

A SUMMATION FORMULA FOR SEQUENCES INVOLVING FLOOR AND CEILING FUNCTIONS

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ABSTRACT. A closed form expression for the N th partial sum of the p th powers of $\|\sqrt{n}\|$ is obtained, where $\|\cdot\|$ denotes the nearest integer function. As a consequence, a necessary and sufficient condition for the divisibility of n by $\|\sqrt{n}\|$ is derived together with a closed form expression for the least nonnegative residue of n modulo $\|\sqrt{n}\|$. In addition an identity involving the zeta function $\xi(s)$ and the infinite series $\sum_{n=1}^{\infty} 1/\|\sqrt{n}\|^{s+1}$ for real $s > 1$ is also obtained.

1. Introduction. In a recent paper, see [3], the author examined the problem of determining a closed form expression for those sequences $\langle b_m \rangle$ formed from an arbitrary sequence of real numbers $\langle a_n \rangle$ in the following manner. Let $d \in \mathbf{N}$ be fixed, and for each $m \in \mathbf{N}$ define b_m to be the m th term of the sequence consisting of nd occurrences in succession of the terms a_n , as follows:

$$(1) \quad \underbrace{a_1, \dots, a_1}_{d, a_1 \text{ terms}}, \underbrace{a_2, \dots, a_2}_{2d, a_2 \text{ terms}}, \underbrace{a_3, \dots, a_3}_{3d, a_3 \text{ terms}}, \dots$$

For example, if $a_n = n$ and $d = 1$ then the resulting sequence $\langle b_m \rangle$ would be

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots$$

Specifically, the problem described above required the construction of a function $f : \mathbf{N} \rightarrow \mathbf{N}$ such that $b_m = a_{f(m)}$. As was shown in [3] the required function $f(\cdot)$ can easily be described in terms of a combination of floor and ceiling functions, that is the functions defined as $[x] = \max\{n \in \mathbf{Z} : n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbf{Z} : x \leq n\}$ respectively. In particular, for the sequence in (1), we have that $b_m = a_{f(m)}$ where

$$(2) \quad f(m) = \left\lceil \sqrt{\left\lceil \frac{2m}{d} \right\rceil} + \frac{1}{2} \right\rceil.$$

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In this note we continue our examination of those sequences defined in (1) by deriving a summation formula for the N th partial sum $S_N = \sum_{m=1}^N b_m$. Our goal here will be to deduce, as a consequence of the aforementioned formula, a closed form expression for the partial sum of the p th powers of $\|\sqrt{n}\|$, where $\|x\|$ denotes the nearest integer to x . In particular, as special cases it will be shown that

$$(3) \quad \sum_{n=1}^N \frac{1}{\|\sqrt{n}\|} = \frac{N}{\|\sqrt{N}\|} + \|\sqrt{N}\| - 1$$

$$(4) \quad \sum_{n=1}^N \|\sqrt{n}\| = \frac{\|\sqrt{N}\|}{3} (3N + 1 - \|\sqrt{N}\|^2).$$

As will be seen, the method used to establish (3) and (4) is quite different from that employed in establishing a closed form expression for $\sum_{n=1}^N \lfloor \sqrt{n} \rfloor$ as demonstrated in [1, p. 86]. In addition, as a consequence of (3), a necessary and sufficient condition will be derived for the divisibility of N by $\|\sqrt{N}\|$, together with a closed form expression for the least nonnegative residue of N modulo $\|\sqrt{N}\|$.

2. Main result. We begin with a technical result which will help facilitate the calculation of the N th partial sum of the sequences defined in (1).

Lemma 2.1. *Suppose $\langle a_n \rangle$ is an arbitrary sequence of real numbers, and let $d \in \mathbf{N}$. Then, for the sequence $\langle b_m \rangle$ defined in (1), we have*

$$(5) \quad S_N = \sum_{m=1}^N b_m = \left(N - \frac{d}{2} (f(N) - 1) f(N) \right) a_{f(N)} + d \sum_{n=1}^{f(N)-1} n a_n,$$

where $f(\cdot)$ is the function in (2).

Proof. For the sequence defined in (1), we have $b_m = a_n$, whenever $n(n-1)d/2 < m \leq n(n+1)d/2$, that is, $f(m) = n$ when

$$m \in I_n = \left[\frac{n(n-1)}{2} d + 1, \frac{n(n+1)}{2} d \right].$$

Now defining the mapping $S : \mathbf{N} \rightarrow \mathbf{N}$ by $S(N) = \max\{n \in \mathbf{N} : N \notin \cup_{r=1}^n I_r\}$ and noting that each interval I_n contains nd integers, observe the following

$$\begin{aligned}
 (6) \quad S_N &= \sum_{n=1}^{S(N)} \sum_{r \in I_n} a_{f(r)} + \sum_{\substack{r \in I_{S(N)+1} \\ r \leq N}} a_{f(r)} \\
 &= \sum_{n=1}^{S(N)} n d a_n + \sum_{\substack{r \in I_{S(N)+1} \\ r \leq N}} a_{S(N)+1}.
 \end{aligned}$$

Our task is thus reduced to determine a closed form expression for $S(N)$ in terms of N and so evaluate the second summation in (6). Suppose $N \in I_n$ for some $n \in \mathbf{N}$, then by definition of I_n ,

$$(7) \quad \frac{n(n-1)}{2} d < N \leq \frac{n(n+1)}{2} d.$$

Now if, for some $x \in \mathbf{R}^+$ we have $n_1 < x \leq n_2$ for $n_1, n_2 \in \mathbf{N}$, then $n_1 + 1 \leq \lceil x \rceil \leq n_2$. Consequently, from the inequality in (7) we have $n(n-1) + 1 \leq \lceil 2N/d \rceil \leq n(n+1)$. However, as $\sqrt{n(n+1)} < n + 1/2$ and $n - 1/2 < \sqrt{n(n-1)} + 1$, one in turn deduces that

$$n < \sqrt{\left\lceil \frac{2N}{d} \right\rceil} + \frac{1}{2} < n + 1.$$

Thus, we have $n = f(N)$ and so $S(N) = f(N) - 1$. Finally as the number of integers $r \in I_{S(N)+1} = I_{f(N)}$ with $r \leq N$ is given by

$$N - f(N)(f(N) - 1) \frac{d}{2} - 1 + 1,$$

one sees that the second summation in (6) is equal to

$$\sum_{\substack{r \in I_{f(N)} \\ r \leq N}} a_{f(N)} = \left(N - \frac{d}{2} (f(N) - 1) f(N) \right) a_{f(N)}.$$

Hence (6) yields (5) as required. □

Before establishing the main result it should be noted that the mapping $x \mapsto \|x\|$ is strictly, by definition, multi-valued at $x = (2n + 1)/2$, where $n \in \mathbf{N}$, since $(2n + 1)/2$ lies a distance of $1/2$ units from n and $n + 1$. In such cases the convention, as in [2, p. 78], is to set $\|(2n + 1)/2\| = n + 1$. However this ambiguity does not arise for the mapping $N \mapsto \|\sqrt{N}\|$, where $N \in \mathbf{N}$, as $\sqrt{N} \neq (2n + 1)/2$ for any $n \in \mathbf{N}$. We now prove our main result for summing the p th powers of $\|\sqrt{n}\|$, from which (3) and (4) will follow as a corollary.

Theorem 2.1. *Suppose $p \in \mathbf{R}$, then*

$$(8) \quad \sum_{n=1}^N \|\sqrt{n}\|^p = (N - (\|\sqrt{N}\| - 1) \|\sqrt{N}\|) \|\sqrt{N}\|^p + 2 \sum_{n=1}^{\|\sqrt{N}\| - 1} n^{p+1}.$$

In particular, when $p \in \mathbf{N}$, then

$$\begin{aligned} \sum_{n=1}^N \|\sqrt{n}\|^p &= (N - (\|\sqrt{N}\| - 1) \|\sqrt{N}\|) \|\sqrt{N}\|^p \\ &\quad + \frac{2}{p+2} \sum_{k=0}^{p+2} \binom{p+2}{k} B_k \|\sqrt{N}\|^{p+2-k}, \end{aligned}$$

where B_k denotes the k th Bernoulli number.

Proof. We first show that $\|x\| = \lfloor x + 1/2 \rfloor$ for every $x \in \mathbf{R}^+$. Indeed suppose $\|x\| = n$, taking the largest if two are equally distant. Setting $n = x + \theta$ with $-1/2 < \theta \leq 1/2$, observe $\lfloor x + 1/2 \rfloor = n + \lfloor -\theta + 1/2 \rfloor = n$ since $0 \leq -\theta + 1/2 < 1$. Consequently, $\|\sqrt{m}\| = \lfloor \sqrt{m} + 1/2 \rfloor$ and so from (2) we deduce that the sequence $\langle \|\sqrt{m}\|^p \rangle$ corresponds to the sequence $\langle b_m \rangle$ defined in (1), with $a_n = n^p$ and $d = 2$. Hence, in this instance, we see that (5) reduces to (8) as required. Finally if $p \in \mathbf{N}$, then the second equality follows immediately from the identity

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}. \quad \square$$

We now examine (8) in the case when $p = \pm 1$.

Corollary 2.1.

$$\sum_{n=1}^N \|\sqrt{n}\|^p = \begin{cases} N\|\sqrt{N}\|^{-1} + \|\sqrt{N}\| - 1 & \text{for } p = -1 \\ (\|\sqrt{N}\|/3)(3N + 1 - \|\sqrt{N}\|^2) & \text{for } p = 1. \end{cases}$$

Proof. Setting $p = -1$ in (8), observe that

$$\begin{aligned} \sum_{n=1}^N \|\sqrt{n}\|^{-1} &= (N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\|)\|\sqrt{N}\|^{-1} + 2 \sum_{n=1}^{\|\sqrt{N}\|-1} 1 \\ &= N\|\sqrt{N}\|^{-1} + \|\sqrt{N}\| - 1. \end{aligned}$$

Similarly, setting $p = 1$ in (8) and recalling $\sum_{r=1}^n r^2 = n(n + 1)(2n + 1)/6$, one arrives, after some simplification, at the second formula.

□

Using the summation formula in (3) we can deduce the following divisibility property.

Corollary 2.2. *Suppose $N \in \mathbf{N}$. Then $\|\sqrt{N}\|$ divides N if and only if either $N = \|\sqrt{N}\|^2$ or $N = \|\sqrt{N}\|(\|\sqrt{N}\| + 1)$. Moreover, the least nonnegative residue of N modulo $\|\sqrt{N}\|$ is given by*

$$\begin{aligned} N - \|\sqrt{N}\|^2 + \frac{\|\sqrt{N}\|}{2} \left((-1)^{\lfloor (N/\|\sqrt{N}\|) - \|\sqrt{N}\| + 1 \rfloor} \right. \\ \left. + 2(-1)^{\lfloor 1/2((N/\|\sqrt{N}\|) - \|\sqrt{N}\| + 1) \rfloor} - 1 \right). \end{aligned}$$

Proof. From the summation formula in (3) it is immediate that $\|\sqrt{N}\|$ divides N if and only if $\sum_{n=1}^N 1/\|\sqrt{n}\|$ is an integer. Recalling that $\|\sqrt{m}\| = \lfloor \sqrt{m} + 1/2 \rfloor$, we deduce that the sequence $(\|\sqrt{m}\|^{-1})$ corresponds to the sequence (b_m) defined in (1), with $a_n = 1/n$ and $d = 2$. Consequently, from (6) we find that

$$(9) \quad \sum_{n=1}^N \frac{1}{\|\sqrt{n}\|} = 2(\|\sqrt{N}\| - 1) + \sum_{\substack{r \in J_{\|\sqrt{N}\|} \\ r \leq N}} \frac{1}{\|\sqrt{N}\|},$$

and so our task is reduced to determining those $N \in I_{\|\sqrt{N}\|}$ for which the summation on the righthand side of (9) is integer valued. Now since the interval $I_{\|\sqrt{N}\|}$ contains $2\|\sqrt{N}\|$ integers we see that

$$\frac{1}{\|\sqrt{N}\|} \leq \sum_{\substack{r \in I_{\|\sqrt{N}\|} \\ r \leq N}} \frac{1}{\|\sqrt{N}\|} \leq 2.$$

Furthermore, as the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ is equal to $N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1)$, we conclude that the summation in question assumes the integer values of 1 and 2 if and only if $N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1) = \|\sqrt{N}\|$ and $2\|\sqrt{N}\|$, respectively. Hence, $\|\sqrt{N}\|$ divides N if and only if either $N = \|\sqrt{N}\|^2$ or $N = \|\sqrt{N}\|(\|\sqrt{N}\| + 1)$.

Denote the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ by $R(N)$. After equating (3) with (9) and solving for $N/\|\sqrt{N}\|$, observe from the argument above that the least nonnegative residue of N modulo $\|\sqrt{N}\|$ is equal to $R(N)$, when $1 \leq R(N) < \|\sqrt{N}\|$ and $R(N) - \|\sqrt{N}\|$, when $\|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\|$, while zero, when $R(N) = 2\|\sqrt{N}\|$. Thus the desired residue can be calculated from the following formula

$$(10) \quad R(N) - \sigma(N)\|\sqrt{N}\| - 2\phi(N)\|\sqrt{N}\|,$$

where

$$\sigma(N) = \begin{cases} 0 & 1 \leq R(N) < \|\sqrt{N}\| \\ 1 & \|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\| \\ 0 & R(N) = 2\|\sqrt{N}\| \end{cases}$$

and

$$\phi(N) = \begin{cases} 0 & 1 \leq R(N) < 2\|\sqrt{N}\| \\ 1 & R(N) = 2\|\sqrt{N}\|. \end{cases}$$

Via a simple application of the floor function, we see from inspection that the functions $\sigma(N)$ and $\phi(N)$ are given by

$$\sigma(N) = -\frac{1}{2} \left((-1)^{\lfloor R(N)/\|\sqrt{N}\| \rfloor} - 1 \right)$$

and

$$\phi(N) = -\frac{1}{2} \left((-1)^{\lfloor R(N)/2\|\sqrt{N}\| \rfloor} - 1 \right).$$

Finally substituting the previous expressions for $\sigma(N)$ and $\phi(N)$ into (10) produces, after some simplification, the desired residue formula.

□

Remark 2.1. If $N = s^2$ or $N = s(s + 1)$ for some $s \in \mathbf{N}$, then in either case $s = \|\sqrt{N}\|$. Thus, the previous corollary implies that $\|\sqrt{N}\|$ divides N if and only if N is either a square or a product of two consecutive integers.

To close, we establish a curious connection between the zeta function $\zeta(s)$, for real $s > 1$, and the infinite series involving terms of the form $\|\sqrt{n}\|^{-(s+1)}$.

Corollary 2.3. *Suppose $s > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{\|\sqrt{n}\|^{s+1}} = 2\zeta(s).$$

Proof. After setting $p = -(s + 1)$ in (8) we need only show that

$$(N - (\|\sqrt{N}\| - 1) \|\sqrt{N}\|) \|\sqrt{N}\|^{-(s+1)} = o(1)$$

as $N \rightarrow \infty$. Now, by definition of the floor and ceiling functions, observe that

$$\|\sqrt{N}\| = \left\lfloor \sqrt{N} + \frac{1}{2} \right\rfloor = \left\lceil \sqrt{N} + \frac{1}{2} \right\rceil - 1 \geq \sqrt{N} + \frac{1}{2} - 1 = \sqrt{N} - \frac{1}{2}.$$

Consequently, $(\|\sqrt{N}\| - 1)\|\sqrt{N}\| \geq (\sqrt{N} - 3/2)(\sqrt{N} - 1/2) = N - 2\sqrt{N} + 3/4$, and so $N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\| \leq 2\sqrt{N} - 3/4$. Thus,

$$0 < (N - (\|\sqrt{N}\| - 1) \|\sqrt{N}\|) \|\sqrt{N}\|^{-(s+1)} \leq \frac{2\sqrt{N} - (3/4)}{(\sqrt{N} - (1/2))^{s+1}} \rightarrow 0,$$

as $N \rightarrow \infty$ since $s > 1$. □

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