A polynomial approach to cocycles over elementary abelian groups[*]

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This paper is dedicated to Cheryl Praeger, in honour of her inspiring leadership in Australian group theory and combinatorial design theory.

Abstract

We derive bivariate polynomial formulae for cocycles and coboundaries in $Z^2(\mathbb{Z}_p^n, \mathbb{Z}_p^n)$, and a basis for the $(p^n - 1 - n)$-dimensional $GF(p^n)$-space of coboundaries. When $p = 2$ we determine a basis for the $(2^n + \binom{n}{2} - 1)$-dimensional $GF(2^n)$-space of cocycles and show each cocycle has a unique decomposition as a direct sum of a coboundary and a multiplicative cocycle of restricted form.

Keywords: 2-cocycle, coboundary, multiplicative cocycle, basis, Pascal's triangle

1 Introduction

We consider only two-dimensional cocycles between finite groups, with trivial action. These cocycles are functions arising naturally in surface topology, projective representation theory, combinatorial designs and quantum dynamics, amongst other areas. They are used to search for good high-distance error-correcting codes, low-correlation sequences and functions with strong nonlinearity properties for cryptographic applications. Two basic classes of cocycles, the coboundaries and the multiplicative cocycles, have proved very productive for these purposes. The coboundaries are used to find S-box functions with low differential uniformity, such as PN and APN functions, which are thus robust against differential cryptanalysis. The multiplicative cocycles over elementary abelian groups form a structured space within which to find generalised Hadamard matrices and codes, relative difference sets and finite semifields which coordinatise certain projective planes. Cocycles fall into equivalence classes ("bundles") within which these desirable properties are invariant.

Very little is known about the form of individual cocycles or how to find all the cocycles (or even all the coboundaries) from a finite group G to a finite abelian group C. Focus is usually on listing a set of representatives of the second cohomology group (the quotient group of the group of cocycles by the subgroup of coboundaries). Facilities for computation and manipulation of a set of cohomology class representatives

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exist in MAGMA, in particular in a module (due to Flannery and O'Brien [5]) which uses the Universal Coefficient Theorem to list them. Another algorithm uses a smaller homological model to compute such representatives much faster, but requires more precomputation.

The second cohomology group decomposes as an internal direct sum of two groups, so every cocycle is a sum of an "inflation" cocycle, a "transgression" cocycle and a coboundary, but the decomposition is not unique. Computation of a set of generators for the subgroup of coboundaries is usually left as an exercise in linear algebra, though for cocycles mapping to $\mathbb{Z}_2$, an algorithm is known. In the early 1990s, a group theoretic algorithm was developed that lists a set of generators for the group of cocycles for abelian groups G. See [6, Chapter 6.3] for details.

This paper has two main purposes. The first is to present a formula for any cocycle from $\mathbb{Z}_p^n$ to $\mathbb{Z}_p^n$ as a bivariate polynomial over $GF(p^n)$. This provides a new technique for working with these cocycles and a fourth algorithm for computation in this particular case. The second is to exploit an overlying vector space structure of the group of cocycles to extract a basis for the space of coboundaries. When $p = 2$ – the most important case for applications – we then extract a basis for the space of cocycles.

The paper is organised as follows. In Section 2 we use Lagrange Interpolation and the cocycle equation to derive polynomial formulae for coboundaries, cocycles and multiplicative cocycles from $\mathbb{Z}_p^n$ to $\mathbb{Z}_p^n$. In Section 3 we prove two results about coboundaries. The first (Theorem 3.1) is the basis theorem for coboundaries and the second (Theorem 3.5) is a recursive formula for the coboundary basis in the binary case $p = 2$, suitable for computation.

In the final Section 4, we concentrate on cocycles from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^m$, $n \geq m$. We show (Lemma 4.1) that every symmetrisation cocycle is a coboundary (which, when $m = n$, must be defined by a unique Dembowski-Ostrom polynomial). We then derive the basis theorem for cocycles (Theorem 4.4) and, as a consequence, bases and dimensions for several other subspaces of interest. From this we prove that any cocycle over $\mathbb{Z}_2^n$ has a unique decomposition as a direct sum of a coboundary and a multiplicative non-coboundary cocycle of specific form (Corollary 4.6).

2 Cocycles and coboundaries

Let $GF(q)$ be the finite field of order $q = p^n$, where p is prime and $n \in \mathbb{Z}^+$. Let G be a finite group, C be an additively written finite abelian group, and let $C^1(G, C) = \{\phi : G \to C,\ \phi(1) = 0\}$ be the group of all normalised functions from G to C.

Each $\phi \in C^1(G, C)$ determines a *coboundary* $\partial\phi(x, y) = \phi(xy) - \phi(x) - \phi(y)$, which measures how much ϕ differs from a homomorphism from G to C. A coboundary is the simplest form of cocycle. A (2-dimensional normalised) *cocycle* (with trivial action) is a mapping $\psi : G \times G \to C$ satisfying

$$\psi(1,1) = 0; \quad \psi(x, y) + \psi(xy, z) \ = \ \psi(x, yz) + \psi(y, z), \ \forall x, y, z \in G. \quad (1)$$

The set $Z^2(G, C)$ of cocycles over G with values in C is an abelian group under pointwise addition. The subgroup of coboundaries is denoted $B^2(G, C)$, and the *cobound-*

ary mapping $\partial : C^1(G,C) \to B^2(G,C)$ mapping ϕ to $\partial\phi$ is a group homomorphism with kernel ker(∂) = Hom(G,C).

In the reverse direction to ∂ is the *diagonal mapping*

$$D : Z^2(G,C) \to C^1(G,C), \quad D\psi(x) = \psi(x,x),\ x \in G. \tag{2}$$

It follows that $\partial \circ D : Z^2(G,C) \to B^2(G,C)$ and $(D \circ \partial)\phi(x) = \phi(x^2) - 2\phi(x)$. The mappings D and ∂ may be thought of as generalising the mappings between bilinear and quadratic forms.

We are interested two other subgroups of $Z^2(G,C)$, the subgroup $M^2(G,C)$ of multiplicative cocycles and the subgroup $S^2(G,C)$ of symmetric cocycles.[1]

A cocycle is called *multiplicative* if it is a homomorphism on either coordinate (and hence on both coordinates, by (1)). If G is abelian, on defining $\psi^\top(x,y) = \psi(y,x)$ and $\psi^-(x,y) = \psi(x,y) - \psi(y,x)$ for all $x,y \in G$, we have that $\psi^\top$ and ψ^- are cocycles, the decomposition $\psi = \psi^\top + \psi^-$ is unique, and ψ^-, the commutator pairing, is multiplicative [1, Exercises IV.3.8 and V.6.5].

A cocycle ψ is called *symmetric* if $\psi(x,y) = \psi(y,x)$ for all $x,y \in G$. If G is abelian, the coboundaries are all symmetric; that is, $B^2(G,C) \leq S^2(G,C)$. If G is abelian, the *symmetrisation* ψ^+ of ψ, given by $\psi^+(x,y) = \psi(x,y) + \psi(y,x)$, is a symmetric cocycle, and then the *symmetrisation mapping* $S^+ : Z^2(G,C) \to S^2(G,C)$, given by $S^+(\psi) = \psi^+$, is a group homomorphism. We set $S^2_+(G,C) = S^+(Z^2(G,C))$.

If G is abelian the isotype of the finite abelian group $Z^2(G,C)$ may be derived using cohomological techniques [6, Theorem 6.13, Corollary 6.16]. We record the elementary abelian case needed here.

Proposition 2.1 *Let $G \cong \mathbb{Z}_p^n$ and $C \cong \mathbb{Z}_p^m$, $n \geq m \geq 1$ and set $N = p^n + \binom{n}{2} - 1$, with $\binom{n}{2} = 0$ if $n = 1$. Then $Z^2(\mathbb{Z}_p^n, \mathbb{Z}_p^m) \cong C^N \cong (\mathbb{Z}_p^m)^N$.*

2.1 A polynomial formula for cocycles over $GF(q)$

This subsection is devoted to the representation of cocycles in $Z^2(\mathbb{Z}_p^n, \mathbb{Z}_p^n)$ as bivariate polynomials. We treat $\mathbb{Z}_p^n$ as the underlying additive group of the finite field $GF(q)$, where $q = p^n$, so that $G = C = (GF(q), +)$.

By the Lagrange Interpolation Formula [9, Theorem 1.71], if $m \geq 1$, then for m distinct points $a_0, \ldots, a_{m-1}$ of $GF(q)$ and m arbitrary points $b_0, \ldots, b_{m-1}$ of $GF(q)$ there exists a unique polynomial $f \in GF(q)[x]$ of degree $< m$ such that $f(a_i) = b_i$ for $i = 0, \ldots, m-1$. Any function $\pi : GF(q) \times GF(q) \to GF(q)$ can therefore be represented by a unique polynomial $P \in GF(q)[x,y]$ as follows. Let α be a primitive element of $GF(q)$ and order $GF(q)$ as $GF(q) = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = \alpha, \ldots, \alpha_{q-1} = \alpha^{q-2}\}$. Under this ordering a $q \times q$ array A with entries from $GF(q)$ determined by $\pi(\alpha_i, \alpha_j) = a_{ij}$ will have for each row $A_i = [a_{i0}, \ldots, a_{i,q-1}]$, a unique polynomial $f_i(y) = \sum_{k=0}^{q-1} b_{ik} y^k \ \in GF(q)[y]$ such that $f_i(\alpha_j) = a_{ij}$. For fixed k, the coefficients of y^k from each row are $\{b_{ik} |\ i = 0, \ldots, q-1\}$, and for each k

[1] In [6] the subgroup of symmetric cocycles is denoted $S^2_+(G,C)$ but the notation presented here, with $S^2_+(G,C)$ reserved for the subgroup of symmetrisation cocycles, is more consistent.

there is a unique polynomial $g_k(x) = \sum_{l=0}^{q-1} \lambda_{lk} x^l$ such that $g_k(\alpha_i) = b_{ik}$. Since $f_i(y) = \sum_{k=0}^{q-1} g_k(\alpha_i) y^k$, there exists a unique polynomial $P \in GF(q)[x, y]$ such that $\pi(x, y) = P(x, y) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} \lambda_{lk} x^l y^k$.

If $\psi : GF(q) \times GF(q) \to GF(q)$ also satisfies $\psi(x, 0) = \psi(0, y) = 0$ for all $x, y \in GF(q)$ (a necessary condition for the cocycle equation (1) to hold for $G = C = (GF(q), +)$) then it has a unique representation in $GF(q)[x, y]$

$$\psi(x, y) = \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} \lambda_{ij} x^i y^j. \tag{3}$$

For a coboundary, the coefficients in (3) are very restricted. Again by Lagrange interpolation every $\phi \in C^1(G, G)$ may be represented uniquely as a polynomial of degree at most $q - 1$ in $GF(q)[x]$.

Lemma 2.2 *Let $\psi : GF(q) \times GF(q) \to GF(q)$ be given by (3). If $\phi : GF(q) \to GF(q)$ is given by $\phi(x) = \sum_{i=1}^{q-1} \phi_i x^i$, then $\psi = \partial\phi$ if and only if*

1. $\lambda_{ij} = 0$, *for* $j = q - i, \ldots, q - 1$
2. $\lambda_{ij} = \binom{i+j}{i} \phi_{i+j}$, *otherwise.*

Proof.

$$\begin{aligned} \partial\phi(g, h) &= \sum_{i=1}^{q-1} \phi_i \left\{ (g + h)^i - g^i - h^i \right\} = \sum_{i=2}^{q-1} \sum_{j=1}^{i-1} \phi_i \binom{i}{j} g^{i-j} h^j \\ &= \sum_{i=1}^{q-2} \sum_{j=1}^{q-i-1} \phi_{i+j} \binom{i+j}{j} g^i h^j, \end{aligned}$$

and the result follows on equating coefficients. □

Clearly, if $\psi : GF(q) \times GF(q) \to GF(q)$ is given by (3) then ψ is symmetric if and only if $\lambda_{ij} = \lambda_{ji},\ 1 \le i, j \le q - 1$.

Subsequently we will need the following theorem, attributed to Lucas in [4], and its corollaries.

Theorem 2.3 (Lucas) [4, Theorem 1] *Let p be a prime, and let*

$$\begin{aligned} M &= M_0 + M_1 p + M_2 p^2 + \ldots + M_k p^k, \qquad (0 \le M_r < p) \\ N &= N_0 + N_1 p + N_2 p^2 + \ldots + N_k p^k, \qquad (0 \le N_r < p). \end{aligned}$$

Then

$$\binom{M}{N} = \binom{M_0}{N_0} \binom{M_1}{N_1} \binom{M_2}{N_2} \cdots \binom{M_k}{N_k} \textit{ mod } p. \tag{4}$$

Corollary 2.4 [4, Theorem 3] *A necessary and sufficient condition that all the binomial coefficients*

$$\binom{M}{N}, \quad 0 < N < M, \tag{5}$$

be divisible by p is that M be a power of p.

The p-ary *weight* $w_p(k)$ of the natural number k is the weight of the vector of coefficients of its p-ary expansion, so for example, $w_p(p^i) = 1$ and, for $i \neq j$, $w_p(p^i + p^j) = 2$.

Corollary 2.5 *Let $M = 2, \ldots, p^n - 1$ and $N < M$. Then $w_p(M) = 1 \Longrightarrow \binom{M}{N} = 0$ mod p.*

The next function class of interest is that consisting of the "multiplicative" bivariate polynomials (3); that is, those which are homomorphic in each coordinate. Such functions are always cocycles. The following result may be well-known but a proof is provided for completeness.

Theorem 2.6 *Let $\psi : GF(q) \times GF(q) \to GF(q)$ be given by (3). Then ψ is multiplicative if and only if*

$$\psi(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \lambda_{p^i p^j} x^{p^i} y^{p^j}.$$

Proof. It is straightforward to show that if ψ has this form it is multiplicative. To show the converse is true, let ψ be multiplicative in both coordinates, i.e.

$$\begin{aligned} \psi(x+z, y) &= \psi(x, y) + \psi(z, y), & (6)\\ \psi(x, y+z) &= \psi(x, y) + \psi(x, z). & (7) \end{aligned}$$

Expanding the LHS of (6) gives, for each $\lambda_{ij} \neq 0$,

$$\sum_{i=1}^{q-1} \left[\binom{i}{1} x^{i-1} z + \binom{i}{2} x^{i-2} z^2 + \ldots + \binom{i}{i-1} x z^{i-1} \right] = 0 \quad \forall\, x, z \in GF(q).$$

Therefore, we must have that for each $i = 1, 2, \ldots, q-1$,

$$\binom{i}{r} \equiv 0 \bmod p \text{ for } r = 1, 2, \ldots, i-1.$$

From Corollary 2.4, $i = p^s$ for $s = 0, 1, \ldots, \lfloor \log_p(q-1) \rfloor = n-1$. By symmetry, (7) implies $j = p^l$ only, with $l = 0, 1, \ldots, n-1$. □

Our main result for this section is the following formula for any cocycle in $Z^2(\mathbb{Z}_p^n, \mathbb{Z}_p^n)$, in terms of simultaneous linear equations over $GF(q)$ in the bivariate polynomial coefficients λ_{ij}. Proof is a straightforward exercise in tracking limits of summation.

This new approach to studying $Z^2(\mathbb{Z}_p^n, \mathbb{Z}_p^n)$ complements those in [6].

Theorem 2.7 *Let $\psi : GF(q) \times GF(q) \to GF(q)$ be given by (3). Then ψ satisfies (1) if and only if*

1. $\binom{i+l}{l}\lambda_{i+l,j} = 0$, *for* $i = 1, \ldots, j-1$ *and* $l = q-j, \ldots, q-1-i$
2. $\binom{j+l}{l}\lambda_{i,l+j} = 0$, *for* $i = j+1, \ldots, q-1$ *and* $l = q-i, \ldots, q-1-j$
3. $\binom{i+l}{l}\lambda_{i+l,j} = \binom{j+l}{l}\lambda_{i,l+j}$, *otherwise.*

Proof. Now, $\psi(g,h) + \psi(g+h,k) - \psi(g,h+k) - \psi(h,k) = 0$ if and only if

$$\begin{aligned} 0 &= \sum_{j=1}^{q-1}\sum_{i=2}^{q-1}\sum_{l=1}^{i-1}\lambda_{ij}\binom{i}{l}g^{i-l}h^l k^j - \sum_{j=2}^{q-1}\sum_{i=1}^{q-1}\sum_{l=1}^{j-1}\lambda_{ij}\binom{j}{l}g^i h^{j-l}k^l \\ &= \sum_{j=1}^{q-1}\sum_{i=1}^{q-2}\sum_{l=1}^{q-i-1}\lambda_{i+l,j}\binom{i+l}{l}g^i h^l k^j - \sum_{j=1}^{q-2}\sum_{i=1}^{q-1}\sum_{l=1}^{q-j-1}\lambda_{i,j+l}\binom{j+l}{l}g^i h^l k^j. \end{aligned}$$

Splitting this into cases for $i < j$, $i = j$ and $i > j$ gives

$$\begin{aligned} 0 = &\sum_{j=2}^{q-2}\sum_{i=1}^{j-1}\sum_{l=1}^{q-1-j}\left(\lambda_{i+l,j}\binom{i+l}{l} - \lambda_{i,l+j}\binom{j+l}{l}\right)g^i h^l k^j \\ &+\sum_{j=1}^{q-3}\sum_{i=j+1}^{q-2}\sum_{l=1}^{q-1-i}\left(\lambda_{i+l,j}\binom{i+l}{l}g^i h^l k^j - \lambda_{i,l+j}\binom{j+l}{l}\right)g^i h^l k^j \\ &+\sum_{i=1}^{q-2}\sum_{l=1}^{q-1-i}\left(\lambda_{i+l,i}\binom{i+l}{l} - \lambda_{i,l+i}\binom{i+l}{l}\right)g^i h^l k^i \\ &+\sum_{j=2}^{q-1}\sum_{i=1}^{j-1}\sum_{l=q-j}^{q-1-i}\lambda_{i+l,j}\binom{i+l}{l}g^i h^l k^j - \sum_{j=1}^{q-2}\sum_{i=j+1}^{q-1}\sum_{l=q-i}^{q-1-j}\lambda_{i,l+j}\binom{j+l}{l}g^i h^l k^j. \end{aligned}$$

The sum on the RHS contains each term $g^i h^l k^j$ exactly once, so by linear independence it equals 0 if and only if the stated conditions hold. □

Solution of these simultaneous equations using Theorem 2.3 will give the general cocycle formula for each q. Some coefficients will necessarily be zero.

We illustrate Theorem 2.7 for the smallest examples $q = 2, 3, 4, 5, 7, 8$. The number of independent coefficients λ_{ij}, respectively $1, 2, 4, 4, 6, 10$, is the integer N of Proposition 2.1. (See also [6, Examples 6.3.1, 6.3.2].)

Example 1 *Let $G = C = (GF(q), +)$. Let $\psi \in Z^2(G,G)$ have form (3).*

1. *If* $q = 2$, $\psi(x,y) = \lambda_{11}\, xy$.
2. *If* $q = 3$, $\psi(x,y) = \lambda_{11}\, xy + \lambda_{12}\,(xy^2 + x^2y)$.
3. *If* $q = 4$, $\psi(x,y) = \lambda_{11}\, xy + \lambda_{12}\, xy^2 + \lambda_{21}\, x^2y + \lambda_{22}\, x^2y^2$.

4. *If $q = 5$, $\psi(x,y) = \lambda_{11}\, xy + \lambda_{12}\,(xy^2 + x^2y) + \lambda_{22}\,(4xy^3 + x^2y^2 + 4x^3y)$ $+\lambda_{14}\,(xy^4 + 2x^2y^3 + 2x^3y^2 + x^4y)$.*

5. *If $q = 7$, $\psi(x,y) = \lambda_{11}\, xy + \lambda_{12}\,(xy^2 + x^2y) + \lambda_{13}\,(xy^3 + 5x^2y^2 + x^3y)$ $+\lambda_{14}\,(xy^4 + 2x^2y^3 + 2x^3y^2 + x^4y) + \lambda_{15}\,(xy^5 + 6x^2y^4 + x^3y^3 + 6x^4y^2 + x^5y)$ $+\lambda_{16}\,(xy^6 + 3x^2y^5 + 5x^3y^4 + 5x^4y^3 + 3x^5y^2 + x^6y)$.*

6. *If $q = 8$, $\psi(x,y) = \lambda_{11}\, xy + \lambda_{12}\, xy^2 + \lambda_{21}\, x^2y + \lambda_{22}\, x^2y^2 + \lambda_{14}\, xy^4 + \lambda_{41}\, x^4y +$ $\lambda_{24}\, x^2y^4 + \lambda_{42}\, x^4y^2 + \lambda_{44}\, x^4y^4 + \lambda_{16}\,(xy^6 + x^2y^5 + x^3y^4 + x^4y^3 + x^5y^2 + x^6y)$.*

3 Coboundaries over $GF(p^n)$

We will abbreviate by B the finite abelian group of coboundaries $B^2(\mathbb{Z}_p^n, \mathbb{Z}_p^n)$.

Theorem 3.1 (Coboundary Basis Theorem) *For $n > 1$ and $k = 2, \ldots, p^n - 1$, define $c_k \in B$ by*

$$c_k(x,y) = \sum_{i=1}^{k-1} \binom{k}{i} x^i y^{k-i}.$$

Then $\{c_k : k = 2, \ldots, p^n - 1,\ w(k) \geq 2\}$ is a basis for B over $GF(p^n)$, and dim$(B) = p^n - 1 - n$.

Proof. For $\phi(x)$ as in Lemma 2.2, $\partial\phi(x,y) = \sum_{i=1}^{p^n-2} \sum_{j=1}^{p^n-i-1} \phi_{i+j} \binom{i+j}{i} x^i y^j$ $= \sum_{k=2}^{p^n-1} \sum_{i=1}^{k-1} \phi_k \binom{k}{i} x^i y^{k-i} = \sum_{k=2}^{p^n-1} \phi_k\, c_k(x,y)$, so $\{c_k : k = 2, \ldots, p^n - 1\}$ spans B. From Corollary 2.5, $w_p(k) = 1 \Rightarrow c_k \equiv 0$ so there are $p^n - 1 - n$ elements c_k spanning B. These c_k are linearly independent since distinct c_k have no monomial summands in common. □

MAGMA computations using Theorem 3.1 show that the coboundary bases exhibit recursive patterns as n increments. In effect, this recursion occurs because the coefficients of each basis element are the non-trivial binomial coefficients in a row of Pascal's triangle, as we now show. In [11], Wolfram describes the self-similar geometry of Pascal's triangle when the binomial coefficients are taken modulo r, and it is noted that for the case of r a prime, a very regular self-similar pattern is found. We can explain this regularity in terms of Lucas' Theorem and the matrix Kronecker product. The matrix P_n in Theorem 3.2 has the successive rows of Pascal's triangle as its upper diagonals, with the convention $\binom{m}{0} = 1$. The core of P_n, that is, P_n stripped of its first row and column of 1s, has the coefficients of successive basis elements c_k as its upper diagonals.

Theorem 3.2 *Define $P_n = [a_{ij}]$, where $a_{ij} = \binom{i+j}{i}$ mod p, $\ i,j = 0, 1, \ldots, p^n - 1$. Then $P_{n+1} = P_n \otimes P_1 = \otimes^{n+1} P_1$.*

Proof. Now,

$$P_k = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{p^k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{p^k}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \cdots & \binom{p^k+1}{2} \\ \vdots & & & \cdots & \vdots \\ \binom{p^k-1}{p^k-1} & \binom{p^k}{p^k-1} & \binom{p^k+1}{p^k-1} & \cdots & \binom{2p^k-2}{p^k-1} \end{bmatrix}.$$

Consider any entry a_{ij} where $i + j \geq p^k$. From the definition of the binomial coefficients, these entries will all have a factor p^k, and hence $a_{ij} = 0 \bmod p$ for $i + j \geq p^k$. Therefore,

$$P_k = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{p^k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & 0 \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \cdots & 0 \\ \vdots & & & \cdots & \vdots \\ \binom{p^k-1}{p^k-1} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{8}$$

Now consider the $(i,j)^{th}$ entry of P_{k+1}, where $i = up^k + m,\ 0 \leq u < p,\ 0 \leq m < p^k$ and $j = vp^k + n,\ 0 \leq v < p,\ 0 \leq n < p^k$. Then

$$\begin{aligned} a_{ij} &= \binom{up^k + m + vp^k + n}{up^k + m} \\ &= \binom{(m+n) + (u+v)p^k}{up^k + m}. \end{aligned}$$

From (8), if $(m+n) + (u+v)p^k \geq p^{k+1}$, then $a_{ij} = 0 \bmod p$. Two cases can occur if $(m+n) + (u+v)p^k < p^{k+1}$. If $m + n \geq p^k$ then a_{ij} will have a factor $(1 + u + v)p^k$, and hence $a_{ij} = 0 \bmod p$. If $m + n < p^k$, then by Theorem 2.3,

$$\begin{aligned} a_{ij} &= \binom{m + n + (u+v)p^k}{up^k + m} \\ &= \binom{m+n}{m}\binom{u+v}{u} \bmod p. \end{aligned}$$

Since $P_k = [a_{mn}]$ for $0 \leq m, n < p^k$, and $P_1 = [a_{uv}]$ for $0 \leq u, v < p$, we have that $P_{k+1} = P_k \otimes P_1$. □

A collection of consequences of Theorem 2.3 in the binary case follows for convenience; proof is straightforward and is left to the reader.

Lemma 3.3 (Corollaries of Lucas' Theorem) *Let* $i, j, k \in \mathbb{Z}^+$.

1. *For* $1 \leq i < 2^k$, $\binom{i+2^k}{i} = \binom{i+2^k}{2^k} \equiv 1 \bmod 2$.

2. *For* $1 \le j < i < 2^k$, $\binom{j+2^k}{i} \equiv 0 \bmod 2$.
3. *For* $1 \le j < 2^k < i < 2^{k+1}$, $\binom{i+j}{i} \equiv 1 \bmod 2 \Rightarrow \binom{i+j-2^k}{i-2^k} \equiv 1 \bmod 2$.

Next, we use Lemma 3.3 and the fact that, irrespective of n, all coefficients of the basis polynomials c_k found in Theorem 3.1 lie in the ground field $GF(2)$, to describe the recursive formula implied by Theorem 3.2 for the basis of B directly. An intermediate lemma is our key.

Lemma 3.4 *For* $n > 1$, *define* $A_n(x,y) = \sum_{k=2}^{2^n-1} c_k(x,y)$,
$A_n^{(c)}(x,y) = y^{2^n} \sum_{l=1}^{2^n-1} \binom{l+2^n}{l} x^l$ *and* $A_n^{(r)}(x,y) = x^{2^n} \sum_{l=1}^{2^n-1} \binom{l+2^n}{2^n} y^l$.
Then $A_{n+1} = A_n + y^{2^n} A_n + A_n^{(c)} + A_n^{(r)} + x^{2^n} A_n$.

Proof.

$$\begin{aligned} A_{n+1} &= \sum_{k=2}^{2^{n+1}-1} \sum_{i=1}^{k-1} \binom{k}{i} x^i y^{k-i} \;=\; A_n + \sum_{k=2^n+1}^{2^{n+1}-1} \sum_{i=1}^{k-1} \binom{k}{i} x^i y^{k-i} \\ &= A_n + \sum_{l=2}^{2^n-1} \sum_{i=1}^{l-1} \binom{l+2^n}{i} x^i y^{l+2^n-i} + \sum_{l=1}^{2^n-1} \sum_{i=l}^{l+2^n-1} \binom{l+2^n}{i} x^i y^{l+2^n-i}. \end{aligned}$$

By Lemma 3.3.1-3,

$$\begin{aligned} A_{n+1} &= A_n + y^{2^n} A_n + \sum_{l=1}^{2^n-1} \binom{l+2^n}{l} x^l y^{2^n} + \sum_{l=1}^{2^n-1} \sum_{i=l+1}^{l+2^n-1} \binom{l+2^n}{i} x^i y^{l+2^n-i} \\ &= A_n + y^{2^n} A_n + A_n^{(c)} + \sum_{l=1}^{2^n-1} \sum_{i=l+1}^{2^n-1} \binom{l+2^n}{i} x^i y^{l+2^n-i} \\ &\qquad + \sum_{l=1}^{2^n-1} \binom{l+2^n}{2^n} x^{2^n} y^l + \sum_{l=2}^{2^n-1} \sum_{i=2^n+1}^{l+2^n-1} \binom{l+2^n}{i} x^i y^{l+2^n-i} \\ &= A_n + y^{2^n} A_n + A_n^{(c)} + 0 + A_n^{(r)} + \sum_{l=2}^{2^n-1} \sum_{i=1}^{l-1} \binom{l+2^n}{i} x^{l+2^n-i} y^i \\ &= A_n + y^{2^n} A_n + A_n^{(c)} + A_n^{(r)} + x^{2^n} A_n. \qquad \Box \end{aligned}$$

Theorem 3.5 *For* $n > 1$ *and* $2^{n-1} < k < 2^n$, *the coboundaries* c_k *over* $\mathbb{Z}_2^n$ *can be defined recursively:*

$$c_k(x,y) = \begin{cases} (x^{k-2^{n-1}} + c_{k-2^{n-1}})\, y^{2^{n-1}} \;+\; x^{2^{n-1}} (y^{k-2^{n-1}} + c_{k-2^{n-1}}), & w(k) \ge 3 \\ x^{2^r} y^{2^{n-1}} + x^{2^{n-1}} y^{2^r}, & k = 2^{n-1} + 2^r,\; r = 0, \dots, n-2. \end{cases}$$

Proof. By Lemma 3.4, $\sum_{k=2^{n-1}}^{2^n-1} c_k(x,y) = y^{2^{n-1}} A_{n-1} + A_{n-1}^{(c)} + A_{n-1}^{(r)} + x^{2^{n-1}} A_{n-1}$. Hence $c_k(x,y) = y^{2^{n-1}} c_{k-2^{n-1}} + x^{k-2^{n-1}} y^{2^{n-1}} + x^{2^{n-1}} y^{k-2^{n-1}} + x^{2^{n-1}} c_{k-2^{n-1}}$. If $w(k) = 2$ then $w(k - 2^{n-1}) = 1$ and $c_{k-2^{n-1}} \equiv 0$. $\Box$

We distinguish between the basis coboundaries c_k with $w_2(k) = 2$ and those with $w_2(k) \geq 3$, since it is plain from (10) and Corollary 4.3 that the c_k with $w_2(k) = 2$ form a basis for S_+ .

Corollary 3.6 *For $n > 1$, define $C = \operatorname{span}\{c_k,\ k = 2, \ldots, 2^n - 1, w_2(k) \geq 3\}$ and*

$$b_{ij}(x, y) = c_{2^i+2^j}(x, y) = x^{2^i} y^{2^j} + x^{2^j} y^{2^i},\ x, y \in GF(2^n),\ 0 \leq i < j \leq n - 1.$$

Then $S_+ = \operatorname{span}\{b_{ij},\ 0 \leq i < j \leq n - 1\}$ and $B = S_+ \oplus C$. □

4 A polynomial basis for cocycles over $\mathbb{Z}_2^n$

From now on, we consider cocycles in $Z^2(\mathbb{Z}_2^n, \mathbb{Z}_2^m)$, $n \geq m \geq 1$, where we can say more about the subgroup of symmetrisation cocycles S_+^2. Not only is every symmetrisation multiplicative, it is a coboundary.

Lemma 4.1 *Let $G \cong \mathbb{Z}_2^n$ and $C \cong \mathbb{Z}_2^m$, $n \geq m \geq 1$.*

(i) $\ker(S^+) = S^2(G, C)$ *so* $S_+^2(G, C) \leq M^2(G, C) \cap S^2(G, C)$.

(ii) $S^+ = \partial \circ D$ *so* $S_+^2(G, C) = (\partial \circ D)(Z^2(G, C)) \leq M^2(G, C) \cap B^2(G, C)$.

Proof. (i) This follows by definition since $\psi^- = \psi^+$ and ψ^- is multiplicative.
(ii) If $\psi \in Z^2(G, C)$, then $\partial(D\psi)(x, y) = \psi(x + y, x + y) + \psi(x, x) + \psi(y, y)$ = $(\psi(x, y) + \psi(x, y + x + y) + \psi(y, y + x)) + \psi(x, x) + \psi(y, y)$ by (1), which equals $\psi(x, y) + (\psi(y + y, x) + \psi(y, y) + \psi(y, x)) + \psi(y, y)$ again by (1), which equals $S^+(\psi)(x, y)$. □

Hereafter, assume $G = C = (GF(2^n), +) \cong \mathbb{Z}_2^n$. We will abbreviate by Z, B, M, S and S_+ the finite abelian groups of cocycles, coboundaries, multiplicative, symmetric and symmetrisation cocycles respectively. In this case, by Proposition 2.1, Z is a $GF(2^n)$-vector space of dimension $N = 2^n + \binom{n}{2} - 1$.

The function $\phi : G \to G$ is said to be *linearised* or *quadratic*, respectively, if every monomial summand has degree of binary weight ≤ 1 or ≤ 2, respectively. A polynomial for which every monomial summand has degree of weight 2 is called a *Dembowski-Ostrom* (DO) polynomial in [2]. That is, a polynomial ϕ in $GF(2^n)[x]$ is DO if, when reduced modulo $x^{2^n} - x$, it is of the form

$$\phi(x) = \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \lambda_{ij}\ x^{2^i+2^j},\ \lambda_{ij} \in GF(2^n). \tag{9}$$

When $G = C$, Lemma 4.1 can be improved to show $M \cap B = S_+$. We need the following result, proof of which is an easy adaption of that of [2, Theorem 3.2] to the case $p = 2$. Proposition 4.2 has been rediscovered by other authors, eg [7].

Proposition 4.2 (cf. [2]) *Let $G \cong \mathbb{Z}_2^n$, let $f \in C^1(G, G)$ have linearised summand ℓ and set $\phi = f - \ell$. Then $\partial\phi = \partial f \in M \cap B$ if and only if ϕ is DO, if and only if f is quadratic. That is, $M \cap B = \{\partial f : f = \phi + \ell, \phi$ DO, ℓ linearised$\}$.* □

Every DO polynomial is the image under the diagonal mapping of at least one cocycle. For the DO polynomial ϕ in (9), define

$$\varphi(x,y) = \sum_{j=1}^{n-1}\sum_{i=0}^{j-1} \lambda_{ij}\, x^{2^i} y^{2^j}, \tag{10}$$

which is multiplicative by Theorem 2.6. Then $D\varphi = \phi$ and so $\partial\phi = \varphi^+$.

Corollary 4.3 *Let $G \cong \mathbb{Z}_2^n$, let ϕ be a DO polynomial (9) and let φ be the corresponding cocycle (10). Then*

(i) $\partial\phi = \varphi^+ = \partial \circ D(\varphi)$,

(ii) $M \cap B = S_+ = (\partial \circ D)(Z)$. □

In Theorem 3.1 we proved the $GF(2^n)$-subspace B of coboundaries has dimension $2^n - n - 1$. By Proposition 2.1, Z has $GF(2^n)$-dimension $N = 2^n + \binom{n}{2} - 1$, so we need only to identify a further $n + \binom{n}{2}$ basis cocycles which are not coboundaries. We make the surprising and valuable observation that these may all be chosen to be multiplicative. This observation is not apparent from the theory described above. The multiplicative cocycles have already been identified in Theorem 2.6. If $i < j$, then $\lambda_{ij}x^{2^i}y^{2^j} + \lambda_{ji}x^{2^j}y^{2^i} = (\lambda_{ji} + \lambda_{ij})x^{2^j}y^{2^i} + \lambda_{ij}(x^{2^i}y^{2^j} + x^{2^j}y^{2^i}) = (\lambda_{ji} + \lambda_{ij})x^{2^i}y^{2^j} + \lambda_{ji}(x^{2^i}y^{2^j} + x^{2^j}y^{2^i})$ and in either form the second summand is a multiple of the symmetrisation coboundary $b_{ij}(x,y)$. We have two possible representations of $\lambda_{ij}x^{2^i}y^{2^j} + \lambda_{ji}x^{2^j}y^{2^i}$ here, and without loss of generality we choose the former.

Hence $\dim(M) = n^2 = n + 2\binom{n}{2}$ and in M there are n linearly independent multiplicative symmetric cocycles we denote by

$$d_i(x,y) = x^{2^i}y^{2^i},\ 0 \le i \le n-1,$$

and $\binom{n}{2}$ linearly independent multiplicative asymmetric cocycles we denote by

$$a_{ji}(x,y) = x^{2^j}y^{2^i},\ 0 \le i < j \le n-1$$

as well as the $\binom{n}{2}$ linearly independent symmetrisation coboundaries $b_{ij}(x,y),\ 0 \le i < j \le n-1$ already found as a basis for S_+.

Theorem 4.4 (Basis Theorem) *A $GF(2^n)$-basis for $Z^2(\mathbb{Z}_2^n, \mathbb{Z}_2^n)$, $n > 1$, consists of the following $N = 2^n + \binom{n}{2} - 1$ polynomials:*

1. *n multiplicative symmetric non-coboundary cocycles $d_i,\ i = 0, \ldots, n-1$;*
2. *$\binom{n}{2}$ multiplicative asymmetric non-coboundary cocycles $a_{ji},\ 0 \le i < j \le n-1$;*
3. *$\binom{n}{2}$ multiplicative symmetrisation coboundaries $b_{ij},\ 0 \le i < j \le n-1$;*
4. *$2^n - \binom{n}{2} - n - 1$ non-multiplicative symmetric coboundaries $c_k,\ 2 \le k \le 2^n - 1,\ w_2(k) \ge 3$.* □

Corollary 4.5 *Define* $A = \mathrm{span}\{a_{ji},\ 0 \leq i < j \leq n-1\}$ *and* $D = \mathrm{span}\{d_i,\ i = 0, \ldots, n-1\}$. *Then* $\dim(A) = \binom{n}{2}$, $\dim(B) = 2^n - n - 1$, $\dim(C) = 2^n - \binom{n}{2} - n - 1$, $\dim(D) = n$, $\dim(M) = n^2$, $\dim(S) = 2^n - 1$, $\dim(S_+) = \binom{n}{2}$ *and* $\dim(Z) = 2^n + \binom{n}{2} - 1$. *Furthermore,*

1. $Z = A \oplus D \oplus S_+ \oplus C = M \oplus C = A \oplus S$;
2. $S = D \oplus S_+ \oplus C$;
3. $M = A \oplus D \oplus S_+$;
4. $B = S_+ \oplus C$. □

Theorem 3.5, Theorem 4.4 and Corollary 4.5 provide us with a new and effective approach to working with cocycles over $\mathbb{Z}_2^n$. Known algorithms for finding a generating set of cocycles require costly precomputation of representative cocycles in each cohomology class, and for a generating set of coboundaries to be found using linear algebra, on a case by case basis. There are iterative techniques for finding the cohomology class representatives as n increments, but no simple recursive formula for coboundaries, such as we have given in Theorem 3.5.

We illustrate the transformation from the basis found by another algorithm to this basis with a small example.

Example 2 *If* $n = 2$, $Z \cong (\mathbb{Z}_2^2)^4$. *Using Algorithm 1 [6, 6.3.1], each cocycle* ψ *is uniquely defined by the 4 values* $\psi(1,1) = \alpha$, $\psi(\omega,\omega) = \beta$, $\psi(1,\omega) = \gamma$ *and* $\psi(1,\omega) + \psi(\omega,1) = \kappa$, *where* ω *is a primitive element of* $GF(4)$. *These values can be used to identify 4 basis cocycles. By Theorem 4.4, there are 4 basis polynomials* $d_0(x,y) = xy, d_1(x,y) = x^2y^2, a_{10}(x,y) = x^2y$ *and* $b_{01}(x,y) = xy^2 + x^2y$. *Suppose* $\psi = \lambda_1 d_0 + \lambda_2 d_1 + \lambda_3 a_{10} + \lambda_4 b_{01}$. *Then the transform matrix is given by*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} \omega & 1 & \omega^2 & 0 \\ \omega^2 & 1 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & \omega & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \kappa \\ \gamma \end{bmatrix}.$$

As a consequence of the Basis Theorem, every cocycle from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$ has a unique decomposition as a direct sum of a coboundary and a multiplicative cocycle of restricted type, a fact which we do not believe has been previously observed.

Corollary 4.6 *Since* $Z = (A \oplus D) \oplus B$, *every cocycle* $\psi \in Z$ *has a unique decomposition as a direct sum of the form* $\psi = \mu \oplus \partial\phi$ *where* $\mu \in A \oplus D$ *is multiplicative and* $\partial\phi$ *is a coboundary.*

The known but previously unusable unique decomposition $\psi = \psi^\top + \psi^-$, where ψ^- is the commutator pairing, is now revealed as the decomposition $Z = (A \oplus D \oplus C) \oplus S_+$, since $\psi^- = \psi^+ = S^+(\psi)$.

We expect the Basis Theorem and Corollary 4.6 will prove very useful in the search for orthogonal and other cocycles with low differential uniformity in $Z^2(\mathbb{Z}_2^n, \mathbb{Z}_2^m)$, $m \leq n$, for applications in coding and cryptography. For instance, it is conjectured in [8] that all orthogonal cocycles in $Z^2(\mathbb{Z}_2^n, \mathbb{Z}_2^m)$, $n \geq m \geq 2$ are multiplicative, based on computed results for $n \leq 4$. The decomposition above may be the clue to discovering if this is true for all n when $p = 2$. (It cannot be true for odd p [2].)

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