Energy transfers during the interaction between two co-rotating quasi-geostrophic vortices

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Abstract :

We investigate the strong interactions between two co-rotating quasi-geostrophic vortices. We first determine equilibrium states based on an ellipsoidal model. We then address the linear stability of the vortices to ellipsoidal perturbations. An instability will trigger a strong interaction during the nonlinear evolution of the vortex pair. Then we investigate the nonlinear evolution of a large subset of the unstable equilibrium states. The dominant type of interaction is partial merger where only a part of a vortex merges with the other one. For vortices of significantly different initial volumes, the most commonly observed interaction is the a partial straining out, where the small vortex shed some of its volume as filaments. Analysing the energies of the vortices, we show that the net energy transfer is toward large spatial scales, whereas a large number of small spatial scales are produced. Intermediate scales tend to disappear from the flow.

Résumé :

Nous étudions les interactions fortes entre deux tourbillons co-rotatifs quasi-géostrophiques . Nous déterminons d'abord des états d'équilibre basés sur un modèle ellipsoïdal. Puis nous analysons la stabilité linéaire des tourbillons soumis à des perturbations ellipsoïdales. Une instabilité engendrera une interaction forte durant l'évolution non-linéaire da la paire de tourbillons. Nous étudions ensuite l'évolution non-linéaire d'un large sous-ensemble d'états d'équilibre instables. Le type d'interaction dominant est l'appariement partiel où seulement une partie d'un des tourbillons est absorbée par l'autre tourbillon. Par contre, pour des tourbillons avec un rapport de volume initial important, l'interaction la plus souvent observée est la destruction partielle du plus petit tourbillon. En analysant les energies des tourbillons, nous montrons qu'en moyenne l'energie est transférée vers les grandes échelles, bien que l'on observe une génération importante de petites structures. Les structures d'échelle spatiale intermédiaire tendent à disparaitre.

Key-words :

Vortex Interactions, Quasi-geostrophy

1 Introduction

Oceanic and atmospheric meso-scale flows are dynamically dominated by the slow evolution of, and the interaction between vortices – swirling masses of fluid which can be identified as coherent volumes of potential vorticity (PV). Such flows are strongly influenced by both the planetary rotation and the stable density stratification of the fluid.

The simplest dynamical system which takes into account both these two dominant effects is the three-dimensional quasi-geostrophic (QG) model. Until recently, little was known about how two QG-vortices would interact in the general case. Even for equal-PV vortices, the interaction depends on 5 parameters: the vortices' height-to-width aspect ratios, their volume ratio, their vertical offset, and their horizontal separation.

We briefly introduce the QG model in §2. We investigate in §3 the stability of pairs of equal-PV vortices in mutual equilibrium. The margin of stability indicates the onset of a strong interaction. We determine the margin of stability using a simplified approach where vortices are modelled as ellipsoidal volumes of uniform PV, at a minimum numerical cost. Any other approach would be impractical for a large parameter space. In §4, we investigate the nonlinear evolution of a pair of vortices settled just beyond their margin of stability, using the Contour-Advective Semi-Lagrangian Algorithm (CASL). We determine how the self-energy of the vortices is, on average, transferred to larger scales (in physical space). Conclusions are drawn in §5.

2 The quasi-geostrophic model

The inviscid QG model can be derived by an asymptotic expansion of Euler's equations with $H/L \ll 1$, where H and L are respectively the vertical and horizontal characteristic length scales, and $Fr^2 \ll Ro \ll 1$, where $Fr = U/NH$ is the Froude number, and $Ro = U/fL$ is the Rossby number. Here, U is the horizontal velocity scale, N is the buoyancy frequency and f is the Coriolis frequency, see e.g. Gill (1982) for a complete discussion. In this context, the fluid motion is fully determined by the slow evolution of a scalar field, the potential vorticity anomaly q.

In an adiabatic, dissipationless fluid, q is materially conserved i.e.

$$\frac{Dq}{Dt} \equiv \frac{\partial q}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} q = 0, \tag{1}$$

where $\mathbf{u}$ is the two-dimensional advecting velocity field, tangent to isopycnals, which can be derived from a streamfunction ψ

$$\mathbf{u}(\mathbf{x}, t) = \left(-\frac{\partial \psi}{\partial y}\,,\, \frac{\partial \psi}{\partial x}\,,\, 0 \right). \tag{2}$$

For simplicity, and following many previous studies, we take N and f both constant. Stretching the vertical direction z by N/f, typically $\gg 1$, the streamfunction ψ can be recovered from the PV distribution $q(x, y, z, t)$ by the linear inversion relation

$$\Delta \psi = q(x, y, z, t). \tag{3}$$

3 Equilibrium states

Reinaud and Dritschel (2005) determined the shape of two equal-PV, $q = 2\pi$, vortices in mutual equilibrium at the margin of stability in a large parameter space spanned by the height-to-width aspect ratio of the vortices $(h_1/r_1)_{i=1,2}$, their volume ratio V_2/V_1 and their relative vertical offset $\Delta z/(h_1 + h_2)$. A general description of the parameters describing the pair of vortices is provided in figure 1. We set the total volume of PV to $4\pi/3$ such that the volume integral of PV is the same for all cases.

To obtain the equilibrium states, the authors used a numerical technique derived from the Ellipsoidal Model (ELM) described in Dritschel *et al.* (2004), see Reinaud and Dritschel (2005) for details. The margin of stability corresponds to a critical value of the horizontal gap δ that separates the innermost edges of the vortices. Below this critical gap δ_c all states are unstable.

Figure 2 illustrates equilibrium states at the margin of stability for $h_1/r_1 = h_2/r_2 = 0.8$, $V_1/V_2 = 0.3$, and $\Delta z/(h_1 + h_2) = 0.005, 0.25$, and 0.75. As the vertical offset is increased, the vortices tend to tilt toward each other. We use these equilibrium states next as initial conditions for our investigation of the full nonlinear interaction between the vortices.

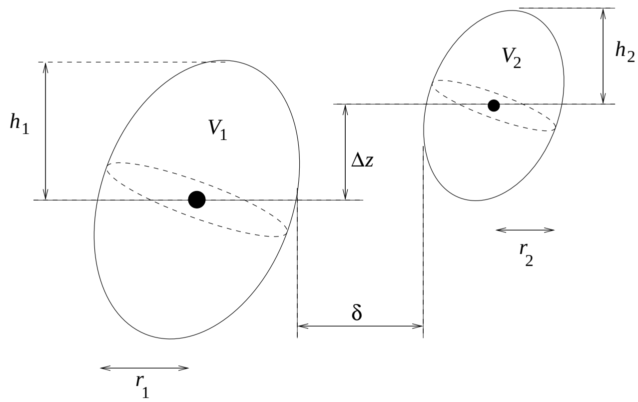

Figure 1: Definition of the parameters describing the vortex interaction.

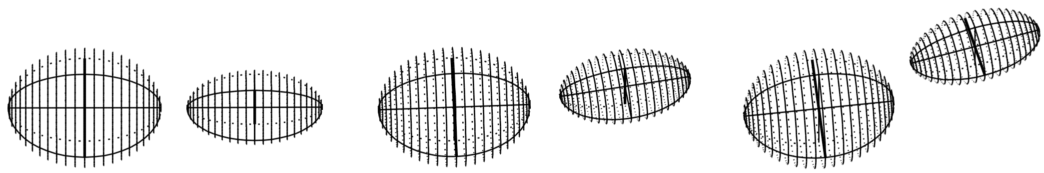

Figure 2: Equilibrium states at the margin of stability for $h_1/r_1 = h_2/r_2 = 0.8$, $V_2/V_1 = 0.3$, and $\Delta z/(h_1 + h_2) = 0.005, 0.25$, and 0.75.

4 Nonlinear evolution

We now address the outcome of the strong interaction between vortices set at the margin of stability. To that end, we perform nonlinear simulations using the CASL (Contour Advective Semi-Lagrangian) algorithm developed by Dristchel and Ambaum (1997). The coarse grid resolution for the velocity is set to 128^3. The vortices are rescaled to fit in a 2^3 box centred in the $(2\pi)^3$ periodic computation box to avoid a significant influence of the periodic images on the interaction. Results are then rescaled back to their original size such that the volume integral of PV is the same for all cases. We have simulated $n = 625$ individual cases. The range of parameters used is given in table 1. We only consider vortices with aspect ratio of the order of unity as Reinaud *et al.* (2003) showed that they represent the dominant population of vortices in QG turbulence. Indeed most of the vortices exhibit a height-to-width $h/r \simeq 0.8$.

We study how the self-energy E of the vortices is redistributed during the strong interactions. Note that the interaction energy is small compared to self-energies, usually a few percent. To that end, we identify vortices from their contour representation as contiguous volumes of PV

Parameter	Min	Max	Step
h_1/r_1	0.4	1.2	0.2
h_2/r_2	0.4	1.2	0.2
ρ_V	0.2	1.0	0.2
$\Delta z/(h_1 + h_2)$	0.0	0.8	0.2

Table 1: Range of parameters considered, h_i/r_i, $i = 1, 2$ are the height to width aspect ratios of each vortex, ρ_V is the volume ratio V_2/V_1 and $\Delta z/(h_1 + h_2)$, the vertical offset.

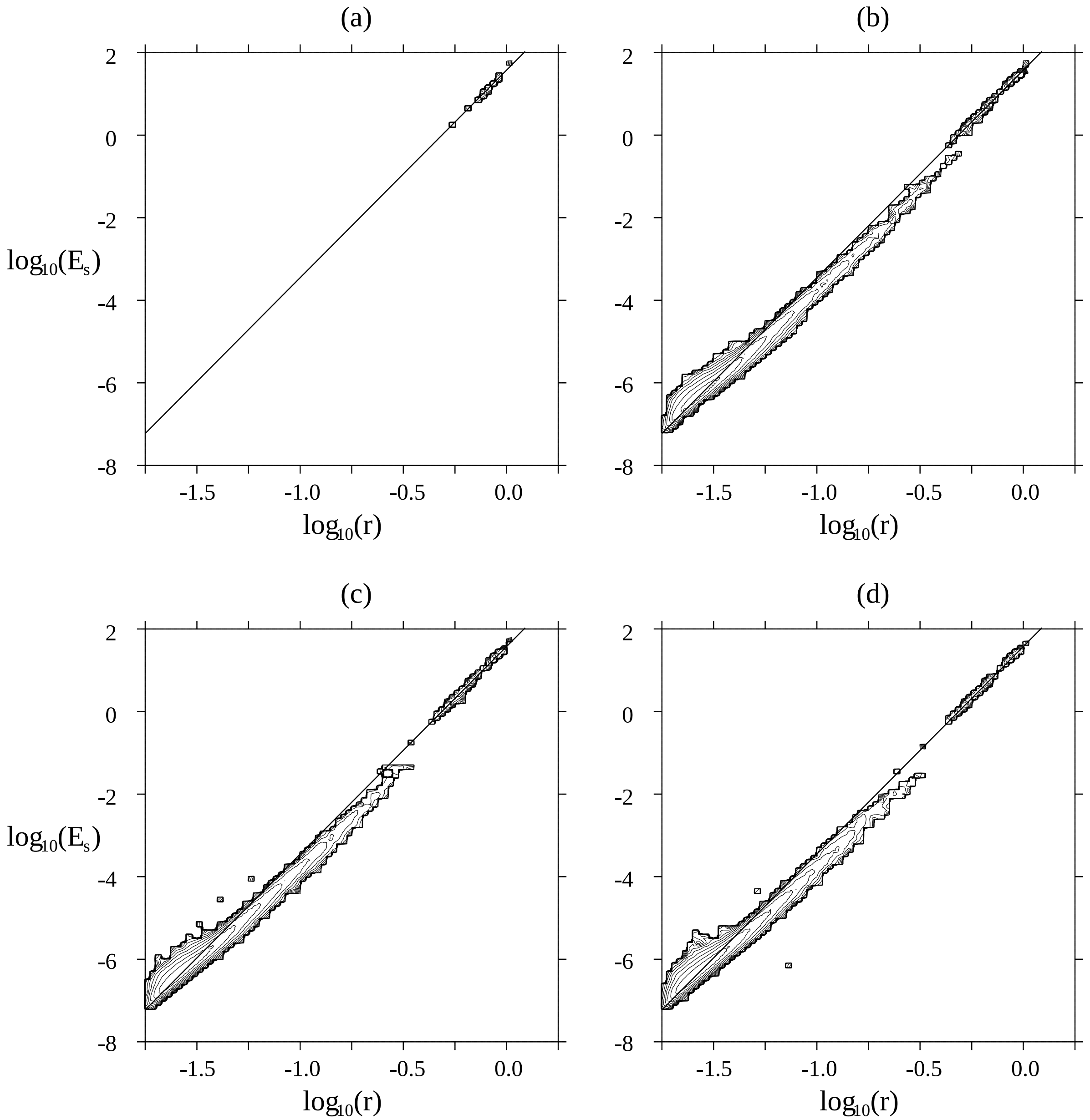

Figure 3: Contours of the number density (n) of vortices (contoured as $\log_{10} n$). The first (outermost) contour has $\log_{10} n = 0$, the innermost contour has $\log_{10} n = 11$, the contour increment is $\log_{10} \Delta n = 0.5$. 100 intervals were used in each direction, equally spaced in logarithmic scales. We add for reference the line corresponding to the energy of a sphere of PV Q, $E_s = (4\pi Q^2/15)r^5$.

and we compute their mean radius $r = (3V/4\pi)^{1/3}$ and their self-energy

$$E = -\frac{1}{2}\iiint q\psi_v \mathrm{d}V, \quad (4)$$

by contour integration, where ψ_v is the streamfunction induced by a given vortex on itself. Note that for a sphere of radius r of uniform PV (Q), the self energy is known analytically,

$$E = \frac{4\pi}{15}Q^2 r^5. \quad (5)$$

Figure 3 shows the number density of vortices having a given radius and a given energy at $t = 0$, 30, 40, and 60. Initially the energy of the vortices follows roughly eq. (5), as the vortices are initially ellipsoids with height-to-width aspect ratio of order of unity. During the strong interaction, many small scale vortices are generated. Interestingly they also roughly follow the trend indicated by eq. (5). A few larger vortices are also created and from their energy (fitting eq. (5)), we can conclude that they remain compact, long-lived structures. On the other hand, we notice that intermediate scale vortices are initially generated at scales $0.25 < r < 0.40$. In this range, for a given value of r, the self energy of each vortex is significantly less than that of a spherical vortex of the same volume. As the self-energy of a contiguous volume of PV generally decreases with its deformation (or departure from spheroidal shape), we can conclude that these vortices are strongly deformed, and consequently highly sensitive to the strain induced by the largest vortex in the flow. This argument can also be backed by considering the conservation of the angular impulse

$$J = \iiint q(x^2 + y^2)\mathrm{d}V. \quad (6)$$

Suppose two vortices initially merge to create a single large distorted vortex. When this structure breaks asymmetrically, generating two main structures, angular impulse conservation implies that if the secondary vortex is of intermediate spatial scale, it must remain close to the largest vortex while a smaller secondary vortex will be expelled further away from the main vortex. Consequently intermediate vortices generated during such complex nonlinear interactions are more likely to be subjected to high values of strain from the largest remaining vortex than a small secondary vortex would be. These two facts explain the disappearance of the intermediate-scale vortices in these interactions.

We next plot the integral F with respect to r of the self-energies of all the vortices from all $n = 625$ simulations,

$$F = \frac{1}{n}\int E(r)\mathrm{d}r. \quad (7)$$

Results are presented in figure 4. From $r = 0.5$ to $r = 0.76$, the amount of energy contained in scales up to r, given by $F(r)$, has increased at $t = 60$. On the other hand for $r > 0.76$, the curve at $t = 60$ is displaced to the right, indicating that most of the energy has shifted to larger spatial scales. We may then say that although there was a small direct energy cascade, the net cascade is inverse, feeding the large scales.

5 Conclusions

In this paper we reviewed recent results from the exploration of strong interactions between two equal-PV QG vortices. We first determined initial conditions consisting of pairs of vortices residing at an approximate margin of stability over an large parameter space. On a subset of this

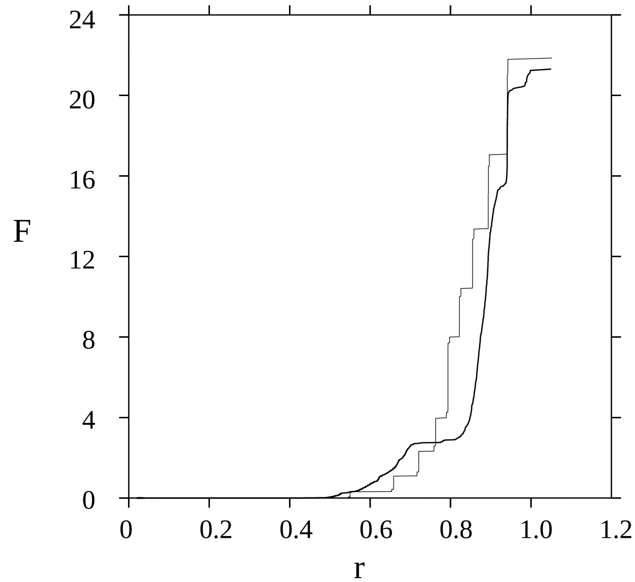

Figure 4: F plotted against r where, $F = (1/n)\int E_s \mathrm{d}r$, at times $t = 0$ (thin line) and $t = 60$ (bold line).

parameter space, we conducted nonlinear simulations using the CASL algorithm. From the data we extracted the self-energies of the vortices. We showed that there was a very large generation of small scale vortices but that the net energy cascade was inverse, i.e. feeding the large scales in physical space, in agreement with the inverse energy cascade observed in spectral space in QG turbulence.

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References

Gill, A.E. 1982 Atmosphere-ocean dynamics. *Academic Press.*

Dritschel D.G., Reinaud, J.N and McKiver, W.J. 2004 The quasi-geostrophic ellipsoidal vortex model. *J. Fluid Mech.* **505** 201-223.

Reinaud, J.N and Dritschel D.G. 2005 The critical merger distance between two co-rotating quasi-geostrophic vortices. *J. Fluid Mech.* **522** 357-381.

Dritschel, D.G. and Ambaum, M.H.P. 1997 A contour-advective semi-lagrangian algorithm for the simulation of fine-scale conservative fields. *Q. J. Roy. Meteorol. Soc.* **123** 1097-1130.

Reinaud, J.N., Dritschel D.G. and Koudella, C.R. 2003 The shape of vortices in quasi-geostrophic turbulence. *J. Fluid Mech.* **474** 175-192.