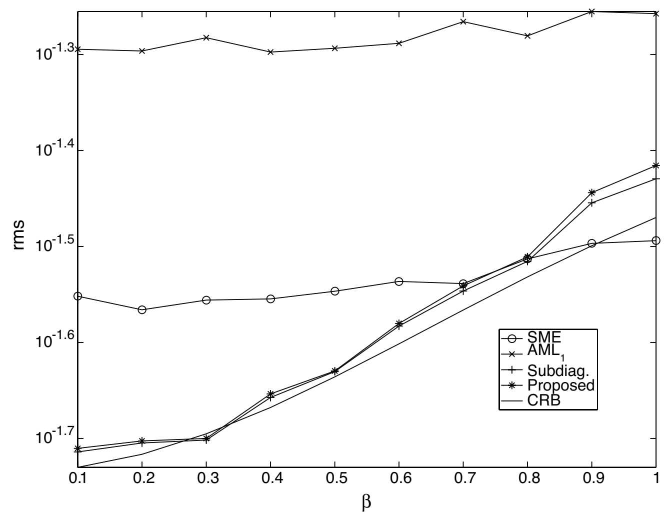

Fig. 4. RMSE of the DOA estimates versus β. $m = 8$, $N = 64$, and $q^2 = 0$ dB.

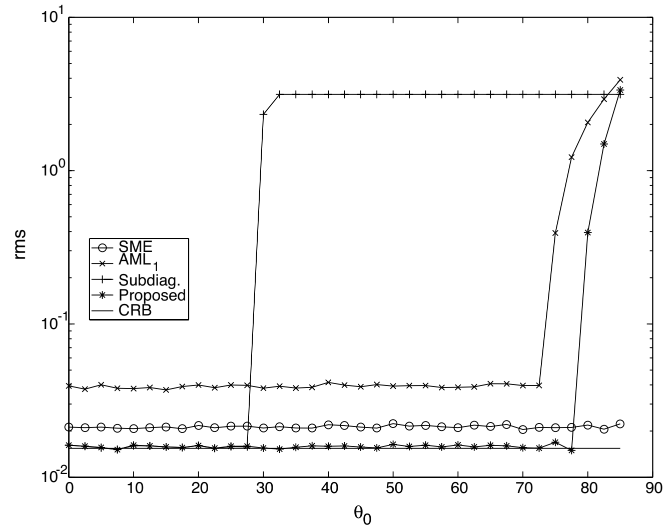

Fig. 5. RMSE of the DOA estimates versus θ_0. $m = 8$, $N = 64$, $q^2 = 0$ dB, and $\beta = 0.25$.

of [1] provides erroneous estimates, which clearly demonstrates the ambiguity problem. In contrast, our method does not suffer from this problem.

V. Conclusions

In this correspondence, we consider the direction-finding problem for an extended target whose power spatial density is not necessarily symmetric with respect to its mass center. Two computationally simple algorithms were proposed. One is based on the spectral moments of the target, which are easily related to its DOA. The second borrows ideas from [1] and extends the range of DOAs that can be estimated unambiguously. Both methods provide robust, simple, yet accurate DOA estimates.

References

[1] O. Besson, P. Stoica, and A. Gershman, "Simple and accurate direction of arrival estimator in the presence of imperfect spatial coherence," *IEEE Trans. Signal Processing*, vol. 49, pp. 730–737, Apr. 2001.

[2] A. Monakov, "Observation of extended targets with antenna arrays," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 36, pp. 297–302, Jan. 2000.

[3] M. Zatman, "How narrowband is narrowband?," *Proc. Inst. Elect. Eng.—Radar, Sonar Navigat.*, vol. 145, no. 2, pp. 85–91, 1998.

[4] K. Pedersen, P. Mogensen, and B. Fleury, "A stochastic model of the temporal and azimuthal dispersion seen at the base station in outdoor propagation environments," *IEEE Trans. Veh. Technol.*, vol. 49, pp. 437–447, Mar. 2000.

[5] A. Paulraj and T. Kailath, "Direction of arrival estimation by eigenstructure methods with imperfect spatial coherence of wavefronts," *J. Acoust. Soc. Amer.*, vol. 83, pp. 1034–1040, Mar. 1988.

[6] A. Gershman, C. Mecklenbräuker, and J. Böhme, "Matrix fitting approach to direction of arrival estimation with imperfect coherence of wavefronts," *IEEE Trans. Signal Processing*, vol. 45, pp. 1894–1899, July 1997.

[7] R. Raich, J. Goldberg, and H. Messer, "Bearing estimation for a distributed source via the conventional beamformer," in *Proc. SSAP Workshop*, Portland, OR, Sept. 1998, pp. 5–8.

[8] R. Moses, T. Söderström, and J. Sorelius, "Effects of multipath-induced angular spread on direction of arrival estimators in array signal processing," in *Proc. IEEE/IEE Workshop Signal Process. Methods Multipath Environments*, Glasgow, U.K., Apr. 20–21, 1995, pp. 6–15.

[9] D. Astély and B. Ottersten, "The effects of local scattering on direction of arrival estimation with MUSIC," *IEEE Trans. Signal Processing*, vol. 47, pp. 3220–3234, Dec. 1999.

[10] M. Bengtsson and B. Ottersten, "Low-complexity estimators for distributed sources," *IEEE Trans. Signal Processing*, vol. 48, pp. 2185–2194, Aug. 2000.

[11] ——, "A generalization of weighted subspace fitting to full-rank models," *IEEE Trans. Signal Processing*, vol. 49, pp. 1002–1012, May 2001.

[12] O. Besson and P. Stoica, "Decoupled estimation of DOA and angular spread for a spatially distributed source," *IEEE Trans. Signal Processing*, vol. 48, pp. 1872–1882, July 2000.

[13] O. Besson, F. Vincent, P. Stoica, and A. Gershman, "Approximate maximum likelihood estimators for array processing in multiplicative noise environments," *IEEE Trans. Signal Processing*, vol. 48, pp. 2506–2518, Sept. 2000.

Comments on "A High-Resolution Quadratic Time-Frequency Distribution for Multicomponent Signals Analysis"

Zahir M. Hussain

***Abstract*—It is shown that the time-frequency distribution (TFD) proposed in the above paper is not well defined in the ordinary sense for power signals, including the single-tone sinusoid, and it needs the introduction of generalized functions and transforms. It is also shown that the proposed TFD does not satisfy the conditions cited by the authors of the paper to justify the claim that it has the instantaneous frequency property.**

***Index Terms*—Generalized functions, instantaneous frequency, multicomponent signals, reduced interference distributions, time-frequency analysis.**

Recently, a time-frequency distribution (TFD) of Cohen's Class, which is known as the B-distribution (BD), was proposed and claimed

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to satisfy the instantaneous frequency (IF) property and have better time-frequency resolution under noise-free and noisy conditions than the spectrogram and Wigner–Ville distribution (WVD) [1]–[3]. The following are four comments about the BD.

I. COMMENT 1

The time-lag kernel of the BD is given by [1]

$$G(t,\tau)=\left[\frac{|\tau|}{\cosh^2(t)}\right]^\alpha \tag{1}$$

where α is a real positive number less than one. Hence, the BD for any time signal $z(t)=a(t)\exp[j\phi(t)]$ is given by the general formula for Cohen's Class [4]–[8]

$$\begin{aligned}\rho_z(t,\omega)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(t-u,\tau)K_z(u,\tau)e^{-j\omega\tau}\,du\,d\tau\\&=\underset{\tau\to\omega}{\mathcal{F}}\{p_z(t,\tau)\}\end{aligned} \tag{2}$$

where $\mathcal{F}$ represents the Fourier transformation, $\omega=2\pi f$ is the radian frequency, $K_z(u,\tau)=z(u+\tau/2)z^*(u-\tau/2)$ is the instantaneous autocorrelation product, $p_z(t,\tau)=G(t,\tau)*_{(t)}K_z(t,\tau)$, and $*_{(t)}$ stands for time convolution. Note that $G(u-t,\tau)=G(t-u,\tau)$. In fact, the structure of the BD is equivalent to a smoothed pseudo-Wigner distribution. The notion of such a distribution was first time defined by Flandrin as detailed in [6, p. 254].

Due to the diverging factor $|\tau|^\alpha$ in the integrand, the above integral does not exist in the ordinary sense unless $z(t)$ is an energy signal with either

1) finite length;
2) infinite length such that there exists an ordinary (nonsingular) function $r(\tau)$ for which the convolution $r(\tau)*_{(\tau)}p_z(t,\tau)$ is defined $\forall\tau$ and $\lim_{|\tau|\to\infty}[r(\tau)*_{(\tau)}p_z(t,\tau)]=0$ [11].

For example, let us consider the sinusoidal signal $z(t)=\exp(j\omega_o t)$. This power signal is of fundamental importance in signal analysis. TheWVD for this signal is given by

$$W(t,\omega)=2\pi\delta(\omega-\omega_o) \tag{3}$$

where δ is the Dirac delta function. This means that the WVD gives the best possible representation of this signal since it gives ideal concentration around the IF of the signal, which is ω_o. The BD for this signal is given by

$$\rho_z(t,\omega)=k_\alpha\int_{-\infty}^{\infty}|\tau|^\alpha\exp[-j(\omega-\omega_o)\tau]\,d\tau \tag{4}$$

where $k_\alpha=\beta(1/2,\alpha)$, β is the beta function [9]. Note that k_α is time independent. The above integral, which is simply the Fourier transform of $|\tau|^\alpha$ (with respect to the frequency variable $F=\omega-\omega_o$) scaled by k_α, is undefined in the ordinary sense for all $\alpha>0$ at every point (t,ω) in the time-frequency plane since the integrand diverges indefinitely when $|\tau|$ increases. To further clarify this point, note that the above integral exists if and only if the integral

$$I(\alpha,F)=\int_0^\infty\tau^\alpha\cos(F\tau)\,d\tau \tag{5}$$

exists for $0<\alpha<1$ and $F>0$. This is so because the Fourier transform of a real and even function is real and even. Using the technique of integration by parts and the tables in [9], it follows that

$$\begin{aligned}I(\alpha,F)&=\tau^\alpha\frac{\sin(F\tau)}{F}\bigg|_{\tau=0}^{\infty}-\frac{\alpha}{F}\int_0^\infty\tau^{\alpha-1}\sin(F\tau)\,d\tau\\&=\lim_{\tau\to\infty}\tau^\alpha\frac{\sin(F\tau)}{F}-\frac{\Gamma(\alpha+1)}{F^{\alpha+1}}\sin\frac{\alpha\pi}{2}.\end{aligned} \tag{6}$$

The above limit does not exist; hence, the BD is not well defined in the ordinary sense for single-tone sinusoidal signals. However, the BD of a sinusoid can be understood in terms of *generalized functions* (or *functionals*) defined on the class S of "good" functions, i.e., functions that are infinitely differentiable and tend to zero as $\omega\to\infty$ faster than the reciprocal of any polynomial function [11, Sec. 7.1]. Only in this sense, it can be shown that [11, pp. 138–139] for $0<\alpha<1$

$$g(\tau)=|\tau|^\alpha\Leftrightarrow G(\omega)=\frac{2\Gamma(\alpha+1)\cos\left[\frac{\pi}{2}(\alpha+1)\right]}{|\omega|^{\alpha+1}}. \tag{7}$$

Note that the right-hand side of (7) is not an ordinary function. Rather, $1/|\omega|^{\alpha+1}$ in (7) is a functional defined on the class S in the sense that

$$1/|\omega|^{\alpha+1}=-\frac{1}{\alpha}\left[\frac{1}{|\omega|^\alpha}\operatorname{sgn}(\omega)\right]' \tag{8}$$

where the derivative $'$ is the *generalized derivative* [11, p. 130]. In this case, the BD is meaningful only in conjunction with the integral $\int_{-\infty}^{\infty}\rho_z(t,\omega)\varphi(\omega)\,d\omega$ for a "good" frequency function $\varphi(\omega)$ [11].

II. COMMENT 2

The authors of [1] state that the BD has better concentration (resolution) than the WVD for stepped FM (a combination of finite-length sinusoids) and linear FM signals (see [1, Figs. 4 and 7]). A more general statement in the Conclusion of [1] confirms that "The proposed distribution outperforms the WVD and the spectrogram in terms of time-frequency resolution and cross term suppression." No reasoning or analysis is provided in [1] to support this claim, knowing that the WVD has the best possible concentration for infinite-length sinusoids (see Comment 1 above) and linear FM signals [4], where it gives the delta function around the IF law in both cases.

III. COMMENT 3

It is claimed in [1] that the first moment of the BD yields the instantaneous frequency of the signal, that is

$$\omega_i=\frac{\int_{-\infty}^{\infty}\omega\rho_z(t,\omega)\,d\omega}{\int_{-\infty}^{\infty}\rho_z(t,\omega)\,d\omega} \tag{9}$$

because its Doppler-lag kernel $g(\nu,\tau)$ satisfies the following conditions:

$$\begin{aligned}\frac{\partial g(\nu,\tau)}{\partial\tau}\bigg|_{(0,0)}&=\frac{\partial g(\nu,\tau)}{\partial\nu}\bigg|_{(0,0)}=0\\g(\nu,0)&=\text{constant}\quad\forall\nu.\end{aligned} \tag{10}$$

First, it must be pointed out that the correct conditions for the IF property are [4]–[8], [10]

$$\begin{aligned}\frac{\partial g(\nu,\tau)}{\partial\tau}\bigg|_{\tau=0}&=0\quad\forall\nu\\g(\nu,0)&=1\quad\forall\nu.\end{aligned} \tag{11}$$

The condition $(\partial g(\nu,\tau))/(\partial\nu)|_{(0,0)}=0$ (or its correct version $(\partial g(\nu,\tau))/(\partial\nu)|_{\nu=0}=0\ \forall\tau$) is inappropriate here as it is related to the time delay property and not the IF property [4]–[8], [10].

Second, I would like to point out that the Doppler-lag kernel of the BD satisfies neither (10) nor (11), as shown below.

Proof: According to [1, (7)], we have

$$\begin{aligned}g(\nu,\tau)&=|\tau|^\alpha\frac{2^{2\alpha-1}}{\Gamma(2\alpha)}\Gamma(\alpha+j\pi\nu)\Gamma(\alpha-j\pi\nu)\\&=g_1(\tau)g_2(\nu)\end{aligned} \tag{12}$$

where $g_1(\tau)=|\tau|^\alpha$, and $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt \tag{13}$$

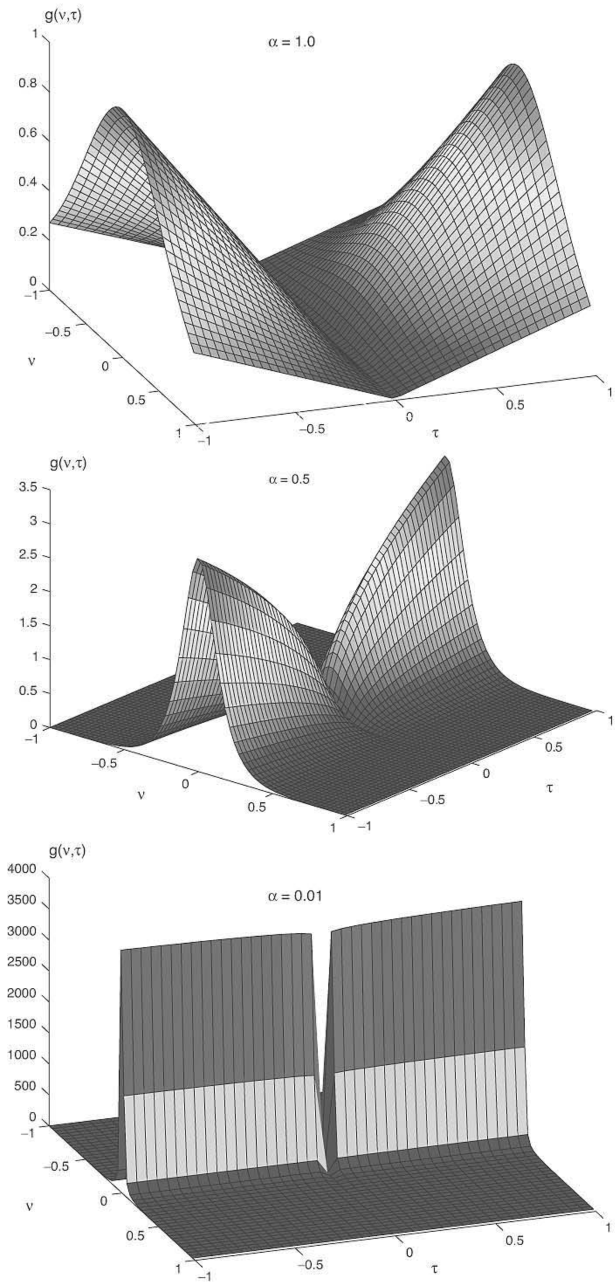

Fig. 1. Doppler-lag kernel $g(\nu, \tau)$ of the BD for different values of α. Contrary to all known quadratic TFDs, the BD kernel has a highpass shape in the lag direction with a minimum (zero) at the origin.

and has simple poles in the complex z-plane at $z = 0, -1, -2, \ldots$, etc. [9]. For the range of α specified in [1], i.e., $0 < \alpha \leq 1$, the derivative $(\partial g(\nu, \tau))/(\partial \tau) = g_2(\nu) d|\tau|^\alpha / d\tau$ does not exist at $\tau = 0$ since

$$\lim_{\tau \to 0} \frac{d|\tau|^\alpha}{d\tau} = \begin{cases} +\infty & \tau \to 0^+, \quad 0 < \alpha < 1 \\ -\infty & \tau \to 0^-, \quad 0 < \alpha < 1 \\ +1 & \tau \to 0^+, \quad \alpha = 1 \\ -1 & \tau \to 0^-, \quad \alpha = 1 \end{cases} \tag{14}$$

while $g_2(\nu) = (2^{2\alpha - 1})/(\Gamma(2\alpha)) |\Gamma(\alpha + j\pi\nu)|^2$ is always positive and finite for all ν when $0 < \alpha \leq 1$. Therefore, it cannot be concluded that the BD has the IF property unless a valid proof is given. □

IV. Comment 4

The authors of [1] state that the kernel of the BD is designed *intuitively* to retain the auto-terms and suppress the cross-terms. However, in order to suppress the cross-terms [that accumulate far away from the origin of the Doppler-lag (a.k.a. ambiguity) domain] while keeping the auto-terms (that accumulate around the origin), it is well known that the Doppler-lag kernel $g(\nu, \tau)$ of any quadratic time-frequency distribution should have a two–dimensional lowpass shape with a maximum at the origin $(\nu, \tau) = (0, 0)$ (see, for example, [5] and [6]). If the preservation of the signal total energy is considered, this maximum should be one, i.e., $g(0, 0) = 1$ (see [5, p. 164] and [6, p. 107]). Contrary to all known TFDs in the quadratic class (a.k.a. Cohen's class), the BD kernel has a highpass shape in the lag direction with a minimum (zero) at the origin, as shown in Fig. 1. No reasoning is provided in [1] for this abnormality.

V. Conclusion

In this correspondence, it is shown that the B-distribution (BD), which was recently proposed as a time-frequency distribution (TFD), is not well-defined in the ordinary sense for power signals, including the pure sinusoid, which is of fundamental importance in time-frequency analysis. A correct definition may be introduced using generalized functions. It is also shown that the BD does not satisfy the conditions cited by the authors of [1] to justify the claim that it has the conventional IF property. In addition, it is pointed out that the BD has an abnormal time-lag kernel with a minimum at the origin, contrary to the original purpose of designing the BD to attenuate the cross-terms and pass the auto-terms.

Acknowledgment

The auhtor would like to thank the Editor-in-Chief, Prof. A. H. Sayed, for his valuable time and sincere efforts in reviewing these comments. He is deeply grateful to the Associate Editor, Dr. T.-H. Li, for his insightful comments, fruitful suggestions, and patience with him throughout this discussion. Many thanks to the reviewers for their constructive comments and to the IEEE for this unique dedication to the truth. He would like also to thank the authors of the BD paper for their reply.

References

[1] B. Barkat and B. Boashash, "A high-resolution quadratic time-frequency distribution for multicomponent signals analysis," *IEEE Trans. Signal Processing*, vol. 49, pp. 2232–2239, Oct. 2001.
[2] B. Barkat, "Design, estimation, and performance of time-frequency distributions," Ph.D. dissertation, Queensland Univ. Technol., Brisbane, Australia, 2000.
[3] B. Barkat and K. Abed-Meraim, "A blind components separation procedure for FM signal analysis," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 2, pp. 1425–1428, May 2002.
[4] L. Cohen, "Time-frequency distributions—A review," *Proc. IEEE*, vol. 77, pp. 941–981, July 1989.
[5] ——, *Time-Frequency Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[6] P. Flandrin, *Time-Frequency/Time-Scale Analysis*. Boston, MA: Academic, 1998.
[7] J. Jeong and W. Williams, "Kernel design for reduced interference distributions," *IEEE Trans. Signal Processing*, vol. 40, pp. 402–412, Feb. 1992.
[8] P. J. Loughlin, J. W. Pitton, and L. E. Atlas, "Bilinear time-frequency representations: New insights and properties," *IEEE Trans. Signal Processing*, vol. 41, pp. 750–767, Feb. 1993.
[9] I. S. Gradshtein, *Table of Integrals, Series, and Products*, 6th ed. Boston, MA: Academic, 2000.

[10] T. C. M. Claasen and W. F. G. Mecklenbrauker, "The Wigner distribution—A tool for time-frequency signal analysis—Part III: Relations with other time-frequency signal transformations," *Philips J. Res.*, vol. 35, no. 6, pp. 372–389, 1980.
[11] *A Handbook of Fourier Theorems*, Cambridge Univ. Press, New York, 1987.

Reply to "Comments on 'A High-Resolution Quadratic Time-Frequency Distribution for Multicomponent Signals Analysis'"

B. Barkat and B. Boashash

I. Reply to Comment 1

The recently proposed quadratic time-frequency distribution (TFD), referred as BD, handles power signals in the same way the Wigner–Ville distribution (WVD) or any other time-frequency distribution (TFD) handles them. These TFDs, being in essence Fourier transforms (FTs) of the signal or quadratic functions of it, cannot be evaluated in an ordinary way but have to be evaluated using functionals. As an illustration, let us reconsider the signal used in the comments, namely, a sinusoid expressed as $z(t) = \exp(j\omega_0 t)$. The WVD of this signal is given by

$$W_z(t,f) = \int_{-\infty}^{\infty} [z(t+0.5\tau) \cdot z^*(t-0.5\tau)] \exp[-j\omega\tau]\, d\tau. \quad (1)$$

Because $[z(t+0.5\tau)\cdot z^*(t-0.5\tau)] = \exp[-j\omega_0\tau]$ is not absolutely integrable over the considered interval, its FT cannot be obtained by direct evaluation, and one has to resort to transforms in the limit or functionals in order to obtain the final result in (3) of the comments. The same discussion applies to the BD or any other TFD. FTs in the limit have existed for decades and are not new concepts for the community. As a consequence, Comment 1 does not provide any new information.

II. Reply to Comment 2

It was clearly stated that the BD can solve *some* problems that the WVD or the spectrogram cannot. It was never claimed that the BD performs better than the two other distributions at all times and all situations. To be more specific, in the paper introduction, the following statement was given: "*This comparison is performed with respect to some criteria detailed later in the paper.*" All the criteria pertaining to the comparisons in the paper were given in detail. In addition, the paper comparisons were basically numerical and not analytical, and in a numerical implementation, it is not possible to use an infinite-length signal, and consequently, the theoretical "best possible concentration" of the WVD is not guaranteed. This point was well illustrated in the paper where *Monte-Carlo analysis* as well as several examples, including real-life data, were provided to support the claim.

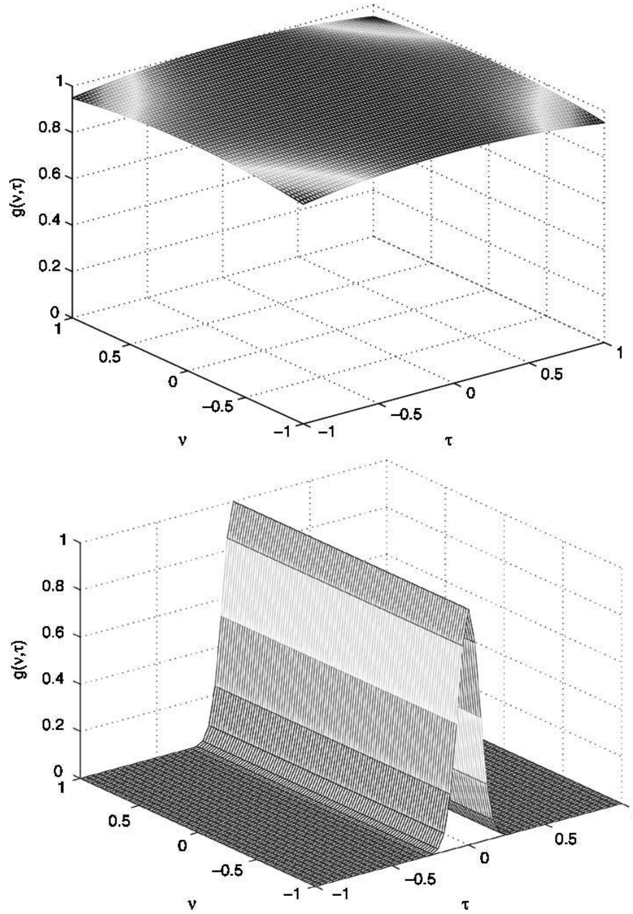

Fig. 1. Doppler-lag kernels of the cone-shape distribution for a small value (top plot) and a large value (bottom plot) of its parameter.

III. Reply to Comment 3

To start with, let us observe that conditions (11) are just *sufficient* conditions, and consequently, there is the following.

i) Other, less restrictive, conditions can also be valid conditions for expression (9). This means that conditions (10) reported in [2, p. 14] cannot be excluded without a detailed demonstration to prove it.
ii) The first moment of a TFD that violates conditions (11) may still be a good estimator of the signal instantaneous frequency (IF).

In addition, in the paper, expression (9) was not used to estimate the signal IF. Instead, the peak of the BD was used as an IF estimator. Therefore, whether the BD verifies or violates conditions (11) does not have any negative implications on the results of the BD paper or on the simulations therein. Furthermore, many known TFDs do not verify (11). One of them is the spectrogram, which is still today one of the most popular and widely used quadratic TFDs. Since this has not limited its application in real-life problems, the same statement can be made about the BD.

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