Frequency-Limited H_∞ Model Reduction for Positive Systems

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Abstract—**In this paper, the problem of frequency-limited H_∞ model reduction for positive linear time-invariant systems is investigated. Specifically, our goal is to find a stable positive reduced-order model for a given positive system such that the H_∞ norm of the error system is bounded over a frequency interval of interest. A new condition in terms of matrix inequality is developed for characterizing the frequency-limited H_∞ performance. Then an equivalent parametrization of a positive reduced-order model is derived, based on which, an iterative algorithm is constructed for optimizing the reduced-order model. The algorithm utilizes coarse reduced-order models resulting from (generalized) balanced truncation as the initial value. Both continuous- and discrete-time systems are considered in the same framework. Numerical examples clearly show the effectiveness and advantages of the proposed model reduction method.**

Index Terms—**Model reduction, positive systems, frequency-limited H_∞ performance, iterative algorithm.**

I. Introduction

Positive systems, whose state variables are always positive or nonnegative, are a class of dynamic systems often encountered in various industrial engineering and social science areas that include biological and chemical reactions, compartmental networks, economics systems and ecosystems [1]. For positive systems, the positivity constraint on system states results in some special properties. On one hand, this kind of particular systems have some beneficial properties that general systems do not have. For instance, without loss of generality, the Lyapunov matrix in the bounded real lemma for positive linear systems can be restricted to be diagonal [2]. On the other hand, to preserve or achieve positivity, many well-established results for general linear systems cannot be directly applied to positive systems or at least need particular but conservative treatment. Due to the significance and particularity, positive systems have received considerable attention during the past decades, and many results on positive systems have been proposed, see [3]–[10].

Model reduction is a basic theme in control theory. Modeling via physical, chemical, social or biological laws often leads to high-order mathematical models, which are inconvenient for system analysis and synthesis, naturally leading to the problem of model reduction. Some classical methods, for instance, balanced truncation (BT) and Hankel norm approximation [11], have been shown to be effective in reducing the order of general linear systems. In recent years, many new approaches such as H_∞ model reduction have been proposed to handle more complicated systems [12]–[14]. Model reduction for positive systems has been specifically addressed by some researchers very recently [15], [16]. For model reduction of positive systems, it is naturally desired that the reduced-order model is also positive. Unfortunately, the aforementioned results on general linear systems do *not* have such a property, and thus cannot be applied to positive systems. To circumvent this difficulty, a generalized BT (GBT) method was proposed in [15], and an iterative approach was constructed for positivity-preserving H_∞ model reduction in [16].

For model reduction, sometimes one may be more concerned about the approximation performance at some frequencies than at others [17], [18]. For instance, the original aircraft model established by analyzing its aerodynamics and considering practical constraints (e.g., actuators have limited bandwidths) are only valid within a specific frequency range [19], therefore, to obtain a reduced-order model for the aircraft, it is only necessary to consider the approximation performance within this frequency range. Some methods exist for enhancing the approximation performance over a limited frequency interval (e.g., frequency weighting [14], [20], [21] and frequency-specific balanced truncation [17], [22]) and at a specific frequency (e.g., moment matching (MM) [23]), but to our knowledge, these methods *cannot* guarantee positivity of the resulting reduced-order model that is expected for model reduction of positive systems. Hence, the problem of model reduction for positive systems over a limited frequency range is still open, which motivates the work of the paper. Especially, note a general fact on positive systems, namely, a positive linear time-invariant (LTI) system always has the maximum gain at zero frequency [2]. This fact reflects that the performance of positive systems around zero frequency is in general more important than that at other frequencies. Therefore, reduced-order models of positive systems are desired to have good approximation performance especially around zero frequency. Actually, since signals in positive systems are all nonnegative, there always exists a remarkable zero-frequency component in the signals, further highlighting the importance of the zero-frequency performance of positive systems.

In the paper, we will consider the problem of frequency-limited H_∞ model reduction for positive LTI systems. For a given stable positive system, the goal of the paper is to find a stable, positive, reduced-order model such that the H_∞ approximation error over a limited frequency interval is minimized. First, a new condition based on the generalized Kalman-Yakubvich-Popov (GKYP) lemma will be derived for characterizing the frequency-limited H_∞ norm of the error system. In this new condition, the product terms between the Lyapunov matrices and the parameter of the reduced-order model have been eliminated. Then, a necessary and sufficient condition will be proposed for parameterizing a positive reduced-order model. It is shown that this new parametrization establishes a connection between a positive reduced-order model and a general one. With coarse reduced-order models resulting from the BT or GBT method as the initial value, an iterative algorithm is constructed to search for a positive reduced-order model with the frequency-limited H_∞ error optimized. Numerical examples will be provided to show the effectiveness and advantages of the proposed method.

Notation: $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrices. Especially, $\mathbb{R}_n$ represents $\mathbb{R}^{n\times n}$ for simplicity, and $\mathbb{R}_+^{m\times n}$ represents $\mathbb{R}^{m\times n}$ with nonnegative elements. A matrix $A\in\mathbb{R}_+^{m\times n}$ is said to be positive; a matrix $A\in\mathbb{R}_n$ is said to be Metzler, if all its off-diagonal elements are nonnegative. $P>0$ (≥ 0) means that matrix P is positive definite (semi-definite). $\mathbf{I}$ denotes an identity matrix with appropriate dimension. $\mathrm{diag}\{A_1,\ldots,A_n\}$ stands for a block-diagonal matrix. For a matrix $A\in\mathbb{R}_n$, $\mathrm{sym}\{A\}$ indicates $A^{\mathrm{T}}+A$.

II. Problem Statement

Consider the following state-space model of the stable system (Σ):

$$\begin{aligned}(\Sigma):\ \lambda\left[x(t)\right]&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\end{aligned}\tag{1}$$

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where $x(t) \in \mathbb{R}^{n_\mathrm{p}}$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ are the state, input and output vectors, respectively. $[A, B; C, D]$ are real constant matrices with appropriate dimensions. The operator $\lambda[x(t)]$ denotes $\dot{x}(t)$ for the continuous-time (CT) case (respectively, $x(t+1)$ for the discrete-time (DT) case). In the frequency domain, we also use λ as Laplace operator s for the CT case (respectively, operator z for the DT case). The system (Σ) is supposed to be positive. The definition of positivity for system (Σ) and its characterization are given as follows:

Definition 1 ([1]): The system (Σ) in (1) is said to be positive if $x(t) \in \mathbb{R}_+^{n_\mathrm{p}}$ and $y(t) \in \mathbb{R}_+^{n_y}$, $t \geq 0$, for all $x(0) \in \mathbb{R}_+^{n_\mathrm{p}}$ and $u(t) \in \mathbb{R}_+^{n_\mathrm{p}}$, $t \geq 0$.

Lemma 1 ([1]): The system (Σ) in (1) is positive if and only if A is Metzler, B, C and D are positive for the CT case (respectively, A, B, C and D are positive for the DT case).

Exactly speaking, Definition 1 corresponds to *internal* positivity, stronger than *external* positivity that is defined only for the input and output (see [1, Definition 1]). Note that internal positivity is state coordinate-specific, that is, when we say a system is internally positive, we must refer to a specific state-space realization. This is because internal positivity in general cannot be preserved under a similarity transformation. This paper is only concerned with internal positivity, and for brevity, we just call it positivity in the sequel.

To approximate the system (Σ), the goal of this paper is to explore a reduced-order stable model (Σ_r) in the following state-space form:

$$(\Sigma_\mathrm{r}): \ \lambda[x_\mathrm{r}(t)] = A_\mathrm{r} x_\mathrm{r}(t) + B_\mathrm{r} u(t)$$
$$y_\mathrm{r}(t) = C_\mathrm{r} x_\mathrm{r}(t) + D_\mathrm{r} u(t) \tag{2}$$

where $x_\mathrm{r}(t) \in \mathbb{R}^{n_\mathrm{r}}$ with $1 \leq n_\mathrm{r} < n_\mathrm{p}$ and $y_\mathrm{r}(t) \in \mathbb{R}^{n_y}$ are the state and output vectors of the reduced-order model. The values of matrices $[A_\mathrm{r}, B_\mathrm{r}, C_\mathrm{r}, D_\mathrm{r}]$ are to be determined later. Augmenting the state vector as $\xi(t) \triangleq \mathrm{col}\{x(t), x_\mathrm{r}(t)\}$, we obtain the state-space dynamics of the approximation error $e(t) \triangleq y(t) - y_\mathrm{r}(t)$ as

$$(\Sigma_\mathrm{e}): \ \lambda[\xi(t)] = A_\mathrm{e}\xi(t) + B_\mathrm{e} u(t)$$
$$e(t) = C_\mathrm{e}\xi(t) + D_\mathrm{e} u(t) \tag{3}$$

where

$$A_\mathrm{e} \triangleq \begin{bmatrix} A & 0 \\ 0 & A_\mathrm{r} \end{bmatrix}, \ B_\mathrm{e} \triangleq \begin{bmatrix} B \\ B_\mathrm{r} \end{bmatrix}$$
$$C_\mathrm{e} \triangleq \begin{bmatrix} C & -C_\mathrm{r} \end{bmatrix}, \ D_\mathrm{e} \triangleq D - D_\mathrm{r}.$$

The transfer function of the error system in (3) is given by $G_\mathrm{e}(\lambda) \triangleq C_\mathrm{e}(\lambda\mathbf{I} - A_\mathrm{e})^{-1} B_\mathrm{e} + D_\mathrm{e}$.

For convenience, we define a set of matrices.

Definition 2: A set of matrices, $\mathbb{P}$, is defined as

$$\mathbb{P} \triangleq \begin{cases} \{[A_\mathrm{r}, B_\mathrm{r}; C_\mathrm{r}, D_\mathrm{r}] : A_\mathrm{r} \text{ is Metzler, } B_\mathrm{r}, C_\mathrm{r} \text{ and } D_\mathrm{r} \text{ are positive}\} & \text{(CT)} \\ \{[A_\mathrm{r}, B_\mathrm{r}; C_\mathrm{r}, D_\mathrm{r}] : A_\mathrm{r}, B_\mathrm{r}, C_\mathrm{r} \text{ and } D_\mathrm{r} \text{ are positive}\} & \text{(DT)} \end{cases}$$

First, as commented in [16], it is expected that the reduced-order model (Σ_r) is also positive since it approximates a positive system (Σ). According to Lemma 1, the matrices $[A_\mathrm{r}, B_\mathrm{r}, C_\mathrm{r}, D_\mathrm{r}]$ are required to belong to the set $\mathbb{P}$. Second, for many practical applications, one may be interested in the approximation performance only in some finite frequency interval [17]. In this paper, the following frequency-limited H_∞ index is employed to describe such a requirement:

$$\|G_\mathrm{e}(\lambda)\|_\infty^\Omega < \gamma \tag{4}$$

where γ is a scalar to be optimized, and

$$\|G_\mathrm{e}(\lambda)\|_\infty^\Omega \triangleq \begin{cases} \sup_{\omega\in\Omega} \sigma_{\max}[G_\mathrm{e}(\mathrm{j}\omega)] & \text{(CT)} \\ \sup_{\omega\in\Omega} \sigma_{\max}[G_\mathrm{e}(\mathrm{e}^{\mathrm{j}\omega})] & \text{(DT)} \end{cases}$$

with $\sigma_{\max}[\cdot]$ denoting the maximum singular value of matrix $[\cdot]$, and $\Omega \triangleq [\omega_1, \omega_2]$ representing the frequency interval of interest. Here $0 \leq \omega_1 < \omega_2 < \infty$ for the CT case (respectively, $0 \leq \omega_1 < \omega_2 < \pi$ for the DT case). In summary, the model reduction problem to be addressed in the paper is formulated as follows.

Problem 1: Given a frequency interval Ω, find a reduced-order model (Σ_r) for the system (Σ), such that

1) the system (Σ_r) is stable, and $[A_\mathrm{r}, B_\mathrm{r}; C_\mathrm{r}, D_\mathrm{r}] \in \mathbb{P}$; and
2) the error system (Σ_e) satisfies (4).

The following result, i.e., the GKYP lemma, will be used later.

Lemma 2 ([24], [25]): Suppose that the error system in (3) is stable, then $\|G_\mathrm{e}(\lambda)\|_\infty^\Omega < \gamma$ if and only if there exist symmetric matrices P, $Q \in \mathbb{R}_{n_\mathrm{p}+n_\mathrm{r}}$ such that $Q > 0$ and

$$\Psi^\mathrm{T}\Phi\Psi < 0. \tag{5}$$

where

$$\Psi \triangleq \begin{bmatrix} A_\mathrm{e}^\mathrm{T} & \mathbf{I} & C_\mathrm{e}^\mathrm{T} & 0 \\ B_\mathrm{e}^\mathrm{T} & 0 & D_\mathrm{e}^\mathrm{T} & \mathbf{I} \end{bmatrix}^\mathrm{T}, \ \omega_\mathrm{c} = \frac{\omega_2 + \omega_1}{2}, \ \omega_\mathrm{a} = \frac{\omega_2 - \omega_1}{2}$$

$$\Phi \triangleq \begin{cases} \mathrm{diag}\left\{\begin{bmatrix} -Q & P + \mathrm{j}\omega_\mathrm{c} Q \\ P - \mathrm{j}\omega_\mathrm{c} Q & -\omega_1\omega_2 Q \end{bmatrix}, \mathbf{I}, -\gamma^2\mathbf{I}\right\} & \text{(CT)} \\ \mathrm{diag}\left\{\begin{bmatrix} P & \mathrm{e}^{\mathrm{j}\omega_\mathrm{c}} Q \\ \mathrm{e}^{-\mathrm{j}\omega_\mathrm{c}} Q & -P - 2\cos\omega_\mathrm{a} Q \end{bmatrix}, \mathbf{I}, -\gamma^2\mathbf{I}\right\} & \text{(DT)} \end{cases}.$$

III. Main Results

A. A New Performance Characterization

Note that the condition in (5) cannot be directly applied to computing the reduced-order model, since the matrices A_r and B_r are coupled with Φ that includes Lyapunov matrices P and Q. Moreover, the structured specification $[A_\mathrm{r}, B_\mathrm{r}; C_\mathrm{r}, D_\mathrm{r}] \in \mathbb{P}$ further increases the difficulty in solving the considered model reduction problem. To circumvent these difficulties, we present the following new result for characterizing the performance specification $\|G_\mathrm{e}(\lambda)\|_\infty^\Omega < \gamma$.

Theorem 1: Let the systems (Σ) and (Σ_r) be given, and suppose that the system (Σ) is stable. The following statements are equivalent.

(i) The system (Σ_r) is stable, and (Σ_e) satisfies (4).

(ii) There exist symmetric matrices P, $Q \in \mathbb{R}_{n_\mathrm{p}+n_\mathrm{r}}$, $P_\mathrm{s} \in \mathbb{R}_{n_\mathrm{r}}$ and diagonal matrices $X \in \mathbb{R}_{n_\mathrm{r}}$, $Y \in \mathbb{R}_{n_y}$ such that $Q > 0$, $P_\mathrm{s} > 0$, $X > 0$, $Y > 0$ and

$$W^\mathrm{T}\Phi W - 2U^\mathrm{T}\mathrm{diag}\{X, Y\}U < 0 \tag{6}$$
$$\Phi_\mathrm{s} - 2U_\mathrm{s}^\mathrm{T} X U_\mathrm{s} < 0 \tag{7}$$

where Φ is defined in (5), and

$$W \triangleq \left[\begin{array}{cc|c|cc} A & 0 & B & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 \\ \hline \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 & 0 \\ \hline C & 0 & D & 0 & -\mathbf{I} \\ \hline 0 & 0 & \mathbf{I} & 0 & 0 \end{array}\right], \ \Phi_\mathrm{s} \triangleq \begin{cases} \begin{bmatrix} 0 & P_\mathrm{s} \\ P_\mathrm{s} & 0 \end{bmatrix} & \text{(CT)} \\ \begin{bmatrix} -P_\mathrm{s} & 0 \\ 0 & P_\mathrm{s} \end{bmatrix} & \text{(DT)} \end{cases}$$

$$U \triangleq \left[\begin{array}{cc|c|cc} 0 & A_\mathrm{r} & B_\mathrm{r} & -\mathbf{I} & 0 \\ 0 & C_\mathrm{r} & D_\mathrm{r} & 0 & -\mathbf{I} \end{array}\right], \ U_\mathrm{s} \triangleq \begin{bmatrix} A_\mathrm{r} & -\mathbf{I} \end{bmatrix}.$$

(iii) There exist symmetric matrices $\tilde{P}$, $\tilde{Q} \in \mathbb{R}_{n_\mathrm{p}+n_\mathrm{r}}$, $\tilde{P}_\mathrm{s} \in \mathbb{R}_{n_\mathrm{r}}$, and diagonal matrices $\tilde{X} \in \mathbb{R}_{n_\mathrm{r}}$, $\tilde{Y} \in \mathbb{R}_{n_u}$ such that $\tilde{Q} > 0$, $\tilde{P}_\mathrm{s} > 0$, $\tilde{X} > 0$, $\tilde{Y} > 0$ and

$$\tilde{W}\tilde{\Phi}\tilde{W}^\mathrm{T} - 2\tilde{U}\mathrm{diag}\left\{\tilde{X}, \tilde{Y}\right\}\tilde{U}^\mathrm{T} < 0 \tag{8}$$
$$\tilde{\Phi}_\mathrm{s} - 2\tilde{U}_\mathrm{s}\tilde{X}\tilde{U}_\mathrm{s}^\mathrm{T} < 0 \tag{9}$$

where $\tilde{\Phi}$ is defined from Φ with P and Q replaced by $\tilde{P}$ and $\tilde{Q}$, respectively, and with the values of n_y and n_u exchanged, $\tilde{\Phi}_{\rm s}$ is defined from $\Phi_{\rm s}$ with $P_{\rm s}$ replaced by $\tilde{P}_{\rm s}$, and

$$\tilde{W} \triangleq \left[\begin{array}{cc|cc|cc} A & 0 & \mathbf{I} & 0 & B & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 \\ \hline C & 0 & 0 & 0 & D & \mathbf{I} \\ \hline 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I} & 0 \end{array}\right],\ \tilde{U} \triangleq \left[\begin{array}{cc} 0 & 0 \\ A_{\rm r} & B_{\rm r} \\ \hline -C_{\rm r} & -D_{\rm r} \\ \hline -\mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{array}\right]$$

$$\tilde{U}_{\rm s} \triangleq \left[\begin{array}{cc} A_{\rm r}^{\rm T} & -\mathbf{I} \end{array}\right]^{\rm T}.$$

Proof: ((i) $\Rightarrow$ (ii)) It is known that $A_{\rm r}$ is stable if and only if the Lyapunov stability inequalities,

$$\begin{cases} A_{\rm r}^{\rm T} P_{\rm s} + P_{\rm s} A_{\rm r} < 0 & \text{(CT)} \\ A_{\rm r}^{\rm T} P_{\rm s} A_{\rm r} - P_{\rm s} < 0 & \text{(DT)} \end{cases} \tag{10}$$

are satisfied for some $P_{\rm s} > 0$. Conditions in (10) can be uniformly written as $\Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} < 0$, where $\Phi_{\rm s}$ is in (7) and $\Psi_{\rm s} \triangleq \left[\begin{array}{cc} \mathbf{I} & A_{\rm r}^{\rm T} \end{array}\right]^{\rm T}$. Note that, since $A_{\rm e}$ is block diagonal and A is supposed to be stable, the error system $(\Sigma_{\rm e})$ in (3) is stable if $A_{\rm r}$ is stable.

From Lemma 2 and the Lyapunov stability inequalities, it follows that the statement (i) is true if and only if some matrices P, Q and $P_{\rm s}$ exist such that $Q > 0$, $P_{\rm s} > 0$ and the conditions in (5) and (10) hold. These conditions imply that

$$\Psi^{\rm T}\Phi\Psi + \varepsilon\Psi^{\rm T}\Phi E E^{\rm T}\Phi\Psi < 0 \tag{11}$$

$$\Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} + \varepsilon_{\rm s}\Psi_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s}E_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} < 0 \tag{12}$$

hold for two sufficiently small scalars $\varepsilon > 0$ and $\varepsilon_{\rm s} > 0$, where

$$E = \left[\begin{array}{cc|cc|c|c} 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{I} & 0 \end{array}\right]^{\rm T},\ E_{\rm s} = \left[\begin{array}{cc} 0 & \mathbf{I} \end{array}\right]^{\rm T}.$$

Using the Schur complement, the inequalities in (11) and (12) can be written as

$$\left[\begin{array}{cc} \Psi^{\rm T}\Phi\Psi & \Psi^{\rm T}\Phi E \\ * & -\varepsilon^{-1}\mathbf{I} \end{array}\right] < 0 \tag{13}$$

$$\left[\begin{array}{cc} \Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} & \Psi_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s} \\ * & -\varepsilon_{\rm s}^{-1}\mathbf{I} \end{array}\right] < 0. \tag{14}$$

Furthermore, we can choose a scalar ϵ such that $\epsilon > 0.5\bar{\lambda}(E^{\rm T}\Phi E + \varepsilon^{-1}\mathbf{I})$ and $\epsilon > 0.5\bar{\lambda}(E_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s} + \varepsilon_{\rm s}^{-1}\mathbf{I})$, where $\bar{\lambda}(\cdot)$ represents the maximum eigenvalue of matrix $(\cdot)$. Thus, we have

$$2\epsilon\mathbf{I} > E^{\rm T}\Phi E + \varepsilon^{-1}\mathbf{I},\ 2\epsilon\mathbf{I} > E_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s} + \varepsilon_{\rm s}^{-1}\mathbf{I}$$

or equivalently,

$$\varepsilon^{-1}\mathbf{I} < 2\epsilon\mathbf{I} - E^{\rm T}\Phi E,\ \varepsilon_{\rm s}^{-1}\mathbf{I} < 2\epsilon\mathbf{I} - E_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s}$$

which combined with (13) and (14) ensure

$$\Upsilon \triangleq \left[\begin{array}{cc} \Psi^{\rm T}\Phi\Psi & \Psi^{\rm T}\Phi E \\ * & E^{\rm T}\Phi E - 2\epsilon\mathbf{I} \end{array}\right] < 0$$

$$\Upsilon_{\rm s} \triangleq \left[\begin{array}{cc} \Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} & \Psi_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s} \\ * & E_{\rm s}^{\rm T}\Phi_{\rm s}E_{\rm s} - 2\epsilon\mathbf{I} \end{array}\right] < 0.$$

Define matrices

$$T_1 \triangleq \left[\begin{array}{cc} \mathbf{I} & 0 \\ \left[\begin{array}{cc|c} 0 & -A_{\rm r} & -B_{\rm r} \\ 0 & -C_{\rm r} & -D_{\rm r} \end{array}\right] & \mathbf{I} \end{array}\right],\ T_{1\rm s} \triangleq \left[\begin{array}{cc} \mathbf{I} & 0 \\ -A_{\rm r} & \mathbf{I} \end{array}\right].$$

By substituting $A_{\rm e}$, $B_{\rm e}$, $C_{\rm e}$ and $D_{\rm e}$ into Ψ, and setting $X = \epsilon\mathbf{I}$ and $Y = \epsilon\mathbf{I}$, it can be verified that

$$W^{\rm T}\Phi W - 2U^{\rm T}\mathrm{diag}\{X, Y\}U = T_1^{\rm T}\Upsilon T_1 < 0$$

$$\Phi_{\rm s} - 2U_{\rm s}^{\rm T}XU_{\rm s} = T_{1\rm s}^{\rm T}\Upsilon_{\rm s}T_{1\rm s} < 0$$

which are the inequalities in (6) and (7), respectively. The statement (ii) is thus proven.

((i) $\Leftarrow$ (ii)) Suppose that the conditions in (6) and (7) hold for some symmetric matrices P, $Q > 0$, $P_{\rm s} > 0$ and diagonal matrices $X > 0$ and $Y > 0$. Define a matrix

$$T_2 \triangleq \left[\begin{array}{c} \mathbf{I} \\ \left[\begin{array}{cc|c} 0 & A_{\rm r} & B_{\rm r} \\ 0 & C_{\rm r} & D_{\rm r} \end{array}\right] \end{array}\right]. \tag{15}$$

Noting that $WT_2 = \Psi$, $UT_2 = 0$ and $U_{\rm s}\Psi_{\rm s} = 0$, we have

$$\Psi^{\rm T}\Phi\Psi = T_2^{\rm T}\left(W^{\rm T}\Phi W - 2U^{\rm T}\mathrm{diag}\{X, Y\}U\right)T_2 < 0$$

$$\Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} = \Psi_{\rm s}^{\rm T}\left(\Phi_{\rm s} - 2U_{\rm s}^{\rm T}XU_{\rm s}\right)\Psi_{\rm s} < 0.$$

Then, by Lemma 2 and the Lyapunov stability condition $\Psi_{\rm s}^{\rm T}\Phi_{\rm s}\Psi_{\rm s} < 0$, we can obtain the statement (i).

The statement (iii) is the dual version of (ii), and can be proven similarly, so its proof is omitted for brevity. ∎

Remark 1: The statement (ii) of Theorem 1 is an equivalent characterization of the index in (4) and stability of the error system $(\Sigma_{\rm e})$ (suppose that the system (Σ) is stable). In this new characterization, the reduced-order model parameters $[A_{\rm r}, B_{\rm r}]$ have been separated from the Lyapunov matrices P and Q in (5). Due to this feature, Theorem 1 is more appealing than the GKYP lemma for reduced-order model synthesis for three reasons. *First*, $[A_{\rm r}, B_{\rm r}]$ multiplied by two matrices P and Q cannot be handled by the commonly used change-of-variable method for reduced-order model synthesis. To illustrate this point, suppose that $P = \mathrm{diag}\{P_{11}, P_{22}\}$ and $Q = \mathrm{diag}\{Q_{11}, Q_{22}\}$, where P_{22} and Q_{22} are $n_{\rm r} \times n_{\rm r}$ matrices. On one hand, if we absorb, for instance, $M = P_{22}A_{\rm r}$ and $N = Q_{22}A_{\rm r}$ as new variables, the condition in (5) is not a linear matrix inequality yet, and thus still cannot be easily tested. On the other hand, it is in general impossible to recover a common matrix $A_{\rm r}$ by reversing the change of variables, that is, the above change of variables is irreversible. *Second*, note that a structured specification $[A_{\rm r}, B_{\rm r}; C_{\rm r}, D_{\rm r}] \in \mathbb{P}$ is required, but the mentioned change of variables does not preserve this requirement for model reduction of positive systems. In other words, even if the mentioned change of variables were applicable, the recovered matrix, for instance, $A_{\rm r} = P_{22}^{-1}M$ would not be guaranteed to be Metzler or positive. *Third*, an extra condition in (10) is needed to guarantee stability of the reduced-order model, which further strengthens the previous two aspects. Nevertheless, the statement (ii) of Theorem 1 includes the multiplication of $(A_{\rm r}, B_{\rm r})$ by one matrix X, rather than by P and Q, overcoming the above difficulties. Specially, X and Y are positive and diagonal, providing flexibility for constructing positive reduced-order models, which will be shown in Section III-B.

B. A Parametrization of Positive Reduced-Order Models

The inequalities in (6)–(9) are not convex conditions with respect to the reduced-order model parameters $[A_{\rm r}, B_{\rm r}, C_{\rm r}, D_{\rm r}]$. In this section, based on Theorem 1, we further develop a necessary and sufficient condition for parameterizing a positive reduced-order model. For convenience, we write the parameters of the reduced-order model together as $K \triangleq [A_{\rm r}, B_{\rm r}; C_{\rm r}, D_{\rm r}]$.

Theorem 2: Given system (Σ), the following statements are equivalent.

(i) There exists a reduced-order model $(\Sigma_{\rm r})$ solving Problem 1.

(ii) There exist symmetric matrices P, $Q \in \mathbb{R}_{n_{\rm p}+n_{\rm r}}$, $P_{\rm s} \in \mathbb{R}_{n_{\rm r}}$, diagonal matrices $X \in \mathbb{R}_{n_{\rm r}}$, $Y \in \mathbb{R}_{n_y}$ and matrices $\mathcal{K} = [\mathcal{A}, \mathcal{B}; \mathcal{C}, \mathcal{D}] \in \mathbb{R}^{(n_{\rm r}+n_y)\times(n_{\rm r}+n_u)}$, $L = [L_A, L_B; L_C, L_D] \in \mathbb{P}$ such that $Q > 0$, $P_{\rm s} > 0$, $X > 0$, $Y > 0$ and

$$\Xi(\mathcal{K}) \triangleq W^{\rm T}\Phi W - \mathrm{sym}\{\mathcal{U}^{\rm T}V\} < 0 \tag{16}$$

$$\Xi_{\mathrm{s}}(\mathcal{A}) \triangleq \Phi_{\mathrm{s}} - \mathrm{sym}\left\{\mathcal{U}_{\mathrm{s}}^{\mathrm{T}} V_{\mathrm{s}}\right\} < 0 \tag{17}$$

where Φ is defined in (5), W in (6), Φ_{s} in (7), and

$$\mathcal{U} \triangleq \left[\begin{array}{cc|c|cc} 0 & \mathcal{A} & \mathcal{B} & -\mathbf{I} & 0 \\ 0 & \mathcal{C} & \mathcal{D} & 0 & -\mathbf{I} \end{array}\right], \ \mathcal{U}_{\mathrm{s}} \triangleq \left[\begin{array}{cc} \mathcal{A} & -\mathbf{I} \end{array}\right]$$

$$V \triangleq \left[\begin{array}{cc|c|cc} 0 & L_A & L_B & -X & 0 \\ 0 & L_C & L_D & 0 & -Y \end{array}\right], \ V_{\mathrm{s}} \triangleq \left[\begin{array}{cc} L_A & -X \end{array}\right].$$

(iii) There exist symmetric matrices $\tilde{P}$, $\tilde{Q} \in \mathbb{R}_{n_{\mathrm{p}}+n_{\mathrm{r}}}$, $\tilde{P}_{\mathrm{s}} \in \mathbb{R}_{n_{\mathrm{r}}}$, diagonal matrices $\tilde{X} \in \mathbb{R}_{n_{\mathrm{r}}}$, $\tilde{Y} \in \mathbb{R}_{n_u}$ and matrices $\mathcal{K} = [\mathcal{A}, \mathcal{B}; \mathcal{C}, \mathcal{D}] \in \mathbb{R}^{(n_{\mathrm{r}}+n_y)\times(n_{\mathrm{r}}+n_u)}$, $L = [L_A, L_B; L_C, L_D] \in \mathbb{P}$ such that $\tilde{Q} > 0$, $\tilde{P}_{\mathrm{s}} > 0$, $\tilde{X} > 0$, $\tilde{Y} > 0$ and

$$\tilde{\Xi}(\mathcal{K}) \triangleq \tilde{W}\tilde{\Phi}\tilde{W}^{\mathrm{T}} - \mathrm{sym}\{\tilde{V}\tilde{\mathcal{U}}^{\mathrm{T}}\} < 0 \tag{18}$$

$$\tilde{\Xi}_{\mathrm{s}}(\mathcal{A}) \triangleq \tilde{\Phi}_{\mathrm{s}} - \mathrm{sym}\left\{\tilde{V}_{\mathrm{s}}\tilde{\mathcal{U}}_{\mathrm{s}}^{\mathrm{T}}\right\} < 0 \tag{19}$$

where $\tilde{\Phi}$, $\tilde{W}$ are defined in (8), $\tilde{\Phi}_{\mathrm{s}}$ in (9), and

$$\tilde{\mathcal{U}} \triangleq \left[\begin{array}{cc|c|cc} 0 & \mathcal{A}^{\mathrm{T}} & -\mathcal{C}^{\mathrm{T}} & -\mathbf{I} & 0 \\ 0 & \mathcal{B}^{\mathrm{T}} & -\mathcal{D}^{\mathrm{T}} & 0 & -\mathbf{I} \end{array}\right]^{\mathrm{T}}, \ \tilde{\mathcal{U}}_{\mathrm{s}} \triangleq \left[\begin{array}{c} \mathcal{A} \\ -\mathbf{I} \end{array}\right]$$

$$\tilde{V} \triangleq \left[\begin{array}{cc|c|cc} 0 & L_A^{\mathrm{T}} & -L_C^{\mathrm{T}} & -\tilde{X}^{\mathrm{T}} & 0 \\ 0 & L_B^{\mathrm{T}} & -L_D^{\mathrm{T}} & 0 & -\tilde{Y}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}, \ \tilde{V}_{\mathrm{s}} \triangleq \left[\begin{array}{c} L_A \\ -\tilde{X} \end{array}\right].$$

Moreover, if the statement (ii) (respectively, (iii)) is true, the parameters of the reduced-order model (Σ_{r}) can be obtained by $K = \mathrm{diag}\{X^{-1}, Y^{-1}\}L$ (respectively, $K = L\mathrm{diag}\{\tilde{X}^{-1}, \tilde{Y}^{-1}\}$).

Proof: ((i) $\Rightarrow$ (ii)) Suppose the statement (i) is true. According to Lemma 1 and Theorem 1, there exist matrices P, Q, P_{s}, X, Y and K such that $Q > 0$, $P_{\mathrm{s}} > 0$, $X > 0$, $Y > 0$, $K \in \mathbb{P}$ and the conditions in (6) and (7) hold. Let $L = \mathrm{diag}\{X, Y\}K$, $\mathcal{K} = K$, then we have $\mathcal{U} = U$, $V = \mathrm{diag}\{X, Y\}U$, $\mathcal{U}_{\mathrm{s}} = U_{\mathrm{s}}$ and $V_{\mathrm{s}} = XU_{\mathrm{s}}$, which imply

$$\begin{aligned}\Xi(\mathcal{K}) &= W^{\mathrm{T}}\Phi W - 2U^{\mathrm{T}}\mathrm{diag}\{X, Y\}U \\ &= W^{\mathrm{T}}\Phi W - \mathrm{sym}\{\mathcal{U}^{\mathrm{T}}V\} < 0 \\ \Xi_{\mathrm{s}}(\mathcal{A}) &= \Phi_{\mathrm{s}} - 2U_{\mathrm{s}}^{\mathrm{T}}XU_{\mathrm{s}} = \Phi_{\mathrm{s}} - \mathrm{sym}\{\mathcal{U}_{\mathrm{s}}^{\mathrm{T}}V_{\mathrm{s}}\} < 0.\end{aligned}$$

That is, the conditions in (16) and (17) hold. Since X and Y are positive and diagonal and K belongs to $\mathbb{P}$, the matrix L from $L = \mathrm{diag}\{X, Y\}K$ also belongs to $\mathbb{P}$. Thus, the statement (ii) is true.

((i) $\Leftarrow$ (ii)) Suppose the statement (ii) is true. Substituting $L = \mathrm{diag}\{X, Y\}K$ into (16) and (17), we have

$$\begin{aligned}\Xi(\mathcal{K}) &= W^{\mathrm{T}}\Phi W - \mathrm{sym}\left\{\mathcal{U}^{\mathrm{T}}\mathrm{diag}\{X, Y\}U\right\} < 0 \\ \Xi_{\mathrm{s}}(\mathcal{A}) &= \Phi_{\mathrm{s}} - \mathrm{sym}\{\mathcal{U}_{\mathrm{s}}^{\mathrm{T}}XU_{\mathrm{s}}\} < 0.\end{aligned}$$

Pre- and post-multiplying the first condition by T_2^{T} and T_2 with T_2 defined in (15), and the second condition by $\Psi_{\mathrm{s}}^{\mathrm{T}}$ and Ψ_{s}, respectively, we further obtain

$$\begin{aligned}\Psi^{\mathrm{T}}\Phi\Psi &= T_2^{\mathrm{T}}(W^{\mathrm{T}}\Phi W - \mathrm{sym}\{\mathcal{U}^{\mathrm{T}}\mathrm{diag}\{X, Y\}U\})T_2 < 0 \\ \Psi_{\mathrm{s}}^{\mathrm{T}}\Phi_{\mathrm{s}}\Psi_{\mathrm{s}} &= \Psi_{\mathrm{s}}^{\mathrm{T}}\left(\Phi_{\mathrm{s}} - \mathrm{sym}\left\{\mathcal{U}_{\mathrm{s}}^{\mathrm{T}}XU_{\mathrm{s}}\right\}\right)\Psi_{\mathrm{s}} < 0.\end{aligned}$$

According to Lemma 2 and the Lyapunov stability theory, $\Psi^{\mathrm{T}}\Phi\Psi < 0$ and $\Psi_{\mathrm{s}}^{\mathrm{T}}\Phi_{\mathrm{s}}\Psi_{\mathrm{s}} < 0$ imply that the statement (i) of the theorem is true for a positive reduced-order model with the parameter K obtained as $K = \mathrm{diag}\left\{X^{-1}, Y^{-1}\right\}L$.

Consequently, the statement (i) is equivalent to (ii). By virtue of the statement (iii) of Theorem 1, the equivalence between statements (i) and (iii) of Theorem 2 can be similarly established. Obviously, K obtained by $K = \mathrm{diag}\{X^{-1}, Y^{-1}\}L$ or $K = L\mathrm{diag}\{\tilde{X}^{-1}, \tilde{Y}^{-1}\}$ is guaranteed to belong to $\mathbb{P}$. ■

From the necessity part of the proof, it is seen that L comes from the change of variable $L = \mathrm{diag}\{X, Y\}K$; thus, requiring $L \in \mathbb{P}$ is not restrictive. According to the statement (ii) of Theorem 2, the reduced-order model is parameterized as $K = \mathrm{diag}\left\{X^{-1}, Y^{-1}\right\}L$. Because $L \in \mathbb{P}$, and X, Y are positive definite and diagonal, it is guaranteed that $K \in \mathbb{P}$, that is, the obtained reduced-order model is also positive. Hence, Theorem 2 circumvents the difficulties of Theorem 1 in handling the multiplication of K by P, Q and P_{s} that are pointed out in Remark 1. Note that the requirement, $L \in \mathbb{P}$, is actually an element-wise convex constraint, implying that it can be easily checked by standard softwares.

Compared with Theorem 1, the conditions in (16) and (17) include an additional matrix $\mathcal{K}$, which satisfies an important property shown by the following theorem.

Theorem 3: For $\mathcal{K} = \left[\begin{array}{cccc} \mathcal{A} & \mathcal{B}; & \mathcal{C} & \mathcal{D} \end{array}\right]$ satisfying the statement (ii) or (iii) of Theorem 2, the state-space model given by

$$\begin{aligned}(\Sigma_{\mathrm{r}}'): \ \lambda\left[x_{\mathrm{r}}(t)\right] &= \mathcal{A}x_{\mathrm{r}}(t) + \mathcal{B}u(t) \\ y_{\mathrm{r}}(t) &= \mathcal{C}x_{\mathrm{r}}(t) + \mathcal{D}u(t)\end{aligned} \tag{20}$$

is a stable n_{r}th-order model approximating the system (Σ) such that the resulting error system (Σ_{e}) satisfies (4).

Proof: Suppose that a matrix $\mathcal{K} = \left[\begin{array}{cccc} \mathcal{A} & \mathcal{B}; & \mathcal{C} & \mathcal{D} \end{array}\right]$ satisfies the statement (ii). Pre- and post-multiplying the condition in (16) by $\mathcal{T}_2^{\mathrm{T}}$ and $\mathcal{T}_2$ with

$$\mathcal{T}_2 \triangleq \left[\begin{array}{c} \mathbf{I} \\ \left[\begin{array}{cc|c} 0 & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{C} & \mathcal{D} \end{array}\right] \end{array}\right]$$

one can obtain (5) for the error system (Σ_{e}) that results from the reduced-order system (Σ_{r}'), that is, (4) is satisfied. From (17), the Lyapunov stability inequality can be obtained through

$$\left[\begin{array}{c} \mathbf{I} \\ \mathcal{A} \end{array}\right]^{\mathrm{T}}\Phi_{\mathrm{s}}\left[\begin{array}{c} \mathbf{I} \\ \mathcal{A} \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \mathcal{A} \end{array}\right]^{\mathrm{T}}\Xi_{\mathrm{s}}(\mathcal{A})\left[\begin{array}{c} \mathbf{I} \\ \mathcal{A} \end{array}\right] < 0.$$

Therefore, the reduced-order system (Σ_{r}') is stable.

The proof for the case of the statement (iii) of Theorem 2 can be completed similarly. ■

Theorem 3 shows that the matrix $\mathcal{K}$ also gives rise to a reduced-order model (Σ_{r}') for the same system (Σ). The difference between systems (Σ_{r}) and (Σ_{r}') is that the latter is not necessary to be positive. In other words, according to Theorem 3, the statement (ii) or (iii) in Theorem 2 establishes a connection between a positive reduced-order model and a common reduced-order one.

C. An Iterative Algorithm for Positive Model Reduction

The inequalities in (16)–(19) are not convex constraints. However, if $\mathcal{K}$ is fixed, these conditions become convex with respect to the remaining variables. Hence, an algorithm is naturally proposed for computing reduced-order models: First fix $\mathcal{K}$ corresponding to some reduced-order model known *a priori*, and then solve the conditions in (16), (17) and/or (18), (19) to obtain $K \in \mathbb{P}$ corresponding to a positive reduced-order model. A merit of the proposed design method is that the design flow can be repeated for further reducing the approximation error. Consequently, the following iterative algorithm is proposed for solving the model reduction problem.

Iterative Algorithm for Positive Model Reduction

S-1 Find the parameter $\mathcal{K}$ of an n_{r}th-order model (Σ_{r}') in (20) via the existing model reduction methods. Set $i = 1$.

S-2 (Primal) Solve the following optimization problem:

$$\begin{aligned}\min\gamma \quad \text{s.t. } &\Xi(\mathcal{K}) < 0, \ \Xi_{\mathrm{s}}(\mathcal{A}) < 0, \ Q > 0 \\ &P_{\mathrm{s}} > 0, \ L \in \mathbb{P}, \ \mathrm{diag}\{X, Y\} > 0 \\ &\text{for } P, \ Q, \ P_{\mathrm{s}}, \ X, \ Y, \ L \text{ and } \gamma.\end{aligned} \tag{21}$$

Set $\mathcal{K} = \mathrm{diag}\left\{X^{-1}, Y^{-1}\right\}L$.

S-3 (Dual) Solve the following optimization problem:

$$\begin{aligned}\min\ \gamma \quad & \text{s.t. } \tilde{\Xi}(\mathcal{K})<0,\ \tilde{\Xi}_{\mathrm{s}}(\mathcal{A})<0,\ \tilde{Q}>0 \\ & \tilde{P}_{\mathrm{s}}>0,\ L\in\mathbb{P},\ \mathrm{diag}\{\tilde{X},\tilde{Y}\}>0 \\ & \text{for } \tilde{P},\ \tilde{Q},\ \tilde{P}_{\mathrm{s}},\ \tilde{X},\ \tilde{Y},\ L \text{ and } \gamma.\end{aligned} \tag{22}$$

Set $\mathcal{K}=L\mathrm{diag}\left\{\tilde{X}^{-1},\tilde{Y}^{-1}\right\}$ and denote the optimum of γ as $\gamma^{(i)}$.

S-4 If $\left|\gamma^{(i)}-\gamma^{(i-1)}\right|/\gamma^{(i)}<\delta$ with δ being a prescribed tolerance or if $i=k$ with k being the prescribed maximum allowable number of iterations, then output $K=\mathcal{K}$ and $\gamma=\gamma^{(i)}$ as the optimized reduced-order model, and EXIT; else, set $i\leftarrow i+1$ and go back to S-2.

Remark 2: Optimization of the frequency-limited H_∞ level $\gamma^{(i)}$ of the error system is realized directly by running the algorithm. It can be shown that $\gamma^{(i)}$ is monotonically non-increasing with respect to i, which will be demonstrated by numerical examples. Note that each $\gamma^{(i)}$ is bounded below by zero. Hence, the sequence $\{\gamma^{(1)},\gamma^{(2)},\cdots\}$ must be converging as i increases. What value $\gamma^{(i)}$ theoretically converges to, however, is unknown. The terminating conditions in the algorithm are just some heuristic criteria for numerical programming.

Remark 3: The algorithm incorporates the primal and dual forms of the derived conditions. One may note that simpler algorithms can be constructed similarly but only using the primal form (S-1, S-2 and S-4) or the dual form (S-1, S-3 and S-4). However, no guarantee exists such that the two simpler algorithms give rise to the same result. Because all these algorithms are heuristic, the algorithms using a single form may produce "early mature" results. In other words, the value of $\gamma^{(i)}$ for one form decreases too slow after very few initial iterations so that one has to terminate computation and accept the obtained result that would be too conservative. According to our experience, combining the two forms together can partly overcome this effect, which will be illustrated by numerical examples.

IV. Numerical Examples

In this section, two numerical examples are presented to illustrate the effectiveness of the proposed model reduction method. Numerical solver SeDuMi [26], invoked through the interface Yalmip [27], will be used to solve the convex optimization problems encountered.

Example 1 (Positive CT System): Consider an example of the positive CT system (Σ) with state-space parameters given by

$$A=\begin{bmatrix}-1.5 & 0.6 & 1.0 & 0 & 0 & 0\\ 0.3 & -1.9 & 0.2 & 0 & 0 & 0\\ 0.2 & 0.5 & -2.7 & 1 & 0 & 0\\ 0 & 0 & 0.5 & -3 & 0.6 & 0.5\\ 0 & 0 & 0 & 0.4 & -1.6 & 0.3\\ 0 & 0 & 0 & 0.6 & 0.5 & -1.6\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

$$C=\begin{bmatrix}\mathbf{I}_{2\times 2} & 0_{2\times 4}\end{bmatrix},\ D=0_{2\times 2}.$$

This numerical example was composed in [16] to describe a compartmental network with two subsystems. Suppose that the limited frequency range Ω takes $[0,2]$ rad/s. By the developed algorithm, the goal of this example is to explore a positive second-order model and optimize the approximation error level γ for the index in (4).

By the GBT method in [15], a reduced-order model is obtained via retaining the first two states of the original model. Using this reduced-order model as the initial value of the developed algorithm, and setting $\delta=0.01$ and $k=50$, we obtain a reduced-order model:

$$\begin{bmatrix}A_{\mathrm{r}} & B_{\mathrm{r}}\\ C_{\mathrm{r}} & D_{\mathrm{r}}\end{bmatrix}=\left[\begin{array}{cc|cc}-1.2546 & 0.6718 & 0.9192 & 0.0054\\ 0.2371 & -1.6161 & 0.0172 & 0.9329\\ \hline 0.9453 & 0.0197 & 0.0330 & 0.0003\\ 0.0180 & 0.8975 & 0.0014 & 0.0396\end{array}\right] \tag{23}$$

and the corresponding optimal γ, denoted as γ^*, is $\gamma^*=0.0154$. Obviously, the obtained reduced-order model in (23) is positive according to Lemma 1. For illustration and comparison, Figure 1 presents the frequency response of the error systems for the proposed method and some existing methods. First, it is seen that, for the reduced-order model in (23), the actual maximum singular value of the error system over $\Omega=[0,2]$ rad/s is effectively upper bounded by $\gamma^*=0.0154$. Second, it is observed that, over the interested frequency range $\Omega=[0,2]$ rad/s, the reduced-order models obtained by GBT, the H_∞ method in [16] and MM have larger approximation errors than the one in (23) in the frequency-limited H_∞ sense. Although MM gives a reduced-order model perfectly matching the original one at zero frequency, it loses such matching at other frequencies and the overall approximation performance over the frequency range Ω could be further improved.

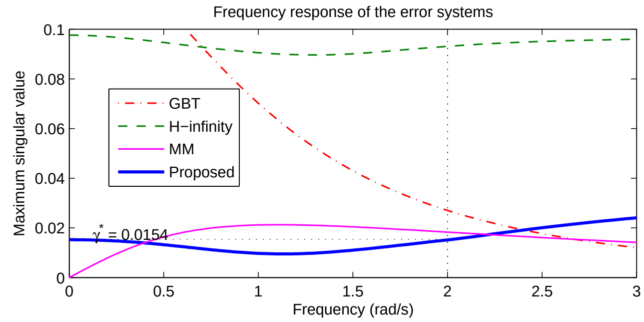

Fig. 1. Frequency response of the error systems in Example 1.

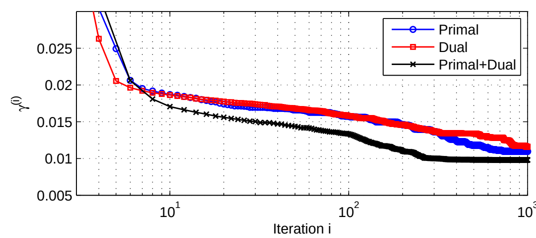

Fig. 2. Evolution of $\gamma^{(i)}$ by different iterative methods in Example 1. For the case of Primal+Dual, the horizontal axis denotes the times solving the problem in (21) or (22).

To illustrate the usefulness of combining the primal and dual forms in the proposed algorithm, we show the evolution of $\gamma^{(i)}$ in Figure 2, which also presents the results on the algorithms only using the primal or dual form. All the algorithms are configured as omitting the terminating condition regarding the tolerance δ and setting $k=1000$; moreover, the iteration number i for the proposed algorithm is scaled by 2, hence the horizontal axis for the proposed method actually denotes the times solving a single problem in (21) or (22). It is shown that the algorithms only using the primal or dual form produce "early mature" results, while the proposed algorithm can provide some improved ones, verifying the comments in Remark 3.

Example 2 (Positive DT System): Consider the following difference evolution equation:

$$x(t+1)=\left[\frac{0.24\ 0.48\ 0.76\ 0.76\ 0.76\ 0.76\ 0.76\ 0.76\ 0.72\ 0.64}{\mathrm{diag}\{0.24,0.30,0.33,0.34,0.33,0.30,0.28,0.24,0.27\}\ 0_{9\times 1}}\right]x(t)$$

which is an example of Leslie's age structured population model [28] specified for the squirrel population data in [1]. It means that the considered population model includes 10 age groups, while $x_i(t)$ denotes the number of individuals of the age group i in the current year. The first equation,

$$x_1(t+1)=[0.24\ 0.48\ 0.76\ 0.76\ 0.76\ 0.76\ 0.76\ 0.76\ 0.72\ 0.64]\,x(t)$$

represents the next year population of the (youngest) age group 1 that are reproduced by all the age groups (the coefficients are computed from survival and fertility rates). Other equations can be written as

$$x_i(t+1) = s_{i-1}x_{i-1}(t),\ i = 2,\dots,10$$

where s_i is the survival rate of age group i in one year. Corresponding to the system (Σ), suppose that other matrices are given by

$$\begin{aligned} B &= \begin{bmatrix} \mathbf{I}_2 & 0_{2\times 8} \end{bmatrix}^{\mathrm{T}},\ D = \begin{bmatrix} 0 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

The matrix B means that the first two age groups are still affected by external input due to immigration, artificial propagation, etc. Matrix C means that the output is the total population of all the age groups. Furthermore, assume that the frequency interval Ω is $[0, \pi/3]$ $\mathrm{rad/year}$. We are interested in constructing a positive second-order model to approximate this population model.

By BT, a second-order model is first obtained as

$$\begin{bmatrix} A_{\mathrm{r}} & B_{\mathrm{r}} \\ C_{\mathrm{r}} & D_{\mathrm{r}} \end{bmatrix} = \left[\begin{array}{cc|cc} 0.6637 & -0.2282 & 1.4723 & 1.8451 \\ -0.0305 & 0.1792 & 0.8133 & -0.5173 \\ \hline 0.5927 & 0.1611 & 0 & 0 \end{array}\right]$$

which, however, is not a positive system according to Lemma 1[1]. Use this model as the initial value and set $\delta = 0.01$ and $k = 50$. The developed algorithm gives rise to a second-order model as

$$\begin{bmatrix} A_{\mathrm{r}} & B_{\mathrm{r}} \\ C_{\mathrm{r}} & D_{\mathrm{r}} \end{bmatrix} = \left[\begin{array}{cc|cc} 0.6645 & 0.0006 & 1.0480 & 1.8138 \\ 0.4625 & 0.0073 & 2.2148 & 0.0159 \\ \hline 0.5360 & 0.1968 & 0.0046 & 0.0081 \end{array}\right] \tag{24}$$

with $\gamma^* = 0.0176$. It can be verified that the model in (24) is stable and positive.

V. Conclusion

This paper has addressed model reduction for positive LTI systems in the frequency-limited H_∞ sense. The cases for CT and DT systems are considered in a unified framework. To preserve positivity of the reduced-order model, a novel necessary and sufficient condition for the existence of a positive reduced-order model with a guaranteed frequency-limited H_∞ approximation error has been proposed, which parameterizes a positive reduced-order model through another common reduced-order one. By virtue of this property, an iterative algorithm has been accordingly developed to search for and optimize the positive reduced-order model. Numerical results show that the proposed algorithm, when initialized by the simple BT or GBT method, can produce satisfactory reduced-order models. It should be noted that the proposed method can only deal with internally positive systems. How to deal with the same problem for externally positive systems is worth future investigation.

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[1] Accidentally, this reduced-order model can be realized by a positive system further through a similarity transformation, for instance, $TA_{\mathrm{r}}T^{-1}$ with $T = [1\ \ -5; 0.3\ \ 1]$. But in general, the BT method is not positivity-preserving. This does not contradict to the above result, because (internal) positivity is state coordinate-specific.

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