Casoratian identities for the discrete orthogonal polynomials in discrete quantum mechanics with real shifts

Satoru Odake*

Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
*E-mail: odake@azusa.shinshu-u.ac.jp

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In our previous papers [S. Odake and R. Sasaki, J. Phys. A **46**, 245201 (2013) and S. Odake and R. Sasaki, J. Approx. Theory **193**, 184 (2015)], the Wronskian identities for the Hermite, Laguerre, and Jacobi polynomials and the Casoratian identities for the Askey–Wilson polynomial and its reduced-form polynomials were presented. These identities are naturally derived through quantum-mechanical formulation of the classical orthogonal polynomials: ordinary quantum mechanics for the former and discrete quantum mechanics with pure imaginary shifts for the latter. In this paper we present the corresponding identities for the discrete quantum mechanics with real shifts. Infinitely many Casoratian identities for the q-Racah polynomial and its reduced-form polynomials are obtained.
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1.   Introduction

New types of orthogonal polynomials, the exceptional and multi-indexed orthogonal polynomials $\{P_{\mathcal{D},n}(\eta)|n \in \mathbb{Z}_{\geq 0}\}$, have seen remarkable progress in the theory of orthogonal polynomials and exactly solvable quantum-mechanical models [1–28]. Our approach to orthogonal polynomials is based on the quantum-mechanical formulations: ordinary quantum mechanics (oQM), discrete quantum mechanics with pure imaginary shifts (idQM) [22–25], and discrete quantum mechanics with real shifts (rdQM) [26–28]. For oQM, the Schrödinger equations are second-order differential equations. In discrete quantum mechanics, they are replaced by second-order difference equations with a continuous variable for idQM or a discrete variable for rdQM. The Askey scheme of the (basic) hypergeometric orthogonal polynomials [29] is well matched to these quantum-mechanical formulations: the Jacobi polynomial etc. in oQM, the Askey–Wilson polynomial etc. in idQM, and the q-Racah polynomial etc. in rdQM. From the exactly solvable quantum-mechanical systems described by the classical orthogonal polynomials in the Askey scheme, we can obtain new exactly solvable quantum-mechanical systems and various exceptional orthogonal polynomials with multi-indices by the multi-step Darboux transformations with appropriate seed solutions. One characteristic feature of these new types of polynomials is the missing degrees. We distinguish the following two cases: the set of missing degrees $\mathcal{I} = \mathbb{Z}_{\geq 0}\backslash\{\deg \mathcal{P}_n|n \in \mathbb{Z}_{\geq 0}\}$ is case (1): $\mathcal{I} = \{0, 1, \ldots, \ell - 1\}$, or case (2): $\mathcal{I} \neq \{0, 1, \ldots, \ell - 1\}$, where ℓ is a positive integer. The situation of case (1) is called stable in Ref. [6]. When the virtual state wavefunctions are used as seed solutions, the deformed systems are

exactly iso-spectral to the original system and the case (1) multi-indexed orthogonal polynomials are obtained [8,10,12,13]. When the eigenstate and/or pseudo virtual state wavefunctions are used as seed solutions, the deformed systems are almost iso-spectral to the original system but some states corresponding to the seed solutions are deleted or added, respectively, and the case (2) multi-indexed orthogonal polynomials are obtained [20,21].

For oQM and idQM, the deformed systems obtained by M-step Darboux transformations in terms of pseudo virtual state wavefunctions (with degrees specified by $\mathcal{D}$) are equivalent to those obtained by multi-step Darboux transformations in terms of eigenstate wavefunctions (with degrees specified by $\bar{\mathcal{D}}$) with shifted parameters [20,21]. The deformed system is characterized by the denominator polynomial $\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})$ and the above equivalence is based on the proportionality of the denominator polynomials for each deformed system:

$$\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) \propto \bar{\Xi}_{\bar{\mathcal{D}}}(\eta; \bar{\boldsymbol{\lambda}}); \tag{1.1}$$

see Refs. [20,21] for details. Here we explain the necessary notation only. Two sets $\mathcal{D}$ and $\bar{\mathcal{D}}$ are defined for positive integers M and $\mathcal{N}$ ($\boldsymbol{\lambda}$ is a set of parameters and $\boldsymbol{\delta}$ is its shift):

$$\mathcal{D} = \{d_1, d_2, \ldots, d_M\} \ \ (d_j \in \mathbb{Z}_{\geq 0} : \text{mutually distinct}),$$

$$\mathcal{N} \geq \max(\mathcal{D}), \quad \bar{d}_j \stackrel{\text{def}}{=} \mathcal{N} - d_j, \quad \bar{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} \boldsymbol{\lambda} - (\mathcal{N}+1)\boldsymbol{\delta}, \quad \bar{\mathcal{N}} \stackrel{\text{def}}{=} \mathcal{N} + 1 - M, \tag{1.2}$$

$$\bar{\mathcal{D}} \stackrel{\text{def}}{=} \{0, 1, \ldots, \mathcal{N}\} \backslash \{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_M\} \stackrel{\text{def}}{=} \{e_1, e_2, \ldots, e_{\bar{\mathcal{N}}}\}.$$

(Exactly speaking, $\mathcal{D}$ and $\bar{\mathcal{D}}$ should be treated as ordered sets. By changing the order of the d_j, $\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})$ changes its overall sign. For the proportional relation (1.1), however, such an overall sign change does not matter). The proportional relation (1.1) gives the Wronskian identity for oQM and the Casoratian identity for idQM. We write down the Casoratian identities for the Askey–Wilson polynomial:

$$\varphi_M(x)^{-1} \, \mathrm{W}_\gamma[\check{\xi}_{d_1}, \check{\xi}_{d_2}, \ldots, \check{\xi}_{d_M}](x; \boldsymbol{\lambda}) \propto \varphi_{\bar{\mathcal{N}}}(x)^{-1} \, \mathrm{W}_\gamma[\check{P}_{e_1}, \check{P}_{e_2}, \ldots, \check{P}_{e_{\bar{\mathcal{N}}}}](x; \bar{\boldsymbol{\lambda}}). \tag{1.3}$$

(See Appendix A.1 for definitions of $\check{\xi}_{\mathrm{v}}(x; \boldsymbol{\lambda})$, $\check{P}_n(x; \boldsymbol{\lambda})$, and $\varphi_M(x)$). Here the Casorati determinant (Casoratian) of a set of n functions $\{f_j(x)\}$ for idQM is defined by

$$\mathrm{W}_\gamma[f_1, f_2, \ldots, f_n](x) \stackrel{\text{def}}{=} i^{\frac{1}{2}n(n-1)} \det\Big(f_k\big(x + i(\tfrac{n+1}{2} - j)\gamma\big)\Big)_{1 \leq j,k \leq n} \tag{1.4}$$

(for $n = 0$, we set $\mathrm{W}_\gamma[\cdot](x) = 1$) and $\gamma = \log q$ for the Askey–Wilson case. Based on the discrete symmetry of the system, the pseudo virtual state polynomial $\check{\xi}_{\mathrm{v}}(x; \boldsymbol{\lambda})$ is obtained from the eigenpolynomial $\check{P}_n(x; \boldsymbol{\lambda})$ by twisting parameters. The Casoratian identities (1.3) represent the relation between Casoratians of the orthogonal polynomials with twisted and shifted parameters, and display the duality between the state adding and deleting Darboux transformations. The shape-invariant properties of the original systems play an important role.

In this paper we consider the Casoratian identities for discrete orthogonal polynomials appearing in rdQM. A natural way to obtain them is the following: (a) define the pseudo virtual state vectors by using discrete symmetries of the system; (b) deform the system by multi-step Darboux transformations in terms of the pseudo virtual state vectors; and (c) compare it with the deformed system obtained by multi-step Darboux transformations in terms of the eigenvectors with shifted parameters. We present a one-step Darboux transformation in terms of the pseudo virtual state vector by taking

the q-Racah case as an example in Appendix B. However, calculation for multi-step cases is rather complicated. So, instead of the natural method mentioned above, we use a "shortcut" method in this paper. We derive the Casoratian identities for the q-Racah polynomial (3.36) from those for the Askey–Wilson polynomial (1.3) by using the relation between the q-Racah and Askey–Wilson polynomials. The discrete orthogonal polynomials in the Askey scheme are obtained from the q-Racah polynomial in appropriate limits. The Casoratian identities for those reduced-form polynomials can be obtained from those for the q-Racah polynomial in the same limits.

The Casoratian identities imply equivalences between the deformed systems obtained by multi-step Darboux transformations in terms of pseudo virtual state vectors and those in terms of eigenvectors with shifted parameters. For each (exactly solvable) rdQM system, we can construct the (exactly solvable) birth and death process [30], which is a stationary Markov chain. The Casoratian identities provide equivalence among such birth and death processes.

Similar Casoratian identities were studied by Curbera and Durán [31]. Their method is different from ours and the identities are presented for the Charlier, Meixner, Krawtchouk, and Hahn polynomials only, which have the sinusoidal coordinate $\eta(x) = x$. In our method various polynomials having five types of sinusoidal coordinates [26] $\eta(x) = x, x(x+d), 1-q^x, q^{-x}-1, (q^{-x}-1)(1-dq^x)$ are covered.

This paper is organized as follows. In Sect. 2 we recapitulate the discrete quantum mechanics with real shifts. Section 3 is the main part of this paper. After presenting the data for the original $(q$-)Racah systems in Sect. 3.1, we discuss their discrete symmetries and present the pseudo virtual state polynomials by using the twist operations in Sect. 3.2. The Casoratian identities for the $(q$-)Racah polynomials are derived starting from those for the Askey–Wilson polynomial in Sect. 3.3. In Sect. 4 the Casoratian identities for the reduced-form polynomials are presented. Section 5 is for the summary and comments. In Appendix A some necessary data for orthogonal polynomials are presented. In Appendix B the pseudo virtual state vectors and one-step Darboux transformation are discussed by taking the q-Racah system as an example.

2. Discrete quantum mechanics with real shifts

In this section we recapitulate the discrete quantum mechanics with real shifts (rdQM) developed in Refs. [26,28].

The Hamiltonian of rdQM $\mathcal{H} = (\mathcal{H}_{x,y})$ is a tri-diagonal real symmetric (Jacobi) matrix and its rows and columns are indexed by integers x and y, which take values in $\{0, 1, \ldots, N\}$ (finite) or $\mathbb{Z}_{\geq 0}$ (semi-infinite):

$$\mathcal{H}_{x,y} \overset{\text{def}}{=} -\sqrt{B(x)D(x+1)}\, \delta_{x+1,y} - \sqrt{B(x-1)D(x)}\, \delta_{x-1,y} + \big(B(x) + D(x)\big)\delta_{x,y}. \tag{2.1}$$

The potential functions $B(x)$ and $D(x)$ are real and positive but vanish at the boundary, $D(0) = 0$ for both cases and $B(N) = 0$ for a finite case. In this paper we consider the case that these $B(x)$ and $D(x)$ are rational functions of x or q^x ($0 < q < 1$). For simplicity in notation, we write the matrix $\mathcal{H}$ as follows:

$$\mathcal{H} = -\sqrt{B(x)D(x+1)}\, e^{\partial} - \sqrt{B(x-1)D(x)}\, e^{-\partial} + B(x) + D(x)$$
$$= -\sqrt{B(x)}\, e^{\partial} \sqrt{D(x)} - \sqrt{D(x)}\, e^{-\partial} \sqrt{B(x)} + B(x) + D(x), \tag{2.2}$$

where the matrices $e^{\pm\partial}$ are

$$e^{\pm\partial} = ((e^{\pm\partial})_{x,y}), \quad (e^{\pm\partial})_{x,y} \overset{\text{def}}{=} \delta_{x\pm1,y}, \quad (e^{\partial})^{\dagger} = e^{-\partial}, \tag{2.3}$$

and we suppress the unit matrix $\mathbf{1} = (\delta_{x,y}) : (B(x) + D(x))\mathbf{1}$ in Eq. (2.2). The notation $f(x)Ag(x)$, where $f(x)$ and $g(x)$ are functions of x and A is a matrix $A = (A_{x,y})$, stands for a matrix whose (x, y)-element is $f(x)A_{x,y}g(y)$. Note that the matrices e^{∂} and $e^{-\partial}$ are not inverse to each other: $e^{\pm\partial}e^{\mp\partial} \neq \mathbf{1}$ for a finite system and $e^{-\partial}e^{\partial} \neq \mathbf{1}$ for a semi-infinite system. This Hamiltonian can be expressed in a factorized form:

$$\mathcal{H} = \mathcal{A}^{\dagger}\mathcal{A}, \quad \mathcal{A} \overset{\text{def}}{=} \sqrt{B(x)} - e^{\partial}\sqrt{D(x)}, \quad \mathcal{A}^{\dagger} = \sqrt{B(x)} - \sqrt{D(x)}\, e^{-\partial}. \tag{2.4}$$

The Schrödinger equation is the eigenvalue problem for the Hermitian matrix $\mathcal{H}$:

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n = 0, 1, 2, \ldots), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots \tag{2.5}$$

$(n = 0, 1, \ldots, N$ for a finite case). The ground state eigenvector $\phi_0(x)$, which is characterized by $\mathcal{A}\phi_0(x) = 0$, is chosen as

$$\phi_0(x) = \prod_{y=0}^{x-1} \sqrt{\frac{B(y)}{D(y+1)}} > 0. \tag{2.6}$$

We use the convention $\prod_{k=n}^{n-1} * = 1$, which means the normalization $\phi_0(0) = 1$. We remark that the boundary condition $D(0) = 0$ and $B(N) = 0$ for a finite case is important for the zero mode equation $\mathcal{H}\phi_0(x) = 0$; cf. Eq. (B.6). For the original systems (not the deformed one) considered in this paper, the eigenvectors have the following factorized form:

$$\phi_n(x) = \phi_0(x)\check{P}_n(x), \quad \check{P}_n(x) \overset{\text{def}}{=} P_n(\eta(x)). \tag{2.7}$$

Here $P_n(\eta)$ is a polynomial of degree n in η, and the sinusoidal coordinate $\eta(x)$ is one of the following [26]: $\eta(x) = x, \epsilon'x(x+d), 1-q^x, q^{-x}-1, \epsilon'(q^{-x}-1)(1-dq^x), (\epsilon' = \pm 1)$, which satisfy the boundary condition $\eta(0) = 0$. We adopt the universal normalization condition [26,28] as

$$P_n(0) = 1 \left(\Leftrightarrow \check{P}_n(0) = 1 \right). \tag{2.8}$$

This $\check{P}_n(x)$ is the eigenvector of the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}} \overset{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial}), \tag{2.9}$$

$$\widetilde{\mathcal{H}}\check{P}_n(x) = \mathcal{E}_n\check{P}_n(x). \tag{2.10}$$

Explicitly, Eq. (2.10) is the difference equation for $\check{P}_n$:

$$B(x)\left(\check{P}_n(x) - \check{P}_n(x+1)\right) + D(x)\left(\check{P}_n(x) - \check{P}_n(x-1)\right) = \mathcal{E}_n\check{P}_n(x). \tag{2.11}$$

Since P_n is a polynomial, $\check{P}_n(x)$ is defined for any $x \in \mathbb{R}$ and the difference equation (2.11) is also valid for $x \in \mathbb{R}$. The eigenvectors are mutually orthogonal ($d_n > 0$):

$$(\phi_n, \phi_m) \overset{\text{def}}{=} \sum_{x=0}^{x_{\max}} \phi_n(x)\phi_m(x) = \sum_{x=0}^{x_{\max}} \phi_0(x)^2\check{P}_n(x)\check{P}_m(x) = \frac{1}{d_n^2}\delta_{nm}, \tag{2.12}$$

where $x_{\max} = N$ for a finite case, ∞ for a semi-infinite case. (Although this notation d_n conflicts with the notation of the label of the pseudo virtual vector d_j in Eq. (1.2), we think this does not cause any confusion because the former appears as $\frac{1}{d_n^2}\delta_{nm}$.)

If we find functions $B'(x)$ and $D'(x)$ satisfying

$$B(x)D(x+1) = \alpha^2 B'(x)D'(x+1), \quad \alpha \neq 0, \tag{2.13}$$

$$B(x) + D(x) = \alpha\big(B'(x) + D'(x)\big) + \alpha', \tag{2.14}$$

where α and α' are real constants, we obtain the following relation:

$$\mathcal{H} = \alpha\mathcal{H}' + \alpha', \tag{2.15}$$

where $\mathcal{H}' = (\mathcal{H}'_{x,y})_{0 \leq x,y \leq x_{\max}}$ is given by

$$\mathcal{H}' = -\frac{|\alpha|}{\alpha}\sqrt{B'(x)D'(x+1)}\,e^{\partial} - \frac{|\alpha|}{\alpha}\sqrt{B'(x-1)D'(x)}\,e^{-\partial} + B'(x) + D'(x). \tag{2.16}$$

For concrete examples, various quantities depend on a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ and q. The parameter q is $0 < q < 1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{(\lambda_1, \lambda_2, \ldots)} = (q^{\lambda_1}, q^{\lambda_2}, \ldots)$. The $\boldsymbol{\lambda}$-dependence is expressed as $\mathcal{H} = \mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda})$, $B(x) = B(x; \boldsymbol{\lambda})$, $\phi_n(x) = \phi_n(x; \boldsymbol{\lambda})$, $\check{P}_n(x) = \check{P}_n(x; \boldsymbol{\lambda}) = P_n\big(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}\big)$, etc. If needed, the q-dependence is also expressed as $\mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda}; q)$, $B(x) = B(x; \boldsymbol{\lambda}; q)$, $\check{P}_n(x) = \check{P}_n(x; \boldsymbol{\lambda}; q) = P_n\big(\eta(x; \boldsymbol{\lambda}; q); \boldsymbol{\lambda}; q\big)$, etc.

The original systems in this paper are shape invariant [26] and they satisfy the relation

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^{\dagger} = \kappa\,\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^{\dagger}\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \quad \kappa > 0, \tag{2.17}$$

which is a sufficient condition for exact solvability. The auxiliary functions $\varphi(x; \boldsymbol{\lambda})$ [26] and $\varphi_M(x; \boldsymbol{\lambda})$ [27] are defined by

$$\varphi(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} \frac{\eta(x+1; \boldsymbol{\lambda}) - \eta(x; \boldsymbol{\lambda})}{\eta(1; \boldsymbol{\lambda})}, \tag{2.18}$$

$$\varphi_M(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} \prod_{1 \leq j < k \leq M} \frac{\eta(x+k-1; \boldsymbol{\lambda}) - \eta(x+j-1; \boldsymbol{\lambda})}{\eta(k-j; \boldsymbol{\lambda})}$$

$$= \prod_{1 \leq j < k \leq M} \varphi\big(x+j-1; \boldsymbol{\lambda} + (k-j-1)\boldsymbol{\delta}\big), \tag{2.19}$$

and $\varphi_0(x; \boldsymbol{\lambda}) = \varphi_1(x; \boldsymbol{\lambda}) = 1$.

For the orthogonal polynomials with Jackson integral measures such as the big q-Jacobi polynomial, the two-component formulation is needed; see Ref. [28].

The symbols $(a)_n$ and $(a; q)_n$ are $(q\text{-})$shifted factorials $((q\text{-})$Pochhammer symbols$)$ [29]. They are defined for a non-negative integer n by $(a)_n = \prod_{j=0}^{n-1}(a+j)$ and $(a; q)_n = \prod_{j=0}^{n-1}(1 - aq^j)$, which are extended to a real n by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}. \tag{2.20}$$

Note that, for $a \neq 0$ and $n \in \mathbb{Z}_{\geq 0}$,

$$(a; q^{-1})_n = (-a)^n q^{-\frac{1}{2}n(n-1)}(a^{-1}; q)_n. \tag{2.21}$$

The hypergeometric series $_rF_s$ and the basic hypergeometric series $_r\phi_s$ are

$$_rF_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array}\middle|\, z\right) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n}\frac{z^n}{n!}, \tag{2.22}$$

$$_r\phi_s\begin{pmatrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{pmatrix} q\,; z \Big) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)\frac{1}{2}n(n-1)} \frac{z^n}{(q;q)_n}, \qquad (2.23)$$

where $(a_1, \ldots, a_r)_n \overset{\text{def}}{=} \prod_{k=1}^{r}(a_k)_n$ and $(a_1, \ldots, a_r; q)_n \overset{\text{def}}{=} \prod_{k=1}^{r}(a_k; q)_n$.

The Casorati determinant (Casoratian) of a set of n functions $\{f_j(x)\}$ for rdQM is defined by

$$\mathrm{W_C}[f_1, f_2, \ldots, f_n](x) \overset{\text{def}}{=} \det(f_k(x+j-1))_{1\le j,k\le n} \qquad (2.24)$$

(for $n = 0$, we set $\mathrm{W_C}[\cdot](x) = 1$).

For well defined quantum systems, the range of parameters λ must be chosen such that the Hamiltonian is real symmetric. On the other hand, our main purpose in this paper is to obtain the Casoratian identities (3.36). Both sides of Eq. (3.36) are polynomials in x or Laurent polynomials in q^x and Eq. (3.36) hold for any parameter range (except for the zeros of the denominators). So we do not bother about the range of parameters, except for in Appendix B.

3. Casoratian identities of the (q-)Racah polynomials

In this section we consider rdQM whose eigenvectors are described by the (q-)Racah polynomials. After discussing some discrete symmetries and pseudo virtual state polynomials, the Casoratian identities of the (q-)Racah polynomials are presented.

3.1. Original systems

Let us consider the Racah (R) and q-Racah (qR) systems [26]. Although there are four possible parameter choices indexed by $(\epsilon, \epsilon') = (\pm1, \pm1)$ in general, we restrict ourselves to the $(\epsilon, \epsilon') = (1, 1)$ case for simplicity of presentation. The set of parameters $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, its shift δ, and κ are

$$\mathrm{R}: \lambda = (a, b, c, d), \quad \delta = (1, 1, 1, 1), \quad \kappa = 1, \qquad (3.1)$$

$$q\mathrm{R}: q^{\lambda} = (a, b, c, d), \quad \delta = (1, 1, 1, 1), \quad \kappa = q^{-1}, \qquad (3.2)$$

and we take

$$\lambda_3 = -N, \quad \text{namely } c = \begin{cases} -N & : \mathrm{R} \\ q^{-N} & : q\mathrm{R}. \end{cases} \qquad (3.3)$$

We list the fundamental data:

$$B(x; \lambda) = \begin{cases} -\dfrac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)} & : \mathrm{R} \\[4mm] -\dfrac{(1-aq^x)(1-bq^x)(1-cq^x)(1-dq^x)}{(1-dq^{2x})(1-dq^{2x+1})} & : q\mathrm{R}, \end{cases} \qquad (3.4)$$

$$D(x; \lambda) = \begin{cases} -\dfrac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)} & : \mathrm{R} \\[4mm] -\tilde{d}\,\dfrac{(1-a^{-1}dq^x)(1-b^{-1}dq^x)(1-c^{-1}dq^x)(1-q^x)}{(1-dq^{2x-1})(1-dq^{2x})} & : q\mathrm{R}, \end{cases} \qquad (3.5)$$

$$\mathcal{E}_n(\lambda) = \begin{cases} n(n+\tilde{d}) & : \mathrm{R} \\ (q^{-n}-1)(1-\tilde{d}q^n) & : q\mathrm{R}, \end{cases} \qquad \eta(x; \lambda) = \begin{cases} x(x+d) & : \mathrm{R} \\ (q^{-x}-1)(1-dq^x) & : q\mathrm{R}, \end{cases} \qquad (3.6)$$

$$\varphi(x;\boldsymbol{\lambda}) = \begin{cases} \dfrac{2x+d+1}{d+1} & :\text{R} \\[2mm] \dfrac{q^{-x}-dq^{x+1}}{1-dq} & :q\text{R}, \end{cases} \qquad \tilde{d} \stackrel{\text{def}}{=} \begin{cases} a+b+c-d-1 & :\text{R} \\ abcd^{-1}q^{-1} & :q\text{R}, \end{cases} \tag{3.7}$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = P_n\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big) = \begin{cases} {}_4F_3\!\left(\begin{matrix} -n,\, n+\tilde{d},\, -x,\, x+d \\ a,\, b,\, c \end{matrix}\,\middle|\, 1\right) & :\text{R} \\[3mm] {}_4\phi_3\!\left(\begin{matrix} q^{-n},\, \tilde{d}q^n,\, q^{-x},\, dq^x \\ a,\, b,\, c \end{matrix}\,\middle|\, q\,;q\right) & :q\text{R} \end{cases}$$

$$= \begin{cases} R_n\big(\eta(x;\boldsymbol{\lambda});a-1,\tilde{d}-a,c-1,d-c\big) & :\text{R} \\ R_n\big(1+d+\eta(x;\boldsymbol{\lambda});aq^{-1},\tilde{d}a^{-1},cq^{-1},dc^{-1}|q\big) & :q\text{R}, \end{cases} \tag{3.8}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \begin{cases} \dfrac{(a,b,c,d)_x}{(d-a+1,d-b+1,d-c+1,1)_x}\dfrac{2x+d}{d} & :\text{R} \\[3mm] \dfrac{(a,b,c,d\,;q)_x}{(a^{-1}dq,b^{-1}dq,c^{-1}dq,q\,;q)_x\,\tilde{d}^x}\dfrac{1-dq^{2x}}{1-d} & :q\text{R}, \end{cases} \tag{3.9}$$

where $R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)$ and $R_n(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta|q)$ are the Racah and q-Racah polynomials in the conventional parametrization [29], respectively. Our parametrization respects the correspondence between the (q-)Racah and (Askey–)Wilson polynomials, and symmetries in (a,b,c,d) are transparent. The expressions on the right-hand sides of $\mathcal{E}_n(\boldsymbol{\lambda})$, $\eta(x;\boldsymbol{\lambda})$, $\varphi(x;\boldsymbol{\lambda})$, $\check{P}_n(x;\boldsymbol{\lambda})$, and $\phi_0(x;\boldsymbol{\lambda})^2$ in Eqs. (3.6)–(3.9) are also valid for real n or real x, and we regard $\mathcal{E}_n(\boldsymbol{\lambda})$ etc. as functions of real n or real x by Eqs. (3.6)–(3.9). Note that $\phi_0(x;\boldsymbol{\lambda})^2 = 0$ for $x \in \mathbb{Z}\backslash\{0,1,\dots,N\}$ due to the factor $(c)_x/(1)_x$ or $(c;q)_x/(q;q)_x$. The R and qR systems are invariant under the exchange $a \leftrightarrow b$.

The potential functions $B(x;\boldsymbol{\lambda})$ and $D(x;\boldsymbol{\lambda})$ have the following symmetries:

$$\text{R}, q\text{R} : \ B(N-x;\boldsymbol{\lambda}') = D(x;\boldsymbol{\lambda}), \quad D(N-x;\boldsymbol{\lambda}') = B(x;\boldsymbol{\lambda}),$$

$$\boldsymbol{\lambda}' = (\lambda_1+\lambda_3-\lambda_4, \lambda_2+\lambda_3-\lambda_4, \lambda_3, 2\lambda_3-\lambda_4), \tag{3.10}$$

$$q\text{R} : \ B(x;\boldsymbol{\lambda};q^{-1}) = \tilde{d}^{-1}B(x;\boldsymbol{\lambda};q), \quad D(x;\boldsymbol{\lambda};q^{-1}) = \tilde{d}^{-1}D(x;\boldsymbol{\lambda};q). \tag{3.11}$$

Corresponding to Eq. (3.10), the identities (1.7.6) and (1.13.26) in Ref. [29] give the relations

$$\check{P}_n(N-x;\boldsymbol{\lambda}') = \check{P}_n(x;\boldsymbol{\lambda}) \times \begin{cases} \dfrac{(a,b)_n}{(1-a+\tilde{d},1-b+\tilde{d})_n} & :\text{R} \\[3mm] \dfrac{c^n(a,b;q)_n}{d^n(a^{-1}\tilde{d}q,b^{-1}\tilde{d}q;q)_n} & :q\text{R}. \end{cases} \tag{3.12}$$

Corresponding to Eq. (3.11), the qR polynomial is invariant under $q \to q^{-1}$:

$$\check{P}_n(x;\boldsymbol{\lambda};q^{-1}) = \check{P}_n(x;\boldsymbol{\lambda};q), \tag{3.13}$$

which is shown by Eq. (2.21). In the conventional notation, this Eq. (3.13) is written as

$$R_n(q^x+\gamma^{-1}\delta^{-1}q^{-x-1};\alpha^{-1},\beta^{-1},\gamma^{-1},\delta^{-1}|q^{-1}) = R_n(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta|q).$$

3.2. Discrete symmetries

Let us consider the twist operation $\mathfrak{t}$, which is an involution acting on x, $\boldsymbol{\lambda}$, and q:

$$\mathfrak{t}(x,\boldsymbol{\lambda},q) = \big(\mathfrak{t}(x),\mathfrak{t}(\boldsymbol{\lambda}),\mathfrak{t}(q)\big), \quad \mathfrak{t}^2 = \text{id}. \tag{3.14}$$

For the R system, the q-part should be ignored. We present twist operations (i)–(ii) and $\widetilde{\text{(i)}}$–$\widetilde{\text{(ii)}}$, which lead to the pseudo virtual state vectors explained in Appendix B.

First let us define twist (ii) as follows:

$$(\text{ii}): \ \mathfrak{t}(x) \stackrel{\text{def}}{=} x + \lambda_3 - 1, \ \ \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (2 - \lambda_1 - \lambda_3 + \lambda_4, 2 - \lambda_2 - \lambda_3 + \lambda_4, 2 - \lambda_3, 2 - 2\lambda_3 + \lambda_4),$$

$$\mathfrak{t}(q) \stackrel{\text{def}}{=} q. \tag{3.15}$$

By using this twist operation, the functions $B'(x)$ and $D'(x)$ satisfying Eqs. (2.13)–(2.14) are obtained:

$$(\text{ii}): \ B'(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} B\big(x - N - 1; \mathfrak{t}(\boldsymbol{\lambda})\big), \ \ D'(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} D\big(x - N - 1; \mathfrak{t}(\boldsymbol{\lambda})\big), \tag{3.16}$$

$$\alpha(\boldsymbol{\lambda}) = \begin{cases} 1 & : \text{R} \\ \tilde{d}q^{-1} & : q\text{R}, \end{cases} \quad \alpha'(\boldsymbol{\lambda}) = \begin{cases} -(\tilde{d} - 1) & : \text{R} \\ -(1 - q)(1 - \tilde{d}q^{-1}) & : q\text{R}. \end{cases} \tag{3.17}$$

Explicitly $B'(x)$ and $D'(x)$ are

$$(\text{ii}): \ B'(x; \boldsymbol{\lambda}) = D(x + 1; \boldsymbol{\lambda}) \times \begin{cases} \dfrac{d + 2x + 2}{d + 2x} & : \text{R} \\ \tilde{d}^{-1} \dfrac{1 - dq^{2x+2}}{1 - dq^{2x}} & : q\text{R}, \end{cases}$$

$$D'(x; \boldsymbol{\lambda}) = B(x - 1; \boldsymbol{\lambda}) \times \begin{cases} \dfrac{d + 2x - 2}{d + 2x} & : \text{R} \\ q^2 \tilde{d}^{-1} \dfrac{1 - dq^{2x-2}}{1 - dq^{2x}} & : q\text{R}. \end{cases} \tag{3.18}$$

We introduce the pseudo virtual state polynomial $\check{\xi}_{\text{v}}(x; \boldsymbol{\lambda})$ $(\text{v} \in \mathbb{Z}_{\geq 0})$,

$$(\text{ii}): \ \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{P}_{\text{v}}\big(x - N - 1; \mathfrak{t}(\boldsymbol{\lambda})\big), \tag{3.19}$$

which is the polynomial part of the pseudo virtual state vector; see Appendix B. It is a polynomial of degree v in $\eta(x; \boldsymbol{\lambda})$, $\check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\text{v}}\big(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}\big)$, because $\eta\big(\mathfrak{t}(x); \mathfrak{t}(\boldsymbol{\lambda})\big) = a\eta(x; \boldsymbol{\lambda}) + b$ $(a, b : \text{constants}, a \neq 0)$. Explicitly it is

$$(\text{ii}): \ \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) = \begin{cases} {}_4F_3\!\left(\begin{matrix} -\text{v}, \text{v} + 2 - \tilde{d}, 1 - x - c, x + a + b - \tilde{d} \\ 1 + a - \tilde{d}, 1 + b - \tilde{d}, 2 - c \end{matrix} \,\middle|\, 1\right) & : \text{R} \\[12pt] {}_4\phi_3\!\left(\begin{matrix} q^{-\text{v}}, \tilde{d}^{-1}q^{\text{v}+2}, c^{-1}q^{1-x}, ab\tilde{d}^{-1}q^{x} \\ qa\tilde{d}^{-1}, qb\tilde{d}^{-1}, q^2c^{-1} \end{matrix} \,\middle|\, q\,; q\right) & : q\text{R}. \end{cases} \tag{3.20}$$

By applying the symmetry (3.10) to this twist (ii), we define a twist (i):

$$(\text{i}): \ \mathfrak{t}(x) \stackrel{\text{def}}{=} -x - 1, \ \ \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (2 - \lambda_1, 2 - \lambda_2, 2 - \lambda_3, 2 - \lambda_4) = 2\boldsymbol{\delta} - \boldsymbol{\lambda}, \ \ \mathfrak{t}(q) \stackrel{\text{def}}{=} q. \tag{3.21}$$

By using this twist (i), the functions $B'(x)$ and $D'(x)$ satisfying Eqs. (2.13)–(2.14) are obtained:

$$(\text{i}): \ B'(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} D\big(-x - 1; \mathfrak{t}(\boldsymbol{\lambda})\big), \ \ D'(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} B\big(-x - 1; \mathfrak{t}(\boldsymbol{\lambda})\big). \tag{3.22}$$

Since the inversion of the coordinate x $(x \to -x - 1)$ means the exchange of the matrices $e^\partial \leftrightarrow e^{-\partial}$, the functions $B(x)$ and $D(x)$ are exchanged in this definition. Explicit forms of $B'(x; \boldsymbol{\lambda})$ and $D'(x; \boldsymbol{\lambda})$ are the same as in Eq. (3.18),

$$B'^{(\text{i})}(x; \boldsymbol{\lambda}) = B'^{(\text{ii})}(x; \boldsymbol{\lambda}), \quad D'^{(\text{i})}(x; \boldsymbol{\lambda}) = D'^{(\text{ii})}(x; \boldsymbol{\lambda}), \tag{3.23}$$

and $\alpha(\lambda)$ and $\alpha'(\lambda)$ are given by Eq. (3.17). The pseudo virtual state polynomial $\check{\xi}_{\mathrm{v}}(x;\lambda)$ ($\mathrm{v} \in \mathbb{Z}_{\geq 0}$) is defined by

$$\text{(i)} : \quad \check{\xi}_{\mathrm{v}}(x;\lambda) \stackrel{\text{def}}{=} \check{P}_{\mathrm{v}}\big(-x-1; \mathfrak{t}(\lambda)\big), \tag{3.24}$$

which should be proportional to the twist (ii) case. In fact, we have

$$\text{(i)} : \quad \check{\xi}_{\mathrm{v}}(x;\lambda) = \begin{cases} {}_4F_3\left(\begin{array}{c} -\mathrm{v},\, \mathrm{v}+2-\tilde{d},\, x+1,\, 1-x-d \\ 2-a,\, 2-b,\, 2-c \end{array} \,\middle|\, 1 \right) & : \mathrm{R} \\[12pt] {}_4\phi_3\left(\begin{array}{c} q^{-\mathrm{v}},\, \tilde{d}^{-1}q^{\mathrm{v}+2},\, q^{x+1},\, d^{-1}q^{1-x} \\ q^2 a^{-1},\, q^2 b^{-1},\, q^2 c^{-1} \end{array} \,\middle|\, q\,;q \right) & : q\mathrm{R}, \end{cases} \tag{3.25}$$

and

$$\check{\xi}_{\mathrm{v}}^{(\text{ii})}(x;\lambda) = \check{\xi}_{\mathrm{v}}^{(\text{i})}(x;\lambda) \times \begin{cases} \dfrac{(2-a, 2-b)_{\mathrm{v}}}{(1+a-\tilde{d}, 1+b-\tilde{d})_{\mathrm{v}}} & : \mathrm{R} \\[12pt] \dfrac{d^{\mathrm{v}}(q^2 a^{-1}, q^2 b^{-1}; q)_{\mathrm{v}}}{c^{\mathrm{v}}(qa\tilde{d}^{-1}, qb\tilde{d}^{-1}; q)_{\mathrm{v}}} & : q\mathrm{R}. \end{cases} \tag{3.26}$$

For the $q\mathrm{R}$ case, we can change the parameter q. By applying the symmetry (3.11) to the twists (i)–(ii), we define the twists $\widetilde{(\text{i})}$–$\widetilde{(\text{ii})}$ for $q\mathrm{R}$ as follows:

$$\widetilde{(\text{i})} : \quad \text{(i) with the replacement } \mathfrak{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{3.27}$$

$$\widetilde{(\text{ii})} : \quad \text{(ii) with the replacement } \mathfrak{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{3.28}$$

which give

$$\widetilde{(\text{i})} : \quad B'(x;\lambda;q) \stackrel{\text{def}}{=} D\big(-x-1; \mathfrak{t}(\lambda); q^{-1}\big),\; D'(x;\lambda;q) \stackrel{\text{def}}{=} B\big(-x-1; \mathfrak{t}(\lambda); q^{-1}\big), \tag{3.29}$$

$$\widetilde{(\text{ii})} : \quad B'(x;\lambda;q) \stackrel{\text{def}}{=} B\big(x-N-1; \mathfrak{t}(\lambda); q^{-1}\big),\; D'(x;\lambda;q) \stackrel{\text{def}}{=} D\big(x-N-1; \mathfrak{t}(\lambda); q^{-1}\big), \tag{3.30}$$

$$\widetilde{(\text{i})}, \widetilde{(\text{ii})} : \quad \alpha(\lambda) = q, \quad \alpha'(\lambda) = -(1-q)(1-\tilde{d}q^{-1}). \tag{3.31}$$

Explicitly $B'(x)$ and $D'(x)$ are

$$\widetilde{(\text{i})}, \widetilde{(\text{ii})} : \quad \alpha(\lambda)B'(x;\lambda) = \alpha^{(\text{i})}(\lambda)B'^{(\text{i})}(x;\lambda), \quad \alpha(\lambda)D'(x;\lambda) = \alpha^{(\text{i})}(\lambda)D'^{(\text{i})}(x;\lambda). \tag{3.32}$$

The pseudo virtual state polynomials $\check{\xi}_{\mathrm{v}}(x;\lambda)$ ($\mathrm{v} \in \mathbb{Z}_{\geq 0}$) are defined by

$$\widetilde{(\text{i})} : \quad \check{\xi}_{\mathrm{v}}(x;\lambda;q) \stackrel{\text{def}}{=} \check{P}_{\mathrm{v}}\big(-x-1; \mathfrak{t}(\lambda); q^{-1}\big), \tag{3.33}$$

$$\widetilde{(\text{ii})} : \quad \check{\xi}_{\mathrm{v}}(x;\lambda;q) \stackrel{\text{def}}{=} \check{P}_{\mathrm{v}}\big(x-N-1; \mathfrak{t}(\lambda); q^{-1}\big), \tag{3.34}$$

and Eq. (3.13) implies

$$\check{\xi}_{\mathrm{v}}^{\widetilde{(\text{i})}}(x;\lambda;q) = \check{\xi}_{\mathrm{v}}^{(\text{i})}(x;\lambda;q), \quad \check{\xi}_{\mathrm{v}}^{\widetilde{(\text{ii})}}(x;\lambda;q) = \check{\xi}_{\mathrm{v}}^{(\text{ii})}(x;\lambda;q). \tag{3.35}$$

The twist operations (i)–(ii) and $\widetilde{(i)}$–$\widetilde{(ii)}$ have essentially the same effects for the R and qR systems, because the relations (3.23), (3.32), (3.26), and (3.35) imply that the pseudo virtual vectors (B.8) obtained by these twists are same (proportional). So they lead to the same Casoratian identities (3.36). However, they may have different effects for the reduced systems in Sect. 4, which are obtained as appropriate limits of R and qR systems, because the symmetries (3.10)–(3.11) may no longer hold for the reduced systems.

Since the R and qR systems are invariant under the exchange $a \leftrightarrow b$, twists (i)–(ii) and $\widetilde{(i)}$–$\widetilde{(ii)}$ can be modified by exchanging $\mathfrak{t}(\lambda_1) \leftrightarrow \mathfrak{t}(\lambda_2)$.

3.3. Casoratian identities

We will present the Casoratian identities for R and qR polynomials. By using the discrete symmetries obtained in Sect. 3.2, the pseudo virtual state vectors are defined; see Appendix B. Then the original systems can be deformed by multi-step Darboux transformations in terms of pseudo virtual state vectors. The original systems can also be deformed by multi-step Darboux transformations in terms of eigenstate vectors [24]. For an appropriate choice of the index sets and shift of parameters, these two deformed systems are found to be equivalent. We present such a calculation for the one-step Darboux transformation in terms of the pseudo virtual state vector for the q-Racah case in Appendix B. However, multi-step Darboux transformations in terms of pseudo virtual state vectors are rather complicated due to the following two facts: (a) the pseudo virtual state vectors do not satisfy the Schrödinger equation at both boundaries, (b) the size of the Hamiltonian increases at each step (namely the $(N+1) \times (N+1)$ matrix becomes the $(N+M+1) \times (N+M+1)$ matrix after M-step). So we present and prove the Casoratian identities for R and qR polynomials in a "shortcut" way.

We take M, $\mathcal{N}$, $\bar{\mathcal{N}}$, d_j, $\bar{d}_j$, e_j, and $\bar{\lambda}$ as in Eq. (1.2). Then the Casoratian identities for the R and qR polynomials are

$$\varphi_M(x-M;\lambda)^{-1} \, \mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{d_2}, \ldots, \check{\xi}_{d_M}](x-M;\lambda)$$
$$\propto \varphi_{\bar{\mathcal{N}}}(x;\bar{\lambda})^{-1} \, \mathrm{W_C}[\check{P}_{e_1}, \check{P}_{e_2}, \ldots, \check{P}_{e_{\bar{\mathcal{N}}}}](x;\bar{\lambda}). \tag{3.36}$$

Note that the variable x in the first line is shifted by $-M$, which corresponds to the range of x in the deformed Hamiltonian $(\mathcal{H}_{d_1 \cdots d_M \, x,y})_{-M \leq x,y \leq N}$; see Appendix B. Since $\check{P}_n(x;\lambda)$, $\check{\xi}_v(x;\lambda)$, and $\varphi_M(x;\lambda)$ are defined for real x, these identities hold for real x. For the qR case, we prove Eq. (3.36) by translating the Casoratian identities (1.3) for the Askey–Wilson polynomial. The identities for the R case are easily obtained from the qR case by taking the $q \to 1$ limit. The necessary data for the Askey–Wilson polynomial are given in Appendix A.1.

The q-Racah polynomial and the Askey–Wilson polynomial are the "same" polynomials [29]. The replacement rule of this correspondence is

$$ix^{\mathrm{AW}} = \gamma(x^{q\mathrm{R}} + \tfrac{1}{2}\lambda_4^{q\mathrm{R}}), \quad \lambda^{\mathrm{AW}} = \lambda^{q\mathrm{R}} - \tfrac{1}{2}\lambda_4^{q\mathrm{R}}\delta^{q\mathrm{R}},$$
$$\text{namely } e^{ix^{\mathrm{AW}}} = q^{x^{q\mathrm{R}}}d^{\frac{1}{2}}, \quad (a_1, a_2, a_3, a_4) = (ad^{-\frac{1}{2}}, bd^{-\frac{1}{2}}, cd^{-\frac{1}{2}}, d^{\frac{1}{2}}). \tag{3.37}$$

Under this replacement rule, we have

$$\check{P}_n^{\mathrm{AW}}(x^{\mathrm{AW}};\lambda^{\mathrm{AW}}) = d^{-\frac{n}{2}}(a,b,c;q)_n \check{P}_n^{q\mathrm{R}}(x^{q\mathrm{R}};\lambda^{q\mathrm{R}}), \tag{3.38}$$

and

$$V(x^{\text{AW}}; \boldsymbol{\lambda}^{\text{AW}}) = -B(x^{q\text{R}}; \boldsymbol{\lambda}^{q\text{R}}), \quad V^*(x^{\text{AW}}; \boldsymbol{\lambda}^{\text{AW}}) = -D(x^{q\text{R}}; \boldsymbol{\lambda}^{q\text{R}}),$$

$$\mathcal{E}_n^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}}) = \mathcal{E}_n^{q\text{R}}(\boldsymbol{\lambda}^{q\text{R}}),$$

$$\eta^{\text{AW}}(x^{\text{AW}}) = \tfrac{1}{2}d^{-\frac{1}{2}}\big(\eta^{q\text{R}}(x^{q\text{R}}; \boldsymbol{\lambda}^{q\text{R}}) + 1 + d\big), \tag{3.39}$$

$$\varphi^{\text{AW}}(x^{\text{AW}}) = i(dq)^{-\frac{1}{2}}(1 - dq)\varphi^{q\text{R}}(x^{q\text{R}} - \tfrac{1}{2}; \boldsymbol{\lambda}^{q\text{R}}).$$

The shifts $\boldsymbol{\delta}^{\text{AW}}$ and $\boldsymbol{\delta}^{q\text{R}}$ are consistent. The twist $\mathfrak{t}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}})$ (A.2) gives the twist (i) $\mathfrak{t}^{q\text{R(i)}}(\boldsymbol{\lambda}^{q\text{R}})$ (3.21). In the following we omit the superscript qR. The twisted potential function of the AW system V' (A.3), the pseudo virtual state energy $\tilde{\mathcal{E}}_{\text{v}}^{\text{AW}}$ (A.4), the pseudo virtual state polynomial $\check{\xi}_{\text{v}}^{\text{AW}}$ (A.5), and the auxiliary function φ_M^{AW} (A.6) become

$$V\big(x^{\text{AW}}; \mathfrak{t}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}})\big) = -D\big(-x - 1; \mathfrak{t}^{\text{(i)}}(\boldsymbol{\lambda})\big) = -B^{'\text{(i)}}(x; \boldsymbol{\lambda}),$$

$$V^*\big(x^{\text{AW}}; \mathfrak{t}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}})\big) = -B\big(-x - 1; \mathfrak{t}^{\text{(i)}}(\boldsymbol{\lambda})\big) = -D^{'\text{(i)}}(x; \boldsymbol{\lambda}), \tag{3.40}$$

$$\tilde{\mathcal{E}}_{\text{v}}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}}) = \tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda}) = \mathcal{E}_{-\text{v}-1}(\boldsymbol{\lambda}), \tag{3.41}$$

$$\check{\xi}_{\text{v}}^{\text{AW}}(x^{\text{AW}}; \boldsymbol{\lambda}^{\text{AW}}) = (qd^{-\frac{1}{2}})^{-\text{v}}(q^2 a^{-1}, q^2 b^{-1}, q^2 c^{-1}; q)_{\text{v}}\check{P}_{\text{v}}\big(-x - 1; \mathfrak{t}^{\text{(i)}}(\boldsymbol{\lambda})\big) \propto \check{\xi}_{\text{v}}^{\text{(i)}}(x; \boldsymbol{\lambda}), \tag{3.42}$$

$$\varphi_M^{\text{AW}}(x^{\text{AW}}) \propto \varphi_M(x - \tfrac{M-1}{2}; \boldsymbol{\lambda}). \tag{3.43}$$

The shifted parameters $\bar{\boldsymbol{\lambda}}^{\text{AW}}$ and $\bar{\boldsymbol{\lambda}}$ are also consistent. For functions $f_j(x^{\text{AW}}; \boldsymbol{\lambda}^{\text{AW}}) = g_j(x; \boldsymbol{\lambda})$, the Casoratian for idQM W_γ (1.4) and that for rdQM W_C (2.24) are related by

$$\text{W}_\gamma[f_1, f_2, \ldots, f_n](x^{\text{AW}}; \boldsymbol{\lambda}^{\text{AW}}) = i^{\frac{1}{2}n(n-1)}\text{W}_\text{C}[g_1, g_2, \ldots, g_n](x - \tfrac{n-1}{2}; \boldsymbol{\lambda}). \tag{3.44}$$

By using Eqs. (3.38), (3.42)–(3.43), and (3.44), the Casoratian identities for the AW polynomial (1.3) are rewritten as

$$\varphi_M\big(x - \tfrac{M-1}{2}; \boldsymbol{\lambda}\big)^{-1}\text{W}_\text{C}[\check{\xi}_{d_1}, \check{\xi}_{d_2}, \ldots, \check{\xi}_{d_M}]\big(x - \tfrac{M-1}{2}; \boldsymbol{\lambda}\big)$$

$$\propto \varphi_{\bar{\mathcal{N}}}\big(x + \tfrac{M+1}{2}; \bar{\boldsymbol{\lambda}}\big)^{-1}\text{W}_\text{C}[\check{P}_{e_1}, \check{P}_{e_2}, \ldots, \check{P}_{e_{\bar{\mathcal{N}}}}]\big(x + \tfrac{M+1}{2}; \bar{\boldsymbol{\lambda}}\big). \tag{3.45}$$

By the replacement $x \to x - \tfrac{M+1}{2}$, this gives the Casoratian identities for the qR polynomial (3.36).

Although the proportionality constants of Eq. (3.36) are not so important, we present them for the qR case. By explicit calculation (we assume $d_1 < d_2 < \cdots < d_M$), we have

$$\varphi_M(x; \boldsymbol{\lambda})^{-1}\text{W}_\text{C}[\check{\xi}_{d_1}^{\text{(i)}}, \check{\xi}_{d_2}^{\text{(i)}}, \ldots, \check{\xi}_{d_M}^{\text{(i)}}](x; \boldsymbol{\lambda})$$

$$= \prod_{j=1}^{M}(qd^{-1})^{d_j}c_{d_j}\big(\mathfrak{t}^{\text{(i)}}(\boldsymbol{\lambda})\big) \cdot q^{-\sum_{j=1}^{M}(j-1)d_j} \prod_{1 \le i < j \le M}(1 - q^{d_j - d_i}) \cdot q^{\binom{M}{3}}\prod_{i=1}^{M}(1 - dq^i)^{M-i}$$

$$\times \big(\text{a monic polynomial of degree } \ell_{\mathcal{D}} \text{ in } \eta\big(x; \boldsymbol{\lambda} + (M-1)\boldsymbol{\delta}\big)\big), \tag{3.46}$$

$$\varphi_M(x; \boldsymbol{\lambda})^{-1}\text{W}_\text{C}[\check{P}_{d_1}, \check{P}_{d_2}, \ldots, \check{P}_{d_M}](x; \boldsymbol{\lambda})$$

$$= \prod_{j=1}^{M}c_{d_j}(\boldsymbol{\lambda}) \cdot q^{-\sum_{j=1}^{M}(j-1)d_j} \prod_{1 \le i < j \le M}(1 - q^{d_j - d_i}) \cdot q^{\binom{M}{3}}\prod_{i=1}^{M}(1 - dq^i)^{M-i}$$

$$\times \big(\text{a monic polynomial of degree } \ell_{\mathcal{D}} \text{ in } \eta\big(x; \boldsymbol{\lambda} + (M-1)\boldsymbol{\delta}\big)\big), \tag{3.47}$$

where $\ell_{\mathcal{D}} = \sum_{j=1}^{M} d_j - \frac{1}{2}M(M-1)$ and $c_n(\lambda)$ is the coefficient of the highest-degree term of $P_n(\eta; \lambda)$, $c_n(\lambda) = \dfrac{(\tilde{d}q^n; q)_n}{(a, b, c; q)_n}$. Then the proportionality constants (we assume $d_1 < d_2 < \cdots < d_M$ and $e_1 < e_2 < \cdots < e_{\bar{\mathcal{N}}}$) are given by

$$\varphi_M(x - M; \lambda)^{-1} \, \mathrm{W_C}[\check{\xi}_{d_1}^{(i)}, \check{\xi}_{d_2}^{(i)}, \ldots, \check{\xi}_{d_M}^{(i)}](x - M; \lambda)$$
$$= A \times \varphi_{\bar{\mathcal{N}}}(x; \bar{\lambda})^{-1} \, \mathrm{W_C}[\check{P}_{e_1}, \check{P}_{e_2}, \ldots, \check{P}_{e_{\bar{\mathcal{N}}}}](x; \bar{\lambda}), \tag{3.48}$$

$$A = \frac{\prod_{j=1}^{M} c_{d_j}\big(\mathfrak{t}^{(i)}(\lambda)\big)}{\prod_{j=1}^{\bar{\mathcal{N}}} c_{e_j}(\bar{\lambda})} \frac{\prod_{1 \le i < j \le M} (1 - q^{d_j - d_i})}{\prod_{1 \le i < j \le \bar{\mathcal{N}}} (1 - q^{e_j - e_i})} \frac{\prod_{i=1}^{M} (1 - dq^i)^{M-i}}{\prod_{i=1}^{\bar{\mathcal{N}}} (1 - dq^{-\bar{\mathcal{N}} - M + i})^{\bar{\mathcal{N}} - i}}$$
$$\times d^{-\sum_{j=1}^{M} d_j} q^{\sum_{j=1}^{M}(M+1-j)d_j + \sum_{j=1}^{\bar{\mathcal{N}}} je_j - \frac{1}{6}M(M-1)(2M-1) - \frac{1}{6}(\bar{\mathcal{N}} - 1)\bar{\mathcal{N}}(\bar{\mathcal{N}} + 1)}. \tag{3.49}$$

Here we have used $\sum_{j=1}^{\bar{\mathcal{N}}} e_j = \sum_{j=1}^{M} d_j + \frac{1}{2}\mathcal{N}(\mathcal{N} + 1) - \mathcal{N}M$, $\ell_{\bar{\mathcal{D}}} = \ell_{\mathcal{D}}$, and $\eta\big(x; \bar{\lambda} + (\bar{\mathcal{N}} - 1)\delta\big) = q^{-M}\eta\big(x - M; \lambda + (M - 1)\delta\big) + (q^{-M} - 1)(1 - dq^{-1})$.

4. Casoratian identities for the reduced-case polynomials

It is well known that the other members of the Askey scheme polynomials of a discrete variable can be obtained by reductions from the (q-)Racah polynomials [29]. Not only the polynomials themselves but also the Hamiltonians are reduced in appropriate limits (overall rescalings may be needed). Some of the twist operations $\mathfrak{t}$ (i)–(ii), $\widetilde{\text{(i)}}$–$\widetilde{\text{(ii)}}$ of R and qR systems are inherited by the reduced systems. The twist operation $\mathfrak{t}$ is Eq. (3.14) and the q-part should be ignored for non-q-polynomials. The twists (i)–(ii) and $\widetilde{\text{(i)}}$–$\widetilde{\text{(ii)}}$ act on x and q as

$$\text{(i)} : \ \mathfrak{t}(x) \stackrel{\text{def}}{=} -x - 1, \qquad \mathfrak{t}(q) \stackrel{\text{def}}{=} q, \tag{4.1}$$

$$\text{(ii)} : \ \mathfrak{t}(x) \stackrel{\text{def}}{=} x - N - 1, \ \ \mathfrak{t}(q) \stackrel{\text{def}}{=} q, \tag{4.2}$$

$$\widetilde{\text{(i)}} : \ \mathfrak{t}(x) \stackrel{\text{def}}{=} -x - 1, \qquad \mathfrak{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{4.3}$$

$$\widetilde{\text{(ii)}} : \ \mathfrak{t}(x) \stackrel{\text{def}}{=} x - N - 1, \ \ \mathfrak{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{4.4}$$

and $\mathfrak{t}(\lambda)$ will be given for each polynomial. For non-q-polynomials, the twists $\widetilde{\text{(i)}}$–$\widetilde{\text{(ii)}}$ are irrelevant. For infinite systems, the twists (i) and $\widetilde{\text{(i)}}$ should be applied. By using the twist operation, the potential functions $B'(x; \lambda)$ and $D'(x; \lambda)$ are defined by Eqs. (3.22), (3.16), and (3.29)–(3.30), and the pseudo virtual state polynomials $\check{\xi}_{\mathrm{v}}(x; \lambda)$ are defined by Eqs. (3.24), (3.19), and (3.33)–(3.34), which are polynomials of degree v in $\eta(x; \lambda)$, $\check{\xi}_{\mathrm{v}}(x; \lambda) \stackrel{\text{def}}{=} \xi_{\mathrm{v}}\big(\eta(x; \lambda); \lambda\big)$ (or $\eta(x; \lambda; q)$, $\check{\xi}_{\mathrm{v}}(x; \lambda; q) \stackrel{\text{def}}{=} \xi_{\mathrm{v}}\big(\eta(x; \lambda; q); \lambda; q\big)$). The pseudo virtual state energies $\tilde{\mathcal{E}}_{\mathrm{v}}(\lambda)$ are defined in Eqs. (B.10)–(B.11) and they satisfy Eq. (B.12). Then the Casoratian identities for these reduced-case polynomials have the same form as in Eq. (3.36),

$$\varphi_M(x - M; \lambda)^{-1} \, \mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{d_2}, \ldots, \check{\xi}_{d_M}](x - M; \lambda)$$
$$\propto \varphi_{\bar{\mathcal{N}}}(x; \bar{\lambda})^{-1} \, \mathrm{W_C}[\check{P}_{e_1}, \check{P}_{e_2}, \ldots, \check{P}_{e_{\bar{\mathcal{N}}}}](x; \bar{\lambda}), \tag{4.5}$$

with the notation (1.2).

The fundamental data for the reduced-case polynomials are listed in Appendixes A.2–A.3. In the following we present twist operations and explicit forms of the pseudo virtual state polynomials.

For the (q-)Hahn, dual (q-)Hahn, (q-)Krawtchouk, and dual q-Krawtchouk cases, we have two twist operations. Two pseudo virtual state polynomials $\check{\xi}_v(x; \lambda)$ obtained by these twists are proportional and two pairs of potential functions $(B'(x; \lambda), D'(x; \lambda))$ are also proportional. Therefore the pseudo virtual state vectors obtained by these twists are proportional and the corresponding Casoratian identities are identical.

Some formulas in Appendix B are written under the condition $\alpha(\lambda) > 0$. For $\alpha(\lambda) < 0$, slight modifications are needed. For example, Eq. (B.3): $B'(x; \lambda) \to \alpha(\lambda) B'(x; \lambda)$ and $D'(x; \lambda) \to \alpha(\lambda) D'(x; \lambda)$, Eq. (B.5): $\tilde{\phi}_0(x; \lambda) \to \tilde{\phi}_0(x; \lambda) \times \left(\mathrm{sgn}\, \alpha(\lambda) \right)^{-x}$, etc.

4.1. Finite cases

4.1.1. Hahn (Ha)

We have two twist operations:

$$\text{(i)} : \mathsf{t}(\lambda) \stackrel{\text{def}}{=} (2 - \lambda_1, 2 - \lambda_2, -2 - \lambda_3) = 2\delta - \lambda, \tag{4.6}$$

$$\text{(ii)} : \mathsf{t}(\lambda) \stackrel{\text{def}}{=} (2 - \lambda_2, 2 - \lambda_1, -2 - \lambda_3). \tag{4.7}$$

The explicit forms of the pseudo virtual state polynomials are

$$\text{(i)} : \check{\xi}_v(x; \lambda) = {}_3F_2 \left(\begin{array}{c} -v,\, v + 3 - a - b,\, x + 1 \\ 2 - a,\, N + 2 \end{array} \,\middle|\, 1 \right), \tag{4.8}$$

$$\text{(ii)} : \check{\xi}_v(x; \lambda) = {}_3F_2 \left(\begin{array}{c} -v,\, v + 3 - a - b,\, N + 1 - x \\ 2 - b,\, N + 2 \end{array} \,\middle|\, 1 \right), \tag{4.9}$$

which are proportional,

$$\check{\xi}_v^{(\text{ii})}(x; \lambda) = \check{\xi}_v^{(\text{i})}(x; \lambda) \times \frac{(2 - a)_v}{(b - v - 1)_v}, \quad \begin{pmatrix} B'^{(\text{ii})}(x; \lambda) \\ D'^{(\text{ii})}(x; \lambda) \end{pmatrix} = \begin{pmatrix} B'^{(\text{i})}(x; \lambda) \\ D'^{(\text{i})}(x; \lambda) \end{pmatrix}, \tag{4.10}$$

and $\alpha^{(\text{i})}(\lambda) = \alpha^{(\text{ii})}(\lambda) = 1$.

4.1.2. Dual Hahn (dHa)

We have two twist operations:

$$\text{(i)} : \mathsf{t}(\lambda) \stackrel{\text{def}}{=} (2 - \lambda_1, 2 - \lambda_2, -2 - \lambda_3), \tag{4.11}$$

$$\text{(ii)} : \mathsf{t}(\lambda) \stackrel{\text{def}}{=} (1 + \lambda_2 + \lambda_3, 1 + \lambda_1 + \lambda_3, -2 - \lambda_3). \tag{4.12}$$

The explicit forms of the pseudo virtual state polynomials are

$$\text{(i)} : \check{\xi}_v(x; \lambda) = {}_3F_2 \left(\begin{array}{c} -v,\, 2 - x - a - b,\, x + 1 \\ 2 - a,\, N + 2 \end{array} \,\middle|\, 1 \right), \tag{4.13}$$

$$\text{(ii)} : \check{\xi}_v(x; \lambda) = {}_3F_2 \left(\begin{array}{c} -v,\, x + a + b + N,\, N + 1 - x \\ b + N + 1,\, N + 2 \end{array} \,\middle|\, 1 \right), \tag{4.14}$$

which are proportional,

$$\check{\xi}_v^{(\text{ii})}(x; \lambda) = \check{\xi}_v^{(\text{i})}(x; \lambda) \times \frac{(2 - a)_v}{(b + N + 1)_v}, \quad \begin{pmatrix} B'^{(\text{ii})}(x; \lambda) \\ D'^{(\text{ii})}(x; \lambda) \end{pmatrix} = \begin{pmatrix} B'^{(\text{i})}(x; \lambda) \\ D'^{(\text{i})}(x; \lambda) \end{pmatrix}, \tag{4.15}$$

and $\alpha^{(\text{i})}(\lambda) = \alpha^{(\text{ii})}(\lambda) = -1$.

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4.1.3.   Krawtchouk (K)
We have two twist operations:

$$(i) : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (\lambda_1, -2 - \lambda_2), \tag{4.16}$$

$$(ii) : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (1 - \lambda_1, -2 - \lambda_2). \tag{4.17}$$

The explicit forms of the pseudo virtual state polynomials are

$$(i) : \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) = {}_2F_1 \left(\begin{matrix} -\text{v}, \, x + 1 \\ N + 2 \end{matrix} \, \middle| \, p^{-1} \right), \tag{4.18}$$

$$(ii) : \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) = {}_2F_1 \left(\begin{matrix} -\text{v}, \, N + 1 - x \\ N + 2 \end{matrix} \, \middle| \, (1 - p)^{-1} \right), \tag{4.19}$$

which are proportional,

$$\check{\xi}_{\text{v}}^{(ii)}(x; \boldsymbol{\lambda}) = \check{\xi}_{\text{v}}^{(i)}(x; \boldsymbol{\lambda}) \times (1 - p^{-1})^{-\text{v}}, \quad \begin{pmatrix} B'^{\,(ii)}(x; \boldsymbol{\lambda}) \\ D'^{\,(ii)}(x; \boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} B'^{\,(i)}(x; \boldsymbol{\lambda}) \\ D'^{\,(i)}(x; \boldsymbol{\lambda}) \end{pmatrix}, \tag{4.20}$$

and $\alpha^{(i)}(\boldsymbol{\lambda}) = \alpha^{(ii)}(\boldsymbol{\lambda}) = -1$.

4.1.4.   q-Hahn (qHa)
We have two twist operations:

$$\widetilde{(i)} : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (2 - \lambda_1, 2 - \lambda_2, -2 - \lambda_3) = 2\boldsymbol{\delta} - \boldsymbol{\lambda}, \tag{4.21}$$

$$(ii) : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (2 - \lambda_2, 2 - \lambda_1, -2 - \lambda_3). \tag{4.22}$$

The explicit forms of the pseudo virtual state polynomials are

$$\widetilde{(i)} : \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}; q) = {}_3\phi_2 \left(\begin{matrix} q^{-\text{v}}, \, a^{-1}b^{-1}q^{\text{v}+3}, \, q^{x+1} \\ q^2 a^{-1}, \, q^{N+2} \end{matrix} \, \middle| \, q \, ; bq^{N-x} \right), \tag{4.23}$$

$$(ii) : \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) = {}_3\phi_2 \left(\begin{matrix} q^{-\text{v}}, \, a^{-1}b^{-1}q^{\text{v}+3}, \, q^{N+1-x} \\ q^2 b^{-1}, \, q^{N+2} \end{matrix} \, \middle| \, q \, ; q \right), \tag{4.24}$$

which are proportional,

$$\check{\xi}_{\text{v}}^{(ii)}(x; \boldsymbol{\lambda}) = \check{\xi}_{\text{v}}^{\widetilde{(i)}}(x; \boldsymbol{\lambda}) \times \frac{(q^2 a^{-1}; q)_{\text{v}}}{(bq^{-\text{v}-1}; q)_{\text{v}}}, \quad \alpha^{(ii)}(\boldsymbol{\lambda}) \begin{pmatrix} B'^{\,(ii)}(x; \boldsymbol{\lambda}) \\ D'^{\,(ii)}(x; \boldsymbol{\lambda}) \end{pmatrix} = \alpha^{\widetilde{(i)}}(\boldsymbol{\lambda}) \begin{pmatrix} B'^{\,\widetilde{(i)}}(x; \boldsymbol{\lambda}) \\ D'^{\,\widetilde{(i)}}(x; \boldsymbol{\lambda}) \end{pmatrix}, \tag{4.25}$$

and $\alpha^{\widetilde{(i)}}(\boldsymbol{\lambda}) = q, \alpha^{(ii)}(\boldsymbol{\lambda}) = abq^{-2}$.

4.1.5.   Dual q-Hahn (dqHa)
We have two twist operations:

$$\widetilde{(i)} : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (2 - \lambda_1, 2 - \lambda_2, -2 - \lambda_3), \tag{4.26}$$

$$\widetilde{(ii)} : \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (1 + \lambda_2 + \lambda_3, 1 + \lambda_1 + \lambda_3, -2 - \lambda_3). \tag{4.27}$$

The explicit forms of the pseudo virtual state polynomials are

$$\widetilde{(i)} : \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}; q) = {}_3\phi_2 \left(\begin{matrix} q^{-\text{v}}, \, a^{-1}b^{-1}q^{2-x}, \, q^{x+1} \\ q^2 a^{-1}, \, q^{N+2} \end{matrix} \, \middle| \, q \, ; bq^{\text{v}+N+1} \right), \tag{4.28}$$

$$(\widetilde{\text{ii}}) : \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = {}_3\phi_2 \left(\begin{matrix} q^{-\text{v}}, \, abq^{x+N}, \, q^{N+1-x} \\ bq^{N+1}, \, q^{N+2} \end{matrix} \, \middle| \, q \, ; a^{-1}q^{\text{v}+2} \right), \qquad (4.29)$$

which are proportional,

$$\check{\xi}_{\text{v}}^{(\widetilde{\text{ii}})}(x;\boldsymbol{\lambda}) = \check{\xi}_{\text{v}}^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \times \frac{(q^2 a^{-1};q)_{\text{v}}}{(bq^{N+1};q)_{\text{v}}}, \quad \begin{pmatrix} B'^{(\widetilde{\text{ii}})}(x;\boldsymbol{\lambda}) \\ D'^{(\widetilde{\text{ii}})}(x;\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} B'^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \\ D'^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \end{pmatrix}, \qquad (4.30)$$

and $\alpha^{(\widetilde{\text{i}})}(\boldsymbol{\lambda}) = \alpha^{(\widetilde{\text{ii}})}(\boldsymbol{\lambda}) = q$.

4.1.6.   Quantum q-Krawtchouk (qqK)
We have one twist operation:

$$(\widetilde{\text{i}}) : \quad \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (-\lambda_1, -2 - \lambda_2). \qquad (4.31)$$

The explicit form of the pseudo virtual state polynomial is

$$(\widetilde{\text{i}}) : \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = {}_2\phi_1 \left(\begin{matrix} q^{-\text{v}}, \, q^{x+1} \\ q^{N+2} \end{matrix} \, \middle| \, q \, ; pq^{N+1-x} \right). \qquad (4.32)$$

4.1.7.   q-Krawtchouk (qK)
We have two twist operations with different $\mathfrak{t}(q)$:

$$(\widetilde{\text{i}}) : \quad \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (2 - \lambda_1, -2 - \lambda_2), \quad \mathfrak{t}(q) = q^{-1}, \qquad (4.33)$$

$$(\widetilde{\text{ii}}) : \quad \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (2 - \lambda_1, -2 - \lambda_2), \quad \mathfrak{t}(q) = q. \qquad (4.34)$$

The explicit forms of the pseudo virtual state polynomials are

$$(\widetilde{\text{i}}) : \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = {}_3\phi_1 \left(\begin{matrix} q^{-\text{v}}, \, -p^{-1}q^{\text{v}+2}, \, q^{x+1} \\ q^{N+2} \end{matrix} \, \middle| \, q \, ; -pq^{N-x-1} \right), \qquad (4.35)$$

$$(\widetilde{\text{ii}}) : \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda}) = {}_3\phi_2 \left(\begin{matrix} q^{-\text{v}}, \, -p^{-1}q^{\text{v}+2}, \, q^{N+1-x} \\ q^{N+2}, \, 0 \end{matrix} \, \middle| \, q \, ; q \right), \qquad (4.36)$$

which are proportional,

$$\check{\xi}_{\text{v}}^{(\text{ii})}(x;\boldsymbol{\lambda}) = \check{\xi}_{\text{v}}^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \times (-p)^{-\text{v}}q^{\text{v}(\text{v}+2)}, \quad \alpha^{(\text{ii})}(\boldsymbol{\lambda}) \begin{pmatrix} B'^{(\text{ii})}(x;\boldsymbol{\lambda}) \\ D'^{(\text{ii})}(x;\boldsymbol{\lambda}) \end{pmatrix} = \alpha^{(\widetilde{\text{i}})}(\boldsymbol{\lambda}) \begin{pmatrix} B'^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \\ D'^{(\widetilde{\text{i}})}(x;\boldsymbol{\lambda}) \end{pmatrix}, \qquad (4.37)$$

and $\alpha^{(\widetilde{\text{i}})}(\boldsymbol{\lambda}) = q, \, \alpha^{(\text{ii})}(\boldsymbol{\lambda}) = -pq^{-1}$.

4.1.8.   Dual q-Krawtchouk (dqK)
We have two twist operations:

$$(\widetilde{\text{i}}) : \quad \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (-\lambda_1, -2 - \lambda_2), \qquad (4.38)$$

$$(\widetilde{\text{ii}}) : \quad \mathfrak{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (\lambda_1, -2 - \lambda_2). \qquad (4.39)$$

The explicit forms of the pseudo virtual state polynomials are

$$(\widetilde{\text{i}}) : \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = {}_3\phi_1 \left(\begin{matrix} q^{-\text{v}}, \, c^{-1}q^{N+1-x}, \, q^{x+1} \\ q^{N+2} \end{matrix} \, \middle| \, q \, ; cq^{\text{v}} \right), \qquad (4.40)$$

$$\widetilde{\text{(ii)}} : \ \check{\xi}_{\text{v}}(x;\lambda;q) = {}_3\phi_1\left(\begin{array}{c} q^{-\text{v}},\, cq^{x+1},\, q^{N+1-x} \\ q^{N+2} \end{array} \middle| \, q\,;c^{-1}q^{\text{v}}\right), \tag{4.41}$$

which are proportional,

$$\check{\xi}_{\text{v}}^{\widetilde{\text{(ii)}}}(x;\lambda) = \check{\xi}_{\text{v}}^{\widetilde{\text{(i)}}}(x;\lambda) \times c^{-\text{v}}, \quad \begin{pmatrix} B'^{\,\widetilde{\text{(ii)}}}(x;\lambda) \\ D'^{\,\widetilde{\text{(ii)}}}(x;\lambda) \end{pmatrix} = \begin{pmatrix} B'^{\,\widetilde{\text{(i)}}}(x;\lambda) \\ D'^{\,\widetilde{\text{(i)}}}(x;\lambda) \end{pmatrix}, \tag{4.42}$$

and $\alpha^{\widetilde{\text{(i)}}}(\lambda) = \alpha^{\widetilde{\text{(ii)}}}(\lambda) = q$.

4.1.9. Affine q-Krawtchouk (aqK)

We have one twist operation:

$$\widetilde{\text{(i)}} : \ \ \mathfrak{t}(\lambda) \stackrel{\text{def}}{=} (-\lambda_1, -2 - \lambda_2). \tag{4.43}$$

The explicit form of the pseudo virtual state polynomial is

$$\widetilde{\text{(i)}} : \ \ \check{\xi}_{\text{v}}(x;\lambda;q) = {}_2\phi_2\left(\begin{array}{c} q^{-\text{v}},\, q^{x+1} \\ p^{-1}q,\, q^{N+2} \end{array} \middle| \, q\,;p^{-1}q^{\text{v}+N+2-x}\right). \tag{4.44}$$

4.2. Semi-infinite cases

4.2.1. Meixner (M)

We have one twist operation:

$$\text{(i)} : \ \ \mathfrak{t}(\lambda) \stackrel{\text{def}}{=} (2 - \lambda_1, \lambda_2). \tag{4.45}$$

The explicit form of the pseudo virtual state polynomial is

$$\text{(i)} : \ \ \check{\xi}_{\text{v}}(x;\lambda) = {}_2F_1\left(\begin{array}{c} -\text{v},\, x + 1 \\ 2 - \beta \end{array} \middle| \, 1 - c^{-1}\right). \tag{4.46}$$

4.2.2. Charlier (C)

We have one twist operation:

$$\text{(i)} : \ \ \mathfrak{t}(\lambda) \stackrel{\text{def}}{=} -\lambda_1. \tag{4.47}$$

The explicit form of the pseudo virtual state polynomial is

$$\text{(i)} : \ \ \check{\xi}_{\text{v}}(x;\lambda) = {}_2F_0\left(\begin{array}{c} -\text{v},\, x + 1 \\ - \end{array} \middle| \, a^{-1}\right). \tag{4.48}$$

4.2.3. Little q-Jacobi (lqJ)

We have one twist operation:

$$\widetilde{\text{(i)}} : \ \ \mathfrak{t}(\lambda) \stackrel{\text{def}}{=} (-\lambda_1, -\lambda_2). \tag{4.49}$$

The explicit forms of the pseudo virtual state polynomials are

$$\widetilde{\text{(i)}} : \quad \check{\xi}_v(x;\boldsymbol{\lambda};q) = (-b)^{-v} q^{\frac{1}{2}v(v+1)} \frac{(a^{-1}q;q)_v}{(b^{-1}q;q)_v} \, {}_2\phi_1 \left(\begin{matrix} q^{-v}, \, a^{-1}b^{-1}q^{v+1} \\ a^{-1}q \end{matrix} \,\Big|\, q\,;bq^{x+1} \right)$$

$$= {}_3\phi_2 \left(\begin{matrix} q^{-v}, \, a^{-1}b^{-1}q^{v+1}, \, q^{x+1} \\ b^{-1}q, \, 0 \end{matrix} \,\Big|\, q\,;q \right). \tag{4.50}$$

4.2.4.   q-Meixner (qM)
We have one twist operation:

$$\widetilde{\text{(i)}} : \quad \mathsf{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} (-\lambda_1, -\lambda_2). \tag{4.51}$$

The explicit form of the pseudo virtual state polynomial is

$$\widetilde{\text{(i)}} : \quad \check{\xi}_v(x;\boldsymbol{\lambda};q) = {}_2\phi_1 \left(\begin{matrix} q^{-v}, \, q^{x+1} \\ b^{-1}q \end{matrix} \,\Big|\, q\,;-b^{-1}c^{-1}q^{-x} \right). \tag{4.52}$$

4.2.5.   Little q-Laguerre/Wall (lqL)
We have one twist operation:

$$\widetilde{\text{(i)}} : \quad \mathsf{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} -\lambda_1. \tag{4.53}$$

The explicit forms of the pseudo virtual state polynomials are

$$\widetilde{\text{(i)}} : \quad \check{\xi}_v(x;\boldsymbol{\lambda};q) = {}_2\phi_1 \left(\begin{matrix} q^{-v}, \, q^{x+1} \\ 0 \end{matrix} \,\Big|\, q\,;a^{-1}q^{v+1} \right)$$

$$= (-a)^{-v} q^{\frac{1}{2}v(v+1)} (aq^{-v};q)_v \, {}_1\phi_1 \left(\begin{matrix} q^{-v} \\ a^{-1}q \end{matrix} \,\Big|\, q\,;a^{-1}q^{v+x+2} \right). \tag{4.54}$$

4.2.6.   Al-Salam–Carlitz II (ASCII)
We have one twist operation:

$$\widetilde{\text{(i)}} : \quad \mathsf{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} -\lambda_1. \tag{4.55}$$

The explicit form of the pseudo virtual state polynomial is

$$\widetilde{\text{(i)}} : \quad \check{\xi}_v(x;\boldsymbol{\lambda};q) = {}_2\phi_1 \left(\begin{matrix} q^{-v}, \, q^{x+1} \\ 0 \end{matrix} \,\Big|\, q\,;a^{-1}q^{-x} \right). \tag{4.56}$$

4.2.7.   q-Bessel (qB) (alternative q-Charlier)
We have one twist operation:

$$\widetilde{\text{(i)}} : \quad \mathsf{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} 2 - \lambda_1. \tag{4.57}$$

The explicit forms of the pseudo virtual state polynomials are

$$
\widetilde{(\text{i})}: \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = q^{\text{v}(x+1)}{}_2\phi_0\left(\begin{array}{c} q^{-\text{v}}, q^{x+1} \\ - \end{array}\middle|\, q\,; -a^{-1}q^{2\text{v}+1-x}\right)
$$

$$
= (-a)^{-\text{v}}q^{\text{v}(\text{v}+2)}{}_2\phi_0\left(\begin{array}{c} q^{-\text{v}}, -a^{-1}q^{\text{v}+2} \\ - \end{array}\middle|\, q\,; -aq^{x-1}\right)
$$

$$
= {}_3\phi_2\left(\begin{array}{c} q^{-\text{v}}, -a^{-1}q^{\text{v}+2}, q^{x+1} \\ 0,0 \end{array}\middle|\, q\,; q\right). \tag{4.58}
$$

4.2.8.   q-Charlier (qC)

We have one twist operation:

$$
\widetilde{(\text{i})}: \quad \mathsf{t}(\boldsymbol{\lambda}) \overset{\text{def}}{=} -\lambda_1. \tag{4.59}
$$

The explicit form of the pseudo virtual state polynomial is

$$
\widetilde{(\text{i})}: \quad \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda};q) = {}_2\phi_0\left(\begin{array}{c} q^{-\text{v}}, q^{x+1} \\ - \end{array}\middle|\, q\,; -a^{-1}q^{-x-1}\right). \tag{4.60}
$$

5.   Summary and comments

In addition to the Wronskian identities for the Hermite, Laguerre, and Jacobi polynomials in oQM [20] and the Casoratian identities for the Askey–Wilson polynomial and its reduced-form polynomials in idQM [21], infinitely many Casoratian identities for the q-Racah polynomial and its reduced-form polynomials in rdQM are obtained. The pseudo virtual state polynomials are defined by using discrete symmetries of the original systems. The derivation of the Casoratian identities in this paper is a "shortcut" way. The pseudo virtual state vectors and the one-step Darboux transformation in terms of it for q-Racah case are discussed in Appendix B. We will report on the multi-step cases and semi-infinite cases elsewhere. The Casoratian identities imply equivalences between the deformed systems obtained by multi-step Darboux transformations in terms of pseudo virtual state vectors and those in terms of eigenvectors with shifted parameters.

Curbera and Durán studied similar Casoratian identities for the Charlier, Meixner, and Hahn polynomials [31], which have the sinusoidal coordinate $\eta(x) = x$. Their method is based on the Krall discrete measure. Let us consider the case $\mathcal{N} = \max(\mathcal{D})$ and $\min(\mathcal{D}) \geq 1$. Then their map I implies $I(\mathcal{D}) = \bar{\mathcal{D}}$ and $I(\bar{\mathcal{D}}) = \mathcal{D}$, and we have $\max(\bar{\mathcal{D}}) = \mathcal{N}$ and $\min(\bar{\mathcal{D}}) \geq 1$. Our identities (4.5) correspond to their Theorems 1.1, 5.1, and 7.1 as follows: Charlier: $F = \bar{\mathcal{D}}$, Meixner: $F_1 = \bar{\mathcal{D}}$ and $F_2 = \emptyset$, Hahn: $F_1 = \bar{\mathcal{D}}$ and $F_2 = F_3 = \emptyset$ (remark: $I(\emptyset) = \emptyset$ and $\max(\emptyset) = -1$). The proportionality constants are also presented.

Among the reduced-form polynomials, the big q-Jacobi family and the discrete q-Hermite II are not mentioned in this paper. The orthogonality relations of the big q-Jacobi family are expressed in terms of the Jackson integral and their rdQM needs the two-component formalism [28]. The rdQM for the discrete q-Hermite II is an infinite system, $x \in \mathbb{Z}$. We have not completed the study of the pseudo virtual state vectors for these two systems. It is plausible that similar Casoratian identities, which are polynomial identities, do exist. In fact R. Sasaki has checked tentative Casoratian identities for the big q-Jacobi family (private communication).

In Sect. 3.2 twist operations (i)–(ii) and $\widetilde{\text{(i)}}$–$\widetilde{\text{(ii)}}$ are presented. There are more discrete symmetries (iii)–(iv) for R and qR, and $\widetilde{\text{(iii)}}$–$\widetilde{\text{(iv)}}$ for qR:

$$(\text{iii}) : \mathsf{t}(x) \stackrel{\text{def}}{=} -x - 1, \quad \mathsf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1 + \lambda_1 - \lambda_4, 1 + \lambda_2 - \lambda_4, 2 - \lambda_3, 2 - \lambda_4), \quad \mathsf{t}(q) \stackrel{\text{def}}{=} q, \quad (5.1)$$

$$(\text{iv}) : \mathsf{t}(x) \stackrel{\text{def}}{=} x + \lambda_3 - 1, \quad \mathsf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1 + \lambda_1 - \lambda_3, 1 + \lambda_2 - \lambda_3, 2 - \lambda_3, 2 - 2\lambda_3 + \lambda_4),$$

$$\mathsf{t}(q) \stackrel{\text{def}}{=} q, \tag{5.2}$$

$$\widetilde{\text{(iii)}} : \text{(iii) with the replacement } \mathsf{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{5.3}$$

$$\widetilde{\text{(iv)}} : \text{(iv) with the replacement } \mathsf{t}(q) \stackrel{\text{def}}{=} q^{-1}, \tag{5.4}$$

all of which give the relation

$$\tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda}) = \mathcal{E}_{\text{v}+N+1}(\boldsymbol{\lambda}). \tag{5.5}$$

It is an interesting problem to clarify whether these twists give new pseudo virtual state vectors and Casoratian identities or not.

Appendix A.   Data for orthogonal polynomials

In this appendix we present some necessary data for the orthogonal polynomials [26,29]. The data for the Askey–Wilson polynomial are given in Appendix A.1. The data for the polynomials appearing in finite rdQM are given in Appendix A.2 and those for semi-infinite rdQM are given in Appendix A.3.

A.1.   Askey–Wilson

The fundamental data for the Askey–Wilson polynomial are [21,29]

$$q^{\boldsymbol{\lambda}} = (a_1, a_2, a_3, a_4) \quad \left(\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \text{ as a set}\right), \quad \gamma = \log q,$$

$$\boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \quad \kappa = q^{-1}, \quad \mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 - b_4 q^{n-1}), \quad b_4 \stackrel{\text{def}}{=} a_1 a_2 a_3 a_4,$$

$$\eta(x) = \cos x, \quad \varphi(x) = 2 \sin x,$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = P_n\big(\eta(x); \boldsymbol{\lambda}\big) = p_n\big(\eta(x); a_1, a_2, a_3, a_4 | q\big) \tag{A.1}$$

$$= a_1^{-n}(a_1 a_2, a_1 a_3, a_1 a_4 \,; q)_n \, {}_4\phi_3 \left(\begin{matrix} q^{-n}, \ b_4 q^{n-1}, \ a_1 e^{ix}, \ a_1 e^{-ix} \\ a_1 a_2, \ a_1 a_3, \ a_1 a_4 \end{matrix} \,\middle|\, q \,; q \right),$$

$$V(x; \boldsymbol{\lambda}) = \frac{\prod_{j=1}^4 (1 - a_j e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad V^*(x; \boldsymbol{\lambda}) = \frac{\prod_{j=1}^4 (1 - a_j e^{-ix})}{(1 - e^{-2ix})(1 - q e^{-2ix})},$$

where $p_n(\eta; a_1, a_2, a_3, a_4 | q)$ is the Askey–Wilson polynomial. Note that the Askey–Wilson system is invariant under the permutation of a_j.

The pseudo virtual state wavefunction is defined by using the twist operation, the discrete symmetry of the Hamiltonian [21]. The twist operation t etc. is

$$\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3, 1 - \lambda_4) = 2\boldsymbol{\delta} - \boldsymbol{\lambda}, \tag{A.2}$$

$$V'(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big), \quad \alpha(\boldsymbol{\lambda}) = b_4 q^{-2}, \quad \alpha'(\boldsymbol{\lambda}) = -(1 - q)(1 - b_4 q^{-2}), \tag{A.3}$$

$$\mathcal{H}(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\lambda})\mathcal{H}'(\boldsymbol{\lambda}) + \alpha'(\boldsymbol{\lambda}), \quad \tilde{\mathcal{E}}_v(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})\mathcal{E}_v\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \alpha'(\boldsymbol{\lambda}) = \mathcal{E}_{-v-1}(\boldsymbol{\lambda}). \tag{A.4}$$

The pseudo virtual state polynomial $\check{\xi}_v(x; \boldsymbol{\lambda}) = \xi_v\big(\eta(x); \boldsymbol{\lambda}\big)$ is defined by

$$\check{\xi}_v(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{P}_v\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big). \tag{A.5}$$

The auxiliary function $\varphi_M(x)$ $(M \in \mathbb{Z}_{\geq 0})$ is defined by

$$\varphi_M(x) \stackrel{\text{def}}{=} \varphi(x)^{[\frac{M}{2}]} \prod_{k=1}^{M-2} \big(\varphi(x - i\tfrac{k}{2}\gamma)\varphi(x + i\tfrac{k}{2}\gamma)\big)^{[\frac{M-k}{2}]}$$

$$= \prod_{1 \leq j < k \leq M} \frac{\eta\big(x + i(\frac{M+1}{2} - j)\gamma\big) - \eta\big(x + i(\frac{M+1}{2} - k)\gamma\big)}{\varphi(i\tfrac{j}{2}\gamma)} \times (-2)^{\frac{1}{2}M(M-1)}, \tag{A.6}$$

and $\varphi_0(x) = \varphi_1(x) = 1$. Here $[x]$ denotes the greatest integer not exceeding x.

A.2. Finite cases

Data for the polynomials appearing in finite rdQM are presented [26,29]. Although there are two possible parameter choices indexed by $\epsilon = \pm 1$ for the Hahn, dual Hahn, q-Hahn, and dual q-Hahn polynomials, we take $\epsilon = 1$ for simplicity of presentation.

A.2.1. Hahn (Ha)

$$\boldsymbol{\lambda} = (a, b, N), \quad \boldsymbol{\delta} = (1, 1, -1), \quad \kappa = 1, \quad \mathcal{E}_n(\boldsymbol{\lambda}) = n(n + a + b - 1),$$

$$\eta(x) = x, \quad \varphi(x) = 1,$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = Q_n\big(\eta(x); a - 1, b - 1, N\big) = {}_3F_2\left(\begin{matrix} -n, n + a + b - 1, -x \\ a, -N \end{matrix} \middle| 1\right), \tag{A.7}$$

$$B(x; \boldsymbol{\lambda}) = (x + a)(N - x), \quad D(x; \boldsymbol{\lambda}) = x(b + N - x),$$

where $Q_n(\eta; \alpha, \beta, N)$ is the Hahn polynomial in the conventional parametrization [29]. The Hahn polynomial is obtained from the Racah polynomial by

$$\boldsymbol{\lambda}^{\mathrm{R}} = (a, b + N + d, -N, d), \quad d \to \infty : \quad \lim_{d \to \infty} \check{P}_n^{\mathrm{R}}(x; \boldsymbol{\lambda}^{\mathrm{R}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.8}$$

A.2.2. Dual Hahn (dHa)

$$\boldsymbol{\lambda} = (a, b, N), \quad \boldsymbol{\delta} = (1, 0, -1), \quad \kappa = 1, \quad \mathcal{E}_n = n,$$

$$\eta(x; \boldsymbol{\lambda}) = x(x + a + b - 1), \quad \varphi(x; \boldsymbol{\lambda}) = \frac{2x + a + b}{a + b},$$

$$\check{P}_n(x;\lambda) = R_n\big(\eta(x;\lambda); a-1, b-1, N\big) = {}_3F_2\left({-n,\, x+a+b-1,\, -x \atop a,\, -N} \,\middle|\, 1\right), \tag{A.9}$$

$$B(x;\lambda) = \frac{(x+a)(x+a+b-1)(N-x)}{(2x-1+a+b)(2x+a+b)}, \quad D(x;\lambda) = \frac{x(x+b-1)(x+a+b+N-1)}{(2x-2+a+b)(2x-1+a+b)},$$

where $R_n(\eta; \gamma, \delta, N)$ is the dual Hahn polynomial in the conventional parametrization [29]. The dual Hahn polynomial is obtained from the Racah polynomial by

$$\boldsymbol{\lambda}^{\mathrm{R}} = (a, b', -N, a+b-1), \ \ b' \to \infty: \ \ \lim_{b'\to\infty} \check{P}_n^{\mathrm{R}}(x;\boldsymbol{\lambda}^{\mathrm{R}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.10}$$

A.2.3. Krawtchouk (K)

$$\boldsymbol{\lambda} = (p, N), \quad \boldsymbol{\delta} = (0, -1), \quad \kappa = 1, \quad \mathcal{E}_n = n, \quad \eta(x) = x, \quad \varphi(x) = 1,$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = K_n\big(\eta(x); p, N\big) = {}_2F_1\left({-n,\, -x \atop -N} \,\middle|\, p^{-1}\right), \tag{A.11}$$

$$B(x;\boldsymbol{\lambda}) = p(N-x), \quad D(x;\boldsymbol{\lambda}) = (1-p)x,$$

where $K_n(\eta; p, N)$ is the Krawtchouk polynomial. The Krawtchouk polynomial is obtained from the Hahn polynomial by

$$\boldsymbol{\lambda}^{\mathrm{Ha}} = (1+pt, 1+(1-p)t, N), \ \ t \to \infty: \ \ \lim_{t\to\infty} \check{P}_n^{\mathrm{Ha}}(x;\boldsymbol{\lambda}^{\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.12}$$

A.2.4. q-Hahn (qHa)

$$q^{\boldsymbol{\lambda}} = (a, b, q^N), \quad \boldsymbol{\delta} = (1, 1, -1), \quad \kappa = q^{-1}, \quad \mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n}-1)(1-abq^{n-1}),$$

$$\eta(x) = q^{-x} - 1, \quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = Q_n\big(1+\eta(x); aq^{-1}, bq^{-1}, N|q\big) = {}_3\phi_2\left({q^{-n},\, abq^{n-1},\, q^{-x} \atop a,\, q^{-N}} \,\middle|\, q\,;q\right), \tag{A.13}$$

$$B(x;\boldsymbol{\lambda}) = (1-aq^x)(q^{x-N}-1), \quad D(x;\boldsymbol{\lambda}) = aq^{-1}(1-q^x)(q^{x-N}-b),$$

where $Q_n(\eta; \alpha, \beta, N|q)$ is the q-Hahn polynomial in the conventional parametrization [29]. The q-Hahn polynomial is obtained from the q-Racah polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{R}}} = (a, bq^N d, q^{-N}, d), \ \ d \to 0: \ \ \lim_{d\to 0} \check{P}_n^{q\mathrm{R}}(x;\boldsymbol{\lambda}^{q\mathrm{R}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.14}$$

A.2.5. Dual q-Hahn (dqHa)

$$q^{\boldsymbol{\lambda}} = (a, b, q^N), \quad \boldsymbol{\delta} = (1, 0, -1), \quad \kappa = q^{-1}, \quad \mathcal{E}_n = q^{-n} - 1,$$

$$\eta(x;\boldsymbol{\lambda}) = (q^{-x}-1)(1-abq^{x-1}), \quad \varphi(x;\boldsymbol{\lambda}) = \frac{q^{-x}-abq^x}{1-ab},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = R_n\big(1 + abq^{-1} + \eta(x;\boldsymbol{\lambda}); aq^{-1}, bq^{-1}, N|q\big) = {}_3\phi_2\left(\begin{array}{c} q^{-n},\, abq^{x-1},\, q^{-x} \\ a,\, q^{-N} \end{array}\middle|\, q\,;q\right),$$

$$B(x;\boldsymbol{\lambda}) = \frac{(q^{x-N}-1)(1-aq^x)(1-abq^{x-1})}{(1-abq^{2x-1})(1-abq^{2x})},$$
$$(A.15)$$
$$D(x;\boldsymbol{\lambda}) = aq^{x-N-1}\frac{(1-q^x)(1-abq^{x+N-1})(1-bq^{x-1})}{(1-abq^{2x-2})(1-abq^{2x-1})},$$

where $R_n(\eta;\gamma,\delta,N|q)$ is the dual q-Hahn polynomial in the conventional parametrization [29]. The dual q-Hahn polynomial is obtained from the q-Racah polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{R}}} = (a, b', q^{-N}, abq^{-1}),\ \ b' \to 0 :\ \ \lim_{b' \to 0} \check{P}_n^{q\mathrm{R}}(x;\boldsymbol{\lambda}^{q\mathrm{R}}) = \check{P}_n(x;\boldsymbol{\lambda}).\qquad (A.16)$$

A.2.6. Quantum q-Krawtchouk (qqK)

$$q^{\boldsymbol{\lambda}} = (p, q^N),\quad \boldsymbol{\delta} = (1, -1),\quad \kappa = q,\quad \mathcal{E}_n = 1 - q^n,\quad \eta(x) = q^{-x} - 1,\quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = K_n^{\mathrm{qtm}}\big(1 + \eta(x); p, N; q\big) = {}_2\phi_1\left(\begin{array}{c} q^{-n},\, q^{-x} \\ q^{-N} \end{array}\middle|\, q\,;pq^{n+1}\right),\qquad (A.17)$$

$$B(x;\boldsymbol{\lambda}) = p^{-1}q^x(q^{x-N}-1),\quad D(x;\boldsymbol{\lambda}) = (1-q^x)(1 - p^{-1}q^{x-N-1}),$$

where $K_n^{\mathrm{qtm}}(\eta; p, N; q)$ is the quantum q-Krawtchouk polynomial. The quantum q-Krawtchouk polynomial is obtained from the q-Hahn polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{Ha}}} = (a, pq, q^N),\ \ a \to \infty :\ \ \lim_{a \to \infty} \check{P}_n^{q\mathrm{Ha}}(x;\boldsymbol{\lambda}^{q\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}).\qquad (A.18)$$

A.2.7. q-Krawtchouk (qK)

$$q^{\boldsymbol{\lambda}} = (p, q^N),\ \boldsymbol{\delta} = (2, -1),\quad \kappa = q^{-1},\quad \mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n}-1)(1+pq^n),$$

$$\eta(x) = q^{-x} - 1,\quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = K_n\big(1 + \eta(x); p, N; q\big) = {}_3\phi_2\left(\begin{array}{c} q^{-n},\, q^{-x},\, -pq^n \\ q^{-N},\, 0 \end{array}\middle|\, q\,;q\right),\qquad (A.19)$$

$$B(x;\boldsymbol{\lambda}) = q^{x-N} - 1,\quad D(x;\boldsymbol{\lambda}) = p(1 - q^x),$$

where $K_n(\eta; p, N; q)$ is the q-Krawtchouk polynomial. The q-Krawtchouk polynomial is obtained from the q-Hahn polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{Ha}}} = (a, -a^{-1}pq, q^N),\ \ a \to 0 :\ \ \lim_{a \to 0} \check{P}_n^{q\mathrm{Ha}}(x;\boldsymbol{\lambda}^{q\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}).\qquad (A.20)$$

A.2.8. Dual q-Krawtchouk (dqK)

$$q^{\boldsymbol{\lambda}} = (c, q^N),\quad \boldsymbol{\delta} = (0, -1),\quad \kappa = q^{-1},\quad \mathcal{E}_n = q^{-n} - 1,$$

$$\eta(x;\boldsymbol{\lambda}) = (q^{-x} - 1)(1 - cq^{x-N}),\quad \varphi(x;\boldsymbol{\lambda}) = \frac{q^{-x} - cq^{1-N}q^x}{1 - cq^{1-N}},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = K_n\big(1 + cq^{-N} + \eta(x;\boldsymbol{\lambda}); c, N|q\big) = {}_3\phi_2\left(\begin{matrix} q^{-n},\, q^{-x},\, cq^{x-N} \\ q^{-N},\, 0 \end{matrix}\,\middle|\, q\,;q\right), \tag{A.21}$$

$$B(x;\boldsymbol{\lambda}) = \frac{(q^{x-N}-1)(1-cq^{x-N})}{(1-cq^{2x-N})(1-cq^{2x+1-N})}, \quad D(x;\boldsymbol{\lambda}) = -cq^{2x-2N-1}\frac{(1-q^x)(1-cq^x)}{(1-cq^{2x-1-N})(1-cq^{2x-N})},$$

where $K_n(\eta; c, N|q)$ is the dual q-Krawtchouk polynomial in the conventional parametrization [29]. The dual q-Krawtchouk polynomial is obtained from the dual q-Hahn polynomial by

$$q^{\boldsymbol{\lambda}^{\mathrm{d}q\mathrm{Ha}}} = (a, a^{-1}cq^{1-N}, q^N), \ \ a \to 0 : \ \ \lim_{a \to 0} \check{P}_n^{\mathrm{d}q\mathrm{Ha}}(x; \boldsymbol{\lambda}^{\mathrm{d}q\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.22}$$

A.2.9.   Affine q-Krawtchouk (aqK)

$$q^{\boldsymbol{\lambda}} = (p, q^N), \quad \boldsymbol{\delta} = (1, -1), \quad \kappa = q^{-1}, \quad \mathcal{E}_n = q^{-n} - 1, \quad \eta(x) = q^{-x} - 1, \quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = K_n^{\mathrm{aff}}\big(1 + \eta(x); p, N; q\big) = {}_3\phi_2\left(\begin{matrix} q^{-n},\, q^{-x},\, 0 \\ pq,\, q^{-N} \end{matrix}\,\middle|\, q\,;q\right), \tag{A.23}$$

$$B(x;\boldsymbol{\lambda}) = (q^{x-N}-1)(1-pq^{x+1}), \quad D(x;\boldsymbol{\lambda}) = pq^{x-N}(1-q^x),$$

where $K_n^{\mathrm{aff}}(\eta; p, N; q)$ is the affine q-Krawtchouk polynomial. The affine q-Krawtchouk polynomial is obtained from the q-Hahn polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{Ha}}} = (pq, b, q^N), \ \ b \to 0 : \ \ \lim_{b \to 0} \check{P}_n^{q\mathrm{Ha}}(x; \boldsymbol{\lambda}^{q\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.24}$$

A.3.   Semi-infinite cases

Data for the polynomials appearing in semi-infinite rdQM are presented [26,29].

A.3.1.   Meixner (M)

$$\boldsymbol{\lambda} = (\beta, c), \quad \boldsymbol{\delta} = (1, 0), \quad \kappa = 1, \quad \mathcal{E}_n = n, \quad \eta(x) = x, \quad \varphi(x) = 1,$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = M_n\big(\eta(x); \beta, c\big) = {}_2F_1\left(\begin{matrix} -n,\, -x \\ \beta \end{matrix}\,\middle|\, 1 - c^{-1}\right), \tag{A.25}$$

$$B(x;\boldsymbol{\lambda}) = \frac{c}{1-c}(x + \beta), \quad D(x;\boldsymbol{\lambda}) = \frac{1}{1-c}x,$$

where $M_n(\eta; \beta, c)$ is the Meixner polynomial. The Meixner polynomial is obtained from the Hahn polynomial by

$$\boldsymbol{\lambda}^{\mathrm{Ha}} = \left(\beta, 1 + \frac{1-c}{c}N, N\right), \ \ N \to \infty : \ \ \lim_{N \to \infty} \check{P}_n^{\mathrm{Ha}}(x; \boldsymbol{\lambda}^{\mathrm{Ha}}) = \check{P}_n(x;\boldsymbol{\lambda}). \tag{A.26}$$

A.3.2.   Charlier (C)

$$\boldsymbol{\lambda} = a, \quad \boldsymbol{\delta} = 0, \quad \kappa = 1, \quad \mathcal{E}_n = n, \quad \eta(x) = x, \quad \varphi(x) = 1,$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = C_n\big(\eta(x); a\big) = {}_2F_0\left(\begin{matrix} -n,\, -x \\ - \end{matrix}\,\middle|\, -a^{-1}\right), \tag{A.27}$$

$$B(x;\boldsymbol{\lambda}) = a, \quad D(x) = x,$$

where $C_n(\eta; a)$ is the Charlier polynomial. The Charlier polynomial is obtained from the Meixner polynomial by

$$\boldsymbol{\lambda}^{\mathrm{M}} = (\beta, \frac{a}{a+\beta}), \quad \beta \to \infty: \quad \lim_{\beta \to \infty} \check{P}_n^{\mathrm{M}}(x; \boldsymbol{\lambda}^{\mathrm{M}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.28}$$

A.3.3. Little q-Jacobi (lqJ)

$$q^{\boldsymbol{\lambda}} = (a, b), \quad \boldsymbol{\delta} = (1, 1), \quad \kappa = q^{-1}, \quad \mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 - abq^{n+1}),$$

$$\eta(x) = 1 - q^x, \quad \varphi(x) = q^x,$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = (-a)^{-n} q^{-\frac{1}{2}n(n+1)} \frac{(aq; q)_n}{(bq; q)_n} p_n\big(1 - \eta(x); a, b|q\big)$$

$$= (-a)^{-n} q^{-\frac{1}{2}n(n+1)} \frac{(aq; q)_n}{(bq; q)_n} \, {}_2\phi_1 \left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array} \middle| q; q^{x+1} \right) \tag{A.29}$$

$$= {}_3\phi_1 \left(\begin{array}{c} q^{-n}, abq^{n+1}, q^{-x} \\ bq \end{array} \middle| q; a^{-1}q^x \right),$$

$$B(x; \boldsymbol{\lambda}) = a(q^{-x} - bq), \quad D(x) = q^{-x} - 1,$$

where $p_n(\eta; a, b|q)$ is the little q-Jacobi polynomial in the conventional parametrization [29]. The little q-Jacobi polynomial is obtained from the q-Hahn polynomial by

$$x^{q\mathrm{Ha}} = N - x, \quad q^{\boldsymbol{\lambda}^{q\mathrm{Ha}}} = (aq, bq, q^N), \quad N \to \infty:$$

$$\lim_{N \to \infty} \check{P}_n^{q\mathrm{Ha}}(x^{q\mathrm{Ha}}; \boldsymbol{\lambda}^{q\mathrm{Ha}}) = (-a)^n q^{\frac{1}{2}n(n+1)} \frac{(bq; q)_n}{(aq; q)_n} \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.30}$$

A.3.4. q-Meixner (qM)

$$q^{\boldsymbol{\lambda}} = (b, c), \quad \boldsymbol{\delta} = (1, -1), \quad \kappa = q, \quad \mathcal{E}_n = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = M_n\big(1 + \eta(x); b, c; q\big) = {}_2\phi_1 \left(\begin{array}{c} q^{-n}, q^{-x} \\ bq \end{array} \middle| q; -c^{-1}q^{n+1} \right), \tag{A.31}$$

$$B(x; \boldsymbol{\lambda}) = cq^x(1 - bq^{x+1}), \quad D(x; \boldsymbol{\lambda}) = (1 - q^x)(1 + bcq^x),$$

where $M_n(\eta; b, c; q)$ is the q-Meixner polynomial. The q-Meixner polynomial is obtained from the q-Hahn polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{Ha}}} = (bq, -b^{-1}c^{-1}q^{-N}, q^N), \quad N \to \infty: \quad \lim_{N \to \infty} \check{P}_n^{q\mathrm{Ha}}(x; \boldsymbol{\lambda}^{q\mathrm{Ha}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.32}$$

A.3.5. Little q-Laguerre/Wall (lqL)

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = 1, \quad \kappa = q^{-1}, \quad \mathcal{E}_n = q^{-n} - 1, \quad \eta(x) = 1 - q^x, \quad \varphi(x) = q^x,$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = (a^{-1}q^{-n}; q)_n \, p_n\big(1 - \eta(x); a|q\big) = {}_2\phi_0 \left(\begin{array}{c} q^{-n}, q^{-x} \\ - \end{array} \middle| q; a^{-1}q^x \right), \tag{A.33}$$

$$B(x; \boldsymbol{\lambda}) = aq^{-x}, \quad D(x) = q^{-x} - 1,$$

where $p_n(\eta; a|q)$ is the little q-Laguerre polynomial in the conventional parametrization [29]. The little q-Laguerre polynomial is obtained from the little q-Jacobi polynomial by

$$q^{\boldsymbol{\lambda}^{\mathrm{lqJ}}} = (a, b), \quad b \to 0 : \quad \lim_{b \to 0} \check{P}_n^{\mathrm{lqJ}}(x; \boldsymbol{\lambda}^{\mathrm{lqJ}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.34}$$

A.3.6. *Al-Salam–Carlitz II (ASCII)*

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = 0, \quad \kappa = q, \quad \mathcal{E}_n = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = (-a)^{-n} q^{\frac{1}{2}n(n-1)} V_n^{(a)}\big(1 + \eta(x); q\big) = {}_2\phi_0 \left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix} \,\middle|\, q \,; a^{-1} q^n \right), \tag{A.35}$$

$$B(x; \boldsymbol{\lambda}) = a q^{2x+1}, \quad D(x; \boldsymbol{\lambda}) = (1 - q^x)(1 - a q^x),$$

where $V^{(a)}(\eta; q)$ is the Al-Salam–Carlitz II polynomial in the conventional parametrization [29]. The Al-Salam–Carlitz II polynomial is obtained from the q-Meixner polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{M}}} = (-ac^{-1}, c), \quad c \to 0 : \quad \lim_{c \to 0} \check{P}_n^{q\mathrm{M}}(x; \boldsymbol{\lambda}^{q\mathrm{M}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.36}$$

A.3.7. *q-Bessel (qB) (alternative q-Charlier)*

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = 2, \quad \kappa = q^{-1}, \quad \mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 + a q^n), \quad \eta(x) = 1 - q^x, \quad \varphi(x) = q^x,$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = (-a)^{-n} q^{-n^2} y_n\big(1 - \eta(x); a; q\big) = q^{nx} \, {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix} \,\middle|\, q \,; -a^{-1} q^{1-n} \right) \tag{A.37}$$

$$= (-a)^{-n} q^{-n^2} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -a q^n \\ 0 \end{matrix} \,\middle|\, q \,; q^{x+1} \right) = {}_3\phi_0 \left(\begin{matrix} q^{-n}, -a q^n, q^{-x} \\ - \end{matrix} \,\middle|\, q \,; -a^{-1} q^x \right),$$

$$B(x; \boldsymbol{\lambda}) = a, \quad D(x) = q^{-x} - 1,$$

where $y_n(\eta; a; q)$ is the q-Bessel polynomial (the alternative q-Charlier polynomial $K_n(\eta; a; q)$) in the conventional parametrization [29]. The q-Bessel polynomial is obtained from the little q-Jacobi polynomial by

$$q^{\boldsymbol{\lambda}^{\mathrm{lqJ}}} = (a', -aa'^{-1}q^{-1}), \quad a' \to 0 : \quad \lim_{a' \to 0} \check{P}_n^{\mathrm{lqJ}}(x; \boldsymbol{\lambda}^{\mathrm{lqJ}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.38}$$

A.3.8. *q-Charlier (qC)*

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = -1, \quad \kappa = q, \quad \mathcal{E}_n = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad \varphi(x) = q^{-x},$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = C_n\big(1 + \eta(x); a; q\big) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix} \,\middle|\, q \,; -a^{-1} q^{n+1} \right), \tag{A.39}$$

$$B(x; \boldsymbol{\lambda}) = a q^x, \quad D(x) = 1 - q^x,$$

where $C_n(\eta; a; q)$ is the q-Charlier polynomial. The q-Charlier polynomial is obtained from the q-Meixner polynomial by

$$q^{\boldsymbol{\lambda}^{q\mathrm{M}}} = (b, a), \quad b \to 0 : \quad \lim_{b \to 0} \check{P}_n^{q\mathrm{M}}(x; \boldsymbol{\lambda}^{q\mathrm{M}}) = \check{P}_n(x; \boldsymbol{\lambda}). \tag{A.40}$$

Appendix B.   Pseudo virtual state vectors and deformed systems

In this appendix we explain pseudo virtual state vectors and deformed systems obtained by the one-step Darboux transformation in terms of pseudo virtual state vectors.

We illustrate them by taking the qR twist (i) case as an example. We consider the following parameter range:

$$c = q^{-N}, \quad 0 < ac < d < 1, \quad qd < b < 1, \tag{B.1}$$

for which the Hamiltonian is well defined, namely real symmetric, and the constant $\alpha(\lambda)$ is positive. Potential functions $B'(x; \lambda)$ and $D'(x; \lambda)$ are given by Eq. (3.23). We restrict the parameter range:

$$ac < dq, \quad b < q, \quad d < q^2. \tag{B.2}$$

Then $B'(x; \lambda)$ and $D'(x; \lambda)$ satisfy

$$B'(x; \lambda) > 0 \ (x = 0, 1, \ldots, N), \ B'(-1; \lambda) = 0,$$
$$D'(x; \lambda) > 0 \ (x = 0, 1, \ldots, N), \ D'(N + 1; \lambda) = 0. \tag{B.3}$$

By further restricting the parameter range, we obtain the positivity of the pseudo virtual state polynomial (3.25) in the extended domain:

$$\check{\xi}_v(x; \lambda) > 0 \ (x = -1, 0, \ldots, N + 1). \tag{B.4}$$

The range of v may be restricted. After Eq. (2.6), let us define $\tilde{\phi}_0(x; \lambda)$ by

$$\tilde{\phi}_0(x; \lambda) \stackrel{\text{def}}{=} \prod_{y=0}^{x-1} \sqrt{\frac{B'(x; \lambda)}{D'(x + 1; \lambda)}} > 0, \tag{B.5}$$

which is an "almost" zero mode of $\mathcal{H}'$ (2.16):

$$\mathcal{H}'(\lambda)\tilde{\phi}_0(x; \lambda) = D'(0; \lambda)\tilde{\phi}_0(0; \lambda)\delta_{x0} + B'(N; \lambda)\tilde{\phi}_0(N; \lambda)\delta_{xN}. \tag{B.6}$$

Explicitly it is

$$\tilde{\phi}_0(x; \lambda) = \frac{1 - dq^{2x}}{1 - d} \frac{1}{q^x \phi_0(x; \lambda)}. \tag{B.7}$$

We define the pseudo virtual state vector $\tilde{\phi}_v(x; \lambda)$ as follows:

$$\tilde{\phi}_v(x; \lambda) \stackrel{\text{def}}{=} \tilde{\phi}_0(x; \lambda)\check{\xi}_v(x; \lambda). \tag{B.8}$$

This pseudo virtual state vector satisfies

$$\mathcal{H}(\lambda)\tilde{\phi}_v(x; \lambda) = \tilde{\mathcal{E}}_v(\lambda)\tilde{\phi}_v(x; \lambda) + \alpha(\lambda)D'(0; \lambda)\tilde{\phi}_0(0; \lambda)\check{\xi}_v(-1; \lambda)\delta_{x0}$$
$$+ \alpha(\lambda)B'(N; \lambda)\tilde{\phi}_0(N; \lambda)\check{\xi}_v(N + 1; \lambda)\delta_{xN}, \tag{B.9}$$

where the pseudo virtual state energy $\tilde{\mathcal{E}}_{\mathrm v}(\boldsymbol{\lambda})$ is defined by (we also present it for twists (ii) and $\widetilde{(\mathrm{i})}$–$\widetilde{(\mathrm{ii})}$)

$$\text{(i), (ii)} : \quad \tilde{\mathcal{E}}_{\mathrm v}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})\mathcal{E}_{\mathrm v}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \alpha'(\boldsymbol{\lambda}), \tag{B.10}$$

$$\widetilde{(\mathrm{i})}, \widetilde{(\mathrm{ii})} : \quad \tilde{\mathcal{E}}_{\mathrm v}(\boldsymbol{\lambda}; q) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda}; q)\mathcal{E}_{\mathrm v}\big(\mathfrak{t}(\boldsymbol{\lambda}); q^{-1}\big) + \alpha'(\boldsymbol{\lambda}; q). \tag{B.11}$$

Namely the pseudo virtual state vector almost satisfies the Schrödinger equation, except for both boundaries $x = 0$ and $x = N$. This is in good contrast to the virtual state vector in rdQM [12,13][1], which fails to satisfy the Schrödinger equation at only one of the boundaries. We remark that the pseudo virtual state energies $\tilde{\mathcal{E}}_{\mathrm v}(\boldsymbol{\lambda})$ for (i)–(ii) and $\widetilde{(\mathrm{i})}$–$\widetilde{(\mathrm{ii})}$ give the same value,

$$\tilde{\mathcal{E}}_{\mathrm v}(\boldsymbol{\lambda}) = \mathcal{E}_{-\mathrm v - 1}(\boldsymbol{\lambda}), \tag{B.12}$$

which is important for the equivalence between the state adding and deleting Darboux transformations [20,21].

Let us introduce potential functions $\hat{B}_{d_1}(x; \boldsymbol{\lambda})$ and $\hat{D}_{d_1}(x; \boldsymbol{\lambda})$ determined by one of the pseudo virtual state polynomials $\check{\xi}_{d_1}(x; \boldsymbol{\lambda})$:

$$\hat{B}_{d_1}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})B'(x; \boldsymbol{\lambda})\frac{\check{\xi}_{d_1}(x + 1; \boldsymbol{\lambda})}{\check{\xi}_{d_1}(x; \boldsymbol{\lambda})}, \quad \hat{D}_{d_1}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})D'(x; \boldsymbol{\lambda})\frac{\check{\xi}_{d_1}(x - 1; \boldsymbol{\lambda})}{\check{\xi}_{d_1}(x; \boldsymbol{\lambda})}, \tag{B.13}$$

which satisfy

$$\hat{B}_{d_1}(x; \boldsymbol{\lambda}) > 0 \ \ (x = 0, 1, \ldots, N), \ \ \hat{B}_{d_1}(-1; \boldsymbol{\lambda}) = 0,$$
$$\hat{D}_{d_1}(x; \boldsymbol{\lambda}) > 0 \ \ (x = 0, 1, \ldots, N), \ \ \hat{D}_{d_1}(N + 1; \boldsymbol{\lambda}) = 0, \tag{B.14}$$
$$B(x; \boldsymbol{\lambda})D(x + 1; \boldsymbol{\lambda}) = \hat{B}_{d_1}(x; \boldsymbol{\lambda})\hat{D}_{d_1}(x + 1; \boldsymbol{\lambda}),$$
$$B(x; \boldsymbol{\lambda}) + D(x; \boldsymbol{\lambda}) = \hat{B}_{d_1}(x; \boldsymbol{\lambda}) + \hat{D}_{d_1}(x; \boldsymbol{\lambda}) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}). \tag{B.15}$$

The original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is rewritten by using them:

$$\mathcal{H}(\boldsymbol{\lambda}) = -\sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})\hat{D}_{d_1}(x + 1; \boldsymbol{\lambda})}\, e^{\partial} - \sqrt{\hat{B}_{d_1}(x - 1; \boldsymbol{\lambda})\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\, e^{-\partial}$$
$$+ \hat{B}_{d_1}(x; \boldsymbol{\lambda}) + \hat{D}_{d_1}(x; \boldsymbol{\lambda}) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}) \tag{B.16}$$
$$\left(= \begin{array}{c} \left(\sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})} - \sqrt{\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\, e^{-\partial}\right)\left(\sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})} - e^{\partial}\sqrt{\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\right) \\ + \hat{D}_{d_1}(0; \boldsymbol{\lambda})\delta_{x0} + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}) \end{array}\right).$$

For this $(N + 1) \times (N + 1)$ matrix $\mathcal{H} = (\mathcal{H}_{x,y})_{0 \le x,y \le N}$, we define a deformed Hamiltonian $\mathcal{H}_{d_1}(\boldsymbol{\lambda})$, which is an $(N + 2) \times (N + 2)$ matrix $\mathcal{H}_{d_1} = (\mathcal{H}_{d_1\, x,y})_{-1 \le x,y \le N}$:

$$\mathcal{H}_{d_1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\sqrt{\hat{B}_{d_1}(x + 1; \boldsymbol{\lambda})\hat{D}_{d_1}(x + 1; \boldsymbol{\lambda})}\, e^{\partial} - \sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\, e^{-\partial}$$
$$+ \hat{B}_{d_1}(x; \boldsymbol{\lambda}) + \hat{D}_{d_1}(x + 1; \boldsymbol{\lambda}) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}) \tag{B.17}$$
$$\left(= \left(\sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})} - e^{\partial}\sqrt{\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\right)\left(\sqrt{\hat{B}_{d_1}(x; \boldsymbol{\lambda})} - \sqrt{\hat{D}_{d_1}(x; \boldsymbol{\lambda})}\, e^{-\partial}\right) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda})\right).$$

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[1] In Refs. [12,13], the type I and type II virtual state vectors do not satisfy the Schrödinger equation at $x = N$ and $x = 0$ respectively. For the type II virtual state vector, see Ref. [13].

Note that, on the RHS, only $\hat{B}_{d_1}(x;\boldsymbol{\lambda})$ and $\hat{D}_{d_1}(x;\boldsymbol{\lambda})$ for $x = 0, 1, \ldots, N$ appear due to $\hat{B}_{d_1}(-1;\boldsymbol{\lambda}) = \hat{D}_{d_1}(N+1;\boldsymbol{\lambda}) = 0$, and $\check{\xi}_{d_1}(x;\boldsymbol{\lambda})$ for $x = -1, 0, \ldots, N+1$ appears. The deformed Hamiltonian $\mathcal{H}_{d_1}(\boldsymbol{\lambda})$ has $N + 2$ eigenvectors; $N + 1$ eigenvectors $\phi_{d_1\,n}(x;\boldsymbol{\lambda})$ $(n = 0, 1, \ldots, N)$ are inherited from the eigenvectors $\phi_n(x;\boldsymbol{\lambda})$ for the original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$, and a new eigenvector $\check{\Phi}_{d_1;d_1}(x;\boldsymbol{\lambda})$ with the pseudo virtual energy $\tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda})$ is added:

$$\mathcal{H}_{d_1}(\boldsymbol{\lambda})\phi_{d_1\,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_{d_1\,n}(x;\boldsymbol{\lambda}) \quad (n = 0, 1, \ldots, N), \tag{B.18}$$

$$\mathcal{H}_{d_1}(\boldsymbol{\lambda})\check{\Phi}_{d_1;d_1}(x;\boldsymbol{\lambda}) = \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda})\check{\Phi}_{d_1;d_1}(x;\boldsymbol{\lambda}). \tag{B.19}$$

Here $\phi_{d_1\,n}(x;\boldsymbol{\lambda})$ and $\check{\Phi}_{d_1;d_1}(x;\boldsymbol{\lambda})$ are given by

$$\phi_{d_1\,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\hat{B}_{d_1}(x;\boldsymbol{\lambda})}\,\phi_n(x;\boldsymbol{\lambda}) - \sqrt{\hat{D}_{d_1}(x+1;\boldsymbol{\lambda})}\,\phi_n(x+1;\boldsymbol{\lambda}), \tag{B.20}$$

$$\check{\Phi}_{d_1;d_1}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x+1;\boldsymbol{\lambda}-\boldsymbol{\delta})}{\sqrt{\check{\xi}_{d_1}(x;\boldsymbol{\lambda})\check{\xi}_{d_1}(x+1;\boldsymbol{\lambda})}}, \tag{B.21}$$

which are defined for $x = -1, 0, \ldots, N$. We remark that Eq. (B.20) is rewritten as

$$\phi_{d_1\,n}(x;\boldsymbol{\lambda}) = \frac{-\sqrt{\alpha(\boldsymbol{\lambda})B'(x;\boldsymbol{\lambda})}\,\tilde{\phi}_0(x;\boldsymbol{\lambda})}{\sqrt{\check{\xi}_{d_1}(x;\boldsymbol{\lambda})\check{\xi}_{d_1}(x+1;\boldsymbol{\lambda})}}\mathrm{W}_{\mathrm{C}}[\check{\xi}_{d_1}, \nu\check{P}_n](x;\boldsymbol{\lambda}), \quad \nu(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda})}{\tilde{\phi}_0(x;\boldsymbol{\lambda})}. \tag{B.22}$$

This deformed Hamiltonian $\mathcal{H}_{d_1}(\boldsymbol{\lambda})$ can be rewritten in the standard form:

$$\begin{aligned}\mathcal{H}_{d_1}(\boldsymbol{\lambda}) &= -\sqrt{B_{d_1}(x;\boldsymbol{\lambda})D_{d_1}(x+1;\boldsymbol{\lambda})}\,e^{\partial} - \sqrt{B_{d_1}(x-1;\boldsymbol{\lambda})D_{d_1}(x;\boldsymbol{\lambda})}\,e^{-\partial} \\ &\quad + B_{d_1}(x;\boldsymbol{\lambda}) + D_{d_1}(x;\boldsymbol{\lambda}) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}) \\ &= \left(\sqrt{B_{d_1}(x;\boldsymbol{\lambda})} - \sqrt{D_{d_1}(x;\boldsymbol{\lambda})}\,e^{-\partial}\right)\left(\sqrt{B_{d_1}(x;\boldsymbol{\lambda})} - e^{\partial}\sqrt{D_{d_1}(x;\boldsymbol{\lambda})}\right) + \tilde{\mathcal{E}}_{d_1}(\boldsymbol{\lambda}).\end{aligned} \tag{B.23}$$

Here potential functions $B_{d_1}(x;\boldsymbol{\lambda})$ and $D_{d_1}(x;\boldsymbol{\lambda})$ are

$$B_{d_1}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})D'(x+1;\boldsymbol{\lambda})\frac{\check{\xi}_{d_1}(x;\boldsymbol{\lambda})}{\check{\xi}_{d_1}(x+1;\boldsymbol{\lambda})}, \quad D_{d_1}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{\lambda})B'(x;\boldsymbol{\lambda})\frac{\check{\xi}_{d_1}(x+1;\boldsymbol{\lambda})}{\check{\xi}_{d_1}(x;\boldsymbol{\lambda})}, \tag{B.24}$$

which satisfy

$$B_{d_1}(x;\boldsymbol{\lambda}) > 0 \;\; (x = -1, 0, \ldots, N-1), \;\; B_{d_1}(N;\boldsymbol{\lambda}) = 0,$$
$$D_{d_1}(x;\boldsymbol{\lambda}) > 0 \;\; (x = 0, 1, \ldots, N), \;\; D_{d_1}(-1;\boldsymbol{\lambda}) = 0. \tag{B.25}$$

Repeating this deformation procedure M times (the parameter range should be restricted appropriately), we can obtain deformed Hamiltonians $\mathcal{H}_{d_1\cdots d_M}(\boldsymbol{\lambda})$. They are $(N+M+1) \times (N+M+1)$ matrices $\mathcal{H}_{d_1\cdots d_M} = (\mathcal{H}_{d_1\cdots d_M\,x,y})_{-M\leq x,y\leq N}$ and their eigenvalues are $\mathcal{E}_n(\boldsymbol{\lambda})$ $(n = 0, 1, \ldots, N)$ and $\tilde{\mathcal{E}}_{d_j}$ $(j = 1, 2, \ldots, M)$. We will report on this subject in detail elsewhere.

Next let us consider the equivalence between the above one-step deformation in terms of the pseudo virtual state vector with $d_1 = \ell$ and the ℓ-step deformation in terms of the eigenvectors with shifted parameters. For simplicity we take $\mathcal{N} = \max(\mathcal{D})$. The multi-step Darboux transformations in terms of the eigenvectors were studied in Ref. [27]. Simple examples, in which the eigenvectors $\phi_1, \phi_2, \ldots, \phi_\ell$ are deleted, are given in its Appendix A. Equations (A.16), (A.24), and (A.26) in Ref. [27] for the q-Racah case give:

$$\varphi_\ell(x;\boldsymbol{\lambda})^{-1}\mathrm{W}_{\mathrm{C}}[\check{P}_1, \check{P}_2, \ldots, \check{P}_\ell](x;\boldsymbol{\lambda}) \propto \check{P}_\ell\big(-x; -\boldsymbol{\lambda} - (\ell-1)\boldsymbol{\delta}\big),$$

namely,

$$\check{\xi}_\ell(x-1;\boldsymbol{\lambda}) \propto \varphi_\ell\big(x;\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big)^{-1} \mathrm{W_C}[\check{P}_1,\check{P}_2,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}). \tag{B.26}$$

This is the special case of the Casoratian identities (3.36) with $M=1$, $\mathcal{N}=\ell$, $\mathcal{D}=\{\ell\}$, and $\bar{\mathcal{D}}=\{1,2,\ldots,\ell\}$. Let us denote the potential functions of the deformed system in Appendix A of Ref. [27] as $B^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda})=B^{\mathrm{Ref.\,[27]}}_\ell(x;\boldsymbol{\lambda})$ and $D^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda})=D^{\mathrm{Ref.\,[27]}}_\ell(x;\boldsymbol{\lambda})$. The Hamiltonian of the deformed system in Appendix A of Ref. [27] is

$$\mathcal{H}^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(\boldsymbol{\lambda}) = \big(\mathcal{H}^{\mathrm{KA}}_{\{1,2,\ldots,\ell\};x,y}(\boldsymbol{\lambda})\big)_{0\leq x,y\leq N-\ell},$$

$$\begin{aligned}
\mathcal{H}^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(\boldsymbol{\lambda}) =\ &-\sqrt{B^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda})D^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x+1;\boldsymbol{\lambda})}\,e^{\partial} \\
&-\sqrt{B^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x-1;\boldsymbol{\lambda})D^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda})}\,e^{-\partial} \\
&+B^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda})+D^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}(x;\boldsymbol{\lambda}).
\end{aligned} \tag{B.27}$$

Then Eqs. (A.26), (A.29), and (A.30) in Ref. [27] give

$$\begin{aligned}
\check{\xi}_\ell(x;\boldsymbol{\lambda}) &= \check{\xi}_\ell^{\mathrm{Ref.\,[27]}}\big(x+1;\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big), \\
B_\ell(x;\boldsymbol{\lambda}) &= \kappa^{-\ell-1}B^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}\big(x+1;\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big), \\
D_\ell(x;\boldsymbol{\lambda}) &= \kappa^{-\ell-1}D^{\mathrm{KA}}_{\{1,2,\ldots,\ell\}}\big(x+1;\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big).
\end{aligned} \tag{B.28}$$

Therefore we have

$$\mathcal{H}_{\ell;x,y}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\delta_{x,y} = \kappa^{-\ell-1}\mathcal{H}^{\mathrm{KA}}_{\{1,2,\ldots,\ell\};x+1,y+1}\big(\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big), \tag{B.29}$$

namely

$$\Big(\mathcal{H}_{\ell;x,y}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\delta_{x,y}\Big)_{-1\leq x,y\leq N} = \kappa^{-\ell-1}\Big(\mathcal{H}^{\mathrm{KA}}_{\{1,2,\ldots,\ell\};x+1,y+1}\big(\boldsymbol{\lambda}-(\ell+1)\boldsymbol{\delta}\big)\Big)_{-1\leq x,y\leq N}. \tag{B.30}$$

This establishes the equivalence of the two systems.

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