Phase-mixing and toroidal cascade in rotating and stratified flows

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Abstract :

Anisotropic statistics are useful in both theory and numerical simulations to explain the morphology and dynamics of rotating stratified flows. The Craya-Herring decomposition allows the incorporation of a toroidal/poloidal decomposition in a tractable geometrical way, and points out two limits: two-dimensional (2D) modes, with horizontal wavevectors, and one-dimensional (1D) "vertically sheared horizontal flow" modes (VSHF, with only vertical wavevectors). In the case of rotation without stratification, both toroidal and poloidal components are affected by inertial waves, thus anisotropic phase mixing is responsible for transient effects, whereas resonant triads control the long-term dynamics. We reproduce the development of triple vorticity correlations by pure linear theory, which is contained in the EDQNM theory, and compare to results from experiments and Direct Numerical Simulations (DNS). A new explanation for the dominance of cyclonic vertical vorticity is given, which avoids arguments based on instabilities (centrifugal or elliptic). For stably stratified flows without rotation, strong nonlinearity only affects the pure toroidal mode, whereas any interaction involving the poloidal mode deals with weak gravity-wave turbulence. We then explore the toroidal cascade. In spite of a strong analogy with a two-dimensional cascade for energy and vertical enstrophy, we show its consistency with a spherical shell-to-shell direct cascade. It can explain the angular energy drain from quasi-2D modes to quasi-1D VSHF modes, which quantifies the horizontal layering of the flow, again without use of 'tall-column' or 'zig-zag' instabilities. This study illustrates the relevance of a refined statistical description, in DNS or theory, of strongly anisotropic homogeneous turbulence, with the inclusion of spectral distribution in terms of both horizontal and vertical wavenumber components.

1 Introduction

The dynamics of rotating stratified flows illustrates the coexistence of weak wave-turbulence and strong turbulence. Homogeneous axisymmetric turbulence is governed by Navier-Stokes equations for the velocity $\boldsymbol{u}$ and buoyancy b fluctuations, within the Boussinesq approximation, in a rotating frame. External parameters are the Brunt-Väisälä frequency N, which expresses the strength of the stabilizing vertical density or temperature gradient, and the Coriolis parameter f. For unbounded flows, an eigenmode decomposition can be used in three-dimensional (3D) Fourier space for identifying the inertio-gravity wave régime Bartello (1995); Cambon (2001); Kaneda (2000). The eigenmodes are also useful for expanding the fully nonlinear equations and analyse them. In the general case, the three eigenmodes include a non-propagating Quasi-Geostrophic $\boldsymbol{N}^{(0)}$ mode, and two A-Geostrophic $\boldsymbol{N}^{(\pm1)}$ wave-like modes. Their definition depends on the type of flow via the ratio f/N, but they can be derived by linear combination of the three unit vectors of the Craya-Herring frame of reference: $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ generate a toroidal/poloidal decomposition for the solenoidal velocity field in physical space; $\boldsymbol{e}^{(3)}$ is aligned with the wavevector $\boldsymbol{k}$ and bears the buoyancy component scaled to be such that its square is the potential energy of gravity waves. Only the two limiting cases of pure rotating ($N = 0$) and pure stratified ($f = 0$) flows are investigated here. For pure rotation, $\boldsymbol{N}^{(0)}$ vanishes, so that the helical modes $\boldsymbol{N}^{(\pm1)} = \boldsymbol{e}^{(2)} + \pm \mathrm{i}\boldsymbol{e}^{(1)}$ Cambon and Jacquin (1989); Waleffe (1992) form a complete basis, useful also without system rotation. Accordingly, turbulence under rapid rotation amounts to inertial wave turbulence at very small Rossby number. For stratification without rotation $f = 0$, the non-propagating mode QG coincides with the toroidal mode for the velocity (*i.e.* $\boldsymbol{N}^{(0)} = \boldsymbol{e}^{(1)}$), whereas the gravity wave mode's kinematic part is the poloidal mode only (so that $\boldsymbol{N}^{(\pm 1)} = \boldsymbol{e}^{(2)} \pm \mathrm{i}\boldsymbol{e}^{(3)}$). In this case, the wave-turbulence regime emerging when lowering the Froude number, consists of nonlinear interactions including at least one poloidal component. As the remaining, pure toroidal, interactions generate strong turbulence, significant nonlinearity only involves these interactions at small Froude number.

Why is it important to distinguish the nonlinear cascade into 'weak' and 'strong' turbulence? We needs to introduce phase mixing: dispersive waves yield a damping of many statistical correlations, by summing up contributions with different oscillation frequencies. Looking at the nonlinear energy transfer mediated by triple correlations, wave-dynamics results in a severe depletion of energy transfer, except for resonant triads (or resonant quartets if the dispersion law does not permit them). These resonant triads span a low dimension manifold and affect only long-term evolution. In strongly stratified fluid, the relevant nonlinearity only involves the toroidal part of the flow at moderate times, nonlinear wave-turbulence dynamics remaining a secondary, delayed, mechanism. Wave-turbulence theory, from linear to weakly nonlinear, is essential in rotating turbulence, and remains useful in various aspects of stratified turbulence (e.g. Lagrangian diffusion), but the horizontal layering can be explained by 'strong' toroidal cascade, excluding wave effects.

2 A new insight to the dominance of cyclonic vertical vorticity in rapidly rotating flows

The statistical impact of the linear operator related to inertial wave can be understood from the basic equation for the fluctuating velocity

$$u_i(\boldsymbol{x}, t) = \sum e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \underbrace{\Re(N_i^{(1)} N_j^{(1)*} e^{\mathrm{i}\sigma_k t}) \mathcal{U}_j(\boldsymbol{k}, \epsilon t)}_{\hat{u}_i}, \quad \sigma_k = f\frac{k_\parallel}{k}, \tag{1}$$

which displays the relevant helical mode $\boldsymbol{N}^{(1)}$ and the dispersion frequency σ_k, under a Green's function in Fourier space. The term $\mathcal{U}_j$ stands for the initial condition, such that $\mathcal{U}_j = \hat{u}_j^0$ in the linear limit, and can incorporate nonlinear dynamics in the general case, with a slow/rapid time-scale decomposition suggested by the small parameter ϵ. Anisotropic phase mixing induced by $e^{\pm \mathrm{i}\sigma_k t}$ terms, and the related breaking of initial isotropy, reflects structure formation, even in pure homogeneous turbulence, but this effect depends on the order and on the degree of complexity of statistical moments. For single-time second-order statistics, isotropy is essentially conserved in the linear limit, since time dependency can cancel out by multiplying $e^{\mathrm{i}\sigma_k t}$ by its complex conjugate. Useful dynamical properties, however, are recovered for two-time second-order statistics, with interesting applications to Lagrangian diffusivity by Kaneda (2000) and by Cambon *et al.* (2004). Finally, isotropy-breaking in third-order single-time correlations is essential, with application to nonlinear transfer terms mediated by triads in Cambon and Jacquin (1989), and to triple vorticity correlations by Gence and Frick (2004).

The latter application is now investigated : the asymmetry of cyclonic and anticyclonic vertical vorticity is found in several physical or numerical experiments, as well as in observations in nature. For instance, Bartello *et al.* (1994) plotted the skewness of vertical vorticity to quantify this effect, showing the rise of a very significant positive value if the Rossby number is not too small. Explanations based on instability theory (*e.g.* centrifugal instability) are not applicable to real turbulence with a 'democratic crowd' of vortices, and the relevance of under-resolved DNS with hyperviscosity (*i.e.* LES implicitely) for calculating vorticity is questionable from a quantitative viewpoint. A recent experimental study by Morize *et al.* (2005) of decaying rotating turbulence showed the relevance of the linear time-scale to collect different cases with the same scaling: the vorticity skewness grows as t^α with $\alpha \in [0.7; 0.75]$ on fig.1(a). On this figure, the final rapid collapse is attributed to the rise of non homogeneous mechanisms, such as Ekmann pumping. Preliminary DNS by van Bokhoven *et al.* (2007), plotted in terms of

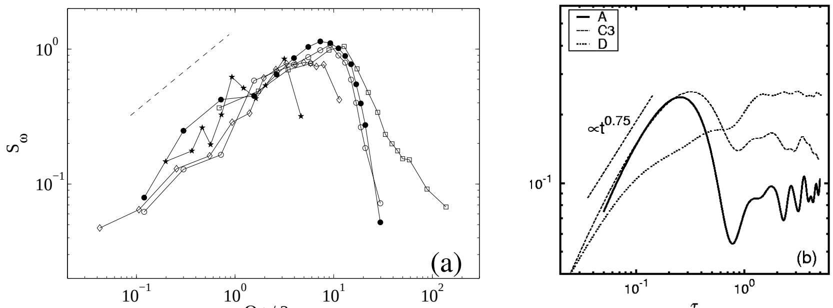


Figure 1: Vertical vorticity skewness: (a) experiment by Morize *et al.* (2005); (b) DNS by van Bokhoven *et al.* (2007)

the linear time-scale on fig.1(b), look similar, but the final collapse can be interpreted as a final stage of linear 'triadic' phase-mixing, in the absence of strong enough nonlinearity in decaying turbulence at moderate initial Reynolds number.

Classically, linear theory (a.k.a. Rapid Distorsion Theory, RDT) is used to predict the evolution of second order statistics, but it can predict correlations of any order. In the following, we will address triple correlations. The formulation in Fourier space allows an accurate description of phase-mixing, either using velocity or vorticity. Triple vorticity correlations in Fourier space are related to a third order spectral tensor as $\langle \hat{\omega}_i(\boldsymbol{q},t)\hat{\omega}_j(\boldsymbol{k},t)\hat{\omega}_m(\boldsymbol{p},t)\rangle = \Phi_{ijm}(\boldsymbol{k},\boldsymbol{p},t)\delta^3(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$, whose 'exact' dynamics comes from eq. (1). The corresponding single-point correlation, is recovered by the integral $\langle \omega_i\omega_j\omega_m\rangle(t) = \int_{\mathbb{R}^6} \Phi_{ijm}(\boldsymbol{k},\boldsymbol{p},t)d^3\boldsymbol{k}d^3\boldsymbol{p}$, so that for instance

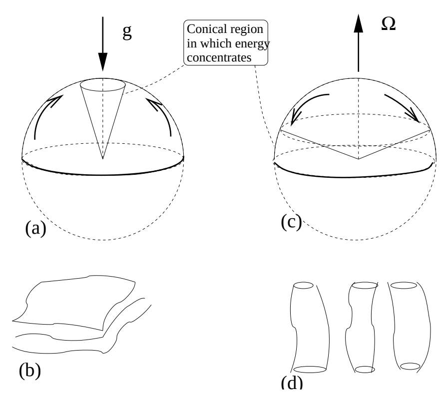
$$\langle \omega_\parallel^3\rangle(t) = \sum_{s,s',s''=\pm 1} \int_{\mathbb{R}^6} \exp\left[\mathrm{i}ft\left(s\frac{k_\parallel}{k} + s'\frac{p_\parallel}{p} + s''\frac{q_\parallel}{q}\right)\right] T_{ss's''}(\boldsymbol{k},\boldsymbol{p},\epsilon t)\, d^3\boldsymbol{k}\, d^3\boldsymbol{p}. \qquad (2)$$

In order to compute this integral, it is necessary to know the contribution from initial, or slowly evolving, triple correlations for all triads[1], but the problem is much better documented than in physical space: robust spectral theories such as EDQNM provide a systematic way to express initial, isotropic, Φ_{ijm} in terms of the initial scalar energy spectrum $E(k)$. More generally, more complex EDQNM$_{2,3}$ versions can be used to solve the full nonlinear problem, not only for generating isotropic initial data in (2) with $\epsilon = 0$. Common to the linear and nonlinear formulations, the phase term controlling phase-mixing appears in eq.(2), and is zero when triads are in exact resonance.

3 Toroidal cascade in strongly stratified flows

Another instance of strongly anisotropic turbulence is stably stratified turbulence, important in the atmosphere and the ocean, in which stable stratification limits vertical motions and renders the flow mainly horizontal. The problem of the sense of the cascade in such flows is still controversial, even if a global consensus is emerging against the idea of a classical 2D inverse cascade. The analogy between quasi-geostrophic and 2D dynamics, with conservation of potential vorticity, was investigated by Charney (1971). This analogy was revisited by Bartello (1995) with a refined analysis using the eigenmode decomposition. Regarding applications, Lilly (1983) proposed that the kinetic energy spectra observed in the atmosphere at mesoscales are a manifestation of this two-dimensional mechanism. Recently, Lindborg (1999) deduced from analysis of third-order statistical moments that the energy cascade is in the direct sense.

[1] Or equivalently, in physical space, for any triple correlation at three points, information which cannot be provided by the third-order structure function alone.



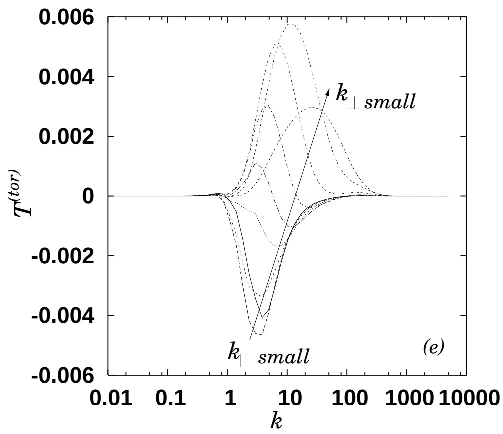


Figure 2: Sketch of anisotropic structure induced by nonlinearity: stratified (left), rotating (middle), spectral (top a-c) and physical (bottom b-d) space, from Godeferd and Cambon (1994). Right: recent anisotropic EDQNM$_2$ result for toroidal energy transfer of stably stratified freely decaying turbulence, at Taylor-microscale based Reynolds number $R_e^\lambda = 145$.

This observational evidence was further supported by a dimensional analysis related to the zig-zag instability Billant and Chomaz (2001), showing that the vertical scale is necessary limited by a local buoyancy length scale $L_B = U/N$, where U is the horizontal velocity scale. Several DNS or LES with hyperviscosity were carried out by Lindborg and coworkers to investigate such a forward cascade. In these computations, only the two-dimensional and two-component modes (2D-2C) are randomly forced, and the horizontal lengthscales are a priori chosen much larger than the vertical ones, using vertically flattened boxes. Even if these studies present interest for atmospheric flows, their contribution to a better conceptual understanding of turbulence is limited by both geometric constraints and artificial forcing.

We propose here to reinterpret the nature of the cascade in strongly stratified flows looking at the basic nonlinear mechanism, following the analyses by Cambon and Jacquin (1989), Waleffe (1992), Godeferd and Cambon (1994), Bartello (1995) and Cambon (2001). Even if the zig-zag instability is an efficient mechanism to break the vertical coherence of the flow and to illustrate the horizontal layering, it is only a modality of a much more general nonlinear mechanism. Layering can be obtained without need of preexisting large coherent 2D vortical structures, and without randomly forcing such structures. In addition, the strongly anisotropic description with angle-dependent spectra allows us to quantify the vertical layering in connection with a toroidal energy cascade, not only in DNS by Liechtenstein *et al.* (2005)—with isotropic initial data, no forcing and no hyperviscosity— but also in statistical theory. For instance, the EDQNM$_2$ model by Godeferd and Cambon (1994) firstly suggested the cartoon in fig.2(a–d), confirmed by our recent studies, and showed that the cascade is essentially direct in terms of interacting spherical shells.

In rotating turbulence, nonlinear interactions drain spectral energy from any wavevector direction towards the horizontal waveplane normal to the system angular velocity, a plane that corresponds to the 2D manifold (fig.2c,d). Conversely, in stably stratified turbulence, an inverse specific energy drain concentrates spectral energy towards vertical wavevectors (2a,b). The limit of vertical wavevectors corresponds to the Vertically Sheared Horizontal Flow mode (term coined by Smith and Waleffe (2002)) and has nothing to do with the 2D mode. On the contrary, it characterizes horizontal layering with only vertical variability in physical space. Accordingly, this nonlinear mechanism illustrates an actually *anti-2D* nonlinear trend, when looking at *angle-to-angle* interactions in wave-space. Of course, this mechanism is not inconsistent with a direct energy cascade looking at *shell-to-shell* interactions. The fact that a cascade can be seen as inverse in terms of $k_\perp$ components, and as direct in terms of $k = |\boldsymbol{k}|$, demonstrates the importance of a detailed description of strongly anisotropic energy and transfer spectra (axisymmetric in our context), in terms of both vertical and horizontal wavevector components.

Omitting viscosity, the toroidal velocity component is governed by the following exact equations

$$\dot{u}^{(1)} + \boldsymbol{e}^{(1)} \cdot \widehat{\boldsymbol{\omega} \times \boldsymbol{u}} = 0, \quad \text{with } \hat{\boldsymbol{u}} = u^{(1)}\boldsymbol{e}^{(1)} + u^{(2)}\boldsymbol{e}^{(2)}, \quad \text{and } \hat{\boldsymbol{\omega}} = \mathrm{i}k\left(u^{(1)}\boldsymbol{e}^{(2)} - u^{(2)}\boldsymbol{e}^{(1)}\right) \quad (3)$$

where $\dot{}$ is a time derivative. Attention is then restricted to a single triad, getting rid of nonlinear contributions such as $u^{(2)}u^{(1)}$ and $u^{(2)}u^{(2)}$, assuming 'weak' gravity wave turbulence. The system of equations

$$\dot{u}_k^{(1)} = (p_\perp^2 - q_\perp^2) G u_p^{(1)*} u_q^{(1)*}, \qquad \dot{u}_p^{(1)} = (q_\perp^2 - k_\perp^2) G u_q^{(1)*} u_k^{(1)*}, \qquad \dot{u}_q^{(1)} = (k_\perp^2 - p_\perp^2) G u_k^{(1)*} u_p^{(1)*},$$

is almost the same as the one obtained by Waleffe (1992) in pure 2D-2C turbulence. It conserves both energy and vertical enstrophy, and suggests than only 'reverse' (R) types of triadic interactions are involved. The latter result derives from Waleffe's instability principle, using the analogy of the former system of equations with Euler's problem for the angular momentum of a solid. Nevertheless, the fact that the completely symmetric factor G depends on both $k_\perp$ and $k_\parallel$ allows a different dynamics over more manifolds than the conventional 'dual' 2D-2C turbulent cascade, inverse for energy, direct for enstrophy.

For comparing the relative amounts of direct and inverse cascades, since the R-type triads allow both senses, exact statistical Lin-type equations are used:

$$\left(\partial_t + 2\nu k^2\right) U^{(tor)} = T^{(tor)} \quad (4)$$

$$\left(\partial_t + 2\nu k^2\right) U^{(w)} = T^{(w)} \quad (5)$$

$$\left(\partial_t + 2\nu k^2 + 2\mathrm{i}N k_\perp / k\right) Z = T^{(z)} \quad (6)$$

in which $U^{(tor)}$ is the spectrum corresponding to $u^{(1)}u^{(1)*}/2$. Naming $U^{(pol)}$ and $U^{(pot)}$ the spectra for $u^{(2)}u^{(2)*}/2$ and $u^{(3)}u^{(3)*}/2$ respectively, $U^{(w)} = U^{(pol)} + U^{(pot)}$ is the total energy of gravity waves, and Z quantifies the unbalance between kinetic (poloidal) and potential (buoyancy) parts of the total wave energy. The real part of Z is $(1/2)(U^{(pol)} - U^{(pot)})$, and its imaginary part contains the poloidal-buoyancy flux (details in Godeferd and Cambon (1994)). The closure of the transfer terms (r.h.s. of eq.4–6) is found in terms of the above mentioned spectra, depending on both $k_\perp$ and $k_\parallel$, or equivalently on k and $\cos\theta = k_\parallel / k$, for the simplest statistical symmetry consistent with the dynamical basic equations which is axisymmetry with mirror symmetry. For instance, $T^{(tor)}$ is an integral of purely toroidal triple correlations $\langle u^{(1)}(\boldsymbol{k}, t) u^{(1)}(\boldsymbol{p}, t) u^{(1)}(\boldsymbol{q}, t)\rangle$, and is modelled by terms such as $\theta_{kpq} U^{(tor)}(\boldsymbol{q}, t)(a(\boldsymbol{k}, \boldsymbol{p}) U^{(tor)}(\boldsymbol{p}, t) - b(\boldsymbol{k}, \boldsymbol{p}) U^{(tor)}(\boldsymbol{k}, t))$ once closed by the anisotropic EDQNM$_2$ procedure. Detailed equations and DNS/EDQNM$_2$ comparisons, including the angle-dependent spectra, are given in Godeferd and Staquet (2003). Here, we simplify the EDQNM$_2$ procedure in order to focus on pure toroidal interactions and to reach ranges of very high Reynolds numbers R_e, very low Froude numbers F_r, and long elapsed times, out of grasp of current DNS.

The simplest run is started with zero poloidal and zero potential energy. In this oversimplified configuration, the flow is purely horizontal but not 2D. Only equation (4) in the full system remains, and $U^{(tor)}$ is initially distributed as in isotropic turbulence, *i.e.* with no angular dependence. Fig.2(e) shows $T^{(tor)}$ as a function of k, parameterized by the angle of the wavevector $\boldsymbol{k}$ to the vertical. An almost uniquely angular transfer is observed, with negative transfer for close-to-horizontal $\boldsymbol{k}$'s (small $k_\parallel$ components), positive for close-to-vertical ones (small $k_\perp$ components).

4 Conclusions and perspectives

The asymmetry in terms of cyclonic and anticyclonic vertical vorticity, induced by rapid rotation, is investigated using strongly anisotropic statistical theory, in addition to previous LES and DNS studies. Phase-mixing is identified in both linear and nonlinear theory as the dominant effect for breaking 3D isotropy and for constructing an increasing positive vertical vorticity skewness. In strongly stratified turbulence, wave phase-mixing mainly affects the poloidal component of the velocity field, so that strong nonlinearity remains relevant for the toroidal component only. The toroidal cascade, investigated separately, presents strong analogies with the dual cascade of purely 2D flows, but is shown to be much more complex and multifold: an angular energy drain towards vertical wavevectors explains and quantifies the horizontal layering, whereas the conventional cascade between spherical shells is essentially direct.

Restricting the dynamical equations to the toroidal mode (eq. 4) yields an informative toy-model, but this cannot exhaust the complexity of the full problem. Once the energy is concentrated in a neighborhood of quasi-vertical wavevectors, it is no longer possible to discard the $\boldsymbol{e}^{(2)}$–related velocity mode, since both $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ modes must contain the same amount of energy in the limit of purely vertical wavevectors, forming an axisymmetric VSHF mode. Similarly, the limit of vanishing Froude number is perhaps not realistic, as shown by Godeferd and Staquet (2003) with simple interpretation by Billant and Chomaz (2001) of vertical Froude number close to unity. In the future, the complete system of dynamical modal equations will be investigated for additional information on the toroidal/poloidal/potential modes interplay.

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