Universidade de Aveiro Departamento de Matemática
2011

VÍTOR LUÍS PEREIRA MORAIS DE SOUSA

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THE RIEMANN-HILBERT METHOD APPLIED TO THE THEORY OF ORTHOGONAL POLYNOMIALS

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dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica da Prof. Dra. Ana Foulquié Moreno, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro, e, do Prof. Dr. Andrei Martínez-Finkelshtein, Professor Catedrático do Departamento de Estatística e Matemática Aplicada da Universidade de Almería (Espanha).

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Dedicações:

Aos meus país pelo empenho em tornar a minha vida mais fácil e serem os melhores exemplos.

À **Mafalda**. Esta tese reflecte o teu esforço, a tua força de vontade e o teu empenho em todos os projectos que abraças. Esta obra também é tua …

À nossa *florzinha* …
.

o júri

presidente

Prof. Dr. António Manuel de Sousa Pereira
Professora Catedrático da Universidade de Aveiro

Prof. Dr. Andrei Martinez-Finkelshtein
Professor Catedrático da Universidade de Almeria - Espanha

Prof. Dr. Arno Kuijlaars
Professor Catedrático da Katholieke Universiteit Leuven - Bélgica

Prof. Dr. Helmuth Robert Malonek
Professor Catedrático da Universidade de Aveiro

Prof. Dr. Semyon Yakubovich
Professor Associado com Agregação da Faculdade de Ciências da Universidade do Porto

Prof. Dra. Ana Foulquié Moreno
Professora Auxiliar da Universidade de Aveiro

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palavras-chave

Polinómios Ortogonais; Assimptótica forte; Método de Riemann-Hilbert; Método do gradiente não-linear; Medida de Jacobi Generalizada; Coeficientes de Recorrência; Coeficientes Principais; Comportamento Local; Zeros; “Clock Behavior”; Funções Hiper-geometricas Confluentes; Núcleos Reprodutores; Núcleo Confluente Hiper-geométrico; Universalidade; Espaços de De Branges.

resumo

Neste trabalho abordamos diferentes áreas do conhecimento Matemático, genericamente, da teoria da Análise Complexa, mais especificamente, da teoria dos Polinómios Ortogonais, da teoria dos Problemas de Valores de Fronteira, da teoria das Funções Especiais, e, apresentamos resultados essencialmente na área dos Polinómios Ortogonais, mas também na área de Problemas de Valores de Fronteira, da teoria das Funções Especiais e até resultados que se enquadram no âmbito da teoria da Análise Funcional.

Apresenta-se resultados analíticos para polinómios ortogonais em contextos que não tinham sido estudados até ao momento. Concretamente, aborda-se o estudo do comportamento assimptótico forte para polinómios ortogonais relativamente à função peso de ortogonalidade com uma singularidade do seguinte tipo:

$$|x-x_0|^\gamma \times \begin{cases} 1, & se\ x\in[-1,\ x_0) \\ c^2, & se\ x\in[x_0,1] \end{cases}, \quad (x_0\in(-1,1),\ \gamma>-1,\ c>0).$$

Apresenta-se o comportamento assimptótico, uniforme, forte dos polinómios ortogonais mónicos em todo plano (fora do intervalo de ortogonalidade, dentro do intervalo de ortogonalidade, e, numa vizinhança da singularidade), bem como os primeiros termos da expansão assimptótica dos seus parâmetros principais: para os coeficientes principais dos polinómios ortonormais e para os coeficientes de recorrência (da fórmula de recorrência a três termos), quando $n\to\infty$. Relativamente aos coeficientes de recorrência prova-se, como caso particular, uma conjectura dada por A. Magnus para este tipo de pesos.

Obtém-se um novo núcleo reprodutor, em termos de funções Hiper-geométricas Confluentes, no caso $\gamma\neq 0$ com $c\neq 1$, que constitui uma generalização do, já conhecido como, segundo núcleo reprodutor de Bessel. Este constitui o primeiro exemplo de núcleo reprodutor correspondente a espaços de De Branges diferente do de Paley-Wiener, assim como, tomando a descontinuidade do tipo degrau ($\gamma=0$, $c\neq 1$), um exemplo explicito da violação da lei de universalidade inclusive quando o peso é distinto de zero.

A técnica que possibilitou a demonstração destes resultados baseia-se na representação dos polinómios ortogonais em termos de um problema de Riemann-Hilbert, e, na aplicação do método do gradiente não-linear, introduzido por Deift e seus colaboradores. Embora o esquema geral do método aqui se mantenha invariante, o mesmo teve que se desenvolver e adaptar, introduzindo as modificações necessárias para a análise local numa vizinhança da singularidade do peso, x_0.

keywords

Orthogonal Polynomials; Strong Assimptotics; Riemann-Hilbert Method; Steepest Descent; Generalized Jacobi Weight; Recurrence Coefficients; Leading Coefficients; Local Behavior; Zeros; Clock Behavior; Confluent Hypergeometric Functions; Reproducing Kernels; Confluent Hypergeometric Kernel; Universality; de Branges Spaces.

abstract

This work deals with different areas of mathematical knowledge in the context of Complex Analysis, particularly the theory of Orthogonal Polynomials, Theory of Boundary Value Problems, of the Theory of Special Functions, and presents results mainly in the area of Orthogonal Polynomials, but also in the area of Boundary Value Problems of the Theory of Special Functions, and even results that fall within the theories of Probability and Statistics and Functional Analysis.

This work presents analytic results for orthogonal polynomials in a context that has not been studied until now. Specifically, it approaches the study of strong uniform asymptotic behavior for orthogonal polynomials with respect to the weight function with a singularity of the following type:

$$|x-x_0|^{\gamma}\times\begin{cases}1, & se\ x\in[-1,\ x_0)\\ c^2, & se\ x\in[x_0,1]\end{cases},\quad (x_0\in(-1,1),\ \gamma>-1,\ c>0).$$

It presents the strong uniform asymptotic behavior for monic orthogonal polynomials in all plane (outside and inside of the orthogonality interval, and, near the singularity point), as well as the first two terms of the asymptotic expansion of the main parameters: to the leading coefficients of the orthonormal polynomials and to the recurrence coefficients (from the three-term recurrence relation), as $n\to\infty$. In what concerns the recurrence coefficients it proves, in particular, the Magnus´s conjecture for this kind of weights.

It obtains a new reproducing kernel, in terms of Confluent Hypergeometric functions, in the case $\gamma\neq 0$ with $c\neq 1$, which is a generalization of the called second Bessel kernel. It provides the first explicit example of a reproducing kernel that belongs in the De Branges space different of those of the classic Paley-Wiener, as well as, taking the step-like discontinuity ($\gamma=0$, $c\neq 1$), an explicit example of violation of the universality even when weight is nonzero. The technique that allows us the proof of the results obtained is based on the characterizing the orthogonal polynomials in terms of a Riemann-Hilbert problem, and, to use the nonlinear steepest descent method, introduced by Deift and his collaborators. Although the general outline of the method here presented has remained invariant, it had to be developed and adapted introducing amendments to the local analysis near the singularity of the weight, x_0.

Contents

Preface

This thesis is written in English, because almost all research in Mathematics and other sciences is published in this language. Although, in this introduction we will write the first three sections: “Agradecimentos”, “Motivações e perspectivas” and “Enquadramento do trabalho e indicação dos resultados” in Portuguese. The first two are a more personal sections and the third one is a outline of the work.

The English readers can skip this first three sections and use the first section of the Chapter 1 as a guidelines and outline of the work, which is more detailed and where we fix the notation and the concepts used. An introduction of the Riemann-Hilbert method is given in Chapter 3.

The fourth subsection is a detailed outline of the thesis. We speak about the way this thesis was written and we do the indication to the principal theorems and results, and where they were published.

In the end of the thesis there is a Index of symbols and notations used throughout the dissertation.

0.1 Agradecimentos

Para a realização deste trabalho foram muito importantes todas as estâncias que fiz em Almería junto do Professor Andrei e do seu grupo. Fica aqui uma palavra de agradecimento por toda a dedicação, empenho, estimulo, visão, personalidade e amizade do Professor Andrei, onde as suas qualidades vão muito além da dimensão científica. As qualidades científicas do Professor Andrei são muitas e bem reconhecidas na comunidade científica, mas, mesmo sendo excelente no plano científico, eu teria muitas dúvidas em identificar qual a sua principal dimensão, se como cientista ou como pessoa. Ao Professor Andrei o meu muito obrigado, por tudo.

Gostava de reforçar o papel que a Professora Ana teve em todo este projecto, foi graças a ela que entrei no mundo da ciência, e, a sua visão foi muito importante, em prever a importância que este assunto despertava na comunidade científica. Gostava de salientar também o papel de orientação da Professora Ana que sempre respeitou o meu ritmo de trabalho, ajudando quando eu mais precisava e libertando-me quando eu estava mais autónomo. Também, pela paciência e compreensão necessária em certas fases deste trajecto, nomeadamente, quando abandonei este projecto, entre 2006 e 2008, devido alterações profissionais, e, consequentemente, pessoais. Nessa fase apoiou-me e respeitou as minhas decisões dando sempre uma palavra de esperança; e, quando surgiu a hipótese de recomeçar o projecto, pela força e vontade com que retomou o trabalho. À Professora Ana o meu muito obrigado, por todos estes anos.

0.2 Motivações e perspectivas

Esta tese é o culminar de um projecto que se iniciou em 2002 durante o curso de Mestrado em Matemática - ramo Análise e Geometria, proposto pela Professora Ana Foulquié. A proposta inicial de estudar a Teoria dos Polinómios Ortogonais rapidamente se transformou no tema que deu origem a esta tese. Começou por ser o objecto de estudo para a tese de mestrado que foi concluída em Junho de 2004, onde o arguente principal da defesa foi o Professor Andrei Martínez. Desde então, e tendo em conta o interesse que o assunto despertava na comunidade científica, ficou projectado um plano para um futuro trabalho na mesma área da tese de Mestrado: "O Problema de Riemann-Hilbert para Polinómios Ortogonais". A tese de mestrado foi um ponto de partida essencial em todo este trajecto, pois aí estudámos quase todos os pré-requisitos necessários para lidar com o método de Riemann-Hilbert, desde conceitos clássicos de Análise Complexa que não foram abordados na licenciatura nem na parte curricular do mestrado, a teoria dos Valores de Fronteira, até à teoria dos Polinómios Ortogonais. Os assuntos abordados no segundo e terceiro capítulo e na primeira secção do primeiro capítulo foram estudádos durante a preparação da tese de mestrado. Graças a este estudo inicial, quando iniciámos os trabalhos de doutoramento começámos quase logo a investigar problemas concretos (depois de um curto tempo a reestudá-los).

Pelo meio passaram vários cursos intensivos e congressos que permitiram o contacto com investigadores seniores e juniores que em muito contribuíram e motivaram a persecução deste trabalho. O curso de verão em OPSF de Coimbra em 2003, o 10^o curso intensivo de Coimbra/Aveiro em 2004, o curso de verão em OPSF de Madrid em 2004, o curso em problemas de RH que o Professor Andrei deu em Aveiro em 2006, o 12^o curso intensivo de Coimbra/Aveiro em 2006, e mais recentemente, o IWOPAT - workshop de Madrid em 2008, o 14^o curso intensivo de Aveiro/Coimbra em 2009, o 11^o EWAGCA - workshop de Coimbra/Aveiro em 2009, o 10^o ISOPSFA - simpósio em Leuven em 2009, e o 1^o JCA em Jaén em 2010, só para mencionar os mais longos e importantes.

Problemas pessoais e burocráticos, relativos ao meu emprego, levaram a que este projecto de doutoramento só se iniciasse em Fevereiro de 2008. Devido à insegurança de manutenção do meu emprego, pelo meio passou uma mudança, em 2006, de Assistente de 2^o triénio na Escola Superior de Tecnologia e Gestão de Bragança para professor do Quadro de Nomeação Definitiva da Escola Básica Dr. Manuel Magro Machado, e, recentemente, em 2009, professor do Agrupamento de Escolas de Escariz, sendo actualmente professor destacado na Escola Secundária João da Silva Correia. Durante estes anos passei de um ambiente de trabalho onde era valorizado, apenas, o trabalho científico para um ambiente de trabalho em que é sobre valorizada a disponibilidade e a relação pedagógica. Na prática o que se verifica é um crescendo de trabalho burocrático em ambos os níveis de ensino. No entanto, como professor do ensino básico e secundário, não antevejo tarefa fácil a compatibilidade entre o trabalho de professor (que poderá ir dos níveis de 7^o ano ao 12^o ano) com a tarefa de investigação em Matemática. No ensino básico e secundário, além das muitas horas dedicadas à docência, 22 horas, acresce ainda uma série de trabalho burocrático que facilmente se transformam em 28 horas de trabalho, sem contar com o tempo necessário para preparar aulas, testes e outras exigências. Além disso, há ainda uma componente de avaliação, do pessoal docente do ensino básico e secundário, que previlegia a disponibilidade para "fazer tudo e mais alguma coisa", mesmo que isso não tenha influência na formação dos alunos. Outro problema é que, nas obrigações do pessoal docente, há uma componente de formação obrigatória creditada. A formação obrigatória disponibilizada é única e exclusivamente centrada nas áreas da psicologia, da educação, e, recentemente, na área da informática designada de novas tecnologias. O docente, mesmo tendo adquirido o grau de Doutor, não tem autonomia de investigação, como a própria legislação lhe atribui, e tem de cumprir todos estes requisitos sob pena de ser avaliado com uma má classificação.

A função de professor deve contemplar sempre duas vertentes, uma vertente educacional e uma vertente científica, e, ao longo dos anos de escolaridade, deveria passar desde mais educacional para mais científica. No entanto, o que se passa no ensino básico e secundário é que parece só existir a vertente educacional, e, o enveredar por uma vertente mais científica não está contemplado na legislação. Ora uma constante actualização/investigação científica parece-me tanto ou mais importante do que uma constante formação em didáctica/pedagógica ou na utilização das novas tecnologias. E, por outro lado, a existência de professores com motivações e aptidões diversas seria o melhor numa comunidade escolar/educativa. Para se sobreviver no mundo da investigação é preciso que haja tempo para investigar e para participar em congressos, pelo menos nas alturas em que não há aulas!

São vários receios sobre a compatibilidade entre a investigação e a docência no ensino básico e secundário, mas, com esperança que a situação se vá modificando e essa compatibilização seja possivel.

0.3 Enquadramento do trabalho e indicação dos resultados

Desde o final da década de 90, Deift e outros colaboradores ([13], [15], [16]), começaram a aplicar o método designado por Método de Riemann-Hilbert a polinómios ortogonais para obter diversas propriedades dos respectivos polinómios. O método de Riemann-Hilbert é a combinação de duas técnicas. A primeira caracteriza os polinómios ortogonais em termos da solução de um problema de Riemann-Hilbert matricial, que foi dada por Fokas, Its e Kitaev [21] em 92. A segunda, designada por método do gradiente não linear foi utilizada por Deift e Zhou [15] em 93 e posteriormente melhorada ([18], [5], [19]).

Uma das principais vantagens deste método é que permite obter resultados sobre o comportamento assimptótico forte dos polinómios ortogonais, quando $n \to \infty$, em todo o plano complexo, incluindo sobre o suporte da medida (intervalo de ortogonalidade) e numa vizinhança das singularidades da medida, e, calcular explicitamente os vários termos da respectiva expansão assimptótica. O método começou por ser aplicado a polinómios ortogonais que são caracterizados por medidas analíticas e suporte em toda a recta real. Mais tarde, Kuijlaars e outros colaboradores [31], aplicaram este método a polinómios ortogonais caracterizados por medidas analíticas com suporte compacto, nomeadamente com suporte em $[-1, 1]$. Até há pouco tempo, esta técnica só havia sido aplicada a medidas analíticas com, quando muito, singularidades algébricas (nos extremos e no interior do intervalo de ortogonalidade), e, apenas recentemente, [26] em 2008, Its e Krasovski, consideraram o caso de uma medida com uma singularidade tipo degrau, com suporte em $\mathbb{R}$.

Uma etapa fulcral do método do gradiente não linear para a obtenção dos resultados assimptóticos fortes é a resolução de um problema de Riemann-Hilbert centrado nas singularidades da medida. A solução deste problema é construída usando funções especiais que determinam o comportamento assimptótico dos correspondentes polinómios ortogonais numa vizinhança das respectivas singularidades da medida de ortogonalidade.

Uma das contribuições que damos na evolução desta teoria (capítulo 7), é a aplicação do método a polinómios ortogonais caracterizados por uma medida, em $[-1, 1]$, com uma descontinuidade não algébrica tipo degrau ($c \neq 1$, $\gamma = 0$), e, também, por uma medida com duas singularidades no mesmo ponto, uma algébrica e outra tipo degrau:

$$w_{c,\gamma}(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}|x_0 - x|^{\gamma} \begin{cases} 1, & x \in [-1, x_0) \\ c^2, & x \in [x_0, 1]. \end{cases}, \qquad x \in [-1,1],$$

onde $x_0 \in (-1,1)$, $\alpha, \beta, \gamma > -1$, h é real analítica e estritamente positiva sobre $[-1,1]$.

Enquanto que para outras medidas e para singularidades algébricas tipicamente as funções especiais usadas, na solução do problema local, são as funções de Airy (no caso de medidas com suporte ilimitado sobre $\mathbb{R}$), e as funções de Bessel (no caso de singularidades algébricas), no caso de singularidades não algébricas tipo degrau as funções especiais que usamos são as funções Hiper-geométricas Confluentes. Na aplicação do método à medida com duas singularidades em x_0, uma algébrica e outra tipo degrau, obtemos um problema local diferente dos que surgiram até ao momento, e, demos a solução desse problema em termos de funções Hiper-geométricas Confluentes de ordens que dependem de c e γ (ver capítulo 6).

Como resultados, obtemos os comportamentos assimptóticos fortes em todo plano: fora do intervalo de ortogonalidade - Teorema 8.1.1, dentro do intervalo de ortogonalidade - Teorema 8.4.1, e, numa vizinhança da singularidade - Teorema 9.1.1 (capítulos 8 e 9), obtendo os dois primeiros termos da respectiva expansão assimptótica, para os polinómios ortogonais mónicos, para os coeficientes principais dos polinómios ortonormais - Teorema 8.2.1, e para os coeficientes de recorrência da fórmula de recorrência a três termos - Teorema 8.3.1, quando $n \to \infty$. Relativamente a estes provamos, como, caso particular, uma conjectura dada por A. Magnus para este tipo de medidas (ver secção 8.3).

Outra contribuição que damos para a teoria dos polinómios ortogonais é, considerando o caso $\gamma = 0$ (medida descontínua) analisar o comportamento dos zeros dos polinómios ortogonais mónicos numa vizinhança da singularidade da medida. Concluímos que os zeros destes polinómios não estão igualmente espaçados, ou seja, que os zeros não têm um "comportamento relógio", enquanto que os polinómios numa vizinhança dos outros pontos não singulares da medida verificam o "comportamento relógio" (ver secção 10.2). Relacionado com o "comportamento relógio" está o problema de universalidade para o núcleo reprodutor, ou, núcleo de Christoffel-Darboux. Este é um problema que teve origem na teoria de matrizes aleatórias e é um dos tópicos que tem despertado maior interesse na comunidade científica actual, nomeadamente na teoria da Análise Funcional e da Estatística e Probabilidade. O caso mais simples da lei de universalidade é que o comportamento limite do núcleo reprodutor corresponde a um núcleo seno (fórmula (1.2.5)) (o qual é obtido para os pontos não singulares da medida), e, nos pontos com singularidades algébricas obtém-se o núcleo Bessel (fórmula(1.2.10)), no caso de polinómios sobre $[-1,1]$, ou, núcleo Airy no caso de polinómios sobre $\mathbb{R}$. Relativamente ao núcleo centrado no ponto de descontinuidade x_0, obtemos um novo comportamento limite para o núcleo reprodutor, em termos de funções Hiper-geométricas Confluentes (corolário 9.2.1). Relacionado com este núcleo, e seguindo recentes trabalhos de D. Lubinsky, provamos que este tipo de núcleo é, o primeiro exemplo explícito de, um núcleo reprodutor dos espaços de "De Branges" diferente do núcleo seno (ver secção 10.3). Para obter estes resultados provamos, também, algumas propriedades das funções Hiper-geométricas, resultados que não conseguimos encontrar na bibliografia sobre estas funções especiais (secção 10.1). Relativamente ao caso geral, com $\gamma \neq 0$ e $c \neq 1$, mostramos que o correspondente núcleo (fórmula 10.3.2) constitui uma generalização do, designado por, segundo núcleo de Bessel (fórmula (1.2.11)) – Teorema 9.2.2.

No último capítulo apresentamos alguns problemas relacionados que poderão ser trabalhados nos próximos tempos.

Com excepção do Teorema 8.3.1 (que publicamos em [23]) e da secção 10.2 (que publicamos em [22]), os resultados dos capítulos 8, 9 e 10 não estão publicados e generalizam resultados de [22], sendo que, todos os resultados apresentados a partir do capítulo 5 são novos.

0.4 Outline of the dissertation

This thesis is divided in three parts. The first part is the background and contains the presentation of notation, of related subjects and the known results. The second part contains the results that we obtained. The third part discusses some problems for future research.

With the exception of the results of Chapter 10 (published in [22]) and the Theorem 8.3.1 (published in [23]), results from Chapters 8 and 9 have not appeared in press, and generalize results in [22].

In the first part we present some background related to Orthogonal Polynomials and the Riemann-Hilbert method, and some known results that are used in the work. In this part the results are presented without proof or with no detailed proofs where, sometimes, only the idea of the proof is given. In fact, almost always, in this thesis we present the proofs and the results in plain text instead of the formal written of the theorem followed by the respective proof.

In the **first** chapter we give the guidelines and we present the problems that we deal with. We present, in the second section, the asymptotic theory and, in particular, results from the Szegő's theory.

In the **second** chapter we show some relevant aspects of the theory of boundary value problems for analytic functions: some classical and useful results from Complex Analysis, the scalar additive and multiplicative Riemann-Hilbert problem, the Sokhotskii-Plemelj formulas, and we deduce the formula for the Szegő function in terms of the Cauchy transform.

In the **third** chapter we speak about the Riemann-Hilbert method. We give the characterization of the orthogonal polynomials in terms of a Riemann-Hilbert problem and we present descriptively the steepest descent method.

In the **fourth** chapter we summarize some useful properties of the special functions that we need, Bessel and Confluent Hypergeometric functions.

The second part of the thesis contains the new results.

In the Chapter **five** we present the explicit expression for the Szegő function for the step-like function, and we compute and verify its boundary values and it relation with the weight function that we are considering.

The following two chapters contains the steps of the steepest descent method. We divide into two chapters because the Chapter six, "RH problem for Confluent Hypergeometric Function", constitutes one of the most important steps in the method and may appear in the analysis of other problems and other orthogonal polynomials (e.g. [13]).

In the **seventh** chapter we apply the steepest descent method to the RH problem established in Chapter three.

In Chapter **eight** we present and prove the results concerning the asymptotic behavior of the monic orthogonal polynomials away from the singular point (on $\mathbb{C}\setminus[-1,1]$ - Theorem 8.1.1, and on $(-1,x_0)\cup(x_0,1)$ - Theorem 8.4.1), for the leading coefficient of the orthonormal polynomial on $\mathbb{C}\setminus[-1,1]$ (Theorem 8.2.1), and, for the recurrence coefficients on $\mathbb{C}\setminus[-1,1]$ (Theorem 8.3.1). All these results are new.

In Chapter **nine** we do the analysis of the local behavior at the singular point (that combines a root-type and a step-type singularity) for the monic orthogonal polynomials (on a neighborhood of x_0 - Theorem 9.1.1), we compute the limit of reproducing kernel at x_0 - Theorem 9.2.1, and we show that this kernel generalizes other: the second Bessel kernel obtained when we have just a root-type singularity at x_0, and the Confluent Hypergeometric kernel obtained when we have just a step-type singularity at x_0. All these results are new also.

In the Chapter **ten** we show some applications of the results of Chapter nine. In the first section of this chapter we prove some properties of the Confluent Hypergeometric functions that we needed to prove the results in this chapter - Proposition 10.1. In the second section we analyze the distribution of the zeros at the jump point x_0 - section 10.2 for the particular case of the strictly positive weight on $(-1, 1)$ only with a step-like singularity at x_0. In the third section we show that the kernel obtained in Chapter nine belongs to a De Branges space - section 10.3. All these results are new also.

In the third part and **eleventh** chapter, we present some related problems and problems for future research.

In the end of the thesis there is a **Index** of the symbols and notations used throughout the dissertation.

Part I

Background

Chapter 1

Elements of the general theory of Orthogonal Polynomials

In this chapter we introduce some elements of the algebraic and analytic theory of orthogonal polynomials (the main object of this dissertation), where we define the orthogonal polynomials, the subjects that we will aboard in the thesis, the main goals and an outline of the thesis. In the second section we present some asymptotic results from the classic Szegő theory to general weights.

1.1 Asymptotics of Orthogonal Polynomials

Let ω be a finite positive Borel measure on $\mathbb{R}$ with all the moments $\int x^j d\omega(x)$, $j \geq 0$, finite, and with infinitely many points in its support. We call, **weight function** w to the almost everywhere existing Radon-Nikodym derivative of ω with respect to the Lebesgue measure,

$$w(x) = \frac{d\omega}{dx}.$$

We denote by $P_n(x) = P_n(x; w)$ the monic polynomial of degree n orthogonal with respect to the weight w on $[-1, 1]$, such that

$$\int_{-1}^{1} P_n(x) x^k w(x)\ dx = 0, \qquad \text{for } k = 0, 1, \cdots, n-1,$$

and use $p_n(x) = p_n(x; w)$ to denote the corresponding orthonormal polynomial,

$$p_n(x) = k_n P_n(x),$$

where $k_n > 0$ is the leading coefficient of p_n.

It is well known (see e.g. [57]) that $\{P_n\}_{n=0}^{\infty}$ satisfy the three-term recurrence relation

$$P_{n+1}(z) = (z - b_n) P_n(z) - a_n^2 P_{n-1}(z), \tag{1.1.1}$$

or its equivalent form, for $\{p_n\}_{n=0}^{\infty}$,

$$x p_n(z) = a_n p_{n+1}(z) + b_n p_n(z) + a_{n-1} p_{n-1}(z). \tag{1.1.2}$$

In this thesis, we do not make the difference between p_n as the polynomial of degree n and as the sequence of polynomials $\{p_n\}_{n=0}^{\infty}$, in the context it is well understood, and it simplifies the notation.

The asymptotic expansion for the polynomials p_n and $P_n(z)$ and for their parameters k_n, a_n, and b_n (as $n \to \infty$) is known for a general class of measures $d\omega$. The most important theory to obtain it was given by Szegő. He is the founder of the modern asymptotic theory of orthogonal polynomials on the unit interval for weights w that satisfy the condition ("Szegő condition")

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx > -\infty. \tag{1.1.3}$$

Are called as classical weights, the weights w that satisfy the first order differential equation, the Pearson equation, $(\phi w)' = \psi w$ for some polynomials on $\mathbb{R}$ ϕ and ψ. For these weights the asymptotic results both on and away from the interval of orthogonality, as well as at its endpoints, can be derived using multiple identities that these orthogonal polynomials satisfy: the differential equation, the Rodrigues formula, integral representation, etcetera. However, in a general situation the problem is much more difficult. Starting from the 80's, many new asymptotic results were found for various classes of weights, and the breakthrough was partially motivated by the development of the tools from potential theory and operator theory.

An important new technique for obtaining asymptotics for orthogonal polynomials in all the regions of the complex plane is the Riemann-Hilbert (RH) method that we will describe in the Chapter 3 and we will apply in the second part of this work to obtain asymptotic results for monic orthogonal polynomials, for the leading coefficient and for the recurrence coefficients in Chapter 8.

1.1.1 On the Szegő theory for Orthogonal Polynomials

Szegő is the founder of the modern asymptotic theory of orthogonal polynomials on the real line for weights that satisfy the Szegő condition (1.1.3). His theory rests on the fact that the simply connected set $\overline{\mathbb{C}} \setminus [-1,1]$ ($\overline{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$) can be applied by a conformal mapping onto the exterior of the unit circle. Thus, his theory is developed for the polynomials on the unit circle, where can be used all theory of the Harmonic functions within the circle, and using the relationship between orthogonal polynomials on the circle and orthogonal polynomials on $[-1,1]$ it is possible to transfer all the results of the circle to the interval $[-1,1]$.

His results are formulated in terms of two functions that will play a relevant role in what follows. Namely,

$$\varphi(z) = z + \sqrt{z^2-1} \tag{1.1.4}$$

with the branch of $\sqrt{z^2-1}$ that is analytic in $\mathbb{C} \setminus [-1,1]$ and behaves like z as $z \to \infty$; it has the following important properties:

1. φ is the conformal map from $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle;

2. $\varphi_+(x)\,\varphi_-(x) = 1$ as $x \in [-1,1]$; (where $\varphi_\pm(x)$ denote the limiting values of $\varphi(z)$ as z approaches x from above $(+)$ and below $(-)$, are defined in (2.1.1))

3. $\varphi(z) = 2z + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Furthermore, let a weight w on $[-1,1]$ that satisfies (1.1.3), we can define the so-called Szegő function $D(z) = D(z;w)$ associated with w,

$$D(z,w) = D(z) = \exp\left(\frac{\sqrt{z^2-1}}{2\pi}\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}\frac{dx}{z-x}\right), \qquad \text{for } z \in \mathbb{C}\setminus[-1,1] \tag{1.1.5}$$

again with $\sqrt{z^2-1} > 0$ for $z > 1$ and $\sqrt{1-x^2} > 0$ on $(-1,1)$. The function $D(z)$ is a non-zero analytic function on $\mathbb{C}\setminus[-1,1]$ such that

$$D_+(x)D_-(x) = w(x), \qquad \text{for a.e. } x \in (-1,1),$$

where $D_+(x)$ and $D_-(x)$ denote the limiting values of $D(z)$ as z approaches x from above and below, respectively. In particular, by (1.1.3), the limit

$$D_\infty = \lim_{z\to\infty} D(z) = \exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\log w_c(x)}{\sqrt{1-x^2}}dx\right)$$

exists and is a positive real number. See the subsection 2.2, where we deduce an equivalent formula to (1.1.5) using its boundary values properties. Consider the generalized Jacobi weight given by $w(x) = (1-x)^\alpha (1+x)^\beta |x-x_0|^\gamma$, on $[-1,1]$ with $\alpha,\beta,\gamma > -1$ and $x_0 \in (-1,1)$, its explicit Szegő function is given by

$$D(z,w) = \frac{(z-1)^{\alpha/2}(z+1)^{\beta/2}(z-x_0)^{\gamma/2}}{\varphi^{(\alpha+\beta+\gamma)/2}(z)}. \tag{1.1.6}$$

For the classical polynomials, Jacobi (with the weight $w(x) = (1-x)^\alpha (1+x)^\beta$ on $[-1,1]$), Laguerre ($w(x) = x^\alpha e^{-x}$ on $[0,\infty)$) and Hermite ($w(x) = e^{-x^2}$) Szegő's theory give all asymptotic properties with all explicit terms of its expansions (see [57, Chapter 8]). However, for more general weights, even those that satisfy (1.1.3):

Theorem 1.1.1 (Szegő) *Uniformly on compact subsets of $\overline{\mathbb{C}}\setminus[-1,1]$,*

$$\frac{p_n(z)}{\varphi(z)^n} = \frac{\varphi(z)^{1/2}}{\sqrt{2\pi}\,(z^2-1)^{1/4}\,D(z;w_{c,\gamma})}\left[1+o(1)\right], \quad \text{as } n\to\infty, \tag{1.1.7}$$

$$\frac{k_n}{2^n} = \frac{1}{\sqrt{\pi}D_\infty}\left[1+o(1)\right], \quad \text{as } n\to\infty, \tag{1.1.8}$$

$$\frac{2^n P_n(z)}{\varphi(z)^n} = \frac{D_\infty}{D(z;w_{c,\gamma})}\frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2-1)^{1/4}}\left[1+o(1)\right], \quad \text{as } n\to\infty. \tag{1.1.9}$$

From Szegő (see [57, pp.309-310]), we also have, for the recurrence coefficients (1.1.1) or (1.1.2)

$$\lim_{n\to+\infty} a_n = \frac{1}{2} \tag{1.1.10}$$

$$\lim_{n\to+\infty} b_n = 0 \tag{1.1.11}$$

The results obtained are valid for a general kind of weights which satisfy the Szegő condition (1.1.3), although, give us only the first term of the asymptotic expansion and asymptotics on $\overline{\mathbb{C}}\setminus[-1,1]$. The weight that we will consider $w_{c,\gamma}$ defined in (1.4.1) belongs to the Szegő's class, so these results apply.

1.1.2 Asymptotic distribution of zeros

Another important subject in the theory of Orthogonal Polynomials is the asymptotic distribution of zeros. This subject attracts not only among people of OP but within the General Theoretical and Mathematical Physics communities that study the "eigenvalue statistics", because the connections that we will describe in the next section.

Polynomial P_n has n simple zeros, all lying on $\mathbb{R}$, more precisely on the convex hull of *supp* $(d\omega)$. It is well known that under mild assumptions they distribute asymptotically in the *weak-* sense* according to the *equilibrium measure* (see [55] for the definitions), ρ, of the interval. In particular, if *supp*$(w) = [-1, 1]$ and $w > 0$ a.e. on $[-1, 1]$, then the normalized "zero counting measure"(the positive measure that has at every zero of P_n a mass equal to the multiplicity of the zero) for the sequence P_n weakly tends to the absolutely continuous measure for *supp* $(d\omega)$ given by $\rho(x)\,dx$, with, for measures on $[-1, 1]$ is,

$$\rho(x) \stackrel{\text{def}}{=} \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \tag{1.1.12}$$

see [55] for further details. As it follows from several works of Deift and collaborators (and also from a recent series of papers of Lubinsky and Levin and Lubinsky, see e.g. [41], [36]), a much stronger statement holds: at any point of $t \in (a, b)$ where $w\,(x)$ is continuous and strictly positive they distribute very precisely in accordance with $\rho(x)$, complying with the so-called clock behavior, see e.g. [51]. If, following [51], we enumerate the zeros of P_n near the fixed point t inside *supp*, which we label $x_j^{(n)}\,(t) = x_j^{(n)}$, as follows,

$$\cdots < x_{-k}^{(n)} < \cdots < x_{-1}^{(n)} < t \leq x_0^{(n)} < \cdots < x_k^{(n)} < \ldots \tag{1.1.13}$$

then **clock behavior** at t means

$$\lim_{n\to\infty} n\left(x_{j+1}^{(n)} - x_j^{(n)}\right) = \frac{1}{\rho\,(t)}, \quad j \in \mathbb{Z}. \tag{1.1.14}$$

The continuity at an interior point t can be replaced by the *Lebesgue condition*

$$\lim_{\delta\to 0}\frac{1}{2\delta}\int |w\,(x) - w\,(t)|\,dx = 0 \tag{1.1.15}$$

and $\lim\limits_{\delta\to 0}(2\delta)^{-1}\,\omega_s\,(t-\delta, t+\delta) = 0$.

1.2 Reproducing kernels

Very much related with this problem is the universality problem for the **reproducing kernel**, or, also, called as Christoffel-Darboux (CD) kernel, or kernel polynomial:

$$K_n\,(x, y) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} p_k\,(x)\,p_k\,(y)\,, \tag{1.2.1}$$

where p_n are the orthonormal polynomials with respect to the weight w. Using the Christoffel-Darboux formula [57, Section 3.2], we can write the kernel (1.2.1) as

$$\begin{aligned} K_n\,(x, y) &= \frac{k_{n-1}}{k_n}\frac{p_n\,(x)\,p_{n-1}\,(y) - p_n\,(y)\,p_{n-1}\,(x)}{x-y} \\ &= k_{n-1}^2\frac{P_n\,(x)\,P_{n-1}\,(y) - P_n\,(y)\,P_{n-1}\,(x)}{x-y}, \quad x \neq y. \end{aligned}$$

We also define the so called **normalized kernel**

$$\widetilde{K}_n(x,y) \stackrel{\text{def}}{=} (w(x))^{1/2}(w(y))^{1/2} K_n(x,y). \tag{1.2.2}$$

It satisfies

$$\int \widetilde{K}_n(x,y)\,dx = n,$$

and it has the property

$$\int \widetilde{K}_n(x,s)\,\widetilde{K}_n(s,y)\,ds = \widetilde{K}_n(x,y). \tag{1.2.3}$$

In many problems from Random Matrix Theory and mathematic physics (see next section) we need to study the limit of the rescaled kernels

$$\lim_{n\to\infty} \frac{1}{n\rho(t)} \widetilde{K}_n\left(t+\frac{u}{n\rho(t)}, t+\frac{v}{n\rho(t)}\right) = K(u,v). \tag{1.2.4}$$

The so-called **universality property** is that under very general conditions the right hand side (r.h.s.) in (1.2.4) is the **sine kernel**:

$$K(u,v) = \mathbb{S}(u,v) \stackrel{\text{def}}{=} \begin{cases} \dfrac{\sin(\pi(u-v))}{\pi(u-v)}, & u\neq v, \\ 1, & u=v, \end{cases} \tag{1.2.5}$$

this phenomenon is also known as **universality law**, and it is independent of the weight that we are considering. The universality problem has been attracting lately close attention of many researchers. A recent series of remarkable contributions of Lubinsky (see e.g. [36]–[42] and [53]) allowed to weaken considerably the conditions on the weight to be able to assure the **universality law**, that is (1.2.4) with (1.2.5). Now we know that for t within the support of the weight where it is positive and continuous, the right hand side of (1.2.5) holds uniformly (with u, v in compact subsets of the real line).

Another formulation of the universality is

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(t+\frac{u}{\widetilde{K}_n(t,t)}, t+\frac{v}{\widetilde{K}_n(t,t)}\right)}{\widetilde{K}_n(t,t)} = \mathbb{S}(u,v). \tag{1.2.6}$$

In [37], Lubinsky showed that for $t \in J \subset \mathit{supp}(\omega)$, both compact sets (with ω absolutely continuous in a open set containing J), where w is positive and continuous (or t is a Lebesgue point (1.1.15)), (1.2.6) holds for u, v in a compact sets of $\mathbb{R}$, it is equivalent to

$$\lim_{n\to\infty} \frac{K_n\left(t+\frac{u}{\widetilde{K}_n(t,t)}, t+\frac{v}{\widetilde{K}_n(t,t)}\right)}{K_n(t,t)} = \mathbb{S}(u,v), \tag{1.2.7}$$

for u, v in a compact sets of $\mathbb{C}$, and it is also equivalent to (called as *Lubinsky wiggle condition*)

$$\lim_{n\to\infty} \frac{K_n\left(t+\frac{u}{n}, t+\frac{u}{n}\right)}{K_n(t,t)} = 1, \tag{1.2.8}$$

for u in a compact sets of $\mathbb{R}$.

Levin rediscovered the follows result of Freud (see [36]):

Theorem 1.2.1 (Levin-Freud) *Universality (1.2.6) at t implies clock spacing, (1.1.14).*

Under some additional growth assumptions on $\widetilde{K}_n$, clock spacing implies universality (see [37]).

There are several methods for establishing universality, but the RH method is one of the best, because it yields for many other things, as also to obtain asymptotic expansions (see [32], [33]). Though it is required some analyticity for the measure. Levin and Lubinsky observed that first order asymptotics for OP are sufficient to establish universality (see [38]).

The limit kernels obtained in the establishment of the universality are closely related with the local behavior of the orthogonal polynomials (see [32] and [31], [33] and [59], and [22]). From the perspective of the RH method, the functions that solve the local problem and give the local behavior, are related with the functions used to describe the limit of the correspondent kernel $K(u,v)$.

In general, on the *bulk* (inner points) of the support of equilibrium measure the universality sine kernel (1.2.5) holds. Although, there are other three kernels that routinely appear as limit in the r.h.s. of (1.2.4). For instance, for weights on $\mathbb{R}$, $K(u,v)$ at the *soft edge* (end points of the support of the equilibrium measure where it vanishes), is the **Airy kernel**,

$$\mathbb{A}(u,v) = \frac{Ai(u)\,Ai'(v) - Ai'(u)\,Ai(v)}{u-v}, \tag{1.2.9}$$

which is expressed in terms of the Airy functions $Ai(\cdot)$ (solution of the differential equation $y'' = xy$, see [2, formula 10.4.1]), and appears, for instance, in the asymptotics of Hermite OP. For the *hard edge* (end points of the support of the equilibrium measure where it goes to infinity), $K(u,v)$ is expressed in terms of Bessel functions (for instance to Jacobi and Laguerre OP). For instance, for the Jacobi weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, on $x \in [-1,1]$ and $\alpha,\beta > -1$, with $t = \pm 1$, is given by the **Bessel kernel**:

$$\mathbb{J}_\alpha(u,v) = \frac{J_\alpha(\sqrt{u})\,\sqrt{v}J'_\alpha(\sqrt{v}) - J_\alpha(\sqrt{v})\,\sqrt{u}J'_\alpha(\sqrt{u})}{2(u-v)}, \tag{1.2.10}$$

where J_α is the usual Bessel function of the first kind and order α.

At the origin of the spectrum related with the varying weight $w_n(x) = |x|^{2\alpha} e^{-nV(x)}$ on $\mathbb{R}$ (see [33]), the limit kernel, $K(u,v)$, is described in terms of the **second Bessel kernel**:

$$\mathbb{J}^o_\alpha(u,v) = \pi\sqrt{u}\sqrt{v}\frac{J_{\alpha+\frac{1}{2}}(\pi u)\,J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha+\frac{1}{2}}(\pi v)\,J_{\alpha-\frac{1}{2}}(\pi u)}{2(u-v)}, \tag{1.2.11}$$

this kernel are related with the local behavior at algebraic singularities within the support of the weight.

1.3 Connections with Random Matrix theory

Orthogonal polynomials play an important role in random matrix theory, especially in the case of so called unitary ensembles, see [12] (and, e.g. [20], [32] and [33]).

Let $\mathcal{M}(n)$ denote the space of $n \times n$ Hermitian matrices $M = (M_{ij})_{1\le i,j\le n}$, and consider the probability distribution on $\mathcal{M}(n)$,

$$\mathcal{P}^{(n)}(M) = \frac{1}{Z_n} e^{F(M)} dM$$
$$= Ce^{-F(M)} \left(\prod_{j=1}^{n} dM_{jj}\right) \left(\prod_{j<k} d\left(\operatorname{Re} M^{jk}\right) d\left(\operatorname{Im} M_{jk}\right)\right),$$

where $F(M)$ is a function defined on $\mathcal{M}(n)$, and Z_n is a normalization constant such that

$$\frac{1}{Z_n}\int_{\mathbb{R}^{n^2}} \frac{1}{Z_n} e^{F(M)} dM = 1$$

($\mathcal{M} \simeq \mathbb{R}^{n^2}$ as a real vector space). Under assumptions that these distributions are invariant under unitary transformations (unitary ensembles), we get that F can be of the form

It consider measures that are invariant under any unitary matrix - UE - unitary ensembles. Here $F(M)$ can be either, the most important are the

$$F(M) = -tr\ Q(M), \quad \text{with } Q(M) = M^2 \text{ (GUE - Gaussian unitary ensemble)},$$

where tr is the trace, or,

$$Q(M) = -\log(w(M)) \text{ (JUE - Jacobi unitary ensemble)},$$

where w is the Jacobi weight. Using the properties of the Hermitian matrices, such as the eigenvalues are simple, and its representation as a diagonal matrix of its eigenvalues, it can be proven that the $\mathcal{P}^{(n)}$ is identified with the probability density function of the n eigenvalues $x_1 \le x_2 \le \cdots \le x_n$ of M,

$$P^{(n)}(x) = \frac{1}{Z_n}\prod_{j=1}^{n} e^{-\sum_{i=1}^{n} Q(x_i)} \prod_{i<j} |x_i - x_j|^2,$$

with $x = (x_1, x_2, \cdots, x_n)$, and, also, that

$$P^{(n)}(x) = \frac{1}{n!}\det\left(\widetilde{K}_n(x_i, x_j)\right)_{1\le i,j\le n}, \tag{1.3.1}$$

where

$$\int_{\mathbb{R}^n} P^{(n)}(x)\, d^n x = 1,$$

with $d^n x = dx_1 \cdots dx_n$.

The connection between random matrices and OP is given by (1.3.1), where $\widetilde{K}_n$ is the kernel defined in (1.2.2) with respect to the weight $w(x) = e^{-Q(x)}dx$ on $\mathbb{R}$.

This is used to compute many of statistical quantities. For instance, let

$$A_m(\delta) = \Pr(\text{a matrix in } \mathcal{M}(n) \text{ has precisely } m \text{ eigenvalues in the interval } (-\delta, \delta),\ \delta > 0),$$

where Pr denote the probability. It has the following expression,

$$\begin{aligned} A_0(\delta) &= \int_{\{|x_i|>\delta:1\le i\le n\}} P^{(n)}(x_1, x_2, \cdots, x_n)\, d^n x \\ &= \frac{1}{n!} \int_{\{|x_i|>\delta:1\le i\le n\}} \det\left(\widetilde{K}_n(x_i, x_j)\right)_{1\le i,j\le n} d^n x \\ &= \sum_{m=0}^{n} \frac{(-1)^m}{m!} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} R_{m,n}(x_1, x_2, \cdots, x_m)\, d^m x \\ &= \det\left(\mathbf{I} - \widetilde{K}_n\right), \end{aligned} \tag{1.3.2}$$

where $R_{m,n}$ is the $m-$ *point correlation function* for $P^{(n)}(x)\, d^n x$,

$$\begin{aligned} R_{m,n}(x_1, x_2, \cdots, x_m) &= \frac{n!}{(n-m)!} \underbrace{\int \cdots \int}_{n-m} P^{(n)}(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)\, dx_{m+1} \cdots dx_n \\ &= \det\left(\widetilde{K}_n(x_i, x_j)\right)_{1\le i,j\le m}. \end{aligned}$$

In particular,

$$\int_B R_{1,n}(x_1)\, dx_1 = \mathcal{E}\left(\#\ of\ eigenvalues\ in\ B\right),$$

where $\mathcal{E}$ denotes the *expected value*, and (x_1, x_2),

$$\int_B R_{2,n}(x_1, x_2)\, dx_1 dx_2 = \mathcal{E}\left(\#\ of\ pairs\ of\ eigenvalues\ in\ B\right).$$

For $A_m(\delta)$,

$$A_m(\delta) = \frac{1}{m!}\left(-\frac{d}{ds}\right)^m \det\left(\mathbf{I} - s\widetilde{K}_n\right)\Big|_{s=1}. \tag{1.3.3}$$

Another quantity of basic interest is the spacing distribution of the eigenvalues of a unitary ensembles random matrix. For $s > 0$, to compute the *expected value*, $\mathcal{E}$, of

$$S(s, M) = \#\left\{1 \le i, j \le n-1 : x_{j+1}(M) - x_j(M) \le s\right\}$$

where $x_1(M) \le \cdots \le x_n(M)$ are the ordered eigenvalues of M. This expectation is given by

$$\mathcal{E}(S(s)) = \sum_{m\ge 2} (-1)^m \int_{x_1\le\cdots\le x_m;\, x_m - x_1 \le s} R_{m,n}(x_1, x_2, \cdots, x_m)\, d^m x.$$

The universality limit in the bulk asserts that for a fixed $m \ge 2$, t in interior of $supp(\omega)$ and real $u_1, u_2, \cdots, u_m$ we have

$$\lim_{n\to\infty} \frac{1}{\widetilde{K}_n(t,t)} R_{m,n}\left(t + \frac{u_1}{\widetilde{K}_n(t,t)}, t + \frac{u_2}{\widetilde{K}_n(t,t)}, \ldots, t + \frac{u_m}{\widetilde{K}_n(t,t)}\right) = \det\left(\mathbb{S}(u_i, u_j)\right)_{1\le i,j\le m}$$

where $\mathbb{S}$ is the sine kernel defined in (1.2.5), and, as $u_i = u_j$, we interpret $\mathbb{S}(u_i, u_{ij}) = 1$. This limit is the same for all weights $\frac{1}{Z_n} e^{F(M)} dM$.

Because m is fixed in this limit, under certain assumptions, the analyze for general m can be reduced to the case $m = 2$, like in (1.2.6).

Both OP and random matrices are related with the **Determinantal Point Processes** or, in Physics, with **Fermionic Point Process** (see [7] and [28]). Here the link is made by the called *correlation functions* ρ_m, which plays the same role than $R_{m,n}$ above, such that, $\rho_m \equiv R_{m,n}$, and the kernels $K(x,y)$ are called *correlation kernels*. The determinant in (1.3.2) and (1.3.3) are known in functional analysis as **Fredholm determinant** (see [9] for details), $\det\left(\mathbf{I} - \widetilde{K}_n\right)$, of the (trace class) operator $K_B : L^2(B) \to L^2(B)$ such that, $f \mapsto K_B f$ defined by

$$K_B f(x) = \int_B K(x,y) f(y)\, dy, \qquad x \in B.$$

It is shown the connection between OP and random matrices, which had a crucial contribution to develop the RH method (see, e.g. in [12] and [20]).

1.4 Goals and contributions

Taking in to account all these questions and remarks, it remains to know what happen when the weight function has some singular behavior of not necessarily algebraic type. The first natural extension is a jump discontinuity.

Namely, we consider polynomials that are orthogonal on a finite interval $[-1,1]$ with respect to a generalized Jacobi weight of the form

$$w_{c,\gamma}(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}|x_0 - x|^{\gamma}\Xi_c(x), \qquad x \in [-1,1], \tag{1.4.1}$$

where $x_0 \in (-1,1)$, $\alpha, \beta, \gamma > -1$, h is real analytic and strictly positive on $[-1,1]$, and Ξ_c is a step-like function defined by

$$\Xi_c(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \in [-1, x_0) \\ c^2, & x \in [x_0, 1]. \end{cases}$$

In this thesis we answer several questions presented above, related to $w_{c,\gamma}$

- Modifications of the RH method to handle this weight? – see Chapter 7.
- Which is the associated Szegő function for the step-like function? – see Chapter 5.
- Which kind of local problem at x_0 we obtain? We are able to solve it explicitly? – see Chapter 6.
- What is the asymptotic expansion for the polynomials P_n? And for its parameters k_n, a_n and b_n? – see Chapter 8.
- Does the universality (1.2.5) hold at x_0? – see section 9.2.
- Do the zeros of P_n at x_0 satisfy the clock behavior (1.1.14)? – see section 10.2.

The weight $w_{c,\gamma}$ is non analytic in a neighborhood of $[-1,1]$, because it has both a jump and an algebraic singularity at the same point x_0. Since it satisfies the Szegő condition (1.1.3), then the asymptotic results presented in the subsection 1.1.1 holds for this weight.

The particular cases $h \equiv 0$ with $\gamma = 0$ and $c = 1$ correspond to the classical Jacobi weight. For this case almost all is known: asymptotic expansions for polynomials and its parameters, with explicit terms, and also distribution of zeros and universality at the edge and at the bulk. At ± 1 the corresponding kernel is the Bessel $\mathbb{J}$, and in the bulk the universality law (1.2.5) holds (see [57]).

The case with $\gamma = 0$ and $c = 1$, was considered in [31], which was the first work where RH method was applied in OP with compact support at $[-1,1]$. In [31] and in [32], was given several asymptotic expansions and the related kernels, where at ± 1 holds, yet, the Bessel kernel $\mathbb{J}$.

The case $c = 1$, in particular, was considered in [59], where the asymptotic expansion for the recurrence relation a_n and b_n was computed. Another works considered this kind of weights but for weights with other supports: in [33] for varying weights on the real line, and, in [47]) for weights on the circle. The work [59] was the first where it was considered algebraic singularities on the bulk from the point of view the RH method. The local problem, at x_0, is solved using the Bessel functions of other orders different from α or β (like happen at ± 1), and the related kernel is the second Bessel $\mathbb{J}^o$.

The case $\gamma = 0$, provides different behavior of the weight, from discontinuity like a jump in the bulk, at x_0. This case has different problems when it applies the RH method, because, at x_0, the weight is not analytic with an not algebraic discontinuity. Until now, asymptotics expansions were not known. A. Magnus [43] made in 1994 a conjecture for the recurrence coefficients a_n and b_n, see subsection 8.3.1. An important contribution was given by Its and Krasovski [26], in 2008, even considering another kind of weights, with unbounded support and aiming at other applications. They considered a weight with step-like discontinuities and showed that the local problem obtained is solved using the Confluent Hypergeometric functions. They found also important relations and properties of these functions that we partially present in Chapter 4. For this kind of singularity, many questions still open. In [22], we consider the case $w_{c,0}$ (and $x_0 = 0$, without loss of generality) and we obtained some important results. We show that the local problem for this weight is the same than the considered in [26], we obtain asymptotic expansions for P_n and its parameters, and we show that this kind of weights yield a new kernel, at $x_0 = 0$, constructed in terms of Confluent Hypergeometric functions. For this weight we show that at $x_0 = 0$ the clock behavior (1.1.14) of the distribution of zeros fails in the bulk even when the weight does not vanish (see section 10.2). We show, also, that the universality law (1.2.5) fails in a bulk even when the weight does not vanish. Furthermore the Confluent Hypergeometric kernel obtained is the first explicit example of reproducing kernel correspondent to the De Branges space different than the Paley-Wiener (see section 10.3).

The general case $w_{c,\gamma}$ was considered by us, in [23], where we apply the RH method to obtain the asymptotic expansion of the recurrence coefficients a_n and b_n, which, in particular, proves the Magnus conjecture, with some additions (see section 8.3). To the general case $w_{c,\gamma}$ the local problem obtained is a generalization of the one obtained in [26] or [22] and [59], and the solution is constructed in terms of Confluent Hypergeometric functions of orders that depend on c and γ multiplied by a power root (see Chapter 6).

Here we also consider the general case $w_{c,\gamma}$, and we give all the asymptotic expansions for P_n, k_n, a_n and b_n, on $\overline{\mathbb{C}} \setminus [-1,1]$, and for P_n over $(-1,1) \setminus \{x_0\}$, in the Chapter 8, and we study the local behavior of P_n at x_0 and the related questions, like universality and clock spacing in the Chapter 9. There we obtain a new kernel, which generalizes $\mathbb{J}^o$ and the kernel that we obtained in [22]. Additionally we present with detail all steps of RH method in Chapter 3, 6 and 7.

Chapter 2

Boundary Value Problems in the Theory of Analytic Functions

The Riemann-Hilbert Problem is a kind of the classical field Boundary Value Problems and the Complex Analysis. In this chapter we will explore several kinds of Riemann-Hilbert Problems and the correspondent solutions. In the first section we present all notations and basic results from Complex Analysis, the Cauchy transform, the Sokhotskii-Plemelj's formulas, the formulation and solution for additive and multiplicative RH problem and the behavior of the Cauchy transform of some useful functions. In the second section we deduce the formula of the Szegő function on $[-1, 1]$ in terms of the Cauchy transform.

2.1 Riemann-Hilbert Problem

2.1.1 Orientation and Boundary values

Suppose Σ is a system of oriented arcs or contours. Induced by the orientation, denote the $+$ side (on the left) and a $-$ side on Σ. All curves we consider are sufficiently smooth but they may have points of self-intersection or endpoints: at such points the $+$ and $-$ sides are not defined. We also denote by

$$\Sigma^o = \Sigma \setminus \{\text{points of self intersection and end points}\}.$$

Let U be a domain, $\Sigma \subset U$, then for f analytic in $U \setminus \Sigma$ we denote

$$f_+(t) = \lim_{\substack{z \to t \\ z \text{ on } + \text{ side}}} f(z) \qquad \text{and} \qquad f_-(t) = \lim_{\substack{z \to t \\ z \text{ on } - \text{ side}}} f(z). \tag{2.1.1}$$

We consider $f_\pm(t)$ for $t \in \Sigma^o$ and we say that f has **continuous boundary values** on Σ if $f_\pm$ are, at least, continuous functions on Σ^o. It is possible to study boundary values in other senses, like L^p -sense (see [12]), but we will always consider boundary values in the sense of continuous boundary values.

Let Σ_0 be a subarc of Σ^o such that divides U in two domains: U_+ (on the left) and U_-. If f is analytic on $U \setminus \Sigma$ and $f_+(t) = f_-(t)$, $t \in \Sigma_0$, then for any simple close curve Γ cut by Σ_0, on U, we have $\int_\Gamma f = \int_{\Gamma \cap U_+} f + \int_{-\Sigma'} f + \int_{+\Sigma'} f + \int_{\Gamma \cap U_-} f = 0$ (where Σ' is the subarc of Σ_0 that closes $(\Gamma \cap U_\pm)$); by the *Morera's Theorem* we conclude the following.

Proposition 2.1 *If f has continuous boundary values $f_\pm$ on a subarc Σ_0 of Σ^o, and $f_+ = f_-$ there, then f is analytic across Σ_0.*

This result is known as *Analytic Continuation Principle* .

2.1.2 Sokhotskii-Plemelj's Theorem

Under mild smoothness conditions, the Sokhotskii-Plemelj's Theorem gives us an answer for f if we know the difference between its boundary values. We will introduce some definitions and notations.

We said that $v(s)$, $s \in \Sigma^o$, is **Hölder continuous** if for all $s_1, s_2 \in \Sigma^o$,

$$|v(s_1) - v(s_2)| < K\,|s_1 - s_2|^\lambda,\ (0 < \lambda \leq 1) \tag{2.1.2}$$

for some positive constants K and λ.

This continuity is a little stronger than the usual continuity. The class of Hölder continuous functions is close for the addition, product, quotient and composition, and, the functions with limited derivative are Hölder too. For instance $(1-x)^\alpha$ is Hölder on $(-1, 1)$, for $\alpha > -1$.

For z in a domain U, and v defined for $s \in \Sigma$, we denote the **Cauchy transform** of v over Σ (positively oriented), as

$$\mathcal{C}(v)(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_\Sigma \frac{v(s)}{s-z}ds \qquad (z \in U \setminus \Sigma). \tag{2.1.3}$$

The $\mathcal{C}(v)$ define an analytic function on $U \setminus \Sigma$.

By the Cauchy formula, let Σ be a simple close oriented positively contour and U_0 the part of U encircled by Σ, if v is analytic on U_0 and continuous on Σ, then $\mathcal{C}(v)(z) = v(z)$ for $z \in U_0$, and 0 for $z \notin U_0$.

For $z = t \in \Sigma$, the integral (2.1.3) is singular. We define the **principal value of singular Cauchy integral** as

$$\fint_\Sigma \frac{v(s)}{s-t}ds \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{\Sigma - c_\varepsilon} \frac{v(s)}{s-t}ds \tag{2.1.4}$$

where c_ε is the portion of the circle $|z-t| = \varepsilon$ removed from the contour Σ, around t.

Let $t \in \Sigma^o$, rewriting $\mathcal{C}(v)(z)$ as

$$\frac{1}{2\pi i}\int_\Sigma \frac{v(s)}{s-z}ds = \frac{1}{2\pi i}\int_\Sigma \frac{v(s)-v(t)}{s-z}ds + \frac{v(t)}{2\pi i}\int_\Sigma \frac{1}{s-z}ds$$

and denoting by $f(z)$ the first integral on the right hand side, it can be proven (see [24, section I.4]) that the first integral $f(z)$ on passing through the point $z = t \in \Sigma^o$ behaves like a continuous function. Using that

$$\int_\Sigma \frac{1}{s-z}ds = \begin{cases} 2\pi i, & z \text{ on } + \text{ side} \\ 0 & z \text{ on } - \text{ side} \\ \pi i & z \text{ on } \Sigma \end{cases}$$

we obtain the following relations

$$\begin{aligned} f_+(t) &= \mathcal{C}(v)_+(t) - v(t) \\ f_-(t) &= \mathcal{C}(v)_-(t) \\ f(t) &= \mathcal{C}(v)(t) - \frac{1}{2}v(t), \end{aligned}$$

and using the continuity of f, we can deduce the central result of the theory of boundary values problems, the Sokhotskii-Plemelj formula:

Theorem 2.1.1 (Sokhotskii-Plemelj) *Let Σ be a smooth contour (close or open) and v a Hölder continuous function on Σ. Then the Cauchy transform $\mathcal{C}(v)(z)$ (defined by 2.1.3) has continuous boundary values $\mathcal{C}(v)_+(t)$, $\mathcal{C}(v)_-(t)$, for all $t \in \Sigma^o$, given by*

$$\mathcal{C}(v)_+(t) = \frac{1}{2}v(t) + \mathcal{C}(v)(t) \tag{2.1.5}$$

$$\mathcal{C}(v)_-(t) = -\frac{1}{2}v(t) + \mathcal{C}(v)(t) \tag{2.1.6}$$

being the integral

$$\mathcal{C}(v)(t) = \frac{1}{2\pi i} \fint_{\Sigma} \frac{v(s)}{s-t} ds \ (t \in \Sigma^o)$$

*understood in the sense of the principal value. Furthermore, its boundary values satisfying the **Sokhotskii-Plemelj formulas** ($t \in \Sigma^o$)*

$$\mathcal{C}(v)_+(t) - \mathcal{C}(v)_-(t) = v(t) \tag{2.1.7}$$

$$\mathcal{C}(v)_+(t) + \mathcal{C}(v)_-(t) = 2\mathcal{C}(v)(t). \tag{2.1.8}$$

2.1.3 Additive Riemann-Hilbert Problem

Let Σ be a system of smooth oriented arcs or contours and v Hölder continuous on Σ, the **scalar additive Riemann-Hilbert problem** is to find a function f that satisfies the following conditions:

(RH1) f is analytic on $\mathbb{C} \setminus \Sigma$;

(RH2) $f_+(t) = f_-(t) + v(t)$, $t \in \Sigma^o$;

(RH3) $f(z) \longrightarrow 0$ as $z \to \infty$.

By (2.1.7), $f(z) = \mathcal{C}(v)(z)$ satisfies the conditions (RH1) and (RH2). But $\mathcal{C}(v)(z)$ is not unique; if g is an *entire function* another solution could be $h(z) = f(z) + g(z)$, because $h_+ = f_+ + g = f_- + v + g = h_- + v$.

The condition (RH3) is known as the normalization condition, and it is needed precisely to assure the uniqueness of the solution. But it is enough just when Σ is closed, $\Sigma = \Sigma^o$.

In our case we will consider $\Sigma = [-1, 1]$, and $\Sigma \neq \Sigma^o$. For this situation, only three conditions (RH1)-(RH3) do not assure uniqueness. For instance, $f(z) = \mathcal{C}(v)(z) + \frac{1}{z^2-1}$ satisfies (RH1)-(RH3), too. We need to impose additional conditions to ensure (2.1.9) for each point $a \in \Sigma \setminus \Sigma^o$. The *Riemann's Theorem on Removable Singularities* assures us that:

Proposition 2.2 *Let f be analytic in $U \setminus \{a\}$ and*

$$\lim_{z \to a} (z - a) f(z) = 0 \tag{2.1.9}$$

then a is a removable singularity of f.

With this result we can prove the following Lemma.

Lemma 2.1 (Uniqueness of RHP) *If f satisfies (RH1)-(RH3) and (2.1.9) for all $a \in \Sigma \setminus \Sigma^o$, then f is the unique solution.*

Proof. Suppose that f and g are two solutions of (RH1)-(RH3) then $h(z) = f(z) - g(z)$ satisfies $h_+(t) = h_-(t)$ on $\Sigma \setminus \Sigma^o$, and we conclude that it is analytic on $\mathbb{C} \setminus \{a : a \in \Sigma \setminus \Sigma^o\}$. Furthermore, by (2.1.9), the singularities of h are removable for all $a \in \Sigma \setminus \Sigma^o$. Then h is entire on $\mathbb{C}$. As $h(z) \to 0$, for $z \to \infty$, by Liouville's Theorem, $h \equiv 0$ and $f = g$. ∎

Throughout the text it will appear to us a scalar RHP with a different (RH3) condition, non constant. To solve this, we need another classical result, an *extension of Liouville's Theorem*:

Proposition 2.3 *Let f be an entire function (analytic single valued function on $\mathbb{C}$) such that $f(z) = \mathcal{O}(z^n)$, as $z \to \infty$, then f is a polynomial function, at most, of degree n.*

2.1.4 Behavior of the Cauchy Transform

To guarantee the uniqueness of the $\mathcal{C}(v)$ as a solution of the RHP on $\Sigma \neq \Sigma^o$, we need to know the behavior of $\mathcal{C}(v)$ in these singular points $a \in \Sigma \setminus \Sigma^o$, and to verify that this behavior is compatible with (2.1.9). The proof of the following results can be found in section I.8 of [24].

1. If $a \in \Sigma \setminus \Sigma^o$ is an end point of Σ and v is Hölder continuous there,
$$\mathcal{C}(v)(z) = \mathcal{O}(\log(z-a)), \text{ as } z \to a.$$
2. If $a \in \Sigma \setminus \Sigma^o$ is an end point of Σ and $v(t) = |t-a|^\alpha$, $-1 < \alpha < 0$,
$$\mathcal{C}(v)(z) = \mathcal{O}((z-a)^\alpha), \text{ as } z \to a.$$
3. If v have a discontinuity at $a \in \Sigma^o$ (no Hölder continuous there),
$$\mathcal{C}(v)(z) = \mathcal{O}(\log(z-a)), \text{ as } z \to a.$$
4. If $a \in \Sigma^o$ and $v(t) = |t-a|^\alpha \times \begin{cases} 1, & t>a \\ c, & t<a \end{cases}$, $-1 < \alpha < 0$,
$$\mathcal{C}(v)(z) = \mathcal{O}((z-a)^\alpha), \text{ as } z \to a.$$

2.1.5 Multiplicative Riemann-Hilbert Problem

Let Σ be a simple Jordan contour and v Hölder continuous on Σ, the **scalar multiplicative Riemann-Hilbert problem** is to find a function f that satisfies the following conditions:

(RH1) f is analytic on $\mathbb{C} \setminus \Sigma$;

(mRH2) $f_+(t) = f_-(t)v(t)$, $t \in \Sigma^o$;

(mRH3) $f(z) \longrightarrow 1$ as $z \to \infty$.

If the *index* or the *winding number* of v with respect to Σ (increment of the argument divided by 2π) is $Ind_\Sigma v = 0$ this problem can be reduced to the additive RHP. Taking logarithmics the last two conditions are rewritten as

$$(\log f)_+(t) = (\log f)_-(t) + \log v(t), \ \ , t \in \Sigma^o$$
$$\log f(z) \longrightarrow 0, \text{ as } z \to \infty.$$

Then the solution is given by

$$\log f(z) = \mathcal{C}(\log v)(z)$$

and

$$f(z) = \exp(\mathcal{C}(\log v)(z)),$$

of course that the uniqueness is guaranteed if $\Sigma = \Sigma^o$ or if we add conditions for all points of $\Sigma \setminus \Sigma^o$.

If $Ind_\Sigma v = n \in \mathbb{N}$ there is $n+1$ independent solutions, and $Ind_\Sigma v = -n$ ($n \in \mathbb{N}$) there is no solution, but we will not explore these cases here.

2.2 Riemann-Hilbert Problem of the Szegő Function on [-1,1]

A multiplicative RH problem that play a crucial role in the theory of Orthogonal Polynomials is the one defining the Szegő function, $D(z)$. Let w be a non negative weight function ($w > 0$, a.e. $(-1,1)$), that has the following properties:

(D1) $D(z)$ is a non-zero analytic function on $\mathbb{C} \setminus [-1,1]$;

(D2) $D_+(x)D_-(x) = w(x)$, for a.e. $x \in (-1,1)$;

(D3) $D_\infty = \lim_{z\to\infty} D(z) > 0$

Taking logarithmics in (D2),

$$(\log D)_+(x) = -(\log D)_-(x) + \log w(x), \ a.e.\ x \in (-1,1);$$

but, now, we can not apply the Sokhotskii-Plemelj formulas because the minus of the $-(\log D)_-(x)$. Multiplying it by an analytic function on $\mathbb{C}\setminus[-1,1]$, such that $f_+(x) = -f_-(x)$ and $\lim_{z\to\infty} f(z) = 0$, for instance, $\frac{1}{\sqrt{z^2-1}}$ (with the branch $\sqrt{z^2-1}$ that is analytic on $\mathbb{C} \setminus [-1,1]$ and behaves like z as $z \to \infty$), $\left(\frac{1}{\sqrt{z^2-1}}\right)_+ = \frac{1}{i\sqrt{1-x^2}}$, $\left(\frac{1}{\sqrt{z^2-1}}\right)_- = -\frac{1}{i\sqrt{1-x^2}}$. Then for $F(z) = \frac{1}{\sqrt{z^2-1}} \log D(z)$, we have

$$\begin{aligned} F_+(x) &= \frac{1}{i\sqrt{1-x^2}} (\log D)_+(x) \\ &= \frac{1}{i\sqrt{1-x^2}} \left(-(\log D)_-(x) + \log w(x)\right) \\ &= \frac{-1}{i\sqrt{1-x^2}} (\log D)_-(x) + \frac{\log w(x)}{i\sqrt{1-x^2}} \\ &= F_-(x) + \frac{\log w(x)}{i\sqrt{1-x^2}}, \text{ a.e. } x \in (-1,1); \end{aligned}$$

and, now by Sokhotskii-Plemelj formulas, $F(z) = \frac{\log D(z)}{\sqrt{z^2-1}} = \mathcal{C}\left(\frac{\log w(x)}{i\sqrt{1-x^2}}\right)(z)$ and the solution for $D(z)$ is

$$\begin{aligned} D(z) &= \exp\left(\sqrt{1-z^2}\mathcal{C}\left(\frac{\log w(x)}{\sqrt{1-x^2}}\right)(z)\right), \qquad \text{for } z \in \mathbb{C} \setminus [-1,1] \qquad (2.2.1) \\ &= \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} \frac{dx}{z-x}\right), \qquad \text{for } z \in \mathbb{C} \setminus [-1,1] \end{aligned}$$

where $\sqrt{z^2-1}$ with main branch and $\sqrt{1-z^2} \stackrel{\text{def}}{=} -i\sqrt{z^2-1}$, this is, $\left(\sqrt{1-z^2}\right)_+ > 0$ for $z \in (-1,1)$.

Taking limit, $D_\infty = \lim_{z\to\infty} D(z) = \exp\left(\frac{1}{2\pi}\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx\right) \in (0,\infty)$, and (D3) is satisfied.

As $[-1,1]$ is no closed we don't expect unique solution for D.

Chapter 3

The Riemann-Hilbert Method

The Riemann-Hilbert method applied to Orthogonal Polynomials is a combination of two techniques. One is the characterization of the Orthogonal Polynomials by means of a Riemann-Hilbert problem for 2×2 matrix-valued functions and was found by Fokas, Its and Kitaev (see [21]). The second is the steepest descent method of Deift and Zhou, introduced in [15] and further developed in [5], [18], [16] and [19], to mention a few. Initially, this method was applied for orthogonal polynomials with respect to weights with support on unbounded intervals on $\mathbb{R}$. For orthogonal polynomials with compact support, $[-1, 1]$, the first and the best contribution was given by Kuijllars et al. in [31]. After, many applications of this method appeared, to cite some, for orthogonal polynomials on the unit circle, see [47], and, for discrete orthogonal polynomials, see [6].

A crucial contribution to this method is [31], from 2003, where the complete asymptotic expansion for the orthogonal polynomials with respect to a Jacobi weight modified by a real analytic and strictly positive function is obtained (the case $w = w_{1,0}$ in our notation defined in (1.4.1)). This was the first time that the method was applied to orthogonal polynomials on $[-1, 1]$.

The solution of this RHP involves the monic orthogonal polynomials of degree n, P_n, and $n-1$, P_{n-1}, and the leading coefficient k_{n-1}, and their Cauchy transform on the second. Making some manipulations with the solution we can rewrite classical properties from orthogonal polynomials in terms of this Riemann-Hilbert solution. Combining this with the results obtained with the steepest descent method, like asymptotics on all regions of the plane, this is one of the most powerful techniques to obtain asymptotic expansions for P_n, p_n, k_n (and for all things related with them), for the recurrence coefficients, for the limit of reproducing kernels, etc.; these are only some applications that we will show in the second part of this text.

The restrictions of the method are that we need analyticity of the weight. In [31], the weight considered ($w_{1,0}$) is analytic on $(-1, 1)$; [59], considers a weight (general case of $w_{1,\gamma}$) with singularities at some finite set of points in $(-1, 1)$. However, so far not much was known for the case when the weight has a step-type discontinuity on the interval of orthogonality. The only contribution is [26], where the authors considered an exponential weight on $\mathbb{R}$ with a jump at the origin, although from a different perspective to obtain asymptotics of Hankel determinants, without a detailed analysis of OP.

In [22] we study, the asymptotic properties of orthogonal polynomials with respect to $w = w_{c,0}$ (with a jump at the origin). In [23] we considered the case $w = w_{c,\gamma}$ (that combines an algebraic and a step-type singularity at the origin) but we only gave the asymptotic expansion for recurrence coefficients. In this thesis we will consider the most general case $w = w_{c,\gamma}$ and we will give the complete asymptotic expansion for the orthogonal polynomials (see the second part of the thesis).

In this chapter, in the first section we characterize the Orthogonal Polynomials by means of a Riemann-Hilbert problem. In the second section we describe the several steps and motivations of the steepest descent method, which is developed in Chapter 7.

3.1 Riemann-Hilbert Problem for Orthogonal Polynomials

This formulation uses the weight function w for the jump relation, that characterizes the associated orthogonal polynomials; and it does, at infinity, a non "constant" normalization, of the Riemann-Hilbert problem. These two changes allows us to obtain a solution given in terms of orthogonal polynomials associated to the weight w. Also the problem is given as a 2×2 matrix form, although, we can see it as a system of four scalar RH problems.

In all thesis we use the notation for the third Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and, for $a, b \in \mathbb{C} \setminus \{0\}$, we define

$$a^{b\sigma_3} \stackrel{\text{def}}{=} \begin{pmatrix} a^b & 0 \\ 0 & a^{-b} \end{pmatrix};$$

and by $\mathcal{O}\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ we mean $\mathcal{O}(\cdot)$ for each entry of the matrix, and, $(\mathbf{I} + \mathcal{O}(1/z))$ means $(1 + \mathcal{O}(1/z))$ in the diagonal entries and $(\mathcal{O}(1/z))$ in the entries 12 and 21.

We characterize the orthogonal polynomials in terms of the unique solution $\mathbf{Y}$ of the following 2×2 matrix valued Riemann-Hilbert (RH) problem, for $n \in \mathbb{N}$:

(Y1) $\mathbf{Y}$ is analytic in $\mathbb{C} \setminus [-1, 1]$.

(Y2) On $(-1, x_0) \cup (x_0, 1)$, $\mathbf{Y}$ possesses continuous boundary values $\mathbf{Y}_+$ (from the upper half plane) and $\mathbf{Y}_-$ (from the lower half plane), and

$$\mathbf{Y}_+(x) = \mathbf{Y}_-(x) \begin{pmatrix} 1 & w_{c,\gamma}(x) \\ 0 & 1 \end{pmatrix}.$$

(Y3) $\mathbf{Y}(z) = (\mathbf{I} + \mathcal{O}(1/z))\, z^{n\sigma_3}$, as $z \to \infty$;

(Y4) if $(\zeta, s) \in \{(-1, \beta), (x_0, \gamma), (1, \alpha)\}$ then for $z \to \zeta$, $z \in \mathbb{C} \setminus [-1, 1]$,

$$\mathbf{Y}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-\zeta|^s \\ 1 & |z-\zeta|^s \end{pmatrix}, & \text{if } s < 0; \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-\zeta| \\ 1 & \log|z-\zeta| \end{pmatrix}, & \text{if } s = 0; \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } s > 0. \end{cases}$$

Standard arguments show that this RH problem has an unique solution given by

$$\mathbf{Y}(z,n) = \begin{pmatrix} P_n(z) & \mathcal{C}(P_n w_{c,\gamma})(z) \\ -2\pi i k_{n-1}^2 P_{n-1}(z) & -2\pi i k_{n-1}^2 \mathcal{C}(P_{n-1} w_{c,\gamma})(z) \end{pmatrix}, \tag{3.1.1}$$

(P_n is the monic orthogonal polynomial, p_n is the orthonormal polynomial, $k_n > 0$ is the leading coefficient, and $\mathcal{C}(\cdot)$ is the Cauchy transform on $[-1,1]$ defined by (2.1.3)).

Note that $\mathbf{Y}$ and other matrices introduce hereafter depend on n; however, to simplify notation we omit the explicit reference to n.

This is not a classical Riemann-Hilbert Problem, because we have not a (constant) normalization to infinity. Remark that by (Y3), $\mathbf{Y} \sim \mathbf{I} \times z^{n\sigma_3}$, instead $\mathbf{Y} \sim$ "constant ". But this condition is the key of the link between Riemann-Hilbert Problems and Orthogonal Polynomials.

The jump relation (Y2), in fact, is a system of four jump relations, two of them are the classical jump relations of the Riemann-Hilbert problem, where the solution is typically the Cauchy transform of the jump ($Y_{i,1}(x)\, w_{c,\gamma}(x),\ i = 1,2$).

The fourth condition is given to ensure uniqueness. As we expect, the solution where the first column are polynomials and the second are Cauchy transforms, (Y4) is fixed by the corresponding behaviors (see subsection 2.1.4).

Remark 3.1 *Here we take the boundary values (Y2) in terms of the continuous boundary values. It is possible statement boundary values in terms of the L^p $(p > 1)$ sense.*

Although, for weight based in Jacobi weight, where appear power roots as $(1-x)^\alpha$ with $\alpha > -1$, the integrals in L^p $(p > 1)$ sense doesn't converge for $\alpha \in (-1, -1/2)$.

For weights with unbounded support, on the real line, the boundary values has been taken in L^2 sense, see works of Deift [12].

The proof of uniqueness and existence is standard and we can find it in the literature (see the books [12], [29], or many other articles). But we will present the proofs here, for the sake of completeness. We prove only the statement for the first row, because the second is analogous.

Theorem 3.1.1 *The solution of RHP (Y1)-(Y4), is unique.*

Proof. Suppose that Y is a solution. Consider $\det Y(z) = Y_{11}(z)\, Y_{22}(z) - Y_{12}(z)\, Y_{21}(z)$, from (Y1), it is analytic on $\mathbb{C} \setminus [-1,1]$. From (Y2),

$$(\det Y)_+(x) = (\det Y_-(x)) \det \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} = (\det Y)_-(x)\,,\ x \in (-1,1) \setminus \{x_0\}, \tag{3.1.2}$$

and by the Analytic Continuation Principle (Proposition 2.1) $\det Y$ is analytic on $(-1,1)\setminus\{x_0\}$, and, consequently, on $\mathbb{C} \setminus \{-1, x_0, 1\}$. Let $a \in \{-1, x_0, 1\}$, from (Y4), $\lim_{z \to a} (z-a) \det Y(z) = 0$ then, as in the proof of Lemma 2.1, $\det Y$ is entire on $\mathbb{C}$.

From (Y3), $\det Y(z) \to 1$ as $z \to \infty$, then, by Liouville's Theorem, $\det Y(z) = 1$ on $z \in \mathbb{C}$. Furthermore, $Y(z)$ is invertible on $\mathbb{C}$, and $Y^{-1}(z) = \begin{pmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{pmatrix}$ is analytic on $\mathbb{C} \setminus [-1,1]$, too.

Suppose, now, that $\widetilde{Y}$ is another solution. Defining $M(z) = \widetilde{Y}(z)\, Y^{-1}(z)$, it is analytic on $\mathbb{C} \setminus [-1,1]$. As $(\det M_+) = \left(\det \widetilde{Y}_+\right)\left(\det Y_+^{-1}\right)$, applying (3.1.2) to $\widetilde{Y}$ and Y^{-1} we obtain

$$(\det M)_+ = (\det M)_-\,, \text{ as } x \in (-1,1).$$

As

$$M=\begin{pmatrix}\widetilde{Y}_{11}Y_{22}-\widetilde{Y}_{12}Y_{21} & \widetilde{Y}_{12}Y_{11}-\widetilde{Y}_{11}Y_{12}\\ \widetilde{Y}_{21}Y_{22}-\widetilde{Y}_{22}Y_{21} & \widetilde{Y}_{22}Y_{11}-\widetilde{Y}_{21}Y_{12}\end{pmatrix},$$

also, each entry of M, M_{ij} $(1\le i,j\le 2)$ satisfies (Y4) for each $a\in\{-1,x_0,1\}$. Using the same argument than above, M is entire on $\mathbb{C}$, and, as $M(z)\to I_{2\times 2}$ for $z\to\infty$, we conclude that $M=I$, and then $\widetilde{Y}=Y$. ■

Theorem 3.1.2 *(3.1.1) is the solution of the RHP (Y1)-(Y4).*

Proof. Let $Y(z)=Y_{ij}(z)$ $(1\le i,j\le 2)$ be analytic on $\mathbb{C}\setminus[-1,1]$. From (Y2),

$$(Y_{11})_+(x)=(Y_{11})_-(x)\ ,\ \text{ as } x\in(-1,1)\setminus\{x_0\},$$

then by the Analytic Continuation Principle (Proposition 2.1) Y_{11} is analytic on $(-1,1)\setminus\{x_0\}$. From (Y4), we have $Y_{11}(z)=O(1)$ as $z\to a$ $(a\in\{-1,x_0,1\})$, and, consequently, $Y_{11}(z)$ is entire. From (Y3)

$$Y_{11}(z)=z^n+O\left(z^{n-1}\right)\ ,\ \text{ as }\ z\to\infty,$$

by the extension of the Liouville's Theorem, it follows that Y_{11} is a monic polynomial function of degree n. Denote it by $Y_{11}=Q_n$.

For Y_{12}, from (Y2) we have

$$(Y_{12})_+(x)=(Y_{12})_-(x)+(Y_{11})_+(x)\,w(x)\ ,\ \text{ as } x\in(-1,1)\setminus\{x_0\},$$

then, by the Sokhotskii-Plemelj formula $Y_{12}(z)=\mathcal{C}(Q_nw)(z)$. Writing the Cauchy kernel of the Cauchy transform as

$$\frac{1}{x-z}=-\sum_{k=0}^{n-1}\frac{x^k}{z^{k+1}}+\frac{x^n}{z^n(x-z)}$$

then

$$\begin{aligned}Y_{12}(z)&=\frac{1}{2\pi i}\int_{-1}^{1}Q_n(x)\,w(x)\left[-\sum_{k=0}^{n-1}\frac{x^k}{z^{k+1}}+\frac{x^n}{z^n(x-z)}\right]dx\\&=-\sum_{k=0}^{n-1}\frac{1}{2\pi i}\left[\int_{-1}^{1}Q_n(x)\,x^kw(x)\,dx\right]\frac{1}{z^{k+1}}+O\left(z^{-n-1}\right),\end{aligned}$$

and, by (Y3), as $Y_{12}(z)=O\left(z^{-n-1}\right)$, for $z\to\infty$, we conclude that

$$\int_{-1}^{1}Q_n(x)\,x^kw(x)\,dx=0\ ,\qquad k=0,1,...,n-1.$$

This implies that Q_n is the monic orthogonal polynomial $P_n(z)$.

Analogously we deduce Y_{21} and Y_{22}.

The entries Y_{11} and Y_{21} are polynomials then the behavior of the first column of (Y4) is satisfied. The second column of (Y4) corresponds to the behavior of the Cauchy transform of the weight function $w_{c,\gamma}(x)$ times a polynomial function, then its behavior is given in the subsection 2.1.4, and is compatible with (Y4).

Furthermore, P_n is analytic on $\mathbb{C}$, and the Cauchy transform $\mathcal{C}(P_nw_{c,\gamma})(z)$ is analytic on $\mathbb{C}\setminus[-1,1]$, then $Y(z)$ satisfies (Y1). ■

The recurrence coefficients a_n and b_n, from the three-term recurrence relation (1.1.1) or (1.1.2), also can be expressed in terms of the solution of the RH problem for $\mathbf{Y}$. One way to prove this is defining $\mathbf{M}(z,n) = \mathbf{Y}(z,n+1)\,\mathbf{Y}^{-1}(z,n)$; and, using the same arguments than the proof of the Theorem 3.1.1, it is entire. From (Y3) and $\det \mathbf{Y} = 1$,

$$\mathbf{M}(z,n) = \begin{pmatrix} z + \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}\left(z^{-1}\right) \end{pmatrix}, \quad \text{as } z \to \infty.$$

By the Liouville Theorem extension, for certain constants r_n, s_n, t_n,

$$\mathbf{M}(z,n) = \begin{pmatrix} z - r_n & s_n \\ t_n & 0 \end{pmatrix}, \tag{3.1.3}$$

and

$$\mathbf{Y}(z,n+1) = \begin{pmatrix} z - r_n & s_n \\ t_n & 0 \end{pmatrix} \mathbf{Y}(z,n).$$

The $(1,1)$ and $(2,1)$ entries of this matrix gives

$$\begin{aligned} P_{n+1}(x) &= (x - r_n)\,P_n(x) - 2\pi i k_{n-1}^2 s_n P_{n-1}(x) \\ -2\pi i k_n^2 P_n(x) &= t_n P_n(x) \end{aligned}$$

and from (1.1.1) we conclude

$$a_n^2 = -s_n t_{n-1}, \qquad b_n = r_n. \tag{3.1.4}$$

Then from the definition of $\mathbf{M}$, (3.1.3) and (3.1.4)

$$\begin{aligned} a_n^2 = {} & \left(\mathbf{Y}_{11}(z,n+1)\,\mathbf{Y}_{12}(z,n) - \mathbf{Y}_{12}(z,n+1)\,\mathbf{Y}_{11}(z,n)\right) \\ & \times \left(\mathbf{Y}_{21}(z,n)\,\mathbf{Y}_{22}(z,n-1) - \mathbf{Y}_{22}(z,n)\,\mathbf{Y}_{21}(z,n-1)\right), \end{aligned} \tag{3.1.5}$$

$$b_n = z - \mathbf{Y}_{11}(z,n+1)\,\mathbf{Y}_{22}(z,n) + \mathbf{Y}_{12}(z,n+1)\,\mathbf{Y}_{21}(z,n), \tag{3.1.6}$$

for every $z \in \mathbb{C} \setminus [-1,1]$. From (Y3),

$$\mathbf{Y}(z,n) = \begin{pmatrix} z^n + \mathcal{O}\left(z^{n-1}\right) & \mathcal{O}\left(z^{-n-1}\right) \\ \mathcal{O}\left(z^{n-1}\right) & z^{-n} + \mathcal{O}\left(z^{-n-1}\right) \end{pmatrix}, \quad \text{as } z \to \infty,$$

we obtain the following: in (3.1.5) just the first term behaves as $\mathcal{O}(1)$; in (3.1.6) just the first term behaves as $z + \mathcal{O}(1)$. Then, taking limit as $z \to \infty$, we obtain

$$a_n^2 = \lim_{z\to\infty} z^2 \mathbf{Y}_{12}(z,n)\,\mathbf{Y}_{21}(z,n). \tag{3.1.7}$$

$$b_n = \lim_{z\to\infty} \left(z - \mathbf{Y}_{11}(z,n+1)\,\mathbf{Y}_{22}(z,n)\right) \tag{3.1.8}$$

3.2 The nonlinear Steepest Descent Method

The goal of the method of steepest descent for Riemann-Hilbert problems is to change the original $\mathbf{Y}$-RH problem, that characterizes the OP, into an equivalent RH problem for which both the jump matrices and the behavior at infinity are asymptotically closed to the identity matrix, $\mathbf{I}$, for large n. Then we conclude that the solution is also asymptotically close to $\mathbf{I}$.

Depending on the kind of OP, the steps of the method may be different. Here we apply the method for the weight $w_{c,\gamma}(x)$ on the compact support $[-1,1]$.

More precisely, the key idea of the steepest descent analysis is the following: instead of dealing directly with the sequence of orthogonal polynomials, we may start with the corresponding RH for $\mathbf{Y}$ that characterizes these polynomials.

Then we will try to perform a number of explicit and invertible transformations,

$$\mathbf{Y} \to \mathbf{T} \to \cdots \to \mathbf{R}$$

that changes the original $\mathbf{Y}$-RH problem into equivalents RH problems, with the goal that the last $\mathbf{R}$-RH problem satisfies

- $\mathbf{R}$ is analytic on $\mathbb{C} \setminus \Sigma_R$
- $\mathbf{R}_+(z) = \mathbf{R}_-(z)\mathbf{J}_R(z)$ for $z \in \Sigma_R$
- $\lim_{z\to\infty} \mathbf{R}(z) = \mathbf{I}$

where Σ_R are certain contours on $\mathbb{C}$, $\mathbf{Y}$ and all matrices depend on n, and

$$\mathbf{J}_R(z) = (\mathbf{I} + \mathbf{\Delta}(z))$$

with

$$\mathbf{\Delta}(z) = \mathbf{\Delta}(z,n) = \mathcal{O}\left(\frac{1}{n}\right), \text{ as } n \to \infty, \tag{3.2.1}$$

valid uniformly on a neighborhood of Σ_R. At this point, using results like Theorem 7.10 of [19] or Lemma 8.3 of [31], we obtain that

$$\mathbf{R}(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right), \text{ as } n \to \infty,$$

also valid uniformly, in $\mathbb{C} \setminus \Sigma_R$.

After tracing back the transformations

$$\mathbf{Y} \leftarrow \mathbf{T} \leftarrow \cdots \leftarrow \mathbf{R}$$

we can obtain an asymptotic expression for $\mathbf{Y}$ valid uniformly in $\mathbb{C}$, and, consequently, for each entry of $\mathbf{Y}$, recalling that the $\mathbf{Y}_{11}$ entry is the corresponding monic orthogonal polynomial P_n.

The good and the bad of this method is that we either have all or nothing about the strong asymptotics of P_n, everywhere. Unlike some other approaches we cannot obtain the strong asymptotics just on a specific subset of the plane. But, this method will work only if we could achieve (3.2.1) uniformly in Σ_R.

The achievement of $\mathbf{\Delta}(z)$ usually depends on our ability to solve certain local RH problems typically at singular points of the weight. In our case, for the weight $w_{c,\gamma}$, and as the local problem is solved at ± 1 by [31], our major difficulty was the local problem at x_0.

We will describe here the outline of the steepest descent method for the weights $w_{c,\gamma}$ (1.4.1). The detailed analysis will be done in the Chapter 7. Comparing the method applied in [31] (for $w_{1,0}$ of) with our for $w_{c,\gamma}$, a crucial step is finding the solution of the local RH problem; for that reason we will analyze it separately in the Chapter 6.

The main steps of the steepest descent method for our weight $w_{c,\gamma}$ (1.4.1) that we will perform will be:

1. First transformation $\mathbf{Y} \to \mathbf{T}$: normalization of the condition (Y3) at infinity (as a constant matrix).

   We use the conformal mapping $\varphi(z)$ from $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle (defined in (1.1.4)), that behaves like $2z$ as $z \to \infty$, and have as a multiplicative jump on $(-1,1)$: $\varphi_+(x)\varphi_-(x) = 1$. To do it we multiply $\mathbf{Y}$ by $\varphi(z)^{-n\sigma_3}$ from the right.

   Remark: a multiplication of $\mathbf{Y}$ from the left by a constant matrix affects only the behavior at infinity (condition 3), while multiplying $\mathbf{Y}$ by a matrix from the right has impact on both conditions (jump - condition 2, and normalization - condition 3).

2. Second transformation $\mathbf{T} \to \mathbf{S}$: involves a factorization of the jump matrix and a contour deformation, also called "opening lenses".

   This step actually gives the name of "steepest descent method", in analogy with the classical procedure.

   With the first transformation we win the normalization at ∞ but lose the nice behavior of the jump (T2), now with oscillatory behavior. The key is, using factorization of the jump matrix to split the jumps into several curves where they either approach the identity matrix for large n, or are independent of n. However, we need analytic extension of the weight, but, as $w_{c,\gamma}$ has a singularity at x_0, the curves need to meet at x_0. This is the first difference comparing with the case $w_{1,0}$ from [31].

   With this transformation we obtain a RH problem with two kind of jumps. One, with the exponential decay jump (as we want), for large n, away from $[-1,1]$, because $\left|\varphi(z)^{-1}\right| < 1$ on $\mathbb{C} \setminus [-1,1]$. The other, with dependence of $w_{c,\gamma}$ on $[-1,1]$.

   The main purpose for the next step is to construct the "outer parametrix ($\mathbf{N}$)" that solves the "stripped" or model RH problem, obtained from the RH problem for $\mathbf{S}$ by ignoring all jumps asymptotically close to identity. Unfortunately, this property of $\mathbf{N}$ is not uniform on the whole plane: the jumps of $\mathbf{N}$ and $\mathbf{S}$ are no longer close in the neighborhoods of ± 1 and x_0. The analysis at these points ($\mathbf{P}$) requires a separate treatment, called "local parametrices", that we perform later.

3. Outer parametrix ($\mathbf{N}$): the parametrix away from the singular points $\{\pm 1, x_0\}$.

   This parametrix depends on the weight $w_{c,\gamma}$, with the jump on $[-1,1]$. This parametrix is standard (comparing with $w_{1,0}$ from [31]) and is based on the Szegő function (1.1.5) D, where $D_+ D_- = w_{c,\gamma}$ on $[-1,1]$.

4. Local parametrices ($\mathbf{P}$): we construct the local parametrix near the endpoints ± 1 and near the x_0.

   The construction of the local parametrix near the endpoints ± 1 is the same than for $w_{1,0}$ in [31], and it involves modified Bessel functions of orders α or β.

   The construction of the local parametrix near the x_0 is more complicated, since in the model local local problem we obtain a system of 8 jumps. For the particular case $w_{1,\gamma}$ the solution involves modified Bessel functions of orders $\frac{\gamma \pm 1}{2}$ (see [59]). For the particular case $w_{c,0}$ (that we analyzed in [22]), we obtain a system of 6 jumps, and the solution involves Confluent Hypergeometric functions of orders $\left(i\frac{\log c}{\pi}; 1\right)$ (see also [26] where they obtained a similar solution for the case with unbounded support, on the real line). For general $w_{c,\gamma}$, we obtain a system of 8 jumps, and the solution was given by us in [23], and the solution involves Confluent Hypergeometric functions of orders $\left(i\frac{\log c}{\pi} + \frac{\gamma}{2}; \gamma + 1\right)$.

5. Last transformation $\mathbf{S} \to \mathbf{R}$: we define a RH problem without jumps on $[-1, 1]$, such that the remaining jumps are close to identity for large n.

   Defining $\mathbf{R} = \mathbf{S}\mathbf{N}^{-1}$, for z outside the neighborhoods of $\zeta \in \{\pm 1, x_0\}$, and $\mathbf{R} = \mathbf{S}\mathbf{P}_\zeta^{-1}$, for z near the ζ, as the jumps of $\mathbf{N}$ and $\mathbf{P}_\zeta$ are, in each region, the same than the $\mathbf{S}$, the bad jumps from $\mathbf{S}$ are killed in that regions, and the remaining jumps are: on the boundary of neighborhoods of ζ, that (by construction) behaves as identity, for large n; and, the exponential decaying jumps, away from $[-1, 1]$, on the lenses of the $\mathbf{S}$.

   It can be pointed out that at this stage it can be seen that the order of the correction to the leading term of asymptotics depends on the leading term of asymptotics depends on the character of the singularities of the weight (i.e., from the matching condition at this singularities).

Chapter 4

Special functions

In this chapter we provide a minimum background needed for what follows. We will present some special functions and some of their relations and properties. The results presented here are contained in [2], [54], [3] and in the appendix of [26].

In the first section we give some motivation. In the second section, we introduce some special functions that we use in this thesis. In the third section we present the Confluent Hypergeometric functions and their relations and properties, essential for the solution of the local problem in Chapter 6.

4.1 Introduction

Orthogonal Polynomials are closely connected with trigonometric, hypergeometric, Bessel and Elliptic functions, which belong to the class of the so-called Special Functions. Newton and Leibniz explored their role in the solution of differential equations, that also characterize the classical orthogonal polynomials.

One of the steps of the steepest descent method is to find a local parametrix as a solution of a system of normalized Riemann-Hilbert Problems. The solutions of these problems involve special functions. In general, special functions are **multiple-valued** analytic functions.

Relatively to the multiple-valued analytic functions, we usually fix the standard **determination** or **main branch**. For instance, for $(z)^{1/2}$ we consider the branch that is analytic on $\mathbb{C} \setminus [-\infty, 0]$ and such that $z^{1/2} > 0$ for $z > 0$; in this case, $[-\infty, 0]$ is the **branch cut** and we call it **main branch** of $(z)^{1/2}$ (∞ and 0 are the **branch points**). To the $\log(z)$ we consider the branch that is analytic on $\mathbb{C} \setminus [-\infty, 0]$ and such that $\log z = \log|z| + i \arg(z)$ is real for $z > 0$ (with $-\pi < \arg z < \pi$); in this case, $[-\infty, 0]$ is the **branch cut** and we call it by **main branch** of $\log(z)$.

In this thesis we expect to seek solutions for a RH problem using special functions. The key idea is to find special functions that have compatible jump relations across the branch cut that we are considering. In general, the cut for the selected branch will be a line of discontinuity. The kind of discontinuity of the branch of multiple-valued analytic functions on the approaches of a cut is used in the solution of the boundary value problems, and, consequently in the solution of the RH problems. The discontinuity of the branch corresponds to the relation with another different branch, when we consider the argument, of the multiple-valued function, varying continuously. Sometimes, in the literature, this discontinuity or (jump) relation is known as **continuation formulae** (see [54]).

Considering the function $(z)^{1/2}$ with main branch such that is positive for $z > 0$, taking the usual orientation on $[-\infty, 0]$, then for $x \in (-\infty, 0)$, we have

$$\left(e^{2\pi i} x\right)^{1/2} = e^{\pi i} x^{1/2},$$

and we say that the continuation formulae of the determination of $(z)^{1/2}$, on $(-\infty, 0)$, has a multiplicative constant jump of $e^{\pi i}$. To the $\log(\cdot)$, taking the main branch and $[-\infty, 0]$ with the usual orientation, then on $(-\infty, 0)$, we have

$$\log\left(e^{2\pi i} x\right) = \log x + 2\pi i,$$

and we say that the continuation formulae of the determination of $\log z$ on $(-\infty, 0)$ has an additive constant jump of $2\pi i$.

In the next two sections we present some special functions, some of their properties and their continuation formulae or jump relations.

4.2 Some important functions

The Gamma function, which generalizes the natural factorial, is defined by (see [2, Chapter 6])

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad (\operatorname{Re} z > 0),$$

and satisfies the following identities:

$$\Gamma(z+1) = z\Gamma(z) = z!, \tag{4.2.1}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{4.2.2}$$

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} \frac{4^z}{\sqrt{2}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{4.2.3}$$

$$\Gamma(\overline{z}) = \overline{\Gamma(z)}, \tag{4.2.4}$$

$$|\Gamma(x+iy)| \le |\Gamma(x)|, \tag{4.2.5}$$

$$\Gamma(iy)\Gamma(iy) = |\Gamma(iy)|^2 = \frac{\pi}{y\sinh(\pi y)}, \tag{4.2.6}$$

$$\Gamma(1+iy)\Gamma(1-iy) = |\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh(\pi y)}. \tag{4.2.7}$$

Related with this is the Pochhammer's Symbol:

$$\begin{aligned} &(z)_0 = 1, \\ &(z)_n = z(z+1)(z+2)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}, \end{aligned} \tag{4.2.8}$$

note that $(1)_n = n!$.

The Bessel functions are defined as a solution of the second order differential equation (see [2, Chapter 9]),

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + \left(z^2 - \alpha^2\right) w = 0$$

being $J_{\pm\alpha}(z)$ the first kind ($Y_\alpha(z)$ the second kind, and $H_\alpha^{(1)}(z)$ and $H_\alpha^{(2)}(z)$ the third kind). $J_{\pm\alpha}(z)$ is single-valued analytic function of z on $\overline{\mathbb{C}}\setminus[-\infty,0]$, and, is an entire function of z when $\alpha\in\mathbb{Z}$. The Bessel function (of first kind) is defined by

$$J_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}z\right)^{\alpha+2n}}{n!\Gamma(\alpha+n+1)}. \tag{4.2.9}$$

The modified Bessel functions I_α and K_α satisfy the differential equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-\left(z^2+\alpha^2\right)w=0,$$

and, are defined by

$$I_\alpha(z)=\left(\frac{z}{2}\right)^\alpha\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^n}{n!\Gamma(\alpha+n+1)}, \tag{4.2.10}$$

which is single-valued analytic function of z on $\overline{\mathbb{C}}\setminus[-\infty,0]$, and, is an entire function of z when $\alpha\in\mathbb{Z}$, and

$$K_\alpha(z)=\frac{\pi}{2}\frac{I_{-\alpha}(z)-I_\alpha(z)}{\sin(\alpha\pi)}, \tag{4.2.11}$$

where the right hand side of this equation is replaced by its limiting value if $\alpha\in\mathbb{Z}$.

4.3 Confluent Hypergeometric Function

The general Hypergeometric function can be defined in terms of a power series:

$${}_nF_m\left(\begin{matrix}a_1,a_2,\cdots,a_n\\ b_1,b_2,\cdots,b_m\end{matrix};z\right)\stackrel{\text{def}}{=}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_n)_k}{(b_1)_k(b_2)_k\cdots(b_m)_k}\frac{z^k}{k!}, \tag{4.3.1}$$

where the *parameters* $a_i,b_j\in\mathbb{C}$ with $b_j\notin\mathbb{Z}^-$ ($\mathbb{Z}^-=\mathbb{Z}\setminus(\mathbb{N}\cup\{0\})$), and $(a_i)_k$ defined in (4.2.8). It converges absolutely for all z if $n\le m$, and for $|z|<1$ if $n=m+1$, and diverges for all $z\neq 0$ if $n>m+1$.

We will consider the Confluent Hypergeometric functions

$$\phi(a,b;z)\stackrel{\text{def}}{=}{}_1F_1(a;b;z)\quad\text{and}\quad\psi(a,b;z)\stackrel{\text{def}}{=}z^{-a}{}_2F_0(a,a-b+1;-;-1/z), \tag{4.3.2}$$

that are independent solutions of the confluent hypergeometric equation (see e.g. [2, formula (13.1.1)]),

$$zw''+(b-z)w'-aw=0. \tag{4.3.3}$$

The function ϕ is an analytic function of z, and is undefined for both a,b negative integers. The function ψ is a multiple-valued function, being single-valued in some domain in $\mathbb{C}$ with a cut from 0 to ∞. Integral representations for this functions are

$$\phi(a,b;z)=\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}\,dt,\quad \operatorname{Re}b>\operatorname{Re}a>0;$$

$$\psi(a,b;z)=\frac{1}{\Gamma(a)}\int_0^{\infty e^{-i\theta}}e^{-zt}t^{a-1}(1-t)^{b-a-1}\,dt,\quad \operatorname{Re}a>0,$$
$$-\pi<\theta<\pi,\quad -\frac{\pi}{2}+\theta<\arg z<\frac{\pi}{2}+\theta.$$

Using this representations and taking different values for θ, Its and Krasovski, in [26, Appendix] (formulas (7.18), (7.26), (7.29), (7.30), (7.27)), proved the following continuation formulae for ϕ and ψ:

$$\phi\left(a,b;e^{\pm 2\pi i}z\right)=\phi\left(a,b;z\right), \tag{4.3.4}$$

$$\phi\left(a,b;z\right)=\frac{\Gamma\left(b\right)}{\Gamma\left(b-a\right)}e^{-i\pi a}\psi\left(a,b;e^{-2\pi i}z\right)+\frac{\Gamma\left(b\right)}{\Gamma\left(a\right)}e^{i\pi(b-a)}\psi\left(b-a,b;e^{-\pi i}z\right)e^{z}, \tag{4.3.5}$$

$$\phi\left(a,b;z\right)=\frac{\Gamma\left(b\right)}{\Gamma\left(b-a\right)}e^{i\pi a}\psi\left(a,b;z\right)+\frac{\Gamma\left(b\right)}{\Gamma\left(a\right)}e^{-i\pi(b-a)}\psi\left(b-a,b;e^{-\pi i}z\right)e^{z}, \tag{4.3.6}$$

$$\psi\left(b-a,b;e^{i\pi}z\right)e^{z}=\frac{-\Gamma\left(a\right)}{\Gamma\left(b-a\right)}e^{-i\pi b}\psi\left(a,b;z\right)+\frac{\Gamma\left(a\right)}{\Gamma\left(b\right)}e^{-i\pi(b-a)}\phi\left(a,b;z\right), \tag{4.3.7}$$

$$\psi\left(a,b;e^{2\pi i}z\right)=e^{-2i\pi a}\psi\left(a,b;z\right)+e^{-i\pi a}\frac{2\pi i}{\Gamma\left(a\right)\Gamma\left(1+a-b\right)}\psi\left(b-a,1;e^{i\pi}z\right)e^{z}, \tag{4.3.8}$$

which hold for all values of $\arg z$. Combining the last two formulas and 4.2.2, we obtain:

$$\psi\left(a,b;e^{2\pi i}z\right)=\psi\left(a,b;z\right)e^{-2\pi ib}+\phi\left(a,b;z\right)\frac{2\pi i}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}e^{-\pi ib}. \tag{4.3.9}$$

We list also several properties and relations of ϕ that we can be found in [2, Chapter 13]:

$$\phi\left(a,b;z\right)=e^{z}\phi\left(b-a,b;-z\right), \tag{4.3.10}$$

Kummer transformation:

$$\left(b-a\right)\phi\left(a-1,b;z\right)+\left(2a-b+z\right)\phi\left(a,b;z\right)-a\phi\left(a+1,b;z\right)=0; \tag{4.3.11}$$

$$\left(1+a-b\right)\phi\left(a,b;z\right)-a\phi\left(a+1,b;z\right)+\left(b-1\right)\phi\left(a,b-1;z\right)=0; \tag{4.3.12}$$

$$b\phi\left(a,b;z\right)-b\phi\left(a-1,b;z\right)-z\phi\left(a,b+1;z\right)=0; \tag{4.3.13}$$

and, for the derivatives of ϕ,

$$\phi'\left(a,b;z\right)=\frac{a}{b}\phi\left(a+1,b+1;z\right); \tag{4.3.14}$$

$$a\phi\left(a+1,b;z\right)=a\phi\left(a,b;z\right)+z\phi'\left(a,b;z\right); \tag{4.3.15}$$

and, for ψ,

$$\psi'\left(a,b;z\right)=-a\psi\left(a+1,b+1;z\right); \tag{4.3.16}$$

$$a\left(1+a-b\right)\psi\left(a+1,b;z\right)=a\psi\left(a,b;z\right)+z\psi'\left(a,b;z\right). \tag{4.3.17}$$

We have the following relation for the wronskian of ϕ and ψ:

$$\begin{vmatrix}\phi\left(a,b;z\right) & \psi\left(a,b;z\right)\\ \phi'\left(a,b;z\right) & \psi'\left(a,b;z\right)\end{vmatrix}=-\frac{\Gamma\left(b\right)e^{z}}{z^{b}\Gamma\left(a\right)}. \tag{4.3.18}$$

The behavior as $|z|\to\infty$ and a,b fix, is

$$\begin{aligned}\phi\left(a,b;z\right)=&\tfrac{\Gamma(b)}{\Gamma(b-a)}e^{a\pi i}z^{-a}\left[\textstyle\sum_{n=0}^{R-1}\tfrac{(a)_n(1+a-b)_n}{n!}\left(-z\right)^{-n}+\mathcal{O}\left(|z|^{-R}\right)\right]\\ &+\tfrac{\Gamma(b)}{\Gamma(a)}z^{a-b}\left[\tfrac{1}{z}\textstyle\sum_{n=0}^{S-1}\tfrac{(b-a)_n(1-a)_n}{n!}z^{-n}+\mathcal{O}\left(|z|^{-S-1}\right)\right]e^{z}\end{aligned} \tag{4.3.19}$$

holds for $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, and, as $|z| \to \infty$ and a, b fixed,

$$\psi(a,b;z) = z^{-a}\left[\sum_{n=0}^{R-1} \frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + \mathcal{O}\left(|z|^{-R}\right)\right], \quad \frac{-3\pi}{2} < \arg z < \frac{3\pi}{2}. \tag{4.3.20}$$

The behavior as $|z| \to 0$ and a, b fixed, is

$$\phi(a,b;z) = 1, \qquad b \neq -n \quad (n \in \mathbb{N}); \tag{4.3.21}$$

and,

$$\psi(a,b;z) = \mathcal{O}\left(|z|^{1-b}\right), \qquad \operatorname{Re} b > 1; \tag{4.3.22}$$

$$\psi(a,b;z) = \mathcal{O}(|\log z|), \qquad b = 1; \tag{4.3.23}$$

$$\psi(a,b;z) = \mathcal{O}(1), \qquad \operatorname{Re} b < 1. \tag{4.3.24}$$

The relations above are a basis for a characterization of a confluent hypergeometric function in terms of a RH problem, see Chapter 6.

The Bessel function J_α and the modified Bessel function I_α, can be seen as a particular case of the confluent hypergeometric function ϕ:

$$\phi\left(\alpha + \frac{1}{2}, 2\alpha + 1; 2iz\right) = \Gamma(1+\alpha)\, e^{iz}\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z), \tag{4.3.25}$$

$$\phi\left(\alpha + \frac{1}{2}, 2\alpha + 1; 2z\right) = \Gamma(1+\alpha)\, e^{z}\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z). \tag{4.3.26}$$

The exponential and trigonometric functions also can be seen as a particular case of the confluent hypergeometric function ϕ:

$$\phi(\alpha,\alpha;z) = e^z, \tag{4.3.27}$$

$$\phi(1,2;-2iz) = \frac{e^{-iz}}{z}\sin z, \tag{4.3.28}$$

$$\phi(1,2;2z) = \frac{e^z}{z}\sinh z. \tag{4.3.29}$$

Part II

New Results

Chapter 5

The Szegő function for a step-like function

In this chapter we present the Szegő function for Ξ_c defined in (1.4.1), and using the known Szegő function for the Jacobi weight in subsection 1.1.1 and its deduction in section 2.2, we will obtain the Szegő function for $w_{c,\gamma}(x)$ that was defined in (1.4.1). In the second section we will check its properties and deduce a Lemma that relates $w_{c,\gamma}(x)$ with its Szegő function.

5.1 Szegő function for $w_{c,\gamma}(x)$

Due to the multiplicative property of the Szegő function, we have that for $w_{c,\gamma}$ defined in (1.4.1),

$$D(z, w_{c,\gamma}) = D(z, w_{1,\gamma}) D(z, \Xi_c), \tag{5.1.1}$$

where, using the definition (2.2.1),

$$D(z, h) = \exp\left(\sqrt{1-z^2}\,\mathcal{C}\left(\frac{\log h(t)}{\sqrt{1-t^2}}\right)(z)\right), \tag{5.1.2}$$

$D(z, w_{1,\gamma})$ was given in (1.1.6) and,

$$D(z, \Xi_c) = c\ \exp\left(-\lambda \log\left(\frac{1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)}}{z - x_0}\right)\right), \tag{5.1.3}$$

with

$$\lambda \overset{\text{def}}{=} i\frac{\log c}{\pi} \tag{5.1.4}$$

where we take the main branches of $(z-1)^{\alpha/2}$, $(z+1)^{\beta/2}$, $(z-x_0)^{\gamma/2}$, $\sqrt{z^2-1}$ that are positive for $z > 1$, as well as the main branch of the logarithm, and $\left(\sqrt{1-z^2}\right)_+ > 0$ on $(-1, 1)$. Finally,

$$\begin{aligned} D(\infty, \Xi_c) &= c \exp\left[-\lambda \log\left(-x_0 - i\sqrt{1-x_0^2}\right)\right] \\ &= c \exp\left(i\lambda \arccos\left(-x_0\right)\right) \\ &= c \exp\left(i\lambda\left[\frac{\pi}{2} - \arcsin\left(-x_0\right)\right]\right) \\ &= c^{1/2} \exp\left(i\lambda \arcsin x_0\right) \end{aligned}$$

and as $\varphi(z) \sim 2z$, as $z \to \infty$, and from (5.1.1) we obtain that

$$D_\infty \stackrel{\text{def}}{=} D(\infty, w_{c,\gamma}) = \sqrt{c}\, D(\infty, h)\, 2^{-(\alpha+\beta+\gamma)/2}\, e^{i\lambda \arcsin x_0} > 0. \tag{5.1.5}$$

In comparison with the case $c = 1$, for $c \neq 1$ there is an extra factor, corresponding to the Szegő function of the pure jump Ξ_c.

5.2 Boundary behavior of the Szegő function

Let us study the boundary behavior of the Szegő function on the interval. By (1.1.6),

$$\lim_{\substack{z\to x\in(x_0,1),\\ \operatorname{Im} z>0}} D(z, w_{1,\gamma}) = e^{\frac{\alpha\pi i}{2}}\,(1-x)^{\frac{\alpha}{2}}\,(1+x)^{\frac{\beta}{2}}\,(x-x_0)^{\frac{\gamma}{2}}\,\varphi_+(x)^{-\frac{\alpha+\beta+\gamma}{2}} \lim_{\substack{z\to x\in(x_0,1),\\ \operatorname{Im} z>0}} D(z,h),$$

$$\lim_{\substack{z\to x\in(-1,x_0),\\ \operatorname{Im} z>0}} D(z, w_{1,\gamma}) = e^{\frac{\alpha+\gamma}{2}\pi i}\,(1-x)^{\frac{\alpha}{2}}\,(1+x)^{\frac{\beta}{2}}\,(x_0-x)^{\frac{\gamma}{2}}\,\varphi_+(x)^{-\frac{\alpha+\beta+\gamma}{2}} \lim_{\substack{z\to x\in(-1,x_0),\\ \operatorname{Im} z>0}} D(z,h),$$

where

$$\varphi_+(x) = x + i\sqrt{1-x^2} = e^{i\arccos x}, \quad x\in(-1,1), \tag{5.2.1}$$

with $(1-x)^{\alpha/2}$, $(1+x)^{\beta/2}$ and $\sqrt{1-x^2}$ positive for $x \in (-1,1)$, and $(x-x_0)^{\gamma/2}$ positive for $x \in (x_0, 1)$ and $(x_0 - x)^{\gamma/2}$ positive for $x \in (-1, x_0)$.

Analogously,

$$\lim_{\substack{z\to x\in(x_0,1),\\ \operatorname{Im} z<0}} D(z, w_{1,\gamma}) = e^{-\pi i\alpha/2}\,(1-x)^{\alpha/2}\,(1+x)^{\beta/2}\,(x-x_0)^{\gamma/2}\,\varphi_+^{\frac{\alpha+\beta+\gamma}{2}}(x) \lim_{\substack{z\to x\in(x_0,1),\\ \operatorname{Im} z<0}} D(z,h),$$

$$\lim_{\substack{z\to x\in(-1,x_0),\\ \operatorname{Im} z<0}} D(z, w_{1,\gamma}) = e^{-\frac{\alpha+\gamma}{2}\pi i}\,(1-x)^{\alpha/2}\,(1+x)^{\beta/2}\,(x_0-x)^{\gamma/2}\,\varphi_+^{\frac{\alpha+\beta+\gamma}{2}}(x) \lim_{\substack{z\to x\in(-1,x_0),\\ \operatorname{Im} z<0}} D(z,h).$$

We can be more specific about the limit values of $D(z,h)$ on $(-1,1)$ if we use the Sokhotskii-Plemelj formulas (2.1.7):

$$\mathcal{C}_\pm\left(\frac{\log h(t)}{\sqrt{1-t^2}}\right)(z) = \pm\frac{1}{2}\,\frac{\log h(z)}{\sqrt{1-z^2}} + \frac{1}{2\pi i}⨍_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}}\,\frac{dt}{t-z},$$

where $⨍$ is the integral understood in terms of its principal value. So, if we define $\hbar(x)$ as in (5.2.5), then using (5.1.2) we get

$$\lim_{\substack{z\to x\in(-1,1),\\ \operatorname{Im} z>0}} D(z,h) = \sqrt{h(x)}\, e^{-i\hbar(x)},$$
$$\lim_{\substack{z\to x\in(-1,1),\\ \operatorname{Im} z<0}} D(z,h) = \sqrt{h(x)}\, e^{i\hbar(x)}.$$

Observe that $\hbar(x)$ is real-valued on $(-1,1)$, so that $\left|e^{\pm i\hbar(x)}\right| = 1$. So, if we define on $(-1,1)$ the real-valued function $\Phi(x)$ and $\widehat{\Phi}(x)$ as in (5.2.4) and (5.2.3) then

$$\lim_{\substack{z\to x\in(-1,1),\\ \pm\operatorname{Im} z>0}} D(z, w_{1,\gamma}) = \sqrt{w_{1,\gamma}(x)}\, \exp\left(\pm i\widehat{\Phi}(x)\right).$$

On the other hand, it is easy to check that with the specified selection of the branch of the square root,

$$z \mapsto \frac{1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)}}{z - x_0}$$

is a conformal mapping of $\mathbb{C} \setminus [-1,1]$ onto the lower half plane, such that the lower shore of $(-1,1)$ is mapped onto itself, while the upper boundary is mapped onto $(-\infty,-1) \cup (1,\infty)$. In particular,

$$\lim_{\substack{z \to x \in (x_0,1), \\ \operatorname{Im} z \neq 0}} \arg \left(\frac{1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)}}{z - x_0} \right) = 0,$$

$$\lim_{\substack{z \to x \in (-1,x_0), \\ \operatorname{Im} z \neq 0}} \arg \left(\frac{1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)}}{z - x_0} \right) = -\pi.$$

Hence,

$$\begin{aligned} \lim_{\substack{z \to x \in (x_0,1), \\ \pm \operatorname{Im} z > 0}} D(z, \Xi_c) &= c \exp \left(-\lambda \log \left| \frac{1 - xx_0 \pm \sqrt{(1-x^2)\left(1-x_0^2\right)}}{x - x_0} \right| \right) \\ &= c \exp \left(\mp\lambda \log \left| \frac{1 - xx_0 + \sqrt{(1-x^2)\left(1-x_0^2\right)}}{x - x_0} \right| \right), \end{aligned}$$

with $\sqrt{1-x^2} > 0$ on $(-1,1)$. Taking into account that $e^{-\lambda\pi i} = c$, we also get

$$\begin{aligned} \lim_{\substack{z \to x \in (-1,x_0), \\ \pm \operatorname{Im} z > 0}} D(z, \Xi_c) &= \exp \left(-\lambda \log \left| \frac{1 - xx_0 \pm \sqrt{(1-x^2)\left(1-x_0^2\right)}}{x - x_0} \right| \right) \\ &= \exp \left(\mp\lambda \log \left| \frac{1 - xx_0 + \sqrt{(1-x^2)\left(1-x_0^2\right)}}{x - x_0} \right| \right). \end{aligned}$$

Both identities can be summarized by

$$\lim_{\substack{z \to x \in (-1,x_0) \cup (x_0,1), \\ \pm \operatorname{Im} z > 0}} D(z, \Xi_c) = \sqrt{\Xi_c(x)} \exp \left(\mp i \, \frac{\log c}{\pi} \, \log \left| \frac{1 - xx_0 + \sqrt{(1-x^2)\left(1-x_0^2\right)}}{x - x_0} \right| \right).$$

In order to clarify the local behavior of $D(z, \Xi_c)$ at the origin we observe that for $z \in \mathbb{C} \setminus (-\infty, 1]$ the function $D(z, \Xi_c)$ coincides with

$$c \exp \left(-\lambda \log(1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)}) + \lambda \log(z - x_0) \right),$$

if we take there the main branch of $\log(z - x_0)$, so that

$$e^{-\lambda \log(z - x_0)} D\left(z, \Xi_c\right) = c \exp \left(-\lambda \log \left(1 - zx_0 - i\sqrt{(z^2-1)\left(1-x_0^2\right)} \right) \right).$$

Since

$$\lim_{\substack{z\to x_0,\\ \operatorname{Im} z>0}} \log\left(1 - zx_0 - i\sqrt{\left(z^2-1\right)\left(1-x_0^2\right)}\right) = \log(2)\,,$$

it yields

$$D\left(z,\Xi_c\right) = c^{1+\frac{i}{\pi}\log\left(\frac{z-x_0}{2\left(1-x_0^2\right)}\right)}\left[1+o\left(1\right)\right],\ \text{as } z\to x_0,\ \operatorname{Im} z>0.$$

The case $\operatorname{Im} z<0$ can be deduced using the symmetry of $D\left(\cdot, w_{c,\gamma}\right)$ with respect to $\mathbb{R}$.

We can summarize our findings in the following lemma:

Lemma 5.1 *The Szegő function $D(\cdot,w)$ for the weight $w_{1,\gamma}$ defined in (7.2.2) exhibits the following boundary behavior:*

$$\lim_{\substack{z\to x\in(-1,1),\\ \pm\operatorname{Im} z>0}} D(z,w_{1,\gamma}) = \sqrt{w_{1,\gamma}(x)}\,\exp\left(\pm i\widehat{\Phi}(x)\right), \tag{5.2.2}$$

with

$$\widehat{\Phi}\left(x\right) \stackrel{def}{=} \begin{cases} \Phi\left(x\right)+\frac{\pi\gamma}{2}, & -1<x<x_0 \\ \Phi\left(x\right), & x_0<x<1 \end{cases} \tag{5.2.3}$$

$$\Phi\left(x\right) \stackrel{def}{=} \frac{\pi\alpha}{2} - \frac{\alpha+\beta+\gamma}{2}\arccos x - \hbar(x),\ x\in\left(-1,1\right), \tag{5.2.4}$$

$$\hbar(x) \stackrel{def}{=} \frac{\sqrt{1-t^2}}{2\pi}\fint_{-1}^{1}\frac{\log h\left(t\right)}{\sqrt{1-t^2}}\frac{dt}{t-x},\ x\in\left(-1,1\right), \tag{5.2.5}$$

Furthermore, for the step function Ξ_c,

$$\lim_{\substack{z\to x\in(-1,x_0)\cup(x_0,1),\\ \pm\operatorname{Im} z>0}} D(z,\Xi_c) = \sqrt{\Xi_c(x)}\,\exp\left(\mp i\,\frac{\log c}{\pi}\,\log\left|\frac{1-x_0x+\sqrt{\left(1-x^2\right)\left(1-x_0^2\right)}}{x-x_0}\right|\right),$$

and

$$D\left(z,\Xi_c\right) = c^{1\pm\frac{i}{\pi}\log\left(\frac{z-x_0}{2(1-x_0^2)}\right)}\left[1+o\left(1\right)\right],\ \textit{as } z\to x_0,\ \pm\operatorname{Im} z>0. \tag{5.2.6}$$

Obviously, the boundary behavior of the Szegő function $D(\cdot,w_{c,\gamma})$ at $(-1,1)$ can be deduced from this Lemma and (5.1.1).

Chapter 6

Riemann-Hilbert problem for the Confluent Hypergeometric function

In this chapter, we present the Riemann Hilbert problem (RHP) which has confluent hypergeometric functions as a solution. This problem appears near a step-like singularities in the bulk of the support of the measure, being independent of the weight considered (see [23], [26] and [22], [13]). The solution obtained here is used to solve the local parametrix at x_0 performed in the steepest descent method, in Chapter 7.

First we will find a solution of a RH problem. After that we will prove the uniqueness given by its asymptotic expansion as $\zeta \to \infty$.

6.1 Associated Riemann-Hilbert problem

Set $\Gamma \overset{\text{def}}{=} \bigcup_{j=1}^{8} \Gamma_j$ a system of unbounded oriented straight lines converging at the origin, like in Fig. 6.1.1 (don't confuse this system of straight lines Γ with the gamma function denoted by $\Gamma(\cdot)$).

More precisely,

$$\Gamma_1 = \left\{t\, e^{i\pi/2} : t > 0\right\}, \Gamma_2 = \left\{t\, e^{3i\pi/4} : t > 0\right\}, \Gamma_3 = \{-t : t > 0\}, \Gamma_4 = \left\{t\, e^{5i\pi/4} : t > 0\right\}, \tag{6.1.1}$$

$$\Gamma_5 = \left\{t\, e^{3i\pi/2} : t > 0\right\}, \Gamma_6 = \left\{t\, e^{-i\pi t/4} : t > 0\right\}, \Gamma_7 = \{t : t > 0\}, \Gamma_8 = \left\{e^{i\pi t/4} : t > 0\right\}.$$

These lines split the plane into 8 sectors, enumerated anti-clockwise from ① to ⑧ as in Fig. 6.1.1.

Consider the, 2×2 matrix valued function, RH problem for $\mathbf{\Psi}$, that satisfies the following conditions:

(Ψ1) $\mathbf{\Psi}$ is analytic in $\mathbb{C} \setminus \Gamma$.

(Ψ2) for $k = 1, \cdots, 8$, $\mathbf{\Psi}$ satisfies the jump relation $\mathbf{\Psi}_+(\zeta) = \mathbf{\Psi}_-(\zeta)\mathbf{J}_k$ as $\zeta \in \Gamma_k$, with

$$\mathbf{J}_1 = \begin{pmatrix} 0 & e^{-\lambda\pi i} \\ -e^{\lambda\pi i} & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} 1 & 0 \\ e^{-(\gamma-\lambda)\pi i} & 1 \end{pmatrix}, \quad \mathbf{J}_3 = \mathbf{J}_7 = \begin{pmatrix} e^{i\gamma\pi/2} & 0 \\ 0 & e^{-i\gamma\pi/2} \end{pmatrix},$$

$$\mathbf{J}_4 = \begin{pmatrix} 1 & 0 \\ e^{(\gamma-\lambda)\pi i} & 1 \end{pmatrix}, \mathbf{J}_5 = \begin{pmatrix} 0 & e^{\lambda\pi i} \\ -e^{-\lambda\pi i} & 0 \end{pmatrix}, \mathbf{J}_6 = \begin{pmatrix} 1 & 0 \\ e^{-(\gamma+\lambda)\pi i} & 1 \end{pmatrix}, \mathbf{J}_8 = \begin{pmatrix} 1 & 0 \\ e^{(\gamma+\lambda)\pi i} & 1 \end{pmatrix},$$

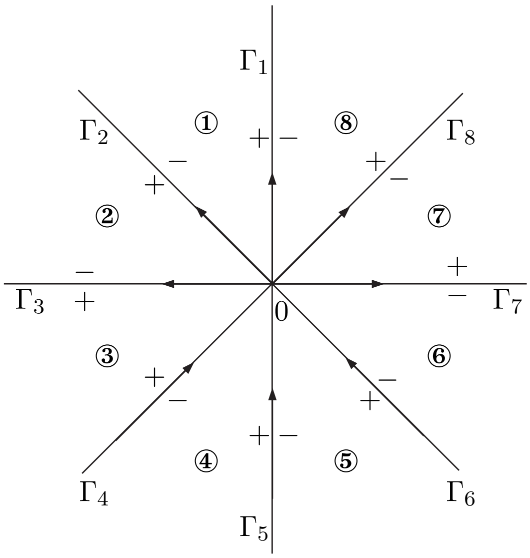

FIGURE 6.1.1: Auxiliary contours Γ.

where $\lambda = i\frac{\log c}{\pi}$, as in (5.1.4).

(Ψ3) as $\zeta \to \infty$, $\mathbf{\Psi}(\zeta) = \left[\mathbf{I} + \mathcal{O}\left(\frac{1}{|\zeta|}\right)\right] \zeta^{-\lambda\sigma_3} e^{\frac{-\zeta\sigma_3}{2}} \begin{cases} e^{-i\pi\lambda\sigma_3} e^{\pm\frac{\gamma}{4}\pi i\sigma_3}, & \text{as} \operatorname{Re}\zeta > 0, \\ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) e^{\pm\frac{\gamma}{4}\pi i\sigma_3}, & \text{as} \operatorname{Re}\zeta < 0, \end{cases}$ for $\pm \operatorname{Im}\zeta > 0$,

(Ψ4) the behavior of $\mathbf{\Psi}$ as $\zeta \to 0$ is

- for $\gamma < 0$:
$$\mathbf{\Psi}(\zeta) = \mathcal{O}\left(|\zeta|^{\gamma/2}\right),$$

- for $\gamma = 0$
$$\mathbf{\Psi}(\zeta) = \begin{cases} \mathcal{O}\left(\log|\zeta|\right), & \text{for } \zeta \in ① \cup ④ \cup ⑤ \cup ⑧, \\ \mathcal{O}\begin{pmatrix} 1 & \log|\zeta| \\ 1 & \log|\zeta| \end{pmatrix}, & \text{for } \zeta \in ② \cup ③ \cup ⑥ \cup ⑦, \end{cases}$$

- for $\gamma > 0$:
$$\mathbf{\Psi}(\zeta) = \begin{cases} \mathcal{O}\left(|\zeta|^{-\gamma/2}\right), & \text{for } \zeta \in ① \cup ④ \cup ⑤ \cup ⑧, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{\gamma/2} & |\zeta|^{-\gamma/2} \\ |\zeta|^{\gamma/2} & |\zeta|^{-\gamma/2} \end{pmatrix}, & \text{for } \zeta \in ② \cup ③ \cup ⑥ \cup ⑦. \end{cases}$$

The solution for $\mathbf{\Psi}$ is constructed, explicitly, using the confluent hypergeometric functions G and H defined by

$$G(a, b; \zeta) \stackrel{\text{def}}{=} \zeta^{b/2}\phi\,(a, b+1; \zeta)\, e^{-\zeta/2}, \quad H(a, b; \zeta) \stackrel{\text{def}}{=} \zeta^{b/2}\psi\,(a, b+1; \zeta)\, e^{-z/2}. \tag{6.1.2}$$

They form a basis of solutions of the confluent equation (see e.g. [2, formula (13.1.35)])

$$4\zeta^2 w'' + 4\zeta w' + \left[-b^2 + 2\zeta\,(b+1-2a) - \zeta^2\right] w = 0. \tag{6.1.3}$$

We can relate G and H with the Whittaker functions: $G(a,b;z)=M_{\kappa,\varrho}(z)/\sqrt{z}$ and $H(a,b;z)=W_{\kappa,\varrho}(z)/\sqrt{z}$ with $\varrho=b/2$ and $\kappa=1/2+\varrho-a$ (see [2, formula (13.1.32)]). Remark that from (6.1.2) and (4.3.1),

$$G\left(a,0;z\right)={}_1F_1\left(a;1;z\right)e^{-z/2}=e^{-z/2}\sum_{k=0}^{\infty}\frac{(a)_k}{(k!)^2}\,z^k, \tag{6.1.4}$$

is an entire function of z for any value of the parameter $a\in\mathbb{C}$, and $G(a,0;0)=1$, and $G\left(1,0;z\right)=e^{z/2}$.

Observe that, in general, $G(a,b;\zeta)$ and $H(a,b;\zeta)$ from (6.1.3) are multiple-valued, and we take its **principal branch** in $-\frac{\pi}{2}<\arg\zeta<\frac{3\pi}{2}$. For these values of ζ we define

$$\widehat{\mathbf{\Psi}}\left(\zeta\right)\stackrel{\text{def}}{=}\begin{pmatrix}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right)\\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)\end{pmatrix}e^{\frac{\gamma\pi i}{4}\sigma_3}.$$

By (Ψ2), if we set

$$\mathbf{\Psi}\left(\zeta\right)\stackrel{\text{def}}{=}\begin{cases}\widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_8\mathbf{J}_1, & \text{for } \zeta\in①;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_8\mathbf{J}_1\mathbf{J}_2, & \text{for } \zeta\in②;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_8\mathbf{J}_1\mathbf{J}_2\mathbf{J}_3, & \text{for } \zeta\in③;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_8\mathbf{J}_1\mathbf{J}_2\mathbf{J}_3\mathbf{J}_4^{-1}, & \text{for } \zeta\in④;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_7^{-1}\mathbf{J}_6, & \text{for } \zeta\in⑤;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_7^{-1}, & \text{for } \zeta\in⑥;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right), & \text{for } \zeta\in⑦;\\ \widehat{\mathbf{\Psi}}\left(\zeta\right)\mathbf{J}_8, & \text{for } \zeta\in⑧.\end{cases} \tag{6.1.5}$$

then $\mathbf{\Psi}$ has the jumps across Γ specified in (Ψ2). Explicitly, $\mathbf{\Psi}\left(\zeta\right)=$

$$\begin{pmatrix}c^{-1}H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right)\\ -c^{-1}\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right)\end{pmatrix}e^{-\frac{\gamma\pi i}{4}\sigma_3},\ \zeta\in①, \tag{6.1.6}$$

$$\begin{pmatrix}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{2}} & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right)\\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{2}} & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right)\end{pmatrix}e^{-\frac{\gamma\pi i}{4}\sigma_3},\ \zeta\in②, \tag{6.1.7}$$

$$\begin{pmatrix}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right)e^{-\frac{\gamma\pi i}{2}}\\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right)e^{-\frac{\gamma\pi i}{2}}\end{pmatrix}e^{-\frac{\gamma\pi i}{4}\sigma_3},\ \zeta\in③, \tag{6.1.8}$$

$$\begin{pmatrix}c\,H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-2\pi i}\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right)\\ -c\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-2\pi i}\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right)\end{pmatrix}e^{\frac{\gamma\pi i}{4}\sigma_3},\ \zeta\in④, \tag{6.1.9}$$

$$\begin{pmatrix}-\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\lambda+\frac{\gamma}{2}\right)}H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{\pi i}\right)e^{-\lambda\pi i} & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right)\\ H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{\pi i}\right)e^{-\lambda\pi i} & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)\end{pmatrix}e^{-\frac{\gamma\pi i}{4}\sigma_3},\ \zeta\in⑤, \tag{6.1.10}$$

$$\begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in ⑥, \tag{6.1.11}$$

$$\begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in ⑦, \tag{6.1.12}$$

$$\begin{pmatrix} -c^{-1}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\ c^{-1}\, H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in ⑧. \tag{6.1.13}$$

6.2 Proof of the solution

We will prove the solution in two times. First proving that (6.1.5) satisfies (Ψ1), (Ψ2), (Ψ4), and after determining the asymptotic expansion of (6.1.5) and proving (Ψ3).

Theorem 6.2.1 *The solution of the RH problem* (Ψ1), (Ψ2), (Ψ3), (Ψ4) *is given by (6.1.5) and* $\det \boldsymbol{\Psi}(z) = 1$, *for* $z \in \mathbb{C} \setminus \Gamma$.

Proof. If we take the branch cut across $\arg\zeta = -\pi/2$ oriented towards the origin (we consider $-\pi/2 < \arg\zeta < 3\pi/4$), we have that the matrix $\boldsymbol{\Psi}$ has on this cut the following jump (using 5.1.4):

$$\boldsymbol{\Psi}_+(\zeta) = \boldsymbol{\Psi}_-(\zeta)\mathbf{J}_5, \quad \zeta \in \Gamma_5, \tag{6.2.1}$$

$$\widehat{\boldsymbol{\Psi}}_+(\zeta) = \widehat{\boldsymbol{\Psi}}_-(\zeta)\begin{pmatrix} e^{i\pi\gamma} & -e^{-i\pi\lambda}+e^{i\pi\lambda}e^{-i\pi\gamma} \\ 0 & e^{-i\pi\gamma} \end{pmatrix}, \quad \zeta \in \Gamma_5. \tag{6.2.2}$$

Set

$$\widehat{\boldsymbol{\Psi}}_{11}(\zeta) = \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}\zeta^{\gamma/2}\phi\left(\lambda+\frac{\gamma}{2},\gamma+1;\zeta\right)e^{-\zeta/2}e^{i\pi\gamma/4},$$

$$\widehat{\boldsymbol{\Psi}}_{12}(\zeta) = -\zeta^{\gamma/2}\psi\left(\lambda+\frac{\gamma}{2},\gamma+1;\zeta\right)e^{-\zeta/2}e^{-i\pi\gamma/4}.$$

Then from (4.3.4) and (4.3.9) if follows that for $\zeta \in \Gamma_5$,

$$\begin{aligned}\left(\widehat{\boldsymbol{\Psi}}_{11}\right)_+(\zeta) &= \left(e^{2\pi i}\zeta\right)^{\gamma/2}\phi\left(\lambda+\frac{\gamma}{2},\gamma+1;e^{2\pi i}\zeta\right)e^{-\zeta/2}e^{i\pi\gamma/4}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} \\ &= e^{i\pi\gamma}\left(\widehat{\boldsymbol{\Psi}}_{11}\right)_-(\zeta),\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{\boldsymbol{\Psi}}_{12}\right)_+(\zeta) &= -\left(e^{2\pi i}\zeta\right)^{\gamma/2}\psi\left(\lambda+\frac{\gamma}{2},\gamma+1;e^{2\pi i}\zeta\right)e^{-\zeta/2}e^{-i\pi\gamma/4} \\ &= \frac{2\pi i e^{-\pi i}e^{-i\pi\gamma/4}\Gamma(\gamma+1)}{\Gamma\left(\lambda-\frac{\gamma}{2}\right)\Gamma(\gamma+1)\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)e^{i\pi\gamma/4}}\left(\widehat{\boldsymbol{\Psi}}_{11}\right)_-(\zeta) + e^{\pi i(\gamma-2\gamma-2)}\left(\widehat{\boldsymbol{\Psi}}_{12}\right)_-(\zeta) \\ &= \left[-e^{-i\pi\lambda}+e^{i\pi\lambda}e^{-i\pi\gamma}\right]\left(\widehat{\boldsymbol{\Psi}}_{11}\right)_-(\zeta) + e^{-i\pi\gamma}\left(\widehat{\boldsymbol{\Psi}}_{12}\right)_-(\zeta),\end{aligned}$$

in accordance with (6.2.1). Analogously, we can satisfy the second row of (6.2.1) if we take

$$\widehat{\boldsymbol{\Psi}}_{21} = \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\gamma+1\right)}\zeta^{\gamma/2}\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+1;\zeta\right)e^{-\zeta/2}e^{i\pi\frac{\gamma}{4}},$$

$$\widehat{\boldsymbol{\Psi}}_{22} = \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}\zeta^{\gamma/2}\psi\left(1+\lambda+\frac{\gamma}{2},\gamma+1;\zeta\right)e^{-\zeta/2}e^{-i\pi\frac{\gamma}{4}}.$$

By construction, $\boldsymbol{\Psi}$ satisfies the jumps relations in (Ψ2). Using formulas (4.3.5), (4.3.7) and (4.3.6), we can write explicitly the matrix $\boldsymbol{\Psi}$ in all regions. Since the local behavior of $\psi(a,b;z)$ depends only on the value of the parameter b (see (4.3.22)–(4.3.24)), by construction, all rows of $\widehat{\boldsymbol{\Psi}}$ have the same asymptotics as $\zeta \to 0$. Hence, it is sufficient to analyze the first row.

From formulas (4.3.22)–(4.3.24) it follows that for $\zeta \in$ ⑦, $\widehat{\boldsymbol{\Psi}}$ has the behavior described in (Ψ4), as $\zeta \to 0$. Indeed, for $\gamma > 0$,

$$\widehat{\boldsymbol{\Psi}}_{11} = \mathcal{O}\left(\zeta^{\gamma/2}\right), \quad \widehat{\boldsymbol{\Psi}}_{12} = \mathcal{O}\left(\zeta^{-\gamma/2}\right);$$

for $\gamma = 0$,

$$\widehat{\boldsymbol{\Psi}}_{11} = \mathcal{O}\left(1\right), \quad \widehat{\boldsymbol{\Psi}}_{12} = \mathcal{O}\left(\log\zeta\right);$$

and for $-1 < \gamma < 0$,

$$\widehat{\boldsymbol{\Psi}}_{11} = \mathcal{O}\left(\zeta^{\gamma/2}\right), \quad \widehat{\boldsymbol{\Psi}}_{12} = \mathcal{O}\left(\zeta^{\gamma/2}\right).$$

Analogously we can check that $\boldsymbol{\Psi}$ satisfies (Ψ4) in all regions of the plane.

Finally, using formula (4.3.18),

$$\begin{vmatrix} \phi\left(a,b;\zeta\right) & \psi\left(a,b;\zeta\right) \\ \phi'\left(a,b;\zeta\right) & \psi'\left(a,b;\zeta\right) \end{vmatrix} = -\frac{\Gamma\left(b\right)e^{\zeta}}{\zeta^{b}\Gamma\left(a\right)},$$

as well as the differential relations (4.3.17) and (4.3.15), we easily get that

$$\begin{vmatrix} \frac{\Gamma(b-a)}{\Gamma(b)}\zeta^{\frac{b-1}{2}}\phi\left(a,b;\zeta\right)e^{-\zeta/2}e^{\frac{i\pi\gamma}{4}} & -\zeta^{\frac{b-1}{2}}\psi\left(a,b;\zeta\right)e^{-\zeta/2}e^{-\frac{i\pi\gamma}{4}} \\ \frac{\Gamma(1+a)}{\Gamma(b)}\zeta^{\frac{b-1}{2}}\phi\left(a+1,b;\zeta\right)e^{-\zeta/2}e^{\frac{i\pi\gamma}{4}} & \frac{\Gamma(1+a)}{\Gamma(-(1+a-b))}\zeta^{\frac{b-1}{2}}\psi\left(a+1,b;\zeta\right)e^{-\zeta/2}e^{-\frac{i\pi\gamma}{4}} \end{vmatrix} = 1.$$

This implies that det $\widehat{\boldsymbol{\Psi}} = 1$, and, by construction, det $\boldsymbol{\Psi} = 1$.

The condition (Ψ4) will be verified, detailed, bellow, which concludes the proof. ■

Remark 6.1 *To find the solution we work first with the condition* (Ψ2). *The conditions* (Ψ3), (Ψ4) *just gives uniqueness. Considering functions with the form* $z^c\psi(a,b;z)$, (Ψ2) *and* (Ψ4) *will determine the relation between parameter* b *and* c. *The parameter* a *just acts on the behavior at* ∞, *and it is* (Ψ3) *that determines* a.

In fact (Ψ3) *is fixed aftermost. Other analytic functions and constants that appear in the solution are to obtain* det $= 1$ *and to satisfy* (Ψ3).

In [13] they solve an equivalent problem and it is given another solution.

Now we will study the asymptotic behavior of $\boldsymbol{\Psi}$ at infinity to verify the condition (Ψ3). This behavior is needed, also, in order to construct the analytic function $\mathbf{E}_n$ in (7.4.4) and the asymptotic expansion for $\mathbf{R}$ and, consequently, for $\mathbf{Y}$. Let us introduce the notation (see the definition of $(a)_n$ in (4.2.8))

$$\mu_n \stackrel{\text{def}}{=} \mu_n(\lambda) = \frac{\left(\lambda + \frac{\gamma}{2}\right)_n \left(\lambda - \frac{\gamma}{2}\right)_n}{n!}, \tag{6.2.3}$$

$$\tau_\lambda \stackrel{\text{def}}{=} \frac{\Gamma\left(-\lambda + \frac{\gamma}{2}\right)}{\left(-\frac{\gamma}{2} - \lambda\right)\Gamma\left(\frac{\gamma}{2} + \lambda\right)} = -\frac{\Gamma\left(\frac{\gamma}{2} - \lambda\right)}{\Gamma\left(\frac{\gamma}{2} + \lambda + 1\right)}; \tag{6.2.4}$$

$$\mu \stackrel{\text{def}}{=} -\mu_1 = -\left(\lambda^2 - \frac{\gamma^2}{4}\right) = \frac{\log^2 c}{\pi^2} + \frac{\gamma^2}{4} \in \mathbb{R}^+ \tag{6.2.5}$$

observe that

$$\tau_{-\lambda} = \overline{\tau_\lambda}, \quad \mu_n(-\lambda) = \overline{\mu_n(\lambda)}. \tag{6.2.6}$$

Lemma 6.1 *As $\zeta \to \infty$, $\zeta \in \mathbb{C} \setminus \Gamma$,*

$$\mathbf{\Psi}(\zeta) = \left[\mathbf{I} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} (-1)^n \mu_n & n\tau_\lambda \overline{\mu_n} \\ (-1)^n n\overline{\tau_\lambda}\mu_n & \overline{\mu_n} \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] \zeta^{-\lambda\sigma_3} e^{\frac{-\zeta\sigma_3}{2}} M^{-1}(\zeta) \tag{6.2.7}$$

with μ_n defined by (6.2.3), τ_λ defined by (6.2.4), $\lambda = i\log(c)/\pi$, and

$$M(\zeta) \stackrel{def}{=} \begin{cases} e^{\frac{\gamma}{4}\pi i\sigma_3} e^{-\lambda\pi i\sigma_3}, & \frac{\pi}{2} < \arg\zeta < \pi, \\ e^{-\frac{\gamma}{4}\pi i\sigma_3} e^{-\lambda\pi i\sigma_3}, & \pi < \arg\zeta < \frac{3\pi}{2}, \\ e^{\frac{\gamma}{4}\pi i\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg\zeta < 0, \\ e^{-\frac{\gamma}{4}\pi i\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 0 < \arg\zeta < \frac{\pi}{2}, \end{cases}$$

where we use the main branch of $\zeta^{-\lambda} = e^{-\lambda\log\zeta}$ with the cut along $i\mathbb{R}^-$.

Proof. We use the classical formulas (4.3.19) and (4.3.20) for the confluent hypergeometric functions. If we take $b = \gamma + 1$, and multiply ϕ and ψ by $(z^{\gamma/2}e^{-z/2})$, using (6.1.2), we have that, as $|z| \to \infty$,

$$G(a, \gamma; z) = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(a)} z^{a-\gamma/2} \left[\frac{1}{z}\left(1 + \sum_{n=1}^{R-1} \frac{(\gamma+1-a)_n(1-a)_n}{n!z^n} + \mathcal{O}\left(|z|^{-R}\right)\right)\right] e^{z/2}, & \operatorname{Re} z > 0, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-a)} e^{a\pi i} z^{\gamma/2-a} \left[1 + \sum_{n=1}^{R-1} \frac{(a)_n(a-\gamma)_n}{(-1)^n n! z^n} + \mathcal{O}\left(|z|^{-R}\right)\right] e^{-z/2}, & \operatorname{Re} z < 0, \end{cases} \tag{6.2.8}$$

$$H(a, \gamma; z) = z^{\gamma/2-a} \left[1 + \sum_{n=1}^{R-1} \frac{(a)_n(a-\gamma)_n}{(-1)^n n! z^n} + \mathcal{O}\left(|z|^{-R}\right)\right] e^{-z/2}. \tag{6.2.9}$$

Replacing these expansions in the expression for $\mathbf{\Psi}$ for $\zeta \in$ ①, $\frac{\pi}{2} < \arg\zeta < \frac{3\pi}{4}$ and $\frac{-\pi}{2} < \arg\left(e^{-\pi i}\zeta\right) < \frac{-\pi}{4}$, we get for $|\zeta| \to \infty$,

$$\mathbf{\Psi}(\zeta) = \begin{pmatrix} \zeta^{-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\lambda+\frac{\gamma}{2}\right)_n \left(\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2} e^{\lambda\pi i} & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} \left(e^{-\pi i}\zeta\right)^{-1+\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1-\lambda+\frac{\gamma}{2}\right)_n \left(1-\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n(-1)^n \zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{\zeta/2} \\ \frac{-\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} \zeta^{-1-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1+\lambda+\frac{\gamma}{2}\right)_n \left(1+\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2} e^{\lambda\pi i} & \left(e^{-\pi i}\zeta\right)^{\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\frac{\gamma}{2}-\lambda\right)_n \left(-\frac{\gamma}{2}-\lambda\right)_n}{(-1)^n(-1)^n \zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{\zeta/2} \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3},$$

which can be rewritten using notation (6.2.3)–(6.2.4) as

$$= \left[\mathbf{I} + \begin{pmatrix} \left[\sum_{n=1}^{R-1} (-1)^n \frac{\mu_n}{\zeta^n}\right] & \tau_\lambda \left[\sum_{n=1}^{R-1} \frac{n\overline{\mu_n}}{\zeta^n}\right] \\ \overline{\tau_\lambda} \left[\sum_{n=1}^{R-1} (-1)^n \frac{n\mu_n}{\zeta^n}\right] & \left[\sum_{n=1}^{R-1} \frac{\overline{\mu_n}}{\zeta^n}\right] \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right) \right] \left(\zeta^{-\lambda} e^{-\zeta/2} e^{\lambda\pi i} e^{-\frac{\gamma\pi i}{4}}\right)^{\sigma_3}.$$

This yields (6.2.7) for $\pi/2 < \zeta < 3\pi/4$; this expansion is also valid for $\zeta \in$ ②. A comparison of (6.1.7) with (6.1.8) shows that the behavior for $\zeta \in$ ③, $\pi < \arg\zeta < \frac{5\pi}{4}$, can be obtained from the expansion in ② by multiplying by $e^{i\frac{\gamma}{2}\pi\sigma_3}$, which again yields 6.2.7 for $\pi < \zeta < 5\pi/4$. It is easy to see that asymptotics in ③ is also valid in ④.

Using (6.1.10), (6.2.8), (6.2.9) and comparing the expression for $\mathbf{\Psi}$ in ① and ⑤, we conclude that for $\zeta \in$ ⑤, $-\frac{\pi}{2} < \arg\zeta < -\frac{\pi}{4}$ ($\frac{\pi}{2} < \arg(\zeta) e^{\pi i} < \frac{3\pi}{4}$ and $\operatorname{Re}\zeta > 0$), as $|\zeta| \to \infty$,

$$\mathbf{\Psi}(\zeta) = \begin{pmatrix} -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} \left(e^{\pi i}\zeta\right)^{-1+\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1-\lambda+\frac{\gamma}{2}\right)_n \left(1-\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n(-1)^n\zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\lambda\pi i} e^{\zeta/2} & -\zeta^{-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\lambda+\frac{\gamma}{2}\right)_n \left(\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n!\zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2} \\ \left(e^{\pi i}\zeta\right)^{\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\frac{\gamma}{2}-\lambda\right)_n \left(-\frac{\gamma}{2}-\lambda\right)_n}{(-1)^n(-1)^n\zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\lambda\pi i} e^{\zeta/2} & -\frac{-\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} \zeta^{-1-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1+\lambda+\frac{\gamma}{2}\right)_n \left(1+\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n!\zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2} \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}$$

$$= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} n\tau_\lambda \overline{\mu_n} & -(-1)^n \mu_n \\ \overline{\mu_n} & -(-1)^n n\overline{\tau_\lambda}\mu_n \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right) \right]$$

$$\times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \zeta^{\lambda\sigma_3} e^{-\frac{\gamma}{4}\pi i\sigma_3} e^{\frac{\zeta}{2}\sigma_3}$$

$$= \left[\mathbf{I} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} (-1)^n \mu_n & n\tau_\lambda \overline{\mu_n} \\ (-1)^n n\overline{\tau_\lambda}\mu_n & \overline{\mu_n} \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right) \right] \zeta^{-\lambda\sigma_3} e^{\frac{\gamma}{4}\pi i\sigma_3} e^{-\frac{\zeta}{2}\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This expression is valid in ⑥ as well. Finally, comparing (6.1.11) with (6.1.12) we see that the behavior for $\zeta \in$ ⑦, $0 < \arg\zeta < \frac{\pi}{4}$, corresponds to that in ⑥ times the constant factor $e^{i\frac{\gamma}{2}\pi\sigma_3}$, which yields (6.2.7). Since the asymptotics for $\zeta \in$ ⑧ is the same than in ⑦, this concludes the proof of Lemma. ■

Chapter 7

The Nonlinear Steepest Descent Method

We apply the Deift-Zhou method of steepest descent to the RH problem for $\mathbf{Y}$ formulated in section 3.1. The description of the main steps was given in section 3.2. Here we will give the detailed procedure. This Chapter is divided in seven sections corresponding to five main steps described in section 3.2, the sixth where we determine the asymptotic expansion of the solution $\mathbf{R}$, and the seventh, showing the relation between the $\mathbf{Y}$-RH problem and the $\mathbf{R}$-RH problem where we present the solution of $\mathbf{Y}$ in terms of the matrix $\mathbf{R}$, on each region of the plane. The fourth section is divided in two subsections, one for the the local analysis at the points ± 1, and, the second, for the local analysis at x_0 to get the RH problem for the Confluent Hypergeometric function, which was solved in Chapter 6.

7.1 First transformation: Y - T

In the first transformation we will do a normalization at infinity, changing the condition (Y3), that is non constant, for a “constant ”normalization condition. To do it we use the conformal mapping $\varphi(z) = z + \sqrt{z^2-1}$ (defined in (1.1.4)), from $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle, that behaves as $2z$ as $z \to \infty$, and satisfy $\varphi_+(x)\varphi_-(x) = 1$ on $(-1,1)$.

Set

$$\mathbf{T}(z) \stackrel{\text{def}}{=} 2^{n\sigma_3}\mathbf{Y}(z)\,\varphi(z)^{-n\sigma_3}. \tag{7.1.1}$$

The jump matrix for $\mathbf{T}$, $\mathbf{J_T}$, given in (T2) is obtained by:

$$\begin{aligned}(\mathbf{T}_-(x))^{-1}\mathbf{T}_+(x) &= \varphi_-(z)^{n\sigma_3}(\mathbf{Y}_-(z))^{-1}\,2^{-n\sigma_3}2^{n\sigma_3}\mathbf{Y}_+(z)\,\varphi_+(z)^{-n\sigma_3}\\ &= \varphi_-(z)^{n\sigma_3}\,\mathbf{J_Y}\varphi_+(z)^{-n\sigma_3},\end{aligned}$$

and the condition (T3) is $\mathbf{T}(z) \sim 2^{n\sigma_3}(\mathbf{I}+\mathcal{O}(1/z))\,z^{n\sigma_3}(2z)^{-n\sigma_3} = \mathbf{I}+\mathcal{O}(1/z)$. This transformation does not change the other conditions.

Then $\mathbf{T}$ is the unique solution of the following, equivalent to $\mathbf{Y}$-RHP, RH problem:

(T1) $\mathbf{T}$ is analytic in $\mathbb{C} \setminus [-1,1]$.

(T2) $\mathbf{T}_+(x) = \mathbf{T}_-(x)\mathbf{J_T}(x)$, as $x \in (-1,x_0) \cup (x_0,1)$, where

$$\mathbf{J_T}(x) = \begin{pmatrix} \varphi_+^{-2n}(x) & w_{c,\gamma}(x) \\ 0 & \varphi_-^{2n}(x) \end{pmatrix}.$$

(T3) $\mathbf{T}(z) = \mathbf{I} + \mathcal{O}(1/z)$, as $z \to \infty$;

(T4) $\mathbf{T}$ has the same asymptotic behavior as $\mathbf{Y}$ at ± 1 and x_0 .

7.2 Second transformation: T - S

In order to introduce a contour deformation we need to extend the definition of the weight of orthogonality to a neighborhood of the interval $[-1,1]$.

In the first transformation the jump matrix $\mathbf{J_T}$ has rapidly oscillating entries, because on $(-1,1)$, $|\varphi(x)| = 1$. To solve this the next transformation is based upon the factorization of the jump matrix for $\mathbf{T}$, $\mathbf{J_T}$, and $\varphi_+(x)\varphi_-(x) = 1$ on $(-1,1)$:

$$\begin{pmatrix} \varphi_+^{-2n} & w_{c.\gamma} \\ 0 & \varphi_-^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w_{c,\gamma}^{-1}\varphi_-^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 0 & w_{c,\gamma} \\ -1/w_{c,\gamma} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w_{c,\gamma}^{-1}\varphi_+^{-2n} & 1 \end{pmatrix}. \tag{7.2.1}$$

In order to perform this transformation we need to extend the definition of the weight of orthogonality to a neighborhood of the interval $[-1,1]$. By assumptions, h is a holomorphic function in a neighborhood U of $[-1,1]$, and positive on this interval. For any Jordan arc $\widetilde{\Sigma}$, intersecting $[-1,1]$ transversally at x_0 and dividing U into two connected components, we denote by Σ_5 its intersection with the upper half plane, and by Σ_6 its intersection with the lower half plane, oriented as shown in Figure 7.2.1. Contours $\Sigma_5 \cup \Sigma_6 \cup \mathbb{R}$ divide U into four open domains ("quadrants"), that we denote by $Q_\pm^{L,R}$ as depicted. Finally, let Q^L (resp., Q^R) be the connected component of $U \setminus \widetilde{\Sigma}$ containing -1 (resp., $+1$).

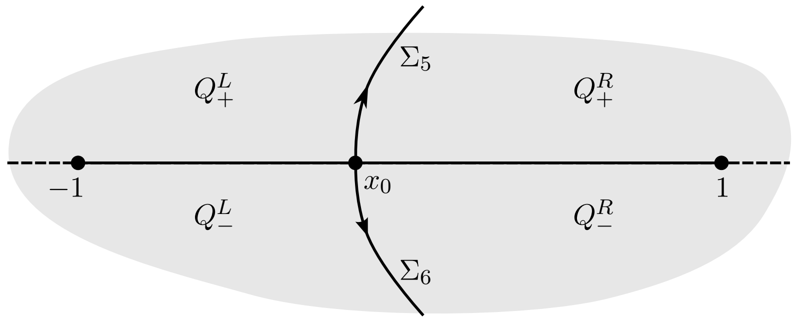

FIGURE 7.2.1: Division of the neighborhood of $[-1,1]$ in four regions.

Now we set

$$w(z) \stackrel{\text{def}}{=} h(z)\,(1-z)^\alpha\,(1+z)^\beta \times \begin{cases} (x_0 - z)^\gamma\,, & z \in Q^L \setminus (-\infty,-1]\,, \\ (z-x_0)^\gamma\,, & z \in Q^R \setminus [1,+\infty)\,, \end{cases} \tag{7.2.2}$$

where the principal branches of the power functions are taken. In this way, w is defined and holomorphic in

$$\widetilde{U} \stackrel{\text{def}}{=} U \setminus \left((-\infty,-1] \cup [1,+\infty) \cup \widetilde{\Sigma}\right), \tag{7.2.3}$$

and $w(x) > 0$ for $x \in (-1,1) \setminus x_0$. Setting

$$\Xi_c(z) = \begin{cases} 1 & z \in Q^L \\ c^2 & z \in Q^R, \end{cases}$$

we extend also

$$w_{c,\gamma}(z) \stackrel{\text{def}}{=} w(z)\,\Xi_c(z), \tag{7.2.4}$$

to a holomorphic function in $\widetilde{U}$.

We describe now the next transformation consisting in opening of lenses or contour deformation.

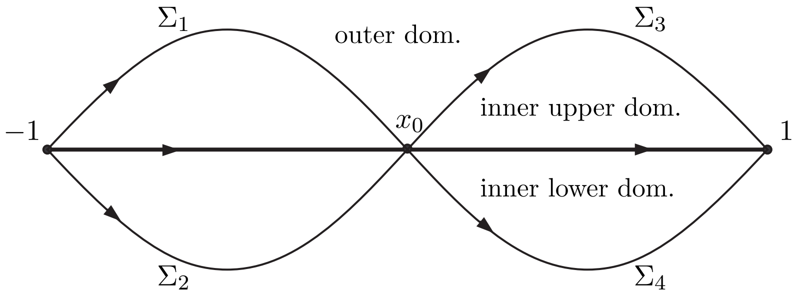

FIGURE 7.2.2: First lens opening.

We build the four contours Σ_i lying in $\widetilde{U}$ (except for their end points) such that Σ_1 and Σ_3 are in the upper half plane, and Σ_1 and Σ_2 are in the left half plane, and oriented "from -1 to 1" but now through x_0 (see Fig. 7.2.2). This construction defines three domains: the inner upper domain, bounded by $[-1,1]$ and the curves Σ_1 and Σ_3; the inner lower domain, bounded by $[-1,1]$ and the curves Σ_2 and Σ_4, and finally the outer domain, bounded by curves Σ_i and containing the infinity. Denote

$$\Sigma \stackrel{\text{def}}{=} [-1,1] \cup \bigcup_{k=1}^{4} \Sigma_k. \tag{7.2.5}$$

Using the matrix $\mathbf{T}$ from (7.1.1) we define

$$\mathbf{S}(z) \stackrel{\text{def}}{=} \begin{cases} \mathbf{T}(z), & \text{for } z \text{ in the outer domain,} \\ \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ -\left(w_{c,\gamma}(z)\varphi^{2n}(z)\right)^{-1} & 1 \end{pmatrix}, & \text{for } z \text{ in the inner upper domain,} \\ \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ \left(w_{c,\gamma}(z)\varphi^{2n}(z)\right)^{-1} & 1 \end{pmatrix}, & \text{for } z \text{ in the inner lower domain.} \end{cases} \tag{7.2.6}$$

Then $\mathbf{S}$ is a solution of a new RH problem, now with jumps on Σ, that are easy to compute explicitly.

(S1) $\mathbf{S}$ is analytic in $\mathbb{C} \setminus \Sigma$;

(S2) $\mathbf{S}_+(z) = \mathbf{S}_-(z)\mathbf{J}_{\mathbf{S}}(z)$ as $z \in \Sigma$, where:

$$\mathbf{J}_{\mathbf{S}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \left(w_{c,\gamma}(z)\varphi^{2n}(z)\right)^{-1} & 1 \end{pmatrix}, & \text{for } z \in \left(\bigcup_{i=1}^{4} \gamma_i\right) \setminus \{-1, x_0, 1\} \\ \begin{pmatrix} 0 & w_{c,\gamma}(z) \\ -1/w_{c,\gamma}(z) & 0 \end{pmatrix}, & \text{for } z \in (-1, x_0) \cup (x_0, 1). \end{cases}$$

(S3) $\mathbf{S}(z) = \mathbf{I} + \mathcal{O}(1/z)$, as $z \to \infty$;

(S4) if $(\zeta, s) \in \{(-1,\beta),(x_0,\gamma),(1,\alpha)\}$ then for $z \to \zeta$, $z \in \mathbb{C}\setminus[-1,1]$,

- for $-1 < s < 0$:

$$\mathbf{S}(z) = \mathcal{O}\begin{pmatrix} 1 & |z-\zeta|^s \\ 1 & |z-\zeta|^s \end{pmatrix}, \quad \text{as } z \to \zeta;$$

- for $s = \alpha = 0$ or $s = \beta = 0$,

$$\mathbf{S}(z) = \mathcal{O}\begin{pmatrix} \log|z-\zeta| & \log|z-\zeta| \\ \log|z-\zeta| & \log|z-\zeta| \end{pmatrix}, \quad \text{as } z \to \zeta;$$

and for $s = \gamma = 0$,

$$\mathbf{S}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix}, & \text{as } z \to x_0 \text{ from the outer domain,} \\ \mathcal{O}\begin{pmatrix} \log|z-x_0| & \log|z-x_0| \\ \log|z-x_0| & \log|z-x_0| \end{pmatrix}, & \text{as } z \to x_0 \text{ from the inner domains,} \end{cases}$$

- for $s > 0$:

$$\mathbf{S}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to \zeta \text{ from the outer domain;} \\ \mathcal{O}\begin{pmatrix} |z-\zeta|^{-s} & 1 \\ |z-\zeta|^{-s} & 1 \end{pmatrix}, & \text{as } z \to \zeta \text{ from the inner domains.} \end{cases}$$

Using standard arguments it can be proven that if $\mathbf{S}$ is the solution of (S1)-(S4) then $\mathbf{T}$ defined from (7.2.6) is solution of (T1)-(T4). The way to prove this is define $\mathbf{M}(z) = \widetilde{\mathbf{T}}(z)\,\mathbf{T}(z)$ where $\mathbf{T}$ is the unique solution of (T1)-(T4) and $\widetilde{\mathbf{T}}$ defined from (7.2.6), and, use the same arguments than the proof of the Theorem 3.1.1.

This step is called as opening lenses. The goal of this step is transform the oscillatory entries of the jump $\mathbf{J_T}$ into a jump with exponentially decaying entries like the first matrix of the $\mathbf{J_S}$ ($|\varphi(z)| > 1$ on $\mathbb{C}\setminus[-1,1]$).

Remark 7.1 *The way to opening lenses can be different depending of the weight that we are considering. For instance, for $w_{c,\gamma}(x)$ with $c \neq 1$ and/or $\gamma \neq 0$ the lenses need to be closed at x_0, because $w_{c,\gamma}(x)$ has a discontinuity (as $c \neq 1$ and $\gamma = 0$) or has a zero (as $\gamma \neq 0$). One problem is that we can not extend analytically $w_{c,\gamma}(x)$ to $w_{c,\gamma}(z)$ on all plane. But the great problem is that the Szegő function, $D(z)$, that perform the next step, doesn't satisfy D_+D_- at x_0 if x_0 is a zero; and, the behavior of $D(z)$ doesn't satisfies (S4) (compare S4 at x_0 with Lemma 5.1).*

For the weight $w_{1,0}(x)$ this problem doesn't happen and the lenses doesn't close at $(-1,1)$, see [31].

For the weight $w_{1,0}(x)$ and with $\alpha = \beta = \pm\frac{1}{2}$ because of its symmetry the system of lens could be different and the steepest descent method is simplified, this case was analyzed by us in [60] (see also [29]).

7.3 Outer parametrix

In the next step, which is also standard, we build the so-called outer parametrix for the RH problem for $\mathbf{S}$ in terms of the Szegő function $D(\cdot, w_{c,\gamma})$ corresponding to the weight $w_{c,\gamma}$. Namely, we construct the 2×2 matrix $\mathbf{N}$ that satisfies

(N1) $\mathbf{N}$ is analytic in $\mathbb{C} \setminus [-1, 1]$;

(N2) $\mathbf{N}_+(x) = \mathbf{N}_-(x) \begin{pmatrix} 0 & w_{c,\gamma}(x) \\ -w_{c,\gamma}(x)^{-1} & 0 \end{pmatrix}, \quad x \in (-1, x_0) \cup (x_0, 1)$;

(N3) $\mathbf{N}(z) = \mathbf{I} + \mathcal{O}(1/z)$, as $z \to \infty$.

The solution is standard and given by

$$\mathbf{N}(z) \stackrel{\text{def}}{=} D_\infty^{\sigma_3} \mathbf{A}(z) D(z, w_{c,\gamma})^{-\sigma_3}, \tag{7.3.1}$$

and we can find it in [60], [31], or [12]. We will describe the three factors appearing in the r.h.s. of (7.3.1). Matrix $\mathbf{A}$ is

$$\mathbf{A}(z) \stackrel{\text{def}}{=} \begin{pmatrix} A_{11} & A_{12} \\ -A_{12} & A_{11} \end{pmatrix} \tag{7.3.2}$$

where,

$$A_{11}(z) = \frac{a(z) + a(z)^{-1}}{2} = \frac{\varphi(z)^{1/2}}{\sqrt{2}\,(z^2 - 1)^{1/4}} = \frac{\varphi(z)}{\left(\varphi(z)^2 - 1\right)^{1/2}} \tag{7.3.3}$$

with $a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}$, and

$$A_{12}(z) = \frac{a(z) - a(z)^{-1}}{2i} = \frac{i\varphi(z)^{-1/2}}{\sqrt{2}\,(z^2 - 1)^{1/4}} = \frac{i}{\varphi(z)} A_{11}(z), \tag{7.3.4}$$

with the main branches of the roots, in such a way that A_{11} is analytic in $\mathbb{C} \setminus [-1, 1]$ with $A_{11}(z) \to 1$, and $A_{12}(z) \to 0$, as $z \to \infty$. The Szegő function $D(\cdot, w_{c,\gamma})$ for $w_{c,\gamma}$ is given in 5.1.1.

7.4 Local parametrices

We fix a $\delta > 0$ small enough such that discs $U_\zeta \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |x - \zeta| < \delta\}$, $\zeta \in \{-1, x_0, 1\}$ are mutually disjoint and lie in the domain of analyticity of the function h. We construct a 2×2 matrix-valued function $\mathbf{P}_\zeta$ in $U_\zeta \setminus \Sigma$ (Σ defined by (7.2.5)) that exhibits the same jumps on $\Sigma \cap U_\zeta$ and the same local behavior at $z = \zeta$ as $\mathbf{S}$, and that matches the matrix $\mathbf{N}$ on the boundary ∂U_ζ.

The local RH problem at $\zeta \in \{\pm 1, x_0\}$, is,

($\mathrm{P}_\zeta 1$) $\mathbf{P}_\zeta$ is analytic in $U_\zeta \setminus \Sigma$.

(Pζ2) $\mathbf{P}_{\zeta+}(z) = \mathbf{P}_{\zeta-}(z)\mathbf{J}_{\mathbf{P}_\zeta}(z)$, where:

$$\mathbf{J}_{\mathbf{P}_\zeta}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{1}{w_{c,\gamma}(z)\varphi(z)^{2n}} & 1 \end{pmatrix}, & \text{for } z \in U_\zeta \cap \left(\bigcup_{i=1}^4 \gamma_i\right) \setminus \{\zeta\}; \\ \begin{pmatrix} 0 & w_{c,\gamma}(z) \\ -\frac{1}{w_{c,\gamma}(z)} & 0 \end{pmatrix}, & \text{for } z \in U_\zeta \cap ((-1, x_0) \cup (x_0, 1)). \end{cases} \tag{7.4.1}$$

(Pζ3) $\mathbf{P}_\zeta(z)\mathbf{N}^{-1}(z) = \mathbf{I} + \mathcal{O}(1/n)$, as $n \to \infty$, uniformly for $z \in \partial U_\zeta \setminus \Sigma$.

(Pζ4) if $(\zeta, s) \in \{(-1, \beta), (x_0, \gamma)\,(1, \alpha)\}$ then for $z \to \zeta$, $z \in U_\zeta \setminus \Sigma$, $\mathbf{P}_\zeta$ has the same behavior than (S4).

In the first subsection we show the solution for the parametrix at ± 1, but we skip the details of construction of $\mathbf{P}_{\pm 1}$, that can be found in [31, section 6].

In the second subsection we construct detailing all parametrix at the singular point x_0.

7.4.1 Local parametrices at the end points of the interval

This solution is built in terms of Bessel functions, and, for $\mathbf{P}_1$, it is given by

$$\mathbf{P}_1(z) = \mathbf{E}_{1,n}(z)\mathbf{B}_1\left(n^2 f_1(z)\right) W_1(z)^{-\sigma_3}\varphi(z)^{-n\sigma_3} \tag{7.4.2}$$

where φ is defined in (1.1.4);

$$\mathbf{E}_{1,n}(z) = \frac{1}{\sqrt{2}}\mathbf{N}(z)\,W_1(z)^{\sigma_3}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} f_1(z)^{\sigma_3/4}(2n\pi)^{\sigma_3/2},$$

is holomorphic on U_1;

$$W_1(z) = \left((z-1)^\alpha (z+1)^\beta (z-x_0)^\gamma c^2 h(z)\right)^{1/2}$$

is defined and analytic for $z \in U_1 \setminus (-\infty, 1]$, with the branch of the square root chosen which is positive for $z > 1$; $\mathbf{N}$ is defined by (7.3.1);

$$f_1(z) = \frac{\log^2 \varphi(z)}{4},$$

defined and analytic on $z \in \mathbb{C} \setminus (-\infty, 1]$;

$$\mathbf{B}_1(\zeta) = \begin{cases} \widetilde{\mathbf{B}}_1(\zeta), & |\arg\zeta| < \frac{2\pi}{3} \\ \widetilde{\mathbf{B}}_1(\zeta)\begin{pmatrix} 1 & 0 \\ -e^{\alpha\pi i} & 1 \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi \\ \widetilde{\mathbf{B}}_1(\zeta)\begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3} \end{cases}$$

with

$$\widetilde{\mathbf{B}}_1(\zeta) = \begin{pmatrix} I_\alpha\left(2\sqrt{\zeta}\right) & \frac{i}{\pi}K_\alpha\left(2\sqrt{\zeta}\right) \\ 2\pi i\sqrt{\zeta}I'_\alpha\left(2\sqrt{\zeta}\right) & -2\sqrt{\zeta}K'_\alpha\left(2\sqrt{\zeta}\right) \end{pmatrix}$$

where I_α and K_α are defined in (4.2.10) and (4.2.11) are the modified Bessel functions which satisfy the differential equation

$$y'' + \frac{1}{\zeta} y' - \left(1 - \frac{\alpha^2}{\zeta^2}\right) y = 0.$$

The solution for $\mathbf{P}_{-1}$ is

$$\mathbf{P}_{-1}(z) = \mathbf{E}_{-1,n}(z) \mathbf{B}_{-1}\left(n^2 f_{-1}(z)\right) W_{-1}(z)^{-\sigma_3} (-\varphi(z))^{-n\sigma_3} \tag{7.4.3}$$

where φ is defined in (1.1.4);

$$\mathbf{E}_{-1,n}(z) = \frac{1}{\sqrt{2}} \mathbf{N}(z) W_{-1}(z)^{\sigma_3} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} f_{-1}(z)^{\sigma_3/4} (2n\pi)^{\sigma_3/2},$$

is holomorphic on U_{-1};

$$W_{-1}(z) = \left((1-z)^\alpha (-1-z)^\beta (z-x_0)^\gamma h(z)\right)^{1/2}$$

is defined and analytic for $z \in U_{-1} \setminus [-1, \infty)$, with the branch of the square root chosen which is positive for $z < -1$; $\mathbf{N}$ is defined by (7.3.1);

$$f_{-1}(z) = \frac{\log^2(-\varphi(z))}{4},$$

defined and analytic on $z \in \mathbb{C} \setminus [-1, \infty)$;

$$\mathbf{B}_{-1}(\zeta) = \begin{cases} \widetilde{\mathbf{B}}_{-1}(\zeta), & |\arg \zeta| < \frac{2\pi}{3} \\ \widetilde{\mathbf{B}}_{-1}(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{\alpha\pi i} & 1 \end{pmatrix}, & \frac{2\pi}{3} < \arg \zeta < \pi \\ \widetilde{\mathbf{B}}_{-1}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}, & -\pi < \arg \zeta < -\frac{2\pi}{3} \end{cases}$$

with

$$\widetilde{\mathbf{B}}_{-1}(\zeta) = \begin{pmatrix} I_\beta\left(2\sqrt{\zeta}\right) & -\frac{i}{\pi} K_\beta\left(2\sqrt{\zeta}\right) \\ -2\pi i \sqrt{\zeta} I'_\beta\left(2\sqrt{\zeta}\right) & -2\sqrt{\zeta} K'_\beta\left(2\sqrt{\zeta}\right) \end{pmatrix}.$$

7.4.2 Local parametrix at x_0

The solution for $\mathbf{P}_{x_0}$ is given by

$$\mathbf{P}_{x_0}(z) = \mathbf{E}_n(z) \mathbf{P}^{(1)}(z) W(z)^{-\sigma_3} \varphi(z)^{-n\sigma_3}, \tag{7.4.4}$$

where $\mathbf{E}_n$ is defined in (7.4.29), $\mathbf{P}^{(1)}$ in (7.4.27) using (6.1.5) and (7.4.22), W by (7.4.5) and φ by (1.1.4).

Following a standard procedure, we obtain this solution of the $\mathbf{P}_{x_0}$-RH problem in two steps:

1. getting first a matrix $\mathbf{P}^{(1)}$ that satisfies conditions ($P_0 1$, $P_0 2$, $P_0 4$) but with constant jump relation;

2. using an additional freedom in the construction, we take care of the matching condition ($P_0 3$).

To do the first step we need: 1st) to find a function such that reduces to constant jumps, the $\mathbf{P}_{x_0}$-RH problem; 2nd) to rescale the RH problem, defining a conformal map from U_{x_0} to $\mathbb{C}$; 3rd) to find the solution of the equivalent RH problem, on $\mathbb{C}$ (using the Confluent Hypergeometric functions that are treated in Chapter 6). To execute the second step we construct the holomorphic matrix $\mathbf{E}_n$, such that ($P_0 3$) will be satisfied.

We will spare this construction in four subsubsections.

Reduction to constant jumps

We will transform the RH problem for $\mathbf{P}_{x_0}$ into another equivalent one with constant jumps.

We define an auxiliary function W, analytic in $\widetilde{U}\setminus\mathbb{R}$ ($\widetilde{U}$ is defined in (7.2.3)), based on the square root of $w_{c,\gamma}$. In the next formula we understand by $\left(h(z)(1-z)^{\alpha}(1+z)^{\beta}(z-x_0)^{\gamma}c\right)^{1/2}$ the analytic branch of this function in $U_{x_0}\setminus((-\infty,x_0]\cup[1,+\infty))$, positive on $(x_0,1)$. Analogously, $\left(h(z)(1-z)^{\alpha}(1+z)^{\beta}(x_0-z)^{\gamma}c\right)^{1/2}$ stands for the analytic branch in $U\setminus((-\infty,-1]\cup[x_0,+\infty))$, positive on $(-1,x_0)$. With this convention we set (using the notation introduced in the subsection 7.2, $Q_{\pm}^{R/L}$),

$$W(z)=\begin{cases}\left(h(z)(1-z)^{\alpha}(1+z)^{\beta}(z-x_0)^{\gamma}c\right)^{1/2}, & z\in Q_+^L\cup Q_-^L,\\ \left(h(z)(1-z)^{\alpha}(1+z)^{\beta}(x_0-z)^{\gamma}c\right)^{1/2}, & z\in Q_+^R\cup Q_-^R.\end{cases}\tag{7.4.5}$$

It is easy to see from (7.2.2)-(7.2.4) that

$$W^2(z)=\begin{cases}w_{c,\gamma}(z)e^{-\gamma\pi i}c^{-1}, & z\in Q_+^R,\\ w_{c,\gamma}(z)e^{\gamma\pi i}c, & z\in Q_+^L,\\ w_{c,\gamma}(z)e^{-\gamma\pi i}c, & z\in Q_-^L,,\\ w_{c,\gamma}(z)e^{\gamma\pi i}c^{-1}, & z\in Q_-^R.\end{cases}\tag{7.4.6}$$

This shows that W satisfies the following jump relations:

$$W_+(x)=\begin{cases}W_-(x)e^{i\gamma\pi}, & -1<x<x_0,\\ W_-(x)e^{-i\gamma\pi}, & x_0<x<1,\end{cases}\tag{7.4.7}$$

and

$$W_+(z)=e^{i\gamma\pi/2}W_-(z), \quad z\in\Sigma_5\cup\Sigma_6.\tag{7.4.8}$$

Moreover,

$$W_+(x)=\begin{cases}\sqrt{w_{c,\gamma}(x)c}\,e^{i\frac{\gamma\pi}{2}}=\sqrt{w_{1,\gamma}(x)c}\,e^{i\frac{\gamma\pi}{2}}, & -1<x<x_0,\\ \sqrt{w_{c,\gamma}(x)c^{-1}}\,e^{-i\frac{\gamma\pi}{2}}=\sqrt{w_{1,\gamma}(x)c}\,e^{-i\frac{\gamma\pi}{2}}, & x_0<x<1,\end{cases}\tag{7.4.9}$$

and

$$W_+(x)W_-(x)=\begin{cases}w_{c,\gamma}(x)c, & -1<x<x_0,\\ w_{c,\gamma}(x)c^{-1}, & x_0<x<1.\end{cases}\tag{7.4.10}$$

We construct the matrix function $\mathbf{P}^{(1)}$ using the following form:

$$\mathbf{P}_{x_0}(z)=\mathbf{E}_n(z)\mathbf{P}^{(1)}(z)W(z)^{-\sigma_3}\varphi(z)^{-n\sigma_3},$$

where $\mathbf{E}_n$ is a holomorphic matrix-valued function in U_{x_0} (to be determined). Then matrix $\mathbf{P}^{(1)}$ is analytic in $U_{x_0} \setminus (\Sigma \cup \Sigma_5 \cup \Sigma_6)$. The jumps for $\mathbf{P}^{(1)}$ are

$$\begin{aligned}\mathbf{J}_{\mathbf{P}^{(1)}} &= \left(\mathbf{P}^{(1)}(z)\right)_-^{-1}\left(\mathbf{P}^{(1)}(z)\right)_+ \\ &= W_-(z)^{-\sigma_3}\,\varphi_-(z)^{-n\sigma_3}\,\mathbf{J}_{\mathbf{P}_0}(z)\,\varphi_+(z)^{n\sigma_3}\,W_+(z)^{\sigma_3}\end{aligned}$$

that is

$$\begin{pmatrix} 0 & \frac{w_{c,\gamma}(x)}{\varphi_+(x)^n\varphi_-(x)^nW_-(x)W_+(x)} \\ -\frac{\varphi_+(x)^n\varphi_-(x)^nW_-(x)W_+(x)}{w_{c,\gamma}(x)} & 0 \end{pmatrix}, \text{ as } x \in (-1,1)\cap U_{x_0}, \tag{7.4.11}$$

$$\begin{pmatrix} \frac{\varphi_+(z)^nW_+(z)}{\varphi_-(z)^nW_-(z)} & 0 \\ \frac{\varphi_+(z)^n\varphi_-(z)^nW_-(z)W_+(z)}{w_{c,\gamma}(z)\varphi_+(z)^{2n}} & \frac{\varphi_-(z)^nW_-(z)}{\varphi_+(z)^nW_+(z)} \end{pmatrix}, \text{ as } z \in \left(\bigcup_{i=1}^4 \Sigma_i\right)\cap U_{x_0}, \tag{7.4.12}$$

and, where $\mathbf{P}_{x_0}$ had no jumps,

$$\left(\frac{\varphi_+(z)^n\,W_+(z)}{\varphi_-(z)^n\,W_-(z)}\right)^{\sigma_3}, \text{ as } z \in U_{x_0}\setminus\Sigma. \tag{7.4.13}$$

Denote by

$$\mathbf{J}_1 = \begin{pmatrix} 0 & c \\ -1/c & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} 1 & 0 \\ e^{-\gamma\pi i}c^{-1} & 1 \end{pmatrix}, \quad \mathbf{J}_3 = \mathbf{J}_7 = \begin{pmatrix} e^{i\gamma\pi/2} & 0 \\ 0 & e^{-i\gamma\pi/2} \end{pmatrix}, \tag{7.4.14}$$

$$\mathbf{J}_4 = \begin{pmatrix} 1 & 0 \\ e^{\gamma\pi i}c & 1 \end{pmatrix}, \mathbf{J}_5 = \begin{pmatrix} 0 & 1/c \\ -c & 0 \end{pmatrix}, \mathbf{J}_6 = \begin{pmatrix} 1 & 0 \\ e^{-\gamma\pi i}c & 1 \end{pmatrix}, \mathbf{J}_8 = \begin{pmatrix} 1 & 0 \\ e^{\gamma\pi i}c^{-1} & 1 \end{pmatrix}. \tag{7.4.15}$$

Using the properties of W (7.4.10) and φ (defined by (1.1.4)) it is easy to show that by (7.4.11)

$$\mathbf{P}_+^{(1)}(x) = \mathbf{P}_-^{(1)}(x)\begin{cases}\mathbf{J}_5, & x \in (x_0-\delta, x_0),\\ \mathbf{J}_1, & x \in (x_0, x_0+\delta),\end{cases} \tag{7.4.16}$$

and, by (7.4.6) and (7.4.12),

$$\mathbf{P}_+^{(1)}(z) = \mathbf{P}_-^{(1)}(z)\begin{cases}\mathbf{J}_4, & z \in \Sigma_1\cap U_{x_0}\setminus\{x_0\},\\ \mathbf{J}_6, & z \in \Sigma_2\cap U_{x_0}\setminus\{x_0\},\\ \mathbf{J}_2, & z \in \Sigma_3\cap U_{x_0}\setminus\{x_0\},\\ \mathbf{J}_8, & z \in \Sigma_4\cap U_{x_0}\setminus\{x_0\},\end{cases} \tag{7.4.17}$$

and, by (7.4.13), as W has a jump on $\Sigma_5 \cup \Sigma_6$, by (7.4.8), we have two additional jumps on $\Sigma_5 \cup \Sigma_6$:

$$\mathbf{P}_+^{(1)}(z) = \mathbf{P}_-^{(1)}(z)\begin{cases}\mathbf{J}_3, & z \in \Sigma_5\cap U_{x_0}\setminus\{x_0\},\\ \mathbf{J}_7, & z \in \Sigma_6\cap U_{x_0}\setminus\{x_0\}.\end{cases} \tag{7.4.18}$$

Taking into account that $W(z) = \mathcal{O}\left(|z-x_0|^{\gamma/2}\right)$ and $\varphi(z) = \mathcal{O}(1)$ as $z \to x_0$, we conclude also from (P_04)/(S4) that $\mathbf{P}^{(1)}$ has the following behavior at x_0: as $z \to x_0$, $z \in \mathbb{C}\setminus(\Sigma\cup\Sigma_5\cup\Sigma_6)$,

- for $\gamma < 0$:

$$\mathbf{P}^{(1)}(z) = \mathcal{O}\begin{pmatrix} |z-x_0|^{\gamma/2} & |z-x_0|^{\gamma/2} \\ |z-x_0|^{\gamma/2} & |z-x_0|^{\gamma/2} \end{pmatrix}, \tag{7.4.19}$$

- for $\gamma = 0$

$$\mathbf{P}^{(1)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} \log|z-x_0| & \log|z-x_0| \\ \log|z-x_0| & \log|z-x_0| \end{pmatrix}, & \text{from inside the lens,} \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-x_0| \\ 1 & \log|z-x_0| \end{pmatrix}, & \text{from outside the lens,} \end{cases} \tag{7.4.20}$$

- for $\gamma > 0$:

$$\mathbf{P}^{(1)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z-x_0|^{\gamma/2} & |z-x_0|^{-\gamma/2} \\ |z-x_0|^{\gamma/2} & |z-x_0|^{-\gamma/2} \end{pmatrix}, & \text{from outside the lens,} \\ \mathcal{O}\begin{pmatrix} |z-x_0|^{-\gamma/2} & |z-x_0|^{-\gamma/2} \\ |z-x_0|^{-\gamma/2} & |z-x_0|^{-\gamma/2} \end{pmatrix}, & \text{from inside the lens.} \end{cases} \tag{7.4.21}$$

Then, the equivalent to ($\mathrm{P}_0$1)-($\mathrm{P}_0$2)-($\mathrm{P}_0$4) RH problem, for $\mathbf{P}^{(1)}$ is the following:

($\mathbf{P}^{(1)}$1) $\mathbf{P}^{(1)}$ is analytic in $U_{x_0} \setminus \Sigma$;

($\mathbf{P}^{(1)}$2) $\mathbf{P}^{(1)}_+(z) = \mathbf{P}^{(1)}_-(z)\,\mathbf{J}_k$ $(k = 1, \cdots, 8)$ explicitly in (7.4.16)-(7.4.18);

($\mathbf{P}^{(1)}$3) As $z \to x_0$, $z \in \mathbb{C} \setminus (\Sigma \cup \Sigma_5 \cup \Sigma_6)$, the behavior of $\mathbf{P}^{(1)}$ is given by (7.4.19)-(7.4.21).

Rescale the RH problem

We will define a conformal map that maps the neighborhood U_{x_0} onto $\mathbb{C}$.

Using the properties of φ we define an analytic function f in a neighborhood of x_0,

$$f(z) \stackrel{\text{def}}{=} \begin{cases} 2i \arccos x_0 - 2\log\varphi(z), & \text{for } \operatorname{Im} z > 0, \\ 2i \arccos x_0 + 2\log\varphi(z), & \text{for } \operatorname{Im} z < 0, \end{cases} \tag{7.4.22}$$

where we take the main branch of the logarithm. Using that $\varphi_+(x)\,\varphi_-(x) = 1$ on $(-1, 1)$ we conclude that f can be extended to a holomorphic function in a neighborhood of x_0, $U_{x_0} \subset \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$. For $|z| < 1$ we have

$$f(z) = \frac{2i}{\sqrt{1-x_0^2}}(z - x_0) + \mathcal{O}\left((z-x_0)^2\right), \text{ as } z \to x_0. \tag{7.4.23}$$

Hence, for $\delta > 0$ sufficiently small, f is a conformal mapping of U_{x_0}. Moreover, using (5.2.1)

$$\varphi_+(x) = x + i\sqrt{1-x^2} = e^{i \arccos x}, \quad x \in (-1, 1),$$

then

$$f(x) = 2i\left(\arccos x_0 - \arccos x\right), \quad x \in (-1, 1), \tag{7.4.24}$$

so that f maps the real interval $(-1, x_0)$ one-to-one onto the purely imaginary interval $(2i(\arccos x_0 - \pi), 0) \subset \Gamma_5$, as well as $(x_0, 1)$ one-to-one onto the purely imaginary interval

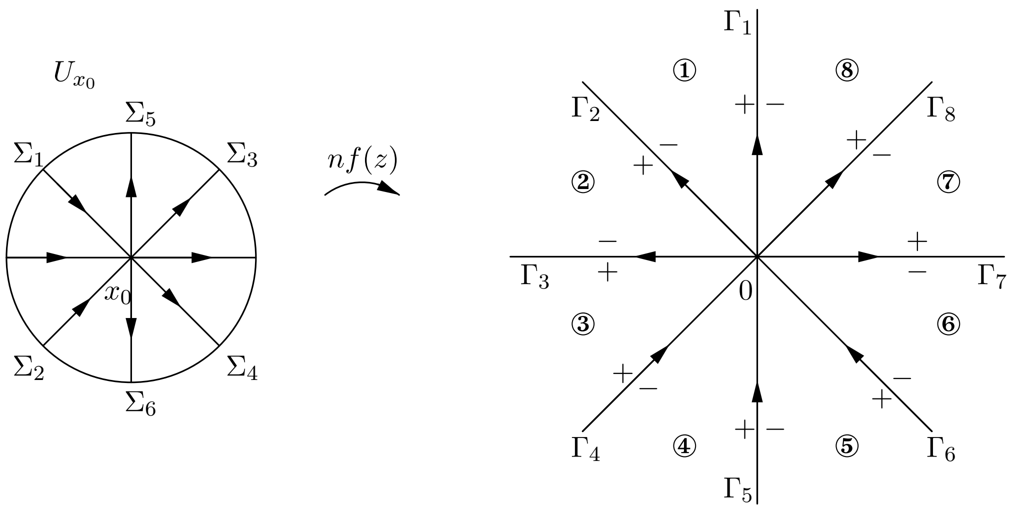

FIGURE 7.4.1: Conformal mapping f, of Σ contours onto Γ contours.

$(0, 2i \arccos x_0) \subset \Gamma_1$, see figure 7.4.1. We can always deform our contours Σ_k $(k = 1, \cdots, 6)$ close to $z = x_0$ in such a way that

$$\begin{aligned} &f(\Sigma_1 \cap U_{x_0}) \subset \Gamma_4, \quad f(\Sigma_2 \cap U_{x_0}) \subset \Gamma_6, \quad f(\Sigma_3 \cap U_{x_0}) \subset \Gamma_2, \\ &f(\Sigma_4 \cap U_{x_0}) \subset \Gamma_8, \quad f(\Sigma_5 \cap U_{x_0}) \subset \Gamma_3, \quad f(\Sigma_6 \cap U_{x_0}) \subset \Gamma_7, \end{aligned}$$

where Γ_i $(k = 1, \cdots, 8)$ are the straight lines defined in (6.1.1). With this convention, set

$$\zeta \stackrel{\text{def}}{=} nf(z), \quad z \in U_{x_0}, \tag{7.4.25}$$

as $n \to \infty$, ζ is a conformal map of the U_{x_0} onto all complex plane.

Solution to the equivalent RH problem on $\mathbb{C}$

Comparing the $\mathbf{P}^{(1)}$-RH problem with the RH problem for the Confluent Hypergeometric function in Chapter 6, from (Ψ1)–(Ψ4) and (7.4.23), the $\mathbf{\Psi}$ matrix-valued function (in (6.1.5)) has the same jumps and the local behavior than $\mathbf{P}^{(1)}$. The jumps (7.4.16)–(7.4.18) are the same as those specified in (Ψ2), taking as in (5.1.4),

$$\lambda = i\frac{\log c}{\pi} \tag{7.4.26}$$

and the local behavior at x_0 (7.4.19)–(7.4.21) is the same than (Ψ4) (where the region "inside lens" corresponds to ①∪④∪⑤∪⑧ and the region "outside lens" corresponds to ②∪③∪⑥∪⑦). The condition (Ψ3) will be used to construct $\mathbf{E}_n$ bellow. We conclude that both RH problems are equivalent, and, then the solution for $\mathbf{P}^{(1)}$ is given by

$$\mathbf{P}^{(1)}(z) \stackrel{\text{def}}{=} \mathbf{\Psi}(nf(z)), \quad z \in U_{x_0}, \tag{7.4.27}$$

with $\mathbf{\Psi}$ defined in (6.1.5).

Constructing $\mathbf{E}_n$

Taking into account the definition (7.4.22) we get that

$$e^{nf(z)} = \varphi_+^{2n}(x_0)\,\varphi^{\mp 2n}(z), \quad \text{for } \pm\operatorname{Im} z > 0,$$

and for $[nf(z)]^\lambda$ we take the cut along $(-\infty, x_0]$. Since

$$[f(z)]^\lambda = |f(z)|^\lambda \exp\left(-\frac{\log c}{\pi}\arg(f(z))\right),$$

straightforward computations show that

$$[f(x)]_\pm^\lambda = \begin{cases} |f(x)|^\lambda c^{-1/2}, & \text{for } x_0 < x < 1, \\ |f(x)|^\lambda c^{-1/2\mp 1}, & \text{for } -1 < x < x_0, \end{cases} \tag{7.4.28}$$

where we assume the natural orientation of the interval.

In order to satisfy ($P_0$3) above, we define

$$\mathbf{E}_n(z) \stackrel{\text{def}}{=} \mathbf{N}(z)\,W(z)^{\sigma_3} \begin{cases} (nf(z))^{\lambda\sigma_3}\varphi_+^{n\sigma_3}(x_0)\,e^{i\frac{\gamma\pi}{4}\sigma_3}c^{\sigma_3}, & \text{if } z \in Q_+^R, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(nf(z))^{\lambda\sigma_3}\varphi_+^{n\sigma_3}(x_0)\,e^{i\frac{\gamma\pi}{4}\sigma_3}, & \text{if } z \in Q_-^R, \\ (nf(z))^{\lambda\sigma_3}\varphi_+^{n\sigma_3}(x_0)\,e^{-i\frac{\gamma\pi}{4}\sigma_3}c^{\sigma_3}, & \text{if } z \in Q_+^L, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(nf(z))^{\lambda\sigma_3}\varphi_+^{n\sigma_3}(x_0)\,e^{-i\frac{\gamma\pi}{4}\sigma_3}, & \text{if } z \in Q_-^L. \end{cases} \tag{7.4.29}$$

By construction, $\mathbf{E}_n$ is analytic in $U_{x_0} \setminus (\mathbb{R} \cup \Sigma_5 \cup \Sigma_6)$. Furthermore, by (N2) and (7.4.10), for $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$,

$$\begin{aligned} W_-(x)^{-\sigma_3}\,\mathbf{N}_-^{-1}(x)\mathbf{N}_+(x)W_+(x)^{\sigma_3} &= \begin{pmatrix} 0 & \frac{w_{c,\gamma}(x)}{W_-(x)W_+(x)} \\ -\frac{W_-(x)W_+(x)}{w_{c,\gamma}(x)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c^{\pm 1} \\ -c^{\mp 1} & 0 \end{pmatrix}, \quad \text{for } \pm x > x_0; \end{aligned}$$

and, by (7.4.8), for $z \in \Sigma_6 \cap U_{x_0}$ (oriented from above to bellow) and for $z \in \Sigma_5 \cap U_{x_0}$ (oriented from bellow to above) we have,

$$W_-(z)^{-\sigma_3}\,\mathbf{N}_-^{-1}(z)\mathbf{N}_+(z)W_+(z)^{\sigma_3} = \begin{pmatrix} W_+(z)/W_-(z) & 0 \\ 0 & W_-(z)/W_+(z) \end{pmatrix} = e^{i\frac{\gamma\pi}{2}\sigma_3}.$$

From (7.4.28) and (7.4.29) it follows that

$$\mathbf{E}_{n-}^{-1}(z)\,\mathbf{E}_{n+}(z) = \mathbf{I}, \quad \text{for } z \in U_{x_0} \setminus \{x_0\}. \tag{7.4.30}$$

In this form, x_0 is the only possible isolated singularity of $\mathbf{E}_n$ in U_{x_0}. The following proposition shows that this is in fact a removable singularity of $\mathbf{E}_n$:

Proposition 7.1

$$\lim_{z\to x_0} \mathbf{E}_n(z) = \frac{\sqrt{2}}{2} D_\infty^{\sigma_3} \begin{pmatrix} e^{-i\arcsin(x_0)/2} & e^{i\arcsin(x_0)/2} \\ -e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix} e^{i\eta_n\sigma_3},$$

with η_n defined by

$$\eta_n \overset{def}{=} \frac{\log c}{\pi} \log\left(4n\sqrt{1-x_0^2}\right) + n\arccos(x_0) - \frac{\gamma\pi}{4} - \Phi\left(x_0\right) \tag{7.4.31}$$

and Φ given by (5.2.4). In particular, $\mathbf{E}_n$ is holomorphic in U_{x_0}.

Proof. Since by (7.4.30) $\mathbf{E}_n$ is analytic in a neighborhood of x_0, it is sufficient to analyze its limit as $z\to x_0$ from the first quarter of the plane, $z\in Q_+^R$. By (5.2.6) and (7.4.23),

$$\lim_{\substack{z\to x_0\\ z\in Q_+^R}} D\left(z,\Xi_c\right) f\left(z\right)^{-\lambda} = \lim_{\substack{z\to x_0\\ z\in Q_+^R}} c^{1+\frac{i}{\pi}\log(z/2)-\frac{i}{\pi}\log(f(z))} = c^{3/2}\left(4\sqrt{1-x_0^2}\right)^{-\lambda}.$$

On the other hand, by (5.2.2) and (7.4.9) (notice that $w_{1,\gamma}$ defined in (7.2.4) coincides with w defined in (7.2.2)),

$$\lim_{\substack{z\to x_0\\ z\in Q_+^R}} D\left(z,w\right) W\left(z\right)^{-1} = c^{-1/2} e^{i\Phi(x_0)} e^{i\gamma\pi/2}.$$

Summarizing,

$$\lim_{\substack{z\to x_0\\ z\in Q_+^R}} D\left(z,w_{c,\gamma}\right)^{-1} W\left(z\right) f\left(z\right)^{\lambda} = \frac{\left(4\sqrt{1-x_0^2}\right)^{\lambda}}{c} e^{-i\Phi(x_0)-i\gamma\pi/2}.$$

Using (7.3.1) and (7.4.29), we rewrite $\mathbf{E}_n(z)$ for $\operatorname{Im} z>0$, as

$$\mathbf{E}_n(z) = D_\infty^{\sigma_3}\,\mathbf{A}(z)\, m_n(z)^{\sigma_3}, \tag{7.4.32}$$

with,

$$m_n(z) \overset{\text{def}}{=} \begin{cases} \dfrac{W\left(z\right) f\left(z\right)^{\lambda}}{D\left(z,w_{c,\gamma}\right)} \varphi_+^n(x_0) n^{\lambda} e^{i\gamma\pi/4} c, & z\in Q_+^R, \\ \dfrac{W\left(z\right) f\left(z\right)^{\lambda}}{D\left(z,w_{c,\gamma}\right)} \varphi_+^n(x_0) n^{\lambda} e^{-i\gamma\pi/4} c, & z\in Q_+^L, \end{cases} \tag{7.4.33}$$

and

$$\lim_{\substack{z\to x_0\\ z\in Q_+^R}} m_n(z) = e^{\eta_n}$$

where η_n is defined in (7.4.31).

Gathering the limits computed above and using that

$$\lim_{\substack{z\to x_0\\ z\in Q_+^R}} A_{11}(z) = e^{-i\arcsin(x_0)/2} = \lim_{\substack{z\to x_0\\ z\in Q_+^R}} \overline{A_{12}(z)}$$

and by definition of η_n, the statement follows. ■

Therefore, by construction the matrix-valued function $\mathbf{P}_{x_0}$ given by 7.4.4 satisfies conditions ($P_0$1)–($P_0$4). Moreover, as $\det\mathbf{\Psi}=1$, it is easy to check that

$$\det \mathbf{P}_{x_0}\left(z\right) = 1 \quad \text{for every } z\in U_{x_0}\setminus\Sigma.$$

7.5 Last transformation: S - R

Recall that matrices $\mathbf{N}$ and $\mathbf{P}_\zeta$, $\zeta \in \{-1, x_0, 1\}$ have det $= 1$ in their domains of definition. We may define

$$\mathbf{R}(z) \overset{\text{def}}{=} \begin{cases} \mathbf{S}(z)\,\mathbf{N}^{-1}(z), & z \in \mathbb{C} \setminus \{\Sigma \cup U_{-1} \cup U_{x_0} \cup U_1\}; \\ \mathbf{S}(z)\,\mathbf{P}_\zeta^{-1}(z), & z \in U_j \setminus \Sigma,\ j \in \{-1, x_0, 1\}. \end{cases} \tag{7.5.1}$$

$\mathbf{R}$ is analytic in $\mathbb{C} \setminus \{\Sigma \cup \partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1\}$. In fact, since $\mathbf{N}$ matches the jump of $\mathbf{S}$ on $(-1, 1)$, and $\mathbf{P}_\zeta$ matches the jumps of $\mathbf{S}$ within U_ζ, $\zeta \in \{-1, x_0, 1\}$, we conclude that $\mathbf{R}$ is analytic in the complement of the contours Σ_R depicted in Fig. 7.5.1, with additional possible singularities at $\{-1, x_0, 1\}$. But taking into account (S4) and the local behavior of $\mathbf{P}_\zeta$ at these points (see ($\mathrm{P}_\zeta 4$)), we conclude that these singularities are removable.

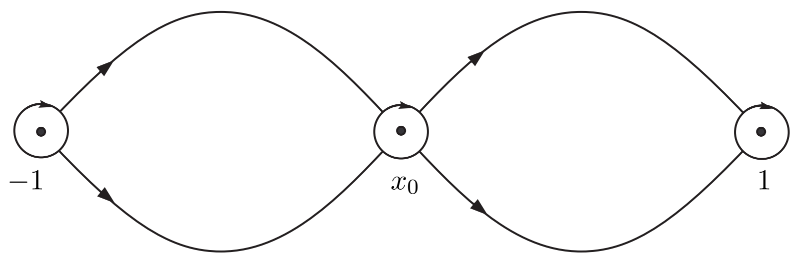

FIGURE 7.5.1: Contours Σ_R.

Now we compute the jumps of $\mathbf{R}$. For the sake of brevity, we denote

$$\Sigma_R^{out} \overset{\text{def}}{=} \Sigma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1).$$

Then by (S2) and (7.5.1), for $z \in \Sigma_R^{out}$,

$$\mathbf{R}_+(z) = \mathbf{R}_-(z)\,\mathbf{N}(z) \begin{pmatrix} 1 & 0 \\ w_c(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix} \mathbf{N}^{-1}(z). \tag{7.5.2}$$

On the other hand, for ∂U_j ($j \in \{-1, x_0, 1\}$) oriented clockwise, we have that $\mathbf{R}_+(z) = \mathbf{S}_+(z)\,\mathbf{N}^{-1}(z)$ and $\mathbf{R}_-(z) = \mathbf{S}_-(z)\,\mathbf{P}_j^{-1}(z)$. Hence,

$$\mathbf{R}_+(z) = \mathbf{R}_-(z)\,\mathbf{P}_j(z)\,\mathbf{N}^{-1}(z), \quad z \in \partial U_j, \quad j \in \{-1, x_0, 1\}. \tag{7.5.3}$$

Summarizing, $\mathbf{R}$ defined in (7.5.1) is analytic in $\mathbb{C} \setminus \Sigma_R$, satisfies the jump relations (7.5.2)–(7.5.3) on Σ_R, and has the following behavior as $z \to \infty$:

$$\mathbf{R}(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right).$$

By (7.5.3) and ($\mathrm{P}_0 3$), as $n \to \infty$,

$$\mathbf{R}_+(z) = \mathbf{R}_-(z)\left(\mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad \text{uniformly on } \partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1. \tag{7.5.4}$$

On the other hand, there exists a constant $0 < q < 1$ such that $|\varphi(z)|^{-1} \le q < 1$ uniformly on Σ_R^{out}. Since $\mathbf{N}$ does not depend on n, we conclude from (7.5.2) that as $n \to \infty$,

$$\mathbf{R}_+(z) = \mathbf{R}_-(z)\left(\mathbf{I} + \mathcal{O}\left(q^{2n}\right)\right) \quad \text{uniformly on } \Sigma_R^{out}, \tag{7.5.5}$$

and the jumps on Σ_R^{out} are negligible. Motivated by (7.5.2)–(7.5.3) we define

$$\mathbf{\Delta}(s) \stackrel{\text{def}}{=} \begin{cases} \mathbf{N}(s)\begin{pmatrix} 1 & 0 \\ w_{c,\gamma}(s)^{-1}\varphi(s)^{-2n} & 1 \end{pmatrix}\mathbf{N}^{-1}(s) - \mathbf{I}, & \text{for } s \in \Sigma_R^{out}; \\ \mathbf{P}_\zeta(s)\mathbf{N}^{-1}(s) - \mathbf{I}, & \text{for } s \in \partial U_\zeta,\ j \in \{-1, x_0, 1\}. \end{cases} \tag{7.5.6}$$

So, $\mathbf{R}$ defined by (7.5.1) satisfy the following RH problem:

(R1) $\mathbf{R}$ is analytic in $\mathbb{C} \setminus \Sigma_R$;

(R2) $\mathbf{R}_+(z) = \mathbf{R}_-(z)(\mathbf{I} + \mathbf{\Delta}(z))$, $z \in \Sigma_R$, with $\mathbf{\Delta}$ defined by (7.5.6);

(R3) $\mathbf{R}(z) = \mathbf{I} + \mathcal{O}(1/z)$, as $z \to \infty$.

7.6 Asymptotics for R

The jump for the matrix RH problem for $\mathbf{R}$, is $(\mathbf{I} + \mathbf{\Delta}(z))$. We can show that $\mathbf{\Delta}$ (7.5.6) has an asymptotic expansion in powers of $1/n$ of the form

$$\mathbf{\Delta}(s) \sim \sum_{k=1}^{\infty} \frac{\mathbf{\Delta}_k(s,n)}{n^k}, \quad \text{as } n \to \infty, \text{ uniformly for } s \in \Sigma_R, \tag{7.6.1}$$

and, where, for $k \in \mathbb{N}$, from (7.5.5),

$$\mathbf{\Delta}_k(s) = 0, \quad \text{for } s \in \Sigma_R^{out}. \tag{7.6.2}$$

Furthermore, near the ± 1, it is given like in [31, formulas (8.5)–(8.6)], by

$$\begin{aligned} \mathbf{\Delta}_k(s) = {} & \frac{(\alpha, k-1)}{2^k[\log\varphi(s)]^k}\mathbf{N}(s)\left[e^{\pm\frac{i\pi\alpha}{2}}c^{\frac{1}{2}}W(s)\right]^{\sigma_3}\begin{pmatrix} \frac{(-1)^k}{k}(\alpha^2 + \frac{1}{2}k - \frac{1}{4}) & -(k-\frac{1}{2})i \\ (-1)^k(k-\frac{1}{2})i & \frac{1}{k}(\alpha^2 + \frac{1}{2}k - \frac{1}{4}) \end{pmatrix} \\ & \times \left[e^{\pm\frac{i\pi\alpha}{2}}c^{\frac{1}{2}}W(s)\right]^{-\sigma_3}\mathbf{N}^{-1}(s), \qquad \text{for } \pm\operatorname{Im} s > 0 \text{ and } s \in \partial U_1; \end{aligned}$$

and

$$\begin{aligned} \mathbf{\Delta}_k(s) = {} & \frac{(\beta, k-1)}{2^k[\log(-\varphi(s))]^k}\mathbf{N}(s)\left[e^{\mp\frac{i\pi\beta}{2}}c^{-\frac{1}{2}}W(s)\right]^{\sigma_3}\begin{pmatrix} \frac{(-1)^k}{k}(\beta^2 + \frac{1}{2}k - \frac{1}{4}) & (k-\frac{1}{2})i \\ (-1)^{k+1}(k-\frac{1}{2})i & \frac{1}{k}(\alpha^2 + \frac{1}{2}k - \frac{1}{4}) \end{pmatrix} \\ & \times \left[e^{\mp\frac{i\pi\beta}{2}}c^{-\frac{1}{2}}W(s)\right]^{-\sigma_3}\mathbf{N}^{-1}(s), \qquad \text{for } \pm\operatorname{Im} s > 0 \text{ and } s \in \partial U_{-1}; \end{aligned}$$

where $(\alpha, 0) \stackrel{\text{def}}{=} 1$, and

$$(\alpha, k) \stackrel{\text{def}}{=} \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)\cdots(4\alpha^2 - (2k-1)^2)}{2^{2k}k!}.$$

Each $\mathbf{\Delta}_k$ on the small contours encircling ± 1 is independent of n and possesses a meromorphic continuation to U_{-1} and U_1 with the only pole at ± 1 of order at most $[(k+1)/2]$. However, unlike in the case analyzed in [31], the existence of a jump in the weight is revealed through the contribution of the local parametrix $\mathbf{P}_{x_0}$, and hence, each $\mathbf{\Delta}_k$ is in general not independent on n, although uniformly bounded in n.

So, it remains to determine $\mathbf{\Delta}_k$ on ∂U_{x_0}. Here we calculate explicitly only the first term, $\mathbf{\Delta}_1$.

Using (7.3.1), (7.4.5), (7.4.4), (7.4.22), (6.2.7), (7.4.25), (7.4.29), (6.2.6) and (6.2.5), we obtain

$$\mathbf{\Delta}\left(s\right) = \mathbf{E}_n(s)\left[\frac{-\mu}{nf(s)}\begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]\mathbf{E}_n^{-1}(s), \quad s \in \partial U_{x_0}, \quad n \to \infty.$$

Let us define

$$\mathbf{\Delta}_1\left(s\right) \stackrel{\text{def}}{=} \frac{-\mu}{f(s)}\,\mathbf{E}_n(s)\begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix}\mathbf{E}_n^{-1}(s), \quad s \in \partial U_{x_0}. \tag{7.6.3}$$

Remark 7.2 *(7.6.1) is in fact an asymptotic expansion in powers of $1/n$. Observe that $\mathbf{\Delta}_1$ in (7.6.3) is uniformly bounded in n. Using that by (7.4.29),*

$$\mathbf{E}_n(s) = \mathbf{F}(s)\,\left(\varphi_+(x_0)^n n^\lambda\right)^{\sigma_3} = \mathbf{F}(s)\,\left(e^{in\arccos(x_0)}c^{\frac{i}{\pi}\,\log n}\right)^{\sigma_3},$$

with

$$\mathbf{F}\left(s\right) \stackrel{def}{=} \begin{cases} \mathbf{N}\left(s\right)W\left(s\right)^{\sigma_3}\,c^{\sigma_3}e^{\pm\frac{\gamma\pi}{4}\sigma_3}f\left(s\right)^{\lambda\sigma_3}, & \text{if } \operatorname{Im} s > 0, \\ \mathbf{N}\left(s\right)W\left(s\right)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}e^{\pm\frac{\gamma\pi}{4}\sigma_3}f\left(s\right)^{\lambda\sigma_3}, & \text{if } \operatorname{Im} s < 0, \end{cases}$$

where we take $\pm$ for $\pm\operatorname{Re} s > x_0$, we conclude that, for $s \in \partial U_{x_0}$, $\Delta_1(z,n)$ is uniformly bounded in n; indeed, $\mathbf{F}$ does not depend on n and

$$\left|e^{in\arccos x_0}c^{\frac{i\log n}{2\pi}}\right| = 1, \quad \forall n \in \mathbb{N}.$$

So $\mathbf{\Delta}_1$ in (7.6.3) is genuinely the first coefficient in the expansion (7.6.1). Similar analysis can be performed for $\mathbf{\Delta}_k(\cdot,n)$, $k \geq 2$, taking higher order terms in the expansion of $\mathbf{\Psi}$ in (6.2.7).

As $\mathbf{E}_n$ and f are holomorphic on U_{x_0} (see (7.4.22) and (7.1)), the explicit expression (7.6.3) and the local behavior of f (7.4.23) show that $\mathbf{\Delta}_1(s,n)$ has an analytic continuation to U_{x_0} except for x_0, where it has a simple pole. Again, similar conclusion is valid for other $\mathbf{\Delta}_k(s,n)$, except that now the pole has order k.

Now, using Theorem 7.10 of [19] we obtain from (7.6.1) that

$$\mathbf{R}\left(z\right) \sim \mathbf{I} + \sum_{j=1}^{\infty}\frac{\mathbf{R}^{(j)}\left(z,n\right)}{n^j}, \quad \text{as } n \to \infty, \tag{7.6.4}$$

uniformly for $z \in \mathbb{C}\setminus\{\partial U_{-1}\cup\partial U_{x_0}\cup\partial U_1\}$, where each $\mathbf{R}^{(j)}(z)$ is analytic, uniformly bounded in n, and

$$\mathbf{R}^{(j)}\left(z,n\right) = \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty.$$

This (7.6.4) is a bona fide asymptotic expansion near infinity, since

$$\forall l \in \mathbb{N}\ \exists C > 0 : |z| \geq 2 \Rightarrow \left\| \mathbf{R}(z) - \mathbf{I} - \sum_{j=1}^{l} \frac{\mathbf{R}^{(j)}(z,n)}{n^j} \right\| \leq \frac{C}{|z|\, n^{l+1}},$$

for any matrix norm $\|\cdot\|$. The proof is based on the integral representation for $\mathbf{R}$, where, as $s \in \Sigma_R$, (R3) is equivalent to $(\mathbf{R}(s) - \mathbf{I})_+ = (\mathbf{R}(s) - \mathbf{I})_- + \mathbf{R}_-(s)\,\boldsymbol{\Delta}(s)$, then based in the Sokhotskii-Plemelj formulas (2.1.1),

$$\mathbf{R}(z) = \mathbf{I} + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{\mathbf{R}_-(s)\boldsymbol{\Delta}(s,n)}{s-z}\, ds, \quad z \in \mathbb{C} \setminus \Sigma_R$$

(see [19]); although in our case the coefficients $\boldsymbol{\Delta}_k$ and $\mathbf{R}_k$ in (7.6.1) and (7.6.4) depend on n, their uniform boundedness allows to follow the steps of the proof of Lemma 8.3 in [31]. In particular, expanding the jump relation $\mathbf{R}_+ = \mathbf{R}_-(\mathbf{I} + \boldsymbol{\Delta})$ up to order $1/n$ we find that

$$\mathbf{R}_+^{(1)}(s,n) - \mathbf{R}_-^{(1)}(s,n) = \boldsymbol{\Delta}_1(s,n), \quad \text{for } s \in \partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1.$$

Since $\mathbf{R}^{(1)}$ is analytic in the complement of $\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1$ (see 7.6.2) and vanishes at infinity, by the Sokhotskii-Plemelj formulas (2.1.1),

$$\mathbf{R}^{(1)}(z,n) = \frac{1}{2\pi i} \int_{\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1} \frac{\boldsymbol{\Delta}_1(s,n)}{s-z}\, ds.$$

Recall that $\boldsymbol{\Delta}_1$ can be extended analytically inside U_j's with simple poles at ± 1 and x_0; let us denote by $\mathbf{A}^{(1)}(n)$, $\mathbf{B}^{(1)}(n)$ and $\mathbf{C}^{(1)}(n)$ the residues of $\boldsymbol{\Delta}_1(\cdot, n)$ at 1, -1 and x_0, respectively. Then the residue calculus gives

$$\mathbf{R}^{(1)}(z,n) = \begin{cases} \dfrac{\mathbf{A}^{(1)}(n)}{z-1} + \dfrac{\mathbf{B}^{(1)}(n)}{z+1} + \dfrac{\mathbf{C}^{(1)}(n)}{z-x_0}, & \text{for } z \in \mathbb{C} \setminus \{U_{-1} \cup U_{x_0} \cup U_1\}; \\ \dfrac{\mathbf{A}^{(1)}(n)}{z-1} + \dfrac{\mathbf{B}^{(1)}(n)}{z+1} + \dfrac{\mathbf{C}^{(1)}(n)}{z-x_0} - \boldsymbol{\Delta}_1(z,n), & \text{for } z \in U_{-1} \cup U_{x_0} \cup U_1. \end{cases} \tag{7.6.5}$$

Residues $A^{(1)}(n)$ and $B^{(1)}(n)$ are in fact independent of n; they have been determined in [31, Section 8]:

$$\begin{aligned} \mathbf{A}^{(1)}(n) = \mathbf{A}^{(1)} &= \frac{4\alpha^2 - 1}{16} D_\infty^{\sigma_3} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} D_\infty^{-\sigma_3}, \\ \mathbf{B}^{(1)}(n) = \mathbf{B}^{(1)} &= \frac{4\beta^2 - 1}{16} D_\infty^{\sigma_3} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} D_\infty^{-\sigma_3} \end{aligned} \tag{7.6.6}$$

(notice however that the constant D_∞ is different with respect to [31], is now defined by (5.1.5)). The value of the remaining residue $\mathbf{C}^{(1)}(n)$ is given in the following

Proposition 7.2 *If we denote*

$$\mathbf{C}^{(1)}(n) = \begin{pmatrix} C_{11}^{(1)}(n) & C_{12}^{(1)}(n) \\ C_{21}^{(1)}(n) & C_{11}^{(1)}(n) \end{pmatrix}$$

then the entries are given explicitly by:

$$C_{11}^{(1)}(n) = -\frac{\mu}{2}x_0 + \frac{\sqrt{\mu}}{2}\sin\theta_n \tag{7.6.7}$$

$$C_{12}^{(1)}(n) = iD_\infty^2\left(\frac{\mu}{2} - \frac{\sqrt{\mu}}{2}\cos\left(\arcsin(x_0) - \theta_n\right)\right) \tag{7.6.8}$$

$$C_{21}^{(1)}(n) = \frac{i}{D_\infty^2}\left(\frac{\mu}{2} + \frac{\sqrt{\mu}}{2}\cos\left(\arcsin(x_0) + \theta_n\right)\right) \tag{7.6.9}$$

$$C_{22}^{(1)}(n) = \frac{\mu}{2}x_0 - \frac{\sqrt{\mu}}{2}\sin\theta_n \tag{7.6.10}$$

where

$$\theta_n = 2\eta_n + \varsigma, \tag{7.6.11}$$

with η_n *defined by (7.4.31),* μ *by (6.2.5) and*

$$\varsigma \overset{def}{=} -2\arg\Gamma\left(\frac{\gamma}{2} + \lambda\right) - \arg\left(\frac{\gamma}{2} + \lambda\right). \tag{7.6.12}$$

Proof. Taking into account (7.4.23) and (7.6.3) we conclude that the residue $\mathbf{C}^{(1)}(n)$ of $\mathbf{\Delta}_1(z,n)$ at $z = x_0$ is given by

$$\mathbf{C}^{(1)}(n) = \frac{-\mu\sqrt{1-x_0^2}}{2i}\mathbf{E}_n(x_0)\begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix}\mathbf{E}_n^{-1}(x_0). \tag{7.6.13}$$

Since $\mathbf{E}_n$ is analytic in a neighborhood of x_0 (see Proposition 7.1),

$$\mathbf{E}_n(x_0) = \lim_{\substack{z\to x_0 \\ z\in Q_+^R}}\mathbf{E}_n(z) = \frac{1}{\sqrt{2}\sqrt[4]{1-x_0^2}}D_\infty^{\sigma_3}\begin{pmatrix} e^{-i\arcsin(x_0)/2} & e^{i\arcsin(x_0)/2} \\ -e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix}e^{i\eta_n\sigma_3}, \tag{7.6.14}$$

so that

$$\mathbf{E}_n^{-1}(x_0) = \frac{1}{\sqrt{2}\sqrt[4]{1-x_0^2}}e^{-i\eta_n\sigma_3}\begin{pmatrix} e^{-i\arcsin(x_0)/2} & -e^{i\arcsin(x_0)/2} \\ e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix}D_\infty^{-\sigma_3}. \tag{7.6.15}$$

From (7.6.14) we obtain

$$\mathbf{C}^{(1)}(n) = \frac{-\mu}{4i}D_\infty^{\sigma_3}\begin{pmatrix} e^{-i\arcsin(x_0)/2} & e^{i\arcsin(x_0)/2} \\ -e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix}e^{i\eta_n\sigma_3}$$
$$\times\begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix}e^{-i\eta_n\sigma_3}\begin{pmatrix} e^{-i\arcsin(x_0)/2} & -e^{i\arcsin(x_0)/2} \\ e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix}D_\infty^{-\sigma_3}.$$

Using formulas (4.2.1), (4.2.4) and (6.2.4) we can rewrite

$$\tau_\lambda = -\frac{\overline{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}}{\left(\frac{\gamma}{2}+\lambda\right)\Gamma\left(\frac{\gamma}{2}+\lambda\right)} = -\left|\frac{\overline{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}}{\left(\frac{\gamma}{2}+\lambda\right)\Gamma\left(\frac{\gamma}{2}+\lambda\right)}\right|e^{i\varsigma} = -\frac{e^{i\varsigma}}{\sqrt{\gamma^2/4+|\lambda|^2}}, \tag{7.6.16}$$

where

$$\varsigma = \arg\left(\frac{\overline{\Gamma\left(\frac{\gamma}{2}+\lambda\right)}}{\left(\frac{\gamma}{2}+\lambda\right)\Gamma\left(\frac{\gamma}{2}+\lambda\right)}\right).$$

Then,

$$\mathbf{C}^{(1)}(n) = \frac{-\mu}{2} D_{\infty}^{\sigma_3} \begin{pmatrix} x_0 - \frac{\sin\theta_n}{\sqrt{\gamma^2/4+|\lambda|^2}} & -i + i\frac{\cos(\arcsin(x_0)-\theta_n)}{\sqrt{\gamma^2/4+|\lambda|^2}} \\ -i - i\frac{\cos(\arcsin(x_0)+\theta_n)}{\sqrt{\gamma^2/4+|\lambda|^2}} & -x_0 - \frac{\sin(-\theta_n)}{\sqrt{\gamma^2/4+|\lambda|^2}} \end{pmatrix} D_{\infty}^{-\sigma_3}.$$

We can simplify this expression using, the definition of μ (6.2.5), that

$$\mu = -\left(\lambda^2 - \frac{\gamma^2}{4}\right) = \left(\frac{\log^2 c}{\pi^2} + \frac{\gamma^2}{4}\right) = \left(\sqrt{\frac{\gamma^2}{4} + |\lambda|^2}\right)^2,$$

and this settles the proof. ■

7.7 Equivalence between Y-RHP and R-RHP

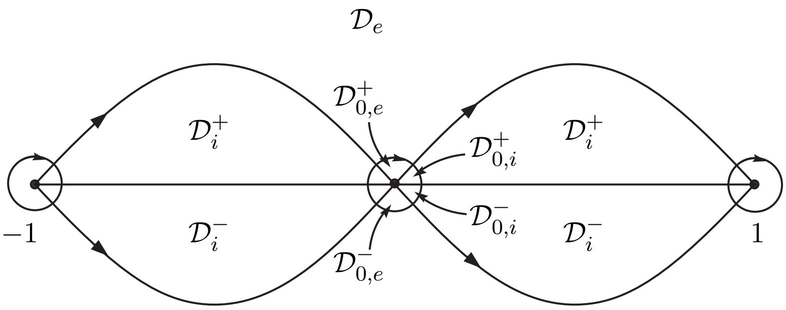

FIGURE 7.7.1: Domains for $\mathbf{Y}$

Unraveling the (invertible) transformations $\mathbf{Y} \to \mathbf{T} \to \mathbf{S} \to \mathbf{R}$ we can obtain an, equivalent, expression for $\mathbf{Y}$ and for its solution (3.1.1). We specify the following domains (see Fig. 7.7.1):

- $\mathcal{D}_e$ is the unbounded component of $\mathbb{C} \setminus \Sigma_R$;
- $\mathcal{D}_i^{\pm}$ corresponds to the portion of the inner domain exterior to U_ζ, $\zeta \in \{-1, x_0, 1\}$, lying in the upper (resp., lower) half-plane;
- $\mathcal{D}_{\zeta,e}^{\pm}$ is the subset of U_ζ in the outer domain and upper (resp., lower) half plane;
- $\mathcal{D}_{\zeta,i}^{\pm}$ is the subset of U_ζ in the inner domain and upper (resp., lower) half plane.

From (7.1.1), (7.2.6), and (7.5.1),

$$\mathbf{Y}(z,n) = \begin{cases} 2^{-n\sigma_3}\mathbf{R}(z)\,\mathbf{N}(z)\,\varphi(z)^{n\sigma_3}, & z \in \mathcal{D}_e; \\ 2^{-n\sigma_3}\mathbf{R}(z)\,\mathbf{N}(z)\begin{pmatrix} 1 & 0 \\ \pm\left(w_{c,\gamma}(z)\,\varphi(z)^{2n}\right)^{-1} & 1 \end{pmatrix}\varphi(z)^{n\sigma_3}, & z \in \mathcal{D}_i^{\pm}; \\ 2^{-n\sigma_3}\mathbf{R}(z)\,\mathbf{P}_\zeta(z)\,\varphi(z)^{n\sigma_3}, & z \in \mathcal{D}_{\zeta,e}^{\pm}; \\ 2^{-n\sigma_3}\mathbf{R}(z)\,\mathbf{P}_\zeta(z)\begin{pmatrix} 1 & 0 \\ \pm\left(w_{c,\gamma}(z)\,\varphi(z)^{2n}\right)^{-1} & 1 \end{pmatrix}\varphi(z)^{n\sigma_3}, & z \in \mathcal{D}_{\zeta,i}^{\pm}; \end{cases} \tag{7.7.1}$$

with $\zeta \in \{-1, x_0, 1\}$.

Thus, using the steepest descent method, we obtain, for the **Y**-RH problem, two equivalent solutions (3.1.1) and (7.7.1). Now we can obtain several results, that involve the entries of the solution (3.1.1), using the results that can be obtained from (7.7.1). We will use asymptotic expression for **R** derived above, and we will obtain information about the behavior of **Y** in different domains of the plane.

Chapter 8

Asymptotic Results for Orthogonal Polynomials

In this chapter we present several asymptotic results. In the first section the asymptotic behavior of the monic orthogonal polynomials away from $[-1,1]$ is described. In the second section we analyze the leading coefficients of the orthonormal polynomials. In the third section we study the recurrence coefficients and compare the result with the Magnus conjecture. In the fourth section the asymptotic behavior of the monic orthogonal polynomials, away from the singular point, on $(-1,x_0)\cup(x_0,1)$, is given. In both sections we compare the results with the those previously known, as [31, Theorem 1.4].

All these results are new and unpublished, and generalize the results that appeared in [22], with the exception of the result from section 3 that are contained in [23].

8.1 Monic orthogonal polynomials away from the interval of orthogonality

Let $P_n(z)$ be the monic orthogonal polynomial, of degree n, with respect to the weight $w_{c,\gamma}$ defined in (1.4.1).

Theorem 8.1.1 *We have that*

$$\frac{2^n P_n(z)}{\varphi(z)^n} = \frac{D_\infty}{D(z)} \frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2-1)^{1/4}} \left[1 + \frac{\mathcal{H}_n(z)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right], \qquad n\to\infty, \tag{8.1.1}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus[-1,1]$. *The function* $\mathcal{H}(z)$ *is analytic on* $\mathbb{C}\setminus[-1,1]$, *and given by*

$$\begin{aligned}\mathcal{H}_n(z) = &-\frac{4\alpha^2-1}{8(\varphi(z)-1)} + \frac{4\beta^2-1}{8(\varphi(z)+1)} + \frac{\sqrt{\mu}}{2(z-x_0)}\\ &\times\left(\sin\theta_n + \sqrt{\mu}\left(\frac{1}{\varphi(z)} - x_0\right) - \frac{\cos(\arccos x_0 - \theta_n)}{\varphi(z)}\right),\end{aligned} \tag{8.1.2}$$

with

$$\theta_n = 2n\arccos x_0 + 2\frac{\log c}{\pi}\log(n) - \Theta$$

$$\mu = \frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}$$

and

$$\begin{aligned}\Theta = &\left(\alpha+\frac{\gamma}{2}\right)\pi-(\alpha+\beta+\gamma)\arccos x_0-2\frac{\log c}{\pi}\log\left(4\sqrt{1-x_0^2}\right)\\ &-2\arg\Gamma\left(\frac{\gamma}{2}-i\frac{\log c}{\pi}\right)-\arg\left(\frac{\gamma}{2}-i\frac{\log c}{\pi}\right)-\frac{\sqrt{1-x_0^2}}{\pi}\fint_{-1}^{1}\frac{\log h(t)}{\sqrt{1-t^2}}\frac{dt}{t-x_0},\end{aligned} \tag{8.1.3}$$

where $\fint$ denotes the integral understood in terms of its principal value.

Szegő gave also asymptotic results for a general $w_{c,\gamma}$ (see it in subsection 5, Theorem 1.1.1).

Comparing with the Theorem 1.1.1, from Szegő, we give the exact first term of the asymptotic expansion, and also the second term. Observe that the first factor in the right hand side of (8.1.1) follows from the Szegő's theory (see subsection 5, Theorem 1.1.1), so our contribution here is in the next asymptotic term.

In [22] we considered the case $w_{c,0}$, while, here we give the result for the general $w_{c,\gamma}$.

The full asymptotic expansion for the classical Jacobi polynomials ($w_{1,0}$ with $h\equiv 1$) can be found in [57, Theorem 8.21.9], while for $w_{1,0}$ with general real analytic and positive h it was established in [31, Theorem 1.4].

Comparing the result given in [31, Theorem 1.4] for $w_{1,0}$ with that for $w_{c,\gamma}$ we see that the first term of the asymptotic expansion is the same and does not depend on $\alpha,\beta,\gamma,c,x_0$, but, the second one does: $\mathcal{H}_n(z)=\mathcal{H}^{\alpha}(z)+\mathcal{H}^{\beta}(z)+\mathcal{H}^{c,\gamma}(z,n)$ (the only influence of γ and c appears in $D(z)$ and D_∞). Furthermore, in $\mathcal{H}_n$ the first two terms $(\mathcal{H}^{\alpha}(z),\mathcal{H}^{\beta}(z))$ are the same than for $w_{1,0}$, but, the third term $(\mathcal{H}^{c,\gamma}(z,n))$ has an oscillatory behavior. This oscillatory term appears for $c\neq 1$ or $\gamma\neq 0$, vanishing for $c=1$ and $\gamma=0$.

Remark 8.1 *For $\alpha,\beta=\pm\frac{1}{2}$ (Chebychev polynomials), the terms $\mathcal{H}^{\alpha}(z)$ and $\mathcal{H}^{\beta}(z)$ vanish, and the second term of the asymptotic expansion just depends on c and γ.*

8.1.1 Proof of the theorem

If K is a compact subset of $\mathcal{D}_e$, then by (7.7.1),

$$\mathbf{Y}(z,n)=2^{-n\sigma_3}\mathbf{R}(z)\mathbf{N}(z)\varphi^{n\sigma_3}(z),\quad z\in K. \tag{8.1.4}$$

Since $P_n(z)=\mathbf{Y}_{11}(z,n)$, we get by (7.3.1)–(7.3.4) that

$$\frac{2^nP_n(z)}{\varphi(z)^n}=\frac{D_\infty}{D(z)}A_{11}(z)\mathfrak{R}(z), \tag{8.1.5}$$

with

$$\mathfrak{R}(z)\overset{\text{def}}{=}\mathbf{R}_{11}(z)-\frac{i}{D_\infty^2\varphi(z)}\mathbf{R}_{12}(z),\ z\in\mathbb{C}. \tag{8.1.6}$$

By (7.6.4), we can rewrite (8.1.6) as

$$\mathfrak{R}(z)=1+\frac{\mathcal{R}_n(z)}{n}+\mathcal{O}\left(\frac{1}{n^2}\right),\quad \text{as } n\to\infty, \tag{8.1.7}$$

where,

$$\mathcal{R}_n(z)\overset{\text{def}}{=}\left(\mathbf{R}^{(1)}\right)_{11}(z)-\frac{i}{D_\infty^2\varphi(z)}\left(\mathbf{R}^{(1)}\right)_{12}(z),\ z\in\mathbb{C}. \tag{8.1.8}$$

From (7.6.5) we see that $\mathcal{R}_n(z)$ is different on each part of the plane. We define

$$\mathcal{H}_n(z) \stackrel{\text{def}}{=} \mathcal{R}_n(z), \quad \text{as } z \in \mathbb{C} \setminus \{U_{-1} \cup U_{x_0} \cup U_1\}$$

Then, by (8.1.5) and (8.1.7), uniformly on K,

$$\frac{2^n P_n(z)}{\varphi(z)^n} = \frac{D_\infty}{D(z)} \frac{\varphi(z)^{1/2}}{\sqrt{2}\,(z^2-1)^{1/4}} \left[1 + \frac{\mathcal{R}_n(z)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right], \quad \text{as } n \to \infty.$$

Taking into account the expression for $\mathbf{R}^{(1)}$ in (7.6.5), as well as (7.6.6) and Proposition 7.2, we get that

$$\mathbf{R}_{11}^{(1)}(z) = \frac{1-4\alpha^2}{16(z-1)} + \frac{4\beta^2-1}{16(z+1)} + \frac{1}{z-x_0}\left[-\frac{\mu}{2}x_0 + \frac{\sqrt{\mu}}{2}\sin\theta_n\right], \tag{8.1.9}$$

$$\mathbf{R}_{12}^{(1)}(z) = iD_\infty^2\left(\frac{4\alpha^2-1}{16(z-1)} + \frac{4\beta^2-1}{16(z+1)} + \frac{1}{z-x_0}\left[\frac{\mu}{2} - \frac{\sqrt{\mu}}{2}\cos\left(\arcsin(x_0) - \theta_n\right)\right]\right). \tag{8.1.10}$$

The trivial identity yields

$$\frac{1}{z \pm 1}\left(1 \pm \frac{1}{\varphi(z)}\right) = \frac{2}{\varphi(z) \pm 1}, \tag{8.1.11}$$

and we conclude that in $K \subset \mathbb{C} \setminus \{U_{-1} \cup U_{x_0} \cup U_1\}$, $\mathcal{R}_n(z) = \mathcal{H}_n(z)$, with $\mathcal{H}_n(z)$ defined in (8.1.2).

8.2 Leading coefficient

For the leading coefficient k_n of the orthonormal polynomial $p_n(z) = k_n P_n(z)$ we have an asymptotic expansion in powers of $1/n$.

Theorem 8.2.1 *As $n \to \infty$,*

$$k_n = \frac{2^n}{\sqrt{\pi}D_\infty}\left[1 - \frac{1}{n}\left(\frac{4\alpha^2-1}{16} + \frac{4\beta^2-1}{16} + \frac{\sqrt{\mu}}{2}\right.\right.$$
$$\left.\left.\times\left(\sqrt{\mu} + \cos\left(\arcsin(x_0) + \theta_{n+1}\right)\right)\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right],$$

with

$$\theta_n = 2n\arccos x_0 + 2\frac{\log c}{\pi}\log(n) - \Theta$$
$$\mu = \frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}$$

where $\Theta = \Theta(\alpha, \beta, \gamma, c, x_0, h)$ is defined in (8.1.3).

Szegő gave also asymptotic results for a general $w_{c,\gamma}$ (see it in subsection 5, Theorem 1.1.1). Comparing with the Theorem 1.1.1, from Szegő, we give the exact first term of the asymptotic expansion, and also the second term. Observe that the first factor in the right hand side follows from the Szegő's theory (see subsection 5, Theorem 1.1.1), so our contribution here is in the next asymptotic term.

In [22] we considered the case $w_{c,0}$, here we give the result for the general $w_{c,\gamma}$.

Comparing the result given in [31, Theorem 1.6] for $w_{1,0}$ with that for $w_{c,\gamma}$ we see that the first term of the asymptotic expansion is the same and does not depend on $\alpha, \beta, \gamma, c, x_0$, but, the second one does: $\mathcal{H}_n = \mathcal{H}_\alpha + \mathcal{H}_\beta + \mathcal{H}_{c,\gamma}(n)$ (the only influence of γ and c appears in D_∞). Furthermore, in $\mathcal{H}_n$ the first two terms ($\mathcal{H}_\alpha, \mathcal{H}_\beta$) are the same than for $w_{1,0}$, but, the third term ($\mathcal{H}_{c,\gamma}(n)$) has has an oscillatory behavior. This oscillatory term appears for $c \neq 1$ or $\gamma \neq 0$, vanishing for $c = 1$ and $\gamma = 0$.

8.2.1 Proof of the theorem

By (3.1.1),

$$k_n^2 = -\frac{1}{2\pi i} \lim_{z\to\infty} z^{-n} \mathbf{Y}_{21}(z, n+1),$$

and with (7.7.1),

$$k_n^2 = -\frac{1}{2\pi i} \lim_{z\to\infty} \left[\left(\frac{2\varphi(z)}{z} \right)^{n+1} \left(z\mathbf{R}_{21}(z, n+1)\, \mathbf{N}_{11}(z) + z\mathbf{R}_{22}(z, n+1)\, \mathbf{N}_{21}(z) \right) \right].$$

Taking into account (N3), (7.3.2)–(7.3.4), and (7.6.5),

$$\mathbf{R}_{21}^{(1)}(z,n) = \frac{i}{D_\infty^2} \left(\frac{4\alpha^2-1}{16(z-1)} + \frac{4\beta^2-1}{16(z+1)} + \frac{1}{z-x_0} \left(\frac{\mu}{2} + \frac{\sqrt{\mu}}{2} \cos(\arcsin(x_0) + \theta_n) \right) \right) \tag{8.2.1}$$

$$\mathbf{R}_{22}^{(1)}(z,n) = \frac{4\alpha^2-1}{16(z-1)} - \frac{4\beta^2-1}{16(z+1)} + \frac{1}{z-x_0} \left(\frac{\mu}{2} x_0 - \frac{\sqrt{\mu}}{2} \sin\theta_n \right) \tag{8.2.2}$$

we see that,

$$\lim_{z\to\infty} z\mathbf{R}_{21}(z, n+1) = \frac{i}{nD_\infty^2} \left(\frac{2\alpha^2 + 2\beta^2 - 1}{8} + \frac{\mu}{2} + \frac{\sqrt{\mu}}{2} \cos(\arcsin(x_0) + \theta_{n+1}) \right) + \mathcal{O}\left(\frac{1}{n^2} \right),$$

and

$$\lim_{z\to\infty} \mathbf{N}_{11}(z) = 1, \qquad \lim_{z\to\infty} z\mathbf{N}_{21}(z) = -\frac{i}{2D_\infty^2}, \qquad \lim_{z\to\infty} \mathbf{R}_{22}(z, n+1) = 1.$$

Thus, we obtain

$$k_n^2 = \frac{4^n}{\pi D_\infty^2} \left[1 - \frac{1}{n} \left(\frac{4\alpha^2-1}{16} + \frac{4\beta^2-1}{16} + \frac{\mu}{2} + \frac{\sqrt{\mu}}{2} \cos(\arcsin(x_0) + \theta_{n+1}) \right) + \mathcal{O}\left(\frac{1}{n^2} \right) \right],$$

and this proves Theorem 8.2.1.

8.3 Recurrence coefficients

The monic polynomials P_n satisfy the three term recurrence relation

$$P_{n+1}(x) = (x - b_n)\, P_n(x) - a_n^2 P_{n-1}(x), \quad n = 0, 1, \ldots,$$

with $P_{-1}(x) = 0$ and $a_n > 0$, or its equivalent form (1.1.2). For the asymptotic behavior of the recurrence coefficients a_n and b_n we have the result above. This result, for general $w_{c,\gamma}$, was considered by us in [23].

Theorem 8.3.1 *The recurrence coefficients a_n and b_n of the orthogonal polynomials corresponding to the generalized Jacobi weight $w_{c,\gamma}$ (1.4.1) have a complete asymptotic expansion of the form*

$$a_n = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{A_k(n)}{n^k}, \quad b_n = -\sum_{k=1}^{\infty} \frac{B_k(n)}{n^k},$$

as $n \to \infty$, where for every $k \in \mathbb{N}$ the coefficients $A_k(n)$ and $B_k(n)$ are bounded in n. In particular,

$$A_1(n) = -\frac{\sqrt{1-x_0^2}}{2}\sqrt{\frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}} \cos\left[2n \arccos x_0 + 2\frac{\log c}{\pi}\log(n) - \Theta\right], \tag{8.3.1}$$

$$B_1(n) = -\sqrt{1-x_0^2}\sqrt{\frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}} \cos\left[(2n+1) \arccos x_0 + 2\frac{\log c}{\pi}\log(n) - \Theta\right], \tag{8.3.2}$$

where $\Theta = \Theta(\alpha, \beta, \gamma, c, x_0, h)$ defined in (8.1.3).

In theory, this approach allows us to compute all the coefficients A_k and B_k in (8.3.1)–(8.3.2). However, the computations are cumbersome and their complexity increases with k.

This theorem shows that $n(2a_n - 1)$ and nb_n have an oscillatory component with amplitude dependent on γ, c and x_0, vanishing for $\gamma = 0$ and $c = 1$; the wave is not periodic if there is a step-like singularity, $c \neq 1$, in this case the frequency depends on x_0. The influence of the all singularities (at x_0 and others) appears also in the phase shift Θ.

Theorem 8.3.1 generalizes some previous known results about the asymptotics of the recurrence coefficients. To mention a few, the weight $w_{1,0}$ was considered in [31, Theorem 1.10] (giving the first four terms), $w_{1,\gamma}$ is a particular case of the weight studied in [59].

For pure Jacobi weight ($w_{1,0}$ and $h = 1$), explicit formulas ([57] or [11]) show that the order $1/n$ term vanishes:

$$a_n = 1/2 + \mathcal{O}\left(1/n^2\right), \qquad b_n = \mathcal{O}\left(1/n^2\right).$$

In [31] it was proved that the same happens for the case $w_{1,0}$ (with general h). This theorem generalizes the case considered by [59]. From (8.3.1)–(8.3.2) it follows that for the general case $c \neq 1$, in contrast with the modified Jacobi weight ($w_{1,0}$), the order $1/n$ terms not vanish, and depends on $c \neq 1$ and $\gamma \neq 0$.

8.3.1 On Magnus conjecture

A. Magnus considered in [43] a weight function smooth and positive on the whole interval of orthogonality up to a finite number of points where algebraic singularities occur. His primary goal was to investigate the influence of these singular points on the asymptotic behavior of the recurrence coefficients of the corresponding orthogonal polynomials (generalized Jacobi polynomials). Based on numerical evidence, he conjectured explicit formulas for the asymptotics of these coefficients (as the degree of the polynomial grows) for the weights of the form

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}|x_0 - x|^{\gamma} \times \begin{cases} B, & \text{for } x \in [-1, x_0), \\ A, & \text{for } x \in [x_0, 1], \end{cases} \tag{8.3.3}$$

with A and $B > 0$ and α, β and $\gamma > -1$, and $x_0 \in (-1,1)$. He conjectured that:

Conjecture 8.3.2 (of Magnus [43]) *The recurrence coefficients of the corresponding orthogonal polynomials exhibit the following behavior $n \to \infty$:*

$$a_n = \frac{1}{2} - \frac{M}{n}\cos\left(2nt_0 - 2v\log(4n\sin t_0) - \widetilde{\Theta}\right) + o(1/n), \tag{8.3.4}$$

$$b_n = -\frac{2M}{n}\cos\left((2n+1)t_0 - 2v\log(4n\sin t_0) - \widetilde{\Theta}\right) + o(1/n), \tag{8.3.5}$$

where

$$x_0 = \cos(t_0),\ 0 < t_0 < \pi, \quad v = \frac{1}{2\pi}\log\frac{B}{A}, \quad M = \frac{1}{2}\sqrt{\frac{\gamma^2}{4} + v^2}\,\sin t_0,$$

$$\widetilde{\Theta} = \left(\alpha + \frac{\gamma}{2}\right)\pi - (\alpha+\beta+\gamma)t_0 - 2\arg\Gamma\left(\frac{\gamma}{2} + iv\right) - \arg\left(\frac{\gamma}{2} + iv\right).$$

This conjecture is proven if we rewrite Theorem 8.3.1 above as

Proposition 8.1 *As $n \to \infty$,*

$$a_n = \frac{1}{2} - \frac{M}{n}\cos\left[2n\arccos x_0 - 2v\log\left(4n\sqrt{1-x_0^2}\right) - \widetilde{\Theta}\right] + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$b_n = -\frac{2M}{n}\cos\left[(2n+1)\arccos x_0 - 2v\log\left(4n\sqrt{1-x_0^2}\right) - \widetilde{\Theta}\right] + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where

$$v = -\frac{\log c}{\pi}, \quad M = \frac{\sqrt{1-x_0^2}}{2}\sqrt{\frac{\gamma^2}{4} + v^2},$$

$$\widetilde{\Theta} = \left(\alpha + \frac{\gamma}{2}\right)\pi - (\alpha+\beta+\gamma)\arccos x_0 - 2\arg\Gamma\left(\frac{\gamma}{2} - i\frac{\log c}{\pi}\right) - \arg\left(\frac{\gamma}{2} - i\frac{\log c}{\pi}\right) - \frac{\sqrt{1-x_0^2}}{\pi}\fint_{-1}^{1}\frac{\log h(t)}{\sqrt{1-t^2}}\frac{dt}{t - x_0}.$$

A comparison of these formulas with those (8.3.4) and (8.3.5), setting $h(x) \equiv B$ and $c^2 = A/B$, shows that Magnus' conjecture on the asymptotic behavior of the recurrence coefficients holds true for weights of the form (8.3.3) (the singular integral holds 0 for constant h).

Observe that Theorem 8.3.1 is a slight extension of the original statement of Magnus:

1. we allow an extra real analytic and strictly positive factor h in the weight, and
2. we can replace the error term $o(1/n)$ in (8.3.4) and (8.3.5) (of [43]) by a more precise $\mathcal{O}(1/n^2)$.

8.3.2 Proof of the theorem

From [21] (see also [12] and [31]) it follows that the coefficients can be found directly in terms of the matrix $\mathbf{Y}$ in (7.7.1). Thus, to express formulas a_n and b_n, given in (3.1.7) and (3.1.8) respectively, in terms of $\mathbf{R}$ we use (7.7.1) for $z \in \mathcal{D}_e$, and from (3.1.7) and (3.1.8) we obtain

$$a_n^2 = \lim_{z\to\infty} z^2 \left(\mathbf{R}_{11}(z)\,\mathbf{N}_{12}(z) + \mathbf{R}_{12}(z)\,\mathbf{N}_{22}(z)\right)\left(\mathbf{R}_{21}(z)\,\mathbf{N}_{11}(z) + \mathbf{R}_{22}(z)\,\mathbf{N}_{21}(z)\right),$$

$$\begin{aligned} b_n = \lim_{z\to\infty} z\,[1 - &\left(\mathbf{R}_{11}(z,n+1)\,\mathbf{N}_{11}(z,n+1) + \mathbf{R}_{12}(z,n)\,\mathbf{N}_{21}(z,n+1)\right) \\ &\times \left(\mathbf{R}_{21}(z,n)\,\mathbf{N}_{12}(z,n) + \mathbf{R}_{22}(z,n)\,\mathbf{N}_{22}(z,n)\right)]. \end{aligned}$$

and from the behavior at infinity of $\mathbf{R}$ and $\mathbf{N}$, we obtain:

$$a_n^2 = \lim_{z\to\infty}\left(-\frac{D_\infty^2}{2i} + z\mathbf{R}_{12}(z,n)\right)\left(z\mathbf{R}_{21}(z,n) + \frac{1}{2iD_\infty^2}\right), \tag{8.3.6}$$

and

$$b_n = \lim_{z\to\infty} z\left(1 - \mathbf{R}_{11}(z,n+1)\,\mathbf{R}_{22}(z,n)\right). \tag{8.3.7}$$

Now, from (8.3.6), taking into account the expression for $\mathbf{R}^{(1)}$ in (8.1.10) and (8.2.1), we obtain for a_n:

$$a_n^2 = \frac{1}{4} - \frac{1}{n}\frac{\sqrt{\mu}}{2}\cos(\arcsin x_0)\cos(\theta_n) + O\left(\frac{1}{n^2}\right), \quad n\to\infty,$$

where θ_n is given by (7.6.11). It also can be rewritten in the form

$$\theta_n = \frac{2\log c}{\pi}\log(n) + 2n\arccos x_0 - \Theta,$$

with Θ defined in (8.1.3). This proves (8.3.1) (with μ defined in 6.2.5).

Analogously, from (8.3.7), taking into account the expression for $\mathbf{R}^{(1)}$ in (8.1.9) and (8.2.2), we obtain for b_n:

$$b_n = -\frac{\sqrt{\mu}\left(\sin\theta_{n+1} - \sin\theta_n\right)}{2n} + O\left(\frac{1}{n^2}\right), \quad n\to\infty.$$

By the definition of θ_n (7.6.11),

$$\theta_{n+1} - \theta_n = 2\arccos(x_0) + 2\frac{\log c}{\pi}\log\left(1 + \frac{1}{n}\right),$$

and

$$\begin{aligned} \sin\theta_{n+1} - \sin\theta_n &= \sin(\theta_n + 2\arccos x_0) - \sin\theta_n + \mathcal{O}\left(\frac{1}{n}\right) \\ &= 2\cos(\theta_n + \arccos x_0)\sin(\arccos x_0) + \mathcal{O}\left(\frac{1}{n}\right), \quad n\to\infty. \end{aligned}$$

Thus we obtain

$$b_n = -\frac{1}{n}\sqrt{\mu}\sin(\arccos x_0)\cos(\theta_n + \arccos x_0) + O\left(\frac{1}{n^2}\right),$$

which proves (8.3.2).

8.4 Monic orthogonal polynomials over the interval of orthogonality, away from singularities

For the monic orthogonal polynomials $P_n(z)$, we have also for $z \in (-1, x_0) \cup (x_0, 1)$ an equivalent result to Theorem 8.1.1 for general $w_{c,\gamma}$. In [22] we considered the case $w_{c,0}$, here we give the result for general $w_{c,\gamma}$.

Theorem 8.4.1 *For x on compact subsets of $(-1, x_0) \cup (x_0, 1)$, the following asymptotic formula holds uniformly, as $n \to \infty$,*

$$P_n(x) = \frac{2^{-n+1/2} D_\infty}{\sqrt{w_{c,\gamma}(x)}(1-x^2)^{1/4}} \operatorname{Re}\left[e^{i(\Omega(x)+n\arccos x - \arcsin(x)/2)} \left(1 + \frac{\mathcal{H}_n(x)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\right],$$

with

$$\begin{aligned}\mathcal{H}_n(x) = &-\frac{4\alpha^2-1}{8(e^{i\arccos x}-1)} + \frac{4\beta^2-1}{8(e^{i\arccos x}+1)} + \frac{\sqrt{\mu}}{2(x-x_0)} \\ &\times \left(\sin\theta_n + \sqrt{\mu}\left(\frac{1}{e^{i\arccos x}} - x_0\right) - \frac{\cos(\arccos x_0 - \theta_n)}{e^{i\arccos x}}\right),\end{aligned} \tag{8.4.1}$$

and

$$\Omega(x) \stackrel{def}{=} -\frac{\log c}{\pi} \log\left|\frac{1-x_0x+\sqrt{(1-x^2)(1-x_0^2)}}{x-x_0}\right| + \begin{cases} \Phi(x) + \frac{\pi\gamma}{2}, & -1 < x < x_0, \\ \Phi(x), & x_0 < x < 1, \end{cases} \tag{8.4.2}$$

$$\Phi(x) = \frac{\pi\alpha}{2} - \frac{\alpha+\beta+\gamma}{2}\arccos x - \frac{\sqrt{1-x^2}}{2\pi} \fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t-x}, \ x \in (-1,1),$$

where $\theta_n = \theta_n(x_0, c, \gamma, \alpha, \beta, h)$ and $\mu = \mu(c, \gamma)$ defined in (7.6.11) and (6.2.5), respectively.

Observe that a full asymptotic expansion for the usual Jacobi polynomials ($w_{1,0}$ with $h \equiv 1$) can be found in [57, Theorem 8.21.9], while for $w_{1,0}$ with general real analytic and positive h was established in [31, Theorem 1.12]. Szegő also gave asymptotic results for general weights like $w_{c,\gamma}$ [57, Theorem 12.1.4].

In [31, Theorem 1.12] for $w_{1,0}$, in this case as $x \in (-1, 1)$, only the first term was given explicitly. Here, for $w_{c,\gamma}$, we see that the first term of the asymptotic expansion is the same, and, does not depend on $\alpha, \beta, \gamma, c, x_0$, the only influence of γ and c appears in $D(z)$ and in the weight. The second term depends on $\alpha, \beta, \gamma, c, x_0$. If we write $\mathcal{H}_n(x) = \mathcal{H}^\alpha(x) + \mathcal{H}^\beta(x) + \mathcal{H}^{c,\gamma}(x, n)$ then the two first terms $(\mathcal{H}^\alpha(x), \mathcal{H}^\beta(x))$ are the same than for $w_{1,0}$, and the third term $(\mathcal{H}^{c,\gamma}(z, n))$ have an oscillatory behavior. Also this oscillatory term appears for $c \neq 1$ or $\gamma \neq 0$, and, vanishing for $c = 1$ and $\gamma = 0$.

8.4.1 Proof of the theorem

By analyticity of P_n's, it is sufficient to consider $z \in \mathcal{D}_i^+$.

Let K be a compact subset of $(-1, 1) \setminus \{x_0\}$ and take the δ small enough such that $K \subset (-1+\delta, x_0-\delta)$ or $K \subset (x_0+\delta, 1-\delta)$.

By (7.7.1), as $z \in \mathcal{D}_i^+$, inside the upper lens close the interval $(-1, 1) \setminus \{x_0\}$:

$$\mathbf{Y}(z, n) = 2^{-n\sigma_3}\mathbf{R}(z)\mathbf{N}(z)\begin{pmatrix} 1 & 0 \\ w_{c,\gamma}(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}\varphi(z)^{n\sigma_3}, \quad z \in K.$$

Since $P_n(z) = \mathbf{Y}_{11}(z, n)$ then, by (7.3.1)–(7.3.4)

$$2^n P_n(z) = \varphi(z)^n \frac{D_\infty}{D(z)} A_{11}(z)\mathfrak{R}(z) + \frac{\varphi(z)^{-n}}{w_{c,\gamma}(z)} D_\infty D(z) A_{12}(z)\widetilde{\mathfrak{R}}(z), \tag{8.4.3}$$

with $\mathfrak{R}$ defined by (8.1.6), and

$$\widetilde{\mathfrak{R}}(z) \stackrel{\text{def}}{=} \mathbf{R}_{11}(z) - \frac{i\varphi(z)}{D_\infty^2}\mathbf{R}_{12}(z), \ z \in \mathbb{C}. \tag{8.4.4}$$

By (7.6.4), we can rewrite (8.4.4) as

$$\widetilde{\mathfrak{R}}(z) = 1 + \frac{\widetilde{\mathcal{R}}_n(z)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty, \tag{8.4.5}$$

where,

$$\widetilde{\mathcal{R}}_n(z) \stackrel{\text{def}}{=} \left(\mathbf{R}^{(1)}\right)_{11}(z) - \frac{i\varphi(z)}{D_\infty^2}\left(\mathbf{R}^{(1)}\right)_{12}(z), \ z \in \mathbb{C}. \tag{8.4.6}$$

Then, from (8.4.3),

$$\begin{aligned} \frac{2^n P_n(z)}{D_\infty} &= \frac{\varphi(z)^n}{D(z)} A_{11}(z) \left(1 + \frac{\widetilde{\mathcal{R}}_n(z)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &\quad + \frac{D(z)}{w_{c,\gamma}(z)} \frac{A_{12}(z)}{\varphi(z)^n} \left(1 + \frac{\widetilde{\mathcal{R}}_n(z)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \end{aligned} \tag{8.4.7}$$

Taking into account the expression for $\mathbf{R}_{11}^{(1)}$ in (8.1.9) and $\mathbf{R}_{12}^{(1)}$ in (8.1.10), and using the identity (8.1.11) replacing $\varphi(z)$ by $\varphi(z)^{-1}$:

$$\frac{1}{z \pm 1}\,(1 \pm \varphi(z)) = \frac{2}{\varphi(z)^{-1} \pm 1},$$

we conclude that $\widetilde{\mathcal{R}}_n(z) = \widetilde{\mathcal{H}}_n(z), \ z \in \mathbb{C} \setminus \{U_{-1} \cup U_{x_0} \cup U_1\}$, letting $\widetilde{\mathcal{H}}_n(z)$ be defined by the formula for $\mathcal{H}_n(z)$ replaced $\varphi(z)$ by $\varphi(z)^{-1}$. Then, when we take limit $z \to x + 0i$, by (8.1.2), it is easy to see that

$$\overline{\mathcal{H}_n(x)} = \widetilde{\mathcal{H}}_n(x), \tag{8.4.8}$$

because, $\varphi(x)$ is the only one complex term in $\mathcal{H}_n(x)$, and $\overline{\varphi(x)} = \varphi(x)^{-1} = e^{-i\arccos x}$.

Furthermore, from Lemma 5.1, taking limit $z \to x + 0i$ we have

$$\frac{(D(x))_+}{w_{c,\gamma}(x)} = \frac{\exp(i\Omega(x))}{\sqrt{w_{c,\gamma}(x)}} = \overline{\frac{1}{(D(x))_+}},$$

where, using (5.2.4), we define Ω in (8.4.2).

Also, by (7.3.3) and (7.3.4), taking limit $z \to x + 0i$

$$(A_{11}(x))_+ = \frac{\varphi_+(x)^{1/2}}{\sqrt{2}(x^2-1)_+^{1/4}} = \overline{\frac{i\varphi_+(x)^{-1/2}}{\sqrt{2}(x^2-1)_+^{1/4}}} = \overline{(A_{12}(x))_+} = \frac{e^{-i\arcsin(x)/2}}{\sqrt{2}(1-x^2)^{1/4}}. \tag{8.4.9}$$

Gathering, all this information in (8.4.7), and taking limit $z \to x + 0i$ ($z \in \mathcal{D}_i^+$), we obtain the Theorem 8.4.1.

Chapter 9

Local Behavior

In this chapter we present the analysis of local behavior at the singular point x_0 which combine a root-type and a step-type singularity. In the first section we present the asymptotic behavior of the monic orthogonal polynomials in a neighborhood of x_0. In the second section we compute the limit of the reproducing kernel at x_0, which generalizes both the kernel obtained in [22] and the second Bessel kernel (1.2.11). All these results are new and unpublished, and generalize the results of [22].

9.1 Local asymptotic behavior for the monic orthogonal polynomials near the singular point in the bulk

For the monic orthogonal polynomials $P_n(x)$, we have also for $x \in (x_0 - \delta, x_0 + \delta)$ an equivalent result to Theorem 8.1.1 for a general $w_{c,\gamma}$. In [22] we considered the case $w_{c,0}$, here we give the result for a general $w_{c,\gamma}$.

Theorem 9.1.1 *For $0 < \delta < 1 - |x_0|$, locally uniformly on compact subsets of $(x_0 - \delta, x_0 + \delta)$, the following asymptotic formula holds, as $n \to \infty$,*

$$\begin{aligned} P_n(x) = {} & \frac{2^{-n+1/2} D_\infty \left|2ng(x)\right|^{\gamma/2}}{\sqrt{w_{1,\gamma}(x)c}(1-x^2)^{1/4}} \\ & \times \operatorname{Re}\left[e^{i(q(x)-ng(x)-\arcsin(x)/2)} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} \right. \\ & \left. {}_1F_1\left(\lambda+\frac{\gamma}{2},\gamma+1;2ing(x)\right)\left(1+\frac{\mathcal{R}_n(x)}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\right], \end{aligned} \tag{9.1.1}$$

where

$$q(x) \overset{def}{=} -\tfrac{\gamma\pi}{4} - \Phi(x) + \tfrac{\log c}{\pi}\log\left|\frac{2ng(x)\left(1-x_0x+\sqrt{(1-x^2)(1-x_0^2)}\right)}{x-x_0}\right| + n\arccos x_0, \tag{9.1.2}$$

$$g(x) = \arccos x_0 - \arccos x, \tag{9.1.3}$$

$$\mathcal{R}_n(x) = \mathcal{H}_n(x) + \frac{i\mu + \sqrt{\mu}e^{-i\left(\arccos x + 2q(x) - 2\arg\left(\Gamma\left(\frac{\gamma}{2}+\lambda\right)\right) - \arg\left(\frac{\gamma}{2}+\lambda\right)\right)}}{2ng(x)},$$

with $\lambda = i\log c/\pi$, $\mathcal{H}_n(x)$, $\mu = \mu(c,\gamma)$ and $\Phi(x) = \Phi(x,\alpha,\beta,\gamma,h)$, defined in (5.1.4), (8.4.1), (6.2.5) and (5.2.4), respectively.

This theorem shows that the local behavior at x_0 depends on the Confluent Hypergeometric function ${}_1F_1$ that is used in the solution of the local parametrix (6.1.5). Recall from [31] that at a singular points on the edge, ± 1, the local behavior depends on Bessel functions, J_a of first kind and order α (at 1) and (β at -1). Here we see that in the bulk the local behavior is given in terms of ${}_1F_1\left(i\frac{\log c}{\pi}+\frac{\gamma}{2},\gamma+1;\cdot\right)$ if $c\neq 1$, and, if $c=1\wedge\gamma\neq 0$, in terms of ${}_1F_1\left(\frac{\gamma}{2},\gamma+1;\cdot\right)$ which can be rewritten in terms of the Bessel functions J_a of orders $\gamma\pm\frac{1}{2}$ (see [59], [33] and [47]).

From the proof of Theorem 9.1.1, using (9.1.10), we have an asymptotic expression, for P_n, on a small disk of $\mathbb{C}$ centered at x_0 and radius δ:

Corollary 9.1 *Locally uniformly for $u\in(-\delta,\delta)$, with $0<\delta<1-|x_0|$, as $n\to\infty$,*

$$\begin{aligned}P_n\left(x_0+u_n\right)=&\frac{2^{-n+1/2}D_\infty\left|2n\right|^{\gamma/2}}{\sqrt{\left(1-x_0\right)^\alpha\left(1+x_0\right)^\beta\left(1-x_0^2\right)^{\frac{1+\gamma}{2}}c}}\\&\times\operatorname{Re}\left[e^{i(\eta_n-\pi u-\arcsin(x_0)/2)}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\gamma+1\right)}\right.\\&\left.{}_1F_1\left(\lambda+\frac{\gamma}{2},\gamma+1;2\pi ui\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\right],\end{aligned}\tag{9.1.4}$$

where η_n is defined in (7.4.31), and

$$u_n=\frac{u}{n\rho\left(x_0\right)}=\frac{u\pi\sqrt{1-x_0^2}}{n}.\tag{9.1.5}$$

9.1.1 Proof of the results

By analyticity of P_n's, it is sufficient to consider $z\in Q_+^L\cap\mathcal{D}_{x_0,i}^+$. Using formulas (7.4.4), (7.4.27) and (7.7.1) we get

$$\mathbf{Y}\left(z,n\right)=2^{-n\sigma_3}\mathbf{R}\left(z\right)\mathbf{E}_n\left(z\right)\mathbf{\Psi}\left(nf\left(z\right)\right)W\left(z\right)^{-\sigma_3}\varphi\left(z\right)^{-n\sigma_3}\begin{pmatrix}1&0\\\frac{\varphi(z)^{-2n}}{w_{c,\gamma}(z)}&1\end{pmatrix}\varphi\left(z\right)^{n\sigma_3}.\tag{9.1.6}$$

We are interested in the first column of $\mathbf{Y}$, which is obtained multiplying the r.h.s. of (9.1.6) from the right by the column vector $(1,0)^T$. Observe that using (7.4.5) and (7.4.9),

$$\frac{W\left(z\right)}{w_{c,\gamma}\left(z\right)}=\frac{\sqrt{w_{c,\gamma}\left(z\right)c}e^{i\frac{\pi\gamma}{2}}}{w_{c,\gamma}\left(z\right)}=\frac{e^{i\pi\gamma}c}{W\left(z\right)},$$

$$\begin{aligned}W\left(z\right)^{-\sigma_3}\varphi\left(z\right)^{-n\sigma_3}\begin{pmatrix}1&0\\\frac{\varphi(z)^{-2n}}{w_{c,\gamma}(z)}&1\end{pmatrix}\varphi\left(z\right)^{n\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}&=\begin{pmatrix}1/W(z)\\W(z)/w_{c,\gamma}(z)\end{pmatrix}\\&=\frac{1}{\sqrt{w_{c,\gamma}\left(z\right)c}e^{i\frac{\pi\gamma}{2}}}\begin{pmatrix}1\\e^{i\pi\gamma}c\end{pmatrix}.\end{aligned}$$

Thus,

$$\sqrt{w_{c,\gamma}c}e^{i\frac{\pi\gamma}{2}}\mathbf{Y}\left(z,n\right)\begin{pmatrix}1\\0\end{pmatrix}=2^{-n\sigma_3}\mathbf{R}\left(z\right)\mathbf{E}_n\left(z\right)\mathbf{\Psi}\left(nf\left(z\right)\right)\begin{pmatrix}1\\e^{i\pi\gamma}c\end{pmatrix}.\tag{9.1.7}$$

Notice that $Q_+^L \cap \mathcal{D}_{x_0,i}^+$ is mapped by f onto the sector denoted by ④ in Figure 6.1.1, and the vector $(1, e^{i\pi\gamma}c)^T$ corresponds to the first column of the jump matrix $\mathbf{J}_4$ in (7.4.14). Taking into account (Ψ2) we conclude that the product of the last two matrices in the right hand side of (9.1.7) is equal to the first column of $\mathbf{\Psi}$ in (6.1.8):

$$\mathbf{\Psi}\left(nf\left(z\right)\right)\begin{pmatrix}1\\ e^{i\pi\gamma}c\end{pmatrix}=\begin{pmatrix}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{4}}\\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{4}}\end{pmatrix}. \tag{9.1.8}$$

By (7.4.32),

$$\begin{aligned}\sqrt{w_{c,\gamma}}ce^{i\frac{\pi\gamma}{2}}\mathbf{Y}\left(z,n\right)\begin{pmatrix}1\\0\end{pmatrix}&=2^{-n\sigma_3}\mathbf{R}\left(z\right)D_\infty^{\sigma_3}\mathbf{A}(z)\,m_n(z)^{\sigma_3}\\&\times\begin{pmatrix}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{4}}\\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)e^{-\frac{\gamma\pi i}{4}}\end{pmatrix},\end{aligned} \tag{9.1.9}$$

with $\mathbf{A}$ and m_n defined in (7.3.2) and (7.4.33), respectively. Taking into account formulas (7.3.3)–(7.3.4), we conclude that

$$\begin{aligned}2^n\sqrt{w_{c,\gamma}}cP_n(z)&=D_\infty\\&\times\left\{A_{11}\left(z\right)\mathfrak{R}(z)\,m_n(z)\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;nf\left(z\right)\right)e^{-\frac{3\gamma\pi i}{4}}\right.\\&\left.+A_{12}\left(z\right)\widetilde{\mathfrak{R}}(z)\,m_n(z)^{-1}\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;nf\left(z\right)\right)e^{-\frac{3\gamma\pi i}{4}}\right\}\end{aligned}$$

where we used notation (8.1.6) and (8.4.4). Inserting again (7.6.4) we obtain the asymptotic expansion for $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ in (8.1.7) and (8.4.5) valid uniformly on compact subsets of U_{x_0}. Using the function $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_n$ defined in (8.1.8) and (8.4.6) we rewrite this identity for P_n as

$$\begin{aligned}&2^n\sqrt{w_{c,\gamma}}cP_n(z)=D_\infty\\&\times\left[A_{11}(z)\left(1+\frac{\mathcal{R}_n(z)}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)m_n(z)\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;nf\left(z\right)\right)e^{-\frac{3\gamma\pi i}{4}}\right.\\&\left.+A_{12}\left(z\right)\left(1+\frac{\widetilde{\mathcal{R}}_n(z)}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}\frac{G\left(1+\lambda+\frac{\gamma}{2},\gamma;nf\left(z\right)\right)e^{-\frac{3\gamma\pi i}{4}}}{m_n(z)}\right].\end{aligned} \tag{9.1.10}$$

Let us simplify this expression for the case that z is on the real line, taking limit $z \to x + 0i \in (x_0 - \delta, x_0)$ from the upper half plane.

Using m_n defined in (7.4.33), from (7.4.9), Lemma 5.1, (7.4.24), (5.2.1), (7.4.28) and (7.4.31), we can define the same way for z such that $\operatorname{Im} z > 0$,

$$\lim_{z\to x+i0,\ z\in\mathcal{D}_{x_0,i}^+\setminus\{x_0\}} m_n\left(z\right)=m_n(x)=e^{iq(x)} \tag{9.1.11}$$

with, q defined for $x \in (-1, x_0) \cup (x_0, 1)$ in (9.1.2).

Remark 9.1 *Recalling the definition of η_n in (7.4.31) we have $\lim_{x\to x_0} q\left(x\right) = \eta_n$, and, then we can extend the definition of $q\left(x\right)$ for all $x \in (-1, 1)$.*

Thus, from (9.1.11), on $(x_0-\delta,x_0)$, we have

$$\overline{m_n(x)} = m_n(x)^{-1}. \tag{9.1.12}$$

The function $G(\cdot,\cdot;ix)$ defined in (6.1.2) is not analytic on $\mathbb{C}$, because there is a power root factor. This factor should be understand as $(ix)^{\gamma/2} = i^{\gamma/2}(x)^{\gamma/2}$ with the main branch of the power root. Thus, we have

$$(ix)_+^{\gamma/2} = \begin{cases} |x|^{\frac{\gamma}{2}} e^{\frac{3\gamma\pi}{4}i}, & x<0, \\ |x|^{\frac{\gamma}{2}} e^{\frac{\gamma\pi}{4}i}, & x>0, \end{cases} \tag{9.1.13}$$

and, the factor $e^{-\frac{3\gamma\pi i}{4}}$ in formula (9.1.10) disappear when we take limit $z\to x+0i$ for $G(\cdot,\cdot;\cdot)$. Additionally, we have $\overline{\lambda}=-\lambda$ and for $x<0\in\mathbb{R}_-$, with account of formulas (4.2.4) and (4.3.10), we obtain,

$$ {}_1F_1\left(1+\lambda+\tfrac{\gamma}{2},\gamma+1;2xi\right)e^{-ix} = \overline{{}_1F_1\left(\lambda+\tfrac{\gamma}{2},\gamma+1;2xi\right)e^{-ix}}, \quad \text{and} \quad \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} = \overline{\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}}, \tag{9.1.14}$$

and, by (9.1.13),

$$e^{-\frac{3\pi}{4}\gamma i}G\left(1+\lambda+\frac{\gamma}{2},\gamma;ix\right) = \overline{e^{-\frac{3\pi}{4}\gamma i}G\left(\lambda+\frac{\gamma}{2},\gamma;ix\right)}. \tag{9.1.15}$$

From (8.4.9) we have too $(A_{11}(x))_+ = \overline{(A_{12}(x))_+}$.

Let us check what happens with $\mathcal{R}_n(z)$ and $\widetilde{\mathcal{R}}_n(z)$ on $(x_0-\delta,x_0)$. Observe however that now the explicit expression for $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_n$ defined in (8.1.8) and (8.4.6), on $\mathcal{D}_{0,i}^+$, differs from $\mathcal{H}_n$ defined in (8.1.2), and $\widetilde{\mathcal{H}}_n$, respectively. From (7.6.5) on $\mathcal{D}_{x_0}\subset\{U_{-1}\cup U_{x_0}\cup U_1\}$,

$$\mathcal{R}_n(z) = \mathcal{H}_n(z) - \left((\mathbf{\Delta}_1(z,n))_{11} - \frac{i}{D_\infty^2\varphi(z)}(\mathbf{\Delta}_1(z,n))_{12}\right),$$

$$\widetilde{\mathcal{R}}_n(z) = \widetilde{\mathcal{H}}_n(z) - \left((\mathbf{\Delta}_1(z,n))_{11} - \frac{i\varphi(z)}{D_\infty^2}(\mathbf{\Delta}_1(z,n))_{12}\right),$$

where $\mathcal{H}_n(z)$ is defined in (8.1.2) and $\widetilde{\mathcal{H}}_n(z)$ is obtained from $\mathcal{H}_n(z)$ replacing $\varphi(z)$ by $\varphi(z)^{-1}$. By (8.4.8), these are conjugates on $(-1,1)$. From (7.6.3), (7.4.29), (7.4.33) and (7.3.2)–(7.3.4), we obtain

$$(\mathbf{\Delta}_1(z,n))_{11} = \frac{-\mu}{nf(z)}\frac{\varphi(z)^2}{\varphi(z)^2-1}\left(-\frac{1}{\varphi^2(z)}-1+\frac{i}{\varphi(z)}\left(\tau_\lambda m_n(z)^2-\frac{\overline{\tau_\lambda}}{m_n(z)^2}\right)\right),$$

$$(\mathbf{\Delta}_1(z,n))_{12} = \frac{-\mu}{nf(z)}\frac{\varphi(z)^2}{\varphi(z)^2-1}D_\infty^2\left(\frac{2i}{\varphi(z)}+\tau_\lambda m_n(z)^2-\frac{\overline{\tau_\lambda}}{m_n^2(z)\varphi^2(z)}\right),$$

and, consequently,

$$\mathcal{R}_n(z)-\mathcal{H}_n(z) = \frac{\mu}{nf(z)}\left[-1-i\frac{\overline{\tau_\lambda}}{\varphi(z)m_n(z)^2}\right],$$

$$\widetilde{\mathcal{R}}_n(z)-\widetilde{\mathcal{H}}_n(z) = \frac{\mu}{nf(z)}\left[1-i\varphi(z)\tau_\lambda m_n(z)^2\right].$$

Thus, taking limit $z \to x + 0i$ on $(x_0 - \delta, x_0)$, by (7.4.24), (6.2.5), (5.2.1), (7.6.16), (9.1.12), and (8.4.8),

$$\overline{\mathcal{R}_n(x)} = \widetilde{\mathcal{R}}_n(x), \tag{9.1.16}$$

with

$$\mathcal{R}_n(x) = \mathcal{H}_n(x) + \frac{i\mu + \sqrt{\mu} e^{-i\left(\arccos x + 2q(x) - 2\arg\left(\Gamma\left(\frac{\gamma}{2}+\lambda\right)\right) - \arg\left(\frac{\gamma}{2}+\lambda\right)\right)}}{2n\left(\arccos x_0 - \arccos x\right)}. \tag{9.1.17}$$

Summarizing, from (9.1.10) taking limit $z \to x + 0i$ on $(x_0 - \delta, x_0)$,

$$\begin{aligned} 2^n \sqrt{w_{c,\gamma}(x)\, c} P_n(x) &= D_\infty \frac{2}{\Gamma(\gamma+1)} \\ &\times \operatorname{Re}\left[A_{11}(x)\left(1 + \frac{\mathcal{R}_n(x)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) m_n(x) \Gamma\left(1 - \lambda + \frac{\gamma}{2}\right) G\left(\lambda + \frac{\gamma}{2}, \gamma; nf(x)\right) e^{-\frac{3\gamma\pi i}{4}}\right]. \end{aligned}$$

Proceeding in the same way, for $x \in (x_0, x_0 + \delta)$, we obtain:

$$\begin{aligned} 2^n \sqrt{w_{c,\gamma}(x)\, c^{-1}} P_n(x) &= \frac{2D_\infty}{\Gamma(\gamma+1)} \\ &\times \operatorname{Re}\left[A_{11}(x)\left(1 + \frac{\mathcal{R}_n(x)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) m_n(x) \Gamma\left(1 - \lambda + \frac{\gamma}{2}\right) G\left(\lambda + \frac{\gamma}{2}, \gamma; nf(x)\right) e^{-\frac{\gamma\pi i}{4}}\right]. \end{aligned}$$

Using the definition of G in (6.1.2) with the observation (9.1.13), (1.4.1), (8.4.9), (9.1.11) and (7.4.24), and defining

$$g(x) \stackrel{\text{def}}{=} \frac{f(x)}{2i}$$

we obtain (9.1.3), and the proof of the Theorem 9.1.1 is complete.

Now, in order to prove the Corollary 9.1, we take into account the uniformly limit in (9.1.1). Let us take the following rescaling

$$x \longmapsto x_0 + \frac{u}{n\rho(x_0)} \tag{9.1.18}$$

with $u \in (-\delta, \delta)$ and $\rho(x)\,dx$, the absolutely continuous measure on $[-1, 1]$ defined in (1.1.12). Looking to (9.1.1), from (1.4.1) we have for $x > x_0$, $\sqrt{w_{c,\gamma}(x)\, c^{-1}} = \sqrt{w_{1,\gamma}(x)\, c}$ and for $x < x_0$, $\sqrt{w_{c,\gamma}(x)\, c} = \sqrt{w_{1,\gamma}(x)\, c}$, then

$$\sqrt{w_{1,\gamma}\left(x_0 + \frac{u}{n\rho(x_0)}\right) c} \underset{n\to\infty}{\longrightarrow} h(x_0)^{1/2} (1 - x_0)^{\alpha/2} (1 + x_0)^{\beta/2} \left|\frac{u}{n\rho(x_0)}\right|^{\gamma/2} c^{1/2}; \tag{9.1.19}$$

from (7.4.24) and (9.1.3),

$$nf(x) = 2ing(x) = 2in\left(\arccos x_0 - \arccos x\right) \underset{n\to\infty}{\longrightarrow} 2\pi u i; \tag{9.1.20}$$

and, from (9.1.2) and (7.4.31) $q(x) \underset{n\to\infty}{\longrightarrow} q(x_0) = \eta_n$. Gathering this in (9.1.1) we obtain:

$$\begin{aligned} P_n(x_0 + u_n) = \frac{2^{-n+1/2} D_\infty |2\pi u|^{\gamma/2}}{(1 - x_0^2)^{1/4} \sqrt{w_{1,0}(x_0)\, |u_n|^\gamma\, c}} &\operatorname{Re}\left[e^{i(\eta_n - \pi u - \arcsin(x_0)/2)} \frac{\Gamma\left(1 - \lambda + \frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}\right. \\ &\left. \times\ {}_1F_1\left(\lambda + \frac{\gamma}{2}, \gamma + 1; 2\pi u i\right)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)\right], \end{aligned} \tag{9.1.21}$$

and, taking into account the definition of u_n (9.1.5) and ρ (1.1.12) we obtain (9.1.4).

9.2 Limit behavior of the reproducing kernel

We show that the jump discontinuity combined with the algebraic singularity in the weight, both at x_0, lead to a different kernel, constructed in terms of the confluent hypergeometric function:

Theorem 9.2.1 *For $c > 0$, $c \neq 1$, and $\gamma > -1$, locally uniformly for u and v on $(-\delta, \delta)$, $0 < \delta < 1 - |x_0|$, when $u \neq v$,*

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \widetilde{K}_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \mathbb{K}_\infty^{c,\gamma}(u,v), \tag{9.2.1}$$

with

$$\mathbb{K}_\infty^{c,\gamma}(u,v) = c^\epsilon e^{i\pi\gamma(\frac{\epsilon}{2}-1)} \left| \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} \right|^2 \frac{\left[G\left(1+\lambda+\frac{\gamma}{2},\gamma;2\pi i u\right); G\left(\lambda+\frac{\gamma}{2},\gamma;2\pi i v\right)\right]}{2\pi i (u-v)} \tag{9.2.2}$$

where

$$\epsilon = \epsilon(u,v) \stackrel{def}{=} \frac{sgn(u) + sgn(v)}{2}, \tag{9.2.3}$$

$\lambda = i\log(c)/\pi$, G was introduced in (6.1.2), and as usual the Lie brackets, $[f(x); g(y)] = f(x)g(y) - f(y)g(x)$.

Kernel (9.2.2) is written in the so-called integrable form. Taking into account the properties of the functions in the right hand side, we can rewrite (9.2.2) alternatively as

$$\mathbb{K}_\infty^{c,\gamma}(u,v) = (2\pi)^\gamma c^\epsilon |uv|^{\frac{\gamma}{2}} \left| \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} \right|^2 \frac{\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iu)\, \widetilde{G}_{\lambda+\frac{\gamma}{2}}(iv) - \widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iv)\, \widetilde{G}_{\lambda+\frac{\gamma}{2}}(iu)}{2\pi i (u-v)} \tag{9.2.4}$$

where

$$\widetilde{G}_a(z) \stackrel{\text{def}}{=} \frac{G(a,\gamma;2\pi z)}{(2\pi z)^{\gamma/2}} = {}_1F_1(a;\gamma+1;2\pi z)\, e^{-z\pi}, \tag{9.2.5}$$

(note that $\widetilde{G}_a(z)$ is entire, and, for $\gamma = 0$, $\widetilde{G}_a(z) = G(a,0;2\pi z)$).

Taking into account (9.1.14), we can rewrite (9.2.4) in a totally real form, as:

$$\mathbb{K}_\infty^{c,\gamma}(u,v) = (2\pi)^\gamma c^\epsilon |uv|^{\frac{\gamma}{2}} \left| \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} \right|^2 \frac{\operatorname{Im}\left(\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iu)\, \widetilde{G}_{\lambda+\frac{\gamma}{2}}(iv)\right)}{\pi(u-v)}. \tag{9.2.6}$$

The confluent hypergeometric functions appeared in the scaling limit (as the number of particles goes to infinity) of the correlation functions of the pseudo-Jacobi ensemble in [8]. This ensemble corresponds to a sequence of weights of the form

$$(1+x^2)^{-n-\operatorname{Re}(s)} e^{2\operatorname{Im}(s)\arg(1+ix)}, \quad x \in \mathbb{R}, \tag{9.2.7}$$

where n is the degree of the polynomial and s is a complex parameter. The connection between both problems becomes apparent if we perform the inversion $x \mapsto 1/(\pi x)$ in (9.2.7); this creates at the origin an algebraic singularity with the exponent $\operatorname{Re}(s)$ and a jump depending on

$\mathrm{Im}(s)$. $\mathbb{K}_\infty^{c;\gamma}$ is a particular case of the reproducing kernel obtained by Borodin and Olshansky in Theorem 2.1 of [8] when $\mathrm{Re}(s) = \gamma/2$.

Let us analyze some particular cases of the kernel (9.2.2) or, equivalently, (9.2.4). For $\gamma = 0$, for a weight $w_{c,0}$ with a step-like singularity at x_0, the kernel at x_0 obtained could be written in the following way:

Corollary 9.2 *For $c > 0$, $c \neq 1$, and $\gamma = 0$, locally uniformly for u and v on $(-\delta, \delta)$, $0 < \delta < 1 - |x_0|$, when $u \neq v$,*

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \widetilde{K}_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \mathbb{K}_\infty^{c,0}(u,v),$$

with

$$\mathbb{K}_\infty^{c,0}(u,v) = c^{1+\epsilon} \frac{\log c}{c^2 - 1} \frac{[G(1+\lambda, 0; 2\pi i u); G(\lambda, 0; 2\pi i v)]}{\pi i (u-v)} \tag{9.2.8}$$

where λ, G, $\epsilon = \epsilon(u,v)$, were introduced in (5.1.4), (6.1.2) and (9.2.3), respectively, and, $[f(x); g(y)]$ is the usual Lie brackets.

This corollary generalizes for an arbitrary $x_0 \in (-1,1)$ the result given by us in [22] (see also [56]).

For $c = 1$, for a weight $w_{1,\gamma}$ with a power root singularity at x_0, the kernel at x_0 obtained could be written in the following way:

Theorem 9.2.2 *For $c = 1$, $\gamma > -1$, locally uniformly for u and v on $(-\delta, \delta)$, $0 < \delta < 1 - |x_0|$, when $u \neq v$,*

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \widetilde{K}_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \mathbb{K}_\infty^{1,\gamma}(u,v),$$

and,

$$\mathbb{K}_\infty^{1,\gamma}(u,v) \equiv e^{i\pi\gamma\left(\frac{\epsilon-1}{2}\right)} \mathbb{J}_{\gamma/2}^{o}(u,v),$$

with $\mathbb{J}_{\gamma/2}^{o}$ given by (1.2.11), ϵ by (9.2.3), and $\mathbb{K}_\infty^{1,\gamma}$ given by (9.2.2) or (9.2.4).

This theorem shows that $\mathbb{K}_\infty^{c;\gamma}$ generalizes the second Bessel kernel $\mathbb{J}_{\gamma/2}^{o}$ that appears, e.g. in [33] and [47].

For $c = 1$ and $\gamma = 0$, since, by definition of G (6.1.2) and (4.3.27), $G(1,0;xi) = \exp(ix/2)$, and $G(0,0;ix) = \overline{G(1,0;ix)} = \exp(-ix/2)$, by (9.2.6) we obtain

$$\mathbb{K}_\infty^{1,0}(u,v) = \frac{\mathrm{Im}\left(e^{i\pi u} e^{-i\pi v}\right)}{\pi(u-v)} = \frac{\sin(\pi(u-v))}{\pi(u-v)} = \mathbb{S}(u,v).$$

This shows that as $c \to 1$ and $\gamma \to 0$, $\mathbb{K}_\infty^{c,\gamma}$ reduces to the sine kernel (1.2.5).

9.2.1 Proofs of the theorems

Using the Christoffel-Darboux formula [57, Section 3.2], we can write the kernel in (1.2.1) as

$$\begin{aligned} K_n(x,y) &= \frac{k_{n-1}}{k_n}\frac{p_n(x)\,p_{n-1}(y) - p_n(y)\,p_{n-1}(x)}{x-y} \\ &= k_{n-1}^2\frac{P_n(x)\,P_{n-1}(y) - P_n(y)\,P_{n-1}(x)}{x-y}, \quad x \neq y, \end{aligned}$$

where by (3.1.1) we obtain

$$\begin{aligned} K_n(x,y) &= \frac{-1}{2\pi i}\frac{\mathbf{Y}_{11}^{(n)}(x)\,\mathbf{Y}_{21}^{(n)}(y) - \mathbf{Y}_{11}^{(n)}(y)\,\mathbf{Y}_{21}^{(n)}(x)}{x-y} \\ &= \frac{1}{2\pi i}\frac{1}{x-y}\begin{pmatrix}0, & 1\end{pmatrix}\mathbf{Y}^{-1}(y,n)\,\mathbf{Y}(x,n)\begin{pmatrix}1\\0\end{pmatrix}. \end{aligned} \tag{9.2.9}$$

By analyticity, it is obviously sufficient to compute K_n when $x, y \in (x_0 - \delta, x_0)$. From (9.1.9),

$$\begin{aligned} \sqrt{w_{c,\gamma}(x)\,c}\,e^{\frac{\gamma\pi i}{2}}\mathbf{Y}_+(x,n)\begin{pmatrix}1\\0\end{pmatrix} &= 2^{-n\sigma_3}\mathbf{R}(x)\,D_\infty^{\sigma_3}\mathbf{A}_+(x)\,m_n(x)^{\sigma_3} \\ \times &\begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;nf(x)\right)e^{-\frac{\gamma\pi i}{4}} \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;nf(x)\right)e^{-\frac{\gamma\pi i}{4}} \end{pmatrix}, \end{aligned} \tag{9.2.10}$$

On the other hand, since $\det \mathbf{Y} = 1$, we have by (9.1.6), (7.4.29), (7.3.1), (7.4.33),

$$\begin{aligned} \mathbf{Y}(y,n)^{-1} &= \varphi(y)^{-n\sigma_3}\begin{pmatrix}1 & 0\\ -\frac{1}{w_{c,\gamma}(y)}\varphi^{-2n} & 1\end{pmatrix}\varphi(y)^{n\sigma_3}\,W(y)^{\sigma_3}\left[\boldsymbol{\Psi}(nf(y))\right]^{-1} \\ &\times m_n(y)^{-\sigma_3}\mathbf{A}_+(y)^{-1}D_\infty^{-\sigma_3}\mathbf{R}(y)^{-1}\,2^{n\sigma_3}. \end{aligned}$$

Matrix $\boldsymbol{\Psi}$ built in (6.1.6)–(6.1.11) also satisfies $\det\boldsymbol{\Psi} = 1$, so that considerations that lead us to (9.1.9) show that

$$\begin{aligned} \sqrt{w_{c,\gamma}(y)\,c}\,e^{i\frac{\pi\gamma}{2}}\begin{pmatrix}0, & 1\end{pmatrix}\mathbf{Y}(y,n)^{-1} &= \begin{pmatrix} -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;nf(y)\right)e^{-\frac{\gamma\pi i}{4}} \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;nf(y)\right)e^{-\frac{\gamma\pi i}{4}} \end{pmatrix}^T \\ &\times m_n(y)^{-\sigma_3}\mathbf{A}_+(y)^{-1}D_\infty^{-\sigma_3}\mathbf{R}(y)^{-1}\,2^{n\sigma_3}. \end{aligned} \tag{9.2.11}$$

Observe also that locally uniformly for $z \in (x_0 - \delta, x_0 + \delta)$, $\mathbf{R}(z) = \mathbf{I} + \mathcal{O}(1/n)$, which implies that

$$\mathbf{R}(y)^{-1}\,\mathbf{R}(x) = \mathbf{I} + \mathcal{O}(1/n), \quad \text{locally uniformly for } x, y \in (x_0 - \delta, \delta + x_0).$$

Let be, using ρ defined in (1.1.12),

$$v_n \stackrel{\text{def}}{=} \frac{v}{n\rho(x_0)}.$$

Gathering (9.2.10) and (9.2.11) in (9.2.9) and taking the limits as (9.1.18), we conclude that for $u \neq v$, and $u, v < 0$,

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \sqrt{w_{c,\gamma}(x_0+v_n)}\sqrt{w_{c,\gamma}(x_0+u_n)}ce^{i\frac{3\pi\gamma}{2}} K_n(x_0+u_n, x_0+v_n) =$$
$$= \frac{1}{2\pi i}\frac{1}{u-v}\left(-\tfrac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right), \tfrac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right)\right)$$
$$\times \begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \end{pmatrix}.$$

Analogously, for $u \neq v$, and $u, v > 0$ we obtain:

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \sqrt{w_{c,\gamma}(x_0+v_n)}\sqrt{w_{c,\gamma}(x_0+u_n)}c^{-1}e^{i\frac{\pi\gamma}{2}} K_n(x_0+u_n, x_0+v_n) =$$
$$= \frac{1}{2\pi i}\frac{1}{u-v}\left(-\tfrac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right), \tfrac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right)\right)$$
$$\times \begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \end{pmatrix},$$

and for $u \neq v$, and $sgn(u) \neq sgn(v)$,

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \sqrt{w_{c,\gamma}(x_0+v_n)}\sqrt{w_{c,\gamma}(x_0+u_n)}e^{i\gamma\pi} K_n(x_0+u_n, x_0+v_n) =$$
$$= \frac{1}{2\pi i}\frac{1}{u-v}\left(-\tfrac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right), \tfrac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\tfrac{\gamma}{2},\gamma;2\pi iv\right)\right)$$
$$\times \begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}G\left(1+\lambda+\frac{\gamma}{2},\gamma;2\pi iu\right) \end{pmatrix}.$$

Using (9.1.20) and using the normalized kernel (1.2.2), we obtain (9.2.2).

To obtain formula (9.2.4) we need just to take into account the definition of G (6.1.2) and (9.1.13); and formula (9.2.6) follows from (9.2.4) and (9.1.15).

We obtain (9.2.8) from formula (9.2.2), if we take $\gamma = 0$ and we use the following identity; by formula (4.2.7) and the definition of λ in (5.1.4),

$$\Gamma(1+\lambda)\Gamma(1-\lambda) = \frac{\log c}{\sinh(\log c)} = \frac{2c\log c}{c^2-1}.$$

To prove the Theorem 9.2.2, observe that, using the Lie Brackets, let be

$$K = \left[\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+1;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma+1;2\pi iv\right)\right],$$

then, from formula (4.3.12) solved in order to $\phi(a,b;z)$,

$$K = \frac{\gamma}{\frac{\gamma}{2}-\lambda}\left[\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+1;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iv\right)\right],$$

using formula (4.3.13) solved in order to $\phi(a+1,b;z)$ and replacing a by $a+1$,

$$K = \frac{\gamma}{\frac{\gamma}{2}-\lambda}\frac{2\pi i}{\gamma+1}\left[u\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+2;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iv\right)\right]$$
$$+\frac{\gamma}{\frac{\gamma}{2}-\lambda}\left[\phi\left(\lambda+\frac{\gamma}{2},\gamma+1;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iv\right)\right],$$

and, using formula (4.3.12) solved in order to $\phi(a,b-1;z)$, in the second Lie bracket,

$$K = \frac{\gamma}{\frac{\gamma}{2}-\lambda}\frac{2\pi i}{\gamma+1}\left[u\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+2;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iv\right)\right]$$
$$+\frac{\gamma}{\frac{\gamma}{2}-\lambda}\frac{\lambda+\frac{\gamma}{2}}{\gamma}(-K),$$

then,

$$K = 2i\left[\frac{\pi u}{\gamma+1}\phi\left(1+\lambda+\frac{\gamma}{2},\gamma+2;2\pi iu\right);\phi\left(\lambda+\frac{\gamma}{2},\gamma;2\pi iv\right)\right].$$

Now, being $\lambda=0$, by (4.3.25) as

$$\phi\left(1+\frac{\gamma}{2},\gamma+2;2\pi iu\right)e^{-i\pi u} = J_{\frac{\gamma+1}{2}}(\pi u)\,\Gamma\left(1+\frac{\gamma+1}{2}\right)\left(\frac{\pi u}{2}\right)^{-\frac{\gamma+1}{2}}, \tag{9.2.12}$$

and

$$\phi\left(\frac{\gamma}{2},\gamma;2\pi iv\right)e^{-i\pi u} = J_{\frac{\gamma-1}{2}}(\pi v)\,\Gamma\left(1+\frac{\gamma-1}{2}\right)\left(\frac{\pi v}{2}\right)^{-\frac{\gamma-1}{2}}. \tag{9.2.13}$$

Using formula (4.2.3) we obtain,

$$\Gamma\left(\frac{\gamma+1}{2}\right) = \sqrt{\pi}2^{-\gamma}\frac{\Gamma(\gamma+1)}{\Gamma\left(\frac{\gamma}{2}+1\right)}. \tag{9.2.14}$$

Thus, from (9.2.12), by (9.2.14) and (4.2.1), we obtain

$$\frac{\Gamma\left(1+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}(2\pi iu)^{\frac{\gamma}{2}}\frac{(\pi u)}{\gamma+1}\phi\left(1+\frac{\gamma}{2},\gamma+2;2\pi iu\right)e^{-i\pi u} = J_{\frac{\gamma+1}{2}}(\pi u)\sqrt{\pi u}\sqrt{\frac{\pi}{2}}i^{\frac{\gamma}{2}}, \tag{9.2.15}$$

and, from (9.2.13), by (9.2.14), we obtain

$$\frac{\Gamma\left(1+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}(2\pi iv)^{\frac{\gamma}{2}}\phi\left(\frac{\gamma}{2},\gamma;2\pi iv\right)e^{-i\pi v} = J_{\frac{\gamma-1}{2}}(\pi v)\sqrt{\pi v}\sqrt{\frac{\pi}{2}}i^{\frac{\gamma}{2}}.$$

Gathering all this in formula (9.2.2), with $\lambda=0$, we obtain,

$$\mathbb{K}_\infty^{1,\gamma}(u,v) = e^{i\pi\gamma\left(\frac{\epsilon}{2}-1\right)}i^\gamma\pi\sqrt{u}\sqrt{v}\frac{J_{\frac{\gamma+1}{2}}(\pi u)\,J_{\frac{\gamma-1}{2}}(\pi v) - J_{\frac{\gamma+1}{2}}(\pi v)\,J_{\frac{\gamma-1}{2}}(\pi u)}{2(u-v)}$$
$$= e^{i\pi\gamma\left(\frac{\epsilon-1}{2}\right)}\mathbb{J}^o_{\frac{\gamma}{2}}(u,v),$$

where we used (1.2.11). Theorem 9.2.2 is proved.

Chapter 10

Consequences

In this chapter we give some applications of the local analysis at x_0 for the weight $w_{c,\gamma}$ performed in the previous chapter. We analyze the distribution of zeros for the particular case corresponding to $w_{c,0}$, where the weight has a step-like discontinuity, only. This is a relevant case because provides an example where the weight is nonzero in the bulk of the support of the equilibrium measure (see the discussion of the work of Levin and Lubinsky, sections 1.1.2 and 1.2).

The study of the zeros of P_n's at the origin, the analysis of the clock behavior and the connection with the De Branges spaces requires some further properties of the confluent hypergeometric function ${}_1F_1(\cdot;\cdot;z)$, which we were unable to find in the literature and which might have an independent interest.

In the first section we summarize and prove some properties of the confluent hypergeometric function ${}_1F_1\left(\lambda+\frac{\gamma}{2};1+\gamma;z\right)$.

In the second section we analyze the asymptotic distribution of the zeros at x_0 for the weight $w_{c,0}$.

In the third section we show that the kernel obtained, at x_0 for the weight $w_{c,\gamma}$, is related to a De Branges space.

All these results are new and, with exception of those in the second section, are unpublished, generalizing the results that were published in [22].

10.1 Properties of the Confluent Hypergeometric Function

The following result was needed to prove the results in the next two sections. Let us denote by $\overline{HB}$, the *Hermite-Biehler class*, the set of entire functions E with no zeros in the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and such that $|E(z)| \geq |E(\overline{z})|$ for $z \in \mathbb{C}^+$.

Proposition 10.1 *Let $a, b \in \mathbb{R}$ with $a \neq 0$ and $b > -1/2$. Then*

(i) The functions

$$f_1(z) = \widetilde{G}_{ai+b}(iz) = e^{-iz/2}{}_1F_1(ai+b;1+2b;iz)$$
$$f_2(z) = \overline{\widetilde{G}_{ai+b}(i\overline{z})} = \overline{e^{-i\overline{z}/2}{}_1F_1(1+ai+b;1+2b;i\overline{z})}$$

(see (9.2.5)) belong to the Hermite-Biehler class $\overline{HB}$.

(ii) *For* $x \in \mathbb{R}$,

$$ {}_1F_1\left(ai+b;1+2b;ix\right) \neq 0 \quad \textit{and} \quad {}_1F_1\left(1+ai+b;1+2b;ix\right) \neq 0. \tag{10.1.1}$$

In particular, all zeros of ${}_1F_1\left(ai+b;1+2b;iz\right)$ *lie in the lower half plane* $\mathbb{C}^-$*, while the zeros of* ${}_1F_1\left(1+ai+b;1+2b;iz\right)$ *lie in the upper half plane* $\mathbb{C}^+$*. Additionally,*

$$ \left|{}_1F_1\left(1+ai+b;1+2b;iz\right)\right| \leq \left|{}_1F_1\left(ai+b;1+2b;iz\right)\right|, \quad \operatorname{Im} z \geq 0, \tag{10.1.2}$$

and the equality holds only for $z \in \mathbb{R}$.

(iii) *The function*

$$ y(x) \stackrel{def}{=} \arg {}_1F_1\left(ai+b;1+2b;ix\right), \quad y(0)=0, $$

is real-analytic and is the solution of the following initial value problem:

$$ xy' = a\left(\cos\left(x-2y\right)-1\right) + b\sin\left(x-2y\right), \quad y(0)=0. \tag{10.1.3}$$

In particular for $b=0$*, if* $a>0$*, the function, is real-analytic and non-positive, strictly increasing on the negative and strictly decreasing on the positive semiaxis; if* $a<0$ *the same assertion is valid replacing* $y(x)$ *by* $-y(x)$*.*

(iv) *For* $a \in \mathbb{R}$*, the function*

$$ \mathfrak{G}(x) \stackrel{def}{=} x - 2\arg\left({}_1F_1\left(ai+b;1+2b;ix\right)\right) = x - 2y(x), \quad \mathfrak{G}(0)=0, $$

is strictly increasing in $\mathbb{R}$.

Remark 10.1 *The assertion in* (i) *does not imply that* ${}_1F_1\left(ai+b;1+2b;iz\right) \in \overline{HB}$*, and in general, this is not true.*

Interesting enough, the proof of (i) *is based on some properties of the Christoffel-Darboux kernel observed by Lubinsky in [41]. In this sense, the theory of the confluent hypergeometric functions has benefited from the properties of the reproducing kernels. In the opposite direction, the strict inequality in (10.1.2) implies that* $\mathbb{K}_\infty(z,\overline{z}) > 0$ *for* $z \in \mathbb{C}\setminus\mathbb{R}$ *and* $c = e^{a\pi} \neq 1$*, see (10.1.6) below.*

The next result follows from the solution of the local problem at x_0.

10.1.1 Proofs of the results

(i) Let K_n be the Christoffel-Darboux kernel defined in (1.2.1). Then for $z \in \mathbb{C}$,

$$ K_n(z,\overline{z}) = \sum_{k=0}^{n-1} \left|p_k\left(z\right)\right|^2 > 0. $$

This property is obviously inherited in the limit (9.2.4) and (10.3.2), which implies that

$$ \mathbb{K}_\infty^{c,\gamma}(z,\overline{z}) \geq 0, \quad z \in \mathbb{C}. $$

On the other hand, from formula (4.3.10) and definition of $\widetilde{G}$ in (9.2.5) it follows that

$$\overline{{}_1F_1\left(ia+b;1+2b;i\overline{z}\right)} = {}_1F_1\left(-ia+b;1+2b;-iz\right) = e^{-zi}{}_1F_1\left(1+ia+b;1+2b;iz\right), \quad (10.1.4)$$

$$\overline{\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(i\overline{z}\right)} = \widetilde{G}_{1+\lambda+\frac{\gamma}{2}}\left(iz\right). \quad (10.1.5)$$

Thus, for $z\in\mathbb{C}\setminus\mathbb{R}$, $\lambda = ia$, $a\in\mathbb{R}\setminus\{0\}$, $b=\gamma/2>-1/2$,

$$\begin{aligned}\mathbb{K}_{\infty}^{c;\gamma}\left(z,\overline{z}\right) &= const\,\frac{\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(iz\right)\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}\left(i\overline{z}\right)-\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}\left(iz\right)\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(i\overline{z}\right)}{\operatorname{Im} z}\\ &= const\,\frac{\left|\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(iz\right)\right|^2-\left|\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(i\overline{z}\right)\right|^2}{\operatorname{Im} z}\geq 0, \quad (10.1.6)\end{aligned}$$

which yields that

$$\left|\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(iz\right)\right| \geq \left|\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(i\overline{z}\right)\right|, \quad \operatorname{Im} z>0. \quad (10.1.7)$$

Assume that for $\zeta\in\mathbb{C}^+$, $\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(i\zeta\right)=0$; then by (10.1.5) and (10.1.7),

$$\widetilde{G}_{\lambda+\frac{\gamma}{2}}(i\overline{\zeta}) = \widetilde{G}_{1+\lambda+\frac{\gamma}{2}}\left(i\zeta\right)=0.$$

Hence,

$${}_1F_1\left(ia+b;1+2b;i\zeta\right) = {}_1F_1\left(1+ia+b;1+2b;i\zeta\right)=0.$$

By induction and recurrence relation (4.3.11) we conclude that every ${}_1F_1\left(ia+b+n;1+2b;i\zeta\right)$, with $n\in\mathbb{Z}$, vanishes. But this is impossible, as follows from the addition formula

$${}_1F_1\left(\lambda+\frac{\gamma}{2};1+\gamma;z+\zeta\right) = \left(\frac{\zeta}{z+\zeta}\right)^{\lambda+\frac{\gamma}{2}}\sum_{n=0}^{\infty}\frac{(\lambda+\gamma/2)_n\,z^n}{n!\,(z+\zeta)^n}\,{}_1F_1\left(\lambda+\frac{\gamma}{2}+n;1+\gamma;\zeta\right) \quad (10.1.8)$$

(see [54, formula (2.3.4)]).

Thus, we conclude that $f_1(z)=\widetilde{G}_{\lambda+\frac{\gamma}{2}}\left(iz\right)\in\overline{HB}$. The assertion for f_2 is obtained by means of formula (10.1.4).

(ii) In order to prove (10.1.1) assume that $a\in\mathbb{R}\setminus\{0\}$ and $x\in\mathbb{R}$. Then by (10.1.4),

$$\overline{{}_1F_1\left(ia+b;1+2b;ix\right)} = {}_1F_1\left(-ia+b;1+2b;-ix\right) = e^{-xi}{}_1F_1\left(1+ia+b;1+2b;ix\right).$$

Hence, an assumption that, for $x\in\mathbb{R}$, ${}_1F_1\left(ia+b;1+2b;ix\right)=0$ implies also that

$${}_1F_1\left(1+ia+b;1+2b;ix\right)=0,$$

and we arrive at a contradiction reasoning as above and using the addition formula (10.1.8).

Furthermore, the location of the zeros in the corresponding half planes and the inequality (10.1.2) is a direct consequence of *(i)*. In particular, the function

$$h(z)=\frac{{}_1F_1\left(1+ia+b;1+2b;iz\right)}{{}_1F_1\left(ia+b;1+2b;iz\right)}$$

is holomorphic in $\mathbb{C}^+$, continuous in $\overline{\mathbb{C}^+}$, and satisfies $|h(z)|\leq 1$ for $z\in\overline{\mathbb{C}^+}$ and $|h(z)|=1$ for $z\in\mathbb{R}$. In consequence, by the maximum principle, $|h(z)|<1$ for $z\in\mathbb{C}^+$, which proves that the inequality in (10.1.2) for z in the upper half plane is strict.

(iii) Due to *(ii)*, $y(x)$ is correctly defined and real-analytic on $\mathbb{R}$, in particular, y' can vanish only at a discrete set of points that can accumulate only at infinity. Again by (4.3.10),

$$\frac{{}_1F_1\left(1+ia+b;1+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)} = e^{ix}\,\frac{{}_1F_1\left(-ia+b;1+2b;-ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)} = e^{i(x-2y(x))}\,. \tag{10.1.9}$$

With the straightforward identity (4.3.13),

$${}_1F_1\left(1+ia+b;1+2b;ix\right) - {}_1F_1\left(ia+b;1+2b;ix\right) = \frac{ix}{1+2b}\,{}_1F_1\left(1+ia+b;2+2b;ix\right)$$

we rewrite (10.1.9) as

$$e^{i(x-2y(x))} = 1 + \frac{ix}{1+2b}\,\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}. \tag{10.1.10}$$

and we obtain

$$\begin{aligned}\cos\left(x-2y\right) &= 1 - \frac{x}{1+2b}\,\mathrm{Im}\left(\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}\right),\\ \sin\left(x-2y\right) &= \frac{x}{1+2b}\,\mathrm{Re}\left(\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}\right).\end{aligned}$$

On the other hand, note as $y\left(x\right) = \arg f\left(x\right) = \mathrm{Im}\left(\log\left(f\left(x\right)\right)\right)$, by (4.3.14),

$$\begin{aligned}y'(x) &= \mathrm{Im}\left(\frac{d}{dx}\log\left({}_1F_1\left(ia+b;1+2b;ix\right)\right)\right)\\ y'\left(x\right) &= \mathrm{Im}\left(i\frac{ia+b}{1+2b}\,\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}\right)\\ y'\left(x\right) &= \frac{1}{1+2b}\left(-a\,\mathrm{Im}\left(\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}\right) + b\,\mathrm{Re}\left(\frac{{}_1F_1\left(1+ia+b;2+2b;ix\right)}{{}_1F_1\left(ia+b;1+2b;ix\right)}\right)\right) \qquad (10.1.11)\\ y'\left(x\right) &= -a\left(\frac{1-\cos\left(x-2y\right)}{x}\right) + b\frac{\sin\left(x-2y\right)}{x},\end{aligned}$$

and we obtain (10.1.3).

Observe that, for $b=0$, since the real part of the left hand side of (10.1.10) is ≤ 1, this implies that

$$\mathrm{Im}\left(\frac{{}_1F_1\left(1+ia;2;ix\right)}{{}_1F_1\left(ia;1;ix\right)}\right)\begin{cases}\geq 0, & \text{for } x>0,\\ \leq 0, & \text{for } x<0,\end{cases} \tag{10.1.12}$$

and, (10.1.11) is now

$$y'(x) = \mathrm{Im}\left(\frac{d}{dx}\log\left({}_1F_1\left(ia;1;ix\right)\right)\right) = -a\,\mathrm{Im}\left(\frac{{}_1F_1\left(1+ia;2;ix\right)}{{}_1F_1\left(ia;1;ix\right)}\right), \tag{10.1.13}$$

and by inequality (10.1.12), the second part of the statement of *(iii)* follows.

In order to prove *(iv)* we observe that $\mathfrak{G}$ satisfies the following initial value problem:

$$x\,\mathfrak{G}'(x) = x + 2a\left(1-\cos\mathfrak{G}(x)\right) - 2b\sin\mathfrak{G}(x), \quad \mathfrak{G}(0)=0. \tag{10.1.14}$$

For $a, b = 0$ the statement is trivial.

Since $\mathfrak{G}$ is also real analytic, expanding it at $x = 0$

$$\mathfrak{G}'(x) = 1 + 2a\left(\frac{x}{2}\mathfrak{G}'(0) + \mathcal{O}\left(x^2\right)\right) - 2b\left(\mathfrak{G}'(0) + \frac{x}{2}\mathfrak{G}''(0) + \mathcal{O}\left(x^2\right)\right)$$

and we readily conclude from (10.1.14) that

$$\mathfrak{G}'(0) = \frac{1}{1+2b} > 0.$$

Hence, $\mathfrak{G}$ is locally increasing at the origin. Differentiating (10.1.14) we obtain that

$$x\,\mathfrak{G}''(x) = 1 + \mathfrak{G}'(x)\left(-1 + 2a\sin\mathfrak{G}(x) - 2b\cos\mathfrak{G}(x)\right). \tag{10.1.15}$$

If for $x = \zeta < 0$, $\mathfrak{G}'(\zeta) = 0$, then by (10.1.15),

$$\zeta\,\mathfrak{G}''(\zeta) = 1. \tag{10.1.16}$$

In particular, $\mathfrak{G}''(\zeta) < 0$, which shows that every critical point of $\mathfrak{G}$ in the negative semi-axis is a strict local maximum, which is incompatible with the behavior at the origin (because at 0, $\mathfrak{G}'$ is positive and this implies that there should be a minimum for $\zeta < 0$). Thus, $\mathfrak{G}'$ is sign-invariant on $(-\infty, 0)$, and in consequence, $\mathfrak{G}'(x) > 0$ there.

For $x = \zeta > 0$, if we have $\mathfrak{G}'(\zeta) = 0$, then we get (10.1.16), which shows that every critical point of $\mathfrak{G}$ in the positive semi-axis is a strict local minimum, which is again incompatible with the behavior at the origin. We conclude that $\mathfrak{G}'$ is sign-invariant on $(0, \infty)$, and in consequence, $\mathfrak{G}'(x) > 0$ for all $x \in \mathbb{R}$. With this we end the proof of Proposition 10.1.

10.2 Clock-Behavior and Universality Problem

See Figure 10.2.1 for a typical behavior of the function in the right hand side of (9.1.4) close to the origin, taking $\gamma = 0$.

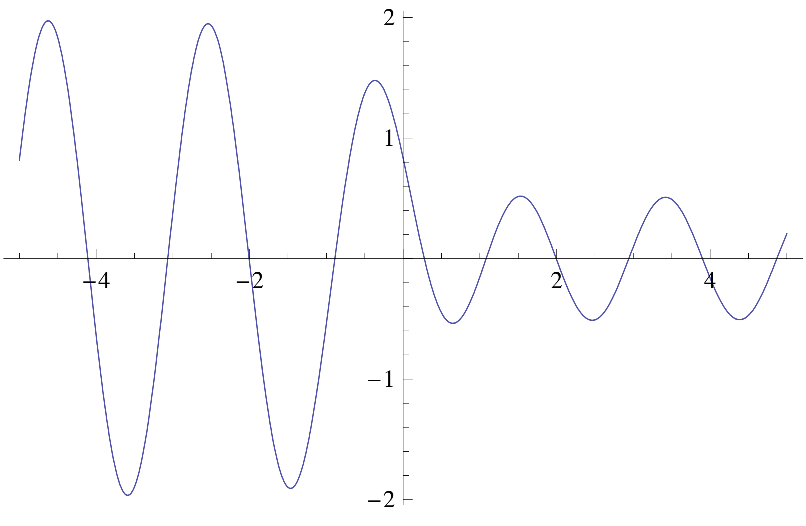

FIGURE 10.2.1: Typical graphics of the r.h.s. of (9.1.4) near the origin.

Recall that P_n has n simple zeros, all lying on $(-1,1)$. It is well known that they distribute asymptotically in the weak-* sense according to the equilibrium measure of the interval. In other words, the normalized zero counting measure for the sequence P_n weakly tends to the absolutely continuous measure on $[-1,1]$ given by $\rho(x)\,dx$, with $\rho(x) = \left(\pi\sqrt{1-x^2}\right)^{-1}$ defined in (1.1.12). It is known that at any point of $(-1,x_0)\cup(x_0,1)$ they distribute very precisely in accordance with $\rho(x)$, complying with the so-called "clock behavior", see e.g. [51]. If, we enumerate the zeros of P_n near the $x_0 \in (-1,1)$ by $x_j^{(n)}(x_0) = x_j^{(n)}$ as in (1.1.13), it follows,

$$\cdots < x_{-k}^{(n)} < \cdots < x_{-1}^{(n)} < x_0 \le x_0^{(n)} < \cdots < x_k^{(n)} < \ldots$$

then "clock behavior" at the origin (where $\rho(x_0) = \left(\pi\sqrt{1-x_0^2}\right)^{-1}$) means as in (1.1.14):

$$\lim_{n\to\infty} \frac{n}{\rho(x_0)^{-1}} \left(x_{j+1}^{(n)} - x_j^{(n)}\right) = 1, \quad j \in \mathbb{Z}.$$

Proposition 10.2 *If $c > 1$, then the sequence $\left\{\frac{n\,x_0^{(n)}}{\pi\sqrt{1-x_0^2}}\right\}$ is dense in an interval of the form $[x_0, t]$, where $t = t(c) < 1$. Furthermore,*

$$0 < \liminf_n \frac{n}{\rho(x_0)^{-1}} \left(x_k^{(n)} - x_{k-1}^{(n)}\right) \le \limsup_n \frac{n}{\rho(x_0)^{-1}} \left(x_k^{(n)} - x_{k-1}^{(n)}\right) < 1, \quad k \in \mathbb{N},$$

and

$$\liminf_n \frac{n}{\rho(x_0)^{-1}} \left(x_k^{(n)} - x_{k-1}^{(n)}\right) > 1, \quad -k \in \mathbb{N}.$$

In particular, the clock behavior of the zeros of P_n at x_0 does not hold.

If $0 < c < 1$, the same inequalities hold inverting the roles of k and $-k$.

This result is not surprising, taking into account that $x = x_0$ is not even a Lebesgue point for the weight $w_{c,0}$, that is, regardless of the meaning we give to $w_{c,0}(x_0)$,

$$\lim_{s\to 0^+} \frac{1}{s} \int_{-s}^{s} |w_{c,0}(x) - w_{c,0}(x_0)|\, dx \ne 0.$$

However, the weight $w_{c,0}$ with $c \ne 1$ provides the first instance of an explicit orthogonality measure for which the clock behavior fails in the bulk.

Remark 10.2 *A weaker condition than (1.1.14) is the quasi-clock behavior (see [51]), namely*

$$\lim_{n\to\infty} \frac{x_{j+1}^{(n)} - x_j^{(n)}}{x_1^{(n)} - x_0^{(n)}} = 1, \quad j \in \mathbb{Z}.$$

This limit is violated in our situation too. However,

$$\lim_{j\to\pm\infty} \liminf_{n\to\infty} \frac{n}{\rho(x_0)^{-1}} \left(x_{j+1}^{(n)} - x_j^{(n)}\right) = \lim_{j\to\pm\infty} \limsup_{n\to\infty} \frac{n}{\rho(x_0)^{-1}} \left(x_{j+1}^{(n)} - x_j^{(n)}\right) = 1, \tag{10.2.1}$$

which shows a smooth transition to the genuine clock behavior as we move away from the jump of the weight.

10.2.1 Proof of the result

In order to prove Proposition 10.2, using (10.1.1), we can rewrite (9.1.4) as

$$P_n\left(x_0+\frac{x}{n\rho(x_0)}\right)=\frac{2^{-n+1/2}D_\infty\,\Upsilon(c)}{\sqrt[4]{\left(1-x_0^2\right)^{1+\gamma}}\sqrt{c\,w_{1,0}\left(x_0\right)}}\left|{}_1F_1\left(\lambda;1;2\pi ix\right)\right|$$
$$\times\operatorname{Im}\left[e^{\frac{i}{2}\left(\widetilde{\theta}_n-\mathfrak{G}(2\pi x)\right)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\right],\tag{10.2.2}$$

where $\mathfrak{G}$ is the function introduced in *(iii)* of Proposition 10.1, corresponding to $a=\log(c)/\pi$, and the factor

$$\Gamma\left(1-\lambda\right)=\overline{\Gamma\left(1+\lambda\right)}=\overline{\lambda\Gamma\left(\lambda\right)}=-i\frac{\log c}{\pi}\left|\Gamma\left(\lambda\right)\right|e^{-i\arg\Gamma(\lambda)}$$
$$=-i\frac{\log c}{\sqrt{\log c\sinh\left(\log c\right)}}e^{-i\arg\Gamma(\lambda)}=-i\Upsilon(c)e^{-i\arg\Gamma(\lambda)},$$

with $\Upsilon(c)$ given by

$$\Upsilon(c)\stackrel{\text{def}}{=}sgn(\log(c))\sqrt{\frac{2c\log c}{c^2-1}},\quad c\neq 1,\quad \Upsilon(1)=1,\tag{10.2.3}$$

and

$$\widetilde{\theta}_n=2\eta_n-2\arg\Gamma\left(\frac{\gamma}{2}+\lambda\right)-\arcsin x_0\tag{10.2.4}$$
$$=\theta_n+\arg\left(\frac{\gamma}{2}+\lambda\right)-\arcsin x_0.$$

Without lost of generality, we will consider $x_0=0$, and only the case $c>1$ (the other case can be easily reduced to $c>1$ by a change of variables $x\mapsto -x$). Then $\mathfrak{G}$ is strictly increasing in $\mathbb{R}$. If we denote by

$$\dots<\zeta_{-k}^{(n)}<\dots<\zeta_{-1}^{(n)}<0\leq\zeta_0^{(n)}<\dots<\zeta_k^{(n)}<\dots$$

the solutions of

$$\frac{1}{2\pi}\,\mathfrak{G}(2\pi x)\equiv\frac{\widetilde{\theta}_n}{2\pi}\operatorname{mod}\left(\mathbb{Z}\right),$$

then by (10.2.2),

$$\lim_n\left(\frac{n}{\pi}\,x_k^{(n)}-\zeta_k^{(n)}\right)=0,\quad k\in\mathbb{Z},\tag{10.2.5}$$

where we have used notation (1.1.13). Since $\mathfrak{G}(0)=0$, we have that $\zeta_0^{(n)}$ is given by

$$\frac{1}{2\pi}\,\mathfrak{G}(2\pi x)=\left\{\frac{\widetilde{\theta}_n}{2\pi}\right\},$$

where $\{\cdot\}$ is the fractional part of the number, which by strict monotonicity of $\mathfrak{G}$ shows that

$$\frac{1}{2\pi}\,\mathfrak{G}\left(2\pi\zeta_k^{(n)}\right)=\left\{\frac{\widetilde{\theta}_n}{2\pi}\right\}+k,\quad k\in\mathbb{Z}.\tag{10.2.6}$$

In particular,

$$[k,k+1)\ni\frac{1}{2\pi}\mathfrak{G}\left(2\pi\zeta_k^{(n)}\right)=\frac{1}{2\pi}\left(2\pi\zeta_k^{(n)}-2\arg\left({}_1F_1\left(\lambda;1;2\pi i\zeta_k^{(n)}\right)\right)\right)\geq\zeta_k^{(n)},$$

where we have used *(iv)* of Proposition 10.1. Hence,

$$0\leq\zeta_0^{(n)}<1\quad\text{and}\quad\zeta_{k-1}^{(n)}<\zeta_k^{(n)}<k+1,\quad k\in\mathbb{Z}.$$

By compactness and diagonal argument, we can always select a subsequence $\Lambda\subset\mathbb{N}$ such that the following limits exist:

$$\lim_{n\in\Lambda}\zeta_k^{(n)}=\zeta_k,\quad k\in\mathbb{Z}.$$

By (10.2.6),

$$\frac{1}{2\pi}\mathfrak{G}\left(2\pi\zeta_k^{(n)}\right)-\frac{1}{2\pi}\mathfrak{G}\left(2\pi\zeta_{k-1}^{(n)}\right)=1,$$

and taking limits we conclude that

$$\zeta_k-\zeta_{k-1}=1+\frac{1}{\pi}\left(\arg\left({}_1F_1\left(\lambda;1;2\pi i\zeta_k\right)\right)-\arg\left({}_1F_1\left(\lambda;1;2\pi i\zeta_{k-1}\right)\right)\right).\tag{10.2.7}$$

Let $k\in\mathbb{N}$; since, by *(iii)* of Proposition 10.1, $\arg\left({}_1F_1\left(\lambda;1;2\pi i\zeta_k\right)\right)$ is strictly decreasing in $[0,+\infty)$, the second term in the right hand side of (10.2.7) is <0, so we conclude that

$$0<\zeta_k-\zeta_{k-1}<1,\quad k\in\mathbb{N}.$$

By (10.2.5), we obtain that

$$0<\liminf_n\frac{n}{\pi}\left(x_k^{(n)}-x_{k-1}^{(n)}\right)\leq\limsup_n\frac{n}{\pi}\left(x_k^{(n)}-x_{k-1}^{(n)}\right)<1,\quad k\in\mathbb{N}.$$

In the same vein, since $\arg\left({}_1F_1\left(\lambda;1;2\pi i\zeta_k\right)\right)$ is strictly increasing in $(-\infty,0)$, by (10.2.7),

$$\zeta_k-\zeta_{k-1}>1,\quad -k\in\mathbb{N},$$

so that

$$\liminf_n\frac{n}{\pi}\left(x_k^{(n)}-x_{k-1}^{(n)}\right)>1,\quad -k\in\mathbb{N}.$$

Furthermore, observe that for $c\neq1$, the accumulation points of the sequence $\zeta_0^{(n)}$ is dense in the interval $\mathfrak{G}^{-1}([0,2\pi])$. Indeed, by (7.4.31), (7.6.11) and (10.2.4),

$$\frac{\widetilde{\theta}_n}{2\pi}=n\frac{\arccos x_0}{\pi}+\frac{\log c}{\pi^2}\log n+\upsilon,$$
$$\upsilon\stackrel{\text{def}}{=}\frac{\log c\log\left(4\sqrt{1-x_0^2}\right)}{\pi^2}-\frac{2\alpha+\gamma}{2}+\frac{\alpha+\beta+\gamma}{2}\frac{\arccos x_0}{\pi}+\frac{\hbar(x_0)}{\pi}-\frac{\arg\Gamma(\lambda)}{\pi}+\frac{\arg(\lambda)-\arcsin x_0}{2\pi}.$$

Since we are taking $x_0=0$, and $c\neq1$, we can always take $b\in\{2,3\}$ such that $(\log c)(\log b)/\pi^2\notin\mathbb{Q}$ (indeed, otherwise we would have that $\log 3/\log 2$ is rational, which is obviously impossible). For such a b, with $n=2\,b^m$, $m\in\mathbb{N}$, equation (10.2.6) is rewritten as

$$\frac{1}{2\pi}\mathfrak{G}\left(2\pi\zeta_0^{(n)}\right)=\left\{\frac{\log c}{\pi^2}\log 2+m\frac{\log c}{\pi^2}\log b+\upsilon\right\}.$$

By Kronecker-Weyl Theorem (see, e.g. [10, Chapter III]), the sequence

$$\left\{\frac{\log c}{\pi^2}\log 2 + m\frac{\log c}{\pi^2}\log 2 + v\right\}$$

is dense in $(0,1)$, and it remains to use the strict monotonicity of $\mathfrak{G}$. This finishes the proof of Proposition 10.2.

Remark 10.3 *Obviously, if* $c=1$, $\lambda=0$, ${}_1F_1(\lambda;1;2\pi iy)\equiv 1$, *and we obtain the clock behavior via (10.2.5) and (10.2.7).*

Regarding Remark 10.2, it is easy to check numerically that in general

$$\zeta_k - \zeta_{k-1} \neq \zeta_{k+1} - \zeta_k, \quad k \in \mathbb{Z},$$

which shows that the quasi-clock behavior fails too. Nevertheless, for function $y(x)$ introduced in Proposition 10.1 we obtain applying (10.1.3) that $y'(x) = \mathcal{O}(1/|x|)$, $|x|\to\infty$, from where the "clock behavior in the limit" (10.2.1) follows by (10.2.7).

10.3 Reproducing Kernel of the De Branges Space

The reproducing kernel of the a De Branges space is different than standard reproducing kernel (1.2.1) for an orthogonal polynomial system. It involves involves a conjugate variable (10.3.3). Following a recent paper of Lubinsky [41], we need establish the standard reproducing kernel defined in (1.2.1).

In Theorem 9.2.1 we use the normalized reproducing kernel (1.2.2), and, first we will give an equivalent theorem but for the standard reproducing kernel defined in (1.2.1).

Theorem 10.3.1 *For* $c>0$, $c\neq 1$, *locally uniformly for* u *and* v *on* $(-\delta,\delta)$, $0<\delta<1-|x_0|$,

$$\lim_{n\to\infty}\frac{1}{n\rho(x_0)}K_n\left(x_0+\frac{u}{n\rho(x_0)}, x_0+\frac{v}{n\rho(x_0)}\right) = K_\infty(u,v), \tag{10.3.1}$$

with

$$\begin{aligned} K_\infty(u,v) &= \frac{(2n)^\gamma}{w_{1,0}(x_0)\left(1-x_0^2\right)^{\gamma/2}c}\left|\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}\right|^2 \\ &\times\begin{cases}\dfrac{\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iu)\,\widetilde{G}_{\lambda+\frac{\gamma}{2}}(iv)-\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iv)\,\widetilde{G}_{\lambda+\frac{\gamma}{2}}(iu)}{2\pi i(u-v)}, & u\neq v,\\ \left(\widetilde{G}'_{1+\lambda+\frac{\gamma}{2}}(iu)\,\widetilde{G}_{\lambda+\frac{\gamma}{2}}(iu)-\widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iu)\,\widetilde{G}'_{\lambda+\frac{\gamma}{2}}(iu)\right), & u=v,\end{cases}\end{aligned} \tag{10.3.2}$$

where $\lambda = i\log(c)/\pi$, $\widetilde{G}_a(\cdot)$ *was introduced in (9.2.5) and, by the definition (1.4.1),* $w_{1,0}(x_0) = h(x_0)(1-x_0)^\alpha(1+x_0)^\beta$.

Several remarks are in order.

Since $\widetilde{G}'_{1+\lambda+\frac{\gamma}{2}}(0) = 2\pi i\frac{\lambda+1/2}{\gamma+1}$, $\widetilde{G}'_{\lambda+\frac{\gamma}{2}}(0) = 2\pi i\frac{\lambda-1/2}{\gamma+1}$ and $\widetilde{G}_{\lambda+\frac{\gamma}{2}}(0) = 1$, evaluating $K_\infty(x_0, x_0)$ in (10.3.2) we conclude that

$$\lim_{n\to\infty} \frac{K_n\left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{u}{n\rho(x_0)}\right)}{K_n(x_0, x_0)} = \frac{\widetilde{G}'_{1+\lambda+\frac{\gamma}{2}}(iu)\,\widetilde{G}_{\lambda+\frac{\gamma}{2}}(iu) - \widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iu)\,\widetilde{G}'_{1+\lambda+\frac{\gamma}{2}}(iu)}{2\pi i\,(\gamma+1)^{-1}},$$

locally uniformly in $(-\delta, \delta)$. This shows that even the weak Lubinsky's wiggle condition (1.2.8) is not satisfied in a neighborhood of the jump of the weight.

A recent paper of Lubinsky [41] revealed an interesting connection of K_∞ with the theory of entire functions. Namely, in accordance with Theorem 1.6 of [41], K_∞ is a reproducing kernel of a De Branges space, equivalent to a classical Paley-Wiener space. More precisely and following the notation of [41], the *Hermite-Biehler class* $\overline{HB}$ is the set of entire functions E with no zeros in the upper half plane $\mathbb{C}^+$ and such that $|E(z)| \geq |E(\overline{z})|$ for $z \in \mathbb{C}^+$. The *De Branges space* $\mathcal{H}(E)$ corresponding to $E \in \overline{HB}$ is comprised of entire functions g such that both $g(z)/E(z)$ and $\overline{g(\overline{z})}/E(z)$ belong to the Hardy class $H^2(\mathbb{C}^+)$. A reproducing kernel for $\mathcal{H}(E)$ is different than the standard kernel, and involves a conjugate variable:

$$\mathcal{K}(x, y) = \frac{i}{2\pi}\,\frac{E(x)\overline{E(y)} - \overline{E(\overline{x})}E(\overline{y})}{x - \overline{y}}, \quad x \neq y. \tag{10.3.3}$$

Comparing this expression with K_∞ in (10.3.2) we conclude that (for purely imaginary λ)

$$\mathcal{K}(x, \overline{y}) = K_\infty(x, y),$$

with $\lambda = i\log(c)/\pi$ and

$$E(z) = \left(\frac{(2n)^\gamma}{h(x_0)(1-x_0)^\alpha(1+x_0)^\beta\left(1-x_0^2\right)^{\gamma/2}c}\right)^{1/2} \left|\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}\right| \widetilde{G}_{1+\lambda+\frac{\gamma}{2}}(iz) \in \overline{HB}$$

(see Proposition 10.1 (i) and (10.1.5)). Lubinsky showed that reproducing kernels, different from the right hand side in (1.2.5), can appear for sequences of measures (cf. [8]). To the best of our knowledge, this is the first explicit example of a non-sine reproducing kernel of a De Branges space that arises as a universality limit in the bulk of a fixed measure of orthogonality.

10.3.1 Proof of theorem

Observe, that the standard kernel defined in (1.2.1) can be obtained from (9.2.4) divided it by $\sqrt{w_{c,\gamma}(x_0 + u_n)}\sqrt{w_{c,\gamma}(x_0 + v_n)}$. Thus, from the Theorem 9.2.1 in form (9.2.4) and (9.1.19) and (1.4.1), we obtain (10.3.2). Finally, the confluent form of the kernel in (10.3.2) is obtained from the expression for $u \neq v$ by taking limit $v \to u$.

Part III

Open problems and further research

Chapter 11

Open problems and further research

In this chapter we present some open problems of interest that will constitute an object of our further research.

Problem 1 *Jacobi weight with non-symmetric algebraic singularities.*

A natural generalization of the weights based on the Jacobi weight is to consider an algebraic singularity at x_0 of the following form:

$$w_{\gamma,\delta}(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta} \left\{ \begin{array}{ll} (x_0 - x)^{\gamma}, & x \in [-1, x_0) \\ (x - x_0)^{\delta}, & x \in [x_0, 1]. \end{array} \right. , \qquad x \in [-1,1],$$

where $x_0 \in (-1,1)$, $\alpha, \beta, \gamma, \delta > -1$, h is real analytic and strictly positive on $[-1,1]$. A new analysis is needed at x_0, since the system of jumps obtained is very different, dependent on $\gamma - \delta$. This indicates that we might need a different local parametrix (not necessarily expressible in terms of known special functions) that realizes a "transition" between Bessel-type behavior with different indices at both sides.

Problem 2 *Non-symmetric Jacobi weight with jump singularities.*

Taking into account the previous problem and the results of this thesis, a relevant question is what happens when the algebraic singularity at x_0 is non symmetric and also with a step-like singularity:

$$w_{c,\gamma,\delta}(x) = h(x)\Xi_c(x)(1-x)^{\alpha}(1+x)^{\beta} \left\{ \begin{array}{ll} (x_0 - x)^{\gamma}, & x \in [-1, x_0) \\ (x - x_0)^{\delta}, & x \in [x_0, 1]. \end{array} \right. , \qquad x \in [-1,1],$$

where $x_0 \in (-1,1)$, $\alpha, \beta, \gamma, \delta > -1$, h is real analytic and strictly positive on $[-1,1]$, and Ξ_c is a step-like function defined by

$$\Xi_c(x) = \left\{ \begin{array}{ll} 1, & x \in [-1, x_0) \\ c^2, & x \in [x_0, 1]. \end{array} \right.$$

After we solve the first problem, we can expect to solve this problem applying the ideas of the Riemann-Hilbert analysis from this work.

Problem 3 *Varying weights: Jacobi weight with varying jump.*

A problem that we would like to analyze is related to a varying weight based on the Jacobi weight with a jump:

$$w_n(x) = h(x)\Xi_{c_n}(x)\,(1-x)^{\alpha}(1+x)^{\beta}, \qquad x \in [-1,1],$$

($\alpha, \beta > -1$, h is real analytic and strictly positive on $[-1,1]$), and Ξ_{c_n} defined by

$$\Xi_{c_n}(x) = \begin{cases} 1, & x \in [-1,0) \\ c_n, & x \in [0,1]. \end{cases}$$

where c_n depends on n, for instance $c_n \to 1$. We are especially interested in the double scaling limit $p_n(z_n, c_n)$, where $z_n \to 1$ is chosen appropriately to reveal non-standard phenomenon. Obviously, the simplest case is $w_n = \Xi_{c_n}$. The appearance of nonlinear special functions, such as Painlevé, in the local asymptotics, could occur.

Problem 4 *Varying weights: Jacobi weight with varying jump and gap.*

Another problem could be analyze the case of a weight varying based on $w_{c,0}$:

$$w_n(x) = h(x)\Xi_{c_n}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\Delta_n(x), \qquad x \in [-1,1],$$

where

$$\Delta_n(x) = \begin{cases} 1, & x \in [-1,-x_n] \cup [x_n,1], \\ 0, & \textit{otherwise}, \end{cases}$$

and $x_n > 0$, $x_n \to 0$ ("closing the gap"), and, with $\alpha, \beta > -1$, h is real analytic and strictly positive on $[-1,1]$, and Ξ_{c_n} is a step-like function defined by

$$\Xi_{c_n}(x) = \begin{cases} 1, & x \in [-1,x_n), \\ c_n, & x \in [x_n,1]. \end{cases}$$

with c_n depends on n, for instance $c_n \to 1$.

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Index