Estimation and testing in M-quantile Regression with applications to small area estimation

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Abstract

In recent years M-quantile regression has been applied to small area estimation (SAE) to obtain reliable and outlier robust estimators without recourse to strong parametric assumptions. In this paper, after a review of M-quantile regression, we cover several topics related to model specification and selection for M-quantile regression that received little attention so far. Specifically, a pseudo-R^2 goodness of fit measure is proposed, along with likelihood ratio and Wald type tests for linear hypotheses on the M-quantile regression parameters. A new estimator of the scale, motivated by a parametric representation of the M-quantile regression estimation, is also proposed. This parametric representation, that generalizes the Asymmetric Laplace distribution, often associated to quantile regression can be exploited to solve specific problems in M-quantile regression. For instance, when the Huber loss function is adopted, it provides the basis for a data driven choice of the tuning parameter. Finally a test to assess the presence of actual area heterogeneity in the data is also proposed. The properties of the tests are theoretically studied and their finite sample properties empirically assessed in Monte-Carlo simulations. The use of the proposed methods is illustrated in a well-known real data application in SAE.

Keywords: Generalized Asymmetric Least Informative distribution; goodness-of-fit; likelihood ratio type test; loss function; robust regression

1 Introduction

In sample surveys, estimates of population descriptive quantities for a target variable Y are usually needed both for the population as a whole and for subpopulations, known as domains or areas. Provided that large enough domain-specific sample sizes are available, statistical agencies can perform domain estimation by using the same design-based methods used for the estimation of population level quantities (direct estimation). In the case of small domain sample sizes, direct estimation may lead to estimates with large sampling

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variability. When direct estimation is not reliable in all or most of the domains, there is need to use small area estimation (SAE) techniques.

Area-level and unit-level linear mixed models have been studied in the literature to obtain empirical best linear unbiased predictors (EBLUP) of small area means (Rao and Molina , 2015). Empirical best estimation is useful for estimating the small area means efficiently when normality holds, otherwise, its properties can be deteriorated by the presence of outliers in the data. Consequently, it is of interest to see how robust survey estimation can be adapted to small area estimation.

In recent years, Chambers and Tzavidis (2006) and Sinha and Rao (2009) addressed the issue of outlier robustness in SAE proposing robust techniques that can be used to down-weight any outliers when fitting the underlying model. Sinha and Rao (2009) addressed this issue from the perspective of linear mixed models. Chambers and Tzavidis (2006) proposed to apply the M-quantile regression models to small area estimation with the aim to obtain reliable and outlier robust estimators without recourse to parametric assumptions for the residuals distribution. This approach involves weaker parametric assumptions than the linear mixed model and is robust to outliers in the response variable because of its use of M-estimation theory. A comparison of these two alternative approaches can be found in Chambers et al. (2014a). The distinguishing features of the approach by Chambers and Tzavidis (2006) include the protection that a careful choice of a loss function $\rho(\cdot)$ offers against the effect of outliers and the characterization of domain heterogeneity in terms of domain-specific M-quantiles. The M-quantiles can be viewed as an alternative to random effects for measuring area-specific unobserved heterogeneity. Whenever there is insufficient evidence of this heterogeneity, a prediction based on a simpler linear regression model would be more efficient. A number of papers on M-quantile regression that focus on theoretical developments (Tzavidis et al., 2010; Fabrizi et al., 2012; Salvati et al., 2012; Bianchi and Salvati, 2015; Chambers et al., 2014a; Fabrizi et al., 2014a; Tzavidis et al., 2016; Alfò et al., 2017), extensions to non-linear models (Pratesi et al., 2009; Chambers et al., 2014b; Dreassi et al., 2014; Tzavidis et al., 2015; Chambers et al., 2016) and various small area applications (Tzavidis et al., 2008; Pratesi et al., 2008; Salvati et al., 2011; Tzavidis et al., 2012; Fabrizi et al., 2014b) has been published in recent years. In view of this growing number of studies, in this paper we review M-quantile linear regression with special focus on its application to small area estimation.We complement the review discussing model comparison and testing tools that received so far little attention in literature.

In particular, for model specification, we propose likelihood ratio and Wald type tests in line with those proposed by Koenker and Machado (1999) for quantile regression for testing linear hypotheses on the vector of regression coefficients. These tests can be used for variable selection and to define a pseudo-R^2 goodness-of-fit measure. Second, we propose a test based on M-quantile coefficients to assess the need of incorporating area-specific heterogeneity in small area prediction. Third, we consider the parametric distribution associated to a general loss $\rho_\tau(\cdot)$, that we will call Generalized Asymmetric Least Informative (GALI) distribution that relates to M-quantile regression in the same

way that the normal distribution is associated with quadratic loss function and the Asymmetric Laplace (AL) distribution in quantile regression (Yu and Moyeed, 2001): the log of its density coincide with the loss function to be minimized to obtain the estimator of the location parameter. In line with most of the applications we quoted, a special attention will be devoted to the tilted version of the popular Huber loss function,

$$\rho_\tau(u) = 2 \begin{cases} (c|u| - c^2/2)|\tau - I(u \leq 0)| & |u| > c \\ u^2/2|\tau - I(u \leq 0)| & |u| \leq c, \end{cases} \tag{1}$$

where $I(\cdot)$ is an indicator function, $0 < \tau < 1$ represents the quantile and c is a cutoff constant. We note that if we set $\tau = 0.5$, a well-defined distribution, the so-called Least Informative (LI) distribution, is associated to this function (Huber, 1981, Section 4.5). Further, we use the distribution associated to this loss function to propose an estimator for the tuning constant c, using a method than can be generalized to other loss functions involving tuning constants. This procedure could be very useful as a model selection method because M-quantile regression allows us to trade robustness for efficiency by properly tuning the constant c: robustness is increased as c decreases, while efficiency is increased as c increases. We are fully aware of the limitations and pitfalls of inference based on pseudo-likelihoods in quantile regression (Yang et al. , 2015) and for this reason we are not going to treat the GALI as an actual likelihood for the data.

The paper is organized as follows. In Section 2 we review M-quantile regression and introduce a new estimator for the scale parameter based on the GALI distribution. In Section 3 we introduce the pseudo-R^2 goodness-of-fit measure and likelihood ratio and Wald type tests for linear hypotheses on the M-quantile regression parameters. Section 4 presents the review on how M-quantile regression can be applied to SAE problems. Section 5 presents the heuristic procedure for assessing the presence of specific-area effects. In Section 6 we present simulation studies aimed at assessing the finite sample properties of the proposed tests and estimators. In Section 7 we present the application of the methods to real data. Finally, Section 8 concludes the paper with some final remarks.

2 M-quantile regression

Quantile regression (Koenker and Bassett, 1978; Koenker, 2005) represents a useful generalization of median regression whenever the interest is not limited to the estimation of a location parameter at the centre of the conditional distribution of the target variable y given a set of predictors $\mathbf{x}$ but extends to location parameters (quantiles) at other parts of this conditional distribution. Similarly, expectile regression (Newey and Powell, 1987) generalizes least squares regression at the centre of a distribution to estimation of location parameters at other parts of the target conditional distribution namely, expectiles. Breckling and Chambers (1988) introduce M-quantile regression that extends the ideas of M-estimation (Huber, 1964; Huber and Ronchetti, 2009) to a different set of location parameters of the target conditional distribution that lie between quantiles and expectiles. M-quantiles aim at combining the robustness properties of quantiles with the efficiency

properties of expectiles.

Given a random variable y with cdf $F(y)$ and a (a.e.) continuously differentiable convex loss function $\rho(u)$, $u \in \mathcal{R}$, we define the tilted version of the loss function as

$$\rho_\tau(u) = |\tau - I(u < 0)|\rho(u), \tag{2}$$

with $\tau \in (0, 1)$. The τ-th M-quantile θ_τ is obtained as the minimizer of,

$$\int \rho_\tau(y - \theta_\tau)F(dy). \tag{3}$$

Depending on the choice of the loss function, M-quantiles may reduce to ordinary quantiles ($\rho(u) = |u|$) and expectiles ($\rho(u) = u^2$) while other choices are also possible (Dodge and Jureckova, 2000). However, as it is well known, quantiles and expectiles should be treated separately due to different properties of the corresponding influence functions. In regression the argument in the loss functions is defined by standardized residuals $u = \sigma_\tau^{-1}(y - \mathbf{x}^T\boldsymbol{\beta}_\tau)$, where σ_τ is a scale parameter for the residuals' distribution.

Let y be a random variable and $\mathbf{x}$ a p-dimensional random vector with first component $x_1 = 1$. The observed data $\{(\mathbf{x}_i, y_i), i = 1, \ldots, n\}$ is assumed to be a random sample of size n drawn from the population; thus $(\mathbf{x}_i, y_i)$ are independent and identically distributed random variables. Assuming a linear model, for any $\tau \in (0, 1)$, the M-quantile (hereafter, MQ) of order τ of y_i given $\mathbf{x}_i$ is defined by

$$MQ_\tau(y_i|\mathbf{x}_i) = \mathbf{x}_i^T\boldsymbol{\beta}_\tau, \tag{4}$$

where $\boldsymbol{\beta}_\tau \in \Theta \subset \mathcal{R}^p$ is the solution to

$$\min_{\boldsymbol{\beta}\in\Theta} E\left[\rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\sigma_\tau}\right)\right], \tag{5}$$

and σ_τ is a scale parameter that characterizes the distribution of $\varepsilon_{\tau i} = y_i - \mathbf{x}_i^T\boldsymbol{\beta}_\tau$. The linear specification in (4) can be alternatively written as

$$y_i = \mathbf{x}_i^T\boldsymbol{\beta}_\tau + \varepsilon_{\tau i},$$

where $\{\varepsilon_{\tau i}\}$ is a sequence of independent and identically distributed errors with unknown distribution function F_τ satisfying, by definition, $MQ_\tau(\varepsilon_{\tau i}|\mathbf{x}_i) = 0$. The estimator of the MQ regression coefficients (Breckling and Chambers, 1988) is defined as

$$\hat{\boldsymbol{\beta}}_\tau = \text{argmin} \sum_{i=1}^{n} \rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\hat{\sigma}_\tau}\right), \tag{6}$$

where $\hat{\sigma}_\tau$ is a consistent estimator of σ_τ. Since ρ is (a.e.) continuously differentiable and convex, the vector $\hat{\boldsymbol{\beta}}_\tau$ can equivalently be obtained as the solution of the following system

of equations

$$\sum_{i=1}^{n} \psi_\tau \left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}}{\hat{\sigma}_\tau} \right) \mathbf{x}_i = \mathbf{0}, \tag{7}$$

where $\psi_\tau(u) = d\rho_\tau(u)/du = |\tau - I(u < 0)|\psi(u)$, with $\psi(u) = d\rho(u)/du$. An iterative method is needed here to obtain a solution, like an iteratively re-weighted least squares algorithm or the Newton-Raphson algorithm.

Regarding the scale parameter σ_τ, it may generally be defined by an implicit relation of the form

$$E\Big[\chi\left(\frac{\varepsilon_{\tau i}}{\sigma_\tau}\right)\Big] = 0, \tag{8}$$

where the expectation is taken with respect to the distribution of $\varepsilon_{\tau i}$. In MQ regression, a typical choice for χ is $\chi(u) = sgn(|u - Med(u)| - 1)$, which leads to the scaled population median absolute deviation $\sigma_\tau = \frac{Med\{|\boldsymbol{\varepsilon}_\tau - \xi_{1/2,\tau}|\}}{q}$, $\xi_{1/2,\tau} = Med(F_\tau(\boldsymbol{\varepsilon}_\tau))$, $q = \Phi^{-1}(3/4) = 0.6745$, with Φ denoting the distribution function of the standard normal distribution. The corresponding estimator is the scaled sample median absolute deviation (MAD)

$$\hat{\sigma}_\tau = \frac{Med\{|\hat{\boldsymbol{\varepsilon}}_\tau - Med(\hat{\boldsymbol{\varepsilon}}_\tau)|\}}{q}, \tag{9}$$

where $\hat{\boldsymbol{\varepsilon}}_\tau = (\hat{\varepsilon}_{\tau 1}, \ldots, \hat{\varepsilon}_{\tau n})$, $\hat{\varepsilon}_{\tau i} = y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_\tau$.

The asymptotic theory for MQ regression with i.i.d. errors and fixed regressors can be derived from the results in Huber (1973), as pointed out in Breckling and Chambers (1988). Bianchi and Salvati (2015) show the consistency and the asymptotic normality of the estimator of $\boldsymbol{\beta}_\tau$ and the consistency of its asymptotic variance estimator,

$$\widehat{Var}(\hat{\boldsymbol{\beta}}_\tau) = (n-p)^{-1} n \hat{\mathbf{W}}_\tau^{-1} \hat{\mathbf{G}}_\tau \hat{\mathbf{W}}_\tau^{-1} \tag{10}$$

where

$$\hat{\mathbf{W}}_\tau = (n\hat{\sigma}_\tau)^{-1} \sum_{i=1}^{n} \hat{\psi}'_{\tau i} \mathbf{x}_i \mathbf{x}_i^T,$$

$$\hat{\mathbf{G}}_\tau = n^{-1} \sum_{i=1}^{n} \hat{\psi}^2_{\tau i} \mathbf{x}_i \mathbf{x}_i^T,$$

with $\hat{\psi}'_{\tau i} := \psi'_\tau(\hat{\varepsilon}_{i\tau}/\hat{\sigma}_\tau)$, $\hat{\psi}_{\tau i} = \psi_\tau(\hat{\varepsilon}_{i\tau}/\hat{\sigma}_\tau)$ in case of stochastic regressors and in the presence of heteroskedasticity.

2.1 A likelihood perspective for M-quantiles: the Generalized Asymmetric Least Informative distribution

Yu and Moyeed (2001) show the relationship between the loss function for quantile regression and the maximization of a likelihood function formed by combining independently distributed Asymmetric Laplace densities. In this Section we show a similar relationship for MQ regression models.

Given a loss function ρ_τ, we can define the GALI random variable with density function

$$f_\tau(y;\mu_\tau,\sigma_\tau) = \frac{1}{\sigma_\tau B_\tau} exp\left\{-\rho_\tau\left(\frac{y-\mu_\tau}{\sigma_\tau}\right)\right\}, \quad -\infty < y < +\infty. \tag{11}$$

where $B_\tau = \int_{-\infty}^{+\infty} \frac{1}{\sigma_\tau} exp\left\{-\rho_\tau\left(\frac{y-\mu_\tau}{\sigma_\tau}\right)\right\} dy < +\infty$ and μ_τ and σ_τ are location and scale parameters. We note that μ_τ coincides with the τ^{th} MQ of the distribution; in fact μ_τ can be obtained as the solution of

$$\int_{-\infty}^{+\infty} \psi_\tau\Big(\frac{y-\mu_\tau}{\sigma_\tau}\Big) f_\tau(y;\mu_\tau,\sigma_\tau)dy = 0,$$

that defines the MQ of the distribution.

For linear MQ regression, that is when $\mu_\tau = \mu_{\tau i} = \mathbf{x}_i^T\boldsymbol{\beta}_\tau$, the estimators of the unknown regression parameters $\boldsymbol{\beta}_\tau$ and the scale σ_τ may be obtained by maximizing the log-likelihood function:

$$l_\tau(y) = -n\log\sigma_\tau - n\log B_\tau - \sum_{i=1}^{n} \rho_\tau\Big(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}_\tau}{\sigma_\tau}\Big). \tag{12}$$

The estimating equations for the regression coefficients $\boldsymbol{\beta}_\tau$ are the same as those of equation (7). The estimating equation for σ_τ is

$$-\frac{n}{\sigma_\tau} + \frac{1}{\sigma_\tau^2}\sum_{i=1}^{n} \psi_\tau\Big(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}_\tau}{\sigma_\tau}\Big)(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_\tau) = 0, \tag{13}$$

and its solution defines a new estimator for σ_τ alternative to (9). With respect to (8) in this case $\chi(u) = -u\psi_\tau(u) - 1$ and the parameter is defined as the solution of

$$E\left[-\varepsilon_{\tau i}\psi_\tau\left(\frac{\varepsilon_{\tau i}}{\sigma_\tau}\right)\right] = \sigma_\tau.$$

This choice is in line with what Koenker and Machado (1999) and Yu and Zhang (2005) propose for quantile regression, considering the maximum likelihood estimator under the asymmetric Laplace distribution.

Solving equations (7) and (13) requires an iterative algorithm. The steps of this algorithm are as follows:

1. For specified τ define initial estimates $\hat{\boldsymbol{\beta}}_\tau^{(0)}$ and $\hat{\sigma}_\tau^{(0)}$.
2. At each iteration t calculate $w_{\tau i}^{(t-1)} = \psi_\tau(u_{\tau i}^{(t-1)})/u_{\tau i}^{(t-1)}$ with $u_{\tau i}^{(t-1)} = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_\tau^{(t-1)})/\hat{\sigma}_\tau^{(t-1)}$.
3. Compute the new weighted least squares estimates from

$$\hat{\boldsymbol{\beta}}_\tau^{(t)} = \Big\{\sum_{i=1}^{n}(w_{\tau i}^{(t-1)}\mathbf{x}_i\mathbf{x}_i^T)\Big\}^{-1}\Big\{\sum_{i=1}^{n}(y_i w_{\tau i}^{(t-1)}\mathbf{x}_i)\Big\}. \tag{14}$$

4. Compute the new estimate of $\hat{\sigma}_\tau$ by

$$\hat{\sigma}_\tau^{(t)} = \Big\{ n^{-1} \sum_{i=1}^{n} w_{\tau i}^{(t-1)} (y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_\tau^{(t-1)})^2 \Big\}^{1/2}. \tag{15}$$

5. Repeat steps 2-4 until convergence. Convergence is achieved when the difference between the estimated model parameters obtained from two successive iterations is less than a small pre-specified value.

The consistency of the scale estimators (MAD and MLE) can be proved by standard theory of M-estimators (Wooldridge, 2010), assuming that (8) has a unique solution.

If $\rho_\tau(\cdot)$ is the Huber loss function defined in (1) the normalizing constant is given by

$$B_\tau = \sqrt{\frac{\pi}{\tau}} \Big[\Phi(c\sqrt{2\tau}) - 1/2 \Big] + \sqrt{\frac{\pi}{1-\tau}} \Big[\Phi(c\sqrt{2(1-\tau)}) - 1/2 \Big]$$

$$+ \frac{1}{2c\tau} exp\{-c^2\tau\} + \frac{1}{2c(1-\tau)} exp\{-c^2(1-\tau)\}, \tag{16}$$

where $\tau \in (0,1)$ and Φ is the cumulative distribution function of the standard Normal distribution. In this case we call (11) the Asymmetric Least Informative (ALI) distribution. This distribution is essentially a modified standard normal distribution with heavier tails (when $y > c$). For $\tau = 0.5$, this distribution was derived by Huber (1981, Section 4.5) as the one minimizing the Fisher information in the ε-contaminated neighborhood of the normal distribution. Formulae for the cumulative distribution function and moments of the ALI distribution ($\tau \in (0,1)$) are in the Appendix A.

The ALI distribution depends on the tuning constant c. In M-regression, the tuning constant is defined by the data analyst such that the M-estimate has a specified asymptotic efficiency (generally 95%) under normality (Huber, 1981). Alternatively, Wang et al. (2007) propose a data-driven method, based on efficiency arguments.

In this paper, we propose to interpret c as a parameter of the density f_τ and estimate $\boldsymbol{\beta}_\tau$, σ_τ and c by maximizing the log-likelihood function (12). For estimating the tuning constant there is no closed form. In this case the compass search algorithm or the Nelder-Mead (Griva et al., 2008) can be used. The final estimating procedure works by adding to the proposed iterative algorithm the new step 4′ below:

4′ Given $\hat{\boldsymbol{\beta}}_\tau^{(t)}$ and $\hat{\sigma}_\tau^{(t)}$ maximize the log-likelihood function (12) with respect to c using the compass search algorithm (Bottai et al., 2015) or the Nelder-Mead algorithm.

An `R` function that implements an iterative algorithm for estimating the parameters is available from the authors.

The idea of estimating the tuning constant using likelihood equations can be applied to other loss functions as well whenever they include an additional parameter or tuning constant.

3 Goodness-of-fit and likelihood ratio type tests in M-quantile regression

In this section we present a pseudo-R^2 goodness-of-fit statistic for MQ regression and likelihood ratio and Wald type tests for linear hypotheses on the regression parameters. The asymptotic theory for $\hat{\beta}_\tau$ has been developed according to standard M-estimation theory, as in Gourieroux and Monfort (1989) and Wooldridge (2010).

3.1 A goodness-of-fit measure

For a given quantile, the introduction of the pseudo-R^2 is motivated by the need for a measure analogous to the ordinary R^2 used in least squares regression. Since this goodness-of-fit statistic will be quantile-dependent, it is also useful to study its variation across quantiles. We start by partitioning MQ regression as follows,

$$MQ_\tau(y_i|\mathbf{x}_i) = \mathbf{x}_{i1}^T\boldsymbol{\beta}_{1\tau} + \mathbf{x}_{i2}^T\boldsymbol{\beta}_{2\tau}, \tag{17}$$

where $\boldsymbol{\beta}_\tau = (\boldsymbol{\beta}_{1\tau}^T, \boldsymbol{\beta}_{2\tau}^T)^T$, $\boldsymbol{\beta}_{1\tau}$ is a $(p-k)\times 1$ vector and $\boldsymbol{\beta}_{2\tau}$ is a $k\times 1$ $(0<k<p)$ vector. We are interested in testing the null hypothesis:

$$H_0 : \boldsymbol{\beta}_{2\tau} = \mathbf{0}. \tag{18}$$

Let $\hat{\boldsymbol{\beta}}_\tau$ denote the MQ estimator of the full model and let $\tilde{\boldsymbol{\beta}}_\tau = (\tilde{\boldsymbol{\beta}}_{1\tau}^T, \mathbf{0}^T)^T$ denote the MQ estimator under the null hypothesis specified in (18).

A relative goodness-of-fit measure comparing the full to the reduced MQ regression model is defined as

$$R_\rho^2(\tau) = 1 - \frac{\sum_{i=1}^n \rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_\tau}{\hat{\sigma}_\tau}\right)}{\sum_{i=1}^n \rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\tilde{\boldsymbol{\beta}}_\tau}{\hat{\sigma}_\tau}\right)}. \tag{19}$$

When the reduced model includes only the intercept, this measure is the natural analog of the usual R^2 goodness-of-fit measure used in mean regression. It varies between 0 and 1 and it represents a measure of goodness-of-fit for a specified τ.

To explore the behaviour of the index $R_\rho^2(\tau)$ introduced in this section we use a range of artificial data as in Koenker and Machado (1999). We consider a simple bivariate regression settings under three different scenarios with $n = 100$:

- Gaussian noise: the data are generated with y_i iid standard normal distribution an independent of x. The value of x_i are generated as iid $N(5, 1)$.
- Gaussian location shift: the data are generated according the model

  $$y_i = x_i + \epsilon_i$$

  with ϵ_i iid $N(0, 1)$, x_i iid $N(5, 1)$.

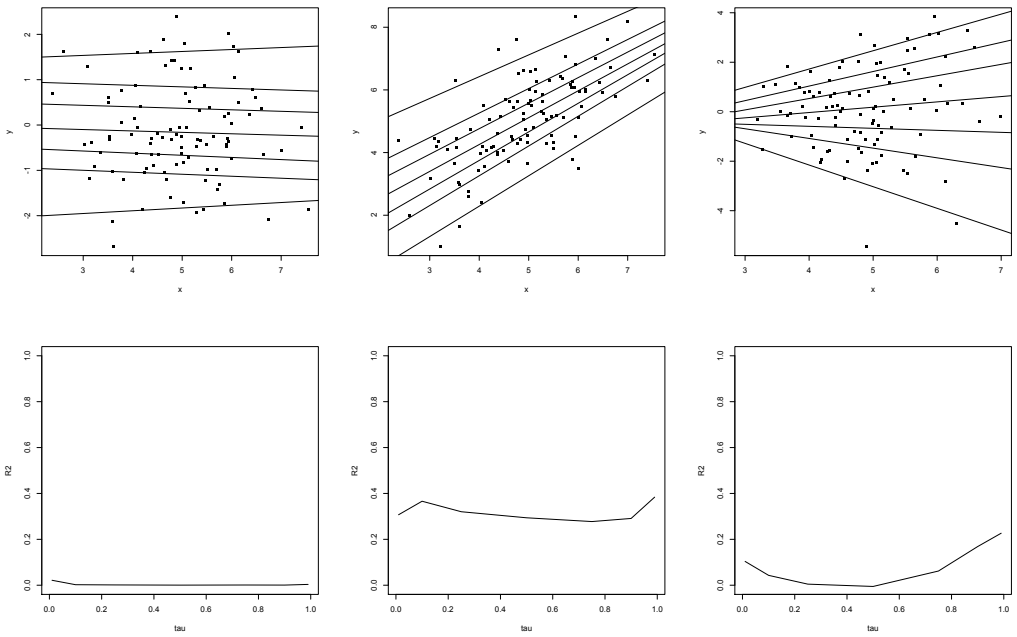

Figure 1: The figure shows three different scenarios and their associated $R^2_\rho(\tau)$. The top row presents the data and in solid font the M-quantile model lines fitted at $\tau = (0.01, 0.10, 0.25, 0.50, 0.75, 0.90, 0.99)$. The second row depicts the values of $R^2_\rho(\tau)$ at different quantiles.

- Gaussian scale shift: a heteroskedastic version of the regression model is given by

$$y_i = \left(x_i + \frac{1}{4}x_i^2\right)\epsilon_i$$

with ϵ_i iid $N(0, 1/100)$, x_i iid $N(3, 1)$.

Figure 3.1 illustrates in the top row of the panels the M-quantile model lines fitted at $\tau = (0.01, 0.10, 0.25, 0.50, 0.75, 0.90, 0.99)$. The bottom row of the figure shows the values of $R^2_\rho(\tau)$ at different values of τ. As we expected, under Gaussian noise, the values of $R^2_\rho(\tau)$ are nearly 0 over the entire range $\tau \in (0, 1)$. Under the Gaussian location shift scenario the values of $R^2_\rho(\tau)$ show a flat relationship between y and x for each τ. This indicates that all the conditional M-quantiles are equally successfull in reducing variability (Koenker and Machado, 1999). In the case of heterosketasiticity (scenario 3) the conditional median and the conditional median are equal. For the other values of τ there is a clear benefit from the quantile form of the conditional M-quantile specification.

3.2 Hypothesis testing

For testing the null hypothesis (18), the following theorem presents the distribution of the likelihood ratio statistic when the residuals follow a general distribution. This leads to a likelihood ratio type test. In the following, let

$$\hat{V}(\tau) = \sum_{i=1}^{n} \rho_\tau \left(\frac{y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_\tau}{\sigma_\tau} \right), \qquad \tilde{V}(\tau) = \sum_{i=1}^{n} \rho_\tau \left(\frac{y_i - \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}_\tau}{\sigma_\tau} \right).$$

Consider the following regularity conditions:

(C1) Θ compact set in $\mathcal{R}^p$;

(C2) ρ is (a.e.) twice continuously differentiable;

(C3) $|\sup_{\boldsymbol{\beta}_\tau \in \Theta} \rho\left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}_\tau}{\sigma_\tau}\right)| < h(\mathbf{x}_i, y_i)$ and $|\sup_{\boldsymbol{\beta}_\tau \in \Theta} \psi'\left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}_\tau}{\sigma_\tau}\right) \mathbf{x}_i \mathbf{x}_i^T| < g(\mathbf{x}_i, y_i)$, with h and g are P-integrable functions;

(C4) $E\left[\mathbf{x}_i \mathbf{x}_i^T \psi'((y_i - \mathbf{x}_i^T \boldsymbol{\beta})/\sigma_\tau)\right]$ is uniformly nonsingular for $\boldsymbol{\beta} \in \Theta$.

(C5) the errors $\varepsilon_{\tau i}$ are independent of $\mathbf{x}_i$.

Assumption (C3) guarantees the applicability of the Uniform Law of Large Numbers. In case of the Huber loss function, (C3) is satisfied provided $E|\mathbf{x}_i|^2 < +\infty$ and $E|y_i| < +\infty$. Assumption (C5) is required for the validity of the generalized information equality. This would hold also if the $\mathbf{x}_i$'s are fixed regressors. The information equality is needed for the validity of the likelihood ratio type test. It can be relaxed for the Wald test.

Theorem 1. *Provided conditions (C1)-(C5) are satisfied under the null hypothesis* H_0

$$-2\frac{E\psi'_{\tau i}}{E\psi^2_{\tau i}}(\hat{V}(\tau) - \tilde{V}(\tau)) \xrightarrow{d} \chi^2_k, \tag{20}$$

where $\psi'_{\tau i} = \psi'_\tau(\varepsilon_{\tau i}/\sigma_\tau)$, $\psi_{\tau i} = \psi_\tau(\varepsilon_{\tau i}/\sigma_\tau)$.

Proof. Using a second order Taylor expansion

$$2[\tilde{V}(\tau) - \hat{V}(\tau)] = \sqrt{n}(\tilde{\boldsymbol{\beta}}_\tau - \hat{\boldsymbol{\beta}}_\tau)^T(\boldsymbol{\Psi}_\tau/\sigma_\tau)\sqrt{n}(\tilde{\boldsymbol{\beta}}_\tau - \hat{\boldsymbol{\beta}}_\tau) + o_p(1), \tag{21}$$

where, by using (C4), $\boldsymbol{\Psi}_\tau = \sigma_\tau^{-1} E(\psi'_{\tau i}) E(\mathbf{x}_i \mathbf{x}_i^T)$. Theorem 1 in Bianchi and Salvati (2015) ensures that

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau) = \boldsymbol{\Psi}_\tau^{-1} n^{-1/2} \sum_{i=1}^n \psi_{\tau i} \mathbf{x}_i + o_p(1). \tag{22}$$

Similarly, a standard mean value expansion (under H_0) gives

$$n^{-1/2} \sum_{i=1}^n \tilde{\psi}_{\tau i} \mathbf{x}_i = n^{-1/2} \sum_{i=1}^n \psi_{\tau i} \mathbf{x}_i - \boldsymbol{\Psi}_\tau \sqrt{n}(\tilde{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau) + o_p(1),$$

where $\tilde{\psi}_{\tau i} = \psi_\tau(\tilde{\varepsilon}_{\tau i}/\sigma_\tau)$, $\tilde{\varepsilon}_{\tau i} = y_i - \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}_\tau$. Hence,

$$\sqrt{n}(\tilde{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau) = \boldsymbol{\Psi}_\tau^{-1} n^{-1/2} \left[-\sum_{i=1}^n \tilde{\psi}_{\tau i} \mathbf{x}_i + \sum_{i=1}^n \psi_{\tau i} \mathbf{x}_i \right] + o_p(1). \tag{23}$$

Substituting (22) and (23) into (21), we obtain

$$2[\tilde{V}(\tau) - \hat{V}(\tau)] = \left(n^{-1/2} \sum_{i=1}^n \tilde{\psi}_{\tau i} \mathbf{x}_i \right)^T \left(E(\psi'_{\tau i}) E(\mathbf{x}_i \mathbf{x}_i^T) \right)^{-1} \left(n^{-1/2} \sum_{i=1}^n \tilde{\psi}_{\tau i} \mathbf{x}_i \right) + o_p(1).$$

Following Wooldridge (2010), we introduce the $k \times p$ full rank matrix $\mathbf{R} = [\mathbf{0} : \mathbf{I}_k]$ and write H_0 as $\mathbf{R}\boldsymbol{\beta}_\tau = \mathbf{0}$. Since $\mathbf{R}\sqrt{n}(\tilde{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau) = \mathbf{0}$, it can be proved (multiplying equation (23) by $\mathbf{R}\boldsymbol{\Psi}_\tau^{-1}$) that

$$\mathbf{R}\boldsymbol{\Psi}_\tau^{-1} n^{-1/2} \sum_{i=1}^{n} \tilde{\psi}_{\tau i}\mathbf{x}_i \xrightarrow{d} N(\mathbf{0}, \mathbf{R}\boldsymbol{\Sigma}_\tau\mathbf{R}^T),$$

where

$$\boldsymbol{\Sigma}_\tau = \sigma_\tau^2 \frac{E\psi_{\tau i}^2}{E\psi'_{\tau i}} E\left[\mathbf{x}_i\mathbf{x}_i^T\right]^{-1}, \tag{24}$$

so that

$$\left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right)^T \boldsymbol{\Psi}_\tau^{-1}\mathbf{R}^T\left(\mathbf{R}\boldsymbol{\Sigma}_\tau\mathbf{R}^T\right)^{-1}\mathbf{R}\boldsymbol{\Psi}_\tau^{-1}\left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right) \xrightarrow{d} \chi_k^2.$$

The previous expression can be simplified to

$$\left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right)^T \left(E(\psi_{\tau i}^2)E(\mathbf{x}_i\mathbf{x}_i^T)\right)^{-1}\left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right) \xrightarrow{d} \chi_k^2$$

and therefore we have that

$$\begin{aligned} &2\frac{E\psi'_{\tau i}}{E\psi_{\tau i}^2}[\tilde{V}(\tau) - \hat{V}(\tau)] \\ &\quad = \left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right)^T \left(E(\psi_{\tau i}^2)E(\mathbf{x}_i\mathbf{x}_i^T)\right)^{-1}\left(n^{-1/2}\sum_{i=1}^{n}\tilde{\psi}_{\tau i}\mathbf{x}_i\right) + o_p(1) \xrightarrow{d} \chi_k^2. \end{aligned} \tag{25}$$

□

A hypothesis test for H_0 is obtained by substituting the unknown quantities in (20) with consistent estimators leading to,

$$-2\frac{(n-p)^{-1}\sum_{i=1}^{n}\hat{\psi}'_{\tau i}}{n^{-1}\sum_{i=1}^{n}\hat{\psi}_{\tau i}^2}\left[\sum_{i=1}^{n}\rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_\tau}{\hat{\sigma}_\tau}\right) - \sum_{i=1}^{n}\rho_\tau\left(\frac{y_i - \mathbf{x}_i^T\tilde{\boldsymbol{\beta}}_\tau}{\hat{\sigma}_\tau}\right)\right], \tag{26}$$

where $\hat{\psi}'_{\tau i}$ and $\hat{\psi}_{\tau i}$ have been previously defined and the nuisance parameter σ_τ is estimated under the full model. This is to ensure that the test statistic is nonnegative. Even though the asymptotic distribution of (26) is not exactly asymptotically χ_k^2, simulations show that χ_k^2 is still a good approximation for it (see Section 6.2). This is due to the fact that the contribution of the estimation of σ to the asymptotic variance is negligible, as it was noticed in Bianchi and Salvati (2015). The same approach was adopted in Schrader and Hettmansperger (1980). This test is more commonly known as likelihood ratio (LR) type test since the density of the $\varepsilon_{\tau i}$ does not have to correspond to the loss function. Notice also that the proposed test can be easily extended to test more general linear hypotheses for example, $H_0 : \mathbf{R}\boldsymbol{\beta}_\tau = \mathbf{r}$, where $\mathbf{R}$ is a $k \times p$ full rank matrix and $\mathbf{r}$ is a $k \times 1$ vector. Similar results for M-regression estimators are provided by Schrader and Hettmansperger

(1980) in the case of fixed regressors, and for quantile regression with fixed regressors by Koenker and Machado (1999).

An alternative to the LR-type test is to use a Wald type test. The test statistic is derived by using Theorem 1 in Bianchi and Salvati (2015). Let $\mathbf{R} = [\mathbf{0} : \mathbf{I}_k]$. It follows that under H_0

$$n(\mathbf{R}\hat{\boldsymbol{\beta}}_\tau)^T[\mathbf{R}\boldsymbol{\Sigma}_\tau\mathbf{R}]^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}}_\tau) \xrightarrow{d} \chi^2_k,$$

where $\boldsymbol{\Sigma}_\tau$ is defined in (24). Replacing $\boldsymbol{\Sigma}_\tau$ with a consistent estimator

$$\hat{\boldsymbol{\Sigma}}_\tau = \hat{\sigma}^2_\tau \frac{(n-p)^{-1}\sum_{i=1}^n \hat{\psi}^2_{\tau i}}{n^{-1}\sum_{i=1}^n \hat{\psi}'_{\tau i}} \left[\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T\right]^{-1},$$

the statistic

$$W \equiv n(\mathbf{R}\hat{\boldsymbol{\beta}}_\tau)^T[\mathbf{R}\hat{\boldsymbol{\Sigma}}_\tau\mathbf{R}]^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}}_\tau)$$

follows asymptotically a χ^2 distribution. A major difference between the LR-type test and the Wald type test is that the latter can be made robust to the presence of heteroskedasticity by using a robust estimator of the covariance matrix in place of $\hat{\boldsymbol{\Sigma}}_\tau$.

4 An overview of M-quantile models for small area estimation

In this section we review the use of M-quantile regression in SAE. Let us suppose that a population is divided into D non-overlapping small areas of size N_j, $j = 1, ..., D$, so that $\sum_{j=1}^D N_j = N$. Suppose that a sample of size $n_j > 0$ is drawn from each small area. For simplicity of exposition, we do not consider the case $n_j = 0$, although the theory can be easily extended to it. In what follows we assume that unit record data are available at small area level. For the sampled units in the population this records consists of indicators of small area affiliation, values y_i of the variable of interest, values $\mathbf{x}_{ij}$ of a $p \times 1$ vector of individual level covariates. For the non-sampled population units we do not know the values of y_i. Let's introduce a second subscript in our notation for indicating the hierarchical nature of the data, $\{(\mathbf{x}_{ij}, y_{ij}), i = 1, \ldots, n_j; j = 1, \ldots, D\}$. We also assume that sampling is non-informative for the small area distribution of y_i given $\mathbf{x}_{ij}$, allowing us to use population level models with the sample data.

The papers by Chambers and Tzavidis (2006) and Aragon et al. (2005) were the first to introduce the idea of measuring heterogeneity in the data via M-quantiles. In particular, Chambers and Tzavidis (2006) characterize the variability across the population of interest by introducing the idea of MQ-coefficients. At the population level the MQ-coefficient for a unit within a small area is defined as the value τ_{ij} such that $MQ_{\tau_{ij}}(y_{ij}|\mathbf{x}_{ij}) = y_{ij}$. If a hierarchical structure does explain part of the variability, after accounting for the effect of covariates, units within small area are expected to have similar MQ-coefficients. Chambers and Tzavidis (2006) propose to characterize each small area j by the average of the MQ-coefficients of the units that belong to that small area. The small area-specific

MQ-coefficient, denoted by τ_j, identifies the most characteristic MQ regression line for that small area. We can think of this in the context of linear mixed models as the group-specific regression line that is distinguished from population-average line by the random effect. The aim is to use this data to predict various area specific quantities, including (but not only) the area j mean m_j of y. When (4) holds, and $\boldsymbol{\beta}_\tau$ is a sufficiently smooth function of τ, Chambers and Tzavidis (2006) suggest a predictor of m_j of the form:

$$\hat{m}_j^{MQ} = N_j^{-1}\Big\{\sum_{i\in s_j} y_{ij} + \sum_{i\in r_j} \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\hat{\tau}_j}\Big\}, \tag{27}$$

where we use indices s and r to denote sample and non-sample quantities, respectively. Thus, the set s_j contains the n_j indices of the units drawn from the population and the set r_j contains the $N_j - n_j$ indices of the non-sampled units in small area j. Here, $\hat{\tau}_j$ is an estimate of the average value of the MQ-coefficients of the units in area j. The case of $n_j = 0$, mentioned above, can be easily dealt with by using a synthetic M-quantile predictor, which is obtained by setting $\hat{\tau}_j = 0.5$ ($\hat{m}_j^{MQ/SYN}$). Chambers et al. (2014a) defined such method as robust projective since it projects sample non-outlier (i.e. working model) behaviour onto the non-sampled part of the survey population.

Chambers et al. (2014a) proposed methods to address a representative outlier (Chambers , 1986), i.e. a sample outlier that is potentially drawn from a group of population outliers and hence cannot be unit weighted in estimation. This method allows for contributions from representative sample outliers and it is defined as robust predictive methods since it attempts to predict the contribution of the population outliers to the population quantity of interest. A bias-corrected version of estimator (27) is given by

$$\hat{m}_j^{MQ-BC} = N_j^{-1}\Big\{\sum_{i\in s_j} y_{ij} + \sum_{i\in r_j} \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\hat{\tau}_j} + \frac{N_j - n_j}{n_j}\sum_{i\in s_j} \omega_{ij}^{MQ}\phi\Big\{\frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\hat{\tau}_j}}{\omega_{ij}^{MQ}}\Big\}\Big\}, \tag{28}$$

where ω_{ij}^{MQ} is a robust estimator of the scale of the residual $y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\hat{\tau}_j}$ in area j. We replace the robust influence function ψ used to define $\hat{\boldsymbol{\beta}}_{\hat{\tau}_j}$ above by one that is still bounded, but more accommodating of sample outliers, i.e. such that $|\psi| \leq |\phi|$. Its purpose is to define an adjustment for the bias caused by the fact that the first two terms on the right hand side of (28) treat sample outliers as not representative. See for details Chambers et al. (2014a). If the tuning constant in the ϕ function tends to infinity, the predictor (28) becomes a Chambers and Dunstan estimator (Tzavidis et al., 2010).

Two different analytic methods of Mean Squared Error (MSE) estimation for M-quantile-based robust predictors of small area means under the robust-projective and robust-predictive approaches have been proposed in the literature. Both are developed on the assumption that the working model for inference conditions on the realized values of the area effects, and so the proposed MSE estimators are conditional estimators. Chambers et al. (2011) define a pseudo-linearization estimator of the conditional MSE of predictor (27) they label as CCT estimator. Chambers et al. (2014a) use first-order approximations to the variances of solutions of estimating equations to develop conditional

MSE estimators for predictors (27) and (28) (labelled as CST). The MSE estimator for predictor (28) is based on the approximation

$$mse(\hat{m}_j^{MQ-BC}) = \Big(1-\frac{n_j}{N_j}\Big)^2\Big[\{\bar{\mathbf{x}}_{rj}-\bar{\mathbf{x}}_{sj}\}^T\hat{V}(\hat{\boldsymbol{\beta}}_{\hat{\tau}_j})\{\bar{\mathbf{x}}_{rj}-\bar{\mathbf{x}}_{sj}\}+\hat{V}(\bar{e}_{rj})+\frac{1}{n_j^2}\sum_{i\in s_j}\Big\{\omega_{ij}^{MQ}\phi\Big\{\frac{y_{ij}-\mathbf{x}_{ij}^T\hat{\boldsymbol{\beta}}_{\hat{\tau}_j}}{\omega_{ij}^{MQ}}\Big\}\Big\}^2\Big], \tag{29}$$

where $\hat{V}(\hat{\boldsymbol{\beta}}_{\hat{\tau}_j})$ is the estimated variance of the fitted M-quantile regression coefficients at $\tau=\hat{\tau}_j$, $\hat{V}(\bar{e}_{rj}) = (N_j-n_j)^{-1}(n-1)^{-1}\sum_k\sum_{i\in s_k}(y_{ki}-\mathbf{x}_{ki}^T\hat{\boldsymbol{\beta}}_{\hat{\tau}_k})^2$ and $\bar{\mathbf{x}}_{rj}$, $\bar{\mathbf{x}}_{rs}$ denote the vectors of average values of $\mathbf{x}_{ij}$ for the N_i-n_i non-sampled units and the n_i of sampled units, respectively, in area i. The results of the simulation experiments in Chambers et al. (2014a) show that the CST has lower bias than the estimator CCT and is also more stable for both predictors (27) and (28).

Several methodological developments on M-quantile regression in small area estimation have been made in recent years. Here a brief review of the most important papers in this field. Fabrizi et al. (2012) consider two problems relevant to practical small area applications. They propose a solution to guarantee the benchmarking property of small area estimators. The procedure is consistent with the M-quantile regression framework, thus it is theoretically more interesting than a simple ratio adjustment. The second problem is the correction of the under/over-shrinkage of small area estimators. The authors note that the M-quantile small area estimators may under-shrink (under normality) or over-shrink (when the distribution of actual small area parameters is skewed). In line with most literature, notions of under- and over-shrinkage are defined in terms of variance calculated over the ensemble of small area parameters. This may not be robust to the presence of outlying areas, but the method of Fabrizi et al. (2012) can be readily extended to other descriptions of the variability of the ensemble of area parameters.

Fabrizi et al. (2014a) adopt a model-assisted approach for developing design-consistent (weighted) M-quantile small area estimators. The authors assume a working linear M-quantile model and consider only properties with respect to the randomization distribution induced by the sample design. Fabrizi et al. (2014a) note that for the estimation of small area means and totals, the weighted M-quantile based estimators may be expressed in GREG form and can therefore be easily interpreted.

Salvati et al. (2012) incorporate the spatial information in small area predictors based on M-quantile models via Geographically Weighted Regression (GWR). In particular, the authors specify an M-quantile GWR model that is a local model for the M-quantiles of the conditional distribution of the outcome variable given the covariates. This model is then used to define a bias-robust predictor of the small area characteristic of interest that also accounts for spatial association in the data. Another approach to take into account spatial information in small area M-quantile predictors is by using a semiparametric M-quantile regression model as proposed by Pratesi et al. (2008). In this case the response variable depends on the geographical position of the observations through an unknown smooth bivariate function estimated by low-rank thin plate splines. The performance of the non-parametric specification of the conditional M-quantile of y given the covariates

has been investigated in Salvati et al. (2011). From the simulation results, the semiparametric M-quantile models in small area estimation appear to be a useful tool when the functional form of the relationship between the variable of interest and the covariates is left unspecified and the data are characterized by complex patterns of spatial dependence.

Finally, the M-quantile approach to small area prediction has been extended to discrete responses. In particular, Tzavidis et al. (2015) proposed a small area predictor based on a new semiparametric M-quantile model for counts that extends the ideas of Cantoni and Ronchetti (2001) and Chambers and Tzavidis (2006). This predictor can be viewed as an outlier robust alternative to the more commonly used conditional expectation predictor for counts that is based on a Poisson Generalised Linear Mixed Models with Gaussian random effects. Chambers et al. (2014b) introduce a semi-parametric approach to ecological regression for disease mapping, based on modelling the regression M-quantiles of a negative binomial variable. The method is robust to outliers in the model covariates, including those due to measurement error, and can account for both spatial heterogeneity and spatial clustering. Chambers et al. (2016) extend the M-quantile approach to small area estimation for counts (Tzavidis et al., 2015; Chambers et al., 2014b) to the case where the response is binary. Modelling the M-quantiles of a binary outcome presents more challenges than modelling the M-quantiles of a count outcome. A detailed account of these challenges is provided in the paper. With the proposed approach random effects are avoided and between-area variation in the response is characterized by variation in area-specific values of M-quantile indices. Furthermore, outlier robust inference is achieved in the presence of both misclassification and measurement error.

5 A test to assess the presence of area-specific effects

In this section we present a LR-type test for the presence of unobserved heterogeneity (clustering). The proposed test has a similar aim to that of a hypothesis test for the strict positiveness of variance components in the case of a linear mixed (random) effects model. Testing for the presence of significant clustering is a well known problem in literature (Greven et al., 2008; Crainiceanu and Ruppert, 2004; Datta et al., 2011). Clustering can exist either because of the design used to collect the data (i.e. use of a multi-stage cluster design) or because of natural structures that exist in the population (i.e. pupils nested within schools or individuals nested within households). The discussion in this section will pay special attention to the existence of area-effects in small area estimation.

Our aim is to test for the presence of significant area/cluster effects by proposing a testing procedure for the cluster-specific M-quantile coefficients τ_j.

Differently from Chambers and Tzavidis (2006), in the present work we define the MQ-coefficients $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)^T$ by adopting an approach that is explicitly based on the loss function. Within group j, τ_j is defined to be the one that uniquely solves

$$\min_{\tau} E\left[\rho\left(\frac{y_{ij} - \mathbf{x}_{ij}^T \boldsymbol{\beta}_\tau}{\sigma}\right) | j\right].$$

Intuitively, τ_j is defined as the MQ for which the regression plane identified by $\boldsymbol{\beta}_{\tau_j}$ is closest to observations from group j, according to the metrics of $\rho(\cdot)$. Note that $\rho(\cdot)$ is the untilted loss function, i.e. $\rho_{0.5}(\cdot)$, so the scale σ coincides with $\sigma_{0.5}$. The use of the untilted loss function is motivated by the search of the regression plane that best fits the units in a specific sub-group of the population. Testing for the presence of clustering is equivalent to testing whether the group-specific MQ-coefficients are all equal, that is,

$$H_0 : \tau_j = 0.5 \; \forall j = 1, \ldots, d$$

$$H_A : \tau_j \neq 0.5 \text{ for at least one } j.$$

Of course $\tau = 0.5$ represents the global minimizer when considering all groups $j = 1, \ldots, p$.

A natural estimator $\hat{\tau}_j$ for τ_j is obtained by solving

$$\min_{\tau} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\tau}}{\hat{\sigma}} \right),$$

where $\hat{\sigma}$ is an estimator of σ such as the one obtained solving (13) for $\tau = 0.5$. Since ρ is a positive function, the problem may be rewritten as follows. The vector of estimated MQ-coefficients $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_d)^T$ is obtained as the solution of

$$\min_{(\tau_1, \ldots, \tau_d)} \sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\tau_j}}{\hat{\sigma}} \right). \tag{30}$$

Assuming that conditions (C1)-(C5) are satisfied and that $\boldsymbol{\beta}_{\tau}$ is differentiable in τ with $\partial^2 \boldsymbol{\beta}_{\tau} / \partial \tau^2 = 0$ (i.e. $\boldsymbol{\beta}_{\tau}$ linear in τ), it may be shown that under H_0

$$-2 \frac{E\psi'_{ij}}{E\psi_{ij}^2} \left[\sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \boldsymbol{\beta}_{\hat{\tau}_j}}{\sigma} \right) - \sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \boldsymbol{\beta}_{0.5}}{\sigma} \right) \right] \xrightarrow{d} \chi^2_{d-1} \tag{31}$$

where $\psi'_{ij} = \psi'(\varepsilon_{0.5ij}/\sigma)$, $\psi_{ij} = \psi(\varepsilon_{0.5ij}/\sigma)$ and $\varepsilon_{0.5ij} = (y_{ij} - \mathbf{x}_{ij}^T \boldsymbol{\beta}_{0.5})$. For a sketch of the proof, see Appendix B. By simulation (Section 6.3), we show that by substituting the unknown parameters in (31), the asymptotic distribution is still well approximated by a χ^2_{d-1} distribution. Hence, a hypothesis test may be based on

$$-2 \frac{(n-p)^{-1} \sum_{ij} \hat{\psi}'_{ij}}{n^{-1} \sum_{ij} \hat{\psi}_{ij}^2} \left[\sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{\hat{\tau}_j}}{\hat{\sigma}} \right) - \sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho \left(\frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{0.5}}{\hat{\sigma}} \right) \right],$$

where $\hat{\psi}'_{ij} = \psi'(\hat{\varepsilon}_{0.5ij}/\hat{\sigma})$, $\hat{\psi}_{ij} = \psi(\hat{\varepsilon}_{0.5ij}/\hat{\sigma})$ $\hat{\varepsilon}_{0.5ij} = (y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{0.5})$, and $\boldsymbol{\beta}_{\hat{\tau}_j}$ and $\boldsymbol{\beta}_{0.5}$ are replaced by the corresponding consistent estimators.

The proposed test can assist the decision to include or not cluster effects in the model. We note that the asymptotic result holds if $n_j \to +\infty$ for each $j = 1, \ldots, d$. Even though the test is asymptotically valid when the sample size within each group tends to infinity, we empirically show in Section 6 that it provides reasonable results in the small area

estimation context as well. In Section 6 we explore the validity of this asymptotic result for different scenarios of the group-specific sample sizes.

The test we propose has a different aim to that of specification tests such as that recently proposed by Parente and Silva (2013) as we are not testing the assumptions needed for the estimation of $\boldsymbol{\beta}_\tau$ but whether units belonging to the same cluster are characterized by similar quantile coefficients, which is useful in prediction.

6 Simulation study

In this section we present results from three simulation studies used to investigate the method for selecting the tuning constant c proposed in Section 2.1, the finite sample properties of the tests proposed in Section 3 and the test statistic used for testing the presence of clustering in Section 5. Since these tests can be useful in small area estimation we generate data under linear mixed (random) effects models that incorporate area specific variation. The results for the Wald type test are not reported because they are very similar to the likelihood ratio type test. However, they are available to the prospective reader from the authors.

6.1 Choosing the tuning constant

In this Section we present results from a simulation study that is used to evaluate the estimation of the tuning constant c under the ALI distribution as proposed in Section 2.1. At each iteration of the algorithm the equations for $\boldsymbol{\beta}_\tau$, σ_τ, c are re-evaluated until convergence. The data is generated under the following mixed (random) effects model,

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_i + \varepsilon_{ij}, \quad i = 1, \ldots, n_j, \; j = 1, \ldots, d, \tag{32}$$

where $\beta_0 = 1$, $\beta_1 = 2$, x follows a Uniform distribution $(0, 5)$, $d = 100$, $n_j = 5$ ($n = 500$). The error terms of the mixed model, u_i and ε_{ij}, are generated by using different parametric assumptions; the random effects u_i are generated from a Normal distribution with mean 0 and $\sigma_u^2 = 1$ and ε are drawn from different error distributions,

1. Gaussian with mean 0, variance 1;
2. t-student with 3 degrees of freedom (t_3);
3. Contaminated Normal with $\varepsilon \sim (1-\gamma)N(0,1) + \gamma N(0,25)$ where γ is an independently generated Bernoulli random variable with $Pr(\gamma = 1) = 0.1$, i.e. the individual errors are independent draws from a mixture of two normal distributions, with 90% on average drawn from a well-behaved $N(0,1)$ distribution and 10% on average drawn from an outlier $N(0,25)$ distribution;
4. Cauchy with location 0 and scale 1.

As in the previous section, the residuals are rescaled so their variance is equal to 1 and the value of intraclass correlation under different scenarios is always approximately

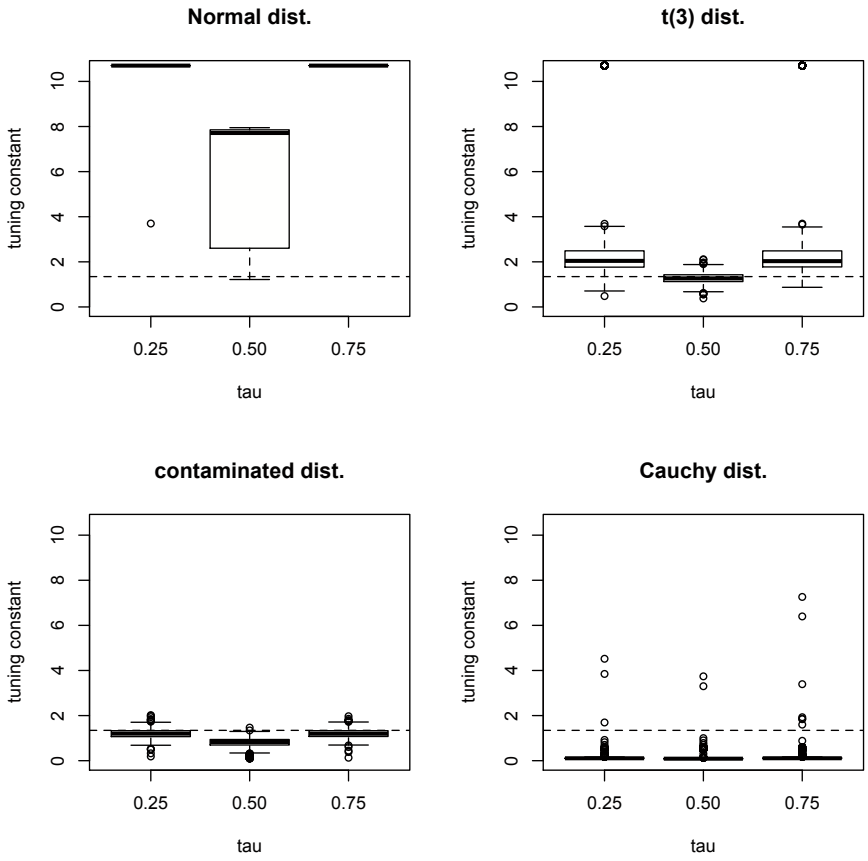

Figure 2: The distribution of the values of the tuning constant over Monte-Carlo samples and different settings for the error distribution at $\tau = 0.25,\ 0.50,\ 0.75$ and $d = 100$. The horizontal dashed line represents the choice of $c = 1.345$.

equal to 0.3. Figure 2 shows the distribution, over 10000 Monte-Carlo samples of the estimated tuning constants for the four scenarios at $\tau = 0.25,\ 0.5,\ 0.75$. The horizontal dashed line represents the usual choice of $c = 1.345$. Under the Gaussian setting, the values of the tuning constants are clearly larger than the value 1.345 (the conventional value used in MQ regression) at each τ. The estimated value of the tuning constant suggests that using a robust estimator in this case is not justified as one would expect under the assumptions we made in scenario 1. In contrast, the values of the estimated tuning constant are smaller than 1.345 in the contaminated and Cauchy scenarios. For instance, in the case of the contaminated scenario, the median value of the estimated tuning constant at $\tau = 0.5$ is 0.794. In the case of the Cauchy scenario the median value of the estimated tuning constant, at each quantile, degenerates to 0 because the Cauchy distribution has heavier tails than the exponential distributions and it should be truncated as the level of influential units becomes higher. For the t-student scenario the median value of the estimated tuning constant is 1.27 at $\tau = 0.5$ and it becomes higher than 1.345 (about 2.0) at $\tau = 0.25,\ 0.75$.

In applications a unique c should be chosen; it can be the optimal one at 0.5 or chosen taking into consideration also optimal values at other quantiles.

6.2 Likelihood Ratio type test

For evaluating the LR and Wald type tests for linear hypotheses on the MQ regression parameters, data is generated under the following mixed (random) effects model,

$$y_{ij} = \beta_0 + \beta_1 x_{ij1} + \beta_2 x_{ij2} + \beta_3 x_{ij3} + u_i + \varepsilon_{ij}, \quad i = 1, \ldots, n_j, \; j = 1, \ldots, d, \tag{33}$$

where j indexes the areas (clusters) and i units within areas. The regression coefficients are set as follows: $\beta_0 = 0$, $\beta_1 = 0.5$ and β_2, β_3 vary pairwise from 0 to 1, i.e. $(\beta_2, \beta_3) = (0, 0)$, $(\beta_2, \beta_3) = (0.25, 0.25)$, $(\beta_2, \beta_3) = (0.5, 0.5)$ and $(\beta_2, \beta_3) = (1, 1)$. The values of x_1, x_2 and x_3 are drawn from a Normal distribution with mean 5, 3 and 2, respectively and variance equal to 1. The number of small areas is set equal to $d = 20,\ 100$ and sample size in each small area $n_j = 5$, so we consider two different overall sample sizes: $n = 100,\ 500$. The error terms of the mixed model, u_i and ε_{ij}, are generated by using different parametric assumptions. Three settings for generating ε_i are considered,

1. Gaussian with mean 0, variance 1;
2. t-student distribution with 3 degrees of freedom (t_3);
3. Chi-squared errors with 2 degrees of freedom ($\chi^2(2)$).

T-students and Chi-squared random variables are re-scaled so to have variance equal to 1; in the case of chi-squared we substract the mean to generate zero-meaned residuals. The random effects are generated from a Normal distribution with mean 0 and $\sigma_u^2 = 0.43$. This entails that for all the scenarios the value of intraclass correlation is approximately equal to 0.3. These choices define a $4 \times 3 \times 2$ design of simulations. Each scenario is independently simulated $T = 10000$ times. MQ regression is fitted at $\tau = 0.5,\ 0.75,\ 0.90$ by using the Huber influence function with $c = 1.345$ for t-student and Chi-squared errors, $c = 100$ for Gaussian errors and the maximum likelihood estimator (15) based on ALI as the estimator of σ_τ. Setting c equal to 1.345 gives reasonably high efficiency under normality and protects against outliers when the Gaussian assumption is violated (Huber, 1981). For the Gaussian scenario the resistance against outliers is not necessary and a large value for the tuning constant is preferred.

The results for the LR-type test for the null hypothesis

$$H_0 : \beta_{2\tau} = \beta_{3\tau} = 0$$

at the significance level $\alpha = 0.10,\ 0.05,\ 0.01$ are presented in Table 1. In all cases when $\beta_2 = \beta_3 = 0$ and the null hypothesis is true, the Type I error is very close to the nominal α, with small deviations in the case of $\tau = 0.9$ in the t_3 and $\chi^2(2)$ scenarios with $d = 20$ ($n = 100$) where the test turns out to be slightly conservative. For the Gaussian scenario, the power of the test tends to 1 as soon as the values of β_2 and β_3 increase, i.e. the null hypothesis is rejected for both sample sizes. In case of departures from normality, for example under the t_3 scenario, the value of the power of the test tends to 1 at $\tau = 0.5$ and 0.75 once the $\beta_2, \beta_3 = 0.25$ especially for $d = 100$ ($n = 500$). At $\tau = 0.9$ the likelihood

ratio type test performs well as regression coefficients increase (as soon as $\beta_2, \beta_3 = 0.5$). Under the Chi-squared setting the test at $\tau = 0.75,\ 0.90$ appears to have lower power in rejecting the null hypothesis especially for the scenario with $d = 20$. Results for this scenario improve as the number of groups, d, and the values of the regression parameters (β_2, β_3) increase.

6.3 Testing for the presence of clustering

In this section we present an empirical evaluation of the properties of the test used for the hypothesis of the presence of clustering and we show how this test can be useful in small area estimation context. For these simulations, data is generated under model (32). Two scenarios for the number of groups, d, are used, $d = 20$ and $d = 100$ and three scenarios for the within group samples size, $n_j = 5$, $n_j = 20$ and $n_j = 50$. The error terms of the mixed model, u_i and ε_{ij}, are generated by using different parametric assumptions. In particular, the random effects are generated from a Normal distribution with mean 0 and different scenarios for the level 2 variance components $\sigma_u^2 = 0,\ 1,\ 2.5,\ 7.5$. For $\sigma_u^2 = 0$, data is generated under the null hypothesis of no clustering. For the values of σ_u^2 other than 0 we start introducing clustering in the simulated data. Individual effects are generated according to Normal distribution with mean 0 and variance 5. When $\sigma_u^2 = 0$, i.e. under the null hypothesis, we empirically study the Type I error by using the proposed test. For all other scenarios of $\sigma_u^2 \neq 0$ we study the power of the proposed test. Each scenario is independently simulated $T = 10000$ times.

In this Monte-Carlo simulation, MQ regression is fitted by using the Huber influence function with $c = 100$ and the maximum likelihood estimator for the scale (15) under the ALI distribution. Table 2 reports the results of the simulation experiment. The Table shows the values of the intraclass correlation, $r = \sigma_u^2/(\sigma_u^2+\sigma_\varepsilon^2)$, the Type I error and power of the proposed test statistic for $\alpha = 0.01,\ 0.05,\ 0.10$. To start with, we note that under the null hypothesis the Type I error is very close to the nominal value of α. As the value of σ_u^2 increases the power of the test increases too. The power increases more sharply for larger within cluster sample sizes. The number of clusters also seems to impact on the power of the test. The power of the test increases fairly sharply when we have a larger number of clusters even if each cluster consists of a small number of units. Under the null hypothesis we have also computed the empirical expected value and variance of the test statistic. We expect that, under the χ^2_{d-1} asymptotic approximation, the expected value of the test statistic will be equal to $d-1$ and the variance equal to $2\times(d-1)$. Results from the simulation studies confirm that the χ^2_{d-1} is a good approximation to the distribution of this test statistic. Finally, we have run a simulation where the individual effects are generated according to t-student with 3 degrees of freedom and the MQ regression is fitted by using the Huber influence function with $c = 1.345$. Also in this case under the null hypothesis the Type I error is very close to the nominal value of α and power of the test increases as the value of σ_u^2 increases. The detailed results are available to the interested reader from the authors.

The test can be used in small area estimation framework to detect the presence of area

Table 1: Type I error and power of the proposed likelihood ratio type test under Gaussian, t_3 and $\chi^2(2)$ distributions at $\tau = 0.50,\ 0.75,\ 0.90$ with β_2, β_3 varying pairwise from 0 to 1, $\alpha = 0.10,\ 0.05,\ 0.01$ and $d = 8,\ 20,\ 100$ with $n_j = 5$.

d	α	Gaussian, $c = 100$			t_3, $c = 1.345$			$\chi^2(2)$, $c = 1.345$		
		$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
		$(\beta_2, \beta_3) = (0, 0)$								
	0.10	0.117	0.132	0.181	0.119	0.143	0.322	0.120	0.159	0.355
8	0.05	0.066	0.079	0.117	0.067	0.087	0.232	0.069	0.096	0.268
	0.01	0.018	0.021	0.044	0.017	0.028	0.128	0.017	0.029	0.150
	0.10	0.110	0.114	0.133	0.103	0.114	0.147	0.109	0.120	0.181
20	0.05	0.059	0.062	0.075	0.050	0.063	0.089	0.057	0.064	0.112
	0.01	0.012	0.015	0.021	0.012	0.016	0.030	0.012	0.016	0.049
	0.10	0.101	0.105	0.109	0.102	0.108	0.122	0.103	0.106	0.126
100	0.05	0.052	0.058	0.058	0.052	0.053	0.063	0.050	0.055	0.069
	0.01	0.010	0.011	0.012	0.013	0.012	0.017	0.010	0.011	0.018
		$(\beta_2, \beta_3) = (0.25, 0.25)$								
	0.10	0.392	0.394	0.400	0.547	0.491	0.490	0.353	0.248	0.188
8	0.05	0.283	0.282	0.301	0.430	0.380	0.401	0.251	0.166	0.102
	0.01	0.130	0.135	0.159	0.229	0.205	0.156	0.104	0.068	0.073
	0.10	0.574	0.547	0.481	0.681	0.605	0.457	0.497	0.313	0.273
20	0.05	0.453	0.430	0.371	0.566	0.488	0.357	0.375	0.215	0.191
	0.01	0.245	0.225	0.192	0.337	0.267	0.191	0.184	0.088	0.082
	0.10	1.000	0.999	0.964	1.000	0.996	0.909	0.984	0.823	0.395
100	0.05	0.991	0.998	0.934	0.998	0.991	0.846	0.967	0.728	0.282
	0.01	0.962	0.989	0.914	0.991	0.968	0.671	0.903	0.498	0.128
		$(\beta_2, \beta_3) = (0.50, 0.50)$								
	0.10	0.849	0.827	0.774	0.952	0.900	0.761	0.779	0.498	0.437
8	0.05	0.776	0.746	0.694	0.919	0.846	0.692	0.689	0.391	0.397
	0.01	0.580	0.554	0.516	0.807	0.702	0.546	0.485	0.212	0.200
	0.10	0.978	0.962	0.920	0.993	0.982	0.852	0.944	0.729	0.449
20	0.05	0.960	0.941	0.873	0.987	0.961	0.784	0.905	0.619	0.352
	0.01	0.883	0.841	0.729	0.953	0.890	0.619	0.774	0.400	0.196
	0.10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.872
100	0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.795
	0.01	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.590
		$(\beta_2, \beta_3) = (1, 1)$								
	0.10	1.000	1.000	0.995	1.000	0.998	0.973	0.994	0.906	0.731
8	0.05	1.000	1.000	0.991	1.000	0.997	0.958	0.990	0.854	0.657
	0.01	0.995	0.991	0.968	0.998	0.991	0.916	0.966	0.708	0.501
	0.10	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.996	0.841
20	0.05	1.000	1.000	1.000	1.000	1.000	0.996	1.000	0.990	0.767
	0.01	1.000	1.000	1.000	1.000	1.000	0.985	1.000	0.965	0.604
	0.10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
100	0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.01	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2: Type I error and power of the proposed test statistic for clustering under Gaussian distribution with r varying between 0 and 0.6, $\alpha = 0.10,\ 0.05,\ 0.01$, $d = 8,\ 20,\ 100$ and $n_j = 5,\ 20,\ 50$.

α	$d = 8$			$d = 20$			$d = 100$		
	$n_j = 5$	$n_j = 20$	$n_j = 50$	$n_j = 5$	$n_j = 20$	$n_j = 50$	$n_j = 5$	$n_j = 20$	$n_j = 50$
					$r = 0$				
0.10	0.114	0.085	0.103	0.141	0.104	0.099	0.120	0.089	0.103
0.05	0.062	0.035	0.048	0.075	0.059	0.047	0.060	0.036	0.042
0.01	0.008	0.007	0.014	0.015	0.012	0.008	0.018	0.009	0.009
					$r = 0.16$				
0.10	0.413	0.910	0.991	0.702	0.999	1.000	0.983	1.000	1.000
0.05	0.213	0.875	0.985	0.565	0.998	1.000	0.969	1.000	1.000
0.01	0.118	0.765	0.971	0.325	0.992	1.000	0.906	1.000	1.000
					$r = 0.33$				
0.10	0.707	0.983	0.998	0.954	1.000	1.000	1.000	1.000	1.000
0.05	0.572	0.981	0.998	0.904	1.000	1.000	1.000	1.000	1.000
0.01	0.330	0.955	0.995	0.763	1.000	1.000	1.000	1.000	1.000
					$r = 0.60$				
0.10	0.933	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
0.05	0.881	0.999	1.000	0.998	1.000	1.000	1.000	1.000	1.000
0.01	0.720	0.995	1.000	0.989	1.000	1.000	1.000	1.000	1.000

effects. If the test rejects H_0 it means that there is unobserved heterogeneity between areas and predictor (27) can be used to estimate the small area mean. Otherwise, if H_0 is not rejected, the synthetic estimator can be used for predicting the small area quantity because, in the case of absence of unobserved heterogeneity between areas, it guarantees less variability and bias than estimator (27). To evaluate the performance of the synthetic predictor and the MQ predictor (27) the absolute relative bias (ARB) and the relative root mean squared error (RRMSE) of estimates of the mean value in each small area are computed. Table 3 reports the average values over areas of these indices for $n_j = 5,\ 20,\ 50$ and $d = 100$. The results for $d = 20$ are not reported because these are very similar to those for $d = 100$, but are available from from the authors upon request. Table 3 shows that the average ARB and RRMSE of the synthetic predictor increase as the intraclass correlation increases. The average values of ARB and RRMSE for estimator (27) remain constant at different values of r given the sample size. From the results in Table 3 it is apparent that when the assumption of significant between area heterogeneity is not rejected, the synthetic estimator offers the best performance. On the other hand, as soon as the intraclass correlation increases the predictor (27) performs best. Thus the LR-type test for the presence of clustering can drive the choice of the M-quantile predictor in small area estimation. The increase in the RRMSE when incorporating the area effect into prediction unnecessarily has been documented by other authors (see Datta et al., 2011). Our work extends these results to the case of small area estimation based on M-quantile regression.

Table 3: Values of the average ARB and average RRMSE over small areas for synthetic and (27) predictors under Gaussian distribution with r varying between 0 and 0.6, $d = 100$ and $n_j = 5,\ 20,\ 50$. Values are expressed as percentages.

Predictor	$n_j = 5$		$n_j = 20$		$n_j = 50$	
	ARB	RRMSE	ARB	RRMSE	ARB	RRMSE
			$r = 0$			
$\hat{m}_j^{MQ}$	11.07	13.62	5.66	7.04	3.55	4.45
$\hat{m}_j^{MQ/SYN}$	1.39	1.74	0.99	1.24	0.86	1.08
			$r = 0.16$			
$\hat{m}_j^{MQ}$	10.63	13.25	5.44	6.82	3.45	4.33
$\hat{m}_j^{MQ/SYN}$	11.41	14.29	11.20	14.02	10.84	13.58
			$r = 0.33$			
$\hat{m}_j^{MQ}$	10.54	13.20	5.60	7.10	3.73	4.87
$\hat{m}_j^{MQ/SYN}$	17.96	22.50	17.67	22.13	17.12	21.44
			$r = 0.60$			
$\hat{m}_j^{MQ}$	11.71	15.10	7.17	10.40	5.46	8.91
$\hat{m}_j^{MQ/SYN}$	31.07	38.92	30.59	38.31	29.65	37.13

7 Application

In this Section we use a dataset well-known in the small area estimation literature for illustrating the proposed model fit, selection and diagnostic criteria. Battese et al. (1988) analyse survey and satellite data for corn and soybean production for 12 counties in North Central Iowa. The dataset comes from the June 1978 Enumerative Survey, consists of 37 observations and includes information on the number of segments in each county, the number of hectares of corn and soybeans for each sample segment, the number of pixels classified by the LANDSAT satellite as corn and soybeans for each sample segment, and the mean number of pixels per segment in each county classified as corn and soybeans. These data were used by Battese et al. (1988) to predict the hectares of corn and soybean by county. We use this dataset to compute the tuning constant c (Huber loss function is going to adopted), the R^2 goodness-of-fit measure, the LR-type test for specifying the explanatory variables to be included in MQ regression, and the likelihood ratio type test for the presence of actual area heterogeneity. County specific random effects were introduced by Battese et al. (1988) to improve prediction, so we would like to use our methodology to test whether there is significant between county variation in the MQ-coefficients, something that would justify the inclusion of county specific quantiles.

The response variable y is the number of hectares of corn and soybeans and the model includes two fixed effects, x_1 and x_2 that represent the number of pixels classified by the LANDSAT satellite as corn and soybeans respectively for each sample segment. Battese et al. (1988) use the following two-level linear mixed model where i denotes the counties and j denotes the segments:

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + u_i + e_{ij}.$$

A random effect u_i is specified at the county level. This model will be used for benchmarking our results. Diagnostic for this model is reported in other papers (see for

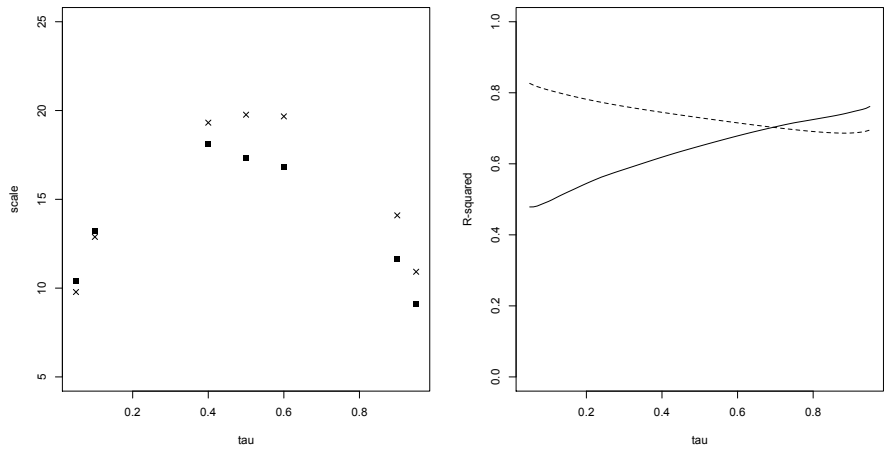

Figure 3: Left plot shows the values of the estimated scale at different value of τ for corn (■) and soybean (×). Right plot presents the R-squared at different value of τ for corn (solid line) and soybean (dashed line).

example Sinha and Rao, 2009). They indicate that for the soybean variable normality of u and e approximately holds. For the corn variable, on the other hand, there is an influential outlier in the Hardin county.

We present results for MQ regression at $\tau = 0.05,\ 0.10,\ 0.25,\ 0.5,\ 0.75,\ 0.90,\ 0.95$. We further compare our results at $\tau = 0.5$ to model diagnostics from the linear mixed model used by Battese et al. (1988). For the analysis of the corn outcome, the estimate of the tuning constant c using the GALI pseudo-likelihood at $\tau = 0.5$ is equal to 1.94, a relatively low value, consistent with the presence of the outlier identified in diagnostic analysis. For the soybean variable the tuning constant c estimate at $\tau = 0.5$ is 7.85. This value suggests that there are no issues with contamination. Using $c = 1.345$, that represents a typical choice in the applications of the Huber loss function, or the value we chose for corn, would increase the robustness unnecessarily at the cost of lower efficiency. Similar conclusions hold for other values of τ.

Estimates of the scale parameter σ_τ obtained with the GALI-based method are shown in Figure 3. We note that these are sensitive to the M-quantile being considered and exhibit an inverted u-shape: for quantiles far from 0.5 the proportion of residuals for which $|u| > c$ is larger and this reduces their average size. When τ is close to 0.5 the estimates we obtain are close to those obtained by using the MAD estimator (9). On the contrary, MAD estimates are larger for quantiles far from 0.5 compared to those obtained in the central part of the distribution. This can be due to the fact that the scaling constant q in (9) should be quantile-adjusted. Looking at the R^2 model fit criterion we note that for the corn outcome this increases as τ increases (see Figure 3 solid line). For the soybean outcome there appears to be an almost constant high value of R^2 at all values of τ (see Figure 3 dashed line). Overall, for both outcomes there appears to be a moderate to strong linear relationship between the outcome and the explanatory variables at the different values of τ.

	$H_0 : (\beta_1, \beta_2) = 0$		$H_0 : \beta_2 = 0$	
τ	LR Test	p-value	LR Test	p-value
0.05	21.4	0.000	1.4	0.4935
0.10	23.8	0.000	0.3	0.8350
0.25	38.4	0.000	0.0	0.9996
0.50	68.3	0.000	0.4	0.7855
0.75	105.1	0.000	0.6	0.7376
0.90	97.1	0.000	0.1	0.9534
0.95	65.8	0.000	0.0	0.9959

Table 4: LR-type test for the model specification of the corn outcome, $H_0 : (\beta_1, \beta_2) = 0$ and $H_0 : \beta_2 = 0$

The LR-type tests results for the corn outcome are presented in Table 4 and for the soybean outcome in Table 5. When testing jointly the significance of x_1 and x_2, the tests suggest that these covariates are significant for explaining the variability in both outcomes. For the corn outcome the tests show that after controlling for the number of pixels classified by the LANDSAT satellite as corn (x_1), the number of pixels classified by the LANDSAT satellite as soybean (x_2) is not significant. Similarly, for the soybean outcome after controlling for the number of pixels classified by the LANDSAT satellite as soybean (x_2), the number of pixels classified by the LANDSAT satellite as corn (x_1) is not significant. Hence, the model specification can be simplified by dropping the non-significant terms. The same conlcusions can be obtained by using the Wald-type test. For validating these results at $\tau = 0.5$, we run the same analysis under the two-level linear mixed model used by Battese et al. (1988). For the corn outcome after controlling for x_1, the p-value for including x_2 is equal to 0.6315 indicating that x_2 can be dropped from the model. For the corn outcome after controlling for x_2, the p-value for including x_1 is equal to 0.6049 indicating that x_1 can be dropped from the model.

We turn our attention to testing the significance of the between county variability. The two scatter plots in Figure 4 show the relationship between the predicted county random effects computed with the mixed model and the MQ county coefficients computed with the MQ model for the corn outcome (scatter plot (a)) and the soybean outcome (scatter plot (b)). For both outcomes the two measures of county effects are well correlated. For testing the significance of the county MQ coefficients we use the proposed LR-type test. For the corn outcome the value of the test statistic is 17.152 and the corresponding p-value= 0.103. We have also conducted the hypothesis test for the presence of significant between county variation by using the linear mixed model. For testing the null hypothesis of a zero between county variation we compute the conditional-AIC (cAIC) value (Vaida and Blanchard , 2005) and compare this to the AIC value for a linear regression model without random effects. The cAIC for the linear mixed model is 327.5109 and the AIC for the linear regression model is 327.4116. This indicates that the linear model without random effects fits almost as well as the more complex model that includes random effects. Hence, random effects may not be needed in the analysis of the corn outcome.

For the soybean outcome the value of the LR-type test for the presence of clustering is 26.791 and the corresponding p-value= 0.0049. As in the case of the corn outcome,

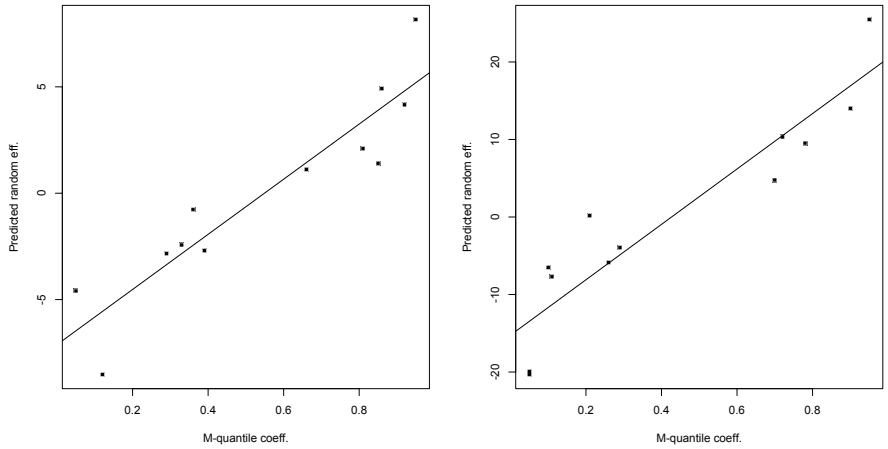

Figure 4: Scatter plots for the relationship between the predicted county random effects (computed with the mixed model) and the MQ county coefficients (computed with the MQ model) for the corn outcome (a) and for the soybean outcome (b).

	$H_0 : (\beta_1, \beta_2) = 0$		$H_0 : \beta_1 = 0$	
τ	LR Test	p-value	LR Test	p-value
0.05	195.7	0.000	2.6	0.2696
0.10	146.6	0.000	1.2	0.5496
0.25	116.0	0.000	0.3	0.8557
0.50	91.8	0.000	0.0	0.9972
0.75	66.7	0.000	0.4	0.8129
0.90	61.9	0.000	1.2	0.5380
0.95	65.3	0.000	01.6	0.4532

Table 5: LR-type test for the model specification of the soybean outcome, $H_0 : (\beta_1, \beta_2) = 0$ and $H_0 : \beta_1 = 0$

we have also conducted the hypothesis test for the presence of significant between county variation by using the linear mixed model. The cAIC for the linear mixed model is 311.8459 and the AIC for the linear regression model is 333.8107. This indicates that the linear model with county random effects fits better than the simpler model that ignores the random effects.

8 Final remarks

In this paper we have reviewed the M-quantile regression model and its application to SAE. We have also extended the available toolkit for inference in M-quantile regression. For given τ we have proposed a pseudo-R^2 goodness-of-fit measure, a likelihood ratio and Wald type tests for testing linear hypotheses on the M-quantile regression parameters.

The cluster-specific M-quantile coefficients have been used for proposing a test for the presence of clustering in the data. The set of tests we present in the paper can be applied in small area estimation framework to validate the M-quantile models used for prediction. For a large class of continuously differentiable convex functions we showed the

relationship between the loss function used in M-quantile regression and the maximization of a likelihood function formed by combining independently distributed GALI densities. Using this parametrization, we further propose an estimator of the scale parameter and a data-driven tuning constant to be used in the loss function. For each test the asymptotic theory has been developed involving recent works on inference by Wooldridge (2010) and Bianchi and Salvati (2015).

The simulation results for studying the finite sample properties of the model-fit criteria and the tests show that the Type I error of the LR-type test and the clustering test is very close to the nominal level α. For both tests, the results also indicate that the power tends to 1 as the values of the regression coefficients and the interclass correlation coefficient increase. In the simulation experiments we have also investigated the behaviour of the method proposed for estimating the tuning constant in the Huber loss function. The tuning constant derived by using the likelihood method is able to reflect different levels of contamination in the data.

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A Properties of the ALI

In this appendix we provide some more properties for special case of the GALI distribution when the $\rho(\cdot)$ is given by (1), that is the ALI we introduced in section 2.1. Suppose that U is a random variable with the standard ALI density ($\mu_\tau = 0, \sigma_\tau = 1$), then its cumulative distribution function is written as

$$F(u) = \begin{cases} \frac{1}{2c(1-\tau)B_\tau} exp\{[2cu + c^2](1-\tau)\} & u \leqslant -c \\ \frac{1}{B_\tau}\left\{\frac{1}{2c(1-\tau)}e^{-c^2(1-\tau)} + \sqrt{\frac{\pi}{1-\tau}}\Big[\Phi(u\sqrt{2(1-\tau)}) - \Phi(-c\sqrt{2(1-\tau)})\Big]\right\} & -c < u \leqslant 0 \\ \frac{1}{B_\tau}\left\{\frac{1}{2c(1-\tau)}e^{-c^2(1-\tau)} + \sqrt{\frac{\pi}{1-\tau}}\Big[\Phi(c\sqrt{2(1-\tau)}) - 1/2\Big] + \sqrt{\frac{\pi}{\tau}}\Big[\Phi(u\sqrt{2\tau}) - 1/2\Big]\right\} & 0 < u \leqslant c \\ \frac{1}{B_\tau}\left\{\frac{1}{2c\tau}e^{-c^2\tau} - \frac{1}{2c\tau}exp\{-2\tau cu + c^2\tau\}\right\} & u > c \end{cases}$$

For obtaining the expected value and the variance of U, the moment generating function is computed and it can be written as:

$$M_\tau(t) = \frac{1}{B_\tau[2c(1-\tau)+t]}exp\{-c^2(1-\tau) - ct\}$$

$$+\frac{exp\{\frac{t^2}{4(1-\tau)}\}}{B_\tau}\sqrt{\frac{\pi}{(1-\tau)}}\Big[\Phi\left(-\frac{t}{\sqrt{2(1-\tau)}}\right) - \Phi\left(\frac{-2c(1-\tau)-t}{\sqrt{2(1-\tau)}}\right)\Big]$$

$$+\frac{exp\{\frac{t^2}{4\tau}\}}{B_\tau}\sqrt{\frac{\pi}{\tau}}\Big[\Phi\left(\frac{2c\tau - t}{\sqrt{2\tau}}\right) - \Phi\left(-\frac{t}{\sqrt{2\tau}}\right)\Big] - \frac{1}{B_\tau(t-2c\tau)}exp\{-c^2\tau + ct\},$$

for $-2c(1-\tau) < t < 2c\tau$.
The first moment then is

$$E(U) = -\frac{1}{4B_\tau c^2(1-\tau)^2}exp\{-c^2(1-\tau)\} + \frac{1}{4B_\tau c^2\tau^2}exp\{-c^2\tau\} + \frac{1-2\tau}{2\tau(1-\tau)B_\tau}$$

and the variance is

$$Var(U) = \frac{1}{B_\tau}\left[e^{-c^2(1-\tau)}\frac{1+2c^2(1-\tau)}{4c^3(1-\tau)^3} + e^{-c^2\tau}\frac{1+2c^2\tau}{4c^3\tau^3} + \frac{1}{2}\frac{\sqrt{\pi}\Big[\Phi(c\sqrt{2\tau}) - 0.5\Big]}{\tau^{3/2}}\right.$$

$$\left. +\frac{1}{2}\frac{\sqrt{\pi}\Big[\Phi(c\sqrt{2(1-\tau)}) - 0.5\Big]}{(1-\tau)^{3/2}}\right].$$

These formulae may be easily generalized to the location and scale case. They can be used to obtain method of moments estimates of c and σ_τ to be used as initial values when minimizing (12) when $\rho_\tau(\cdot)$ is the Huber loss function, in line with Yu and Zhang (2005). The computations for obtaining the moment generating function, the expected value and the variance of U are not reported in the paper, but they are available from the authors upon request.

B Sketch of the proof of equation (31)

Under the assumptions of the theorem, convergence of $\hat{\tau}_j$ to τ_j is verified by using standard Taylor linearization techniques. For the asymptotic distribution of the test statistic, let $Q(\boldsymbol{\tau}) = \sum_{j=1}^{d} \sum_{i=1}^{n_j} \rho\left(\frac{y_{ij}-\mathbf{x}_{ij}^T\boldsymbol{\beta}_{\tau_j}}{\sigma}\right)$, $\mathbf{s}(\boldsymbol{\tau}) = \left\{\frac{1}{\sqrt{n_j}} \sum_{i=1}^{n_j} \frac{\partial\rho}{\partial\tau_j}\left(\frac{y_{ij}-\mathbf{x}_{ij}^T\boldsymbol{\beta}_{\tau_j}}{\sigma}\right)\right\}_{j=1}^{d}$, $\mathbf{H}(\boldsymbol{\tau}) = diag\left\{\frac{1}{n_j}\sum_{i=1}^{n_j}\frac{\partial^2\rho}{\partial\tau_j^2}\left(\frac{y_{ij}-\mathbf{x}_{ij}^T\boldsymbol{\beta}_{\tau_j}}{\sigma}\right)\right\}$, and $\mathbf{n} = (n_1, \ldots, n_d)^T$. Let $\mathbf{A}_0 = diag\{a_j\}$ and $\mathbf{B}_0 = diag\{b_j\}$ with

$$\begin{aligned} a_j &= E\left[\frac{\partial^2\rho}{\partial\tau_j^2}\left(\frac{y_{ij}-\mathbf{x}_{ij}^T\boldsymbol{\beta}_{\tau_j}}{\sigma}\right)\Big|_{0.5}\right] = \sigma^{-2}E\psi'_{ij}E\left(\mathbf{x}_{ij}^T\frac{\partial\boldsymbol{\beta}_{\tau_j}}{\partial\tau_j}\Big|_{0.5}\right)^2 \\ b_j &= E\left[\frac{\partial\rho}{\partial\tau_j}\left(\frac{y_{ij}-\mathbf{x}_{ij}^T\boldsymbol{\beta}_{\tau_j}}{\sigma}\right)\Big|_{0.5}\right]^2 = \sigma^{-2}E\psi_{ij}^2E\left(\mathbf{x}_{ij}^T\frac{\partial\boldsymbol{\beta}_{\tau_j}}{\partial\tau_j}\Big|_{0.5}\right)^2. \end{aligned}$$

Under H_0, a mean value expansion yields

$$\mathbf{0} = \mathbf{s}(\hat{\boldsymbol{\tau}}) = \mathbf{s}(\mathbf{0.5}) + \sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}} - \mathbf{0.5}) + o_p(1),$$

implying $\sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}}-\mathbf{0.5}) \xrightarrow{d} N(\mathbf{0}, \mathbf{A}_0^{-1}\mathbf{B}_0\mathbf{A}_0^{-1})$, as $n_j \to +\infty$, $j = 1, \ldots, d$, where $\cdot$ denotes the Hadamard product.

Then

$$\begin{aligned} Q(\mathbf{0.5}) - Q(\hat{\boldsymbol{\tau}}) &= \frac{1}{2}(\hat{\boldsymbol{\tau}} - \mathbf{0.5})^T H(\dot{\boldsymbol{\tau}})(\hat{\boldsymbol{\tau}} - \mathbf{0.5}) \\ &= \frac{1}{2}[\sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}} - \mathbf{0.5})]^T\mathbf{A}_0[\sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}} - \mathbf{0.5})] + o_p(1), \end{aligned}$$

where $\dot{\boldsymbol{\tau}}$ is a value between $\hat{\boldsymbol{\tau}}$ and $\mathbf{0.5}$. Hence

$$2[Q(\mathbf{0.5}) - Q(\hat{\boldsymbol{\tau}})]\frac{E\psi'_{ij}}{E\psi_{ij}^2} = [\sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}} - \mathbf{0.5})]^T[\mathbf{A}_0^{-1}\mathbf{B}_0\mathbf{A}_0^{-1}]^{-1}[\sqrt{\mathbf{n}}\cdot(\hat{\boldsymbol{\tau}} - \mathbf{0.5})] + o_p(1).$$

Intuitively, for reasons of symmetry, if $\tau_j = 0.5$ for $j = 1, \ldots, d-1$, also $\tau_d = 0.5$ (as the global minimizer is $\tau = 0.5$). The same relationship needs to hold for the corresponding estimators $\hat{\tau}_j$'s. So the previous expression may be reparametrized leading to a χ^2_{d-1} asymptotic distribution.