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Highlights:

- Hencky bar-chain model (HBM) is developed for shape optimization of circular arches.
- HBM is ease to handle minimum cross-sectional area constraint.
- Analytical optimization solutions are obtained via HBM.
Hencky bar-chain model for optimal circular arches against buckling

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*

Abstract

This paper is concerned with the formulation of the Hencky bar-chain model (HBM) for shape optimization of pinned-pinned circular arches under uniform radial pressure for maximum buckling capacity. The so-called HBM is a discrete model which comprises a finite number of rigid curved segments connected by frictionless hinges and elastic rotational springs. The different rotational spring stiffnesses along the arch represent the varying cross-section of the arch. Therefore, the optimization of the rotational spring stiffnesses of a HBM leads to the optimal shape of a circular arch. With a sufficiently large number of springs, one may obtain the optimal continuous shape of the arch. HBM has a great advantage over other numerical methods in seeking the optimal solution because it allows one to obtain the analytical optimality conditions in a set of recursive equations that requires minimal computational effort to solve the problem. Although HBM has been used by Krishna and Ram [1] and Zhang et al. [2] for column shape optimization, this is the first time that HBM has been developed for arch optimization.

Keywords: buckling; Hencky bar-chain; optimization; radial pressure; circular arch.

1. Introduction

About 100 years ago, Hencky [3] proposed a discrete beam model comprising of rigid segments connected by frictionless hinges and elastic rotational springs with stiffnesses $E I a$ where $E I$ is the flexural rigidity of the beam and $a$ the segmental length. Silverman [4] called the discrete beam model as Hencky’s “bar chain” and pointed out that it is the physical structural model for the mathematical finite difference method. Since then, the Hencky bar-chain model (HBM) has been called by different names such as discrete model [5], discrete element model [6, 7], segmented rod/column [8, 9], linked rod [10], discrete link-spring model [1] and microstructured beam model [11-13]. More recent developments on HBM were made by Wang and his associates for beams with elastic internal [14] and end restraints [15], resting on Winkler foundation [16], of varying cross-section [17] and under self-weight [18] in buckling and vibration problems.

The beauty of the HBM lies in its simplicity to model and analyse both articulated structures and continuum structures; the latter is a special case of the HBM when the number of segments is sufficiently large. The equations associated with the HBM are all algebraic equations instead of differential equations found in continuum structural models. Moreover, the HBM being a discrete model where the rotational springs represent the stiffness of the structure in their respective segments allows the analyst to readily handle the effect of local damage, local stiffening, and varying cross-section. So the HBM may be exploited for shape optimization of structures. Krishna and Ram [1] and Zhang et al. [2] did that just for the shape optimization of columns for maximum buckling capacity.

In this paper, the HBM will be used in the shape optimization of circular arches for maximum buckling load. It will be shown herein that analytical optimality conditions in a set of recursive algebraic equations may be obtained by adopting HBM. These recursive equations may be easily solved even for thousands of segments to obtain the optimal shape of the circular arches. It should be mentioned that the shape optimization of circular arches against buckling has been studied earlier by Wu [19] and Budiansky et al. [20] who obtained the solution analytically from continuum mechanics theory. Other studies using optimization techniques for determining the optimal variation of the cross-sectional area of arches include papers by Tadjbakhsh and Farshad [21], Amazigo [22], Błachut and Gajewski [23], Domaszewski et al. [24], Serra [25] and Marano et al. [26]. In this paper, the versatility of the simple HBM will be demonstrated for seeking the optimal designs of pinned ended arches without or with a minimum cross-sectional area constraint.

2. Problem Definition

Consider a uniform circular arch of length $2L$, central angle $2\alpha$, radius $R$ with pinned-pinned ends and subjected to a uniform radial pressure $q$ as shown in Figure 1. The problem at hand is to obtain a circular arch of varying cross-sectional area for maximum buckling load with a given volume, arch length and central angle.

Figure 1 Uniform circular arch under uniform radial pressure $q$
3. Method of Solution via HBM

The optimization process can be conducted with the aid of HBM which comprises 2n rigid curved segments with equal arc length \( a = R\phi = R\pi/n \), subjected to concentrated loads at the joints as shown in Figure 2a. The curved segments are connected by frictionless hinges with rotational springs of stiffness \( C_j = EI/a \), where \( E \) is the Young’s modulus and \( I_j \) the second moment of area at joint \( j \). Owing to the non-uniformity of the arch, the internal spring stiffness \( C_j \) is different for each joint. Considering the symmetry, let the center joint of the HBM be numbered by 0 while the left and right end be numbered by \(-n\) and \(n\), respectively. The positive radial displacement \( v_j \) and tangential displacement \( u_j \) are taken in the direction as shown in Figure 2b.

As a result, the moment for joint \( j \) can be obtained by

\[
M_j = v_j - \alpha = n\phi \quad \alpha = n\phi
\]

For simplicity, the cross-section is assumed to be circular and let \( A_j \) denote the area of the \( j \)th segment of the arch. Accordingly, the internal rotational spring stiffness \( C_j \) of the HBM is related to \( A_j \) by

\[
C_j = \frac{EI_j}{a} = \frac{EA_j^2}{4\pi a} \quad \text{for } j = -n+1, -n+2, \ldots, n-1
\]

As shown in Figure 2c, the arc length of the two end segments are assumed to be \(a/2\) whereas the arc length of the internal segments is \(a\). Consequently, the volume of the right half piecewise arch is given by

\[
V = a\sum_{j=1}^{n+1} A_j + \frac{1}{2} aA_n + \frac{1}{2} aA_1
\]

In the next section, the shape of the uniform arch shown in Figure 1 will be optimized with the aid of HBM through a set of recursive equations. Analytical optimal buckling load and cross-sectional area will be obtained.

4. Optimization Process

We shall start from the derivation of governing equation for the HBM modelling non-uniform circular arches. The rotational angle \( \psi_j \) of a curved segment of HBM from its undeformed state to its deformed state is given by

\[
\psi_j = \frac{v_{j+1} - v_j}{a} + \frac{u_j}{R}
\]

As a result, the moment for joint \( j \) can be obtained by [27]
\[ M_j = -C_j \left( \psi_j - \psi_{j-1} \right) = C_j \left( \frac{v_{j+1} - 2v_j + v_{j-1}}{a} + \frac{u_j - u_{j-1}}{R} \right) \]

(4)

Based on the inextensibility of HBM, which requires
\[ \frac{v_j}{R} = \frac{u_j - u_{j-1}}{a} \quad \text{for} \quad j = n+1, n+2, \ldots, n-1 \]

(5)

the moment for joint \( j \) can be expressed as
\[ M_j = -C_j \left( \frac{v_{j+1} - 2v_j + v_{j-1}}{a} + \frac{v_j}{R^2} a \right) \]

(6)

Since the moment for joint \( j \) is also related to the transversely distributed force as [28, 29]
\[ M_j = qRv_j \]

(7)

the governing equation for HBM with pinned ends can be obtained by substituting Eq. (7) into Eq. (6) and it is given by
\[ \frac{EI}{R^2} \left( v_{j+1} - 2v_j + v_{j-1} + v_j \phi^3 \right) + qRv_j = 0 \]

(8)

Considering the relationship given by Eq. (1), the governing equation (8) can be written as
\[ A_j^2 \left( v_{j+1} - 2v_j + v_{j-1} \right) + \mu \phi^2 v_j = 0 \]

(9)

where \( \mu = 4\pi qR^2/E \).

According to Wu [19], it is known that the lowest buckling mode of circular arches has one node between the two ends. Owing to the symmetry resulting from both pinned restrained ends, the critical buckling of shallow arches with a sufficiently large central angle may be antisymmetric [20]. Therefore, the boundary conditions can be written as
\[ v_0 = 0 \quad \text{and} \quad v_n = 0 \]

(10)

Therefore, the governing equation (9) and boundary conditions (10) can be cast to the following matrix form:
\[ \mathbf{K} = \begin{bmatrix} \varphi^2 - 2 & 1 \\ 1 & \varphi^2 - 2 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varphi^2 - 2 \end{bmatrix}_{(n-1) \times (n-1)} \]

\[ \mathbf{M} = -\varphi^2 \begin{bmatrix} A_1^2 & & \\ & A_2^2 & \vdots \\ \vdots & \vdots & \ddots \\ & & A_n^{-2} \end{bmatrix} 
\begin{bmatrix} A_1^{-2} & & \\ & A_2^{-2} & \vdots \\ \vdots & \vdots & \ddots \\ & & A_n^{-2} \end{bmatrix} 
\]

(11a)

(11b)

\[ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix} \]

(11c)

\[ (\mathbf{K} - \mu \mathbf{M}) \mathbf{v} = 0 \]

(11d)

To obtain the maximum critical pressure \( \mu_c \), with optimized cross-sectional areas \( A_0, A_1, \ldots, A_n \), for a given volume, length and central angle of a circular arch, we shall seek a stationary value of a Lagrange function given by [1]
\[ \Pi = \frac{1}{2} v^T \mathbf{K} v - \frac{1}{2} \mu \varphi^2 v^T \mathbf{M} v + \eta \left( \sum_{j=1}^{n} A_j + \frac{2}{a} a A_0 + \frac{2}{a} a A_n - V \right) \]

(12)

where \( \eta \) is Lagrange multiplier.

The variation of \( \Pi \) with respect to \( v \) leads to the eigenvalue problem (11a) to (11d), variation with respect to \( \eta \) gives the volume constraint (2) and variation with respect to \( A_j \) provides the optimality condition as follows,
\[ A_j^{-1} v_j^2 = A_j^{-2} v_j^2 \quad \text{for} \quad j = 2, \ldots, n-1 \]

(13)

By normalizing Eq. (13) by \( A_1 \) and \( v_1 \), one obtains
\[ \tilde{A}_j = A_j / A_1 \quad \text{and} \quad \tilde{v}_j = v_j / v_1 \quad \text{for} \quad j = 2, \ldots, n-1 \]

(14a)

\[ \tilde{v}_1 = 1 \quad \text{and} \quad \tilde{v}_j = v_j / v_1 \quad \text{for} \quad j = 2, \ldots, n-1 \]

(14b)

\[ \tilde{\mu} = \mu / \tilde{A}_1^2 \quad \text{and} \quad \tilde{V} = V / A_1 \]

(14c)

With the normalization given by Eqs. (14a) and (14b), Eq. (13) can be transformed to
\[ \tilde{v}_j = \tilde{A}_j^{1/2} \quad \text{for} \quad j = 2, \ldots, n-1 \]

(15)

By substituting Eq. (15) into Eqs. (11a) to (11d), we have the following recursive relations:
\[ \begin{align*}
\tilde{A}_1 &= 1 \\
\tilde{A}_2 &= \left( 2 - \varphi^2 - \tilde{\mu} \varphi^2 \right)^{1/3} \\
\tilde{A}_j+1 &= \left[ \left( 2 - \varphi^2 \right) \tilde{A}_{j-1}^{1/2} - \tilde{\mu} \varphi^2 \tilde{A}_{j-1}^{1/2} \right]^{2/3} \quad \text{for} \quad j = 2, \ldots, n-2 \\
\tilde{\mu} &= \left( 2 - \varphi^2 \right) \tilde{A}_{n-1}^{1/2} \tilde{A}_{n-2}^{1/2} \varphi^2
\end{align*} \]

(16)

We first give an initial guess of \( \tilde{\mu} \) and then \( \tilde{A}_2, \ldots, \tilde{A}_{n-1} \), can be calculated from the iteration in Eq. (16) for a given \( \varphi \) and finally \( \tilde{\mu} \) can be improved through the last equation of Eq. (16). In particular, the Bisection Method can be adopted to find a maximum \( \tilde{\mu} \) with a set of optimal values of \( \tilde{A}_2, \ldots, \tilde{A}_{n-1} \) to the required accuracy [1]. With the optimal
values of \( \hat{A}_n \), the cross-sectional area of the first segment \( A/LV \) can be calculated through

\[
A_n = \frac{V}{V} = \frac{A_L}{V} = \frac{n}{\sum_{j=1}^{n} \hat{A}_j / V + \hat{A}_j / 2}
\]

(17)

where \( \hat{A}_n = A_n / A \) and \( \hat{A}_1 = A_1 / A \). Note that \( V \) and \( L \) are the volume and length of half arch. The other areas \( A_2 \ldots A_{n-1} \) and the buckling load \( \mu \) can be thereby known from Eqs. (14a) and (14c). To have an optimal shape design of arches conservatively and practically, we set the areas of the middle point and the right end as \( A_0 = A_2/2 \) and \( A_n = A_{n-1}/2 \), respectively.

Table 1 shows the ratios of optimal critical pressure \( q_o \) over critical pressure \( q_o \) of uniform arch for different segmental numbers \( n \) and central angles \( 2\alpha \) where \( q_o = E/I/2(\pi^2/27\alpha^2 - 1) \) [29]. It is found that the optimal results shown in Table 1 agree well with the solutions obtained by Budiansky et al. [20]. In order to prove the accuracy and convergence of HBM in the shape optimization of circular arches, the optimal buckling load ratios with \( n = 1000 \) are compared to those calculated from Budiansky et al. [20]. The comparison study is shown in Table 2 by selecting different values of \( \omega \) which is a key shape parameter in Budiansky et al. [20]. It can be seen that our optimal buckling loads have a perfect agreement with the optimal results obtained by Budiansky et al. [20].

![Figure 3](image-url)

Figure 3 plots the optimal shape of an arch of central angle \( 2\alpha = \pi/2 \) and 20 stepped segments based on HBM. The radii of segments 0, …, 10 are denoted by \( r_0, \ldots, r_{10} \) while \( r \) represents the radius of uniform arch. When the segmental number \( n \) increases to large enough, the shape of the arch becomes smoother. Owing to the assumption that the arch will buckle in an antisymmetric mode, the areas of middle section and two end sections are the same and smallest. Wu [19] shows the same conclusion for arches of rectangular cross section and central angle \( 2\alpha = \pi \). Note that the optimal shape of circular arch can be easily plotted with HBM while it is difficult to be obtained from continuum mechanics theory.

**Table 1** Critical pressure ratios \( q_o/q_o \) of optimal pinned-pinned circular arches for various segmental numbers \( n \) and central angles \( 2\alpha \).

<table>
<thead>
<tr>
<th>( 2\alpha )</th>
<th>( n = 10 )</th>
<th>( n = 100 )</th>
<th>( n = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/4 )</td>
<td>1.3200</td>
<td>1.3336</td>
<td>1.3339</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1.3212</td>
<td>1.3354</td>
<td>1.3357</td>
</tr>
<tr>
<td>2( \pi/3 )</td>
<td>1.3223</td>
<td>1.3372</td>
<td>1.3375</td>
</tr>
<tr>
<td>5( \pi/3 )</td>
<td>1.3224</td>
<td>1.3621</td>
<td>1.3626</td>
</tr>
</tbody>
</table>

**Table 2** Comparison of optimal critical pressure ratios \( q_o/q_o \) based on HBM and those from Budiansky et al. [20] for different \( \omega \).

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>HBM</th>
<th>Budiansky et al. [20]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01 (( \omega = 0.1483\pi ))</td>
<td>1.3341</td>
<td>1.3342</td>
</tr>
<tr>
<td>0.1 (( \omega = 0.4269\pi ))</td>
<td>1.3403</td>
<td>1.3403</td>
</tr>
<tr>
<td>1 (( \omega = 0.8265\pi ))</td>
<td>1.3621</td>
<td>1.3621</td>
</tr>
<tr>
<td>2 (( \omega = 0.9001\pi ))</td>
<td>1.3684</td>
<td>1.3684</td>
</tr>
</tbody>
</table>

**Table 3** Critical pressure ratios \( q_o/q_o \) of optimal pinned-pinned circular arches with minimum cross-sectional area \( A/\pi L/V = A/(2\pi L/V) = 0.5 \) for various segmental numbers \( n \) and central angles \( 2\alpha \).

<table>
<thead>
<tr>
<th>( 2\alpha )</th>
<th>( n = 10 )</th>
<th>( n = 100 )</th>
<th>( n = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/4 )</td>
<td>1.2784</td>
<td>1.3225</td>
<td>1.3326</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1.2791</td>
<td>1.3243</td>
<td>1.3344</td>
</tr>
<tr>
<td>2( \pi/3 )</td>
<td>1.2799</td>
<td>1.3261</td>
<td>1.3362</td>
</tr>
<tr>
<td>5( \pi/3 )</td>
<td>1.2755</td>
<td>1.3506</td>
<td>1.3613</td>
</tr>
</tbody>
</table>
5. Concluding Remarks

Treated herein is the optimization problem of a pinned-pinned circular arch of varying cross-sectional area subjected to uniform radial pressure against buckling for a given volume, length and central angle based on HBM. The optimality conditions in a set of recursive equations are derived herein so that one can obtain the maximum buckling load and optimal shape of arches analytically. The convergence and accuracy of HBM have been proved by comparing the optimal results to the solutions given in previous papers. An optimal shape of a stepped circular arch with pinned-pinned ends is also presented. Although the shape optimization problem of arches has been investigated before, the optimization method (i.e., HBM) studied herein provides a new perspective from a discrete sense. If the segmental number is taken to very large, say $n = 1000$, the continuum result can be easily obtained. The superiority of HBM against the continuum theory is that HBM allows analysts to easily obtain the optimal shape of arches as well as its ability to handle a minimum cross-sectional area constraint. Local damage or local strengthening and even a point restraint could also be considered by adjusting the spring stiffness properties during the optimization process [30].

References