

Internal Solitary Waves and Fronts in a Weakly Stratified Fluid

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Abstract :

Large amplitude internal solitary waves in a continuously stratified fluid are discussed analytically. Long-wave scaling procedure is used in order to construct approximate plateau-shape solitary waves possessing table-top crests and narrow fronts. Existence of exact solutions is proved near the Boussinesq limit by means of generalized Lyapunov-Schmidt method.

Key-words :

stratification; solitary waves

1 Introduction

Global theory of interfacial solitary waves in a two-layer fluid has been developed by Amick & Turner (1986). Their analytical results predict, in particular, the broadening of solitary waves when the wave speed attains the propagation speed of internal bore. Smooth bore is a steady front-like wave connecting two different conjugate flows at opposite ends of fluid layer. Consequently, broad solitary wave possess almost parallel flow at the middle, and this flow should be conjugated with uniform up- and downstream. It is the case that symmetric solitary wave can be superposed by two internal fronts of opposite polarity.

The broadening effect was confirmed experimentally, numerically and analytically by several authors. Turner & Vanden-Broeck (1988) calculated interfacial plateau-shape waves in the framework of integro-differential formulation for 2D Euler equations. Dias & Vanden-Broeck (2003) computed directly by similar method interfacial fronts of large amplitude, including the overhanging front. Lamb & Wan (1998) found also numerically flat-crested solitary waves in a continuously stratified fluid. Grue et al (2000) observed breaking and broadening of solitary waves in laboratory experiments. Existence of small solitary waves and fronts was proved rigorously via the center manifold technique by Amick & Turner (1989) and by Mielke (1995) for a two-layer fluid, and by James (1997, 2001) for a continuously stratified fluid. Makarenko (1992, 2003b) involved the Lyapunov-Schmidt bifurcation method to prove the existence of fronts.

In the present paper, we extend the Lyapunov-Schmidt procedure to the problem on large amplitude solitary waves in shallow fluid having almost linear density stratification. We seek the solution near the Boussinesq scaling limit. This perturbation scheme was suggested first by Benney & Ko (1978) especially to treat finite amplitude internal waves in a slightly stratified fluid. Derzho & Grimshaw (1997) combined this method with matched expansion technique in order to construct solitary waves with a recirculation core. Maltseva (2003) discussed an alternative limit configuration which includes two-level plateau-shaped solitary waves. In the Section 2, we adjust the Boussinesq approximation to obtain broad solitary waves at leading order. Section 3 sketches the derivation of nonlinear operator equation which provides existence of exact solution by the fixed point principle.

2 The Boussinesq limit for internal solitary waves

2.1 Basic equations

We study the propagation of solitary waves in an inviscid inhomogeneous fluid being stratified under gravity. Coordinates Oxy are used with horizontal x -axis and vertical y -axis, so the flow becomes steady in the frame reference moving with the wave. It is known that steady 2D Euler equations reduce to the Dureuil-Jacotin–Long (DJL) equation for a stream function ψ ,

$$\rho(\psi) \Delta\psi + \rho'(\psi) \left(gy + \frac{1}{2} |\nabla\psi|^2 \right) = \rho'(\psi) \left(\frac{g\psi}{c} + \frac{1}{2} c^2 \right).$$

Here ρ is the fluid density, g is gravity acceleration and c is the wave speed. We consider fluid layer confined between flat bottom $y = 0$ and rigid lid $y = h$. In this case, boundary conditions are $\psi = 0$ ($y = 0$) and $\psi = ch$ ($y = h$). We assume that the flow tends to uniform current having known density profile far upstream, $\psi \rightarrow \psi_\infty = cy$ and $\rho \rightarrow \rho_\infty(y)$ as $x \rightarrow -\infty$. It is supposed no recirculate zones everywhere in the flow region $-\infty < x < +\infty$. Therefore we have $\rho(\psi) = \rho_\infty(\psi/c)$ due to the density conservation along streamlines. Basic dimensionless constants are the Boussinesq parameter σ and inverse densimetric Froude number λ ,

$$\sigma = \frac{N_0^2 h}{\pi g}, \quad \lambda = \frac{\sigma g h}{\pi c^2},$$

where N_0 is the Brunt–Väisälä frequency. We suppose that upstream density $\rho = \rho_\infty(y)/\rho_\infty(0)$ depends on dimensionless variable $\bar{y} = \pi y/h$ as follows:

$$\rho(\bar{y}, \sigma) = 1 - \sigma\bar{y} - \sigma^2 \rho_0(\bar{y}, \sigma). \quad (1)$$

In such a way, we take into account small perturbations by the terms of the order $O(\sigma^2)$ for background linear density $\rho = 1 - \sigma\bar{y}$, as well as for exponential stratification $\rho = \exp(-\sigma\bar{y})$. Anyway, we require the function ρ_0 to be sufficiently smooth.

Condition A. *The function $\rho_0(\bar{y}, \sigma) \in C^l([0, \pi] \times [0, \sigma_0])$, $l \geq 4$, is such that the inequalities $\rho > 0$, $\rho_{\bar{y}} < 0$ are valid by $(\bar{y}, \sigma) \in [0, \pi] \times (0, \sigma_0)$ with some $\sigma_0 > 0$.*

Looking for the solution $\psi(x, y) = (ch/\pi) [\bar{y} + v(\bar{x}, \bar{y})]$ depending on scaled variables \bar{y} and $\bar{x} = \sqrt{\sigma}\pi x/h$, we formulate dimensionless equations with bar omitted in notations. Unknown function $v(x, y)$ is supposed to be even in x due to the symmetry of solitary wave. This function should satisfy nonlinear eigenvalue problem in the strip $\Omega = R \times (0, \pi)$ with spectral parameter $\lambda > 0$,

$$\sigma(\rho v_x)_x + (\rho v_y)_y = \lambda \rho' v + \frac{1}{2} \sigma \rho' (\sigma v_x^2 + v_y^2), \quad (x, y) \in \Omega, \quad (2)$$

$$(a) v(x, 0) = v(x, \pi) = 0; \quad (b) v \rightarrow 0 \quad (x \rightarrow \pm\infty). \quad (3)$$

Here is denoted $\rho = 1 - \sigma\psi - \sigma^2 \rho_0(\psi, \sigma)$ and $\rho' = \sigma^{-1} \rho_\psi(\psi, \sigma)$ with $\psi = y + v$.

2.2 Conjugate flows and plateau-shape solitary waves

We check the broadening limits for solitary waves by fixing horizontal middle flow at $x = 0$ to be close to 1D shear flow conjugated with uniform upstream. Solutions $v(y) \equiv 0$ and $v_{con}(y) \neq 0$ of equations (2)–(3a) form conjugate pair if both these flows possess equal mass, momentum and energy fluxes. The compatibility of total mass and total energy follows immediately from the density conservation under the Bernoulli theorem. In contrast, momentum

equation implies an additional matching relation between conjugate flows. Thus we obtain nonlinear one-dimensional eigenvalue problem for $v = v_{con}(y)$

$$(\rho v_y)_y - \rho' \left(\lambda v + \frac{1}{2} \sigma v_y^2 \right) = 0, \quad v(0) = v(\pi) = 0, \quad (4)$$

and total momentum conserves due to the flow force integral

$$\int_0^\pi L(v; \sigma, \lambda) dy = 0 \quad (5)$$

where L is the Lagrangian of the DJL operator.

In accordance with Makarenko (1999, 2003a), conjugate flows of n -th mode bifurcate from simple eigenvalues $\lambda_n = n^2$ ($n = 1, 2, \dots$) of the problem (4) linearized within the limit $\sigma = 0$. The solution of Eqs. (4)-(5) has the form $v_{con}(y) = b \sin ny + w(y)$ with $(w, \sin y)_{L_2[0, \pi]} = 0$ for λ being close to λ_n . Amplitude bound $|b| < 1/n$ provides absence of a return flow, and bifurcation values of b are generated by the extreme points of the function $\Lambda_n(b) = -2 s_n(b)/b^2$ where

$$s_n(b) = \frac{2n^2}{\pi} \int_0^\pi \int_y^{\pi - y + b \sin ny} \left(\rho_0(y + b \sin ny) - \rho_0(\psi) \right) d\psi dy + \frac{\pi}{4} (nb)^2 + \frac{n}{3\pi} (1 - (-1)^n) b^3.$$

Theorem 1. *Let $b_0 \neq 0$ ($|b_0| < 1/n$) satisfies the conditions $\Lambda'_n(b_0) = 0$, $\Lambda''_n(b_0) \neq 0$. Then there exists unique branch of conjugate flows $(v_{con}, \lambda) \in C^l[0, \pi] \times \mathbb{R}$ such that $(v_{con}(y; \sigma), \lambda(\sigma)) \rightarrow (b_0 \sin ny, n^2)$ as $\sigma \rightarrow 0+$. The eigenvalues have the asymptotics $\lambda(\sigma) = n^2 + \Lambda_n(b_0) \sigma + O(\sigma^2)$.*

Let us note that the number of extreme amplitudes b_0 mentioned in the Theorem 1 depends on the fine-scale density coefficient ρ_0 only. The 1-mode conjugate flow is supercritical with respect to the phase speed of infinitesimal waves while the inequality $\Lambda_1(b_0) < \Lambda_1(0)$ is satisfied.

Further we consider similarly to Maltseva (2003) approximate solutions modelling first-mode plateau-shape solitary waves. We put $\lambda = 1 + \sigma \Lambda_1(b) + O(\sigma^2)$ in (2) with amplitude b bounded by $|b| < 1$. In accordance with the Theorem 1, the broadening values of b are given by local maxima of the wave speed c defined by the formula $c^2 = \sigma gh / \pi (1 + \sigma \Lambda_1(b))$ up to the order $O(\sigma^3)$. Looking for the power expansion $v = v_0 + \sigma v_1 + \dots$ we obtain from (2)–(3a) the set of equations

$$v_{jyy} + v_j = f_j, \quad (x, y) \in \Omega; \quad v_j = 0 \quad (y = 0, y = \pi), \quad j = 0, 1, \dots$$

with $f_0 = 0$. The lowest-order solution has the form $v_0 = a_0(x) \sin y$ where unknown function a_0 should satisfy vanishing condition $(f_1, \sin y)_{L_2[0, \pi]} = 0$ of secular terms in the expression

$$f_1 = -v_{0xx} + (y + v_0) v_{0yy} - \rho'_0(y + v_0) v + v_{0y} + \frac{1}{2} v_{0y}^2 - \Lambda_1(b) v_0.$$

Thus we obtain the equation

$$a_0'' + p'(a_0) = 0 \quad (6)$$

where the function $p(a_0) = (1/2) a_0^2 (\Lambda_1(b) - \Lambda_1(a_0))$ is not polynomial in general case. Solitary wave solution exists for b such that $\Lambda'_1(b) \neq 0$ and $\Lambda_1(a) > \Lambda_1(b)$ as $|a| < |b|$. This solution a_0 has exponential decay $a_0(x) \sim C \exp(-\alpha_0 |x|)$ ($|x| \rightarrow \infty$) with $\alpha_0 > 0$ defined by the formula

$$\alpha_0^2 = \Lambda_1(0) - \Lambda_1(b). \quad (7)$$

The right-hand side of (7) is positive when the midsection conjugate flow is supercritical.

3 Existence of exact solutions of DJL equation near the Boussinesq limit

We begin with function spaces adapted to the solitary wave problem (2)–(3). Let $L_2^\alpha(R)$ be the Hilbert space of even functions $a(x)$ having finite norm

$$\|a\|_\alpha^2 = \int_{-\infty}^{+\infty} e^{2\alpha|x|} |a(x)|^2 dx,$$

where $0 < \alpha < \alpha_0$ is valid with the exponent α_0 defined by (7). For integer $k \geq 1$, we introduce weighted Sobolev space $H_\alpha^k(R)$ of the functions $a(x)$ which have generalized derivatives $D_x^m u$ up to the order $m \leq k$ belonging to the class $L_2^\alpha(R)$. Finally, we define the class

$$X^k = \{u(x, y) \mid D_x^m D_y^j u \in L_2([0, \pi]; H_\alpha^k(R)) \quad (0 \leq m + j \leq 2); \quad u(\cdot, 0) = u(\cdot, \pi) = 0\}$$

and the class $Y^k = L_2([0, \pi]; H_\alpha^k(R))$. Weighted norms in X^k and Y^k will be denoted as $\|\cdot\|_k$ and $|\cdot|_k$ consequently.

Further, we define the domain of the DJL operator in such a way that it includes solitary waves of large amplitude. The reason is that the flow to be determined should be close at $x = 0$ to the conjugate flow $\psi = y + b_0 \sin y + O(\sigma^2)$ where the amplitude b_0 is not small. Let $B_r = \{\|u\|_k < r\}$ be the open ball in the Banach space X^k ($k \geq 1$). Then for a given $r > 0$ there exists $\sigma_1 = \sigma_1(r)$ such that the function $\psi = y + a_0(x) \sin y + \sigma u(x, y)$ is bounded by $0 \leq \psi(x, y) \leq \pi$, $(x, y) \in \Omega$. This is fulfilled for arbitrary $u \in B_r$, $\sigma \in (0, \sigma_1]$ due to the Sobolev embedding theorem. As a consequence, the fluid density $\rho(\psi, \sigma)$ is well defined if σ is sufficiently small.

Thus we can rewrite the Eq. (2) with $v = a_0(x) \sin y + \sigma u(x, y)$ as

$$\sigma u_{xx} + u_{yy} + u = \varphi(u, \sigma), \quad (8)$$

where the operator φ collects nonlinear terms of the DJL operator. The structure of this nonlinearity is important for constructing operator equation in the class X^k . We use some properties of the Fréchet derivative $D_u \varphi$, as well as the estimates of remainder $\tilde{\varphi}$ in the Taylor expansion

$$\varphi(u, \sigma) = \varphi(0, \sigma) + D_u \varphi(0, \sigma)u + \tilde{\varphi}(u, \sigma).$$

Let introduce linear operator $M : X^k \rightarrow Y^k$ acting by the formula $Mu = -(\psi_0 u_y)_y + \theta u$ where the coefficients are $\psi_0 = y + a_0(x) \sin y$ and $\theta = \rho'_0(\psi_0) + (1 + \rho''_0(\psi_0))(\psi_0 - y) + \Lambda_1(b)$.

Lemma 1. For $u, v \in B_r \subset X^k$ ($k \leq l - 2$) and $\sigma \in (0, \sigma_1)$ the estimates

$$\begin{aligned} (i) \quad & |D_u \varphi(0, \sigma)u - \sigma Mu|_k \leq C \sigma^2 \|u\|_k, \\ (ii) \quad & |\tilde{\varphi}(u, \sigma) - \tilde{\varphi}(v, \sigma)|_k \leq C \sigma^2 \|u - v\|_k. \end{aligned}$$

are valid with the constant C independent on σ .

Further, we apply the modified Lyapunov–Schmidt procedure suggested by Beale (1991) for solitary water waves. Separation of variables x and y at leading-order solution term motivate us to introduce direct sums of closed subspaces

$$X^k = \mathcal{X}^k \oplus (I - Q)X^k, \quad Y^k = \mathcal{Y}^k \oplus (I - Q)Y^k$$

with appropriate projection Q onto infinite-dimensional subspaces

$$\mathcal{X}^k = \{u(x, y) = a(x) \sin y \mid a \in H_\alpha^{k+2}(R)\}, \quad \mathcal{Y}^k = \mathcal{X}^{k-2}.$$

Namely, we apply here the projection Q associated with orthogonal basis $\{\sin ny\}$ in $L_2[0, \pi]$. We now identify the solution $u(x, y) = a(x) \sin y + w(x, y)$ with a pair $(w, a) \in (I - Q)X^k \times H_\alpha^{k+2}(R)$. By this way, the operator equation (8) splits to the system for (w, a)

$$\begin{aligned} (a) \quad & \sigma w_{xx} + w_{yy} + w = f(w, a, \sigma), \\ (b) \quad & a'' + p''(a_0)a = q(w, a, \sigma). \end{aligned} \quad (9)$$

Here is denoted $f(w, a, \sigma) = (I - Q)\varphi(a \sin y + w, \sigma)$. The nonlinearity q arises from the projection of (8) onto subspace \mathcal{X}^k , this is the remainder beyond of linear part. In fact, the left-hand side of (9b) presents the equation (6) linearized with respect to solitary wave solution $a_0(x)$. Further we use resolvent estimates for elliptic operator appearing in (9a).

Lemma 2. For a given $f \in (I - Q)Y^k$ a priori estimates

$$\sigma^{m/2} |D_x^m D_y^j w|_k \leq C |f|_k \quad (0 \leq m + j \leq 2)$$

are valid with $0 < \sigma \leq \sigma_1$, where the constant C does not depends on σ .

In addition, we use also the estimates of inverse operator for linear ODE in (9b) with fixed q ,

$$Gq(x) = a_1(x) \int_0^x a_2(s)q(s) ds + a_2(x) \int_x^{+\infty} a_1(s)q(s) ds,$$

where $a_1(x) = a_0'(x)$ and $a_2(x) = a_1(x) \int_{x_0}^x a_1^{-2}(s) ds$ with $x_0 \neq 0$.

Lemma 3. If the function $q \in H_\alpha^k(R)$ ($k \geq 0$) then $Gq \in H_\alpha^{k+2}(R)$ is valid with the estimate $\|Gq\|_{H_\alpha^{k+2}(R)} \leq C \|q\|_{H_\alpha^k(R)}$.

Estimates of Lemma 1-3 provide to apply the contraction fixed point principle to the system (9) with small parameter σ . Note that x -derivatives are estimated by Lemma 2 non-uniformly with respect to σ , however estimates of nonlinear terms formulated in Lemma 1 compensate this irregularity. The following proposition is the main result of the paper.

Theorem 2. Let the density profile (1) satisfies the Condition A. Let b_0 be the amplitude of supercritical 1-mode conjugate flow given by the Theorem 1. Then for $b \neq b_0$ being close to b_0 , there exists exact solitary wave solution of the problem (2)–(3) having the form

$$v(x, y; \sigma) = a_0(x) \sin y + \sigma u(x, y; \sigma)$$

where $u \in X^{l-3}$ is uniformly bounded as $\sigma \rightarrow 0$.

4 Conclusions

In this paper we have considered the problem on steady internal waves in a weakly stratified fluid. The condition providing existence of exact solitary wave solution of DJL equation was obtained in the case of linear background density. In accordance with this condition, the broadening amplitude limits are given by local maxima of the phase speed. The wave speed depending on the fine-scale density can be non-monotone with respect to the wave amplitude, so the broadening limit can be *not unique*. We used the method which permits to consider large amplitude waves being far away from equilibrium state. This is because the perturbation scheme involves small density slope instead of small wave amplitude. However, it is not clear at present how to extend this method to overhanging waves and the waves with recirculated zones.

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