Iterated Sums of Arithmetic Progressions

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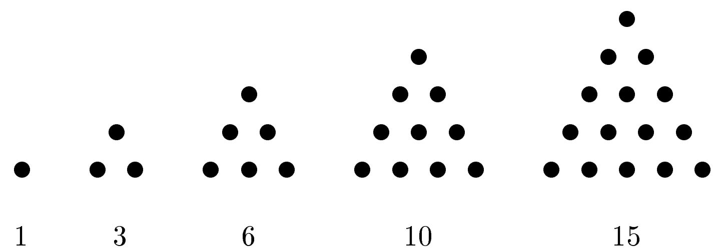
1 Introduction

The triangular numbers namely, $1, 3, 6, 10, 15, \ldots$, whose n-th term is given by the expression $n(n+1)/2$, is an example of a polygonal or figurative number sequence, as each term in the sequence counts the number of dots in an equilateral triangle having n dots in each side as pictured below.



Other examples of figurative number sequences are of course, the perfect squares $1, 4, 9, \ldots$ and the pentagonal numbers $1, 5, 12, \ldots$, which each count the arrangement of dots in ever increasingly larger planer geometrical arrays of squares and pentagons respectively. One can also construct a three dimensional figurative number sequence from the triangular numbers, by stacking the above planer equilateral triangles on top of each other to produce a tetrahedral pyramid of dots. Such a stacking results in what is known as a tetrahedral number sequence $1, 4, 10, 20, \ldots$ which by definition represent the successive partial sums of the triangular number sequence as follows $1, 1+3, 1+3+6, 1+3+6+10, \ldots$. Using some rather ingenious counting devices, Conway and Guy (see [1, pp. 44]) showed that the n-th tetrahedral number is given by the expression $n(n+1)(n+2)/6$.

Clearly the above process of sequentially adding partial sums of the natural number can be repeated infinitum, to produce an infinite family of higher dimensional figurative number sequences. Each such sequence represents an example of what is known as an iterated or k-fold summation of the natural numbers. Formally a k-fold summation of a given sequence $\{a_n\}$ can be defined as follows. Beginning with the n-th partial sum of the sequence $\{a_n\}$ denoted, $S_n^{(0)} = \sum_{i=1}^{n} a_i$, one can proceed with the construction of another sequence $\{S_n^{(1)}\}$, formed from the n-th partial sums of the sequence $\{S_n^{(0)}\}$, that is $S_n^{(1)} = \sum_{i=1}^{n} S_i^{(0)}$. Repeating this procedure a further $k-1$ times, were $k \geq 1$, produces the resulting k-fold summation $S_n^{(k)} = \sum_{i=1}^{n} S_i^{(k-1)}$ of the original sequence $\{a_n\}$. With this definition in mind, one may naturally question whether, like the tetrahedral numbers,

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a closed-form expression for each of the k-fold summations of the natural numbers can be found.

In this article, we shall in fact show that for any arbitrary sequence of real numbers whose terms are in arithmetic progression, a simple algebraic expression exists for $S_n^{(k)}$, and moreover is given in terms of two binomial coefficients involving the parameters n and k. The question of determining such a closed-form expression will be tackled by two contrasting methods, each of which will solve the problem by first finding the k-fold summation of the natural numbers. One of these methods shall involve the use of a generating function, that is a power series and its associated functional representation. In particular for the sequence of natural numbers, we will identify, for each fixed $k \geq 0$, a function denoted $f_k(x)$, having $\sum_{n=0}^{\infty} S_n^{(k)} x^n$ as its corresponding power series expansion. As shall be seen, $f_k(x)$ will be given in terms of the function $(1+x)^\alpha$, from whose Maclaurin expansion the desired closed-form expression for $S_n^{(k)}$ can be derived. The second and more elementary method, which we shall explore first, involves the use of the so-called Hockey Stick Theorem (see [4]), to show that the k-fold summation for the sequence of natural numbers can be identified as an entry in Pascal's Triangle.

2 An Elementary Approach

To begin, we note that the problem of determining the k-fold summation for a general arithmetic progression, can easily be reduced to that of determining the k-fold summation of the sequence of natural numbers. Indeed, if we denote the k-fold summation of the sequences $a_n = a_1 + (n-1)d$ and $b_n = n$ by $S_n^{(k)}$ and $T_n^{(k)}$ respectively, then it is easily proven by an inductive argument that for $k \geq 1$

$$S_n^{(k)} = dT_n^{(k)} + (a_1 - d)T_n^{(k-1)} \ . \tag{1}$$

To construct a closed form expression for $T_n^{(k)}$, let us first examine closely the diagonal rows of Pascal's Triangle pictured below.

Each diagonal labelled $D = k$ for $k = 0, 1, \ldots$ contains a sequence of positive integers whose first term is the number one. Indexing the terms of these sequences with say the variable i, where $i = 1, 2, \ldots$, the Hockey Stick Theorem states that the n-th partial sum of the sequence in any diagonal $D = k$, is equal to the n-th term of the sequence found in the neighbouring diagonal $D = k+1$. For example, the sum of the first four terms in diagonal $D = 3$ is $1 + 6 + 21 + 56 = 84$, but the number 84 is the fourth term of the sequence found in diagonal $D = 4$.

Now as diagonal $D = 0$ contains the sequence $T_n^{(0)} = \frac{n(n+1)}{2}$, the Hockey Stick Theorem implies the the n-th term of the sequence found in diagonal $D = 1$ must be $T_n^{(1)}$, while the n-th term of the sequence in diagonal $D = 2$ must be $T_n^{(2)}$ and so on infinitum. Thus by applying the Hockey Stick Theorem k times, beginning at diagonal $D = 0$, we conclude

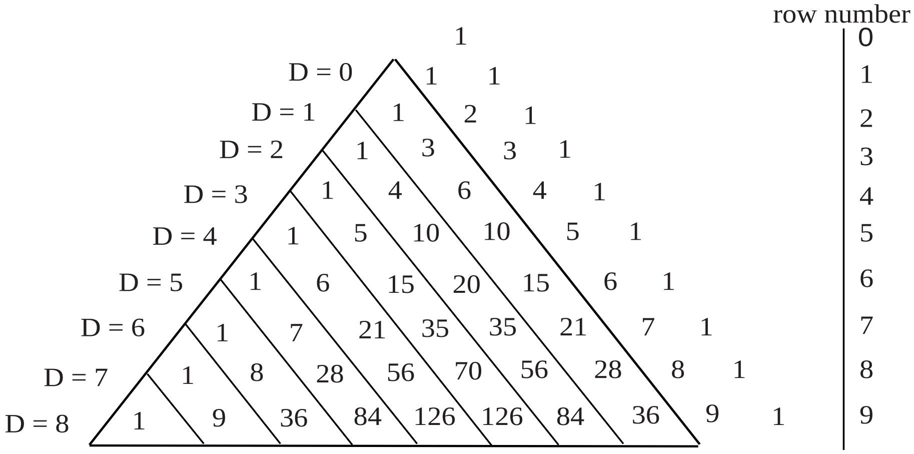

Figure 1: Pascal's Triangle.

that the integer $T_n^{(k)}$ must appear as the n-th term of the sequence found in diagonal $D = k$. Consequently as each entry in the Pascal Triangle is a binomial coefficient, we conclude from the above observation that

$$T_n^{(k)} = \binom{r}{p} = \frac{r(r-1)\cdots(r-p+1)}{p!} \ , \tag{2}$$

where r and p are the row and position number along a row, of the n-th term of the sequence in diagonal $D = k$, noting the convention that $\binom{r}{0} = 1$. Hence, to find a closed form expression for $T_n^{(k)}$, one must be able to determine r and p as functions of the parameters n and k. To this end, note that the first term of the sequence in any diagonal $D = k$, occurs as the leftmost element in row $k+2$ and that the row number of each subsequent term of the sequence in diagonal $D = k$ must increase by one. Thus $r = n + k + 1$ is the row number of the n-th term of the sequence in diagonal $D = k$. Secondly, note that the n-th term of the sequence in diagonal $D = k$, is located as the n-th entry from the left in row $r = n + k + 1$. However, as there are $r + 1$ elements in row r whose position number p, counted from the left, assumes the values $p = 0, 1, \ldots, r$, we deduce that the n-th term of the sequence in diagonal $D = k$ has a position number $p = n - 1$ along the row $r = n + k + 1$. Hence from (2)

$$T_n^{(k)} = \binom{n+k+1}{n-1} = \binom{n+k+1}{k+2} \ , \tag{3}$$

and so finally substituting the second binomial coefficient of (3) into (1) yields that the k-fold summation of an arithmetic progression $a_n = a_1 + (n-1)d$ is given by

$$S_n^{(k)} = d\binom{n+k+1}{k+2} + (a_1 - d)\binom{n+k}{k+1} \ . \tag{4}$$

In view of (3) it is of interest to note that there is a combinatorial interpretation of the k-fold summation of the natural numbers. Recall that two distinct decompositions of a positive integer n into a sum of k non-negative integers is a representation of the form

$$n = a_1 + \cdots + a_k = b_1 + \cdots + b_k \ ,$$

in which $a_i \neq b_i$ for at least one i, $1 \leq i \leq k$. Now the number of distinct decompositions of n into a sum of k non-negative integers is $\binom{n+k-1}{k-1}$ (see [2, pp. 87]). Thus by writing $T_n^{(k)} = \binom{n-1+(k+2)}{k+2}$ we see that the k-fold summation of the natural numbers represents the number of distinct decompositions of $n-1$ into $k+1$ non-negative integers.

3 A Generating Function Approach

In establishing (3) we indirectly applied the Binomial Theorem with regards to the polynomial expansion of $(1+x)^n$ for positive integer n. In contrast, the second method to be employed, will involve the use of generating functions together with an application of the binomial series of the function $(1+x)^\alpha$, where α is an arbitrary real number. We begin by reviewing the concept of a generating function. If one has a sequence $\{a_n\}$ of real or complex numbers, then the function $f(x)$ defined by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ ,$$

is referred to as the generating function of the sequence $\{a_n\}$. Many of the operations one performs on generating functions can be justified rigorously in terms of operations on formal power series, even when the series in question may not be convergent (see [3] for a comprehensive treatment of the theory of generating functions). One of these familiar operations that we shall exploit here, is that of the multiplication of power series. Specifically if given two generating functions of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then one can define their product as

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} a_r b_{n-r} \right) x^n \ .$$

In particular, when $f(x) = 1/(1-x) = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, then the above product $f(x)g(x)$ results in the generating function for the sequence of partial sums of $\{b_n\}$ as follows

$$\frac{1}{1-x} g(x) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} b_{n-r} \right) x^n. \qquad (5)$$

We shall use (5) to first identify, for each $k \geq 0$, the generating function, denoted $f_k(x)$, for the sequence $\{T_n^{(k)}\}$. Indeed, we claim that $f_k(x) = x/(1-x)^{k+3}$ which can be established via the following inductive argument. Setting $g(x) = x/(1-x)^2$ observe after differentiating the series expansion of $1/(1-x)$ that $g(x) = \sum_{n=0}^{\infty} nx^n$. Substituting $g(x) = x/(1-x)^2$ into (5) and defining $T_0^{(k)} = 0$, for all $k \geq 0$, one deduces from definition of $T_n^{(0)}$ that $f_0(x) = x/(1-x)^3$. Assume $f_m(x) = \sum_{n=0}^{\infty} T_n^{(m)} x^n = x/(1-x)^{m+3}$, for some $m \geq 0$, and upon setting $g(x) = f_m(x)$ in (5), observe that

$$\frac{x}{(1-x)^{m+4}} = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} T_{n-r}^{(m)} \right) x^n = \sum_{n=0}^{\infty} T_n^{(m+1)} x^n \ .$$

Consequently $f_{m+1}(x) = x/(1-x)^{m+4}$ and so the result holds for $k = m+1$. Having identified for each $k \geq 0$, the generating function for the sequence $\{T_n^{(k)}\}$, we can determine a closed-form expression for $T_n^{(k)}$ by examining the series expansion of the function $x/(1-x)^{k+3}$ via the binomial series of the function $(1+x)^\alpha$. Recall that if α is an arbitrary real number and $|x| < 1$ then $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$. Thus

$$f_k(x) = x \sum_{n=0}^{\infty} \binom{-k-3}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-k-3}{n} x^{n+1} \ ,$$

and so after equating coefficients of x^n one deduces, for $n \geq 1$, that

$$T_n^{(k)} = (-1)^{n-1} \binom{-k-3}{n-1} \ . \tag{6}$$

However by definition

$$\begin{aligned} \binom{-k-3}{n-1} &= \frac{1}{(n-1)!}(-k-3)(-k-4)\cdots(-k-n-1) \\ &= \frac{(-1)^{n-1}}{(n-1)!}(k+3)(k+4)\cdots(k+n+1) \\ &= (-1)^{n-1}\binom{n+k+1}{k+2} \ , \end{aligned}$$

from which we see (6) reduces down to the required expression in (3), hence (4) follows immediately again from (1).

Having now seen both methods at work, it would appear that the later method could be applied to the problem of constructing closed-form expressions for the n-th partial sum or more generally the k-fold sum of other classes of sequences. Clearly the difficulty in using generating functions is finding, for a given sequence $\{a_n\}$, the correct functional representation for the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

References

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[3] I. Niven, Formal Power Series, *Amer. Math. Monthly* 76 (1969) 871-889.

[4] E. W. Weisstein, Christmas Stocking Theorem, From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/ChristmasStockingTheorem.html

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