Frequency Estimation of Mono- and Multi-component FM Signals Using the T - Distributions

Zahir M. Hussain, *Senior Member, IEEE*
School of Electrical and Computer Engineering, RMIT, Melbourne, Victoria 3000, Australia
Email: zmhussain@ieee.org

***Abstract*— In a recent work we have proposed a subclass of Cohen's Class of quadratic time-frequency distributions (TFD's), the T-class of distributions with time-only Doppler-lag kernels to provide high-resolution and considerable cross-terms reduction for FM signals. In this work we investigate the instantaneous frequency (IF) properties of two members of this class: the hyperbolic and the exponential T-distributions in the presence of Gaussian noise. Both mono- and multi-component FM signals will be considered, with various modulation coefficients. A comparison with two well-known TFD's, Wigner-Ville and Choi-Williams distributions, is presented for performance evaluation.**

I. Introduction

Instantaneous frequency (IF) estimation plays a significant role in signal processing and communication engineering [1], [2], [3]. Methods of IF estimation can be classified into two major categories: parametric and non-parametric. Parametric IF estimation methods are complicated and time-consuming, hence not suitable for real-time applications. Non-parametric IF estimation for multicomponent non-stationary signals is an important (and unresolved) issue in signal processing [4], [5]. The concept of the instantaneous frequency can be found in [4], [1], [2], [3]. Time-frequency analysis is used for IF estimation for multicomponent signals as it is the only reliable tool to reveal the multicomponent nature of such signals by concentrating the signal energy in the time-frequency plane around the component IF laws [1]. These energy concentrations are known as "peaks" or "ridges" of the time-frequency representation or distribution (TFD). However, the quadratic time - frequency distributions of multicomponent signals suffer from the presence of cross-terms [1], [2], [3], which can obscure the real features of interest in the signal. Considerable efforts have been made to define TFD's which reduce the effect of cross-terms while improving the time-frequency resolution [1], [3]. However, there is always a compromise between these two requirements. TFD's have different performances in this respect and the choice of the proper TFD is application dependent.

A class of time-frequency distributions with high time-frequency resolution and strong cross-terms reduction was proposed in [4], [7] and proved to be effective for both mono- and multicomponent FM signals. Also it was recently shown that this class is highly effective in Blind Source Separation (BSS) [8]. Members of this class has kernels that are functions of time only. We shall refer to these TFD's with time-only kernels as the T-distributions (TD's). In this paper we show that this class is also efficient in IF estimation of mono- and multicomponent FM signals in the presence of additive gaussian noise. Its performance is compared to two widely used members of the quadratic class of TFD's: The Choi-Williams Distribution (CWD) and the Wigner-Ville Distribution (WVD).

II. Time-Frequency Distributions for Instantaneous Frequency Estimation

For time-frequency analysis of a real signal $x(t)$, we always consider its analytic associate $z(t) = x(t) + j\hat{x}(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$ [2].

Consider an analytic signal of the form

$$z(t) = ae^{j\phi(t)} + \epsilon(t) \quad (1)$$

where the amplitude a is constant, and $\epsilon(t)$ is a complex-valued white Gaussian noise with independent identically distributed (i.i.d.) real and imaginary parts with total variance σ_ϵ^2. The instantaneous frequency of $z(t)$ is given by

$$f_i(t) = (1/2\pi)d\phi(t)/dt \quad (2)$$

We assume in this analysis that $f_i(t)$ is an arbitrary, smooth and differentiable function of time with bounded derivatives of all orders.

The general equation for quadratic time-frequency representation of a signal $z(t)$ is given by [2]

$$\rho(t, f) = \underset{\tau \to f}{\mathcal{F}}[G(t, \tau) \underset{(t)}{*} K_z(t, \tau)] \quad (3)$$

where $G(t, \tau)$ is the time-lag kernel, $K_z(t, \tau) = z(t + \tau/2)z^*(t - \tau/2)$ and $\underset{(t)}{*}$ denotes time convolution. The kernel could also be expressed in the Doppler-lag domain as $g(\nu, \tau)$, where

$$G(t, \tau) = \underset{\nu \to t}{\mathcal{F}^{-1}}\{g(\nu, \tau)\} \quad (4)$$

In the discrete lag domain $\rho(t, f)$ can be written as follows:

$$\begin{aligned}\rho(t, f) &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K_z(u, 2mT) \\ &\quad \times G(t - u, 2mT)e^{-j4\pi f mT} du \end{aligned} \quad (5)$$

where m is an integer and T is the sampling interval. If $\rho(t, f)$ is discretized over time and frequency then we have

$$\begin{aligned}\rho(n, k) &= \sum_{l=-N}^{N-1} \sum_{m=-N}^{N-1} K_z(lT, 2mT) \\ &\quad \times G(nT - lT, 2mT)e^{-j2\pi \frac{km}{2N}} \end{aligned} \quad (6)$$

where $2N$ is the total number of signal samples. In the frequency domain, the frequency samples are given by $f_k = k/4NT$.

The IF estimate is a solution of the following optimization:

$$\widehat{f}_i(t) = \arg[\max_f \rho(t,f)] \quad ; \; 0 \leq f \leq f_s/2 \tag{7}$$

where $f_s = 1/T$ is the sampling frequency. The frequency estimation error is the difference between the actual value in eq.(2) and the estimate in eq.(7) as follows

$$\Delta\widehat{f}_i(t) = f_i(t) - \widehat{f}_i(t) = \phi'(t)/2\pi - \widehat{f}_i(t). \tag{8}$$

Therefore, the bias and variance are described as follows:

$$\begin{aligned} B(\hat{f}_i(t)) &= \mathcal{E}[\Delta\hat{f}_i(t)] = f_i(t) - \mathcal{E}[\hat{f}_i(t)] \\ V(\hat{f}_i(t)) &= \mathcal{E}[\Delta\hat{f}_i(t)]^2 = \mathcal{E}[\{f_i(t) - \hat{f}_i(t)\}^2] \end{aligned} \tag{9}$$

As we will see later, for the T-Class of TFD's, this bias is zero for single-tone and linear FM (LFM) signals, and therefore a Cramer-Rao bound (CRB) exists for the variance.

III. The T-Class of Quadratic Time-Frequency Distributions

Time-only kernels are a special case of separable time-lag kernels. Suppose we have a separable time-lag kernel as follows

$$G(t,\tau) = g_1(t)g_2(\tau) \tag{10}$$

where g_1 and g_2 are continuous and L^2 integrable functions.

It was shown in [7] that for best time-frequency resolution we should have

$$\begin{aligned} G(t,\tau) &= R(t) = r_1(t)/M \\ g(\nu,\tau) &= r(\nu) = \underset{t\to\nu}{\mathcal{F}}^{-1}\{r_1(t)\}/M \end{aligned} \tag{11}$$

where $M = \int g_1(u)du$ is a constant and $G(t,\tau)$ is now a time-only kernel. This is the formula for all time-only kernels, which are the kernels of the T-distributions.

To examine the behavior of this kind of kernels in terms of resolution and cross-terms reduction, we consider a sum of two complex sinusoids

$$z(t) = a_1\exp\{j(2\pi f_1 t + \theta_1)\} + a_2\exp\{j(2\pi f_2 t + \theta_2)\} \tag{12}$$

where a_1, a_2 are real constants and θ_1 and θ_2 are phase constants. We obtain

$$\begin{aligned} \rho_z(t,f) &= a_1^2\delta(f-f_1) + a_2^2\delta(f-f_2) + 2a_1a_2\,\mathrm{g}(f_1-f_2) \\ &\quad \times\cos\{2\pi(f_1-f_2)t + \theta_1 - \theta_2\}\;\delta(f - \frac{f_1+f_2}{2}) \end{aligned} \tag{13}$$

where there is still *ideal concentration* about the auto-terms, and cross-terms appear with a *controlling factor* $\mathrm{g}(f_1 - f_2)$. The Wigner-Ville distribution, which utilizes a time-only kernel $G(t,\tau) = \mathrm{G}(t) = \delta(t)$ with $g(\nu,\tau) = \mathrm{g}(\nu) = 1$, has significant oscillatory cross-terms without a controlling factor, where the cross-terms can be larger in amplitude than the auto-terms. However, using a low-pass time-only kernel other than $\delta(t)$ will result in controlling the cross-terms by the low-pass function g. In case of two complex sinusoids above we have the controlling factor $\mathrm{g}(f_1 - f_2)$ with cross-terms reduction that depends on the shape of the low-pass function g and the frequency separation $f_1 - f_2$, where better cross-terms reduction is obtained for wider frequency separation.

The Exponential T-Distribution (ETD): the kernel of the Choi-Williams distribution (CWD) in the Doppler - lag domain is $g(\nu,\tau) = \exp(-4\pi^2\nu^2\tau^2/\sigma)$ which can be given in the time-lag domain by [2]

$$G(t,\tau) = \sqrt{\sigma/4\pi\tau^2}\exp(-\sigma t^2/4\tau^2) \tag{14}$$

where σ is a real parameter. In [7], we proposed a time-frequency distribution $T_e(t,f)$ with the following exponential *time-only kernel*

$$G(t,\tau) = R_\sigma(t) = \sqrt{\sigma/\pi}\exp(-\sigma t^2) \tag{15}$$

where σ is a real parameter and $\sqrt{\sigma/\pi}$ is a normalization factor. It was shown in [4] that the resolution of the ETD exceeds that of CWD by far.

The hyperbolic T-distribution (HTD): it has the following time-only kernel [7]

$$G(t,\tau) = R_\sigma(t) = k_\sigma/\cosh^{2\sigma}(t) \tag{16}$$

where σ is a real positive number and k_σ is a normalization factor given by

$$k_\sigma = \int_{-\infty}^{\infty} 1/\cosh^{2\sigma}(t)dt = \Gamma(2\sigma)/2^{2\sigma-1}\Gamma^2(\sigma)$$

in which Γ represents the gamma function.

IV. Frequency Estimation Using the T-Class of Distributions

It can be shown that the T-distributions do not satisfy the time marginal property, hence they do not satisfy the traditional condition for the instantaneous frequency. But in [4] we proposed the following general IF property: at any time t, the time-frequency distribution $\rho_z(t,f)$ should have absolute maximum at $f = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$, which is the actual important characteristic needed for IF estimation. In [4] we have also shown that at any t, the hyperbolic T-distribution *has an absolute maximum* at $f = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$ for linear FM signals. This is general for all T-distributions and constitutes the basis for our IF estimation. For non-linear FM signals this IF estimate is biased, and best IF estimation is achieved in this case by adaptive methods [4]. For an FM signal of the form $z(t) = a\,e^{j\phi(t)}$, a being a constant, the general formula for the T-distributions can be given by

$$\begin{aligned} \rho_z(t,f) &\approx |a|^2\int R_\sigma(t-u)\,\delta\,[\frac{1}{2\pi}\phi'(u) - f]\,du \\ &= |a|^2 R_\sigma(t-\psi(f))\psi'(f) \end{aligned} \tag{17}$$

where ψ is the inverse of $\frac{1}{2\pi}\phi'$, i.e., $\frac{1}{2\pi}\phi'(\psi(f)) = f$ and it is assumed that there is a relatively small effect from higher-order derivatives $\phi^{(k)}(t), k \geq 3$. Assuming that $\psi'(f)$ is not a highly peaked function of f and knowing that $R_\sigma(t - \psi(f))$ is peaked at $t = \psi(f)$ since it is low-pass and even in t, the absolute maximum of $\rho_z(t, f)$ for any time t would be at $\psi(f) = t$, or $f = \frac{1}{2\pi}\phi'(t)$, which is the instantaneous frequency of the FM signal $z(t)$. For non-linear FM signals, the energy peak of $\rho_z(t, f)$ is biased from the instantaneous frequency due to the higher-order phase derivatives. The major contribution in this term is due to $\phi^{(3)}(u)$ [4]. Therefore at the instants of rapid change in the IF law the bias is not negligible and eq.(15) would not be an accurate approximation to $\rho(t, f)$ unless a suitable windowing in the lag direction is used. An adaptive window length would be recommended, but due to significant bias no CRB is applicable.

For linear FM (LFM) signals we have $\phi^{(k)}(t) = 0$ for $k \geq 3$. If $\phi(t) = 2\pi(f_o t + \beta_o t^2/2)$, where f_o, β_o are constants, we have

$$\rho_z(t, f) = \frac{1}{\beta_o}|a|^2 R_\sigma\Big[t - \frac{1}{\beta_o}(f - f_o)\Big] \qquad (18)$$

which has an absolute maximum at $f = f_o + \beta_o t$, the instantaneous frequency. As $\beta_o \rightarrow 0$, the linear-FM signal $z(t)$ will approach a sinusoid, and we have $\rho(t, f) \rightarrow |a|^2\delta(f - f_o)$ for a monocomponent single-tone signal. For a signal composed of the sum of two LFM signals $z(t) = a_1\, e^{j\phi_1(t)} + a_2\, e^{j\phi_2(t)}$ with $\phi_i(t) = 2\pi(f_i t + \beta_i t^2/2)$, $i \in \{1, 2\}$, the T-distribution can be expressed as follows:

$$\begin{aligned}\rho_z(t, f) &= \frac{1}{\beta_1}|a_1|^2 R_\sigma\Big[t - \frac{1}{\beta_1}(f - f_1)\Big] \\ &+ \frac{1}{\beta_2}|a_2|^2 R_\sigma\Big[t - \frac{1}{\beta_2}(f - f_2)\Big] \\ &+ \text{cross} - \text{terms.}\end{aligned} \qquad (19)$$

In the next section we will consider a monocomponent linear FM signal as well as a multicomponent signal with LFM components to test the IF estimation capabilities of the T-Class as compared to WVD and CWD.

V. Simulation Results: A Comparison in Cohen's Class of Distributions

The above time-frequency distributions were simulated and the instantaneous frequency (IF) was estimated according to eqs. (1) and (6-9). First, as a monocomponent signal, a linear FM test signal $z(t) = a\, e^{j\phi(t)}$, $\phi(t) = 2\pi(f_o t + \beta t^2/2)$, with $a = 1$, $f_o = 0.05 f_s$, $\beta = 0.4 f_s$ is used. The instantaneous frequency is given by $f = \frac{1}{2\pi}d\phi/dt = f_o + \beta t$ as shown in Fig.(1). For TFD implementation, the signal length $2N = 512$ samples was selected. The sampling frequency was $f_s = 2N$ Hz, where the total signal duration will be 1 sec. As noise, i.i.d noise samples were added using different SNR's. For each SNR, 1000 Monte Carlo iterations were considered for the purpose of calculating the variance of the IF estimate. Fig.(2) shows the Hyperbolic T-distribution (HTD) of the above monocomponent LFM signal with $\sigma = 0.05$ and signal-to-noise ratio SNR = -5 dB. Even at very low SNR's, the HTD gives a clear concentration around the instantaneous frequency of the signal. Fig.(3) shows the result of applying IF estimation on the above noisy LFM for three TFD's. The performance of the HTD is distinguished as superior to other TFD's, especially at low SNR's. Performance of the ETD is comparable to that of the HTD for monocomponent signals.

Second, to test the performance in IF estimation for multi-component signals, a multicomponent test signal is considered with two linear FM components $z(t) = a_1\, e^{j\phi_1(t)} + a_2\, e^{j\phi_2(t)}$, $\phi_1(t) = 2\pi(f_1 t + \beta_1 t^2/2)$, $\phi_2(t) = 2\pi(f_2 t + \beta_2 t^2/2)$, where $a_1 = a_2 = 1$, $f_{o1} = 0$, $f_{o2} = 0.2 f_s$, $\beta_1 = 0.45 f_s$, $\beta_2 = 0.3 f_s$. The instantaneous frequencies of the individual components are given respectively by (see [4]) $f_1 = \frac{1}{2\pi}d\phi_1/dt = f_o + \beta_1 t$ and $f_2 = \frac{1}{2\pi}d\phi_2/dt = f_{o2} + \beta_2 t$ as shown in Fig. (4). For TFD implementation and robust testing of IF estimation performance, the number of signal points was $2N = 2^9$ points, with $f_s = 2N$ Hz and total signal duration of 1 sec. Noise is applied as above. Fig.(5) shows the exponential T-distribution (ETD) of the above multicomponent FM signal with $\sigma = 0.1$ and signal-to-noise ratio SNR = -5 dB. Despite the fact that IF estimation was at a very low SNR, the ETD is giving a clear concentration (ridge) around the two instantaneous frequencies of the signal. Fig.(6) shows the result of applying IF estimation on the first component of the above noisy multicomponent signal using three TFD's. For each SNR, 1000 Monte Carlo iterations were considered to calculate the variance of the IF estimate. The performance of the ETD is distinguished as superior to other TFD's, including the WVD (which gives ideal concentration for LFM's), especially at low SNR's. The HTD gives a comparable performance, while CWD lags behind these TFDs. It is worth noting that all TFD's approach the same Cramer-Rao bound as SNR increases; this bound is not evident at Fig.(6) as it needs a much larger $2N$ to be revealed, and this will cause computer memory problems when simulated.

VI. Conclusion

It is shown that members the recently developed T-Class of time-frequency distributions (TFD's) outperform other well - known distributions like the Wigner - Ville distribution (WVD) and the Choi - Williams distribution (CWD) in terms of robustness in instantaneous frequency estimation for mono-component and multicomponent signals. For monocomponent IF estimation, the exponential T-distribution (ETD) and the hyperbolic T-distribution (HTD) give a minimal variance for all SNR's, however, the difference in performance is more evident at low SNR's, where the T-Class distributions outperform other TFD's by far. For multicomponent IF estiamtion, the ETD gives the lowest IF variance among all TFD's above.

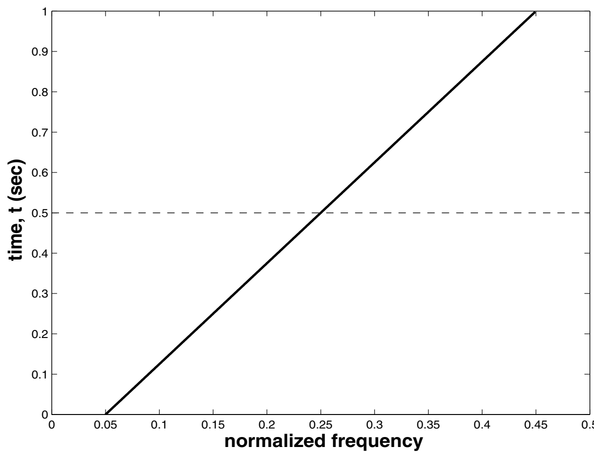

Fig. 1. Instantaneous frequency (IF) of the LFM test signal $z(t) = a\ e^{j\phi(t)}$, $\phi(t) = 2\pi(f_o t + \beta t^2/2)$, with $a = 1$, $f_o = 0.05 f_s$, $\beta = 0.4 f_s$. The instantaneous frequency is given by $f = \frac{1}{2\pi} d\phi/dt = f_o + \beta t$.

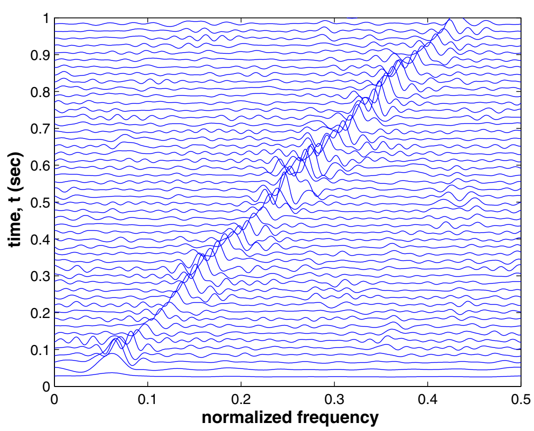

Fig. 2. Hyperbolic T-distribution (HTD) of the linear FM signal in Fig.(1) with $\sigma = 0.05$ and signal-to-noise ratio SNR = -5 dB. The number of points was $2N = 2^9$ points, with $f_s = 2N$ Hz, where the total signal duration will be 1 sec. Despite the very low SNR, the HTD is giving a clear concentration around the instantaneous frequency of the signal.

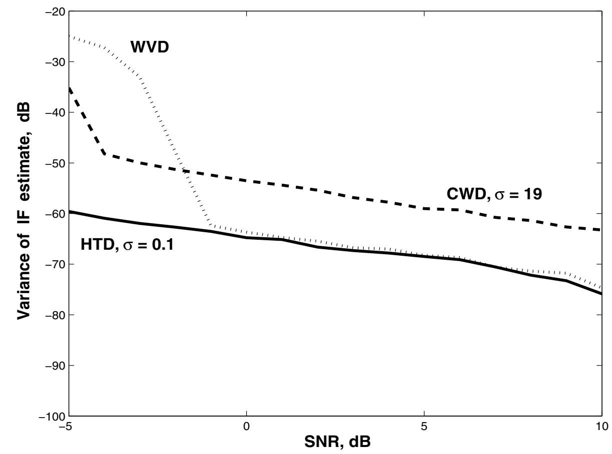

Fig. 3. Performance of various TFD's in IF estimation of a linear FM signal with length $2N = 512$ samples. The sampling frequency was $f_s = 2N$ Hz. It is evident that the recently proposed HTD surpasses other TFD's in robustness where it gives the minimum variance, especially at low SNR's.

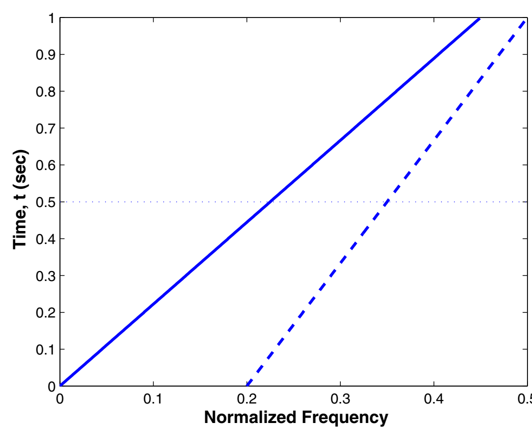

Fig. 4. Instantaneous frequency (IF) components of the multicomponent linear FM test signal $z(t) = a_1\ e^{j\phi_1(t)} + a_2\ e^{j\phi_2(t)}$, $\phi_1(t) = 2\pi(f_1 t + \beta_1 t^2/2)$, $\phi_2(t) = 2\pi(f_2 t + \beta_2 t^2/2)$, where $a_1 = a_2 = 1$, $f_{o1} = 0$, $f_{o2} = 0.2 f_s$, $\beta_1 = 0.45 f_s$, $\beta_2 = 0.3 f_s$. The instantaneous frequencies are given by $f_1 = \frac{1}{2\pi} d\phi_1/dt = f_o + \beta_1 t$ and $f_2 = \frac{1}{2\pi} d\phi_2/dt = f_{o2} + \beta_2 t$.

REFERENCES

[1] Leon Cohen, "Time-frequency distributions - a review" *Proceedings of the IEEE*, vol. 77, no. 7, pp. 941-981, July 1989.

[2] Leon Cohen, *Time-Frequency Analysis*, Prentice Hall PTR, Englewood Cliffs, New Jersey, 1995.

[3] Patrick Flandrin, *Time-Frequency/ Time-Scale Analysis*, Academic Press, Boston, MA, 1998.

[4] Zahir M. Hussain and Boualem Boashash, "Adaptive instantaneous frequency estimation of multi-component FM signals using quadratic time-frequency distributions," *IEEE Transactions on Signal Processing*, vol. 50, no. 8, pp. 1866 -1876, Aug. 2002.

[5] Zahir M. Hussain and Boualem Boashash, "IF estimation for multicomponent signals," in *Time-Frequency Signal Analysis and Processing: A Comprehensive Reference*, pp. 437-445, Elsevier, Oxford, UK, 2003.

[6] Zahir M. Hussain and Boualem Boashash, "Multi-component IF estimation," *Proceedings of the IEEE Signal Processing Workshop on Statistical Signal and array Processing (SSAP'2000)*, Pocono Manor, Pennsylvania, USA, Aug. 14-16, 2000.

[7] Zahir M. Hussain and Boualem Boashash, "Design of time-frequency distributions for amplitude and IF estimation of multicomponent signals," *International Symposium on Signal Processing and Its Applications (ISSPA'2001)*, vol. 1, pp. 339-342, Aug. 2001.

[8] Lei Shang, Zahir M. Hussain, and Richard Harris, "Performance of Quadratic Time-Frequency Distributions in Blind Source Separation," *Proceedings of the Australian Telecommunications Networks and Applications Conference (ATNAC 2003)*, Melbourne, Australia, 8-10 Dec. 2003.

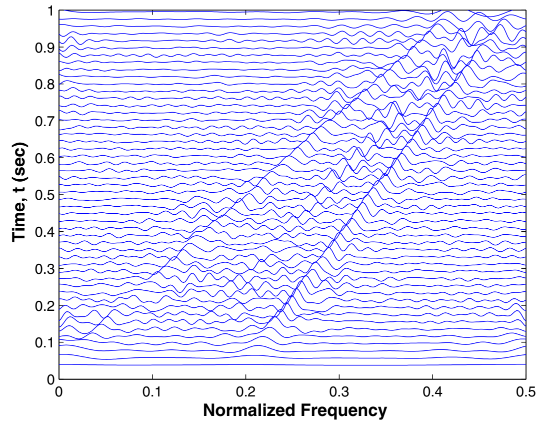

Fig. 5. Exponential T-distribution (ETD) of the multicomponent FM signal in Fig.(4) with $\sigma = 0.1$ and signal-to-noise ratio SNR = -5 dB. The number of points was $2N = 2^9$ points, with $f_s = 2N$ Hz and total signal duration of 1 sec. Although operating at a very low SNR, the ETD is giving a clear concentration around the instantaneous frequencies of the LFM components.

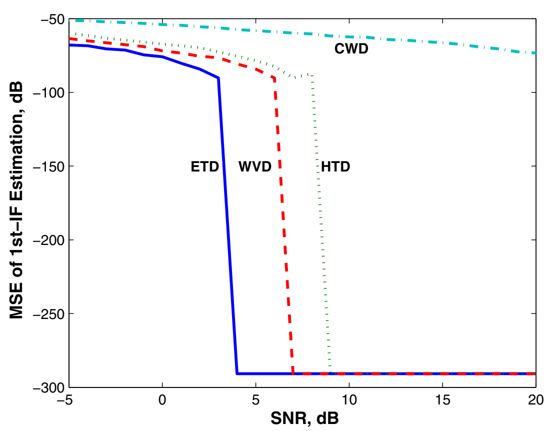

Fig. 6. Performance of various TFD's in IF estimation of the first linear FM component of the multicomponent signal in Fig.(4) with length $2N = 2^9$ samples. The sampling frequency was $f_s = 2N$ Hz. It is evident that the recently proposed ETD surpasses other TFD's in robustness where it gives the minimum variance, especially at low SNR's. The performance of the ETD is near to that of WVD, while CWD lags by far. Theoretically, WVD gives the ideal concentration for LFM (i.e., a delta function around the IF ridge in the t-f plane).