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Nonsingular Terminal Sliding Mode Control of Uncertain Multivariable Systems

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***Abstract*— This paper proposes a nonsingular terminal sliding mode control for uncertain multivariable systems with parameter uncertainties or disturbances. A hierarchical control structure is utilized for simplifying the controller design. Uncertain multivariable linear systems are converted into the block controllable form consisting of two subsystems, an input-output subsystem and a stable internal dynamic subsystem. In order to guarantee fast convergence and better tracking precision, a nonsingular terminal sliding mode manifold is proposed for the input-output subsystem. To eliminate the chattering phenomenon, a continuous nonsingular terminal sliding mode control law is designed using the second-order sliding mode approach. Under the proposed controllers, the states of the input-output subsystem can be driven to converge to zero asymptotically and the stability of the zero-dynamics of the system is guaranteed. The simulation results are presented to validate the design.**

I. Introduction

THE control of uncertain linear multivariable systems with internal parameter uncertainties and external disturbances is a significant issue both theoretically and practically. Two methodologies are commonly used: the state-space feedback control and the optimal control [1]. If a system, however, has a relatively higher dimension, the above two methods may impose severe computational demands in the real time control applications. One approach proposed in [2] alleviates this problem by transforming uncertain linear multivariable systems into a block controllable canonical form (BC-form). It is relatively simple for designing the controllers for this block controllable canonical form. But it does not seem to exhibit robustness.

Variable structure systems (VSS) are well known for their robustness to system parameter variations and external disturbances [3, 4]. An aspect of VSS of particular interest is the sliding mode control, which is designed to drive and constrain the system states to stay in the prescribed switching manifolds that exhibit the desired dynamics. When in the sliding mode, the closed-loop responses of the systems become totally insensitive to certain internal parameter uncertainties and external disturbances. A characteristic of the conventional VSS is that the convergence of the system states to the equilibrium points is usually asymptotical due to the asymptotical convergence of the linear switching manifolds that are commonly chosen. Recently, a terminal sliding mode (TSM) controller was developed [5, 6, 7]. Compared with the linear hyperplane based sliding mode, TSM offers some superior properties such as fast, finite time convergence and better tracking precision. This controller is particularly useful for high precision control as it speeds up the rate of convergence near the equilibrium points. But there exists a singularity problem in TSM control. Based on TSM, a nonsingular terminal sliding mode (NTSM) control has been presented [10]. The novel NTSM manifold is different from TSM manifold and the control methodology can avoid the singularity.

Although sliding mode approaches have many advantages, chattering is a major drawback. Chattering is undesirable because it can excite unmodelled high-frequency dynamics of the system. Chattering is the high-frequency finite amplitude oscillations with finite frequency caused by system imperfections and is produced due to the discontinuity of the sign function. There are two methods of eliminating the chattering. The first method is to introduce a boundary layer. Within the layer, the sign function in the control signal is replaced with a saturation function or a sigmoid-like function at the price of a small deterioration in system performance. The second method is to use a continuous signal instead of switching signal in the control signal, such as the second-order sliding mode approach. The second-order sliding mode control can be used to smoothen the control signal. It is a continuous control, robust to the parameter uncertainties and disturbances. Meanwhile, the characteristics of the traditional sliding mode approaches are guaranteed in the second-order sliding mode control systems [11].

The paper proposes a second-order NTSM decomposed control method for linear multivariable systems, which is robust to certain internal parameter uncertainties and external disturbances. Linear multivariable systems are transformed to the lower controller-Hessenberg forms by the unitary state transformation, then further to the block decoupled BC-forms, in which the coupled state variables are eliminated. As a result, a linear multivariable system is transformed to r (controllability index of the system) block decoupled BC-form system consisting of an input-output subsystem and an internal dynamic subsystem. A special second-order nonsingular TSM manifold is then proposed for the input-output subsystem. The control law is designed to drive the states of the input-output subsystem to zero asymptotically. Then, the stability of the zero-dynamics of the system is guaranteed by the controller. With the advantages of simple controller design and the hierarchical controller

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structure, the proposed method in the paper can be easily applied to higher-dimensional linear multivariable systems. Moreover, the second-order sliding mode control proposed in this paper can eliminate the chattering phenomenon. The simulation results are presented to validate the method.

II. Decomposition of Uncertain Linear MIMO System

Consider the uncertain linear multivariable system given by:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{A}_{per}(t) \ , \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$ is the state variable vector; $\boldsymbol{u}(t) \in R^m$ with $1 \le m < n$ is the control input vector, $\boldsymbol{A} \in R^{n \times n}$ and $\boldsymbol{B} \in R^{n \times m}$ are the known parameters matrices, $\boldsymbol{A}_{per}(t) \in R^n$ represents any uncertainties or nonlinearities. Assume system (1) satisfies the following three assumptions:

1) $\boldsymbol{A}$, $\boldsymbol{B}$ are constant matrices;
2) The pair $\{\boldsymbol{A}, \boldsymbol{B}\}$ is controllable;
3) $\boldsymbol{A}_{per}(t)$ satisfies the following matching condition:

$$\boldsymbol{A}_{per}(t) = \boldsymbol{B}\boldsymbol{d}(t) \ , \tag{2}$$

where $\boldsymbol{d}(t) \in R^{m \times 1}$, is some bounded time-varying matrix and satisfies:. $\|\boldsymbol{d}(t)\| \le l_d \, , l_d > 0$; $\|\dot{\boldsymbol{d}}(t)\| \le l_{dd}$, $l_{dd} > 0$.

The control objective is to force system (1) to converge to zero asymptotically or in a finite time from any initial state $\boldsymbol{x}(0) \neq 0$.

In order to simplify the controller design of system (1), two state transforms are made. First, system (1) is transformed to the lower controller-Hessenberg form by the unitary state transformation:

$$\boldsymbol{x}' = \boldsymbol{F}\boldsymbol{x} \ , \tag{3}$$

where $\boldsymbol{x}' \in R^n$. The state transformation matrix $\boldsymbol{F}$ is constructed by using the staircase algorithm [15].

Using state transformation (3), system (1) is converted into:

$$\dot{\boldsymbol{x}}'(t) = \boldsymbol{A}'\boldsymbol{x}'(t) + \boldsymbol{B}'\boldsymbol{d}(t) + \boldsymbol{B}'\boldsymbol{u}(t) \ , \tag{4}$$

where $\boldsymbol{A}' = \boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T$, $\boldsymbol{B}' = \boldsymbol{F}\boldsymbol{B} = \begin{bmatrix} 0 & \boldsymbol{B}_{1,0}^T \end{bmatrix}^T$, $\boldsymbol{B}_{1,0} \in \boldsymbol{R}^{r \times m}$, rank$\boldsymbol{B}_{10}=r$, $r \le m$, r is the controllable index of system (1), therefore the pseudo-inverse matrix of $\boldsymbol{B}_{1,0}^+$ exists and is given by:

$$\boldsymbol{B}_{1,0}^+ = \boldsymbol{B}_{1,0}^T \left[\boldsymbol{B}_{10}\boldsymbol{B}_{1,0}^T\right]^{-1} \ . \tag{5}$$

Since the pair ($\boldsymbol{A}'$, $\boldsymbol{B}'$) is the lower controller-Hessenberg form, system (4) can be rewritten as:

$$\begin{cases} \dot{\boldsymbol{x}}_i'(t) = \sum_{j=i}^{r} \boldsymbol{A}_{i,j}\boldsymbol{x}_i'(t) + \boldsymbol{B}_{i,i-1}\boldsymbol{x}_{i-1}'(t) \quad , \quad i = 2, \cdots r \\ \dot{\boldsymbol{x}}_1'(t) = \sum_{j=1}^{r} \boldsymbol{A}_{1,j}\boldsymbol{x}_j'(t) + \boldsymbol{B}_{1,0}\boldsymbol{d}(t) + \boldsymbol{B}_{1,0}\boldsymbol{u}(t) \end{cases} \tag{6}$$

where $\boldsymbol{x}'^T = [\boldsymbol{x}_1'^T \cdots \boldsymbol{x}_r'^T]^T$, $\boldsymbol{x}_i' \in \boldsymbol{R}^{n_i}$, i=1, …, r; $\boldsymbol{B}_{i,i-1}$, i= 1, …, r, have full rank; $\boldsymbol{x}_{i-1}'$ is regarded as the virtual control vector of the ith layer of the system (6).

For the convenience of the controller design, system (6) is further transformed to the decoupled BC-form using the state transformation [2]:

$$\boldsymbol{x}' = \boldsymbol{F}'z \ , \tag{7}$$

where $z \in R^n$ and the nonsingular transformation matrix $\boldsymbol{F}'$ is:

$$\boldsymbol{F}' = \begin{bmatrix} \boldsymbol{I} & 0 & 0 & \cdots & 0 & 0 \\ -\boldsymbol{K}_{r-1,r} & \boldsymbol{I} & 0 & \cdots & 0 & 0 \\ -\boldsymbol{K}_{r-2,r} & -\boldsymbol{K}_{r-2,r-1} & \boldsymbol{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\boldsymbol{K}_{2,r} & -\boldsymbol{K}_{2,r-1} & -\boldsymbol{K}_{2,r-2} & \cdots & \boldsymbol{I} & 0 \\ -\boldsymbol{K}_{1,r} & -\boldsymbol{K}_{1,r-1} & -\boldsymbol{K}_{1,r-2} & \cdots & -\boldsymbol{K}_{1,2} & \boldsymbol{I} \end{bmatrix} , \tag{8}$$

where $\boldsymbol{K}_{i,j}$, i= 1, …, r–1; j=2, …, r can be determined as follows:

$$\boldsymbol{K}_{i,i+1} = \boldsymbol{B}_{i+1,i}^+ (\boldsymbol{K}_{i+1,i+2}\boldsymbol{B}_{i+2,i+1} + \boldsymbol{A}_{i+1,i+1} - \boldsymbol{N}_{i+1}(t) \ , \quad i = 1,2, \cdots r-1 \tag{9}$$

$$\boldsymbol{K}_{i,j} = \boldsymbol{B}_{i+1,i}^+ (\boldsymbol{K}_{i+1,j}\boldsymbol{N}_j + \boldsymbol{A}_{i+1,j} + \boldsymbol{K}_{i+1,j+1}\boldsymbol{B}_{j+1,j} - \sum_{k=j-1}^{i+1} \boldsymbol{A}_{i+1,k}\boldsymbol{K}_{k,j}) \ , \quad j = i+2, i+3, \cdots r, \quad i = 1,2, \cdots r-2 \tag{10}$$

where $\boldsymbol{K}_{ij}$=0, $\forall$ $i \notin$ [1, 2, …, r–1] or $j \notin$ [2, 3, …, r], $\boldsymbol{B}_{ij}$=0, $\forall$ $i \notin$ [1, 2, …, r] or $j \notin$ [0, 1, …, r–1]. $\boldsymbol{N}_i$, i= 1, …, r are the designed parameter matrices, which will be determined later.

Then, the lower controller-Hessenberg form (6) can be further transformed to the following form by using the nonsingular state transformation (7):

$$\dot{z}(t) = \boldsymbol{A}''z(t) + \boldsymbol{B}''\boldsymbol{d}(t) + \boldsymbol{B}''\boldsymbol{u}(t) \ , \tag{11}$$

where $\boldsymbol{A}'' = (\boldsymbol{F}')^{-1}\boldsymbol{A}'\boldsymbol{F}'$ and $\boldsymbol{B}'' = (\boldsymbol{F}')^{-1}\boldsymbol{B}' = \begin{bmatrix} 0 & \boldsymbol{B}_{10}^T \end{bmatrix}^T$. System (11) can be rewritten as:

$$\begin{cases} \dot{z}_i(t) = \boldsymbol{N}_i z_i(t) + \boldsymbol{B}_{i,i-1}z_{i-1}(t) \quad , \quad i = 2, \cdots r \\ \dot{z}_1(t) = \sum_{\alpha=1}^{r} \overline{\boldsymbol{A}}_{1,\alpha} z_\alpha(t) + \boldsymbol{B}_{1,0}\boldsymbol{d}(t) + \boldsymbol{B}_{1,0}\boldsymbol{u}(t) \end{cases} , \tag{12}$$

where $z = [z_1^T \ \cdots \ z_r^T]^T$, $z_i \in \boldsymbol{R}^{n_i}$; $\boldsymbol{N}_i$, i=2, …, r, are the matrices. Every eigenvalues of $\boldsymbol{N}_i$ can be designed to have negative real part. After $\boldsymbol{N}_i$ is determined, $\boldsymbol{F}'$ and $\overline{\boldsymbol{A}}_{1,\alpha}$ can be determined also.

Therefore, the uncertain linear multivariable system (1) is transformed to the decoupled BC-form (12) using the two state transformations (3) and (7). It can be seen that it is relatively simple for designing the controller of the decoupled BC-form system (12).

III. Second-order Nonsingular TSM Control

For the convenience of the controller design, the paper assumes that the uncertain linear multivariable system (1) is already in the decoupled BC-form (12) after the state transformations (3) and (7).

A second-order nonsingular TSM control strategy is proposed in this paper. The controller design consists of the following two steps. The first is to design the second-order nonsingular TSM manifold and ensure that the states of the system on the sliding mode manifold can converge to zero asymptotically. The second is to design the robust control law for ensuring that the states of the system always move towards the NTSM manifold and the system is robust to certain internal parameter uncertainties and external disturbances.

First, the paper proposes a linear sliding mode manifold for system (12):

$$\boldsymbol{s}(t) = \boldsymbol{\beta} \boldsymbol{z}_1 + \int_{0_-} z_1 dt \quad , \tag{13}$$

where $s \in R^{n1}$; $z_1 \in R^{n1}$, $\boldsymbol{\beta} \in \mathrm{diag}(\beta_1, \ldots, \beta_{n1})$, $\beta_i > 0$ is a constant, $\int_{0_-} z_1 dt$ is denoted as:

$$\int_{0_-} z_1 dt = \left[\int_{0_-} z_{11} dt, \cdots, \int_{0_-} z_{1n_1} dt \right]^T .$$

In order to guarantee the linear sliding mode manifold $\boldsymbol{s}(t)$ converge to zero in finite time, the paper proposes the following NTSM manifold:

$$\boldsymbol{l}(t) = \gamma^{-1} \dot{\boldsymbol{s}}^{p/q} + \boldsymbol{s} \quad , \tag{14}$$

where $\boldsymbol{l} \in \boldsymbol{R}^m$; $\boldsymbol{\gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)$, $\gamma_i > 0$ are constant. p, q are all odds, $1 < p/q < 2$. $\dot{\boldsymbol{s}}^{q/p}$ is denoted as:

$$\dot{\boldsymbol{s}}^{p/q} = \left[\dot{s}_1^{\,p/q}, \cdots, \dot{s}_m^{\,p/q} \right]^T .$$

The aim of introducing $\boldsymbol{s}(t)$ is to control the input-output subsystem to reach $\boldsymbol{s}(t)$, while introducing $\boldsymbol{l}(t)$ is to realize the second-order sliding mode control and eliminate the chattering phenomena using the second-order approach [11]. Therefore, $\boldsymbol{l}(t)$ is designed as a nonsingular TSM to guarantee the linear sliding mode $\boldsymbol{s}(t)$ to reach zero in finite time and have no singularity.

Suppose t_r is the time when $\boldsymbol{l}(t)$ reaches to 0, that is $\boldsymbol{l}(t)=0$, $\forall t \geq t_r$. From Eq.(14), solving $\gamma^{-1} \dot{\boldsymbol{s}}^{p/q}(t) + \boldsymbol{s}(t) = 0$ gives the time from $\boldsymbol{s}(t_r)$ to $\boldsymbol{s}(t_s)=0$:

$$t_s = t_r + \frac{p}{(p-q) \min\limits_{i=1,\cdots,n_1} \gamma_i} \max_{i=1,\cdots,n_1} (s_i(t_r)^{\frac{p}{p-q}}) . \tag{15}$$

Thus, through the control design, $\boldsymbol{s}(t)$ and $\dot{\boldsymbol{s}}(t)$ can be driven to reach $\boldsymbol{l}(t)=0$ and then remain on $\boldsymbol{l}(t)=0$ to realize the sliding mode motion. Among $\boldsymbol{l}(t)=0$, $\boldsymbol{s}(t)$ will reach zero in finite time t_s (15). After $\boldsymbol{s}(t)$ reaches zero, the system will remain in the linear sliding mode motion (13), that is, the dynamics of system (12) can be determined by the design parameters $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, p, q, and has nothing to do with the system's parameters (14). The relevant control strategy will be given in the theorem below. In order to prove the theorem, the following Lemma is given firstly.

Proposition 1. For any vector $\boldsymbol{x} \in R^n$, the diagonal matrix $\boldsymbol{Q}=diag[q_1, \ldots, q_n]^T$ satisfies

$$-x^T Q \,\mathrm{sgn}(x) \leq -\lambda_{\min}(Q) \|x\| \tag{16}$$

where sgn($\boldsymbol{x}$) is defined as sgn($\boldsymbol{x}$) = [sgn(x_1), … , sgn(x_n)]^T.

Proof: The quadratic form always satisfies the Rayleigh principle, namely [14]:

$$\lambda_{\min}(Q) \|x\|^2 \leq x^T Q x \leq \lambda_{\max}(Q) \|x\|^2$$

therefore,

$$-x^T Q \,\mathrm{sgn}(x) = -x^T Q [\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)]^T$$

$$= -x^T Q diag[\frac{1}{|x_1|}, \cdots, \frac{1}{|x_n|}] x$$

$$\leq -\min_{i \in [1,n]} (\frac{q_i}{|x_i|}) \|x\|^2$$

$$\leq -\min_{i \in [1,n]} (\frac{q_i}{\|x\|}) \|x\|^2$$

$$= -\lambda_{\min}(Q) \|x\|$$

that is, Eq.(16) holds.

Theorem 1 For uncertain linear multivariable system (1), if the linear sliding mode manifold and the second-order nonsingular TSM manifold are chosen as (13) and (14) respectively, and the control law is designed as follows, then system (1) is asymptotically stable:

$$\boldsymbol{u}(t) = \boldsymbol{u}_0(t) + \boldsymbol{u}_1(t) \quad , \tag{17}$$

where:

$$\boldsymbol{u}_0(t) = -\boldsymbol{B}_{10}^{+} \sum_{\alpha=1}^{r} \overline{\boldsymbol{A}}_{1,\alpha} \boldsymbol{z}_\alpha \quad , \tag{18}$$

$\boldsymbol{u}_1(t)$ is obtained through the low-passed filter:

$$\boldsymbol{v}(t) = \boldsymbol{\beta} \dot{\boldsymbol{u}}_1(t) + \boldsymbol{u}_1(t) \quad , \tag{19}$$

where $\boldsymbol{v}(t)$ is the input of the low-passed filter:

$$\boldsymbol{v}(t) = \boldsymbol{v}_{eq}(t) + \boldsymbol{v}_n(t) \quad , \tag{20}$$

$$\boldsymbol{v}_{eq}(t) = -\frac{q}{p} \boldsymbol{B}_{10}^{+} \boldsymbol{\gamma} \dot{\boldsymbol{s}}^{2-p/q} \quad , \tag{21}$$

$$\boldsymbol{v}_n(t) = -\boldsymbol{B}_{1,0}^{+} \left(\|\boldsymbol{B}_{1,0}\| (l_d + \|\boldsymbol{\beta}\| l_{dd}) + \eta \right) \mathrm{sgn}(\boldsymbol{l}) \quad , \tag{22}$$

where $\boldsymbol{l_d}$ and $\boldsymbol{l_{dd}}$ are defined in Eq.(2); $\boldsymbol{\gamma}$, α and η are design parameters, $\alpha > 0$, $0 < \eta < 1$.

Proof: The following Lyapunov function is considered:

$$V(t) = \frac{1}{2} \boldsymbol{l}(t)^T \boldsymbol{l}(t) \ .$$

Differentiating $V(t)$ with respect to time gets:

$$\dot{V}(t) = \boldsymbol{l}^T \dot{\boldsymbol{l}}$$

$$= \boldsymbol{l}^T \left(\frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \ddot{\boldsymbol{s}} + \dot{\boldsymbol{s}} \right)$$

$$= \boldsymbol{l}^T \left(\frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \left(\boldsymbol{B}_{1,0} \boldsymbol{v} + \boldsymbol{B}_{1,0} \left(\boldsymbol{\beta} \dot{\boldsymbol{d}}(t) + \boldsymbol{d}(t) \right) \right) + \dot{\boldsymbol{s}} \right)$$

$$= \boldsymbol{l}^T \left(\frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \left(\boldsymbol{B}_{1,0} \boldsymbol{v}_n + \boldsymbol{B}_{1,0} \left(\boldsymbol{\beta} \dot{\boldsymbol{d}}(t) + \boldsymbol{d}(t) \right) \right) \right)$$

$$= -\boldsymbol{l}^T \frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \left(\|\boldsymbol{B}_{1,0}\| (l_d + \|\boldsymbol{\beta}\| l_{dd}) + \eta \right) \mathrm{sgn}(\boldsymbol{l})$$

$$+ \boldsymbol{l}^T \frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \boldsymbol{B}_{1,0} \left(\boldsymbol{\beta} \dot{\boldsymbol{d}}(t) + \boldsymbol{d}(t) \right)$$

$$\leq -\boldsymbol{l}^T \eta \frac{p}{q} \boldsymbol{\gamma}^{-1} diag(\dot{\boldsymbol{s}}^{p/q-1}) \,\mathrm{sgn}(\boldsymbol{l})$$

Notice that in the above expression, $\boldsymbol{\gamma}^{-1}$ is a diagonal matrix, then $\eta(p/q)\boldsymbol{\gamma}^{-1}diag(\dot{\boldsymbol{s}}^{p/q-1})$ is also a diagonal matrix. From Proposition 1, the above expression can be written as follows:

$$\dot{V}(t) \le -\lambda_{\min}\left(\eta\frac{p}{q}\boldsymbol{\gamma}^{-1}diag(\dot{\boldsymbol{s}}^{p/q-1})\right)\|\boldsymbol{l}\|$$

that is

$$\dot{V}(t) \le -\eta\frac{p}{q}\min_{i\in[1,m]}(\gamma_i^{-1}\dot{s}_i^{p/q-1})\|\boldsymbol{l}\|$$

since p and q are all odds, it can be seen that $\dot{s}_i^{p/q-1} > 0$ for any $\dot{s}_i \neq 0$ and $\dot{s}_i^{p/q-1} = 0$ only for $\dot{s}_i = 0$. Therefore, for $\|\boldsymbol{l}\| \neq 0$, there are two different cases: $\dot{s}_i \neq 0$ for any i and $\dot{s}_i = 0$ for some i.

For the former case, it can be gotten:

$$\dot{V}(t) \le -\eta\frac{p}{q}\min_{i\in[1,m]}(\gamma_i^{-1}\dot{s}_i^{p/q-1})\|\boldsymbol{l}\| < 0 \quad \text{for} \quad \|\boldsymbol{l}\| \neq 0$$

For the latter case, that is, $\dot{s}_i = 0$ but $s_i \neq 0$, the state variables of the system will not always stay on the points ($\dot{s}_i = 0$, $s_i \neq 0$) and will continue to cross the axis $\dot{s}_i = 0$ in the phase plane 0– $s_i\dot{s}_i$ [10].

Therefore, the condition for Lyapunov stability is satisfied. The states of the system can reach the NTSM manifold $\boldsymbol{l}(t)$=0 within finite time.

In the sliding mode $\boldsymbol{l}(t)$=0, from Eq.(14), there is $\boldsymbol{\gamma}^{-1}\dot{\boldsymbol{s}}^{p/q} + \boldsymbol{s} = 0$, or $\dot{\boldsymbol{s}} + \boldsymbol{\gamma}\boldsymbol{s}^{q/p} = 0$, $\boldsymbol{s}(t)$ will reach the zero in t_s (15).

When the states of the system reach and stay on the linear sliding mode manifold $\boldsymbol{s}$=0, there is $\boldsymbol{\beta}\boldsymbol{z}_1 + \int_{0-}^{t} \boldsymbol{z}_1 \mathrm{d}t = 0$. The input-output subsystem of system (12) is stable asymptotically for $t>t_s$, that is z_1 will converge to zero asymptotically. Furthermore, because of $\boldsymbol{N}_i = -\lambda_i \boldsymbol{I}_{ni}$, i=2, …, r, $-\lambda_2 < -\lambda_3 < \ldots < -\lambda_r < 0$, the other states of system (12), z_2, …, z_r, will converge to zero asymptotically.

Since the state transformations between $\boldsymbol{x}$ and z, (3) and (7), are linear, the state $\boldsymbol{x}$ of the original system (1) will converge to zero asymptotically. This completes the proof.

Remark 1 In the controller in Theorem 1, the derivative of sliding mode manifold, $\dot{\boldsymbol{s}}$, should be used. Since it is used only within the closed loop system, it can be directly obtained using a differentiator. It is similar with the derivative operation in the traditional PID controllers.

IV. Simulations

A simulation with a seventh-order system is performed for the purpose of evaluating the performance of the proposed control scheme for uncertain linear multivariable systems in the paper.

Consider the following seventh-order system [13]:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{d}(t) + \boldsymbol{B}\boldsymbol{u} \quad , \qquad (23)$$

where the disturbance is $\boldsymbol{d}(t) = [0.1\sin(2t) \quad 0.1\sin(2t) \quad 0.1\sin(2t)]^T$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

First, system (23) is transformed into the lower Hessenberg form (6):

$$\dot{x}_3' = 2x_3' + [0 \quad 0 \quad 0.5]x_2'$$

$$\dot{x}_2' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_3' + \begin{bmatrix} 2 & 0.5 & 0.5 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_2' + \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} x_1' \quad , (24)$$

$$\dot{x}_1' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_1' + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{u}$$

then, Eq.(24) is further transformed into the following decoupled block controllable canonical form by using the state transformation (7):

$$\dot{z}_3 = -0.6z_3 + [0 \quad 0 \quad 0.5]z_2$$

$$\dot{z}_2 = -z_2 + \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} z_1 \quad , \qquad (25)$$

$$\dot{z}_1 = \overline{\boldsymbol{A}}_{11}z_1 + \overline{\boldsymbol{A}}_{12}z_2 + \overline{\boldsymbol{A}}_{13}z_3 + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} d(t) + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{u}$$

In above transformation, the parameters $\boldsymbol{N}_2$ and N_3 are chosen as: $\boldsymbol{N}_2$= diag(−1, −1, −1), N_3=−0.6. Then, it gets:

$$\boldsymbol{A}_{11} = \begin{bmatrix} 5 & -1.6 & 1 \\ 0 & 5.6 & 0 \\ 0 & 0.8 & 4 \end{bmatrix}, \quad \boldsymbol{A}_{12} = \begin{bmatrix} -9 & 7.5 & 15.66 \\ 0 & 0 & -13.36 \\ 0 & -3 & -6.68 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -17.472 \\ 13.312 \\ 6.656 \end{bmatrix}$$

The state transformation from $\boldsymbol{x}$' to $\boldsymbol{z}$ is:

$$\begin{bmatrix} x_3' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -5.2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -6.72 & -3 & 2.5 & 4.1 & 1 & 0 & 0 \\ 8.32 & 0 & 0 & -4.6 & 0 & 1 & 0 \\ 4.16 & 0 & -1.5 & 2.3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_3 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_1 \end{bmatrix}, \qquad (26)$$

From Eq.(13), three linear sliding mode manifolds are chosen as:

$$s_1 = 0.8z_{11} + (\int_{0-}^{t} z_{11}\mathrm{d}t),$$

$$s_2 = 0.8z_{12} + (\int_{0-}^{t} z_{12}\mathrm{d}t),$$

$$s_3 = 0.8z_{13} + (\int_{0-}^{t} z_{13}\mathrm{d}t).$$

From Eq.(14), the three nonsingular terminal sliding mode manifolds are chosen as:

$$l_1 = \dot{s}_1^{5/3} + s_1,$$
$$l_2 = \dot{s}_2^{5/3} + s_2,$$
$$l_3 = \dot{s}_3^{5/3} + s_3,$$

According to Theorem 1, the NTSM controller of the system is designed as follows:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1,$$

where

$$\boldsymbol{u}_0 = -\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 5 & -1.6 & 1 \\ 0 & 5.6 & 0 \\ 0 & 0.8 & 4 \end{bmatrix} z_1 + \begin{bmatrix} -9 & 7.5 & 15.66 \\ 0 & 0 & -13.36 \\ 0 & -3 & -6.68 \end{bmatrix} z_2 + \begin{bmatrix} -17.472 \\ 13.312 \\ 6.656 \end{bmatrix} z_3 \right).$$

According to Eq.(19), $\boldsymbol{u}_1(t)$ is obtained through the low-passed filter, which is designed as follows:

$$\boldsymbol{v} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \dot{\boldsymbol{u}}_1(t) + \boldsymbol{u}_1(t)$$

where the input of the low-passed filter $\boldsymbol{v}(t)$ is designed as follows according to Eqs. (20), (21) and (22):

$$\boldsymbol{v}(t) = \boldsymbol{v}_{eq}(t) + \boldsymbol{v}_n(t)$$

with

$$\boldsymbol{v}_{eq} = -\frac{3}{5} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \dot{s}_1^{1/3} \\ \dot{s}_2^{1/3} \\ \dot{s}_3^{1/3} \end{bmatrix},$$

$$v_n = -\left(\left\| \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\| (0.1732 + 0.8 \cdot 0.3464) + 0.5 \right) \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \operatorname{sgn}(l)$$

Assume the initial states of the system are: $x_3(0)= 3.26$, $x_{21}(0)= 3.86$, $x_{22}(0)= 4.52$, $x_{23}(0)= 14.64$, $x_{11}(0)= 7.96$, $x_{12}(0)= -10.62$, $x_{13}(0)= -2.26$.

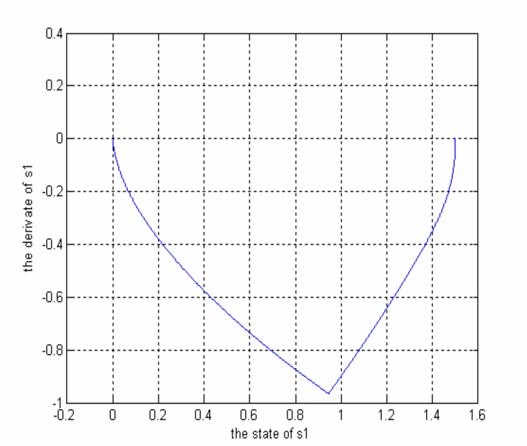

Fig.1. phase plane of $s_1$and its differential.

The simulation results are illustrated in Fig.1 to Fig.7. The phase plane of s_1, s_2, s_3 and their differentials are shown in Fig.1 to Fig.3 respectively. It is seen that s_1, s_2, and s_3 realize the nonsingular terminal sliding mode. The system states $\boldsymbol{z}$ and $\boldsymbol{x}$ are depicted in Fig.4 and Fig.5, respectively. They all

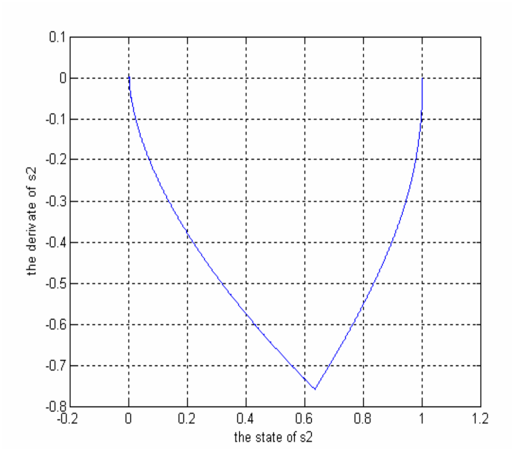

Fig.2. phase plane of $s_2$and its differential.

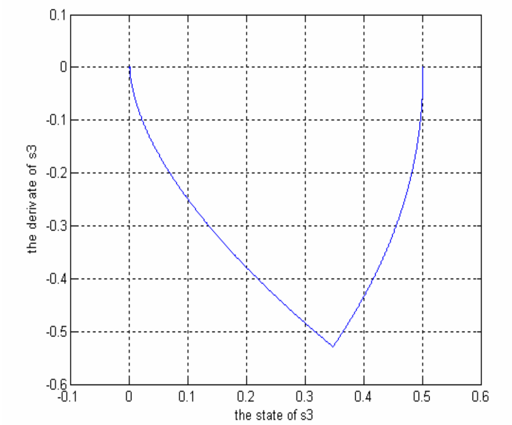

Fig.3. phase plane of s_3 and its differential.

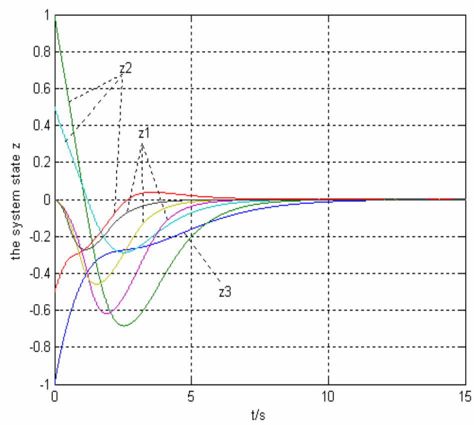

Fig. 4. state variable $\boldsymbol{z}$.

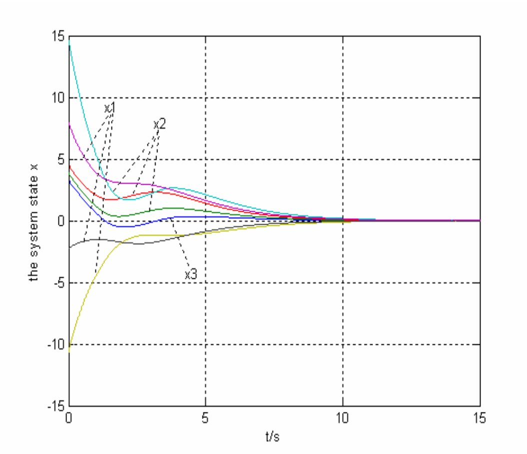

Fig. 5. state variable $\boldsymbol{x}$.

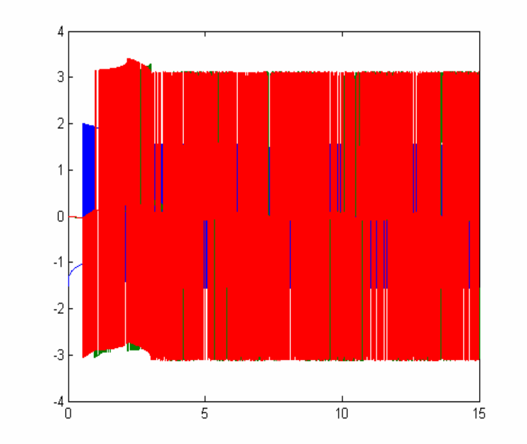

Fig.6. the control signal $\boldsymbol{v}$.

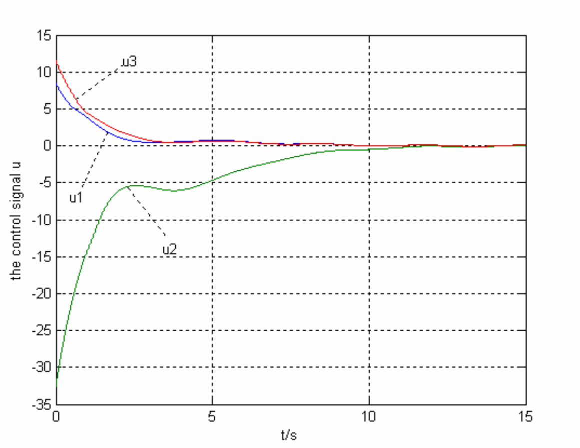

Fig.7. the control signal $\boldsymbol{u}$.

converge to zero asymptotically. Fig.6 shows the input $\boldsymbol{v}$ of the low-passed filter. The control signals are shown in Fig.7. It is seen that no chattering phenomenon occurs.

V. Conclusion

This paper has proposed a special second-order nonsingular TSM decomposed control method for uncertain linear multivariable systems. First, the systems are transformed into the BC–form for design convenience using two state transformations. Then, a second-order nonsingular TSM is proposed for the BC–form system. The proposed control law can drive the states of the input-output subsystem to zero asymptotically and then the states of the stable zero-dynamic subsystem converge to zero asymptotically. The method proposed in the paper can simplify the design of controller and realize hieratical control. The chattering phenomena is eliminated utilizing the second-order sliding mode method. The proposed control method is significant for the high-dimensional uncertain linear multivariable systems.

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