

STABILIZING CONTROL DESIGN OF
A MOTORCYCLE

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Stabilizing Control Design of a Motorcycle

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ABSTRACT

This thesis solves the stabilizing control of an autonomous motorcycle. The control of an autonomous motorcycle is a challenging and interesting problem in the field because the plant is under-actuated, unstable and nonlinear.

Two major problems that have not been considered in the literature are explicitly solved in our work: (i) the robust control problem of the plant subject to uncertainty and exogenous disturbance; (ii) the non-local stabilization of the (nonlinear) plant.

To achieve the first goal, we propose a robust H_∞ controller to the linearized system, which provides a significant improvement in dealing model uncertainty and disturbance attenuation in comparison with classical linear designs.

To achieve the second goal, we propose a nonlinear controller based on the combination of a nonlinear forwarding method with several other methods for the nonlinear plant through identifying an appropriate upper triangular structure of the nonlinear system. This yields a stability region, the whole upper space, such that the trajectory starting from any position in the upper hemi-sphere with arbitrary initial velocities converges to the upright position.

Both results are novel and first results of their kinds in control of an autonomous motorcycle. Computer simulations verify the effectiveness of the proposed controllers.

DECLARATION

I certify that except where due acknowledgement has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and, any editorial work, paid or unpaid, carried out by a third party is acknowledged.

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2007

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Chapter 1

INTRODUCTION

1.1 Overview

The objective of this thesis is to explore the advanced control designs (e.g., robust H_∞ control, nonlinear designs) for a motorcycle which have drawn a little attention in the literature. The research is conducted on the basis of a comprehensive literature review, through Library, Database (for example, IEEE explore, Ei Compendix, ISI), and the internet.

The Chapter is organized as follows: we start with the formulation of the control problem and its motivation in Section 1.2; we recall briefly the development of related theories and carry out the literature review associated with the problem in Section 1.3; next, the contributions of this dissertation are summarized in Section 1.4; finally, the organization of the thesis is outlined in Section 1.5.

1.2 The Control Problem and Motivation

The motorcycle was invented a century ago. It has gained a great popularity since then. However, the motorcycles are easy to fall over even though they are handled by experienced drivers. This results from the “mysterious” dynamics of the motorcycle, which is too fast and too drastic to be dealt with. Furthermore, although energy power assistance motorcycles are used practically, but all of those motorcycles assist human to drive them forward and there are no motorcycle that help to stabilize its posture. This situation has motivated interest in developing automated steering controllers for these two-wheels vehicles. The control of motorcycle presents a challenging task to both the cutting-edge technology and advanced control theory.

The control objectives considered in this thesis are: (1) robustness; (2) non-local stabilization.

Objective 1: The objective for robustness is to explicitly deal with model uncertainty and some exogenous disturbance.

Objective 2: The objective for non-local stabilization is to use the control signal to drive the motorcycle in such a way that the upright position is the asymptotically stable equilibrium with a large domain of attraction.

To the best of our knowledge, no complete solution to the above control objectives has appeared in the literature. This thesis is dedicated to achieve those objectives through applying advanced control design tools. Next, the control literature is briefly reviewed. Moreover, the related results dealing with modeling and control of the spherical inverted

pendulum are recalled.

1.3 Literature Review

1.3.1 Overview of Control Theory

There has been a great success of linear control theory ranging from classical root locus design, frequency domain analysis and PIDs to advanced analysis and design tools, optimal-LQR and Kalman filtering [2], H_∞/H_2 robust control [73], and adaptive control [63] to name just a few. In particular, we have paid much attention to the H_∞ robust control theory in the last two decades because it is a powerful tool to deal with uncertain systems, disturbance rejection and sensitivity minimization problems and quite often, those issues are closely related to a variety of applications in real life. See monographs [2, 14, 74, 73, 63] and many others for the linear control theory.

Meanwhile, nonlinear control also underwent significant progresses such as: the differential geometric approach, the circle and Popov criteria, input-to-state stability (ISS) and small-gain theorems, averaging, singular perturbations, sliding mode control, Lyapunov design and redesign, backstepping, forwarding, high gain observers and nonlinear output regulators . The development of nonlinear control theory is motivated by the fact that real world problems are often inherently nonlinear (see textbooks and monographs [36, 24, 26, 37, 49, 52, 67] for those methods). Nevertheless, it is often not straightforward to apply these nonlinear techniques directly out of the books to a specific nonlinear system. Moreover, because most nonlinear control techniques apply

only to systems of special structure, considerable additional effort is often necessary to successfully employ the existing (non)linear control theory.

Our objective in this thesis is to explore a robust controller and a nonlinear control idea for the motorcycle. The development of the relevant tools is recalled briefly below.

1.3.2 Robust H_∞ Control

When designing robust control laws for linear systems, closed-loop specifications can be formulated in the frequency domain, on peak values of Bode magnitude plots of possibly weighted system transfer functions, this is called H_∞ optimization framework in frequency domain (see, for example, [74, 73]). In a state-space setting, H_∞ control design algorithms boil down to convex optimization over linear matrix inequalities (LMIs), or semi-definite programming. We shall use state space formulation of robust H_∞ control in our design, that is reviewed next.

A dominant aspect of 1990s robust control research was the emergence of a new analysis/synthesis paradigm centered around convex optimization and, in particular, on the linear matrix inequality (LMI). Lyapunov and Riccati equation H_2 and H_∞ problem formulations that dominated the literature of the 1980s came to be seen as special cases of more flexible LMI problem formulations. The interplay of functional analysis and state-space optimal control theory proved to be seductively exotic, reinforcing and giving new impetus to the close links between control and advanced mathematics. Progress in LMI robust control theory has been explosive in recent years (see, for example, [15]).

1.3.3 The Differential Geometric Approach

One of the key ideas in differential geometric control is to transform a system into a linear one by means of feedback and coordinates transformation. This expands the utility of linear control idea considerably. The notion of “zeros” is important in characterizing the limitations of feedback control. In the nonlinear differential geometric approach this is captured by the notion of zero dynamics. This notion plays an important role in the problem of achieving local asymptotic stability, asymptotic tracking, model matching and disturbance decoupling (see [61, 21, 9, 27, 10, 23, 28, 11, 12, 1] for the earlier results). Further results using the differential geometric concepts are devoted to the solution of the problems of output regulation, noninteracting control with stability via static state feedback, and noninteracting control via dynamic feedback, for a broader class of multivariable nonlinear systems (see [6, 41, 62, 8, 25, 34] for the earlier results). We refer to the monograph [24] for an exposition of this approach.

1.3.4 ISS and Small Gain Theorems

The introduction of the concept of ISS and ISS-Lyapunov function by Sontag in [53] brings about a number of new notions and powerful analysis tools for nonlinear systems, for example, small gain theorems, input-to-output stability (IOS) [31] and integral input-to-state stability (iISS) [3]. The seminal paper [53] presented the definition of ISS, established the result on feedback redesign to obtain ISS with respect to actuator errors, and provided the necessary and sufficiency test in terms of ISS-Lyapunov function. The necessity of this Lyapunov-like characterization was given

in [55], which also introduced the “small gain” connection to margins of robustness; the existence of Lyapunov functions then followed from the general result in [40]. The asymptotic gain characterizations of ISS were presented in [59]. Small gain theorems for ISS and IOS notions originated in [31]. The notion of ISS for time-varying systems appeared in the context of asymptotic tracking problems [66]. Many ISS results for continuous time systems, for example, the Lyapunov characterization and ISS small gain theorems, were extended to the discrete time case [30, 32, 38].

We single out an important variation proposed by Teel in [65], who introduces the notion of asymptotic ISS “gains” (see [57] for the equivalence to ISS “gains”) with an associated small gain theorem corresponding to asymptotic ISS “gains”. An application of this result is the well-known forwarding design–nested saturating design for nonlinear systems with an upper triangular structure. Forwarding is reviewed next as it applies to the inverted pendulum.

1.3.5 Forwarding

In the family of recursive control designs for nonlinear systems, nonlinear forwarding and backstepping are two celebrated design procedures for the nonlinear systems with the feedforward structure (also called the forwarding structure or the upper triangular structure) and feedback structure (also called the lower triangular structure) respectively. Backstepping employs aggressive (high gain) controls necessary to suppress finite escape instabilities inherent to strict-feedback systems. In contrast, forwarding exploits cautious (low gain) controllers to the feedforward systems.

Forwarding was mainly studied in the mid- and late-1990's. The theoretical foundation of forwarding was laid out in Teel's dissertation [64], where he introduced the nested saturating designs in which parameters were carefully selected to essentially achieve robustness of linear controllers to nonlinearities. In the light of nonlinear small gain techniques [65], a generalized procedure of nested saturating design for nonlinear systems with the feedforward structure was developed in the same paper [65]. Mazenc and Praly in [45] introduce a Lyapunov approach for the stabilization of feedforward systems. A different Lyapunov approach to the stability of feedforward systems was developed in [29] which constructs an "exact" cross term in the Lyapunov function rather than taking a coordinate change or domination of "cross terms" in [45].

The nested saturating design ideas were extended by [39, 4, 43, 72, 33]. The robustness results to certain classes of unmodelled dynamics associated with some subclasses of forwarding systems were obtained in [4, 43, 72]. The Lyapunov approaches were further developed in [50, 47, 46]. Trajectory tracking was solved under reasonable conditions in [44]. Forwarding tools were successfully applied to several control problems, for example, vertical take-off and landing (VTOL) plane in [65] and the inverted pendulum on the cart in [65, 45].

The Lyapunov based forwarding tools are inherently difficult because one needs find the solution of mostly nonlinear PDEs in each recursive design step. Practically, nested saturating design is more appealing but more conservative because the recursive design steps are based on the linearization of each augmented system.

1.3.6 Control of an autonomous motorcycle

Before exhausting words to portray the control problem of the case study, we would like to cite a news captioned by “*Japanese robot goes bike-riding*”¹ (see Figure 1.1). In the article, the project engineer Shigeki Fukunaga said “*The whole point of developing a robot that rides a bicycle is to show the technology of balancing in the environment, where keeping your balance is tough...*” .

From a control design point of view, designing a high performance control law for the autonomous motorcycle is very challenging because it is nonlinear, unstable and under-actuated.

Strict dynamic model of bicycle (or motorcycle) was proposed in [51]. It is named Sharp model and many researches are based on this model. A problem of this model is that it is complicated and difficult to apply to a bicycle posture controller. The work in [42] derive a simple kinematic and dynamic formulation of an unmanned electric bicycle with load mass balance system which plays important role in stabilization. The work in [68, 60] consider even more simplified models of the motorcycle for their purpose for control, where the inverted pendulum on a rotational arm is taken as the model of dynamics.

As far as the control is concerned, the simplified model which is derived in [20] (see also [19]) prevails over others, which has been used in a variety of papers for a number of applications. By exploitation of the bicycle’s constraints and symmetry, Getz first derives a reduced set of equations of

¹At Tuesday, 11th October, 2005, MX, a local newspaper in Melbourne, Australia, reported this event.



Figure 1.1: “Japanese robot goes bike-riding”. Visitors watch the 20-inch-high Murata Boy robot ride a tiny bicycle without falling during a demonstration Tuesday at the CEATEC Japan 2005 exhibition in Makuhari, east of Tokyo.....The firm developed an earlier version of a bike-riding robot back in 1990, but the latest version can stop without falling over.

motion for the bicycle. Then, he shows how one may use the knowledge of how to steer the kinematics bicycle to construct a controller that allows a leaning bicycle to track planar trajectories without falling. This thesis is based on the model of dynamics to be Getz's model.

The stabilization of the motorcycle about the equilibrium has been extensively studied in the literature. In [51, 16, 60], authors have studied the stability of the linearized model of some motorcycle models and designed some linear controllers.

In [69, 20, 17], the authors consider some nonlinear solutions to the guidance problem on either the path planning or the trajectory tracking, which also include the balance problem about the moving equilibrium because of the moving bicycle (or motorcycle). Due to their different perspectives, the unstable mode of the motorcycle in a non-local region do not stand out as a problem for study. One major part of the thesis will cover this topic.

1.4 Statement of Contributions

In this thesis, the following major contributions have been made:

(1) We design a robust H_∞ controller based on the theory [70] that deals with model uncertainty and exogenous disturbance simultaneously to achieve the so-called quadratic stability with disturbance rejection. To this end, we introduce velocity uncertainty in the model and consider the exogenous disturbance as well.

(2) We design a nonlinear forwarding stabilizing controller to the motorcycle based on the forwarding design idea [65] and several other tools. The controller renders the upper hemisphere (i.e., “global”) the domain of attraction. We achieve this by first transforming the original system to an appropriate upper triangular form, then specifying asymptotic “gains” for the first subsystem and next incorporating the nonlinear forwarding design for the rest. This nonlinear controller is a complete nonlinear controller for the first time associated with the motorcycle.

In addition, the thesis also presents a linear state feedback controller and a dynamic output feedback controller based on well-known state space design techniques.

During the course of my project, a paper has been made based on the work presented in my thesis as follow:

- F. Yuan and L. Wang. Advanced nonlinear design for stabilizing a motorcycle. *Proceedings of the SICE Annual Conference*, Okayama, Japan, 2005, pp.1371–1376

1.5 Outline of the Thesis

Besides Chapter 1 here, the remaining of the thesis is organized into four main chapters and one concluding chapter. Some mathematical notations, classical linear control theory and some Matlab codes are included in Appendix.

Chapter 2. Modelling

We review a model for the motorcycle in the literature that is commonly

used in control designs.

Chapter 3 State variable feedback control

We present a state feedback controller using state feedback based on pole placement and a dynamic output feedback controller based on some observer design for the linearized (nominal) system. The Chapter also points out that classical linear designs do not deal with explicitly the robustness. In addition, only local stability (i.e. small and bounded domain of attraction) are guaranteed. These motivate us to seek the robust controllers and nonlinear controller in the sequel.

Chapter 4 Robust H_∞ control

We propose a robust H_∞ controller to the linearized system, which provides a significant improvement in dealing model uncertainty and disturbance attenuation in comparison with classical linear designs.

Chapter 5 Nonlinear forwarding

ISS, ISS Lyapunov theory [53] and the forwarding tool–nested saturating design [65] are reviewed first. Using certain state and control input transformations we bring the model in an appropriate upper triangular structure that forwarding can be used. The controller is designed in such a way that a high gain part is accountable for the regulation of the angular dynamics and a low gain (forwarding) part takes care of the regulation of other dynamics. The trajectory starting from any position in the upper hemi-sphere and arbitrary initial velocities converges to the upright position.

Chapter 6 Conclusions and future work

The main results of the thesis are summarized. Future research that exploits a number of control problems regarding the autonomous motorcycle is described.

Chapter 2

MODELING

2.1 Overview

The Chapter briefly reviews a model of the autonomous motorcycle which has been used extensively in the control community.

The Chapter is organized as follows: we introduce nomenclatures in Section 2.2 and review the model of dynamics and its simplification derived by Getz [20] in Section 2.3

2.2 Nomenclatures

The nomenclatures for the simple motorcycle model [20] associated with Figure 2.1 are listed as follows

- (x, y) (m), the position of the rear-wheel contact with the ground;
- u (rad/s^2), the normalized input torque exerting on the steering angle ϕ ;

- τ (m/s^2), a normalized reaction force of the ground on the rear wheel;
- v (m/s), the forwarding speed of the motorcycle;
- p (m), the distance from the center of mass to the ground;
- c (m), the horizontal distance from the center of mass to the ground contact point of the rear wheel;
- β (rad), the yaw-angle β of the motorcycle is the angle from the x -axis to the contact line;
- α (rad), the roll-angle of the motorcycle is the angle that the motorcycle frame is rolled away from the vertical line;
- b (m), the horizontal distance from the ground contact point of the front wheel to the ground contact point of the rear wheel;
- ϕ (rad), the angle of the front wheel deflected from the line through the ground contact points of the wheels.

2.3 The Simple Motorcycle Model

With reference to Figure 2.1, the simplified model presented here is proposed by Getz (see [20] for more details of modelling and the full model) and extensively used in [17, 69], which is

$$\dot{x} = \cos(\beta)v$$

$$\dot{y} = \sin(\beta)v$$

$$\dot{\beta} = \psi_\beta v$$

$$\dot{\psi}_\beta = u$$

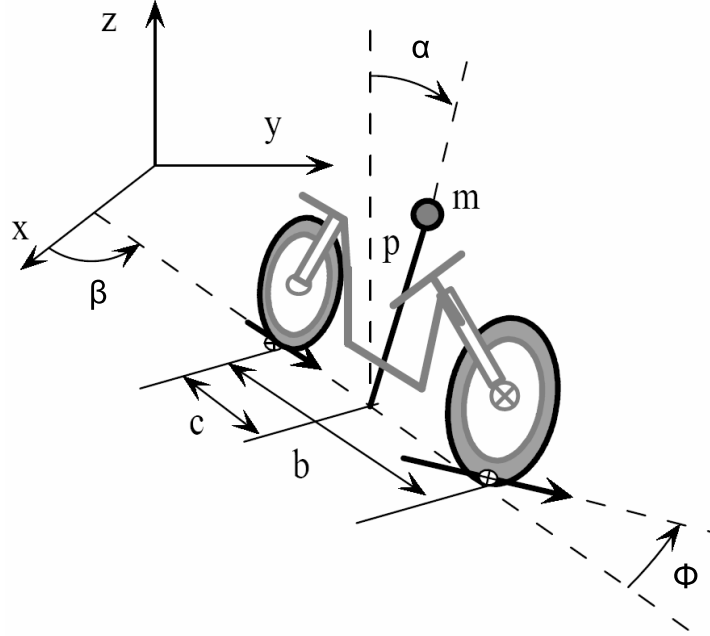


Figure 2.1: The simple motorcycle model

$$\dot{v} = \tau$$

$$\dot{\alpha} = \psi_{\alpha}$$

$$\begin{aligned} \dot{\psi}_{\alpha} = & \frac{1}{p}(g \sin(\alpha) + (1 + p\psi_{\beta} \sin(\alpha)) \cos(\alpha)\psi_{\beta}v^2 \\ & + c \cos(\alpha)(uv + \psi_{\beta}\tau)), \end{aligned} \quad (2.1)$$

where $\psi_{\beta} \triangleq \frac{\tan(\phi)}{b}$, ϕ is dealt with as an internal variable, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and v is the forwarding velocity.

Because we focus on dealing with the unstable dynamics in a short time interval, we assume the forwarding velocity v is a constant. The stabilization of the roll dynamics and the yaw dynamics is considered in

the paper and the position of the motorcycle is assumed to be determined by the forwarding velocity v and the yaw dynamics.

Under the above assumption, the simple motorcycle model (2.1) is reduced to

$$\dot{\beta} = \psi_{\beta}v$$

$$\dot{\psi}_{\beta} = u$$

$$\dot{\alpha} = \psi_{\alpha}$$

$$\begin{aligned} \dot{\psi}_{\alpha} = & \frac{1}{p}(g \sin(\alpha) + (1 + p\psi_{\beta} \sin(\alpha)) \cos(\alpha)\psi_{\beta}v^2 \\ & + c \cos(\alpha)vu), \end{aligned} \quad (2.2)$$

upon which the controller design can be carried out more easily.

Define a state vector

$$\mathbf{x} = (\beta, \psi_{\beta}, \alpha, \psi_{\alpha}). \quad (2.3)$$

We rewrite (2.2) as follows

$$\dot{\mathbf{x}} = f(\mathbf{x}, u). \quad (2.4)$$

Taking the Jacobian of nonlinear model (2.4) about the upper unstable equilibrium

$$(\beta_e, (\phi_{\beta})_e, \alpha_e, (\phi_{\alpha})_e) = (0, 0, 0, 0)$$

gives

$$A = \left. \frac{\partial f(\mathbf{x}, u)}{\partial \mathbf{x}} \right|_{\mathbf{x}=0, u=0}, \quad B = \left. \frac{\partial f(\mathbf{x}, u)}{\partial u} \right|_{\mathbf{x}=0, u=0}.$$

Thus, we obtain a linearized plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{2.5}$$

$$\text{where } A = \begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{v^2}{p} & \frac{g}{p} & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{cv}{p} \end{pmatrix}.$$

The linear control designs in Chapters 3 and 4 are based on the linearized system (2.5) and the nonlinear control design in Chapter 5 is based on the original dynamics (2.2).

Chapter 3

STATE VARIABLE FEEDBACK CONTROL

3.1 Overview

We design a state feedback control law by using pole placement technique and a dynamic output feedback control law by combining pole placement technique with an observer. The pole-placement method, a fundamental tool in state space design, is somewhat similar to the root-locus method in classical control theory because we place closed-loop poles at desired locations. The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations, while in the pole-placement design we place all closed-loop poles at desired locations. Because the linearized system of motorcycle is multi-variable, the state space approaches have some advantages over classical control design methods. In the state feedback design, we assumed that all state variables are available for feedback. In some cases, however, not all state variables are available for feedback. Then, we need to estimate unavailable state variables by designing a state observer, or simply an observer. We use a full-order state observer in the output feedback design here.

The controller presented here is for the purpose of reviewing the classical state space designs and comparison with other controllers in the sequel (e.g., nonlinear controller in Chapter 5).

The Chapter is organized as follows. In Section 3.2, we design a state feedback control law and a dynamic output feedback control law; simulations are carried out in Section 3.3; Section 3.4 summarizes this Chapter.

3.2 State space control designs

3.2.1 State Feedback Controller Design

We recall that the linearized model of simple motorcycle as follows,

$$\dot{x} = Ax + Bu, \quad (3.1)$$

$$\text{where } A = \begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{v^2}{p} & \frac{g}{p} & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{cv}{p} \end{pmatrix}.$$

We assume that all states are measurable and we will assume that some states are unmeasurable in next section. Actually, the design of this section is the first step of output feedback design in next section, where the estimated states replace the actual states here. We shall choose the state feedback gain as follows

$$u = -Kx \quad (3.2)$$

where K is the state feedback gain.

Let $p = 0.6(m)$, $c = 0.5(m)$, $b = 1.2(m)$, $v = 10(m/s)$ in our designs. In

$$\text{this case, } A = \begin{pmatrix} 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 166.7 & 16.3 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 8.3 \end{pmatrix}.$$

One can confirm that the unforced linear system is unstable by verifying that the characteristic polynomial is not Hurwitz, that is, $p(\lambda) = |\lambda I - A|$ is not Hurwitz. Our objective of designing a control law (3.2) is to make the polynomial $p(\lambda) = |\lambda I - (A - BK)|$ Hurwitz.

To apply pole-placement technique, we make sure that the system (3.1) is controllable. This can be verified by checking the rank of the controllability matrix

$$\text{rank} \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = 4. \quad (3.3)$$

which implies that the full rank condition is satisfied such that the system (3.1) is controllable.

To compare the transient response, we choose two sets of desired closed-loop poles P at

$$\text{Set 1 : } s = -10 + j4, \quad s = -10 - j4, \quad s = -20, \quad s = -5 \quad (3.4)$$

$$\text{Set 2 : } s = -20 + j4, \quad s = -20 - j4, \quad s = -4, \quad s = -1 \quad (3.5)$$

Of course, we can calculate the gains by hand. However, it is not necessary since we can use Matlab command to calculate the gains we repeat it here

$$K = \text{place}(A, B, P)$$

which gives the gain matrix

$$K = (-71.0204, -15.5102, 27.8800, 7.2612) \quad (3.6)$$

$$K = (-10.1878, 14.2722, 14.8387, 3.6873) \quad (3.7)$$

respectively.

Then, we can feed back the input (3.2) to linear system (3.1).

3.2.2 Output Feedback Controller Design

In state feedback design, we assumed that all state variables are available for feedback. In some cases, it is likely that only the position of the cart and the deflection angle of the motorcycle are measurable. Then, we need to estimate unavailable state variables and we are here to use the full-order state observer to do the output feedback design.

We proceed to the output feedback controller design through the following steps:

Step 1: Select pole location and develop the control law for the closed-loop system that corresponds to satisfactory dynamic response;

Step 2: Design an estimator;

Step 3: Combine the control law and the estimator.

Before carrying out the design, we reinterpret the linearized model (3.1) of simple motorcycle as follows,

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{3.8}$$

where $A = \begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{v^2}{p} & \frac{g}{p} & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{cv}{p} \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The definition C means that the states x_2 and x_4 are unmeasurable which have to be estimated by designing an estimator with the information of output y . To be consistent with the previous design, we choose the same values to v , c , p .

Step 1: Recall that we have employed the pole placement approach to the design of the system and the obtained gain for the desired closed loop poles $s = -20 + j4$, $s = -20 - j4$, $s = -4$, $s = -1$ are

$$K = (-10.1878, 14.2722, 14.8387, 3.6873).$$

In output feedback design, we suppose that the observed state feedback control is used instead of the actual-state feedback control, that is,

$$u = -K\hat{x} \tag{3.9}$$

where \hat{x} is the observed states to be design in next step and the gain matrix K is the same as above.

Step 2: Since we design a full order state observer, our approach is to

obtain the observer gain matrix K_e by solving pole placement of the dual system

$$\dot{z} = A^T z + C^T v, \quad (3.10)$$

where $v \in R^2$. Therefore, the dual system (3.10) is multiple input and multiple output system (the states z here). Our task is to determine the observer gain matrix K_e corresponding to the control law $v = -K_e^T z$ for system (3.10).

In the design of the state observer, it is desirable to determine several observer gain matrices K_e based on several different desired characteristic equations. For each of the several different matrices K_e , simulation tests must be run to evaluate the resulting system performance. Then, we select the best K_e from the viewpoint of overall system performance. In many practical cases, the selection of the best matrix K_e boils down to a compromise between speedy response and sensitivity to disturbances and noises. Considering that the desired closed loop poles for the state feedback control are $s = -20 + j4$, $s = -20 - j4$, $s = -4$, $s = -1$, we choose two sets of observer poles P_e to be at

$$\text{Set 1 : } s = -40 + j8, \quad s = -40 - j8, \quad s = -8, \quad s = -2 \quad (3.11)$$

$$\text{Set 2 : } s = -80 + j16, \quad s = -80 - j16, \quad s = -16, \quad s = -4 \quad (3.12)$$

Once again, we use Matlab command

$$K_e = \text{place}(A', C', P_e)'$$

to calculate the gain matrix

$$K_e = \begin{pmatrix} 43.8927 & 1.8849 \\ 17.7532 & -4.8797 \\ -6.7377 & 46.1073 \\ 423.7114 & 282.3271 \end{pmatrix} \quad (3.13)$$

$$K_e = \begin{pmatrix} 86.2 & 2.90 \\ 46.8 & -7.10 \\ -55.4 & 93.80 \\ -201.2 & 1222.90 \end{pmatrix} \quad (3.14)$$

for two sets of poles (3.11) and (3.12) respectively.

Step 3: We combine the gain matrices K and K_e acquired from step 1 and step 2 respectively to formulate a dynamic output feedback control law

$$u = -K\hat{x} \quad (3.15)$$

where the estimated states \hat{x} is governed by the dynamics

$$\dot{\hat{x}} = A\hat{x} - BK\hat{x} + K_e(y - C\hat{x}). \quad (3.16)$$

Hence, we obtain the closed loop system with respect to the linearized plant (3.10)

$$\begin{aligned} \dot{x} &= Ax - BK\hat{x} \\ y &= Cx. \end{aligned} \quad (3.17)$$

and the closed loop system with respect to actual plant (nonlinear plant) can be written as follows,

$$\begin{aligned}\dot{x} &= f(x, -K\hat{x}) \\ y &= Cx.\end{aligned}\tag{3.18}$$

3.3 Simulation

First, we carry out a simulation study for the closed loop system with the state feedback control law (3.2). With respect to the gains (3.6) and (3.7), we give a set of initial conditions:

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.1, 0.1, 0.1, 0.1).$$

Our first task is to select a better gain between (3.6) and (3.7). From Figure 3.1, we conclude that the closed loop system (3.1) with linear gain (3.7) performs better, which leads to less overshoot and smooth control force.

Our second task is to evaluate the linear control law with the selected gain matrix (3.7) via nonlinear plant (representing the real plant). From Figure 3.3, we conclude that the linear control law can stabilize the nonlinear system locally around the operating point with reasonable good performance. The reason is that the first approximation of the nonlinear dynamics is quite accurate in a small neighborhood about the operating point. However, in Chapter 5, we will show that the linear control law can not stabilize the nonlinear plant in a non-local region although the closed loop system of linear plant (3.1) perform very well in a local domain. This may motive us to seek some nonlinear controller for better performance.

Next, we carry out a simulation study for the closed loop systems (3.16)

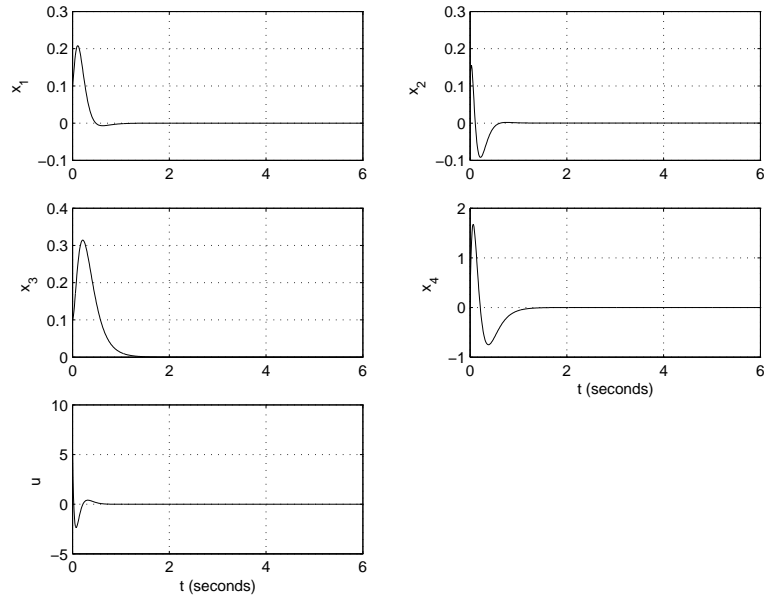


Figure 3.1: Simulation w.r.t case (3.6): $(0.1, 0.1, 0.1, 0.1)$ are initial conditions; the control force is not smooth enough; overshoots are large.

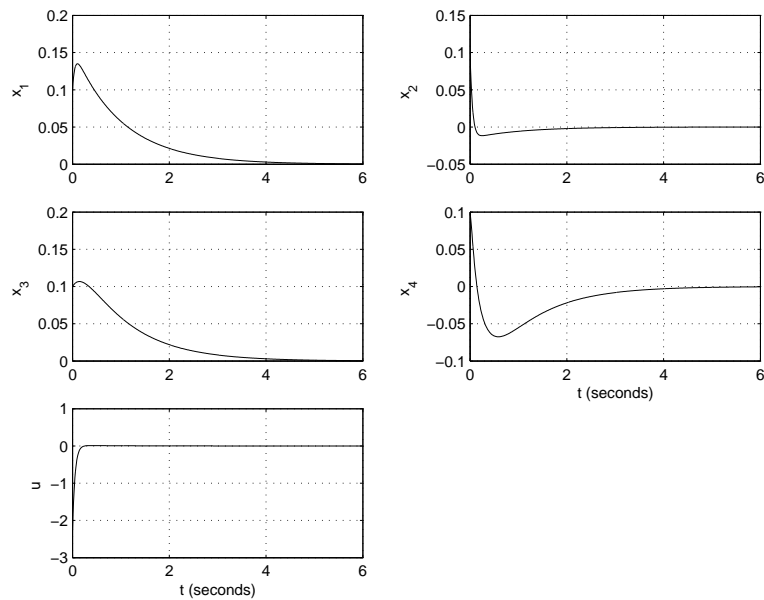


Figure 3.2: Simulation w.r.t case (3.7): $(0.1, 0.1, 0.1, 0.1)$ are initial conditions; the control force is very smooth; overshoots are small.

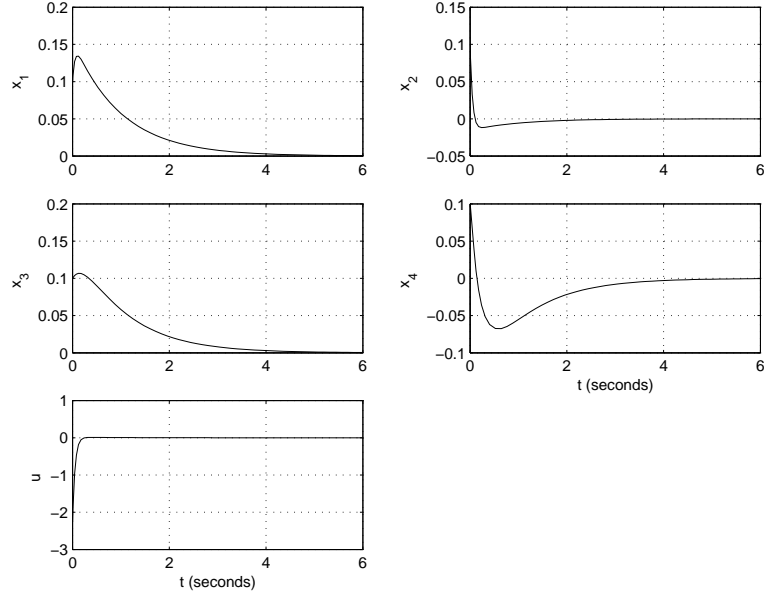


Figure 3.3: Simulation w.r.t case (3.7) based on the nonlinear plant: $(0.1, 0.1, 0.1, 0.1)$ are initial conditions; the nonlinear plant is also stabilized by the linear state feedback controller.

and (3.17) with the output feedback control law (3.15).

Let $e = x - \hat{x}$. With respect to the observer gains (3.10), (3.14), we give two sets of initial conditions:

$$(i) \quad (x_1(0), x_2(0), x_3(0), x_4(0), e_1(0), e_2(0), e_3(0), e_4(0)) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$$

$$(ii) \quad (x_1(0), x_2(0), x_3(0), x_4(0), e_1(0), e_2(0), e_3(0), e_4(0)) = (0.1, 0.1, 0.1, 0.1, 0.2, 0, 0.2, 0)$$

Our first task is to select a better gain between (3.13), (3.14). From Figure 3.4 to 3.7, we conclude that the closed loop system (3.16) with the observer gain (3.14) performs better.

Our second task is to evaluate the performance of the closed loop nonlinear plant (3.18) with respect to the selected gain matrix (3.7), the selected

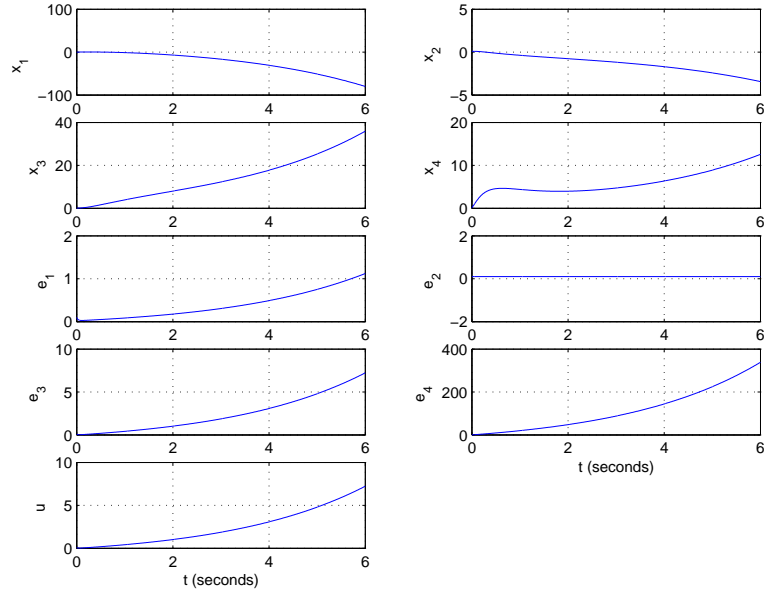


Figure 3.4: Simulation w.r.t case (3.13): $(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$ are initial conditions; some trajectory diverges; the observer gain is not appropriate.

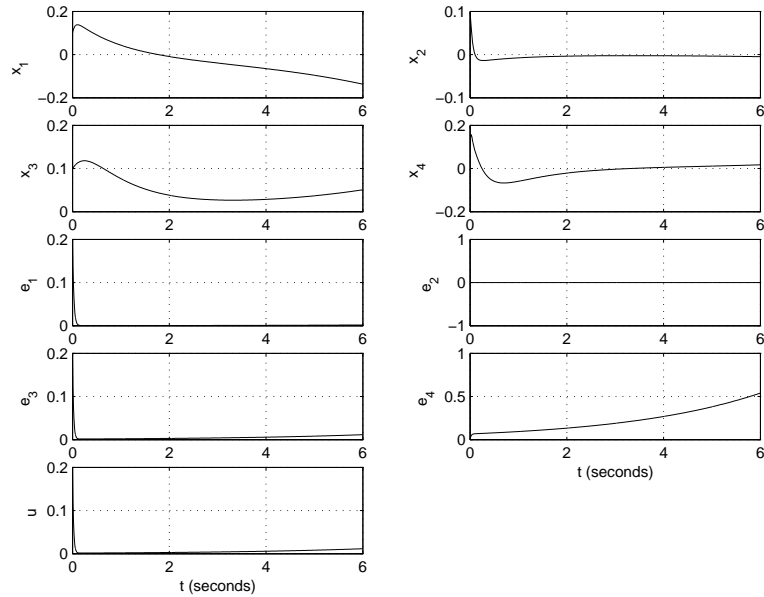


Figure 3.5: Simulation w.r.t case (3.13): $(0.1, 0.1, 0.1, 0.1, 0.2, 0, 0.2, 0)$ are initial conditions; some trajectory diverges; the observer gain is not appropriate.

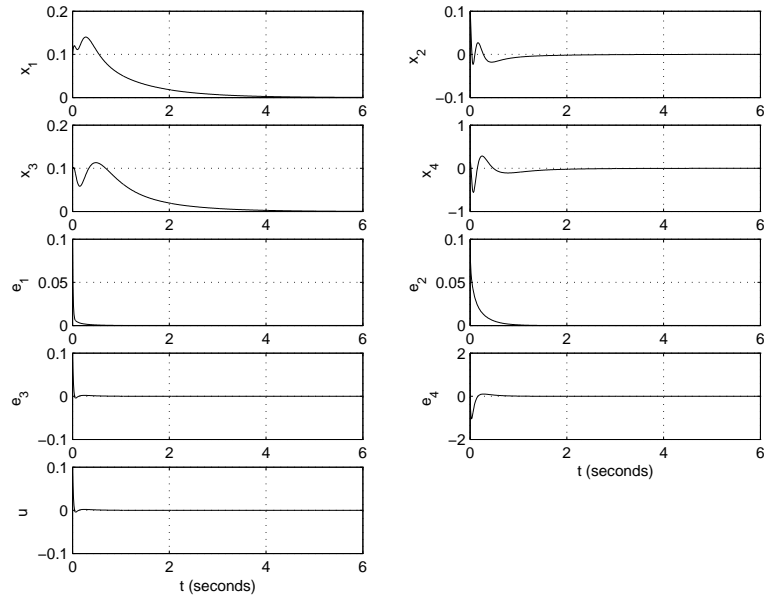


Figure 3.6: Simulation w.r.t case (3.14): $(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$ are initial conditions; all trajectory converges nicely; the observer gain is appropriate.

observer gain (3.14) and the dynamic output feedback law (3.15). From Figure 3.8, we conclude that the dynamic linear output feedback control law can stabilize the nonlinear system locally around the operating point with reasonable good performance. Certainly, this controller also holds locally.

3.4 Conclusion

We design a full state feedback law and a dynamic output feedback control law respectively for the linearized plant of the motorcycle. We take pole-placement technique as the design approach for the plant and the observer because this method is simple and can handle the motorcycle system which is multivariate. To select better closed loop eigenvalues of the closed loop system, we carry out a number of simulations. Then, the control laws

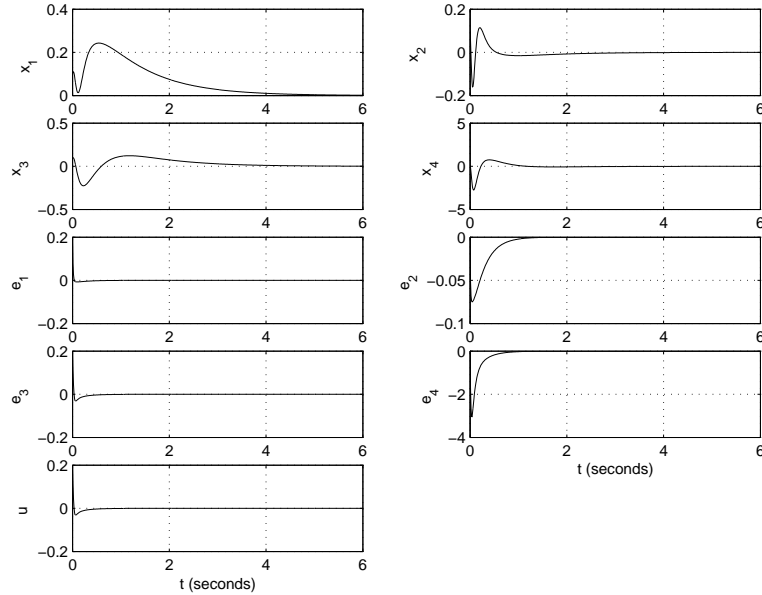


Figure 3.7: Simulation w.r.t case (3.14): $(0.1, 0.1, 0.1, 0.1, 0.2, 0, 0.2, 0)$ are initial conditions; all trajectory converges nicely; the observer gain is appropriate.

with selected state feedback gains and observer gains are applied to the nonlinear plant. Finally, the performance are evaluated through computer simulation. However, the approaches are not optimal and cannot deal with uncertainty and disturbance explicitly. The robust optimal controller will be given in Chapter 4. Furthermore, a linear control law yields only a limited domain (this will be shown in Chapter 5) where its performance is acceptable. Outside this domain, linear controllers are not effective anymore. To overcome the problem, we consider a nonlinear control design in Chapter 5.

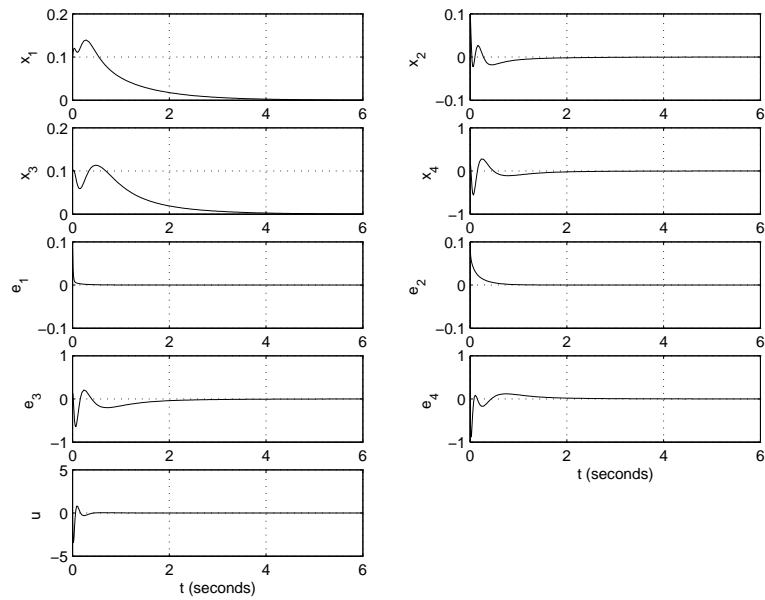


Figure 3.8: Simulation w.r.t case (3.14) based on the nonlinear plant: $(0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$ are initial conditions; the nonlinear plant is also stabilized by the linear output feedback controller.

Chapter 4

ROBUST H_∞ CONTROL

4.1 Overview

The Chapter presents a robust H_∞ state feedback control for the motorcycle which is subject to both time-varying mass uncertainty and exogenous disturbance such that the closed-loop system is quadratically stable and achieves a prescribed level of disturbance attenuation for all admissible parameter uncertainty. Although classical linear controllers in Chapter 3 also deliver certain robustness, the performance of the associated closed loop systems is not guaranteed essentially. The proposed controller in the Chapter will overcome the drawback and yields a good performance in the presence of uncertainty and disturbance.

The Chapter is organized as follows: in Section 4.2, we review some results on quadratic stability with disturbance attenuation; in Section 4.3, we give the motivation of robust control problem and design a robust H_∞ controller to the motorcycle; the effectiveness of the controller is evaluated through computer simulation in Section 4.4; finally, we conclude the work in Section 4.5.

4.2 Preliminary

Definition 4.2.1 [48] *Given a scalar $\gamma > 0$, the system*

$$\begin{aligned}\dot{\mathbf{x}} &= \bar{\mathbf{A}}\mathbf{x} + \mathbf{B}w \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}\tag{4.1}$$

where $\bar{\mathbf{A}} \in R^{n \times n}$, $\mathbf{B} \in R^{n \times q}$, $\mathbf{C} \in R^{p \times n}$, $x \in R^n$, $w \in R^q$ and $y \in R^p$, is said to be stable with disturbance attenuation γ if it satisfies the following conditions:

1. $\bar{\mathbf{A}}$ is a stable matrix;
2. the transfer function from disturbance w to controlled output y satisfies

$$\|\mathbf{C}(sI - \bar{\mathbf{A}})^{-1}\mathbf{B}\|_\infty < \gamma.\tag{4.2}$$

that is a H_∞ bound.

Lemma 4.2.2 [35] *Let $\gamma > 0$ be given. The system (4.1) is stable with disturbance attenuation γ if and only if there exists a symmetric matrix $P > 0$ such that*

$$\bar{\mathbf{A}}^T P + P\bar{\mathbf{A}} + \gamma^{-2} P\mathbf{B}\mathbf{B}^T P + \mathbf{C}^T \mathbf{C} < 0.\tag{4.3}$$

When there is parameter uncertainty $\Delta\bar{\mathbf{A}}(t) \in R^{n \times n}$ in the state matrix of (4.1), the system reads

$$\begin{aligned}\dot{\mathbf{x}} &= (\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}})\mathbf{x} + \mathbf{B}w \\ \mathbf{y} &= \mathbf{C}\mathbf{x}.\end{aligned}\tag{4.4}$$

The parameter uncertainty considered here are norm-bounded and of the

form

$$[\Delta\bar{\mathbf{A}}(t)] = [H][F(t)][E] \quad (4.5)$$

where $H \in R^{n \times i}$, $E \in R^{j \times m}$ corresponding the dimension of some variable $u \in R^m$ and $F \in R^{i \times j}$ that satisfies

$$F^T(t)F(t) \leq \rho^2 I \quad (4.6)$$

where the elements of $F(\cdot)$ being Lebesgue measurable and $\rho > 0$ a given constant.

Definition 4.2.3 [7] *The system (4.4) is said to be quadratically stable if there exists a positive definite symmetric matrix P such that for all admissible uncertainty $\Delta\bar{\mathbf{A}}(t)$, $t \in [0, \infty)$*

$$[\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}}]^T P + P[\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}}] < 0. \quad (4.7)$$

Incorporating Definition 4.2.3 with Lemma 4.2.2 leads to the notion of quadratic stability with disturbance attenuation.

Definition 4.2.4 [71] *Given a scalar $\gamma > 0$, the system (4.4) is said to be quadratically stable with disturbance attenuation γ if there exists a symmetric positive-definite matrix P such that for all admissible uncertainty $\Delta\bar{\mathbf{A}}(t)$, $t \in [0, \infty)$*

$$[\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}}]^T P + P[\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}}] + \gamma^{-2} P \mathbf{B} \mathbf{B}^T P + \mathbf{C}^T \mathbf{C} < 0, \quad (4.8)$$

the resulting closed-loop system is quadratically stable with disturbance attenuation γ .

The notion of quadratic stability with disturbance attenuation implies the following result.

Lemma 4.2.5 [70, 71] *Suppose the system (4.4) is quadratically stable with disturbance attenuation $\gamma > 0$. With zero-initial condition for $\mathbf{x}(t)$, $\|\mathbf{y}\|_2 < \gamma\|w\|_2$ holds for all admissible uncertainty $\Delta\bar{\mathbf{A}}(t)$, $t \in [0, \infty)$ and all nonzero $w \in L_2[0, \infty)$, where $\|\cdot\|_2$ denotes the usual $L_2[0, \infty)$ norm.*

The notion of quadratic stability with disturbance attenuation is a kind of robust H_∞ control, which treats both parameter uncertainty and disturbance input.

Next, we introduce a key result that is useful to solve the robust H_∞ control design associated with the quadratic stability with disturbance attenuation.

Proposition 4.2.6 [70, Lemma 3.1] *Let the constant $\gamma > 0$ be given. Then, there exists a matrix $P > 0$ such that*

$$[\bar{\mathbf{A}} + HF(t)E]^T P + P[\bar{\mathbf{A}} + HF(t)E] + \gamma^{-2}P\mathbf{B}\mathbf{B}^T P + C^T C < 0, \quad (4.9)$$

which is (4.8) with (4.5), for all $F(t)$ satisfying (4.6) if and only if there exists a constant $\epsilon > 0$ such that

$$\bar{\mathbf{A}}^T P + P\bar{\mathbf{A}} + \gamma^{-2}P\mathbf{B}\mathbf{B}^T P + \epsilon\rho^2 P\mathbf{H}\mathbf{H}^T P + \frac{1}{\epsilon}E^T E + C^T C < 0. \quad (4.10)$$

4.3 The model subject to uncertainty

We rewrite the linearized model of the motorcycle:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1 u + \mathbf{B}_2 w \quad (4.11)$$

where $\mathbf{x} \in R^4$ is the state vector, $u \in R$ is the control input, $w \in R^4$ is exogenous disturbance and the system matrix, the input matrix, and the disturbance input matrix are

$$\mathbf{A} = \begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{v^2}{p} & \frac{g}{p} & 0 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{cv}{p} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively.

In most cases, however, the forwarding speed $v(t) \in [\underline{v}, \bar{v}]$ with some constants \underline{v} and \bar{v} is a time varying variable. Furthermore, the dynamics of the body is likely influenced by some unknown exogenous disturbance (e.g., frictions). It is for this kind of problems that we propose a robust controller. Here, we consider $v(t) = v + \Delta v(t)$ where v is the normal speed and $\Delta v(t)$ is an uncertain extra speed. Furthermore, we assume that $|2v| \gg |\Delta v(t)|$. In this case, the model (4.11) can be rewritten as follows

$$\dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} + (\mathbf{B}_1 + \Delta\mathbf{B}_1)u + \mathbf{B}_2w \quad (4.12)$$

where the uncertainty input matrix, and the disturbance input matrix are¹

$$\Delta\mathbf{A} = \Delta v(t) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2v}{p} + \frac{\Delta v(t)}{p} & 0 & 0 \end{pmatrix},$$

¹Notice that $(v + \Delta v)^2/p = (v^2 + 2v\Delta v + (\Delta v)^2)/p$.

and

$$\Delta \mathbf{B} = \Delta v(t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\varepsilon}{p} \end{pmatrix}.$$

Because of $|2v| \gg |\Delta v(t)|$, without loss of generality, we let

$$\Delta \mathbf{A} = \Delta v(t) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2v}{p} & 0 & 0 \end{pmatrix}$$

by ignoring the parasite term $\frac{\Delta v(t)}{p}$ in the entry (4,2) of $\Delta \mathbf{A}$, that is, $\frac{2v}{p} + \frac{\Delta v(t)}{p} \rightarrow \frac{2v}{p}$.

4.4 Robust control design

Our goal is to find a linear state feedback controller in a form

$$u = K\mathbf{x} \tag{4.13}$$

where the feedback gain matrix $K \in R^2 \times R^8$ to be found.

The state space equation (4.12) with the controller (4.13) gives the closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}_1 K + \Delta \mathbf{A} + \Delta \mathbf{B}_1 K)\mathbf{x} + \mathbf{B}_2 w. \tag{4.14}$$

In the context, the matrix $\mathbf{A} + \mathbf{B}_1 K$ is Hurwitz.

Let $\bar{\mathbf{A}} \triangleq \mathbf{A} + \mathbf{B}_1 K$ and $\Delta\bar{\mathbf{A}} \triangleq \Delta\mathbf{A} + \Delta\mathbf{B}_1 K$. By these definitions, the closed loop system (4.14) takes the form

$$\dot{\mathbf{x}} = (\bar{\mathbf{A}} + \Delta\bar{\mathbf{A}})\mathbf{x} + \mathbf{B}_2 w. \quad (4.15)$$

First, let us consider the following system simplified from (4.15):

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \mathbf{B}_2 w. \quad (4.16)$$

Obviously, $\bar{\mathbf{A}}$ is a stable matrix by assumption. We are left to make inequality (4.2) satisfied for some given H_∞ bound γ , that is $\|(sI - \bar{\mathbf{A}})^{-1}B_2\|_\infty < \gamma$ ($C = I$).

Let

$$\Delta\bar{\mathbf{A}} \triangleq HF(t)E \quad (4.17)$$

where

$$H = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2v}{p} & 0 & 0 & \frac{c}{p} \end{array} \right) \in R^{4 \times 5}$$

$$F(t) = \Delta v \mathbf{I}_{5 \times 5} \in R^{5 \times 5}$$

and

$$E = \begin{pmatrix} \mathbf{I}_{4 \times 4} \\ K_{1 \times 4} \end{pmatrix} \in R^{5 \times 4}.$$

With the above preparation, we are ready for applying Proposition 4.2.6.

According to Proposition 4.2.6, the system (4.14) (or (4.15)), $C = \mathbf{I}$, with a so-called robust H_∞ control (4.13) is quadratically stable with disturbance attenuation $\gamma > 0$ if there exists a matrix $P > 0$ such that for all admissible uncertainty $\Delta \mathbf{B}_1 K$ (or (4.17))

$$[\bar{\mathbf{A}} + HF(t)E]^T P + P[\bar{\mathbf{A}} + HF(t)E] + \gamma^{-2} P \mathbf{B}_2 \mathbf{B}_2^T P + \mathbf{I} < 0, \quad (4.18)$$

or equivalently

$$\begin{aligned} [\mathbf{A} + \mathbf{B}_1 K + \Delta \mathbf{A} + \Delta \mathbf{B}_1 K]^T P + P[\mathbf{A} + \mathbf{B}_1 K + \Delta \mathbf{A} + \Delta \mathbf{B}_1 K] \\ + \gamma^{-2} P \mathbf{B}_2 \mathbf{B}_2^T P + \mathbf{I} < 0, \end{aligned} \quad (4.19)$$

if and only if there exists a constant $\epsilon > 0$ such that

$$\bar{\mathbf{A}}^T P + P \bar{\mathbf{A}} + \gamma^{-2} P \mathbf{B}_2 \mathbf{B}_2^T P + \epsilon \rho^2 P H H^T P + \frac{1}{\epsilon} E^T E + \mathbf{I} < 0. \quad (4.20)$$

or equivalently

$$\begin{aligned} (\mathbf{A} + \mathbf{B}_1 K)^T P + P(\mathbf{A} + \mathbf{B}_1 K) + \gamma^{-2} P \mathbf{B}_2 \mathbf{B}_2^T P + \\ \epsilon \rho^2 P H H^T P + \frac{1}{\epsilon} E^T E + \mathbf{I} < 0. \end{aligned} \quad (4.21)$$

Next, we shall find several variables: a symmetric matrix $P > 0$, a gain matrix K and a scalar $\epsilon > 0$ that solve (4.21). To this end, we convert the problem to some linear matrix inequality (LMI) systems and seek LMI tools (e.g., the associated tools in MATLAB) to solve the problem.

Notice that the left hand side of inequality (4.21) contains too many quadratic terms about the variables P , K and ϵ . We define some new variables

$$Q = P^{-1} \text{ and } Y = KQ, \quad (4.22)$$

where $Q = Q^T > 0$ as $P = P^T > 0$ implies $P^{-1} = (P^{-1})^T > 0$. Multiplying matrix Q to each term in inequality (4.21) from both left and right gives

$$\begin{aligned} Q(\mathbf{A} + \mathbf{B}_1 K)^T + (\mathbf{A} + \mathbf{B}_1 K)Q + \gamma^{-2} \mathbf{B}_2 \mathbf{B}_2^T + \\ \epsilon \rho^2 H H^T + \frac{1}{\epsilon} (Q Q + Q E^T E Q) + Q Q < 0, \end{aligned} \quad (4.23)$$

and substituting $Y = KQ$ and $E = K$ to (4.23) obtains

$$\begin{aligned} Q \mathbf{A}^T + Y \mathbf{B}_1^T + \mathbf{A} Q + \mathbf{B}_1 Y + \gamma^{-2} \mathbf{B}_2 \mathbf{B}_2^T + \\ \epsilon \rho^2 H H^T + \frac{1}{\epsilon} (Q Q + Y^T Y) + Q Q < 0. \end{aligned} \quad (4.24)$$

By Schur complement, inequality (4.24) is equivalent to

$$\begin{pmatrix} Q \mathbf{A}^T + Y \mathbf{B}_1^T + \mathbf{A} Q + \mathbf{B}_1 Y + \gamma^{-2} \mathbf{B}_2 \mathbf{B}_2^T + \epsilon \rho^2 H H^T & Y^T & Q & Q \\ & Y & -\epsilon \mathbf{I}_{1 \times 1} & \mathbf{0}_{1 \times 4} & \mathbf{0}_{4 \times 4} \\ & Q & \mathbf{0}_{4 \times 1} & -\mathbf{I}_{4 \times 4} & \mathbf{0}_{4 \times 4} \\ & Q & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} & -\epsilon \mathbf{I}_{4 \times 4} \end{pmatrix} < 0, \quad (4.25)$$

where \mathbf{I} is the identity matrix with appropriate dimensions and $\mathbf{0}$ is a matrix with appropriate dimensions whose entries are zero.

Finally, the static robust H_∞ control gain matrix

$$K = Q^{-1} Y \quad (4.26)$$

can be obtained by solving the following linear matrix inequalities (4.25), $\epsilon > 0$ and $Q > 0$, where Q , Y and ϵ are variables.

For example, let $p = 0.6(m)$, $c = 0.5(m)$, $v = 10(m/s)$, $|\Delta v(t)| < 0.4(m/s)$ and $g = 9.8$ be the parameters of the motorcycle. Because

$F(t)^T F(t) = \Delta v^2 \mathbf{I} \leq 2\mathbf{I}$, we let $\rho^2 = 2$ that ensure the inequality (4.6). Furthermore, let the H_∞ gain be $\gamma = 10$ that is a fairly reasonable number in this case as we assume that the magnitude of the exogenous disturbance is not too large.

Then, using LMI tools in MATLAB solves the linear matrix inequality (4.25) subject to $\epsilon > 0$ and $Q > 0$ (MATLAB codes are attached at Appendix C) and obtain $\epsilon = 0.9555$,

$$Y = \begin{pmatrix} 0.1408 & -1.1566 & -0.0762 & -6.9281 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.3684 & -0.0806 & 0.2339 & 0.6496 \\ -0.0806 & 0.1050 & -0.0318 & -0.5561 \\ 0.2339 & -0.0318 & 0.3581 & -0.4922 \\ 0.6496 & -0.5561 & -0.4922 & 6.5442 \end{pmatrix},$$

$$Q^{-1} = \begin{pmatrix} 59.5790 & -57.5943 & -65.6707 & -15.7474 \\ -57.5943 & 80.1709 & 69.0995 & 17.7267 \\ -65.6707 & 69.0995 & 76.7871 & 18.1659 \\ -15.7474 & 17.7267 & 18.1659 & 4.5886 \end{pmatrix}.$$

Finally, we obtain the gain matrix

$$K = Q^{-1}Y = \begin{pmatrix} 189.1086 & -228.9145 & -220.8756 & -55.8949 \end{pmatrix}, \quad (4.27)$$

such that $u = K\mathbf{x}$ is a static robust H_∞ controller (Because in this chapter, we assume that all states are measurable, we did not use the observer.)

4.5 Simulation

Case 1: We consider the uncertainty $\Delta v(t)$ with respect to time t as shown

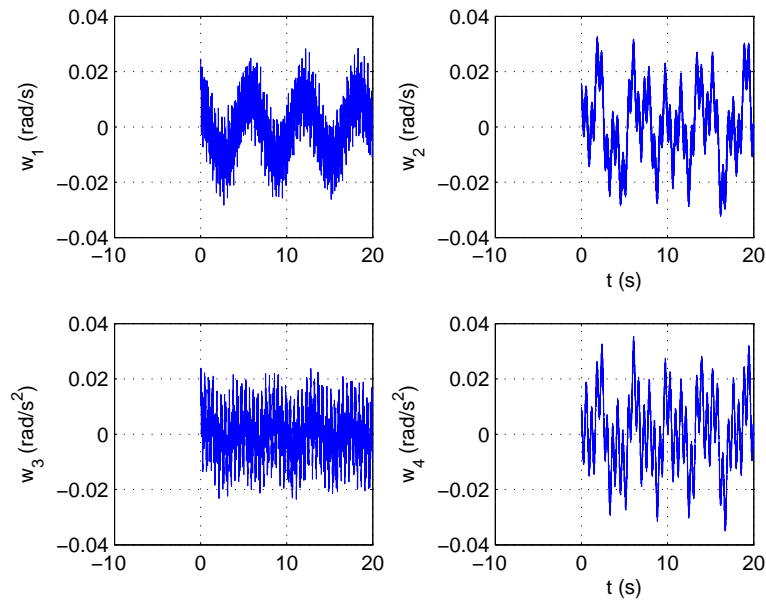


Figure 4.1: Disturbance w in case 1: the disturbance is a collection of exogenous forces that is bounded; the disturbance is not necessarily Gaussian.

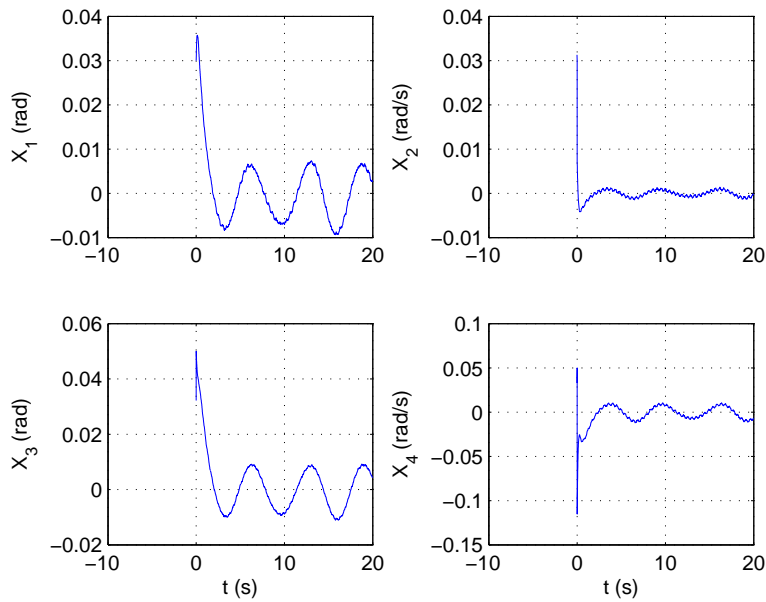


Figure 4.2: State response $\mathbf{x}(t)$ w.r.t uncertainty $\Delta v(t)$ and $w(t)$ in case 1: the trajectory converges to a neighborhood of the origin; the disturbance is rejected.

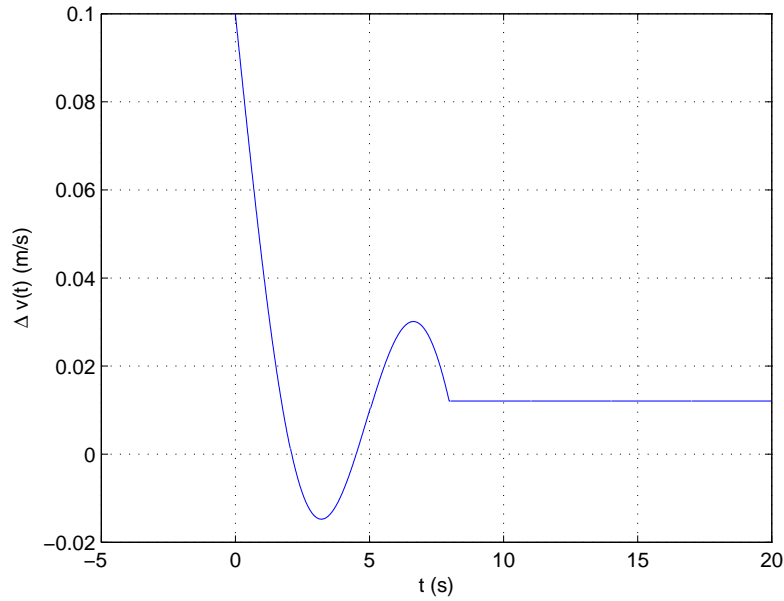


Figure 4.3: Uncertainty $\Delta v(t)$ in case 1: the uncertain forward speed is a function of time.

in Figure 4.3 and the exogenous input illustrated in Figure 4.1. Again, let initial condition be

$$\mathbf{x}(0) = [0.03(\text{rad}); 0.03(\text{rad/s}); 0.05(\text{rad}); 0.05(\text{rad/s})].$$

Figure 4.2 shows the state response. This simulation reflects both the quadratic stability and disturbance attenuation of the system subject to input matrix uncertainty and exogenous input.

Case 2: We consider some larger uncertainty $\Delta v(t)$ and large exogenous input illustrated in Figures 4.4 and 4.5. Let initial condition be

$$\mathbf{x}(0) = [0.03(\text{rad}); 0.03(\text{rad/s}); 0.05(\text{rad}); 0.05(\text{rad/s})].$$

Figure 4.6 shows the state response. This simulation illustrates very good robustness yielded by the controller.

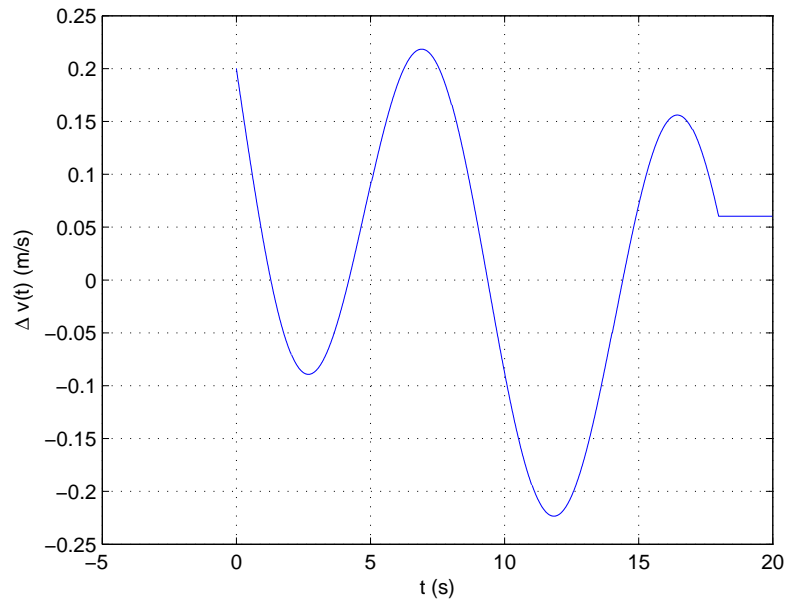


Figure 4.4: Uncertainty $\Delta v(t)$ in case 2: the uncertain forward speed is a function of time.

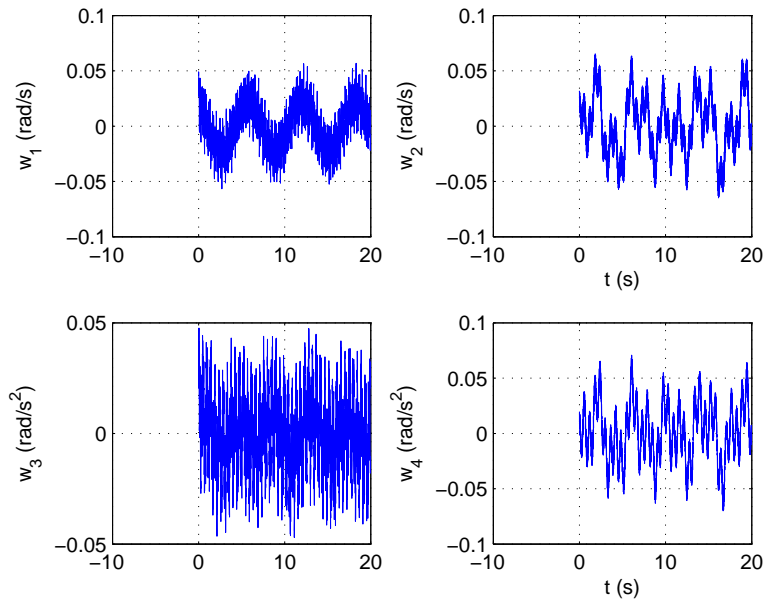


Figure 4.5: Disturbance w in case 2: the disturbance is a collection of exogenous forces that is bounded; the disturbance is not necessarily Gaussian.

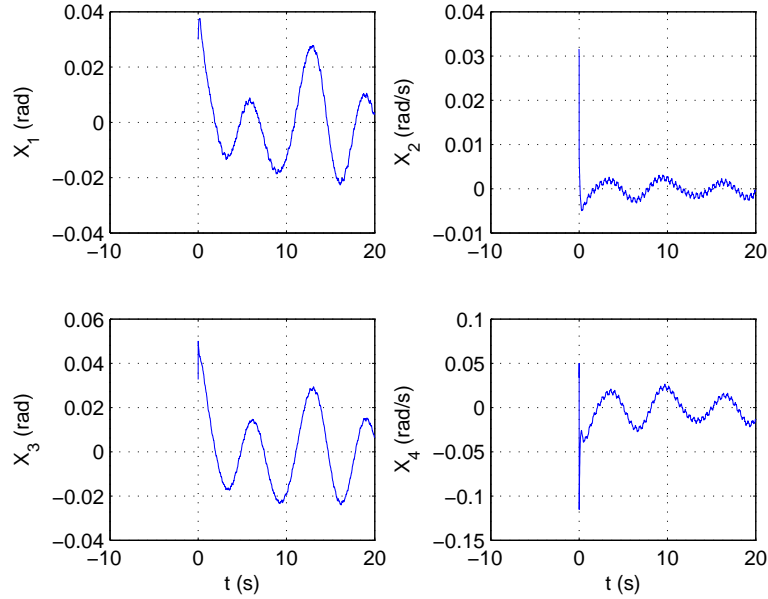


Figure 4.6: State response $\mathbf{x}(t)$ w.r.t uncertainty $\Delta v(t)$ and $w(t)$ in case 2: the trajectory converges to a neighborhood of the origin; the disturbance is rejected.

Case 3: We apply the proposed controller to the nonlinear plant that represents the real plant. We consider additive measurement noises to the states \mathbf{x} as is shown in Figure 4.7. Furthermore, we consider the uncertainty velocity $\Delta v(t)$ as is depicted in Figure 4.8. Let initial condition be

$$\mathbf{x}(0) = [0.3(rad); 0.2(rad/s); 0.3(rad); 0.2(rad/s)].$$

Figure 4.9 shows the state response of the closed loop response. This simulation verifies the robustness of the controller to the nonlinear perturbed plant.

Case 4: We do not consider additive measurement noises and the uncer-

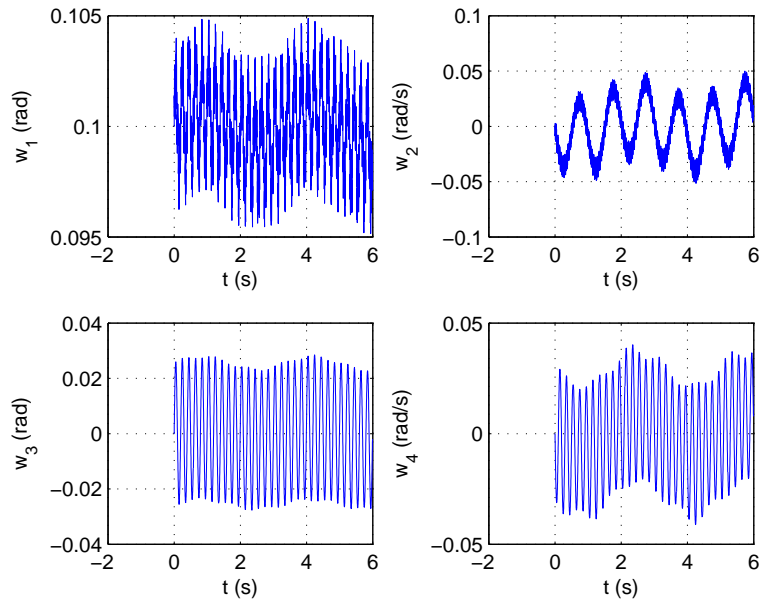


Figure 4.7: Disturbance w in case 3: the disturbance is a collection of exogenous forces that is bounded; the disturbance is not necessarily Gaussian.

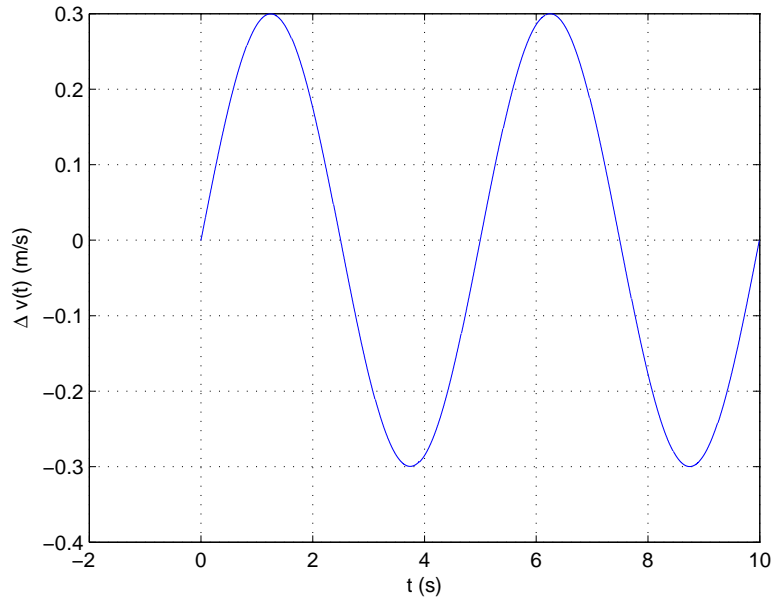


Figure 4.8: Uncertainty speed $\Delta v(t)$ in case 3: the uncertain forward speed is a function of time.

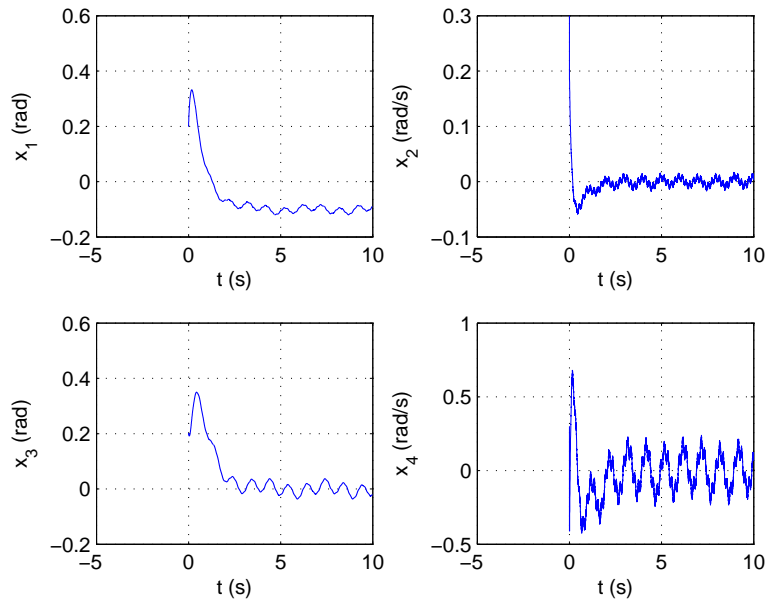


Figure 4.9: State response $\mathbf{x}(t)$ w.r.t uncertainty $\Delta v(t)$ and $w(t)$ in case 3 (nonlinear plant) : the trajectory converges to a neighborhood of the origin; the disturbance is rejected.

tainty velocity $\Delta v(t)$. Let initial condition be

$$\mathbf{x}(0) = [1(\text{rad}); 5(\text{rad/s}); 1(\text{rad}); 5(\text{rad/s})].$$

Figures 4.10 shows the blowing up of the trajectory, that is, the motorcycle is falling down. This simulation shows that, although the proposed linear controller is optimal and robust, it yields also a limited domain of attraction as the linear controllers in Chapter 3 did because all the linear controllers that are relying on the first approximation of the nonlinear model are local controllers.

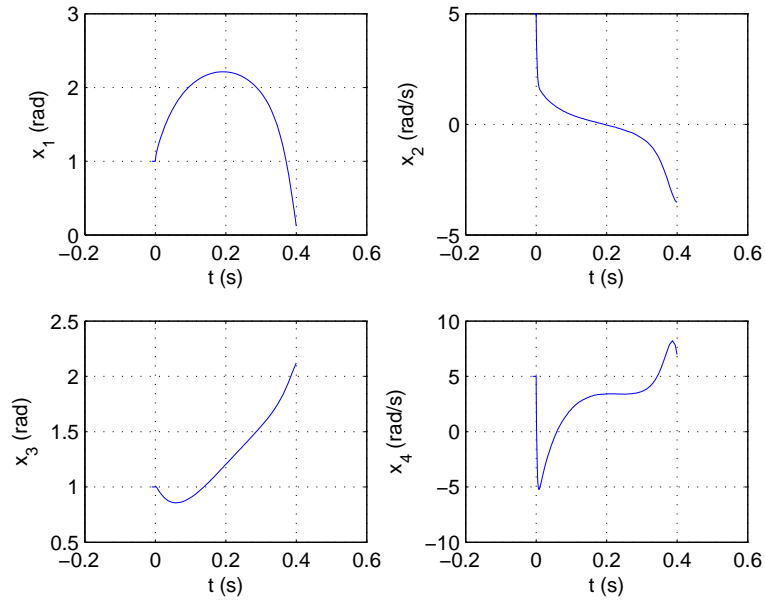


Figure 4.10: Blowing-up of the trajectory $\mathbf{x}(t)$ due to large initial values in case 4 (nonlinear plant)

4.6 Summary

The proposed robust H_∞ control of the motorcycle deals very well with both the fairly large slow time-varying mass uncertainty and large exogenous forces. The simulation verifies the result. Therefore, the good performance of closed loop system is guaranteed. The result is novel. However, the controller also yields a limited domain of attraction as those in Chapter 3. In Chapter 5, we put our efforts to nonlinear control design for the nonlinear plant to achieve much larger domain of attraction.

Chapter 5

NONLINEAR FORWARDING

5.1 Overview

Linear controllers are designed based on the linearization of a nonlinear system about some operating point which only locally regulate the nonlinear plant at a small neighborhood of the operating point. The controllers developed in previous chapters are based on the first approximation of the nonlinear dynamics about the upper unstable equilibrium. Therefore, the static state feedback controller, the dynamic output feedback controller and even linear robust control scheme only regulate the motorcycle locally around the equilibrium.

It is desirable to explore some non-local stabilization because the dynamics of a motorcycle is nonlinear itself. The non-local control of the motorcycle is challenging because the controlled plant is nonlinear, unstable, and underactuated. To the best of our knowledge, there is no result in the literature which achieves the non-local stabilization of the motorcycle.

In this chapter, we propose an advanced nonlinear control law for the motorcycle which yields the domain of attraction up to the upper half

space. In the control design, we first explore some nonlinear state and input transformations to identify an appropriate upper-triangular form and then combine high gains with the low gains where the low gains are obtained by applying Teel's nested saturating design. The performance of the controller is evaluated through computer simulation in comparison with a linear controller. A large domain of attraction yielded by the nonlinear controller can be observed in the simulation results.

The remaining of the chapter is organized as follows: in section II, we briefly review the theories; in section III, we present our main result—the nonlinear control design; in section IV, we show some simulation results; finally, we conclude the chapter in section V.

5.2 Preliminary

5.2.1 Some Useful Concepts

We review the concept of class \mathcal{K} function. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$, the function is said to belong to class \mathcal{K}_∞ .

$$\text{A saturation function } \sigma(s) \triangleq \begin{cases} 1, & s > 1 \\ s, & |s| \leq 1 \\ -1, & s < -1 \end{cases}$$

\mathbb{C}^- denotes the left hand side of the complex plane. We use the concept of “asymptotic gain” (see [65, 26]), which considers only bounds on the asymptotic behavior of the response, as $t \rightarrow \infty$. For a piecewise-continuous

function $u : [0, \infty) \rightarrow \mathbf{R}^m$, define $\|u(\cdot)\|_a = \limsup_{t \rightarrow \infty} \{\max_{1 \leq i \leq m} |u_i(t)|\}$. The quantity thus introduced is referred to as the asymptotic “norm” of $u(\cdot)$. The “asymptotic gain” is used to formulate the nested saturating design tool.

We review a key nonlinear analysis tool, input-to-state stability (ISS) and ISS-Lyapunov function discovered by Sontag et al in [53, 55, 59]. For details of those results, readers can refer to the monograph [26, Chapter 10].

ISS : system

$$\dot{x} = f(x, u) \quad (5.1)$$

is said to be ISS if there exist a class \mathcal{KL} function $\beta(\cdot, \cdot)$ and a class \mathcal{K} function $\gamma(\cdot)$, such that for any input $u(\cdot) \in L_\infty^m$ and any $x(0) \in \mathbf{R}^n$, the response $x(t)$ satisfies $\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_\infty)$.

ISS-Lyapunov function: A C^1 function V is called an ISS-Lyapunov function for system (5.1) if there exist class \mathcal{K}_∞ functions $\underline{a}(\cdot)$, $\bar{a}(\cdot)$, $a(\cdot)$ and a class \mathcal{K} function $\chi(\cdot)$ such that

$$\underline{a}(\|x\|) \leq V(x) \leq \bar{a}(\|x\|) \text{ for all } x \in \mathbf{R}^n \quad (5.2)$$

and

$$\|x\| \geq \chi(\|u\|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leq -a(\|x\|) \text{ for all } x \in \mathbf{R}^n. \quad (5.3)$$

Throughout the paper, the initial conditions are defined as $x_1^\circ \triangleq x_1(0) \in X_1$ and $(x_1, x_2) \triangleq (x_1^T, x_2^T)^T$ is used for convenience.

5.2.2 Input-output Feedback Linearization of MIMO

We recall the method of input-output feedback linearization of MIMO systems [52, Ch.6]. Consider the system

$$\dot{x} = f(x) + G(x)u, \quad y = h(x), \quad (5.4)$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input vector, $y \in R^m$ vector of system outputs, f , h and g_i are smooth vector fields. Assume that r_i is the smallest integer such that at least one of the inputs appears in $\frac{d^{r_i}y_i}{dt^{r_i}}$ for each output y_i . This yields

$$\begin{aligned} \begin{pmatrix} \frac{d^{r_1}y_1}{dt^{r_1}} \\ \vdots \\ \frac{d^{r_m}y_m}{dt^{r_m}} \end{pmatrix} &= \begin{pmatrix} \mathcal{L}_f^{r_1}h_1(x) \\ \vdots \\ \mathcal{L}_f^{r_m}h_m(x) \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^m \mathcal{L}_{g_j}\mathcal{L}_f^{r_1-1}h_j(x)u_j \\ \vdots \\ \sum_{j=1}^m \mathcal{L}_{g_j}\mathcal{L}_f^{r_m-1}h_j(x)u_j \end{pmatrix} \\ &\triangleq \mathcal{L}_f^r h(x) + E(x)u, \end{aligned} \quad (5.5)$$

where $\mathcal{L}_{g_j}\mathcal{L}_f^{r_i-1}h_j(x)u_j \neq 0$, $i = 1, \dots, m$ for at least one j , in a neighborhood χ_i of the point x_0 . Then, the system (5.4) is said to have a vector relative degree (r_1, \dots, r_m) at x_0 . Define χ as the intersection of the χ_i and assume $E(x)$ is invertible over the region χ . Then, the input transformation

$$u = E^{-1}(x)(v - \mathcal{L}_f^r h(x)), \quad (5.6)$$

yields m equations of the simple form

$$\frac{d^{r_i}y_i}{dt^{r_i}} = v_i, \quad (5.7)$$

that is the system is input-output linearized.

5.2.3 Stabilisation of Systems in Upper Triangular Form

We consider systems described by equations having an upper-triangular structure [65, 45, 29, 26]:

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u) \\
 \dot{x}_2 &= f_2(x_2, x_3, \dots, x_n, u) \\
 &\dots \\
 \dot{x}_{n-1} &= f_{n-1}(x_{n-1}, x_n, u) \\
 \dot{x}_n &= f_n(x_n, u), \tag{5.8}
 \end{aligned}$$

in which the functions $f_i(x_i, x_{i+1}, \dots, x_n, u)$ are supposed to satisfy appropriate hypotheses, which will be introduced in the sequel. These systems are often referred to as systems in feedforward form or forwarding form, since they correspond to a cascade interconnection of n subsystems starting with the x_n subsystem of (5.8) and ending with the x_1 subsystem of (5.8), in which the x_i subsystem is fed by the “outputs” x_{i+1}, \dots, x_n of all previous subsystems in the cascade.

Because of this triangular structure, the design of stabilizing feedback law can be achieved in a recursive way. Two ideas are exploited: saturation functions as proposed in [65] (see also Chapter 14 in [26]) and control Lyapunov function [45, 29]) (see Chapter 6 in [49]). Suppose that the feedback law $u = \alpha_n(x_n)$ stabilizes the subsystem

$$\dot{x}_n = f_n(x_n, u), \tag{5.9}$$

and replace u by $\alpha_n(x_n) + u$ (some abuse of notation). Consider now the

subsystem formed by the last two equations, which take the form

$$\begin{aligned}
\dot{x}_{n-1} &= f_{n-1}(x_{n-1}, x_n, \alpha_n(x_n) + u) \\
&\triangleq \tilde{f}_{n-1}(x_{n-1}, x_n, u) \\
\dot{x}_n &= f_n(x_n, \alpha_n(x_n) + u) \\
&\triangleq \tilde{f}_n(x_n, u).
\end{aligned} \tag{5.10}$$

Now suppose that $\alpha_{n-1}(x_n, x_{n-1})$ stabilises (5.10) and then replace u in (5.8) by $\alpha_n(x) + \alpha_{n-1}(x_n, x_{n-1}) + u$.

This whole process can be repeated to construct a state feedback law that stabilise (5.8) provided

a we know how to stabilize the x_n subsystem,

b we know how to stabilize a system of the form

$$\begin{aligned}
\dot{z} &= \psi(z, x, u) \\
\dot{\xi} &= \phi(\xi, u),
\end{aligned} \tag{5.11}$$

given that the lower subsystem is Lyapunov stable when the input is zero.

The Lyapunov function based design method for the forwarding structure [49, 45] requires to find the solution of some general PDE in each recursive step. In this thesis, we employ an alternative approach for the forwarding structure, that is, the nested saturating design proposed by [65].

5.2.4 Nested Saturating Design

Consider the forwarding system of the form (see [65, 26] for details):

$$\begin{aligned}
 \dot{x}_1 &= A_1 x_1 + g_1(x_2, \dots, x_n, u) \\
 \dot{x}_2 &= A_2 x_2 + g_2(x_3, \dots, x_n, u) \\
 &\dots \\
 \dot{x}_{n-1} &= A_{n-1} x_{n-1} + g_{n-1}(x_n, u) \\
 \dot{x}_n &= f_n(x_n, u).
 \end{aligned} \tag{5.12}$$

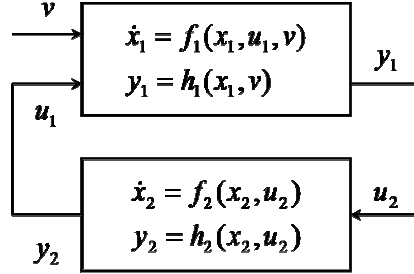
For a system having inputs and outputs, modelled by equations of the form

$$\begin{aligned}
 \dot{x} &= f(x, u) \\
 y &= h(x, u),
 \end{aligned} \tag{5.13}$$

with $x \in R^n$, $u \in R^m$, $y \in R^p$, and $f(0, 0) = 0$, $h(0, 0) = 0$, the notion of asymptotic “gain” is defined next .

Definition 5.2.1 [65, 26] System (5.13) is said to satisfy an asymptotic (input-output) bound, with restriction X on x° and restriction U on $u(\cdot)$, if there exists a class \mathcal{K} function $\gamma_u(\cdot)$, called the gain function, such that, for any $x^\circ \in X$ and for any piecewise-continuous input $u(\cdot)$ satisfying $\|u(\cdot)\|_a < U$, the response $x(t)$ in the initial state $x(0) = x^\circ$ exists for all $t \geq 0$ and is such that $\|y(\cdot)\|_a \leq \gamma_u(\|u(\cdot)\|_a)$.

Suppose now that a nested system shown in Figure (5.1) satisfies an asymptotic input-output bound, with restriction x_1 on x_1° , restriction U_1 on $u_1(\cdot)$ and restriction V on $v(\cdot)$, that is, suppose there exist gain functions $\gamma_1(\cdot)$ and $\gamma_v(\cdot)$ such that, for any $x_1^\circ \in X_1$, the response $x_1(\cdot)$ to piecewise-

Figure 5.1: Feedback connection with input v

continuous $u_1(\cdot)$ and $v_1(\cdot)$ (each one satisfying the indicated restriction) exists for all $t \geq 0$ and

$$\|y_1(\cdot)\|_a \leq \max\{\gamma_1(\|u_1(\cdot)\|_a), \gamma_v(\|v(\cdot)\|_a)\}.$$

Likewise, suppose that the system satisfies an asymptotic input-output bound, with restriction X_2 on x_2° and restriction U_2 on $u_2(\cdot)$, that is, suppose there exist gain functions $\gamma_2(\cdot)$ such that, for any $x_1^\circ \in X_2$, the response $x_2(t)$ to piecewise-continuous $u_2(\cdot)$ (satisfying the indicated restriction) exists for all $t \geq 0$ and $\|y_2(\cdot)\|_a \leq \gamma_2(\|u_1(\cdot)\|_a)$.

The following result shows that, if the gain functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ satisfy the small gain condition, the interconnected system (5.11) satisfies an asymptotic input-output bound, with appropriate restrictions.

Theorem 5.2.2 [65, 26] Consider the interconnected system in Figure (5.1) and suppose both subsystems satisfy asymptotic input-output

bounds with restrictions and gain functions as indicated above. Suppose $U_1 = \infty$. Suppose also that $\lim_{r \rightarrow \infty} \gamma_1(r) < \infty$, $\lim_{r \rightarrow \infty} \gamma_2(r) < U_2$. Let \tilde{V} be any number satisfying $\tilde{V} \leq V$, $\gamma_v(\tilde{V}) \leq U_2$. Suppose that, for all $(x_1^\circ, x_2^\circ) \in X_1 \times X_2$ and any piecewise-continuous $v(\cdot)$ satisfying $\|v(\cdot)\|_a \leq \tilde{V}$, the responses $(x_1(t), x_2(t))$ exist for all $t \geq 0$.

Then, if $\gamma_1 \circ \gamma_2(r) < r$ for all $r > 0$, the interconnected system satisfies an asymptotic input-output bound, with restriction $X_1 \times X_2$ on (x_1°, x_2°) , restriction \tilde{V} on $v(\cdot)$ and $\|y_1(\cdot)\|_a \leq \gamma_v(\|v(\cdot)\|_a)$, $\|y_2(\cdot)\|_a \leq \gamma_2 \circ \gamma_v(\|v(\cdot)\|_a)$.

This particular version of the small-gain theorem will be used to prove the desired stabilisation result for system (5.12). To this end, an auxiliary property is needed.

Corollary 5.2.3 [65, 26] Consider the linear system $\dot{x} = \mathbf{A}x + \mathbf{B}u$. Suppose (\mathbf{A}, \mathbf{B}) is stabilisable and there exists a symmetric matrix $P > 0$ such that $\mathbf{A}^T P + P \mathbf{A} \leq 0$ holds. Then, the matrix $\mathbf{A} - \mathbf{B} \mathbf{B}^T P$ has all eigenvalues in C^- . Let $\sigma(\cdot)$ be any \mathbf{R}^m -valued saturation function and consider the system

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}\sigma(-\mathbf{B}^T P x + v) + w \\ y &= x \end{aligned} \tag{5.14}$$

Then, there exists a number $\delta' > 0$, such that (5.14) satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$.

This corollary means that there exists a matrix \mathbf{K} such that $\mathbf{A} + \mathbf{B} \mathbf{K}$ has

all eigenvalues in C^- and such that

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}\sigma(\mathbf{K}x + v) + w \\ y &= x \end{aligned} \tag{5.15}$$

satisfies an asymptotic (input-output) bound.

The derivation of the next result describes how a control law for a system of the form

$$\begin{aligned} \dot{z} &= \mathbf{A}z + \mathbf{B}u + g(\xi, u) \\ \dot{\xi} &= f(\xi, u). \end{aligned} \tag{5.16}$$

can be effectively designed, simply using linear functions and saturation functions.

Theorem 5.2.4 [65, 26] Consider the system (5.16), in which $z \in \mathbb{R}^n, \xi \in \mathbb{R}^v, u \in \mathbb{R}^m$, $g(\xi, u)$ and $f(\xi, u)$ are locally Lipschitz, and $g(0, 0) = 0, f(0, 0) = 0$. Assume that:

1. (\mathbf{A}, \mathbf{B}) is stabilisable and $P\mathbf{A} + \mathbf{A}^T P \leq 0$ for some symmetric matrix $P > 0$
2. the system

$$\begin{aligned} \dot{\xi} &= f(\xi, u) \\ y &= \xi. \end{aligned} \tag{5.17}$$

satisfies an asymptotic (input-output) bound, with restriction Ξ on ξ° and restriction $U > 0$ on $u(\cdot)$, with linear gain function.

3. the function $g(\xi, u)$ is such that

$$\lim_{\|(\xi, u)\| \rightarrow 0} \frac{\|g(\xi, u)\|}{\|(\xi, u)\|} = 0.$$

Let $\sigma(\cdot)$ be any R^m -valued saturation function. Pick an $n \times m$ matrix K such that $A + BK$ has all eigenvalues in C^- and, for some $\delta' > 0$, system satisfies an asymptotic (input-output) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions. Pick two $m \times m$ matrices Γ and Ω . Then, there exist numbers $\lambda > 0$ and $v > 0$ such that the system with control

$$u^*(z, v) = \lambda \sigma \left(\frac{\mathbf{K}z + \Gamma v}{\lambda} \right) + \Omega v, \quad (5.18)$$

and output $y = (z, \xi)$ satisfies an asymptotic (input-output) bound, with restriction $R^n \times \Xi$ on (z°, ξ°) and restriction V on $v(\cdot)$, with linear gain function $\gamma_v(\cdot)$.

The result of above theorem can also be used for the purpose of recursively stabilizing systems in feedforward form (5.12). Then, the main result for nested saturating design is stated as follows.

Theorem 5.2.5 [65, 26] Consider the system

$$\begin{aligned} \dot{z} &= \mathbf{A}x + g_i(\xi_i, u) \\ \dot{\xi}_i &= f_i(\xi_i, u). \end{aligned} \quad (5.19)$$

in which $z \in R^n$, $\xi_i \in R^v$, $u \in R^m$, $g_i(\xi_i, u)$ and $f_i(\xi_i, u)$ are locally Lipschitz, differentiable at $(\xi_i, u) = (0, 0)$, and $g_i(0, 0) = 0$, $f_i(0, 0) = 0$. Assume that:

- (i) there exists a symmetric matrix $P > 0$ such that $PA + A^T P \leq 0$,
- (ii) the linear approximation of (5.19) at the equilibrium $(z_i, \xi_i, u) = (0, 0, 0)$ is stabilisable.

Moreover, assume that there exists a function

$$\begin{aligned} \alpha_i : R^v \times R^m &\rightarrow R^m \\ (\xi_i, v) &\mapsto \alpha_i(\xi_i, v), \end{aligned}$$

with $\alpha_i(0, 0) = 0$, which is locally Lipschitz, differentiable at $(\xi_i, v) = (0, 0)$, with the following properties:

- (iiia) the matrix $\left[\frac{\partial \alpha_i(\xi_i, v)}{\partial v} \right]_{(0,0)}$ is nonsingular,
- (iiib) the matrix $\left[\frac{\partial f_i(\xi_i, \alpha_i(\xi_i, v))}{\partial \xi_i} \right]_{(0,0)}$ has all eigenvalues in C^- ,
- (iiic) the system

$$\begin{aligned} \dot{\xi}_i &= f_i(\xi_i, \alpha_i(\xi_i, v)) \\ y &= \xi_i \end{aligned}$$

satisfies an asymptotic (input v to output y) bound, with restriction ξ_i on ξ_i° , restriction $V > 0$ on $v(\cdot)$, with linear gain function $\gamma_v(\cdot)$.

Set $\xi_{i+1} = (z, \xi_i)$, $\tilde{v} = n + v$, $f_{i+1}(\xi_{i+1}, u) = \begin{pmatrix} Az + g_i(\xi_i, u) \\ f_i(\xi_i, u) \end{pmatrix}$, and $F_{i+1} = \left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_i(\xi_i, v))}{\partial \xi_{i+1}} \right]_{(0,0)}$, $G_{i+1} = \left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_i(\xi_i, v))}{\partial v} \right]_{(0,0)}$. Then, the pair (F_{i+1}, G_{i+1}) satisfies the hypotheses of Corollary (5.2.3). Let $\sigma(\cdot)$ be any R^m -valued saturation function. Pick a $\tilde{v} \times m$ matrix K_{i+1} such that

$(F_{i+1} + G_{i+1}K_{i+1})$ has all eigenvalues in C^- and, for some $\delta' > 0$, system

$$\begin{aligned} \dot{x} &= F_{i+1}x + G_{i+1}\sigma(K_{i+1}x + v) + w \\ y &= x \end{aligned} \tag{5.20}$$

satisfies an asymptotic (input (v, w) to output y) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$. Pick two $m \times m$ matrices Γ and Ω such that $\Gamma + \Omega$ is nonsingular. Consider the function

$$\begin{aligned} \alpha_{i+1} : R^{\tilde{v}} \times R^m &\rightarrow R^m \\ (\xi_i, v) &\mapsto \alpha_i \left(\xi_i, \lambda \sigma \left(\frac{K_{i+1}\xi_{i+1} + \Gamma v}{\lambda} \right) + \Omega v \right), \end{aligned}$$

Then, there exist numbers $\lambda > 0$ and $\tilde{v} > 0$ such that

- (a) the matrix $\left[\frac{\partial \alpha_{i+1}(\xi_{i+1}, v)}{\partial v} \right]_{(0,0)}$ is nonsingular,
- (b) the matrix $\left[\frac{\partial f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v))}{\partial \xi_{i+1}} \right]_{(0,0)}$ has all eigenvalues in C^- ,
- (c) the system

$$\begin{aligned} \dot{\xi}_{i+1} &= f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v)) \\ y &= \xi_{i+1} \end{aligned}$$

satisfies an asymptotic (input v - output y) bound, with restriction $\xi_{i+1} = R^n \times X_i$ on ξ_{i+1}° , restriction $\tilde{V} > 0$ on $v(\cdot)$, with linear gain function $\gamma_v(\cdot)$.

This result can be repeatedly used to globally asymptotically stabilize a system in feedforward form (5.12) under the hypotheses that each of the upper $n-1$ subsystems, when the corresponding input (i.e. (x_{i+1}, \dots, x_n, u)

for the i -th subsystem) is zero, is stable in the sense of Lyapunov, and the n -th subsystem, by means of some feedback law $u = \alpha_n(x_n, v)$, can be changed into a system which satisfies an asymptotic (input-output) bound, with some nonzero restriction on $v(\cdot)$ and a linear gain function. In the coming chapters, we employ Theorem 5.2.5 to derive a controller stabilizing the motorcycle.

5.3 Nonlinear Controller Design

5.3.1 Problem formulation

The simplified motorcycle model is rewritten as follows,

$$\begin{aligned}\dot{\beta} &= \psi_\beta v \\ \dot{\psi}_\beta &= u \\ \dot{\alpha} &= \psi_\alpha \\ \dot{\psi}_\alpha &= \frac{1}{p}(g \sin(\alpha) + (1 + p\psi_\beta \sin(\alpha)) \cos(\alpha)\psi_\beta v^2 + c \cos(\alpha)vu),\end{aligned}\quad (5.21)$$

where u is the control input and v, p, g, c are constants.

The control objective here is to find a control function u such that the system (5.21) is asymptotically stable in the set $\chi = \{(\alpha, \beta, \psi_\beta, \psi_\alpha) | (-\frac{\pi}{2}, +\frac{\pi}{2}) \times R^3\}$.

5.3.2 Identifying the Upper-triangular Structure

We observe that the system (5.21) is indeed in the forwarding form if the dynamics (β, ψ_β) is regarded as the lower subsystem. Then, we are allowed to proceed the forwarding design via saturation functions (see

appendix I). In this case, we may end up with a nonlinear controller with the high gain corresponding to the dynamics (β, ψ_β) and the low gains corresponding to the dynamics (α, ψ_α) . Notice that the instability of the motorcycle mainly results from the roll dynamics (α, ψ_α) . Therefore, the stabilization of the roll dynamics has a priority over the stabilization of the yaw dynamics.

To this end, we would like to render the subsystem (α, ψ_α) a high gain and then proceed with the forwarding design. We derive an appropriate upper-triangular structure for system (5.21) such that the forwarding tool is applicable.

Lemma 5.3.1 *Consider system (5.21). Let $\chi = \{(\alpha, \beta, \psi_\beta, \psi_\alpha) | (-\frac{\pi}{2}, +\frac{\pi}{2}) \times R^3\}$. There exists a map $T : \chi \rightarrow R^4$ such that using the state transformation*

$$(z_2, z_1, \xi_{11}, \xi_{12}) = T(\beta, \psi_\beta, \alpha, \psi_\alpha) \quad (5.22)$$

and a feedback transformation

$$u = D_{11}^{-1} \cdot (D_{21}^{-1} \cdot (u_2 - D_{22}) - D_{12}), \quad (5.23)$$

where u_2 is the new control and D_{ij} , for $i, j = 1, 2$, are functions of (α, ψ_β) . Then, system (5.21) is transformed to an appropriate upper triangular form

$$\begin{aligned} \dot{z}_i &= A_i z_i + g_i(\xi_i, u_2) \\ \dot{\xi}_1 &= f_1(\xi_1, u_2), \end{aligned} \quad (5.24)$$

for $i = 1, 2$ where $A_i = 0$, $\xi_1 \triangleq (\xi_{11}, \xi_{12})$, $\xi_{j+1} \triangleq (\xi_j, z_j)$ for $j = 1, 2$.

Proof: We prove the result in two steps: first, we take a partial feedback linearization to the subsystem with respect to (α, ψ_α) ; then, we take another feedback transformation and a state transformation to complete the proof.

At step 1, we use the geometric tool to derive the preliminary feedback. It is easy to check that the subsystem (α, ψ_α) has a relative degree two (let $y = \alpha$ be the output and $\mathcal{L}_g \mathcal{L}_f^1 h(x) \neq 0$). Then, we take the input transformation over the region χ ,

$$\begin{aligned} u_1 &= \frac{p}{c \cos(\alpha)v} (u - (g \sin(\alpha) + (1 + p\psi_\beta \sin(\alpha)) \cos(\alpha)\psi_\beta v^2)) \\ &\triangleq D_{11}u + D_{12} \end{aligned} \quad (5.25)$$

where u_1 is the new input. The model (5.21) transforms into

$$\begin{aligned} \dot{\beta} &= \psi_\beta v \\ \dot{\psi}_\beta &= \frac{p}{c \cos(\alpha)v} (u_1 - (g \sin(\alpha) + (1 + p\psi_\beta \sin(\alpha)) \cos(\alpha)\psi_\beta v^2)) \\ \dot{\alpha} &= \psi_\alpha \\ \dot{\psi}_\alpha &= u_1. \end{aligned} \quad (5.26)$$

At step 2, our objective is to make the dynamics (5.24) a global problem. Then, we take a state transformation $T : \chi \rightarrow R^4$

$$(z_2, z_1, \xi_{11}, \xi_{12})^T = (\beta, \psi_\beta, \tan(\alpha), (1 + \tan^2(\alpha))\dot{\psi}_\alpha)^T \quad (5.27)$$

and a change of control input

$$\begin{aligned} u_2 &= (1 + \tan^2(\alpha)) + \left(2(\dot{\psi}_\alpha)^2 \tan(\alpha)(1 + \tan^2(\alpha))\right) u_1 \\ &\triangleq D_{21}u_1 + D_{22} \end{aligned} \quad (5.28)$$

where D_{21} is invertible in the set χ and u_2 is a new input. Then, the dynamics (5.26) transforms to

$$\begin{aligned} \dot{z}_2 &= z_1 v \\ \dot{z}_1 &= \frac{p}{c \cos(\arctan(\xi_{11}))v} (D_{21}^{-1} \cdot (u_2 - D_{22}) \\ &\quad - (g \sin(\arctan(\xi_{11})) + (1 + pz_1 \sin(\arctan(\xi_{11}))) \cos(\arctan(\xi_{11}))z_2 v^2)) \\ \dot{\xi}_{11} &= \xi_{12} \\ \dot{\xi}_{12} &= u_2. \end{aligned} \quad (5.29)$$

where $(z_2, z_1, \xi_{11}, \xi_{12}) \in R^4$. System (5.29) can be rewritten in (5.24). Finally, we obtain the result. \square

5.3.3 Forwarding Design

The forwarding design for the model (5.24) is carried out using the following steps:

Step 1: Derive a high gain controller for the subsystem $\dot{\xi}_1$ such that for the bounded input v_1 , this subsystem satisfies asymptotic input to output stable;

Step 2-3: Use Theorem 5.2.5 twice to design a controller for the augmented subsystem $\xi_{i+1} \triangleq (z_i, \xi_i)$ of the original system for $i = 1, 2$ respectively.

Design Step 1

We develop a controller for subsystem ξ_1 here. In particular, we assign a linear control law as follows

$$\begin{aligned} u_2 &= -L_1\xi_{11} - L_2\xi_{12} + v_1 \\ &\triangleq \alpha_1(\xi_1, v_1), \end{aligned} \tag{5.30}$$

where v_1 is a new input and $L_1 > 0$, $L_2 > 0$ are some constants. Next, we show that the closed loop system ξ_1 with control (5.30) can be made asymptotic input to state stable with the bounded input v_1 by choosing appropriate parameters. The result enables us to apply the forwarding tool in the following steps.

We study the closed loop subsystem ξ_1 ,

$$\dot{\xi}_1 = A\xi_1 + \delta, \tag{5.31}$$

where $A = \begin{pmatrix} 0 & 1 \\ -L_1 & -L_2 \end{pmatrix}$ and input $\delta = (0, v_1)^T \in R^2$. The eigenvalues of A are $-\lambda'_{1,2}$ where $\lambda'_1 = \frac{L_2 + \sqrt{L_2^2 - 4L_1}}{2}$ and $\lambda'_2 = \frac{L_2 - \sqrt{L_2^2 - 4L_1}}{2}$. If we let $L_2^2 > 4L_1$, we have $\lambda'_1 > \lambda'_2 > 0$. Then, we have the following result.

Lemma 5.3.2 *Consider system (5.31). Assume that the following conditions $L_2^2 - 4L_1 > 0$, $\|v_1\| \leq \delta_M$ hold for positive numbers L_1 , L_2 and δ_M . Then, for some class \mathcal{K} functions α_1 and α_2 system (5.31) is asymptotic ISS without restriction on initial states, with restriction on exogenous input*

v_1 and with linear asymptotic gains as follows,

$$\begin{aligned}\|\xi_{11}\|_a &\leq \gamma_1(\|v_1\|_a) \\ \|\xi_{12}\|_a &\leq \gamma_2(\|v_1\|_a)\end{aligned}\tag{5.32}$$

where $\gamma_1(s) = \frac{1}{1-\varepsilon} \left| \frac{4}{(L_2 - \sqrt{L_2^2 - 4L_1})\sqrt{L_2^2 - 4L_1}} \right| s$, $\gamma_2(s) = \frac{1}{1-\varepsilon} \left| \frac{2}{\sqrt{L_2^2 - 4L_1}} \right| s$ for $\varepsilon \in (0, 1)$.

Proof We take state transformation as follows $\xi_1 = Py$ with $P = \begin{pmatrix} 1 & 1 \\ -\lambda'_1 & -\lambda'_2 \end{pmatrix}$.

System (5.31) transforms to

$$\dot{y} = P^{-1}(APy + \delta) \triangleq By + \bar{\delta}\tag{5.33}$$

where $B = \begin{pmatrix} -\lambda'_1 & 0 \\ 0 & -\lambda'_2 \end{pmatrix}$ and $\bar{\delta} = \begin{pmatrix} \frac{-\lambda'_2}{(\lambda'_1 - \lambda'_2)} & \frac{-1}{(\lambda'_1 - \lambda'_2)} \\ \frac{\lambda'_1}{(\lambda'_1 - \lambda'_2)} & \frac{1}{(\lambda'_1 - \lambda'_2)} \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{-v_1}{(\lambda'_1 - \lambda'_2)} \\ \frac{v_1}{(\lambda'_1 - \lambda'_2)} \end{pmatrix}$.

System (5.33) is considered as two decoupled subsystems with external inputs $\bar{\delta}$. Let the Lyapunov candidate $V_1 = \frac{1}{2}y_1^2$ and $V_2 = \frac{1}{2}y_2^2$ for y_1 and y_2 subsystems respectively.

The time derivative of V_1 along the trajectory of system (5.33) is given by

$$\begin{aligned}\frac{\partial V_1}{\partial y_1}(By + \bar{\delta}) &\leq -\left(\lambda'_1 - \frac{\bar{\delta}_1}{|y_1|}\right)y_1^2 \quad (y_1 \neq 0) \\ &\triangleq -\beta_1(y_1^2).\end{aligned}\tag{5.34}$$

For some $\varepsilon \in (0, 1)$,

$$|y_1| \geq \left(\frac{1}{1-\varepsilon} \right) \left| \frac{v_1}{\lambda'_1(\lambda'_1 - \lambda'_2)} \right| \triangleq a_0(|v_1|) \quad (5.35)$$

implies $\beta_1(y_1^2) > 0$. We let $\beta_1(y_1^2) > 0$ such that the right hand side of the inequality (5.34) is negative definite. Clearly, $a_0(\cdot)$ and $\beta_1(\cdot)$ are class \mathcal{K}_∞ functions. Furthermore, $\underline{a}y_1^2 \leq V_1 \leq \bar{a}y_1^2$ hold for $\underline{a} \leq 1$ and $\bar{a} \geq 1$. By definition, V_1 is an ISS-Lyapunov function which implies the ISS property as follows $|y_1(t)| \leq \mathbf{K}|y_1(0)| \exp(-\lambda'_1 t) + \left| \frac{1}{\lambda'_1(\lambda'_1 - \lambda'_2)} \right| (|v_1|)$ for some $\mathbf{K} > 0$. Then, there exists t_1 such that for $t > t_1$, $y_1(t)$ stays in the set $\{y_1(t) \in R \mid |y_1(t)| \leq a_0(|v_1|)\}$. Finally, we conclude an asymptotic gain for y_1 as follows

$$\|y_1\|_a \leq \gamma'_1(\|v_1\|_a), \quad (5.36)$$

where we define $\gamma'_1(s) \triangleq \frac{1}{1-\varepsilon} \left| \frac{1}{\lambda'_1(\lambda'_1 - \lambda'_2)} \right| s$.

Similarly, we take the time derivative of V_2 along the trajectory of system (5.33) and obtain an asymptotic gain for y_2 as follows

$$\|y_2\|_a \leq \gamma'_2(\|v_1\|_a), \quad (5.37)$$

where we define $\gamma'_2(s) \triangleq \frac{1}{1-\varepsilon} \left| \frac{1}{\lambda'_2(\lambda'_1 - \lambda'_2)} \right| s$.

Next, we cast the asymptotic gains for y into the asymptotic gains for ξ_1 . It is easy to check that the inequality $\|\xi_{11}\|_a \leq \|y_1\|_a + \|y_2\|_a \leq 2\gamma'_2(\|v_1\|_a) \triangleq \gamma_1^*(\|v_1\|_a)$ is satisfied because by condition $\lambda_1 > \lambda_2 > 0$, $\gamma'_2 > \gamma'_1$ hold. Furthermore, we have $\|\xi_{12}\|_a \leq \lambda'_1 \|y_1\|_a + \lambda'_2 \|y_2\|_a \leq \frac{2}{\lambda'_1} \gamma'_1(\|v_1\|_a) \triangleq \gamma_2^*(\|v_1\|_a)$. Substituting the appropriate functions of λ'_1 and λ'_2 to class \mathcal{K} functions

$\gamma_1^*(s), \gamma_2^*(s)$ gives the asymptotic gains $\gamma_1(\|v_1\|_a), \gamma_2(\|v_1\|_a)$ for $\|\xi_{11}\|_a$ and $\|\xi_{12}\|_a$ respectively. \square

Design Steps 2

In this step, we apply Theorem 5.2.5 to obtain a nested saturating controller for the augmented system ξ_2 of (5.26),

$$\begin{aligned} \dot{z}_1 &= A_1 z_1 + g_1(\xi_1, \alpha_1(\xi_1, v_1)), \\ \dot{\xi}_1 &= f_1(\xi_1, \alpha_i(\xi_1, v_1)), \end{aligned} \quad (5.38)$$

where $A_1 = 0$, $g_1(\xi_1, \alpha_i(\xi_1, v_1))$ equals to RHS of $\dot{\psi}_\beta$ in (5.26) and $f_1 = \begin{bmatrix} \xi_{12} \\ \alpha_i(\xi_1, v_1) \end{bmatrix}$.

Before doing the design, we have to check all assumptions in Theorem 5.2.5 satisfied otherwise we can not apply the design procedure depicted in Theorem 5.2.5. If all assumptions are satisfied, we can proceed the design of an appropriate saturation function for the external input v_1 . Then, the designed $\alpha_1(\xi_1, v_1)$ ensures that the augmented system

$$f_2(\xi_2, \alpha_1(\cdot, \cdot)) = \begin{pmatrix} A_1 z_1 + g_1(\xi_1, \alpha_1(\cdot, \cdot)) \\ f_1(\xi_1, \alpha_1(\cdot, \cdot)) \end{pmatrix}, \quad (5.39)$$

satisfy an asymptotic input-output bound.

Keeping this procedure in mind, we can start the design now. First, we check all conditions in Theorem 5.2.5 hold. Assumption (i) holds as $A_1 = 0$. Because the linear approximation of the augmented system at the equilibrium $(z_1, \xi_1, v_1) = (0, 0, 0)$ is stabilizable, assumption (ii) holds.

$\left[\frac{\partial \alpha_1(\xi_1, v_1)}{\partial v}\right]_{(0,0)} = 1$ is nonsingular. Eigenvalues of $\left[\frac{\partial f_i(\xi_i, \alpha_i(\xi_i, v))}{\partial \xi_i}\right]_{(0,0)}$ are $-0.5132, -19.4868$ in C^{-1} (we let $L_1 = 10$ and $L_2 = 20$). The Assumptions (iiia-b) are satisfied. Indeed, Assumptions (iiic) is satisfied because subsystem ξ_1 is asymptotic input-to-state stable as shown in Theorem 5.3.2.

Then, we apply the Theorem 5.2.5 to design a control law for (5.38). We have

$$F_2 = \left[\frac{\partial f_2(\xi_2, \alpha_1(\xi_1, v_1))}{\partial \xi_2}\right]_{(0,0)} = \begin{pmatrix} -20 & -3.16 & -2.4 \\ 0 & 0 & 1 \\ 0 & -10 & -20 \end{pmatrix},$$

$$G_2 = \left[\frac{\partial f_2(\xi_2, \alpha_1(\xi_1, v_1))}{\partial v_1}\right]_{(0,0)} = (0.12, 0, 1)^T.$$

Noting that (F_2, G_2) is stabilizable, we employ pole placement technique to design a linear feedback. By applying pole placement technique for the desired poles $P_1 = (-4, -6, -8)$, we obtain the gain matrix

$$k_2 = - \begin{pmatrix} 58.38 & 6.12 & 14.99 \end{pmatrix}.$$

The controller $v_1 = k_2 \xi_2$ place all eigenvalues of $(F_2 + G_2 k_2)$ in C^- , which actually are the desired poles P_1 .

Without loss of generality, we let $\Gamma_1 = 1, \Psi_1 = 0$ in the design and λ_1 be an adjustable parameter. The nested saturating controller for the first

augmented subsystem (5.38) is obtained

$$\begin{aligned} u_2 &= -L_1\xi_{11} - L_2\xi_{12} + \lambda_1\sigma\left(\frac{1}{\lambda_1}(k_2\xi_2 + v_2)\right) \\ &\triangleq \alpha_2(\xi_2, v_2). \end{aligned} \quad (5.40)$$

We can conclude from (5.40): (a) the matrix $\left[\frac{\partial\alpha_2(\xi_2, v_2)}{\partial v_2}\right]_{(0,0)} = 1$ is nonsingular,

(b) the matrix $\left[\frac{\partial f_2(\xi_2, \alpha_2(\xi_2, v_2))}{\partial \xi_2}\right]_{(0,0)}$ has all eigenvalues P_1 in C^- ,

(c) the system $\dot{\xi}_2 = f_2(\xi_2, \alpha_2(\xi_2, v_2))$, $y_2 = \xi_2$ satisfies an asymptotic (input v_2 - output y_2) bound with restriction on v_2 .

Design Steps 3

We again apply Theorem 5.2.5 for the whole system (5.26). Our goal is to design a saturation function for v_2 such that $\alpha_2(\xi_2, v_2)$ with the external input v_2 ensures that the whole closed loop system

$$f_3(\xi_3, \alpha_2(\cdot, \cdot)) = \begin{pmatrix} A_2 z_2 + g_2(\xi_2, \alpha_2(\cdot, \cdot)) \\ f_2(\xi_2, \alpha_2(\cdot, \cdot)) \end{pmatrix}, \quad (5.41)$$

satisfy an asymptotic input-output bound.

We check all assumptions in Theorem 5.2.5 hold. Assumption (i) holds as $A_2 = 0$. Assumption (ii) holds because the linear approximation of each augmented system at the equilibrium $(z_2, \xi_2, v_2) = (0, 0, 0)$ is stabilizable. Assumptions (iiia-c) are automatically satisfied because they are the results of the last step. Therefore, all conditions are satisfied.

Again, we apply Theorem 5.2.5 to design a complete control law for (5.26).

We have

$$F_3 = \left[\frac{\partial f_3(\xi_3, \alpha_2(\xi_2, v_2))}{\partial \xi_3} \right]_{(0,0)} = \begin{pmatrix} 0 & 10 & 0 & 0 \\ 0 & -12.99 & -2.43 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 58.38 & -3.88 & -5.01 \end{pmatrix},$$

$$G_3 = \left[\frac{\partial f_3(\xi_3, \alpha_2(\xi_2, v_2))}{\partial v_2} \right]_{(0,0)} = (0, 0.12, 0, 1)^T.$$

Then, applying pole placement technique for the desired poles

$$P_2 = (-2, -4, -6, -8)$$

obtains the gain matrix

$$k_3 = - \left(19.59 \ 3.96 \ 10.01 \ 2.46 \right).$$

The controller $v_2 = k_3 \xi_3$ place all eigenvalues of $(F_3 + G_3 k_3)$, i.e., P_2 , in C^- .

Finally, a nested saturating controller for the whole system (5.26) is obtained where we let $\Gamma_2 = 0$, $\Psi_1 = 0$,

$$u_2 = -L_1 \xi_{11} - L_2 \xi_{12} + \lambda_1 \sigma \left(\frac{1}{\lambda_1} \left(k_2 \xi_2 + \lambda_2 \sigma \left(\frac{1}{\lambda_2} (k_3 \xi_3) \right) \right) \right) \triangleq \alpha_3(\xi_3) \quad (5.42)$$

5.3.4 The Final Result

We wrap up all design steps and conclude the following result.

Theorem 5.3.3 *Consider system (5.21). Assume that all conditions in*

Lemma 5.3.1 and Lemma 5.3.2 are satisfied. Furthermore, we define $u_2 = \alpha_3(\xi_3)$ as the function (5.42) and we have the nonlinear state transformation $\xi_3 \triangleq (z_2, z_1, \xi_{11}, \xi_{12}) = T(\beta, \psi_\beta, \alpha, \psi_\alpha)$. Then, we obtain a complete feedback controller

$$u = D_{11}^{-1} \cdot (D_{21}^{-1} \cdot (\alpha_3(T(\beta, \psi_\beta, \alpha, \psi_\alpha)) - D_{22}) - D_{12}) , \quad (5.43)$$

where D_{ij} for $i, j = 1, 2$ are functions of (α, ψ_β) such that the closed loop system is asymptotic stable in the set $\chi = \{(\alpha, \beta, \psi_\beta, \psi_\alpha) | (-\frac{\pi}{2}, +\frac{\pi}{2}) \times R^3\}$ which is an estimate of domain of attraction.

Proof The proof is straightforward. Because all conditions in Lemma 5.3.1 and Lemma 5.3.2 are satisfied, our forwarding controller $u_2 = \alpha_3(\xi_3)$ renders the transformed system (5.24) globally asymptotic stable as the result given by Theorem 5.2.5. This implies that the original closed loop system (5.21) is asymptotic stable in the set χ . Then, we can summarize the result. \square

5.4 Simulations

The controller is evaluated through simulation. Let $p = 0.6(m)$, $c = 0.5(m)$, $b = 1.2(m)$, $v = 10(m/s)$, $g = 9.8(m/s^2)$, $\lambda_1 = 10$, $\lambda_2 = 5$, $L_1 = 10(N \cdot s/rad)$, $L_2 = 20(N/rad)$. We carry out the simulations for the closed loop nonlinear system with the nonlinear controller (5.43) and a linear controller respectively such that we can compare with the performance.

Case 1: Let the initial output values:

$$(\beta(0), \alpha(0), \psi_\beta(0), \psi_\alpha(0)) = (30^\circ, 40^\circ, 1, 1).$$

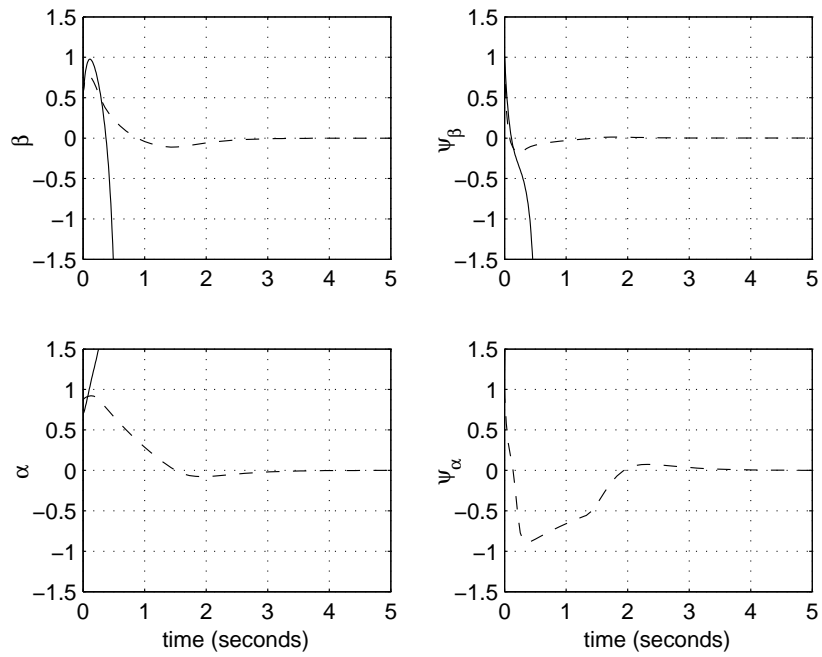


Figure 5.2: Simulation Results: dotted lines represent the nonlinear controller and the solid lines represent the linear controller in Case 1; the linear controller cannot stabilize the nonlinear plant when some large initial condition is given; the proposed nonlinear control can stabilize the nonlinear plant even though the large initial condition is given.

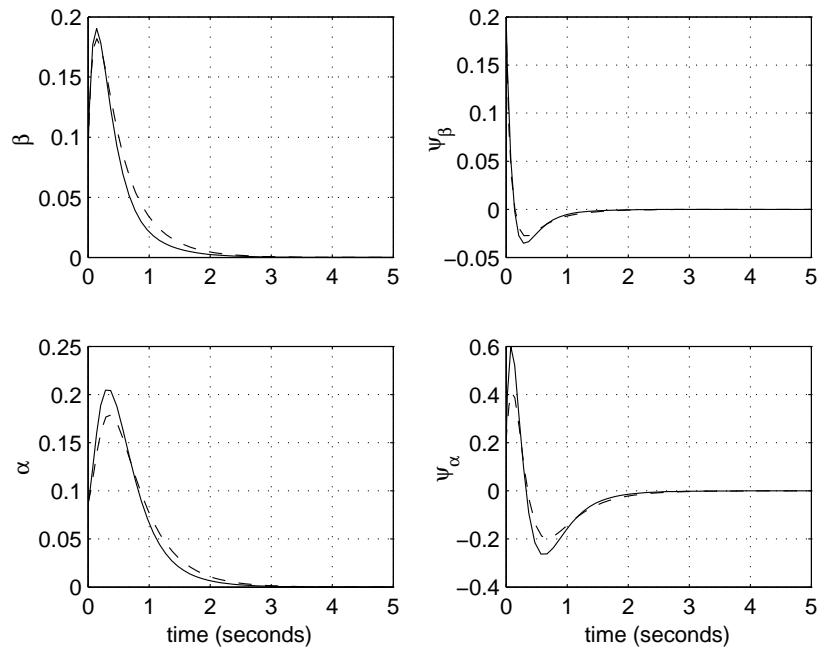


Figure 5.3: Simulation Results: dotted lines represent the nonlinear controller and the solid lines represent the linear controller in Case 2; given some small initial conditions, both the linear control and the nonlinear controller can stabilize the nonlinear plant.

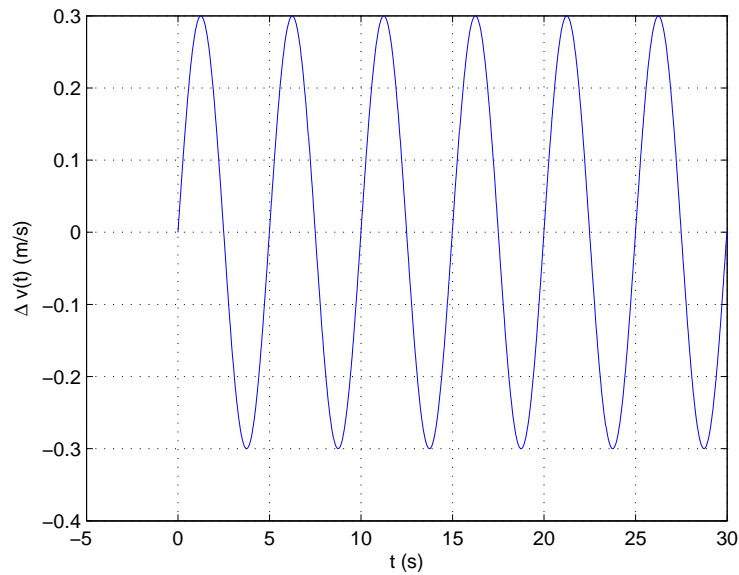


Figure 5.4: Uncertain $\Delta v(t)$ in Case 3: the uncertain forward speed is a function of time.

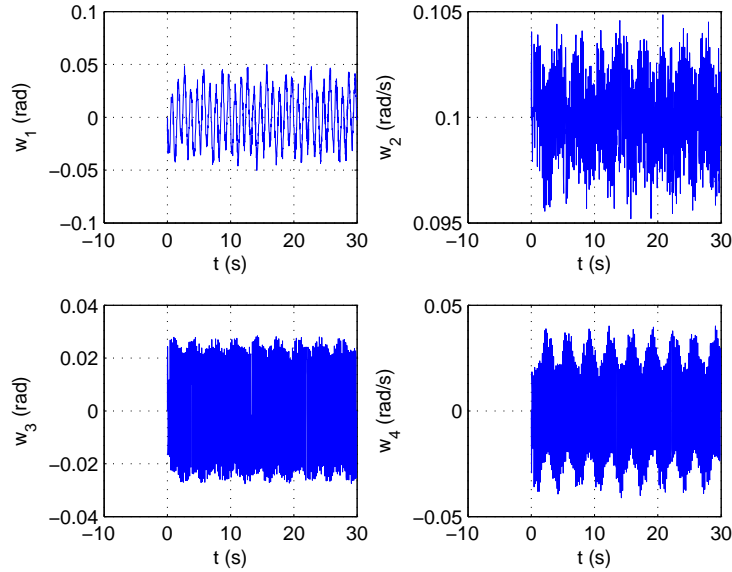


Figure 5.5: Disturbance $\Delta w(t)$ in Case 3: the disturbance is a collection of exogenous forces that is bounded; the disturbance is not necessarily Gaussian.

The simulation results are shown in Figure 5.2. Notice that the trajectories of the system driven by the linear controller are blown up in contrast with the convergence of the trajectories of the closed loop system with the nonlinear controller. This manifests that the nonlinear controller yields a nonlocal domain of attraction.

Case 2: Give another set of the initial output values:

$$(\beta(0), \alpha(0), \psi_\beta(0), \psi_\alpha(0)) = (5^\circ, 5^\circ, 0.2, 0.2).$$

The simulation results are shown in Figure 5.3. The trajectories of the closed loop systems with both controllers converge to the origin. But, the settling time of the nonlinear controller is longer than that of the linear controller due to the inclusion of saturation functions in the nonlinear controller. Therefore, the nonlinear controller is not optimal locally about

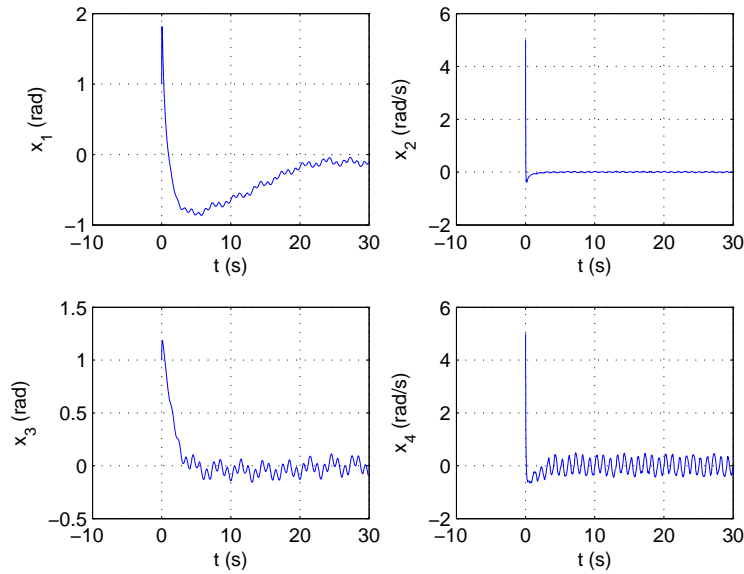


Figure 5.6: Simulations in Case 3 show the robustness of the nonlinear controller: the trajectory converges to a neighborhood of the origin; the disturbance is rejected.

the equilibrium.

Case 3: We consider the additive measurement disturbance $w(t)$ (see Figure 5.4) and the uncertainty speed $\Delta v(t)$ (see Figure 5.5) to test the robustness of the nonlinear controller. Give another set of the initial output values ¹:

$$\mathbf{x} = (1(\text{rad}), 5(\text{rad/s}), 1(\text{rad}), 5(\text{rad/s})).$$

The simulation results are shown in Figure 5.6. The trajectories of the closed loop systems with both controllers remain bounded under the presence of uncertainty and disturbance.

¹The initial values are equivalent to

$$(\beta(0), \psi_\beta(0), \alpha(0), \psi_\alpha(0)) = (57.3248^\circ, 286.6242^\circ/s, 57.3248^\circ, 286.6242^\circ/s),$$

which is very large. By doing this, we can compare the simulation results with those in Chapters 4 and 3.

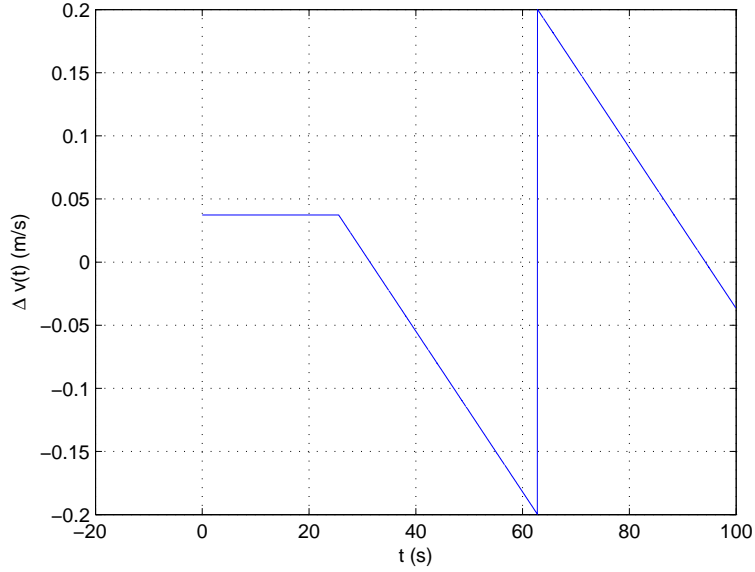


Figure 5.7: Uncertain $\Delta v(t)$ in Case 4: the uncertain forward speed is a function of time.

Case 4: We consider another set of the additive measurement disturbance $w(t)$ (see Figure 5.7) and the uncertainty speed $\Delta v(t)$ (see Figure 5.8) to test the robustness of the nonlinear controller. Give another set of the initial output values ²:

$$\mathbf{x} = (-1(\text{rad}), 5(\text{rad/s}), 1(\text{rad}), 114.6497(\text{rad/s})).$$

The simulation results are shown in Figure 5.9. The trajectories of the closed loop systems with both controllers also remain bounded under the presence of uncertainty and disturbance. So, the proposed nonlinear controller not only offers a “global” domain of attraction, that is, the whole upper space with arbitrary angular velocities, but also yields certain

²The initial values are equivalent to

$$(\beta(0), \psi_\beta(0), \alpha(0), \psi_\alpha(0)) = (-57.3248^\circ, 286.6242^\circ/\text{s}, 57.3248^\circ, 286.6242^\circ/\text{s}),$$

which is very large. By doing this, we can compare the simulation results with those in Chapters 4 and 3.

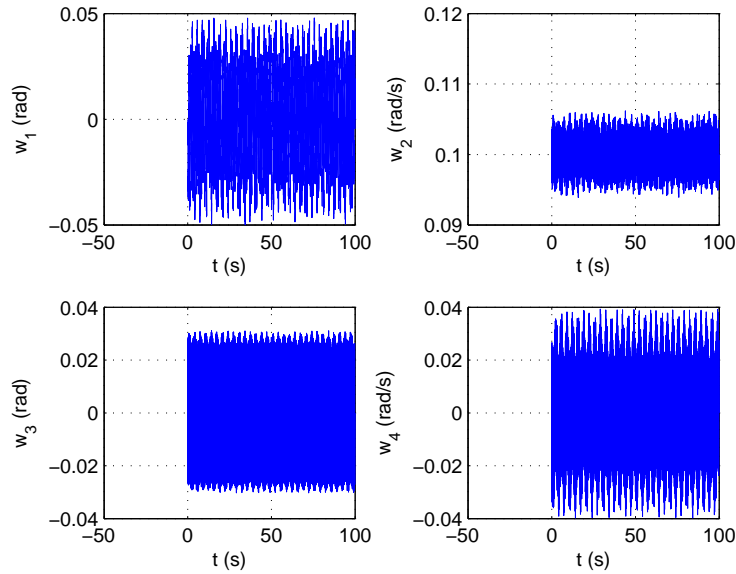


Figure 5.8: Disturbance $\Delta w(t)$ in Case 4: the disturbance is a collection of exogenous forces that is bounded; the disturbance is not necessarily Gaussian.

robustness.

5.5 Summary

We design a nonlinear stabilizing controller for a simple motorcycle model which yields the domain of attraction, the upper half space. The controller is composed of some high gains and some low gains where the low gains are obtained by applying Teel's nested saturating design tool. The performance of the controller is evaluated through computer simulation in comparison with a linear controller. A large domain of attraction yielded by the nonlinear controller is observed in the simulation results, which is consistent with the theoretical development. Furthermore, the simulations show that the proposed nonlinear controller also yields certain robustness. However, analytical analysis of the robustness of the nonlinear controlled

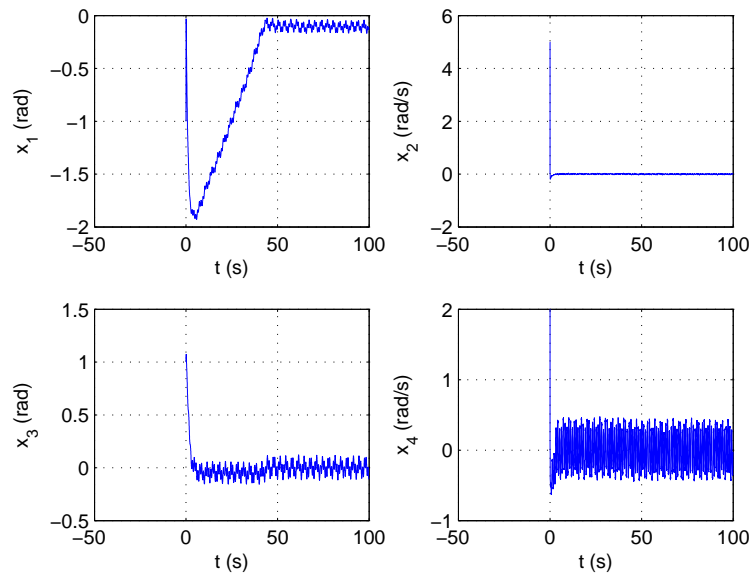


Figure 5.9: Simulations in Case 4 also show the robustness of the nonlinear controller: the trajectory converges to a neighborhood of the origin; the disturbance is rejected.

system with the forwarding controller is a very challenging topic in the community. Furthermore, the controller here is not only depending on the forwarding approach but also relying on other methods (e.g., the coordinate transformation and feedback linearization). The added complexity makes the robustness analysis more difficult.

Chapter 6

CONCLUSIONS AND FUTURE WORK

6.1 Summary of the Thesis

The objective of this thesis was to achieve two control tasks for the autonomous motorcycle: stabilization subject to uncertain model and exogenous disturbance stabilization with a large domain of attraction.

Classical linear designs cannot deal with explicitly the model uncertainty and achieve disturbance attenuation. To this end, we propose a robust H_∞ controller in Chapter 4 to the linearized system, which provides a significant improvement in dealing model uncertainty and disturbance attenuation in comparison with classical linear designs.

A limitation of all classical linear designs is that it only guarantees some small and bounded domain of attraction about the operating point. This motive us to derive a nonlinear forwarding controller in Chapter 5 by first identifying an appropriate under-triangle structure of the nonlinear dynamics and then combining several tools with the forwarding tool [65].

The nonlinear controller yields a domain of attraction as large as the whole upper hemisphere.

The above two designs are novel and have not been considered in the literature.

6.2 Future Work

The autonomous motorcycle is an interesting case study for evaluating both linear and nonlinear control theory because the system is nonlinear, underactuated, unstable and MIMO.

Although a linear robustness controller has been proposed in the thesis, nonlinear robust controller is more appealing because the plant is nonlinear in nature. Nonlinear robust control design is always a difficult task for such a system in the presence of the unmodelled dynamics, the measurement noise, the limitations of actuation forces and exogenous disturbances. One of approaches to make the nonlinear controlled system robust is to derive a Lyapunov function for the nominal controlled system. Then, the Lyapunov function is taken as a controlled Lyapunov function (CLF) for the perturbed system in order to carry out the robustness analysis. However, the forwarding tool used in the thesis is not a Lyapunov based forwarding tool. In this regard, we are urged to apply Lyapunov-based forwarding design tools in the future.

On the other hand, robust tracking or guidance control of the autonomous motorcycle that is still lacking in the literature is of importance in some applications where human riders are unavailable or environments are

harsh for the riders.

Finally, it is desirable that all proposed controllers are implemented in a real system to verify their effectiveness, which has not been done in the thesis. To this end, a test-bench for the motorcycle need to be built where the state variables or at least some of them are measurable via some sensing techniques and a control signal exerted by some actuators (e.g., servomotors) is able to actuate the associated variables. Then, the proposed controllers maybe evaluated based on the experimental set-up. Of course, the amount of the work and its complexity is beyond the scope of this thesis. As such, it is practically appealing to implement the controllers in the future under the support of associated industry.

Appendix A

MATHEMATICAL FUNDAMENTALS

For readers' convenience, we repeat some mathematical fundamentals in textbook [36].

Euclidean Space

The set of all n -dimensional vectors $x = [x_1, \dots, x_n]^T$, where x_1, \dots, x_n are real numbers, defines the n -dimensional Euclidean space denoted by R^n . The one-dimensional Euclidean space consists of all real numbers and is denoted by R . The inner product of two vectors x and y in R^n is $x^T y = \sum_{i=1}^n x_i y_i$.

Vector and Matrix Norms

The norm $\|x\|$ of a vector x is a real-valued function with the properties

- $\|x\| \geq 0$ for all $x \in R^n$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in R^n$.
- $\|\alpha x\| = |\alpha| \|x\|$, for all $\alpha \in R$ and $x \in R^n$.

The second property is the triangle inequality. We consider the class of p -norms, defined by

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_i |x_i|$$

where $p = 2$ is denoted as the Euclidean norm. If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two p -norms, then there exist positive constants c_1 and c_2 such that

$$c_1\|x\|_\alpha \leq \|x\|_\beta \leq c_2\|x\|_\alpha$$

for all $x \in R^n$.

An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m . The induced p -norm of A is defined by

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$$

which for $p = 1, 2$, and ∞ is given by

$$\|A\|_1 = \max_j \sum_{i=1}^m |a_{ij}|, \quad \|A\|_2 = [\lambda_{\max}(A^T A)]^{1/2}, \quad \|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$$

where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$.

Lipschitz Condition

The Lipschitz condition is

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all (t, x) and (t, y) in some neighborhood of (t_0, x_0) where L is called a Lipschitz constant.

The local Lipschitz property is stated in the next lemma.

Lemma A.0.1 *[36, Page 90] If $f(t, x)$ and $[\partial f / \partial x](t, x)$ are continuous on $[a, b] \times D$, for some domain $D \in \mathbb{R}^n$, then f is locally Lipschitz in x on $[a, b] \times D$.*

The global Lipschitz property is stated in the next lemma.

Lemma A.0.2 *[36, Page 91] If $f(t, x)$ and $[\partial f / \partial x](t, x)$ are continuous on $[a, b] \times \mathbb{R}^n$ if and only if $[\partial f / \partial x]$ is uniformly bounded on $[a, b] \times \mathbb{R}^n$.*

Appendix B

LINEAR CONTROL THEORY

Consider a linear, time-invariant, continuous-time system described by

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{B.1}$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$ and $x = 0$ is the equilibrium point.

The following definitions are standard (e.g., see [18, 13]).

Controllability: The state equation (B.1) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1 , there exists an input u that transfers x_0 to x_1 in finite time. Otherwise, (B.1) is said to be uncontrollable.

The system (B.1) is controllable, if and only if the controllability matrix has a full row rank

$$\text{rank}(B, AB, \dots, A^{n-1}B) = n. \tag{B.2}$$

Hence, controllability depends only on the matrix pair (A, B) . We refer to

(A, B) as a controllable pair whenever the system (B.1) is controllable.

State feedback controller: Assume that (A, B) is controllable. A linear time invariant state feedback controller is of the form

$$u = -Kx, \quad (\text{B.3})$$

where $K \in R^{m \times n}$ is the gain matrix. The closed loop system is given by

$$\dot{x} = (A - BK)x. \quad (\text{B.4})$$

When (B.4) is asymptotically stable, we say that $(A - BK)$ is Hurwitz.

Observability: The system (B.1) is said to be observable if for any unknown initial state $x(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, (B.1) is said to be unobservable.

The system (B.1) is observable, if and only if the observability matrix has a full column rank

$$\text{rank}(C^T, A^T C^T, \dots, (A^T)^{n-1} C^T)^T = n. \quad (\text{B.5})$$

Hence, observability depends only on the matrix pair (A, C). We refer to (A, C) as a observable pair whenever the system (B.1) is observable.

Observer: An observer is a system used to reconstruct the state vector of the plant. A full state (Luenberger) observer is defined as

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x}). \quad (\text{B.6})$$

where $\hat{x} \in R^n$ is the observer state and $K_e \in R^{n \times p}$ is the observer gain matrix.

The error between the actual state x and the observer state \hat{x} , $e \triangleq x - \hat{x}$, is governed by the differential equation

$$\dot{e} = (A - K_e C)e. \quad (\text{B.7})$$

If (A, C) is observable, there exist $K_e \in R^{n \times p}$ so that $(A - K_e C)$ is Hurwitz. In this case, \hat{x} converges to x and we can regard \hat{x} as an estimate of x .

Output feedback controller: A dynamic output feedback controller for the system (B.1) can be constructed as

$$u = -K\hat{x}, \quad \dot{\hat{x}} = (A - K_e C - BK)\hat{x} + K_e y. \quad (\text{B.8})$$

Here, we assume that both (A, B) is controllable and (C, A) is observable.

The separation principle: Consider the dynamics of (B.8) together with (B.1), rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - K_e C \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}. \quad (\text{B.9})$$

In view of the block diagonal structure, it is clear that the set of eigenvalues of (B.9) is the union of the eigenvalues due to the state feedback design alone $(A - BK)$ and the eigenvalues due to the observer design alone $(A - K_e C)$. This means that the state feedback and estimator design can be carried out separately.

The idea of assigning desired eigenvalues are called pole placement design, which is based on the following result.

Theorem B.0.3 [5, page 329] *Given $A \in R^{n \times m}$, there exists $K \in R^{m \times n}$ such that the n eigenvalues of $A - BK$ can be assigned arbitrary, real or complex conjugate, locations if and only if (A, B) is controllable (-from-the-origin, or reachable).*

Appendix C

LMI OPTIMIZATION AND SIMULATION CODES

The following MATLAB code is used to find the solutions of LMIs in Chapter 4.

```
%                H_Infinity_Frances
% This is a program that finds the optimal solutions of LMI
% systems for a motorcycle model presented in Chapter 4
% of Fenge Yuan's thesis.
% Copyright ©2007 is held by the author, Ms Fenge Yuan.

clear

clc

%-----define linear model-----
p=0.6;
```

```

c=0.5;

v=10;

g=9.8;

A =[0 v 0 0; 0 0 0 0;0 0 0 1; 0 v^2/p g/p 0];
%-----
B1=[0 ;1;0;c*v/p];
%-----
B2=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

% The following code can be used to verify----
% dA+dB1*K==H*F*[eye(4);K];
%Begin of Verification
syms K k1 k2 k3 k4 dv K=[k1 k2 k3 k4];
%----- dA+dB1*K -----
dA=dv*[0 1 0 0; 0 0 0 0; 0 0 0 0; 0 2*v/p 0 0];
dB1=dv*[0;0;0;c/p]; lhs=dA+dB1*K;
%----- H*F*[eye(4);K]-----
H=[0 1 0 0 0; 0 0 0 0 0; 0 0 0 0 0; 0 2*v/p 0 0 c/p];

F=dv*eye(5);

rhs=H*F*[eye(4);K];

if (lhs-rhs)==0
    disp('\Delta A ==H F E is satisfied: Proceed!!')

```

```

else
    disp('\Delta A ==H F E is not satisfied: Stop!!')
end
%End of Verification

%--constants in H_infinity-----
rho=0.4; %-----rho^2 I>max|\Delta v(t)|^2 I-----
gamma=10; %-----H_infinity bound-----

%-----Construct LMI system-----
setlmis([]);
%-----Define LMI variables-----
Q=lmivar(1,[4 1]);

Y=lmivar(2,[1 4]);

epsilon=lmivar(1,[1 0]);

%-----First LMI System-----
%left hand side
lmiterm([1 1 1 Q],A,1,'s')

lmiterm([1 1 1 Y],B1,1,'s')

lmiterm([1 1 1 epsilon],1,rho^2*H*H')

lmiterm([1 1 1 0],gamma^(-2)*B2*B2')

```

```

lmiterm([1 1 2 -Y],1,1)

lmiterm([1 1 3 Q],1,1)

lmiterm([1 1 4 Q],1,1)

lmiterm([1 2 2 epsilon],-1,1) lmiterm([1 2 3 0],[0 0 0 0])

lmiterm([1 2 4 0],zeros(4))

lmiterm([1 3 3 0],-eye(4))

lmiterm([1 3 4 0],zeros(4))

lmiterm([1 4 4 epsilon],-1,eye(4))

%-----Second LMI System-----
%right hand side
lmiterm([-2 1 1 epsilon],1,1)
%-----Third LMI System-----
%right hand side
lmiterm([-3 1 1 Q],1,1)
%-----Internal Representaion LMISYS of these LMI systems
lmisys=getlmis;
%-----feasibility-----
[tmin,Xfeas]=feasp(lmisys) if tmin<0
    disp('Feasible results are obtained: Proceed!!')
else

```

```

    disp('Infeasible: adjusting gamma and dv, please!')
end
%-----Extract the matrix variables-----
Q=dec2mat(lmisys,Xfeas,Q);
%symmetric matrix Q for quadratic stability\\
Y=dec2mat(lmisys,Xfeas,Y);

epsilon=dec2mat(lmisys,Xfeas,epsilon); %--\epsilon>0-----\\
IQ=inv(Q); %--inversion of Q-----\\
K=Y*inv(Q); %----Gain matrix K=Y*Q^(-1)-----\\
% K =
%   189.1086 -228.9145 -220.8756  -55.8949
% Q
% Q =
%   0.3684  -0.0806   0.2339   0.6496
%  -0.0806   0.1050  -0.0318  -0.5561
%   0.2339  -0.0318   0.3581  -0.4922
%   0.6496  -0.5561  -0.4922   6.5442
% IQ
% IQ =
%   59.5790 -57.5943 -65.6707 -15.7474
%  -57.5943  80.1709  69.0995  17.7267
%  -65.6707  69.0995  76.7871  18.1659
%  -15.7474  17.7267  18.1659   4.5886
% Y
% Y =
%   0.1408  -1.1566  -0.0762  -6.9281

\Delta A ==H F E is satisfied:

```

Proceed!!

Solver for LMI feasibility problems $L(x) < R(x)$

This solver minimizes t subject to $L(x) < R(x) + t*I$

The best value of t should be negative for feasibility

Iteration : Best value of t so far

1	0.255207
2	0.134856
3	0.134856
4	0.067401
5	0.067401
6	0.048225
7	0.048225
8	0.048225
9	0.040901
10	0.040901
11	0.040901
12	0.040901
13	0.017254
***	new lower bound: -0.077746
14	0.017254
15	-6.502181e-004

Result: best value of t : -6.502181e-004

f-radius saturation: 0.000% of $R = 1.00e+009$

tmin =

-6.5022e-004

Xfeas =

0.3684

-0.0806

0.1050

0.2339

-0.0318

0.3581

0.6496

-0.5561

-0.4922

6.5442

0.1408

-1.1566

-0.0762

-6.9281

0.9555

Feasible results are obtained: Proceed!!

The following MATLAB code is used to carry out the simulations of the closed loop linear plant in Chapter 4 subject to uncertainty speed $\Delta v(t)$ and disturbance $w(t)$.

```

%           H_Infinity_Frances_sim
% This is a simulation program of a motorcycle model with
%the optimal robust control obtained from LMI systems.
% Simulations presented in Chapter 4 of Fenge Yuan's thesis
% are based on this program.
% Copyright ©2007 is held by the author, Ms Fenge Yuan.
clear

clc

%-----define linear model-----
p=0.6;

c=0.5;

v=10;

g=9.8;

A =[0 v 0 0; 0 0 0 0;0 0 0 1; 0 v^2/p g/p 0];
%-----
B1=[0 ;1;0;c*v/p];
%-----

B2=eye(4);

```



```

%-----
K =[189.1086 -228.9145 -220.8756 -55.8949];
%----Construct and check the Hurwitz matrices AA---
AA=A+B1*K; megv=max(eig(AA)); if megv<0
    disp('A+B1*K is Hurwitz: proceed!!');
else
    disp('A+B1*K is not Hurwitz: wrong!!')
end
%Constructing external input matrix for closed loop system-
BB=[0 1 0 0 0;0 0 1 0 0;0 0 0 1 0;c/p 0 0 0 1]; CC=eye(4);
DD=zeros(4,5);
%---state space system-----
usys=ss(AA,BB,CC,DD);
%-----give initial conditions-----
X0=[0.03;0.03;0.05;0.05];
%-----generating uncertainty speed dv(t)-----
t=0:.01:20;
[n,m]=size(t);
ii=1800;
dv=0.2-0.3*atan(t(1:ii)/5)*2/pi-0.2*sin(t(1:ii)/1.5);
temp=dv(end);
for jj=ii+1:m
    dv=[dv temp];
end
%-----generating disturbance w(t)---
w1=0.1*sin(43.3*t)+0.04*sin(33.3*t+133)+..
.15*cos(103.3*t+5.87)+.03*cos(31.3*t+.3)...
+.07*sin(15.5*t+.897)+.23*cos(t+.49);

```

```

w2=0.17*sin(43.3*t)+0.09*sin(333.3*t+133)+...
.15*cos(13.3*t+5.87)+.03*cos(110.3*t+.3)...
+.07*sin(1.5*t+.897)+.03*sin(t+.49);
w3=0.17*sin(3.3*t)+0.11*sin(153.3*t+133)+...
.21*cos(3.3*t+125.87)+.13*cos(10.3*t+.3)...
+.17*sin(1.5*t+897)+.13*sin(t+.49);
w4=0.21*sin(3.3*t)+0.071*sin(253.3*t+133)+...
.081*cos(3.3*t+125.87)+.23*cos(10.3*t+.3)...
+.19*sin(1.5*t+897)+.053*sin(t+.49);
scale1=0.1;
scale2=0.1;
uncertainty=1;
%---Construct external disturbance/uncertainty vector
u=[dv*uncertainty;scale1*w1;scale2*w2;scale1*w1;scale2*w2];
%-----Carrying out simulations-----
[y,t,x] = lsim(usys,u,t,X0);
%-----Ploting figures-----
% figure(1), clf
% subplot(2,2,1), plot([-0.01; t],[x(1,1); x(:,1)]),
% grid,ylabel('X_1 (rad)'),
% subplot(2,2,2), plot([-0.01; t],[x(2,1); x(:,2)]),
% grid,ylabel('X_2 (rad/s)'),
% subplot(2,2,3), plot([-0.01; t],[x(3,1); x(:,3)]),
% grid,ylabel('X_3 (rad)'),,xlabel('t (s)')
% subplot(2,2,4), plot([-0.01; t],[x(4,1); x(:,4)]),
% grid,ylabel('X_4 (rad/s)'),xlabel('t (s)'),
%
% figure(2)

```

```
% plot([-0.01;t],[dv(1) dv]),grid on, xlabel('t (s)'),
% ylabel('\Delta v(t) (m/s)')
%
% figure (3), subplot(2,2,1),
% plot([-0.01; t],[w1(1) w1]*scale1),
% grid,ylabel('w_1 (rad/s)'),
% subplot(2,2,2), plot([-0.01; t],[w2(1) w3]*scale2),
% grid,ylabel('w_2 (rad/s)'),xlabel('t (s)'),
% subplot(2,2,3), plot([-0.01; t],[w3(1) w2]*scale1),
% grid,ylabel('w_3 (rad/s^2)'),
% subplot(2,2,4), plot([-0.01; t],[w4(1) w4]*scale2),
% grid,ylabel('w_4 (rad/s^2)'),xlabel('t (s)'),

% control=K*x';
% figure (4), plot([-0.01; t],[control(1) control]),
% grid,ylabel('u (N)'),xlabel('t (s)'),

A+B1*K is Hurwitz: proceed!!
```


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