

A Generalised Hadamard Transform

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Abstract—A Generalised Hadamard Transform for multi-phase or multilevel signals is introduced, which includes the Fourier, Generalised, Discrete Fourier, Walsh-Hadamard and Reverse Jacket Transforms. The jacket construction is formalised and shown to admit a tensor product decomposition. Primary matrices under this decomposition are identified. New examples of primary jacket matrices of orders 8 and 12 are presented.

I. INTRODUCTION

There are many discrete signal transforms whose transform matrices have entries on the unit circle. For instance, the Discrete Fourier Transform (DFT) for signals of length n and the Walsh-Hadamard Transform (WHT) are both special cases of the Fourier Transform (FT) found by interpreting the Cooley-Tukey Fast Fourier Transform in terms of abelian group characters [6]. The family of discrete Generalised Transforms $\{(GT)_r, 0 \leq r \leq m-1\}$ for signals of length 2^m [3, 10.2] includes the WHT as case $r=0$, the complex BIFORE transform (CBT) as case $r=1$ and the 2^m -point DFT as case $r=m-1$. Both the WHT and DFT are suboptimal discrete orthogonal transforms, but each has wide application.

These transform matrices all fall into the class of Butson Hadamard matrices [2].

For signals where entries outside the complex n^{th} roots of unity are needed, Lee has introduced multi-phase or multilevel generalisations of the WHT and of the even-length DFT under the name *Reverse Jacket Transforms* (RJT), see [7]. These are so-called because the border (“jacket”) and the centre of the unitary matrix representing the transform change independently under inversion. The former admit a recursive factorisation into tensor products so represent a fast transform similar to that of the WHT. (Note that the formula [7, Definition 5] is not a reverse jacket transform generalising both the WHT and even-length DFT, as claimed, because it is not unitary.)

The class of Butson Hadamard matrices can be generalised to include all these transform matrices.

II. A GENERALISED HADAMARD TRANSFORM

In this most general situation, we work in a ring R with unity 1. This includes \mathbb{R} , \mathbb{C} and Galois Field alphabets $GF(p^a)$, though if we need to distinguish signal values x and $-x$, the ring must have characteristic $\neq 2$.

Definition 2.1: Suppose R is a ring with unity 1, group of units R^* and that $\text{char } R$ does not divide v . A square matrix

M of order $v \geq 2$, with entries from a subgroup $N \leq R^*$ is a *Generalised Butson Hadamard (GBH)* matrix, if

$$MM^* = M^*M = vI_v,$$

where M^* is the transpose of the matrix of inverse elements of M : $m_{ij}^* = (m_{ji})^{-1}$. It is denoted $GBH(N, v)$, or $GBH(w, v)$ if N is finite of order w .

A *GBH* matrix is always equivalent to a *normalised GBH* matrix, which has first row and column consisting of all 1s. By taking the inner product of any non-initial row of a normalised *GBH* matrix M with the all-1s first column of M^* , we see that the sum of the entries in any row of M , apart from the first, must equal 0, and similarly for rows of M^* (columns of the matrix of inverses $M^{(-1)} = [m_{ij}^{-1}]$). The tensor product of two *GBH* matrices over the same group N is a *GBH* matrix over N .

Definition 2.2: Let \mathbf{x} be a signal of length n from R^* , where $n \in R^*$, let $N \leq R^*$ and let B be a $GBH(N, n)$. A *Generalised Hadamard Transform (GHT)* of \mathbf{x} is

$$\hat{\mathbf{x}} = B \mathbf{x} \quad (1)$$

and an *Inverse Generalised Hadamard Transform (IGHT)* of $\hat{\mathbf{x}}$ is

$$\mathbf{x} = n^{-1} B^* \hat{\mathbf{x}}. \quad (2)$$

The next section describes a construction for *GBH* matrices of even order and additional internal structure, containing the WHT, even-length DFT and reverse jacket transform matrices.

III. THE JACKET MATRIX CONSTRUCTION

Throughout this section, let G be an indexing set of even order $2n$ (sometimes G is a group such as \mathbb{Z}_{2n} or \mathbb{Z}_2^a but G may be non-abelian or the group structure may be irrelevant).

Definition 3.1: Let R be a ring with unity 1. A normalised $GBH(N, 2n)$ matrix K indexed by $G = \{1, \dots, 2n\}$ with entries from $N \leq R^*$ is a *jacket matrix* if it is of the form

$$K = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & * & \dots & * & \pm 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & * & \dots & * & \pm 1 \\ 1 & \pm 1 & \dots & \pm 1 & \pm 1 \end{bmatrix} \quad (3)$$

where the central entries $*$ are from N . The *jacket width* of K is $\tilde{m} \geq 2$ if K is permutation equivalent to a jacket matrix \tilde{K} in which rows $2, \dots, m, 2n-m+1, \dots, 2n$ and

columns $2, \dots, m, 2n - m + 1, \dots, 2n$ all consist of ± 1 and m is maximal for this property. Otherwise the jacket width is $m = 1$.

All non-initial ± 1 rows and columns of a jacket matrix K necessarily sum to 0 in R . If K is a width m jacket matrix it follows that K^* is itself a width m jacket matrix. If also $2n \in R^*$, then K has an inverse $K^{-1} = (2n)^{-1}K^*$ over R . If $(2n)^{\frac{1}{2}} \in R^*$ then $\widehat{K} = (2n)^{-\frac{1}{2}}K$ is unitary ($\widehat{K}\widehat{K}^* = I_{2n}$) and we will (by a slight abuse of terminology) also say K is unitary.

Example 3.2: The matrix \mathcal{S}_t of the WHT of length 2^t

$$\mathcal{S}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{S}_t = \bigotimes^t \mathcal{S}_1, \quad t \geq 2, \quad (4)$$

is an extremal case (jacket width 2^{t-1}) of such jacket matrices (“all jacket”).

Example 3.3: The matrix \mathcal{C}_t of the CBT of length 2^t

$$\begin{aligned} \mathcal{C}_1 &= \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \\ \mathcal{C}_2 &= \begin{bmatrix} \mathcal{S}_1 & \mathcal{S}_1 \\ \mathcal{C}_1 & -\mathcal{C}_1 \end{bmatrix}, \\ \mathcal{C}_m &= \begin{bmatrix} & & \mathcal{C}_{m-1} & \\ & & \mathcal{C}_{m-1} & \\ \mathcal{C}_1 \otimes \mathcal{S}_{m-2} & & -\mathcal{C}_1 \otimes \mathcal{S}_{m-2} & \end{bmatrix}, \quad m \geq 3. \end{aligned} \quad (5)$$

is permutation equivalent to a jacket matrix, for $t \geq 2$.

Proof. For $t \geq 2$, \mathcal{C}_t is normalised and the first two rows of \mathcal{C}_t consist of 2^{t-1} copies of \mathcal{S}_1 . By induction the $(2^{t-1} + 1)^{st}$ column of \mathcal{C}_t is $[1 \ -1]^T$, where $\mathbf{1}$ has length 2^{t-1} . Rotating the second row to the bottom and the $(2^{t-1} + 1)^{st}$ column to the right of \mathcal{C}_t produces a jacket matrix. \square

Example 3.4: The matrix $\mathcal{F}_{2n} = [\omega^{jk}]_{0 \leq j, k \leq 2n-1}$, where $\omega = e^{-\pi i/n}$, $n \geq 1$, of the DFT of length $2n$ is permutation equivalent to a jacket matrix.

Proof. [7, Theorem 1, Definition 1] Represent the indices in mixed radix notation $j = j_1n + j_0 = (j_1, j_0)$, with index set $G = \mathbb{Z}_2 \times \mathbb{Z}_n$. The permutation $(j_1, j_0) \mapsto (j_1, (1 - j_1)j_0 + (n-1-j_0)j_1)$ leaves the first n indices unchanged and reverses the order of the last n indices. Under this permutation on rows and columns, \mathcal{F}_{2n} is equivalent to $\mathcal{K}_n(\omega) =$

$$[\omega^{\{(1-j_1)j_0 + (n-1-j_0)j_1 + j_1n\} \{(1-k_1)k_0 + (n-1-k_0)k_1 + k_1n\}}}], \quad (6)$$

the matrix of Lee’s complex RJT. \square

If K, K' are jacket matrices indexed by G, G' of orders $2n, 2n'$ respectively, with entries from R^* , then the tensor product $K \otimes K'$ is a jacket matrix indexed by $G \times G'$ of order $4nn'$, with entries from R^* , since the border condition is easily seen to be satisfied. In fact, a tensor product of jacket matrices is a jacket matrix of width ≥ 2 .

Lemma 3.5: If K_i is a width m_i jacket matrix with entries from R^* , for $i = 1, 2$, then $K_1 \otimes K_2$ is a jacket matrix of width at least $2m_1m_2$.

Proof. Let K_1 have order $2n$ and K_2 have order $2n'$. Permute K_i to \widetilde{K}_i , so $K_1 \otimes K_2$ is permutation equivalent to $\widetilde{K}_1 \otimes \widetilde{K}_2$. Let $i \in \{1, \dots, m_1, 2n - m_1 + 1, \dots, 2n\}$ be an index of an all- (± 1) s row in \widetilde{K}_1 . The corresponding i^{th} block row in

$\widetilde{K}_1 \otimes \widetilde{K}_2$ consists of $2n$ copies of $\pm 1 \widetilde{K}_2$, so each row indexed $\{2, \dots, m_2, 2n' - m_2 + 1, \dots, 2n'\}$ of each copy consists of all- (± 1) s, contributing $2m_2 - 1$ all- (± 1) rows to the i^{th} block row of $\widetilde{K}_1 \otimes \widetilde{K}_2$. If $i = 1$ the top row is all 1s and if $i > 1$ the top row consists of n 1s and $n - 1$ s. Those in the top m_1 block rows of $\widetilde{K}_1 \otimes \widetilde{K}_2$ may be permuted to occupy the top $2m_1m_2$ rows and those in the bottom m_1 block rows to the bottom $2m_1m_2$ rows, and similarly for columns. \square

IV. TENSOR DECOMPOSITION

If a jacket matrix may be decomposed as a tensor product of two smaller jacket matrices, the decomposition may be repeated until no further tensor product decomposition is possible.

Definition 4.1: A jacket matrix of length $2n$ is a *primary* jacket matrix K_n if it is minimal with respect to tensor product, that is, there are no jacket matrices K, K' such that K_n is permutation equivalent to $K \otimes K'$.

Examples of primary jacket matrices for $n = 1, \dots, 4$ are

$$\begin{aligned} K_1 &= \mathcal{S}_1, \\ K_2(r) &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -r & r & -1 \\ 1 & r & -r & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad r \neq \pm 1 \in R^* \end{aligned} \quad (7)$$

$$K_3(\alpha) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}, \quad (8)$$

where α is a primitive 6^{th} root of unity in an integral domain R , and

$$K_4(i) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & i & -i & 1 & -1 & i & -i & -1 \\ 1 & -i & -1 & i & i & -1 & -i & 1 \\ 1 & 1 & i & i & -i & -i & -1 & -1 \\ 1 & -1 & i & -i & i & -i & 1 & -1 \\ 1 & i & -1 & -i & -i & -1 & i & 1 \\ 1 & -i & -i & -1 & 1 & i & i & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}. \quad (9)$$

The matrix K_1 is the unique 2×2 jacket matrix, and $K_2(1) = K_1 \otimes K_1 = \mathcal{S}_2$, so is not primary. The matrix $K_2(r)$ for $r \neq \pm 1 \in \mathbb{R}^*$ is a “centre-weighted Hadamard transform” (CWHT) matrix [7] and for $r = i \in \mathbb{C}^*$, is $\mathcal{K}_2(i)$. The matrix $K_3(\alpha)$ with $\alpha = e^{i\pi/3}$ is $\mathcal{K}_3(e^{i\pi/3})$ and with α the fourth power of a primitive root in $GF(25)$ is an “extended” complex RJT matrix [7, Example 4]. The matrix $K_4(i)$ is described for the first time here: it is equivalent to the back-circulant matrix derived from a quadriphase perfect sequence [1, Example 2] of length 8; such sequences are rare.

Corollary 4.2: A jacket matrix is permutation equivalent to a tensor product of one or more primary jacket matrices. Conversely, any tensor product of primary jacket matrices is a jacket matrix.

V. CONSTRUCTION OF PRIMARY JACKET MATRICES

By Lemma 3.5 any jacket matrix of width 1 is primary. The tensor product of two jacket matrices is a jacket matrix, but in fact, to construct a jacket matrix it is enough that one factor is a jacket matrix and the other a normalised GBH matrix.

Theorem 5.1: Let B be a normalised GBH matrix and K a jacket matrix, both with entries in R^* . Then $B \otimes K$ is permutation equivalent to a jacket matrix $(B \otimes K)^\dagger$.

Proof. Let B have order m and K have order $2n$. The first $2n$ rows of $B \otimes K$ consist of m blocks K, K, \dots, K , so the 1st row is all 1's and the $2n^{\text{th}}$ row is all ± 1 s, and similarly for columns. Cyclically permute row $2n$ to row $2mn$ and row $2n+i$ to row $2n+i-1$, $i = 1, \dots, 2n(m-1)$, and similarly for columns. This shifts row $2n$ to the bottom of the matrix and column $2n$ to the right of the matrix, leaving the order of the other rows and columns otherwise unchanged. This permuted matrix $(B \otimes K)^\dagger$ is of the form (3). \square

This result explains the generation of some primary jacket matrices and is fundamental to the construction of Generalised Hadamard Transforms.

Example 5.2: Let $\beta \neq 1 \in R^*$ satisfy $\beta^2 + \beta + 1 = 0$, so $\beta^3 = 1$. Let

$$B_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \beta & \beta^2 \\ 1 & \beta^2 & \beta \end{bmatrix},$$

so B_3 is a normalised GBH(3, 3). Then

$$(B_3 \otimes K_1)^\dagger = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \beta & \beta & \beta^2 & \beta^2 & 1 \\ 1 & \beta & -\beta & \beta^2 & -\beta^2 & -1 \\ 1 & \beta^2 & \beta^2 & \beta & \beta & 1 \\ 1 & \beta^2 & -\beta^2 & \beta & -\beta & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

This jacket matrix relates to the DFT matrix of (8) as follows. A second permutation (2543) cycling central rows and columns gives the jacket matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\beta & \beta^2 & -\beta^2 & \beta & -1 \\ 1 & \beta^2 & \beta & \beta & \beta^2 & 1 \\ 1 & -\beta^2 & \beta & -\beta & \beta^2 & -1 \\ 1 & \beta & \beta^2 & \beta^2 & \beta & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

When α is a primitive 6^{th} root of unity in R^* with $\alpha^2 = \beta$ and $\alpha^5 = \gamma$, then this matrix equals $\mathcal{K}_3(\gamma)$.

The following matrix is a new 12×12 primary jacket matrix, since it has width 1.

Example 5.3: Let $r \neq \pm 1 \in R^*$.

Let $K_6(\beta, r) = (B_3 \otimes K_2(r))^\dagger =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -r & r & 1 & -r & r & -1 & 1 & -r & r & -1 & -1 \\ 1 & r & -r & 1 & r & -r & -1 & 1 & r & -r & -1 & -1 \\ 1 & 1 & 1 & \beta & \beta & \beta & \beta^2 & \beta^2 & -r\beta^2 & r\beta^2 & -\beta^2 & 1 \\ 1 & -r & r & \beta & -r\beta & r\beta & -\beta & \beta^2 & -r\beta^2 & r\beta^2 & -\beta^2 & -1 \\ 1 & r & -r & \beta & r\beta & -r\beta & -\beta & \beta^2 & r\beta^2 & -r\beta^2 & -\beta^2 & -1 \\ 1 & -1 & -1 & \beta & -\beta & -\beta & \beta^2 & -\beta^2 & -\beta^2 & -\beta^2 & \beta^2 & 1 \\ 1 & 1 & 1 & \beta^2 & \beta^2 & \beta^2 & \beta^2 & \beta & \beta & \beta & \beta & 1 \\ 1 & -r & r & \beta^2 & -r\beta^2 & r\beta^2 & -\beta^2 & \beta & -r\beta & r\beta & -\beta & -1 \\ 1 & r & -r & \beta^2 & r\beta^2 & -r\beta^2 & -\beta^2 & \beta & r\beta & -r\beta & -\beta & -1 \\ 1 & -1 & -1 & \beta^2 & -\beta^2 & -\beta^2 & \beta^2 & \beta & -\beta & -\beta & \beta & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Those GHT matrices which are jacket matrices have additional structure, by virtue of their tensor product decomposition into primary jacket matrices and their jacket form, which may particularly suit them to specific applications.

Consider the set of jacket matrices with entries in an integral domain R

$$\{K = (\otimes^\ell K_1) \otimes K_2(r)^\epsilon \otimes K_n(\alpha)^\delta; \ell \geq 0, \epsilon, \delta \in \{0, 1\}\}, \quad (10)$$

where $r \neq \pm 1 \in R^*$, α is a primitive $2n^{\text{th}}$ root of unity and where by M^0 we mean the 1×1 identity matrix. When $\ell \geq 1, \epsilon = 0, \delta = 0$, this is the WHT. When $\ell = 0, \epsilon = 0, \delta = 1$ and $R = \mathbb{C}$ this is equivalent to the $2n$ -point DFT. When $\epsilon = 1, \delta = 0$ and $R = \mathbb{R}$, this is the CWHT. When $\epsilon = 0, n = 2, \alpha = i \in \mathbb{C}^*, \delta = 1$, or when $\epsilon = 1, r = i \in \mathbb{C}^*, \delta = 0$, this is the complex RJT, and when $\epsilon = 0, \delta = 1$, this is the extended complex RJT.

This uniform classification may make it possible to recognise common fast algorithms.

Finally, the examples and constructions of primary jacket matrices above are all instances of a class of matrices called *cocyclic*, investigated by the author and colleagues over the past decade [4], [5].

VI. CONCLUSION

To summarise: a generalisation of Butson's Hadamard matrices determines a Generalised Hadamard Transform (GHT). The GHT includes the Fourier and Generalised Transform families (in particular the WHT and DFT) and the centre-weighted Walsh-Hadamard, Complex Reverse Jacket and extended Complex Reverse Jacket Transforms.

In the jacket case, GHT matrices can be permuted into tensor products of primary jacket matrices. New primary jacket matrices may be constructed as tensor products of a Generalised Butson Hadamard matrix which is not a primary jacket matrix, and a primary jacket matrix. New examples in orders 8 and 12 have been given.

Application of the GHT to image processing, error-control coding and decoding and sequence design are obvious directions for future research.

REFERENCES

- [1] K. T. Arasu and W. de Launey, Two-dimensional perfect quaternary arrays, *IEEE Trans. Inform. Theory* 47(4) (2001) 1482–1493.
- [2] A. T. Butson, Generalised Hadamard matrices, *Proc. Amer. Math. Soc.* 13 (1962) 894–898.
- [3] D. F. Elliott and K. R. Rao, Fast Transforms: Algorithms, Analyses, Applications, Academic Press, New York, 1982.
- [4] K. J. Horadam, An introduction to cocyclic generalised Hadamard matrices, *Discrete Appl. Math.* 102 (2000) 115–131.
- [5] K. J. Horadam and P. Udaya, Cocyclic Hadamard codes, *IEEE Trans. Inform. Theory* 46(4) (2000) 1545–1550.
- [6] D. K. Maslen and D. N. Rockmore, Generalised FFTs - a survey of some recent results, *DIMACS Ser. Discr. Math. Theoret. Comp. Sci.* 28 (1997) 183–237.
- [7] Moon Ho Lee, B. Sunder Rajan and J. Y. Park, A generalized reverse jacket transform, *IEEE Trans. Circuits Syst. II*, 48(7) (2001) 684–690.