Diffusion and Clustering of Low-inertia Tracers in Random Hydrodynamic Flow with Fast Rotation

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Abtract :

We consider the diffusion of the low-inertia density field in random divergence-free hydrodynamic flows with fast rotation. The principal feature of this diffusion is the divergence of the density velocity field, which results in clustering of the density field. We calculate the statistical parameters that characterized cluster formation on the ideas of statistical topography.

Key-words :

Diffusion ; Clustering ; Stochastic flow

1 Introduction

In recent time, the interest of both theoreticians and experimenters has been attracted to relation of the behaviour of average statistical characteristics of a problem solution with the behaviour of the solution in certain happenings (realizations). This is especially important for geophysical problems related to the atmosphere and ocean where a respective averaging ensemble is absent and experimenters, as a rule, have to do with individual observations.

Statistical characteristics for the solution to the problem of diffusion of particles and conservative passive tracer density field in random divergent velocity fields may have little in common with the behaviour of concrete realizations. The traditional approach to such problems, based on the moment function description, is of small informational value. However, problems pertaining to diffusion of particles and passive tracer concentration fields in random divergent velocity fields contain statistically coherent physical phenomena that occur with probability unity (clustering of particles and the conservative density field in a divergent velocity field). This means that a given phenomenon occurs in almost all random velocity field realizations.

The coherent phenomena themselves do not depend on a particular model of fluctuating parameters of a dynamic system. In the simplest case, their time-dependent dynamics can be described in terms of simultaneous and one-point probability distributions by the methods of statistical topography. However, the coherent values of the parameters that characterize a given phenomenon (e.g., the characteristic time of cluster formation and spatial scales of cluster) can significantly depend on the model type.

Note that analysis of diffusion of particles and density field indicates that the approximation of the delta-correlated in time random hydrodynamic velocity field is incorrect in the case of a low-inertia tracer. It is therefore necessary to do calculations using the finiteness of the time correlation of hydrodynamic velocity field. This can be made in the diffusion approximation.

Statistical topography yields a different philosophy of statistical analysis of dynamic stochastic systems, which may prove useful for experimenters planning a statistical processing of experimental data.

2 General remarks

The diffusion of the conservative density field $\rho(\mathbf{r},t)$ in random 2-D hydrodynamic flow $\mathbf{u}(\mathbf{r},t)$ with fast rotation is described by continuity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\mathbf{V}(\mathbf{r},t)\right)\rho(\mathbf{r},t)=0,\ \ \rho(\mathbf{r},0)=\rho_0(\mathbf{r}), \quad (1)$$

and the total mass is conserved during the evolution, i.e.,

$$M_0=\int\rho(\mathbf{r},t)d\mathbf{r}=\int\rho_0(\mathbf{r})d\mathbf{r}=\text{const.}$$

The Eulerian tracer velocity field $\mathbf{V}(\mathbf{r},t)$ is described by the quasi-linear equation

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)V_i(\mathbf{r},t)=-\lambda\left[V_i(\mathbf{r},t)-u_i(\mathbf{r},t)\right]+2\Omega\Gamma_{ij}V_j, \qquad (2)$$

where $\Gamma=\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}$, $\Gamma^2=-E$, and E is the unit matrix, which we shall regard as a phenomenological one. In the general case, the nonuniqueness of the solution of Eq. (2), discontinuities, etc. are possible. However, in an asymptotic case of low-inertia particles (parameter $\lambda\to\infty$), which is of special interest to us, there exists a unique solution over a reasonable time interval.

It should be noted that the term $\mathbf{F}(\mathbf{r},t)=\lambda\mathbf{V}(\mathbf{r},t)$ on the right-hand side of Eq. (2), linear in the velocity field $\mathbf{V}(\mathbf{r},t)$, is the well-known Stokes formula} for a resistive force acting on a slowly moving particle. If the particle is approximated by a ball of radius *a*, then one obtains $\lambda=6\pi a\eta/m$, where η is the coefficient of dynamic viscosity, and m is the particle's mass.

The partial differential equations (1) and (2) (Eulerian description) are equivalent to the system of characteristic equations for particles (Lagrangian description) which are Newton equations (for example, without rotation)

$$\frac{d}{dt}\mathbf{r}(t)=\mathbf{V}(t),\ \ \frac{d}{dt}\mathbf{V}(t)=-\lambda\left[\mathbf{V}(t)-\mathbf{u}\left(\mathbf{r}(t),t\right)\right].$$

with the linear friction force described by the Stokes formula for slowly moving particles, under the effect of random force induced by the hydrodynamic flow. The hydrodynamic velocity field $\mathbf{u}(\mathbf{r},t)$ is assumed to be a random Gaussian field, statistically isotropic in space, and stationary in time and has the correlation and spectral tensors

$$B_{ij}(\mathbf{r},t)=\left\langle u_i(\mathbf{r}'+\mathbf{r},t'+t)u_i(\mathbf{r}',t')\right\rangle=\int d\mathbf{k}E_{ij}(\mathbf{k},t)e^{i\mathbf{kr}},$$

$$E_{ij}(\mathbf{k},t)=E^s(k,t)\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)+E^p(k,t)\frac{k_ik_j}{k^2}.$$

Here $E^s(k,t)$, $E^p(k,t)$ are the solenoidal and potential components of the spectral tensor.

For inertialess particles, the parameter $\lambda\to\infty$ we have that $\mathbf{V}(\mathbf{r},t)=\mathbf{u}(\mathbf{r},t)$, and the particle's trajectory in a hydrodynamic flow is described by the simplest equation

$$\frac{d}{dt}\mathbf{r}(t)=\mathbf{u}\left(\mathbf{r}(t),t\right),$$

and we have the pure kinematical motion of particles. In the Fig. 1 we see the results of numerical modeling of evolution of the uniform particles distribution in unit circle by random stationary incompressible flow and of potential steady flow $\mathbf{u}(\mathbf{r})$.

a *b*

FIG. 1 – Diffusion of a system of particles numerically simulated for (*a*)solenoidal and (*b*) divergence-free random steady velocity field $\mathbf{u}(\mathbf{r})$

In the first case total areas enclosed by the deformed contour remain constant but the contour acquires increasing complicated "fractal" form (stochastic advection). In the second case we see demonstrations of the phenomena of *clustering* - regions of high concentration are formed within large rarefaction zones.

Equation (2) can be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)=-\Lambda\left[\mathbf{V}(\mathbf{r},t)-\mathbf{U}(\mathbf{r},t)\right], \qquad (3)$$

where the matrix $\Lambda=\left(\lambda E-2\Omega\Gamma\right)$, and the random velocity field $\mathbf{U}(\mathbf{r},t)$ has the form

$$\mathbf{U}(\mathbf{r},t)=\lambda\Lambda^{-1}\mathbf{u}(\mathbf{r},t), \quad \Lambda^{-1}=\frac{\lambda E+2\Omega\Gamma}{\lambda^2+4\Omega^2}.$$

In the case of $\{\lambda \,\mathrm{or}\, \omega\}\to\infty$ an approximate expression is obtained in the form $\mathbf{V}(\mathbf{r},t)\approx\mathbf{U}(\mathbf{r},t)$. The field $\mathbf{U}(\mathbf{r},t)$ is a divergent one, and for the divergence-free hydrodynamic field $\mathbf{u}(\mathbf{r},t)$ the quantity

$$\operatorname{div}\mathbf{U}(\mathbf{r},t)=\frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}}=\lambda\frac{\partial}{\partial r_k}\Lambda^{-1}_{k\mu}u_\mu(\mathbf{r},t)=\frac{2\lambda\Omega}{\lambda^2+4\Omega^2}\Gamma_{k\mu}\frac{\partial u_\mu(\mathbf{r},t)}{\partial r_k}$$
$$=\frac{2\lambda\Omega}{\lambda^2+4\Omega^2}\left(\frac{\partial u_2(\mathbf{r},t)}{\partial r_1}-\frac{\partial u_1(\mathbf{r},t)}{\partial r_2}\right)$$

is related to the vortex component of the field $\mathbf{u}(\mathbf{r},t)$.

3 Statistical analysis

Now the hydrodynamic divergence-free velocity field $\mathbf{u}(\mathbf{r},t)$ is assumed to be a random Gaussian divergence-free field with zero mean value, statistically isotropic in space, and stationary in time and has the correlation and space-time spectral tensors

$$B^{(\mathbf{u})}_{ij}(\mathbf{r},t)=\int d\mathbf{k}E_{ij}(\mathbf{k},t)e^{i\mathbf{k}\mathbf{r}}, \quad B^{(\mathbf{u})}_{ij}(\mathbf{r},t)=\int d\mathbf{k}\int_{-\infty}^{\infty}d\omega\,\Phi_{ij}(\mathbf{k},\omega)e^{i\mathbf{k}\mathbf{r}+i\omega t}, \qquad (4)$$

where $E_{ij}(\mathbf{k},t) = E(k,t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$, $\Phi_{ij}(\mathbf{k},\omega) = \Phi(k,\omega)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$.

The key feature of the process is the clustering of the tracer field due to the divergence of tracer-velocity field $\mathbf{V}(\mathbf{r},t)$. Introduction of randomness into parameters of the medium gives rise to stochasticity of physical fields themselves. Individual realizations of, say, two-dimensional scalar field $\rho(\mathbf{r},t)$ resemble a complex mountain terrain with randomly distributed peaks, ridges, valleys, passes, etc.

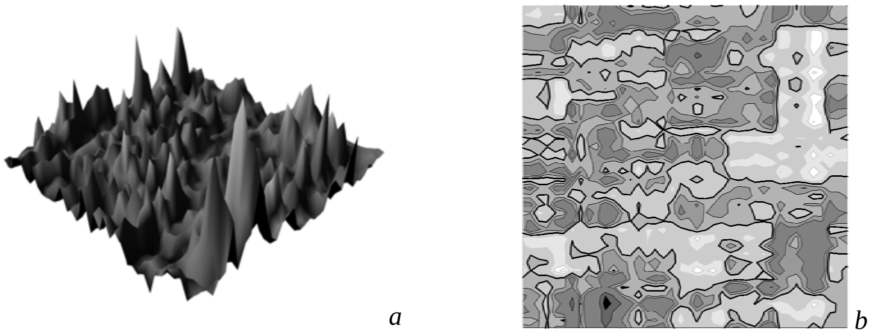

FIG. 2 – One realization of lognormal random field (*a*) and its topographic level lines (*b*)

Given a random field $\mathbf{V}(\mathbf{r},t)$ is Gaussian, statistically homogeneous, spatially isotropic, and stationary in time, with a zero mean value and the correlation tensor

$$\left\langle V_i(\mathbf{r},t)V_j(\mathbf{r}',t')\right\rangle = B_{ij}^{(\mathbf{V})}(\mathbf{r}-\mathbf{r}',t-t')$$

the one-point probability density function (PDF) of the field $\rho(\mathbf{r},t)$ – $P(t,\mathbf{r};\rho)$ for the solution of dynamic equation (1) in both the approximation of the delta-correlated in time field $\mathbf{V}(\mathbf{r},t)$ and in the diffusion approximation solves the equation

$$\frac{\partial}{\partial t}P(t,\mathbf{r};\rho) = \left(D_0\frac{\partial^2}{\partial \mathbf{r}^2} + D_\rho\frac{\partial^2}{\partial \rho^2}\rho^2\right)P(t,\mathbf{r};\rho),$$

where diffusion coefficients are

$$D_0 = \frac{1}{2}\int_0^\infty d\tau\left\langle \mathbf{V}(\mathbf{r},t+\tau)\mathbf{V}(\mathbf{r},t)\right\rangle = \frac{1}{2}\tau_{\mathbf{V}}\left\langle \mathbf{V}^2(\mathbf{r},t)\right\rangle,$$
$$D_\rho = \int_0^\infty d\tau\left\langle \frac{\partial \mathbf{V}(\mathbf{r},t+\tau)}{\partial \mathbf{r}}\frac{\partial \mathbf{V}(\mathbf{r},t)}{\partial \mathbf{r}}\right\rangle = \tau_{\mathrm{div}\mathbf{V}}\left\langle\left(\frac{\partial \mathbf{V}(\mathbf{r},t)}{\partial \mathbf{r}}\right)^2\right\rangle, \quad (5)$$

and the quantities $\tau_{\mathbf{V}}$ and $\tau_{\mathrm{div}\mathbf{V}}$ are the times correlation of the fields $\mathbf{V}(\mathbf{r},t)$ and $\mathrm{div}\mathbf{V}(\mathbf{r},t)$.

The coefficient D_0 characterizes the spatial dispersion of the density field. The process of clustering is controlled by the coefficient D_ρ in the ρ space.

Hence, PDF of a uniform initial density $\rho_0(\mathbf{r}) = \rho_0$ – const is independent of $\mathbf{r}$, and has lognormal behavior in ρ. Its mean value is constant and higher moments grow exponentially

$$\left\langle \rho^n(\mathbf{r},t)\right\rangle = \rho_0^n e^{n(n-1)\tau} \quad \left(\tau = D_\rho t\right).$$

In the ideology of statistical topography some important geometric characteristics of the field $\rho(\mathbf{r},t)$, related to iso-contours $\rho(\mathbf{r},t) = \rho - \mathrm{const}$, and the corresponding statistical means expressed through its PDF.

Examples include the mean area enclosed by contours: $\rho(\mathbf{r},t) > \rho$, and the mean enclosed mass. At large time $\tau = D_\rho t >> 1$, the total area of high-density concentration (above ρ) decreases in time according to the law

$$< S(t,\rho) >= \frac{1}{\sqrt{\pi\tau\rho}} \exp\left(-\tau/4\right) \int \sqrt{\rho_0(\mathbf{R})} d\mathbf{R},$$

whereas the enclosed mass within the ρ-area

$$< M(t,\rho) >= M_0 - \sqrt{\frac{\rho}{\pi\tau}} \exp\left(-\tau/4\right) \int \sqrt{\rho_0(\mathbf{R})} d\mathbf{R}$$

converges monotonically to the total mass of the system M_0. These results confirm conclusion regarding clustering of concentration field and the parameter $t_0 \sim 1/D_\rho$ is the time scale for cluster formation.

Thus, the problem is reduced to the evaluation of diffusion coefficients (5) using the stochastic equations (2, 3) that is, to computing temporal correlation radii $\tau_{\mathbf{V}}$ and $\tau_{\mathrm{div}\mathbf{V}}$ of random fields $\mathbf{V}(\mathbf{r},t)$ and $\mathrm{div}\mathbf{V}(\mathbf{r},t)$, their spatial correlation scales, and variances (Klyatskin & Elperin (2002), Klyatskin (2005)).

We assume that the random velocity field variance $\sigma_{\mathbf{u}}^2 = \left\langle \mathbf{u}^2(\mathbf{r},t) \right\rangle$ in a hydrodynamic flow is sufficiently small and determines the main small parameter of the problem. In what follows, we shall calculate the statistical characteristics of the field $\mathbf{V}(\mathbf{r},t)$ in the first nonvanishing order of smallness in the parameter $\sigma_{\mathbf{u}}^2 = \left\langle \mathbf{u}^2(\mathbf{r},t) \right\rangle$. It is noteworthy that statistics of the field $\mathbf{V}(\mathbf{r},t)$ described by stochastic equations (2) and (3) is not Gaussian in the general case. However, it is easy to see that the highest field cumulants div $\mathbf{V}(\mathbf{r},t)$ are of a higher order of smallness than the second cumulant. It means that the approximation of the Gaussian field $\mathbf{V}(\mathbf{r},t)$ can really be used to derive equation (2).

For large parameters $\{\lambda, \omega\}$, it is possible to linearize equation (3) with respect to the function $\mathbf{u}(\mathbf{r},t)$ and pass to a simpler vector equation

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t) + \left(\mathbf{U}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t) + \left(\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{U}(\mathbf{r},t) = -\Lambda\left[\mathbf{V}(\mathbf{r},t) - \mathbf{U}(\mathbf{r},t)\right].$$

If parameters $\{\lambda, \omega\} \gg \sigma_{\mathbf{u}}^2 \tau_0 / l_{\mathrm{cor}}^2$, where l_{cor} is the spatial correlation scale and τ_0 is the time correlation scale of the field $\mathbf{V}(\mathbf{r},t)$, then we may discard the advective terms and pass to the simple linear equation

$$\frac{\partial}{\partial t} V_i(\mathbf{r},t) + \Lambda_{i\mu} V_\mu(\mathbf{r},t) = \lambda U_\mu(\mathbf{r},t).$$

The approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r},t)$ is incorrect in the case of a low-inertia tracer. It is therefore necessary to do calculations using an arbitrary value of the parameter $\lambda\tau_0$. This can be made in the diffusion approximation.

The resulting expression for $D_\rho(\mathbf{v})$ for $\{\lambda,\Omega\}\tau_0 \gg 1$ is

$$D_\rho = \frac{4\lambda^3\Omega^2 D}{\left(\lambda^2 + 4\Omega^2\right)} = \begin{cases} 4\Omega^2 D/\lambda^3, & \text{if } \lambda \gg \Omega, \\ \lambda^3 D/16\Omega^4, & \text{if } \lambda \ll \Omega, \end{cases} \qquad (6)$$

where τ_0 is time correlation of the hydrodynamic field $\mathbf{u}(\mathbf{r},t)$ and parameter D associated with the vortex component of the field $\mathbf{u}(\mathbf{r},t)$

$$D = -\left\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\right\rangle .$$

Thus, in the framework of the problem under consideration, the generation of the divergent component of field $\mathbf{V}(\mathbf{r},t)$ is described under the conditions $\{\lambda,\Omega\}\tau_0 \gg 1$ in terms of the linear equation without allowance for advective terms. If $\lambda \gg \Omega$ in addition to the above conditions, then one should take into account consequent corrections whose order of magnitude is $\sigma_{\mathbf{u}}^4 = \left\langle \mathbf{u}^2(\mathbf{r},t)\right\rangle^2$ and which can appear sometimes comparable with (6); in this case, we obtain the expression

$$D_\rho = -\frac{4\Omega^2}{\lambda^3}\left\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\right\rangle + \frac{3\tau_0}{2\lambda^2}\left\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\right\rangle^2$$
$$= -\frac{4\Omega^2}{\lambda^3}\left\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\right\rangle \left\{1 - \frac{3\lambda\tau_0}{2\Omega^2}\left\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\right\rangle\right\}.$$

4 Conclusions

We have derived expressions for the diffusion coefficients that characterized the clustering of the low-inertia density field in random hydrodynamic flows. For this problem it is difficult to specify the possible order of smallness for the corresponding diffusion coefficients, because there are three time scales even in the formulation of the problem alone. In addition, two statistical scales emerge, the diffusion coefficients, which also have the dimensions of inverse time. Therefore, we have to perform a detailed analysis of the problem, which was done here.

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