$

where c_i are constants.

→ Let us go back to the variables $\mathbf{w}$. We have

$$\mathbf{w} = \mathbf{B}^{-1}\mathbf{w}' = \mathbf{B}^{-1} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \\ c_4 e^{\lambda_4 t} \end{pmatrix},$$

which can be written as

$$w_i = \sum_{j=1}^{4} C_{ij} e^{\lambda_j t}$$

for $i = 1, \ldots, 4$, where C_{ij} are constants depending on the c_i and on the elements of $\mathbf{B}$.

→ In order to have stability each of the λ_i must be either purely imaginary ($\Longrightarrow$ oscillations) or complex, but with negative real part ($\Longrightarrow$ exponential damping).

→ Let us specialize to our particular system derived from the linearized equations of motion around a Lagrangian point. The matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ U_{xx} & U_{xy} & -\lambda & 2 \\ U_{xy} & U_{yy} & -2 & -\lambda \end{vmatrix} = 0$$

i.e.

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0,$$

which is a biquadratic equation. Defining $s \equiv \lambda^2$, we have

$$s_{1,2} = -\frac{1}{2}(4 - U_{xx} - U_{yy}) \pm \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)\right]^{\frac{1}{2}},$$

so the 4 solutions are

$$\lambda_{1,2} = \pm\left\{-\frac{1}{2}(4 - U_{xx} - U_{yy}) - \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

$$\lambda_{3,4} = \pm\left\{-\frac{1}{2}(4 - U_{xx} - U_{yy}) + \frac{1}{2}\left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

→ The above eigenvalues can be real, complex or imaginary, so in general they can be written as

$$\lambda_{1,2} = \pm(j_1 + \mathrm{i}k_1), \qquad \lambda_{3,4} = \pm(j_2 + \mathrm{i}k_2),$$

where j_1, j_2, k_1, k_2 are real. Therefore for stability we must have $j_1 = j_2 = 0$, i.e. that all the λ_i are purely imaginary.

3.7.5 Stability analysis: collinear points

→ In this case $y_0 = z_0 = 0$, so

$$U_{xz} = U_{yz} = U_{xy} = 0,$$

$$U_{xx} = 1 - \tilde{A} + \tilde{D} = 1 + 2\tilde{A},$$

$$U_{yy} = 1 - \tilde{A},$$

$$U_{zz} = -\tilde{A},$$

because $(r_1^2)_0 = (x_0 - x_1)^2$ and $(r_2^2)_0 = (x_0 - x_2)^2$, so $\tilde{D} = 3\tilde{A}$ (see above definitions of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$).

→ Start from the Z equation, which becomes

$$\ddot{Z} = -\tilde{A}Z,$$

with solution $Z = Ce^{\sqrt{-\tilde{A}}t}$, which is oscillatory because $\tilde{A} > 0$ by definition (recall Euler's formula $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$). C is an arbitrary constant.

→ Let's move now to the $X - Y$ system: the characteristic polynomial becomes

$$\lambda^4 + (2 - \tilde{A})\lambda^2 + (1 + 2\tilde{A})(1 - \tilde{A}) = 0,$$

i.e.

$$s^2 + (2 - \tilde{A})s + (1 + 2\tilde{A})(1 - \tilde{A}) = 0,$$

where $s = \lambda^2$. We know that the solutions s_1 and s_2 satisfy Viète's formula

$$s_1 s_2 = (1 + 2\tilde{A})(1 - \tilde{A}),$$

because in general

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \implies x_1 x_2 = c/a,$$

so

$$(\lambda_1\lambda_2)(\lambda_3\lambda_4) = (1 + 2\tilde{A})(1 - \tilde{A}),$$

i.e.

$$\lambda_1^2\lambda_3^2 = (1 + 2\tilde{A})(1 - \tilde{A}),$$

because $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_3$. For stability all the λ_i must be purely imaginary, so $\lambda_1^2 < 0$ and $\lambda_3^2 < 0$, so a necessary condition for stability is

$$(1 - \tilde{A})(1 + 2\tilde{A}) > 0,$$

i.e. $\tilde{A} < 1$, because $\tilde{A} > 0$. Note that $\tilde{A} < 1$ is a necessary (but not sufficient) condition for stability.

→ Substituting in $\tilde{A}$ the values of x_0 for the three collinear points L_1, L_2 and L_3 (and recalling that $\mu < \frac{1}{2}$ we find in all cases $\tilde{A} > 1$ (see Problem 3.5).

→ So we conclude that all the collinear Lagrangian points are *unstable* for all values of μ.

3.7.6 Stability analysis: triangular points

→ The triangular points L_4 and L_5 have $(r_1)_0 = (r_2)_0 = 1$; $y_0 = \pm\sqrt{3}/2$ and $x_0 = \frac{1}{2} - \mu = \frac{1}{2} - \mu_2$; $z_0 = 0$. Therefore,

$$\tilde{A} = 1, \qquad \tilde{B} = 3, \qquad \tilde{C} = \frac{3}{2}(1 - 2\mu), \qquad \tilde{D} = \frac{3}{4}.$$

and

$$U_{xx} = 1 - \tilde{A} + \tilde{D} = \frac{3}{4},$$

$$U_{yy} = 1 - \tilde{A} + \tilde{B} y_0^2 = \frac{9}{4},$$

$$U_{xy} = \tilde{C} y_0 = \pm \frac{3\sqrt{3}}{4}(1-2\mu),$$

$$U_{zz} = -\tilde{A} + \tilde{B} z_0^2 = -1,$$

$$U_{xz} = \tilde{C} z_0 = 0,$$

$$U_{yz} = \tilde{B} y_0 z_0 = 0.$$

→ The Z equation of motion is

$$\ddot{Z} = -Z,$$

with solution $Z = Ce^{\sqrt{-1}t}$, which is oscillatory (C is an arbitrary constant).

→ Let's move now to the $X - Y$ system: the characteristic polynomial is

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0,$$

so

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0,$$

i.e.

$$s^2 + s + \frac{27}{4}\mu(1-\mu) = 0,$$

where $s = \lambda^2$.

→ The solutions are

$$s_{1,2} = \frac{-1 \pm \sqrt{\Delta}}{2},$$

with

$$\Delta = 1 - 27\mu(1-\mu).$$

Let us consider separately two cases $\Delta \geq 0$ and $\Delta < 0$

→ If $\Delta \geq 0$, $s_{1,2}$ are real and for stability we just have to impose that $s_{1,2} < 0$, i.e.

$$-1 + \sqrt{\Delta} < 0, \quad i.e. \quad 27\mu(1-\mu) > 0, \qquad (always).$$

Note that the condition on s_1, $-1 - \sqrt{\Delta} < 0$, is less restrictive. So for $\Delta \geq 0$ we always have stability.

→ If $\Delta < 0$

$$s_{1,2} = \frac{-1 \pm \mathrm{i}\sqrt{|\Delta|}}{2},$$

so we can write $\lambda_1 = a_1 + \mathrm{i}b_1$ and $\lambda_2 = -a_1 - \mathrm{i}b_1$. Similarly $\lambda_3 = a_2 + \mathrm{i}b_2$ and $\lambda_4 = -a_2 - \mathrm{i}b_2$. We have

$$\lambda_1^2 = (a_1 + \mathrm{i}b_1)^2 = s_1 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{|\Delta|}}{2},$$

so

$$a_1^2 - b_1^2 + \mathrm{i}2a_1b_1 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{|\Delta|}}{2},$$

which cannot be satisfied if $a_1 = 0$. Therefore, $a_1 \neq 0$. If $a_1 > 0$, λ_1 has positive real part ($\implies$ instability); but if $a_1 < 0$, λ_2 has positive real part ($\implies$ instability). So we always have instability for $\Delta < 0$.

→ Summarizing, the necessary and sufficient condition for linear stability is $\Delta \geq 0$, i.e.

$$1 - 27\mu(1-\mu) \geq 0$$

$$\mu^2 - \mu + \frac{1}{27} \geq 0,$$

this is satisfied for

$$\mu \leq \frac{1}{2} - \frac{1}{2}\sqrt{23/27} \simeq 0.03852 \equiv \mu_0.$$

(we recall that by definition $\mu < \frac{1}{2}$).

→ We conclude that for $\mu < \mu_0$ the triangular points are linearly stable. μ_0 is known as Gascheau's value or Routh's value.

→ Linear stability does not necessarily imply non-linear stability, but it has been shown that for $\mu < \mu_0$ the triangular points are also non-linearly stable [S67].

Problem 3.4

Given a 2×2 matrix $\mathbf{A}$ show that $\mathbf{BAB}^{-1} = \mathbf{\Lambda}$ where is the diagonal matrix with the eigenvalue of $\mathbf{A}$ on the diagonal and $\mathbf{B}^{-1}$ is constructed from the column eigenvectors of $\mathbf{A}$.

The inverse of a given 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix}$$

is

$$\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Now let's construct

$$\mathbf{B}^{-1} = \begin{pmatrix} x_1\, y_1 \\ x_2\, y_2 \end{pmatrix},$$

where (x_1, x_2) and (y_1, y_2) are eigenvectors of $\mathbf{A}$, which is a given 2×2 matrix.

$$\mathbf{B} = (\mathbf{B}^{-1})^{-1} = \frac{1}{x_1y_2 - x_2y_1}\begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix},$$

so

$$\mathbf{AB}^{-1} = \begin{pmatrix} \lambda_1 x_1\, \lambda_2 y_1 \\ \lambda_1 x_2\, \lambda_2 y_2 \end{pmatrix},$$

$$\mathbf{BAB}^{-1} = \frac{1}{x_1y_2 - x_2y_1}\begin{pmatrix} \lambda_1(x_1y_2 - x_2y_1) & \lambda_2(y_1y_2 - y_1y_2) \\ \lambda_1(-x_1x_2 + x_1x_2) & \lambda_2(-x_2y_1 + x_1y_2) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{\Lambda}$$

Problem 3.5

Show that for the collinear points L_1, L_2 and L_3 we have $\tilde{A} > 1$ always.

By definition

$$\tilde{A} = \frac{\mu_1}{(r_1^3)_0} + \frac{\mu_2}{(r_2^3)_0} = \frac{1-\mu}{|x_0+\mu|^3} + \frac{\mu}{|x_0+\mu-1|^3}.$$

We know that at the equilibrium points

$$(U_x)_0 = -\frac{(1-\mu)\,(x_0+\mu)}{\left((x_0+\mu)^2\right)^{\frac{3}{2}}} - \frac{\mu\,(x_0+\mu-1)}{\left((x+\mu-1)^2\right)^{\frac{3}{2}}} + x_0 = 0$$

which can be written as

$$1 - \tilde{A} = \frac{\mu(1-\mu)}{x_0}\left[\frac{1}{|x_0+\mu|^3} - \frac{1}{|x_0+\mu-1|^3}\right]$$

so the condition $\tilde{A} < 1$, i.e. $1 - \tilde{A} > 0$ can be written as

$$\frac{1}{x_0}\left[\frac{1}{|x_0+\mu|^3} - \frac{1}{|x_0+\mu-1|^3}\right] > 0,$$

i.e.

$$\frac{1}{x_0}\left[\frac{1}{[(x_0+\mu)^2]^{3/2}} - \frac{1}{[(x_0+\mu-1)^2]^{3/2}}\right] > 0,$$

$$\frac{(x_0+\mu-1)^2 - (x_0+\mu)^2}{x_0} > 0$$

If $x_0 > 0$ (L_1 and L_2) the condition is

$$-2(x_0+\mu) + 1 > 0, \text{ i.e. } x_0 < \frac{1}{2} - \mu,$$

which we have seen is never the case for Lagrangian points L_1 and L_2 (see FIG CM3.6). If $x_0 < 0$ (L_3) the condition is

$$-2(x_0+\mu) + 1 < 0, \text{ i.e. } x_0 > \frac{1}{2} - \mu > 0,$$

which of course is not the case because we are considering $x_0 < 0$.

3.8 Motion around Lagrangian points

3.8.1 Motion near L_4 and L_5

[MD 3.8]

→ Let us consider now the case of stable triangular points L_4 and L_5 ($\mu \leq 0.03852 \equiv \mu_0$)

→ For small (linear) displacements, the characteristic frequencies of oscillation are $|\lambda_{1,2}|$ and $|\lambda_{3,4}|$, i.e. the moduli of the eigenvalues found above, because the time evolution is described by a sum of terms $\propto e^{\lambda_i t}$.

→ Let's write these eigenvalues explicitly:

$$\lambda_{1,2} = \pm\sqrt{s_1} = \pm\sqrt{\frac{-1-\sqrt{1-27\mu(1-\mu)}}{2}}$$

$$\lambda_{3,4} = \pm\sqrt{s_2} = \pm\sqrt{\frac{-1+\sqrt{1-27\mu(1-\mu)}}{2}}$$

In the relevant limit of small μ (expanding λ_i as a function of μ) we have

$$\sqrt{1-27\mu(1-\mu)} \simeq 1 + \frac{1}{2}(-27\mu) = 1 - \frac{27}{2}\mu$$

and, therefore,

$$\lambda_{1,2} = \pm\sqrt{-1+\frac{27}{4}\mu}$$

$$\lambda_{3,4} = \pm\sqrt{-\frac{27}{4}\mu}$$

→ The solution of the linearized equation is in the form $X(t) \propto \sum_i C_i e^{\mathrm{i}|\lambda_i|t}$ (and similarly for Y), so the characteristic periods are $T_i = 2\pi/|\lambda_i|$ and the motion around each of the triangular points is determined by the combination of oscillations with period $T_{1,2} = 2\pi/|\lambda_{1,2}|$ and oscillations with period $T_{3,4} = 2\pi/|\lambda_{3,4}|$. For small μ, $|\lambda_{1,2}| \sim 1$ (short period: $T_{1,2} \sim 2\pi$) and $|\lambda_{3,4}| \ll 1$ (long period, $T_{3,4} \gg 2\pi$).

→ We recall that we have adopted units in which, for the motion of the secondary around the primary, the mean motion is $n = 1$ and the period is 2π. Therefore the motion of the test particle around L_4 or L_5 is described by a short-period oscillation (*epicyclic motion*) with period $\sim 2\pi$ (similar to the period of the primaries) combined with a long-period oscillation (*libration*) with period $\gg 2\pi$.

→ This motion can also be seen as an epicyclic motion, in which the motion of the guiding centre (or epicentre) with period $\gg 2\pi$ is combined with short period oscillations around the guiding centre (see Figs. 3.14 and 3.15 in MD; FIG CM3.7 and FIG CM3.8).

→ The linear and non linear *stability of orbits* around the triangular points has been investigated for various values of μ. For instance, there are stable infinitesimal orbits around L_4 or L_5 not only for $\mu \leq \mu_0$, but also for $\mu_0 < \mu < \mu_1$, where $\mu_1 = 0.044$ [S67].

3.8.2 Tadpole and horseshoe orbits

[MD 3.9; VK]

→ The results obtained from the linear analysis hold only for small displacements around L_4 and L_5. Orbits around these points with larger displacements can be studied by numerical integration of the equations of motion.

→ The numerical result is that there are two kinds of orbits: tadpole orbits (around either L_4 or L_5) and horseshoe orbits (encompassing both L_4 and L_5).

→ By increasing the value of the Jacobi integral C_J we go from a tadpole orbit around, say, L_4 to two joint tadpole orbits (around L_4 and L_5) finally to a full horseshoe orbit.

→ See Figs. 3.16 and 3.17 MD (FIG CM3.9 and FIG CM3.10). See also Fig. 5.4 in VK (FIG CM3.11).

→ The shape of the zero-velocity curves are similar to tadpole and horseshoe orbit (see Fig. 3.9 in MD; FIG CM3.12). However, we recall that zero-velocity curves (ZVCs) just indicate forbidden regions and do not define orbits. In particular ZVCs do not say anything about whether the orbit is stable. See also Fig. 9.11 in MD (FIG CM3.13)

3.8.3 Motion near the collinear points

[S67]

→ We have seen that for all values of μ the collinear points L_1, L_2 and L_3 are linearly unstable.

→ There exist perturbations with specific initial conditions giving trigonometric functions as solutions (i.e. oscillating solutions).

→ There are periodic (2D, in the plane of the primaries, $x - y$) orbits around the collinear points, called "Lyapunov orbits" (e.g. Howell 2001).

→ There are periodic (3D) orbits around the collinear points, called "halo orbits" (Farquhar R.W., 1968 ,PhD thesis; Howell K., 2001).

→ There are quasi periodic (3D) orbits around the collinear points, called "Lissajous orbits" (see Howell & Pernicka 1988, Cel. Mech 41, 107; Howell 2001). Lissajous figures in $x - y$, $x - z$, $y - z$ planes. Lissajous figures, e.g. in the x-y plane are obtained by equations in the form

$$x(t) = A\sin(at), \qquad y(t) = B\sin(bt + c).$$

See Fig. 4 of Howell (2001): FIG CM3.14.

→ All these finite or infinitesimal orbits around the collinear points are also generally found to be unstable [S67]. However, these orbits can be used (and are actually used) by space missions such as space telescopes provided corrections to the orbits are applied to contrast the instability with station-keeping methods.

3.9 Hill's approximation

[MD 3.13]

→ When $\mu \ll 1$ the orbit of the infinitesimal body (test particle) is basically Keplerian (w.r.t. to the primary) when the test particle is far from the secondary. The orbit is significantly perturbed only when the test particle is close to the secondary. It is then useful to derive equations that describe the motion of the test particle near the secondary. These equations were first derived by Hill (1878) for application to lunar theory.

3.9.1 Hill's equations

→ Let us start from the planar ($z = 0$) equations of motion of the circular restricted 3-body problem:

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

where

$$U = \frac{1}{2}(x^2 + y^2) - \Phi,$$

$$\Phi(r_1, r_2) \equiv -\frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = [x - (1-\mu)]^2 + y^2,$$

and we have used $n = 1$. So

$$\ddot{x} - 2\dot{y} - x = -\frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu[x-(1-\mu)]}{r_2^3}$$

$$\ddot{y} + 2\dot{x} - y = -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}.$$

→ Now, let us shift the origin of the coordinate system to the location of the secondary, using $x' = x - 1 + \mu$, $y' = y$ (because the secondary is located at $x = 1 - \mu$). Substituting $x = x' + 1 - \mu$ and $y = y'$ we get:

$$\ddot{x}' - 2\dot{y}' = x' + 1 - \mu - \frac{(1-\mu)(x'+1)}{r_1^3} - \frac{\mu x'}{r_2^3}$$

$$\ddot{y}' + 2\dot{x}' = y' - \frac{y'(1-\mu)}{r_1^3} - \frac{\mu y'}{r_2^3},$$

where

$$r_1^2 = (x'+1)^2 + y'^2, \qquad r_2^2 = x'^2 + y'^2 \equiv \Delta^2.$$

→ We then take the limit $\mu \ll 1$, so $1 - \mu \approx 1 + \mu \approx 1$, but we keep term of the order of $x' \sim \mu^{1/3} > \mu$ ($\mu \ll |x'| \ll 1$). We get

$$\ddot{x}' - 2\dot{y}' = x' + 1 - \frac{x'+1}{r_1^3} - \frac{\mu x'}{r_2^3},$$

$$\ddot{y}' + 2\dot{x}' - y' = -\frac{y'}{r_1^3} - \frac{\mu y'}{r_2^3},$$

where we have also assumed $x' + \mu \approx x'$, because we are interested only in the region close to the secondary, at distances from the secondary of the order of the distances of L_1 and L_2 (of the order of $x' \sim \mu^{1/3}$).

→ We can now expand the above equation in the limit $x' \sim y' \sim \Delta \sim \mathcal{O}(\mu^{1/3}) \ll 1$. Note that the assumption that these quantities are of the order of $\mu^{1/3}$ is justified if we consider distance from the equilibrium point of the order of the distances to L_1 and L_2, which are $\mathcal{O}(\mu^{1/3})$ (see section on location of collinear Lagrangian points). We perform a Taylor expansion in the variables x' and y' of the kind

$$f(x', y') = f(0,0) + f_{x'}(0,0)x' + f_{y'}(0,0)y' + \cdots .$$

For the term

$$-\frac{x'+1}{r_1^3}$$

we have $f(0,0) = -1$, $f_{x'}(0,0) = 2$, $f_{y'}(0,0) = 0$, so we get

$$-\frac{x'+1}{r_1^3} \approx -1 + 2x',$$

For the term

$$-\frac{y'}{r_1^3}$$

we have $f(0,0) = 0$, $f_{x'}(0,0) = 0$, $f_{y'}(0,0) = -1$, so we get

$$-\frac{y'}{r_1^3} \approx -y'.$$

The final equations of motion (Hill's equations) are

$$\ddot{x}' - 2\dot{y}' = \left(3 - \frac{\mu}{\Delta^3}\right) x'$$

$$\ddot{y}' + 2\dot{x}' = -\frac{\mu y'}{\Delta^3}.$$

→ Dropping for simplicity the primes we can write

$$\ddot{x} - 2\dot{y} = \frac{\partial U_{\rm H}}{\partial x},$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U_{\rm H}}{\partial y},$$

where

$$U_{\rm H} \equiv \frac{3}{2}x^2 + \frac{\mu}{\Delta}$$

and $\Delta^2 = x^2 + y^2$.

→ We can find the location of the Lagrangian points L_1 and L_2 in the Hill's approximation by imposing in Hill's equations $\ddot{x} = \dot{x} = 0 = \ddot{y} = \dot{y} = 0$ and $x \neq 0$. We get:

$$y = 0, \qquad x = \pm\Delta_{\rm H}, \qquad \Delta = \Delta_{\rm H},$$

so L_1 and L_2, in this approximation, lie on Hill's sphere (so, at the same distance from the secondary).

→ The force along x vanishes when $3\Delta^3 = \mu$, so we define the Hill's sphere as the sphere around the secondary of radius $\Delta_{\rm H} = \left(\frac{\mu}{3}\right)^{1/3}$ (known as Hill's radius).

→ In Hill's approximation it is straightforward to compute the approximate distance between the secondary and L_1 or L_2 (see Problem 3.6).

Problem 3.6

Compute the distance from the Earth of the Earth-Sun L_1 and L_2 points in Hill's approximation

We have $M_\odot \simeq 1.99 \times 10^{30}\,\mathrm{kg}$, $M_{\mathrm{Earth}} \simeq 5.97 \times 10^{24}\,\mathrm{kg}$, so $\mu = M_{\mathrm{Earth}}/M_\odot \simeq 3 \times 10^{-6}$ and Hill's approximation is justified $\mu \ll 1$.

The average Sun-Earth distance is $a = 1\,\mathrm{AU} = 1.49 \times 10^{11} m$.

Let us define d_1 and d_2 the distances of, respectively, L_1 and L_2 from the Earth. In Hill's approximation

$$d_1 = d_2 = \Delta_{\mathrm{H}} = \left(\frac{\mu}{3}\right)^{1/3},$$

in units such that $a = 1$. In physical units

$$d_1 = d_2 = \Delta_{\mathrm{H}} a = \left(\frac{\mu}{3}\right)^{1/3} a \simeq 0.01a \simeq 1.49 \times 10^9\,\mathrm{m},$$

so the distance to L_1 and L_2 is about 1.5 millions of kilometers.

3.10 Lagrangian points: applications

[S67 5.6; MD 3.11, 3.12]

→ In 1772 Lagrange showed that the five libration points were solutions of the Sun-Jupiter restricted problem (Euler discovered the three collinear points a few years before). In 1906 started the discovery of the Trojan group of asteroids with the detection of the asteroid named "588 Achilles". As of January 2015 there are 6178 known Jupiter Trojan asteroids including both Greeks (leading, L4) and Trojans (trailing, L5).

→ Trojans move on tadpole orbits. Show Fig. 3.23 of MD (FIG CM3.15).

→ There are known Trojan asteroids also in the Sun-Mars (first discovered, Eureka 1990), Sun-Venus, Sun-Uranus and Sun-Neptune systems.

→ Sun-Earth system: 2010 TK7 Trojan (2010), librating around L_5. Another companion of the Earth is Cruithne (discovered in 1986; orbit determined in 1997) in a horseshoe libration, which is not a Trojan (because the orbit is not a tadpole orbit).

→ Coorbital satellites (or Trojan satellites or Trojan moons): located at L_4 or L_5 of planet-satellite system. For example Saturn-Tethys (two known: Telesto and Calipso) or Saturn-Dione (two known: Helene and Polydeuces). There are no known Trojan moons in Jupiter: maybe related to relative width of tadpole and horseshoe orbit (and so to involved mass ratios).

→ *Janus and Epimetheus.* [MD] In 1980 two satellites of Saturn (Janus and Epimetheus) were discovered by the Voyager 1. They were initially thought to be possibly described by a restricted three-body problem Saturn (primary), Janus (secondary) and Epimetheus (test particle). If so Epimetheus was expected to move on a horseshoe orbit. We now know the masses of the two satellites: Janus 1.98×10^{18} kg, Epimetheus 5.5×10^{17} kg, so their masses are actually comparable (mass ratio $\sim 1/4$). Saturn mass is 5.68×10^{26} kg, so for Janus $\mu \sim 10^{-9}$.

→ As the masses are comparable, mutual perturbations are important and a single horseshoe orbit must not be expected. In fact Janus and Epimetheus librate on their horseshoe paths centered on points 180$^{\rm o}$ apart in longitude: these points are fixed in the rotating frame. Show Fig. 3.26 of MD (FIG CM3.16). The longitudinal excursions and the radial widths of the paths are inversely proportional to the mass (Murray & Dermott 1981). They approach each other and reach maximum separation periodically: they are sometimes called "the dancing moons" (see images from Cassini mission).

→ We have seen that with special initial conditions it is possible to find periodic or quasi periodic orbits close to the collinear equilibrium points. These kinds of orbits are used for the artificial satellites placed near L_1 (SOHO, halo orbit) and L_2 of Sun-Earth: WMAP, Herschel, Planck, GAIA and in the future JWST and Euclid. Herschel and GAIA on Lissajous orbits. JWST on halo orbit.

→ Gaia is in a Lissajous Orbit, which describes a Lissajous Curve around the Libration Point with components in the plane of the two primary bodies of the Lagrange System and a component perpendicular to it. The orbit period is about 180 days and the size of the orbit is 263,000 x 707,000 x 370,000 km.

→ Lissajous orbits around a Libration Point are dynamically unstable, requiring some effort to model as small departures from equilibrium grow exponentially as time progresses.

→ Once per month, Gaia has to perform orbit maintenance procedures which will be small engine maneuvers to make sure orbital parameters around L2 stay within predicted models. (Spaceflight101.com)

→ *Close binary stars.* A close binary can be considered in terms of the circular restricted three-body problem, in which the two primaries are the two stars and the test particle is any parcel of material which is in orbit around the stars. Approximations: the stars are considered point masses and the orbit is assumed circular.

→ We have seen that the zero-velocity curves (ZVCs) are, depending on the value of the Jacobi integral $C_{\rm J}$, separate lobes, an 8-shaped curve (crossing in L_1) or a single curve around both stars

→ Though approximately calculated (using the circular restricted 3-body problem), the Roche lobe is actually a physical limit to the size of each star in a binary. The outer layers (or atmosphere) of a star filling the Roche lobe tend to be stripped and accreted through L_1 onto the companion star.

→ *Tidal stripping.* Consider a satellite (galaxy or globular cluster of mass $\mathcal{M}_{\rm sat}$ orbiting within a host galaxy of mass $\mathcal{M}_{\rm host}$. Restricted three-body problem approximation. Primary: host. Secondary: satellite. Test particle: star of the satellite.

→ L_1 and L_2 mark the size of the volume around the satellite within which particles are bound to the satellite. In the limit $\mathcal{M}_{\rm sat} \ll \mathcal{M}_{\rm host}$ the tidal truncation radius is the distance of L_1 and L_2 from the satellite's centre:

$$r_{\rm t} = \left(\frac{\mathcal{M}_{\rm sat}}{3\mathcal{M}_{\rm host}}\right)^{1/3} d, \tag{3.2}$$

where d is the separation between the centres of the satellite and of the host system.

→ In the case of extended host and satellite, if we identify $\mathcal{M}_{\rm sat}$ with the mass of the satellite within $r_{\rm t}$, and $\mathcal{M}_{\rm host}$ with the mass of the host within d, the above equation can be rewritten as

$$\overline{\rho}_{\rm sat} = 3\overline{\rho}_{\rm host}, \tag{3.3}$$

where $\overline{\rho}_{\rm sat}$ is the average density of the satellite within $r_{\rm t}$ and $\overline{\rho}_{\rm host}$ is the average density of the satellite within d.

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