The core and the steady bargaining set for convex games

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Abstract Within the class of zero-monotonic and grand coalition superadditive cooperative games with transferable utility, the convexity of a game is characterized by the coincidence of its core and the steady bargaining set. As a consequence it is proved that convexity can also be characterized by the coincidence of the core of a game and the modified Zhou bargaining set *à la* Shimomura.

Keywords cooperative game · convex games · core · bargaining set

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1 Introduction

Cooperative game theory analyzes how to distribute profits arising from the cooperation of a group of agents by proposing solutions that may consist on a unique allocation of those profits (payoff vector) or on a group of them meeting some stability conditions (set-solution). The core of a game v, $C(N,v)$, is the most natural set-solution concept but it might be empty. The bargaining sets (Davis and Maschler (1963, 1967), Mas-Colell (1989), Zhou (1994) and others) based on objections and counter-objections to payoff proposals offer an alternative solution to the emptiness of the core, at a cost to be rather complex to compute. For this reason, it has been interesting to define non-empty subsolutions of the bargaining sets that were more simply to describe and check, that fulfill some stability conditions and that were related to the core of the game whenever it is non-empty. In this way, the first subsolutions we can find in the literature are the notion of *quasi-core*, introduced by Shapley and Shubik (1966), and the concept of *kernel* of a game (Davis and Maschler, 1965). Years after, Shimomura (1997) introduces the *steady bargaining set* of a game v, $\mathcal{SB}(N,v)$, and a small modification of the Shapley and Shubik quasi-core concept. The steady bargaining set of a game includes its core and it is a subsolution of two well-known variants of bargaining sets, also introduced by Shimomura (1997): the modified Mas-Colell bargaining set, $\mathcal{MB}^*(N,v)$, and the modified Zhou bargaining set, $\mathcal{Z}^*(N,v)$. The relationship among these solutions is as follows:

$$C(N,v) \subseteq \mathcal{SB}(N,v) \subseteq \mathcal{Z}^*(N,v) \subseteq \mathcal{MB}^*(N,v). \tag{1}$$

A sufficient condition that guarantees the non-emptiness of the steady bargaining set and the modified Zhou bargaining set of a game is its grand coalition superadditivity, while grand coalition zero-monotonicity also suffices to check the non-emptiness of the modified Mas-Colell bargaining set.

Convex or supermodular coalitional games were introduced by Shapley (1971). They are an important subclass of games and they model cooperative situations where the marginal contribution of a player to a coalition increases as the coalition becomes larger (the so called snowballing effect). Convex games satisfy important properties from a game theoretical point of view and they have been useful to analyze and capture many economic situations both in cooperative and non-cooperative frameworks.

Einy and Wettstein (1996) opened the question of characterizing the convexity of a game by comparing its bargaining sets with the core, with special reference to the stable bargaining set introduced by Greenberg (1992). Within the domain of zero-monotonic games, Izquierdo and Rafels (2012) give a first answer to that question by means of the coincidence of the core of a game and its modified Mas-Colell bargaining set.

In this paper we focus on enriching the convexity characterization results. Within the domain of zero-monotonic and grand coalition superadditive games, the first characterization requires the coincidence of the the core of a game and its steady bargaining set (Theorem 1). The elaborate proof of our new characterization of the convexity of a game follows a two-step argument: first (see Proposition 1) , we show the characterization within a subclass of almost-convex games (introduced by Núñez and Rafels, 1998); second, in Theorem 1, we tackle the general case.

Finally, by the inclusion relationship given in (1), we also obtain as a corollary of this theorem an additional new characterization of convex games in term of the coincidence of the modified Zhou bargaining set and the core of the game (Corollary 1). We expect this work might serve to both obtain new equivalence theorems and to reanalyze the different convexity notions given for the non-transferable utility case.

2 Notations

Let $N = \{1, 2, \ldots, n\}$ be a set of players. For any coalition $S \subseteq N$, $|S|$ denotes the number of players in S. A cooperative game with player set N is a function $v : 2^N \to \mathbb{R}$ assigning to each coalition $S \subseteq N$ a real number $v(S)$ such that $v(\varnothing) = 0$. The function v is called the *characteristic function* of the game and $v(S)$ is the *worth* of the coalition S. This number is interpreted as what the coalition can obtain on its own. Let $\mathcal{G}^N$ be the class of games with player set N. Given a nonempty coalition $S \subseteq N$, we denote by (S, v_S) the subgame of (N, v) related to coalition S (i.e. $v_S(R) = v(R)$ for all $R \subseteq S$).

A game $v \in \mathcal{G}^N$ is *monotonic* if $v(S) \leq v(T)$, for any $S \subseteq T \subseteq N$. It is *zero-monotonic* if for all $S \subseteq T \subseteq N$ we have $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$ and it is *grand coalition zero-monotonic* if for all $S \subseteq N$ we have $v(S) + \sum_{i \in N \setminus S} v(\{i\}) \leq v(N)$.

A game $v \in \mathcal{G}^N$ is *superadditive* if for all $S, T \subseteq N$ with $S \cap T = \varnothing$ it holds $v(S) + v(T) \leq v(S \cup T)$, and it is *grand coalition superadditive* if for all partition $\mathcal{P}$ of N, $\mathcal{P} = \{S_1, S_2, \ldots, S_m\}$, it holds that $\sum_{j=1}^{m} v(S_j) \leq v(N)$.

A game $v \in \mathcal{G}^N$ is *convex* if, for all $i \in N$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{2}$$

for all $S \subseteq T \subseteq N \setminus \{i\}$. An equivalent definition of convexity states that, for all S,$T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{3}$$

Finally, it is *almost convex* (Núñez and Rafels, 1998) if for all $S \subseteq N$, $S \neq N$ the corresponding subgame (S, v_S) is convex.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors $x = (x_i)_{i \in N}$ where x_i is interpreted as the payoff to player $i \in N$, x_S is the restriction of x to the members of $S \subseteq N$ and $x(S)$ denotes $\sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. Given x and y two vectors in $\mathbb{R}^N$, we write $x \geq y$ to mean that $x_i \geq y_i$, for all $i \in N$.

The *set of preimputations* of a game $v \in \mathcal{G}^N$ is defined by $I^*(N, v) = \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$. Its *set of imputations* is defined by $I(N, v) = \{x \in \mathbb{R}^N \,|\, x(N) = v(N) \text{ and } x_i \geq v(\{i\}),\ \text{for all } i \in N\}$ and its *core* is defined by $C(N, v) = \{x \in \mathbb{R}^N \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game with a non-empty core is called a *balanced* game. Let $\mathcal{B}^N \subseteq \mathcal{G}^N$ be the subclass of balanced games with player set N.

Given a game v, a preimputation $x \in I^*(N, v)$ and a pair of players i and j, $i \neq j$, we define

$$s^v_{ij}(x) = \max\{v(S) - x(S) \mid S \subseteq N,\, i \in S \text{ but } j \notin S\}.$$

We say that player i outweigths player j at x if $s^v_{ij}(x) > s^v_{ji}(x)$. The *prekernel* of the game v, $\mathcal{PK}(N, v)$, is the subset of preimputations such that no player outweights any other player at x. This is

$$\mathcal{PK}(N, v) = \{x \in I^*(N, v) \mid \text{ for all } i, j \in N,\, i \neq j,\, s^v_{ij}(x) = s^v_{ji}(x)\}.$$

For any game , the prekernel is always non-empty.

The kernel was introduced by Davis and Maschler (1965). It is based on the idea of outweighting, but restricting the domain of feasible allocations to imputations. A more general concept was analyzed by Schmeidler (1969) allowing to consider arbitrary payoff domains; in particular we are interested in those satisfying $Y = Y(\ell, u) = \{x \in I^*(N,v) \mid \ell_i \leq x_i \leq u_i, \text{ for all } i \in N\}$, where $\ell = (\ell_i)_{i\in N}$ and $u = (u_i)_{i\in N}$, are the respective vectors of lower and upper bounds for the payoffs of players within Y with $\ell \leq u$. Notice that $Y(\ell, u) \neq \varnothing$ when $\ell(N) \leq v(N) \leq u(N)$. Following Kikuta (1997), a payoff vector x belongs to the *kernel* of v relative to a nonempty box, $\mathcal{K}(N, v, Y(\ell, u))$, when for all pair of distinct players i and j it holds that: if $s^v_{ij}(x) > s^v_{ji}(x)$, then either $x_i = u_i$ or $x_j = \ell_j$. Being $Y(\ell, u)$ a non-empty set it follows that $\mathcal{K}(N, v, Y(\ell, u)) \neq \varnothing$ (see Schmeidler (1969)).

Shimomura (1997) considers modifications of both the Mas-Colell bargaining set (Mas-Colell, 1989) and Zhou bargaining set (Zhou, 1994). As usual the bargaining set is defined by means of an interaction of objections and counterobjections. Let $x \in \mathbb{R}^N$. An *objection* to x is a pair (S, y), $\varnothing \neq S \subseteq N$ and $y \in \mathbb{R}^S$ with $y(S) = v(S)$ such that $y_i > x_i$, for all $i \in S$. A *counterobjection* to (S, y) in the sense of Mas-Colell (*à la* Shimomura) is a pair (T, z), $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$ for all $i \in T \setminus S$. A *counterobjection* to (S, y) in the sense of Zhou (*à la* Shimomura) is a pair (T, z), where $T \setminus S \neq \varnothing$, $S \setminus T \neq \varnothing$, $T \cap S \neq \varnothing$, and $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$ for all $i \in T \setminus S$. Notice the bargaining process represents strictly improvements (strictly higher payoffs) for all players involved in the objections and the counterobjections.

Definition 1 The **Mas-Colell bargaining set** (*à la Shimomura*) is defined as

$$\mathcal{MB}^*(N,v) = \left\{ x \in I(N,v) \;\middle|\; \begin{array}{l} \text{for each objection to } x, \\ \text{there is a Mas-Colell's counterobjection} \end{array} \right\}.$$

Definition 2 The **Zhou bargaining set** (*à la Shimomura*) is defined as

$$\mathcal{Z}^*(N,v) = \left\{ x \in I(N,v) \;\middle|\; \begin{array}{l} \text{for each objection to } x, \\ \text{there is a Zhou's counterobjection} \end{array} \right\}.$$

If no confusion arises we will refer to them simply as the Mas-Colell bargaining set and the Zhou bargaining set. By definition, these sets only consist on imputations (individually rational payoff vectors) and always includes the core. Shimomura (1997) states that a sufficient condition that guarantees the Mas-Colell bargaining set to be nonempty is grand coalition zero-monotonicity, while it is grand coalition superadditivity that ensures the non-emptiness of the Zhou bargaining set.

Shimomura also defines a subset of the Zhou bargaining set (the *steady bargaining set*, $SB(N,v)$) by means of a dominant relationship between coalitions. He claims that the steady bargaining set can be rewritten as follows.

Definition 3 Let $v \in \mathcal{G}^N$ be a game. An imputation $x \in I(N,v)$ is in the **steady bargaining set** $\mathcal{SB}(N,v)$ if for all coalition $S \subseteq N$ with strictly positive excess $v(S) - x(S) > 0$, there exists $M \subseteq N$, such that $S \setminus M \neq \varnothing$, $M \setminus S \neq \varnothing$, $S \cap M \neq \varnothing$ and $v(M) - x(M) \geq v(S) - x(S)$.

For any game v, it can be easily proved the inclusions $C(N,v) \subseteq \mathcal{SB}(N,v) \subseteq \mathcal{Z}^*(N,v) \subseteq \mathcal{MB}^*(N,v)$; let us remark that these inclusions might be strict, even for superadditive games[1]. On the other hand, Izquierdo and Rafels (2012) show that

[1] Let $N = \{1,2,3,4\}$ be the set of players and $v(N) = 2$, $v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 1$, $v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = 1$ and $v(S) = 0$, otherwise. Notice that the core of this game is non-emtpy. The payoff vector $x = (\frac{1}{2}, 1\frac{1}{2}, 0, 0) \in \mathcal{MB}^*(N,v)$, but $x \notin \mathcal{Z}^*(N,v)$. Moreover, the payoff vector $x' = (\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{4}) \in \mathcal{Z}^*(N,v)$, but $x' \notin C(N,v)$. With respect to the steady bargaining set notice $x' = (\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{4}) \in \mathcal{SB}(N,v)$, but $x' \notin C(N,v)$. The Example 2 (with $a = 1$) in Shimomura (1997) provides an example of a

the core of a convex game v and its Mas-Colell bargaining set (à la Shimomura)) do coincide, and thus, $C(N,v) = \mathcal{SB}(N,v) = \mathcal{Z}^*(N,v) = \mathcal{MB}^*(N,v)$.

Let $v \in \mathcal{B}^N$ be a balanced game and $\theta = (i_1, i_2, \ldots, i_n)$ be an ordering of players in N. We denote by Θ_N the set of all orderings in N. A marginal worth vector of the game v relative to θ, $m^\theta(v)$, is defined as

$$m^\theta_{i_1}(v) = v(\{i_1\}) \quad \text{and}$$

$$m^\theta_{i_k}(v) = v(\{i_1, \ldots i_k\}) - v(\{i_1, \ldots i_{k-1}\}), \text{ for all } k = 2, \ldots, n.$$

It is well-known (Shapley (1971), Ichiishi (1981)) that a game is convex if and only the marginal worth vectors of the game are all core elements.

$$v \text{ is convex } \Leftrightarrow m^\theta(v) \in C(N,v), \text{ for all } \theta \in \Theta_N. \tag{4}$$

We say that vector $x \in \mathbb{R}^N$ lexicographically precedes vector $y \in \mathbb{R}^N$ with respect to $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$, $x \prec^\theta_\ell y$, if there exists $k \in \{1, 2, \ldots, n\}$ such that $x_{i_r} = y_{i_r}$ for all $r = 1, \ldots, k-1$ and $x_{i_k} < y_{i_k}$. The *lexmin* solution over the core of a balanced game $v \in \mathcal{B}^N$ relative to $\theta \in \Theta_N$ is defined as the (unique) payoff vector $\ell^\theta(v) \in C(N,v)$ that lexicographically precedes w.r.t. to θ any other vector in the core of the game v, i.e. $\ell^\theta(v) \prec^\theta_\ell x$ for all $x \in C(N,v)$. Let us remark that if a game v is convex then $\ell^\theta(v) = m^\theta(v)$, for each ordering $\theta \in \Theta_N$.

3 Characterization results

In this section we provide two new characterizations of the convexity of a game.

The first one compares the steady bargaining set of the game with its core. To this aim, we first analyze the particular case of almost convex games (games

superadditive game where the steady bargaining set is strictly included in the Zhou bargaining set.

where all proper subgames are convex), since the argument used in the proof of the general case does not apply.

The second characterization is a direct consequence of the first one, and focuses on the coincidence of the Zhou bargaining set with the core.

Proposition 1 *Let (N, v) be a grand coalition superadditive and almost convex game. Then, the following statements are equivalent:*

1. *v is convex.*
2. *$\mathcal{SB}(N, v) = C(N, v)$.*

Proof 1.→ 2.) From Izquierdo and Rafels (2012) it follows that, for any convex game v, $C(N, v) = \mathcal{MB}^*(N, v)$. Hence, since $C(N, v) \subseteq \mathcal{SB}(N, v) \subseteq \mathcal{MB}^*(v)$, we conclude $C(N, v) = \mathcal{SB}(N, v)$.

2.→ 1.) As the game v is grand coalition superadditive, the steady bargaining set is nonempty, i.e. $\mathcal{SB}(N, v) \neq \varnothing$ (Shimomura, 1997), and thus (by hypothesis), $\mathcal{SB}(N, v) = C(N, v) \neq \varnothing$. Therefore, the game is balanced. At this point, the proof is done for the two-person case, $n = 2$, since any two-person balanced game is convex. Hence, from now on let us assume $n \geq 3$. Suppose to the contrary that the game is not convex. Since the game is almost convex but not convex, this means there exists a pair of players, say player 1 and player 2 such that

$$v(N) - v(N \setminus \{1\}) < v(N \setminus \{2\}) - v(N \setminus \{1, 2\}). \tag{5}$$

From Núñez and Rafels (1998) we know there is an extreme point $x \in C(N, v)$ of the core of the game v such that[2]

$$x_1 = v(N) - v(N \setminus \{1\}) \text{ and } x_2 = v(N) - v(N \setminus \{2\}). \tag{6}$$

[2] These authors prove that particular payoff vectors constructed based upon orderings of players $\theta = (i_1, i_2, \ldots, i_n)$ (the reduced marginal worth vector $rm^\theta(v)$) are the extreme core

By (5) and (6) it holds that

$$\begin{aligned} x_1 &= v(N) - v(N \setminus \{1\}) < v(N \setminus \{2\}) - v(N \setminus \{1,2\}) \\ &= v(N) - (v(N) - v(N \setminus \{2\})) - v(N \setminus \{1,2\}) \\ &= v(N) - x_2 - v(N \setminus \{1,2\}) = x(N \setminus \{2\}) - v(N \setminus \{1,2\}). \end{aligned}$$

166 We conclude, $x(N \setminus \{1,2\}) > v(N \setminus \{1,2\})$. Hence, let us remark now that the
167 vector x restricted to $N \setminus \{1,2\}$, $x_{N\setminus\{1,2\}}$, can be viewed as an aspiration of the
168 subgame[3] $(N\setminus\{1,2\}, v_{N\setminus\{1,2\}})$. Since this subgame is convex and any convex game
169 has a large core[4] (Sharkey 1982), there exists $z \in C(N \setminus \{1,2\}, v_{N\setminus\{1,2\}})$ such that
170 $x_{N\setminus\{1,2\}} \geq z$. Moreover, since $x(N \setminus \{1,2\}) > v(N \setminus \{1,2\}) = z(N \setminus \{1,2\})$ there
171 exists a player in $N \setminus \{1,2\}$, say player 3 such that $x_3 > z_3$. This implies that

$$x(S) > z(S) \geq v(S), \text{ for all } S \subseteq N \setminus \{1,2\} \text{ and } 3 \in S. \tag{7}$$

Next define the vector $x' \in \mathbb{R}^N$ as follows:

$$x'_1 = x_1 + \frac{\varepsilon}{2};\ x'_2 = x_2 + \frac{\varepsilon}{2};\ x'_3 = x_3 - \varepsilon \text{ and } x'_k = x_k \text{ for all } k \in N \setminus \{1,2,3\},$$

172 where $0 < \varepsilon < \min_{3\in S\subseteq N\setminus\{1,2\}} \{x(S) - v(S)\}$.

173 By the definition of ε the vector x' is an imputation of the game v, but it is
174 not a core element of (N,v) since $x'_1 > x_1 = v(N) - v(N\setminus\{1\})$ and so $x'(N\setminus\{1\}) <$
175 $v(N \setminus \{1\})$. However, for any coalition $S \subseteq N$ such that $v(S) - x'(S) > 0$ it is easy
176 to check that:

elements of an almost convex balanced game. In particular, if $\theta = (1,2,\ldots,n)$, we have $rm_1^\theta(v) = v(N) - v(N\setminus\{1\})$ and $rm_2^\theta(v) = \min\{v(N\setminus\{1\}) - v(N\setminus\{1,2\}),\ v(N) - v(N\setminus\{2\})\} = v(N) - v(N \setminus \{2\})$, where the last equality follows from (5).

[3] An aspiration of a game (N,v) is a vector $x' \in \mathbb{R}^N$ satisfying all cores inequalities, i.e. $x'(S) \geq v(S)$, for all $S \subseteq N$.

[4] A game has a large core if any aspiration x' of the game can be represented by a core allocation x, i.e. there exists $x \in C(N,v) : x_i \leq x'_i$, for all $i \in N$.

(a) player $3 \in S$;

(b) either player $1 \in S$ or $2 \in S$, but not both;

(c) $v(S) - x'(S) \leq \frac{\varepsilon}{2}$; and

(d) $v(N \setminus \{1\}) - x'(N \setminus \{1\}) = v(N \setminus \{2\}) - x'(N \setminus \{2\}) = \frac{\varepsilon}{2}$.

Taking these remarks into account, let $S \subseteq N$ be an arbitrary coalition with positive excess at x', i.e. $v(S) - x'(S) > 0$. We next show that there exists a coalition $M \subseteq N$ such that $M \cap S \neq \varnothing$, $S \setminus M \neq \varnothing$, $M \setminus S \neq \varnothing$ and $v(M) - x'(M) \geq v(S) - x'(S)$, and so that x' is in the steady bargaining set of (N, v). We consider two cases:

A: If player $1 \in S$, take $M = N \setminus \{1\}$. Notice that, by (c) and (d), $v(M) - x'(M) = \frac{\varepsilon}{2} \geq v(S) - x'(S)$. Moreover, by (a), player $3 \in M \cap S$, player $1 \in S \setminus M$ and, by (b), player $2 \in M \setminus S$.

B: If player $2 \in S$, take $M = N \setminus \{2\}$ and using an analogous reasoning we get that $v(M) - x'(M) = \frac{\varepsilon}{2} \geq v(S) - x'(S)$ with player $3 \in M \cap S$, player $2 \in S \setminus M$ and player $1 \in M \setminus S$.

We conclude the allocation x' is not a core element of the game v but belongs to its steady bargaining set, i.e. $x \in \mathcal{SB}(N, v)$, which contradicts our hypothesis. Hence, the game v must be convex. □

The above characterization result can be now extended to a larger class of cooperative games. The thread of the proof of this result relies on the fact that for a convex game all marginal worth vectors are core elements and coincide with the corresponding lexmin solution relative to the different orderings. As a consequence, if a game is not convex there is at least one marginal worth vector that differs from the corresponding lexmin solution; based upon this, we will construct a particular vector not in the core but in the steady bargaining set. That is, we shall prove that if a game is not convex the steady bargaining set of the game strictly includes its core. The proof of this result is constructive in the sense that, for any non-convex game satisfying conditions of Theorem 1, we built an allocation that turns out to be in the steady bargaining set of the game, but not in its core.

Theorem 1 *Let (N, v) be a zero-monotonic and grand coalition superadditive game. Then, the following statements are equivalent:*

1. *v is convex.*
2. *$\mathcal{SB}(N, v) = C(N, v)$.*

Proof *1.→ 2.*) By convexity of the game v, it holds $C(N, v) = \mathcal{MB}^*(N, v)$ and thus, by (1), we conclude $C(N, v) = \mathcal{SB}(N, v)$.

2.→ 1.) As the game v is grand coalition superadditive, the steady bargaining set is nonempty, i.e. $\mathcal{SB}(N, v) \neq \varnothing$ (Shimomura, 1997), and thus (by hypothesis), $\mathcal{SB}(N, v) = C(N, v) \neq \varnothing$. Therefore, the game is balanced and the lexmin solution $\ell^\theta(v)$ is well-defined for all $\theta \in \Theta_N$. Let us suppose now that the game is not convex. Then, by (4), there must exist at least one ordering $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$ such that $\ell^\theta(v) \neq m^\theta(v)$. Now, if we pairwise compare the lexmin vector $\ell^\theta(v)$ and the marginal worth vector $m^\theta(v)$ corresponding to all orderings we can determine a unique index $t^* \in \{1, \ldots, n\}$ satisfying that:

$$\begin{aligned}&(i)\ \ \ell^\theta_{j_k}(v) = m^\theta_{j_k}(v), \text{ for all } \theta = (j_1, j_2, \ldots, j_n) \in \Theta_N \text{ and } k = 1, \ldots, t^* - 1;\\ &(ii) \text{ there exists } \theta^* = (i_1, i_2, \ldots, i_n) \in \Theta_N \text{ such that } \ell^{\theta^*}_{i_{t^*}}(v) \neq m^{\theta^*}_{i_{t^*}}(v).\end{aligned} \tag{8}$$

Item (i) indicates that, for all ordering, the payoff of players occupying the first $t^* - 1$ positions coincide for both the lexmin vector and the marginal worth vector (notice that this condition does not impose any restriction when $t^* = 1$); item

(ii) states the existence of an ordering where the corresponding lexmin vector and marginal worth vector differ for the first time at position t^*.

Notice that, by Proposition 1, we may assume that (N, v) is not an almost convex game and thus $t^* \neq n$. Moreover, as $\ell^{\theta^*}(v)$ is a core element, it can be checked in condition (ii) that

$$\ell^{\theta^*}_{i_{t^*}}(v) > m^{\theta^*}_{i_{t^*}}(v). \tag{9}$$

To prove it, in other case and by (8), $\ell^{\theta^*}_{i_{t^*}}(v) < m^{\theta^*}_{i_{t^*}}(v) = v(\{i_1, \ldots, i_{t^*}\}) - v(\{i_1, \ldots, i_{t^*-1}\}) = v(\{i_1, \ldots, i_{t^*}\}) - m^{\theta^*}(v)(\{i_1, \ldots, i_{t^*-1}\}) = v(\{i_1, \ldots, i_{t^*}\}) - \ell^{\theta^*}(v)(\{i_1, \ldots, i_{t^*-1}\})$, which involves a contradiction since $\ell^{\theta^*}(v) \in C(N, v)$. Finally, we can also deduce from the above condition (i) that,

$$\text{for all } S \subseteq N \text{ with } |S| < t^*, \text{ the subgame } (S, v_S) \text{ is convex.} \tag{10}$$

Now, define

$$\mathcal{S}^{\theta^*}(v) = \{M \subseteq N \mid M \neq N,\, i_{t^*} \in M \text{ and } \ell^{\theta^*}(v)(M) = v(M)\},$$

where $\theta^* = (i_1, i_2, \ldots, i_n)$ is given in (ii) of (8).

We claim $\mathcal{S}^{\theta^*}(v) \neq \varnothing$. Otherwise, $\ell^{\theta^*}(v)(M) > v(M)$, for all $M \subseteq N$, $M \neq N$, and $i_{t^*} \in M$. Taking this into account we might define the allocation $x \in \mathbb{R}^N$ as $x_{i_{t^*}} = \ell^{\theta^*}_{i_{t^*}}(v) - \varepsilon_1$, $x_{i_{t^*+1}} = \ell^{\theta^*}_{i_{t^*+1}}(v) + \varepsilon_1$ and $x_{i_k} = \ell^{\theta^*}_{i_k}(v)$, else, where $0 < \varepsilon_1 < \min\limits_{M \subsetneq N, i_{t^*} \in M} \{\ell^{\theta^*}(v)(M) - v(M)\}$, and prove that $x \in C(N, v)$. However, this would contradict $\ell^{\theta^*}(v)$ to be the lexmin solution relative to θ^* over the core of the game v.

Let us denote by $\mathcal{S}^{\theta^*}_{min}(v)$ the set of minimal coalitions with respect to the inclusion in the ordered set $(\mathcal{S}^{\theta^*}(v), \subseteq)$ and by T^* the first t^* agents of the ordering

$\theta^* = (i_1, \ldots, i_n)$ given in (8) , i.e. $T^* = \{i_1, \ldots, i_{t^*}\}$. Notice that

$$\ell^{\theta^*}(v)(T^*) > v(T^*), \tag{11}$$

where the strict inequality follows from (8) and (9), since $\ell^{\theta^*}(v)(T^*) = \ell^{\theta^*}(v)(T^* \setminus \{i_{t^*}\}) + \ell^{\theta^*}_{i_{t^*}}(v) = m^{\theta^*}(v)(T^* \setminus \{i_{t^*}\}) + \ell^{\theta^*}_{i_{t^*}}(v) > m^{\theta^*}(v)(T^*) = v(T^*)$.

Next, it can be shown that the set $\mathcal{S}^{\theta^*}_{min}(v)$ contains at least two coalitions. This result is stated in Claim 1 but the rather technical proof is consigned into the Appendix.

Claim 1 $|\mathcal{S}^{\theta^*}_{min}(v)| \geq 2$.

Taking into account this claim, define $\alpha \in \mathbb{R}^N$ as

$$\alpha_i = \begin{cases} \ell^{\theta^*}_i(v) - \varepsilon & \text{if } i = i_{t^*} \\ \ell^{\theta^*}_i(v) & \text{if } i \in N,\ i \neq i_{t^*} \end{cases}$$

where

$$0 < \varepsilon < \min_{\substack{M \subseteq N \\ \ell^{\theta^*}(v)(M) > v(M)}} \{\ell^{\theta^*}(v)(M) - v(M)\}.$$

Notice that the parameter ε is well defined since (11) holds. Moreover, we have that $\alpha(N) < v(N)$.

Take $I^{\theta^*} = \bigcap_{M \in \mathcal{S}^{\theta^*}_{min}(v)} M$ and notice that, by definition, $i_{t^*} \in I^{\theta^*}$ (and thus $I^{\theta^*} \neq \varnothing$), $M \setminus I^{\theta^*} \neq \varnothing$ for all $M \in \mathcal{S}^{\theta^*}_{min}(v)$, and $N \setminus I^{\theta^*} \neq \varnothing$ where the last two assertions follow from Claim 1.

Then define the game $(N \setminus I^{\theta^*}, \omega)$ as follows:

$$\omega(\varnothing) = 0,$$

$$\omega(R) = \max_{\substack{R' \subseteq R \\ Q \subseteq I^{\theta^*}}} \{v(R' \cup Q) - \alpha(R' \cup Q)\} \text{ for all } \varnothing \neq R \subseteq N \setminus I^{\theta^*}.$$

Let us remark that $\omega(R) \in \{0, \varepsilon\}$, for all $R \subseteq N \setminus I^{\theta^*}$, and $\omega(N \setminus I^{\theta^*}) = \varepsilon$. To check this we first claim that, given $M \subseteq N$, we have

$$v(M) - \alpha(M) = \begin{cases} 0 & \text{if } M = \varnothing \\ \varepsilon & \text{if } M \in \mathcal{S}^{\theta^*}(v) \text{ or } M = N \\ \leq 0 \text{ otherwise.} \end{cases} \tag{12}$$

Indeed, if $M = \varnothing$, $v(\varnothing) - \alpha(\varnothing) = 0$; if $M \in \mathcal{S}^{\theta^*}(v)$ or $M = N$ then $v(M) = \ell^{\theta^*}(v)(M)$ and $v(M) - \alpha(M) = v(M) - \ell^{\theta^*}(v)(M) + \varepsilon = \varepsilon$; finally, if $M \notin \mathcal{S}^{\theta^*}(v)$ and $M \neq N, \varnothing$ then either $i_{t^*} \notin M$ or $v(M) < \ell^{\theta^*}(v)(M)$: if $i_{t^*} \notin M$, then $v(M) - \alpha(M) = v(M) - \ell^{\theta^*}(v)(M) \leq 0$, and if $i_{t^*} \in M$ but $v(M) < \ell^{\theta^*}(v)(M)$, then $v(M) - \alpha(M) = v(M) - \ell^{\theta^*}(v)(M) + \varepsilon < 0$, where the last strict inequality follows from the definition of ε.

Taking (12) into account, and since for all $R \subseteq N \setminus I^{\theta^*}$, $\omega(R)$ is the maximum of differences $v(M) - \alpha(M)$, where $M \subseteq R \cup I^{\theta^*}$, it follows that $\omega(R) \in \{0, \varepsilon\}$. To see $\omega(N \setminus I^{\theta^*}) = \varepsilon$ just take $R' = N \setminus I^{\theta^*}$ and $Q = I^{\theta^*}$ in its definition.

Now take an element $\delta \in \mathbb{R}^{N \setminus I^{\theta^*}}$ in the prekernel of the game $(N \setminus I^{\theta^*}, \omega)$, that is $\delta \in \mathcal{PK}(N \setminus I^{\theta^*}, \omega)$. By the monotonicity of the game $(N \setminus I^{\theta^*}, \omega)$ and Theorem

5.6.1 in Peleg and Südholter (2007) it follows[5] that

$$\text{(a) } \delta_i \geq 0, \text{ for all } i \in N \setminus I^{\theta^*}. \tag{13}$$

$$\text{(b) } \delta_i = 0, \text{ for all } i \in N \setminus \bigcup_{M \in \mathcal{S}^{\theta^*}_{min}(v)} M. \tag{14}$$

To see (b), let $i \in N \setminus \bigcup_{M \in \mathcal{S}^{\theta^*}_{min}(v)} M$ and let $R \subseteq N \setminus I^{\theta^*}$ with $i \in R$. Let us check that $\omega(R) - \omega(R \setminus \{i\}) = 0$. If $\omega(R) = 0$, by the monotonicity of the game ω, we are done. If $\omega(R) = \varepsilon$, we know by (12) that $\omega(R) = v(M) - \alpha(M)$, for some coalition $M \in \mathcal{S}^{\theta^*}(v)$ or $\omega(R) = v(N) - \alpha(N)$. In case $\omega(R) = v(M) - \alpha(M)$ for some $M \in \mathcal{S}^{\theta^*}(v)$, we can take, in fact, $M \in \mathcal{S}^{\theta^*}_{min}(v)$; in case $\omega(R) = v(N) - \alpha(N)$, then $R = N \setminus I^{\theta^*}$ and for any $M \in S^{\theta^*}_{min}(v)$ we have that $\varepsilon = w(N \setminus I^{\theta^*}) \geq w(M \setminus I^{\theta^*}) \geq v(M) - \alpha(M) = \varepsilon$, where the first inequality follows from the monotonicity of the game and the last equality by (12). Thus we conclude that, in any of both cases, $\omega(R) = v(M) - \alpha(M)$ with $M \in S^{\theta^*}_{min}(v)$. Now, since by hypothesis of case (b) player i does not belong to any minimal coalition in $\mathcal{S}^{\theta^*}_{min}(v)$, it follows $M \subseteq (R \setminus \{i\}) \cup I^{\theta^*}$ and then we conclude that

$$\varepsilon \geq \omega(R \setminus \{i\}) = \max_{\substack{R' \subseteq R \setminus \{i\} \\ Q \subseteq I^{\theta^*}}} \{v(R' \cup Q) - \alpha(R' \cup Q)\} \geq v(M) - \alpha(M) = \varepsilon,$$

and we are done.

Next define the vector $x \in \mathbb{R}^N$ as follows:

$$x_i = \begin{cases} \alpha_i + \delta_i & \text{if } i \in N \setminus I^{\theta^*} \\ \alpha_i & \text{if } i \in I^{\theta^*}. \end{cases}$$

[5] We are using the fact that for any $\delta \in \mathcal{PK}(N \setminus I^{\theta^*}, \omega)$ and for any $i \in N \setminus I^{\theta^*}$,

$$\min_{R \subseteq N \setminus I^{\theta^*} : i \in R} \{\omega(R) - \omega(R \setminus \{i\})\} \leq \delta_i \leq \max_{R \subseteq N \setminus I^{\theta^*} : i \in R} \{\omega(R) - \omega(R \setminus \{i\})\}.$$

The vector x is efficient, $x(N) = v(N)$, and individually rational in the original game (N, v), i.e. $x_i \geq v(\{i\})$, for all $i \in N$. Only the case $i = i_{t^*}$ deserves some attention. Indeed, by (9), we have $\ell^{\theta^*}_{i_{t^*}}(v) > m^{\theta^*}_{i_{t^*}}(v) \geq v(\{i_{t^*}\})$, where the last inequality comes by zero-monotonicity of the game (N, v). Therefore, $x_{i_{t^*}} = \alpha_{i_{t^*}} = \ell^{\theta^*}_{i_{t^*}} - \varepsilon \geq v(\{i_{t^*}\})$. Thus, x is an imputation of the game (N, v).

However, let us argue that $x \notin C(N, v)$. As $\delta(N \setminus I^{\theta^*}) = \varepsilon > 0$, there must exist $i \in N \setminus I^{\theta^*}$ such that $\delta_i > 0$. Moreover, there also must exist $M \in \mathcal{S}^{\theta^*}_{min}(v)$ such that $i \notin M$ (otherwise $i \in I^{\theta^*}$). Hence, $\delta(M \setminus I^{\theta^*}) < \varepsilon$ and thus $v(M) - x(M) = v(M) - \ell^{\theta^*}(v)(M) + \varepsilon - \delta(M \setminus I^{\theta^*}) > v(M) - \ell^{\theta^*}(v)(M) = 0$, where the last equality comes from $M \in \mathcal{S}^{\theta^*}_{min}(v)$. Hence, we obtain that $v(M) - x(M) > 0$ and we conclude $x \notin C(N, v)$.

At this point, it is also important to notice that

$$\text{for any } M \subseteq N \text{ with } v(M) - x(M) > 0, \text{ we have } M \in \mathcal{S}^{\theta^*}(v). \tag{15}$$

To see this, if $M \notin \mathcal{S}^{\theta^*}(v)$ then, by (12), $v(M) - \alpha(M) \leq 0$. Hence, $v(M) - x(M) = v(M) - \alpha(M) - \delta(M \setminus I^{\theta^*}) \leq 0$, which involves a contradiction.

We finally check that x is in the steady bargaining set of the game v. To this aim take $S \subseteq N$ such that $v(S) - x(S) > 0$. We shall prove there exists $M \subseteq N$ such that $M \setminus S \neq \varnothing$, $S \setminus M \neq \varnothing$, $S \cap M \neq \varnothing$ and $v(M) - x(M) \geq v(S) - x(S)$.

First, recall that $S \in \mathcal{S}^{\theta^*}(v)$ (see (15)). Now, let $S' \in \mathcal{S}^{\theta^*}_{min}(v)$ with $S' \subseteq S$ such that

$$v(S') - x(S') \geq v(P) - x(P), \text{ for all } P \in \mathcal{S}^{\theta^*}_{min}(v) \text{ with } P \subseteq S. \tag{16}$$

Since $v(S) - x(S) = v(S) - \alpha(S) - \delta(S \setminus I^{\theta^*}) = v(S) - \ell^{\theta^*}(v)(S) + \varepsilon - \delta(S \setminus I^{\theta^*}) = \varepsilon - \delta(S \setminus I^{\theta^*}) > 0$, we have $\delta(S \setminus I^{\theta^*}) < \varepsilon$. Therefore, there must exist $j \in N \setminus I^{\theta^*}$,

$j \notin S$ with $\delta_j > 0$. On the other hand, by Claim 1, let us select an arbitrary player $i \in S' \setminus I^{\theta^*}$; notice that $i \neq j$ since $j \notin S'$ because $S' \subseteq S$.

Then, from (12) it follows that

$$\begin{aligned} 0 < v(S) - x(S) = v(S) - \alpha(S) - \delta(S \setminus I^{\theta^*}) &= \varepsilon - \delta(S \setminus I^{\theta^*}) \\ &= v(S') - \alpha(S') - \delta(S \setminus I^{\theta^*}) \\ &\leq v(S') - \alpha(S') - \delta(S' \setminus I^{\theta^*}) = v(S') - x(S'). \end{aligned} \tag{17}$$

That is, the excess of coalition $S \subseteq N$ at x is smaller than the excess of coalition $S' \subseteq S$ at x. Taking this fact into account we also have

$$\begin{aligned} 0 < v(S) - x(S) \leq v(S') - x(S') &= v(S') - \alpha(S') - \delta(S' \setminus I^{\theta^*}) \\ &\leq \omega(S' \setminus I^{\theta^*}) - \delta(S' \setminus I^{\theta^*}) \\ &\leq s^{\omega}_{ij}(\delta) = s^{\omega}_{ji}(\delta) = \omega(R) - \delta(R), \end{aligned} \tag{18}$$

for some $R \subseteq N \setminus I^{\theta^*}$ such that $j \in R$ but $i \notin R$. Finally,

$$\begin{aligned} \omega(R) - \delta(R) &= v(R' \cup Q) - \alpha(R' \cup Q) - \delta(R) \\ &\leq v(R' \cup Q) - \alpha(R' \cup Q) - \delta(R') = v(R' \cup Q) - x(R' \cup Q), \end{aligned} \tag{19}$$

for some $R' \subseteq R$ and $Q \subseteq I^{\theta^*}$. The coalition $M = R' \cup Q$ is precisely the one we next use to prove that x is in the steady bargaining set of v. Notice that $i \notin M$ since $i \notin R \cup I^{\theta^*}$.

First, from (18) and (19), we can deduce that $v(S) - x(S) \leq v(M) - x(M)$. Furthermore, since $v(M) - x(M) > 0$ and $v(S) - x(S) > 0$, by (15), it follows that $I^{\theta^*} \subseteq M \cap S$ which implies $M \cap S \neq \varnothing$. Moreover, $i \in S \setminus M$ since $i \in S' \subseteq S$ and $i \notin R'$ (since $i \notin R$) and $i \notin Q$ (since $i \notin I^{\theta^*}$) .

Finally, if $j \in M$ then $j \in M \setminus S$ and thus $M \setminus S \neq \varnothing$. If $j \notin M$, we still claim that $M \setminus S \neq \varnothing$. To check it, let us suppose on the contrary that $M \subseteq S$ and $j \notin M$. Taking into account that $\delta_j > 0$, $j \in R$ and $j \notin M$ the non-strict inequality in (19) becomes strict; that is $\omega(R)-\delta(R) < v(M)-x(M)$ and thus, by (18) and (19), we obtain

$$0 < v(S') - x(S') < v(M) - x(M). \tag{20}$$

By (15), there exists $M' \subseteq M$ where $M' \in \mathcal{S}^{\theta^*}_{min}(v)$. Therefore,

$$v(M)-x(M) = \varepsilon-\delta(M\setminus I^{\theta^*}) \leq v(M')-\alpha(M')-\delta(M'\setminus I^{\theta^*}) = v(M')-x(M'). \tag{21}$$

By (20) and (21) we conclude that $v(S') - x(S') < v(M') - x(M')$ being $M' \in \mathcal{S}^{\theta^*}_{min}(v)$ and $M' \subseteq M \subseteq S$. However, this contradicts (16) and we conclude that $M \setminus S \neq \varnothing$. This last result proves that x is not in the core of v but in its steady bargaining set, and the proof of this implication ends. □

As far as we know the conditions of zero-monotonicity and gran coalition superadditivity cannot be dropped from Theorem 1 out.

Concerning the zero-monotonicity condition, next example proves its necessity. Let (N, v) be a four-player game where $N = \{1, 2, 3, 4\}$ and $v(S) = 1$, if $|S| = 1$, $v(S) = 2$, if $|S| = 2$ or $|S| = 3$ and $v(N) = 4$. It is easy to see that (N, v) is neither convex nor zero-monotonic, but $C(N, v) = \mathcal{SB}(N, v) = \{(1, 1, 1, 1)\}$.

Grand coalition superadditivity is needed to guarantee that the steady bargaining set is nonempty. It remains an open question whether zero-monotonicity implies the nonemptiness of the steady bargaining set.

As a consequence of Theorem 1 and the characterization result of Izquierdo and Rafels (2012) we get the following corollary.

Corollary 1 *Let $v \in \mathcal{G}^N$ be a zero-monotonic and grand coalitional superadditive game. Then, the following statements are equivalent:*

1. *$\mathcal{Z}^*(N, v) = C(N, v)$.*
2. *v is a convex game.*

4 Conclusions

Bargaining sets face the problem of distributing profits focusing on the negotiation (objections and counterobjections) between agents. Besides this, there are concepts of bargaining sets (e.g. Davis and Maschler (1963, 1967) or Shubik (1984)) that put the stress on the player who leads the objection. For these bargaining sets, there are examples of non-convex cooperative games for whom the core and the bargaining set do coincide (for instance, this is the case of *average monotonic cooperative games* (Izquierdo and Rafels, 2001), or *assignment games* (for the proof of this coincidence see Solymosi(2008)).

In Izquierdo and Rafels (2012), it has been already shown that a modification of the Mas-Colell bargaining set (Shimomura 1997) has been useful to characterize the convexitiy of a game This notion of bargaining set considers objections and counterobjections as proposals made by a group rather than an action led by an specific player. It is also important to remark that agents receive strictly better rewards in objections and counterobjections. Following this idea of group proposals and strictly positive incentives, we have proved in this paper that the modified Zhou bargaining set also characterizes convex games within the class of zero-monotonic and grand coalition superadditive games. The difference between both bargaining sets relies on the qualification of coalitions than might counter-object: while in the Mas-Colell version there are no restrictions on which are the coalitions T that are allowed to react to an objection made by a coalition S, the Zhou's framework requires some conditions. First, there must be at least one player belonging to both coalitions; if not, $S \cap T = \varnothing$, and the counterobjection might be interpreted as a different objection rather than a proper counter-objection.

Second, at least one player involved in the objection must not be involved in the counterobjection; if not, $S \subseteq T$, and the counterobjection might be interpreted as a reinforcement to the objection. Finally, the counterobjecting coalition must involved at least an agent not taking part in the objection; if not, $T \subseteq S$, but this fact might suggest that the original objection should be revised but not rejected. From the point of view of characterizing convex games, our result reveals that it is not so important if we just consider one, two, three or none of the above requirements for the counterobjecting coalitions. Objections and counter-objections made as a group and strictly positive incentives are the important keys to reach these results.

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Appendix

Proof of **Claim 1**.

Let us recall that the claim is under the hypothesis $C(N,v) = \mathcal{SB}(N,v)$. Next, assume $|\mathcal{S}^{\theta^*}_{min}(v)| = 1$, say $\mathcal{S}^{\theta^*}_{min}(v) = \{S^*\}$, where $\theta^* = (i_1, i_2, \ldots, i_n)$. Then, we shall prove that there exists $x \in \mathcal{SB}(N,v)$ but $x \notin C(N,v)$, which contradicts the hypothesis of the coincidence of the core and the steady bargaining set. Along the proof of this claim we will analyze and prove several subclaims.

Subclaim 1.1 *The coalition S^* is included in $T^* = \{i_1, \ldots, i_{t^*}\}$.*

Proof Let us suppose that there exists $i_{t'} \in S^*$ with $t' \in \{t^*+1, t^*+2, \ldots, n\}$. Then, define the vector $x \in \mathbb{R}^N$ as $x_{i_{t^*}} = \ell^{\theta^*}_{i_{t^*}}(v) - \varepsilon_2$, $x_{i_{t'}} = \ell^{\theta^*}_{i_{t'}}(v) + \varepsilon_2$ and $x_{i_k} = \ell^{\theta^*}_{i_k}(v)$, else, where

$$0 < \varepsilon_2 < \min_{\substack{i_{t^*} \in M \subseteq N \\ \ell^{\theta^*}(v)(M) > v(M)}} \{\ell^{\theta^*}(v)(M) - v(M)\}.$$

By (11), the parameter ε_2 is well-defined. To end the proof, we show that $x \in C(N,v)$ contradicting $\ell^{\theta^*}(v)$ to be the lexmin core vector of v relative to θ^*. To see this point, it is straightforward that $x(N) = v(N)$. Moreover, if $M \subseteq N$, $M \neq N$, and $i_{t^*} \notin M$ then $x(M) \geq \ell^{\theta^*}(v)(M) \geq v(M)$. If $i_{t^*} \in M$ and $\ell^{\theta^*}(v)(M) > v(M)$, then $x(M) \geq \ell^{\theta^*}(v)(M) - \varepsilon_2 > \ell^{\theta^*}(v)(M) - (\ell^{\theta^*}(v)(M) - v(M)) = v(M)$. Finally, if $i_{t^*} \in M$ and $\ell^{\theta^*}(v)(M) = v(M)$ then $M \in \mathcal{S}^{\theta}(v)$, and $S^* \subseteq M$ since there is a unique minimal coalition in $\mathcal{S}^{\theta^*}(v)$. Thus $i_{t'} \in M$. Hence, $x(M) = \ell^{\theta^*}(v)(M) \geq v(M)$ and $x \in C(N,v)$, ending the proof of this subclaim. □

Subclaim 1.2 *The number of players in $T^* = \{i_1, \ldots, i_{t^*}\}$ is at least three, i.e. $t^* \geq 3$.*

Proof It is clear that if $t^* = 1$ then $T^* = \{i_1\}$ and the unique minimal coalition in $\mathcal{S}^{\theta^*}(v)$ must be $S^* = \{i_1\}$. Then, by (9), $\ell^{\theta^*}_{i_1}(v) > m^{\theta^*}_{i_1}(v) = v(\{i_1\})$ which contradicts $S^* \in \mathcal{S}^{\theta^*}(v)$. Moreover if $t^* = 2$ then $T^* = \{i_1, i_2\}$, $\ell^{\theta^*}_{i_1}(v) = m^{\theta^*}_{i_1}(v) = v(\{i_1\})$ by (8), and $\ell^{\theta^*}_{i_2}(v) > m^{\theta^*}_{i_2}(v) = v(\{i_1, i_2\}) - v(\{i_1\}) \geq v(\{i_2\})$, where the last inequality comes from zero-monotonicity. From this we deduce $\ell^{\theta^*}_{i_1}(v) + \ell^{\theta^*}_{i_2}(v) > v(\{i_1, i_2\})$, which contradicts the fact that the unique minimal coalition in $\mathcal{S}^{\theta^*}(v)$ must be a subset of T^* (see Subclaim 1.1.). $\square$

Let us recall (see (10)) that any subgame (R, v_R), with $R \subseteq T^*$, $R \neq T^*$, is a convex game. Therefore, the maximal marginal contribution of player $i_{t^*} \in N$ to any subcoalition[6] $Q \subsetneq T^* \setminus \{i_{t^*}\}$ is attained at a coalition containing $t^* - 2$ players; that is, without loss of generality

$$\max_{Q \subsetneq T^* \setminus \{i_{t^*}\}} \{v(Q \cup \{i_{t^*}\}) - v(Q)\} = v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - v(\{i_1, i_2, \ldots, i_{t^*-2}\}).$$

Subclaim 1.3 $\ell^{\theta^*}_{i_{t^*}}(v) = v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - v(\{i_1, i_2, \ldots, i_{t^*-2}\})$.

Proof First, by (8), if $\ell^{\theta^*}_{i_{t^*}}(v) < v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - v(\{i_1, i_2, \ldots, i_{t^*-2}\}) = v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - \ell^{\theta^*}(v)(\{i_1, i_2, \ldots, i_{t^*-2}\})$, then

$$\ell^{\theta^*}(v)(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) < v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}),$$

which contradicts $\ell^{\theta^*}(v) \in C(N, v)$.

On the other hand, if $\ell^{\theta^*}_{i_{t^*}}(v) > v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - v(\{i_1, i_2, \ldots, i_{t^*-2}\})$, then, since $\ell^{\theta^*}(v) \in C(N, v)$,

$$\begin{aligned} \ell^{\theta^*}_{i_{t^*}}(v) &> v(\{i_1, i_2, \ldots, i_{t^*-2}, i_{t^*}\}) - v(\{i_1, i_2, \ldots, i_{t^*-2}\}) \\ &= \max_{Q \subsetneq T^* \setminus \{i_{t^*}\}} \{v(Q \cup \{i_{t^*}\}) - v(Q)\} \geq \max_{Q \subsetneq T^* \setminus \{i_{t^*}\}} \{v(Q \cup \{i_{t^*}\}) - \ell^{\theta^*}(v)(Q)\}. \end{aligned}$$

Thus, $\ell^{\theta^*}(v)(Q \cup \{i_{t^*}\}) > v(Q \cup \{i_{t^*}\})$, for all $Q \subsetneq \{i_1, i_2, \ldots, i_{t^*-1}\}$. However adding this result to (11) we reach a contradiction with Subclaim 1.1. $\square$

Next, let us define

$$J^{\theta^*} = \{i \in T^* = \{i_1, i_2, \ldots, i_{t^*}\} \mid \ell^{\theta^*}(v)(T^* \setminus \{i\}) = v(T^* \setminus \{i\})\}.$$

[6] The symbol $\subsetneq$ between two coalitions $S \subsetneq T$ means $S \subseteq T$ and $S \neq T$.

Notice that, by (8) (i), $\ell^{\theta^*}(v)(T^* \setminus \{i_{t^*}\}) = m^{\theta^*}(v)(T^* \setminus \{i_{t^*}\}) = v(T^* \setminus \{i_{t^*}\})$ and so $i_{t^*} \in J^{\theta^*}$. Furthermore, by Subclaim 1.3 it follows that $i_{t^*-1} \in J^{\theta^*}$ and thus $\{i_{t^*-1}, i_{t^*}\} \subseteq J^{\theta^*}$. Therefore,

$$|J^{\theta^*}| \geq 2. \tag{22}$$

Finally, by zero-monotonicity of the game v, it holds

$$\ell_i^{\theta^*}(v) > v(\{i\}), \text{ for all } i \in J^{\theta^*}. \tag{23}$$

To check this last point, notice that, by zero-monotonicity of v, if $\ell_i^{\theta^*}(v) = v(\{i\})$ then $\ell^{\theta^*}(v)(T^*) = \ell_i^{\theta^*}(v) + \ell^{\theta^*}(v)(T^* \setminus \{i\}) = v(\{i\}) + v(T^* \setminus \{i\}) \leq v(T^*)$ which contradicts (11). Next, let us prove the following subclaim.

Subclaim 1.4 *For all $S \subseteq T^*$ such that $\ell^{\theta^*}(v)(S) = v(S)$, then $J^{\theta^*} \setminus S \neq \varnothing$.*

Proof Since $i_{t^*} \in J^{\theta^*}$, the result is trivial if $i_{t^*} \notin S$. If $i_{t^*} \in S$, let $\kappa \in \{1, \ldots, t^*-1\}$ such that $i_\kappa \notin S$ and $i_{\kappa+1}, i_{\kappa+2}, \ldots, i_{t^*} \in S$. Notice that the index κ is well-defined since, by (11), we have $S \neq T^*$. We next prove that $i_\kappa \in J^{\theta^*} \setminus S$. To see this, first notice that

$$\begin{aligned} v(T^* \setminus \{i_\kappa\}) \leq \ell^{\theta^*}(v)(T^* \setminus \{i_\kappa\}) &= \ell^{\theta^*}(v)(T^* \setminus (S \cup \{i_\kappa\})) + \ell^{\theta^*}(v)(S) \\ &= m^{\theta^*}(v)(T^* \setminus (S \cup \{i_\kappa\})) + \ell^{\theta^*}(v)(S). \end{aligned} \tag{24}$$

Notice that $T^* \setminus (S \cup \{i_\kappa\}) \subseteq \{i_1, i_2, \ldots, i_{\kappa-1}\}$ and thus we describe $T^* \setminus (S \cup \{i_\kappa\}) = \{i_{r_1}, i_{r_2}, \ldots, i_{r_m}\}$, where $r_1 < r_2 < \cdots < r_m < \kappa$. Moreover, and for all $i_{r_j} \in T^* \setminus (S \cup \{i_\kappa\})$, let us denote by $P_{i_{r_j}}^{\theta^*} = \{i_1, i_2, \ldots, i_{(r_j)-1}\} \subseteq N$ the set of predecessors of player i_{r_j} relative to the ordering

$$\theta^* = (i_1, i_2, \ldots, i_{r_1}, \ldots, i_{r_2}, \ldots, i_{r_j}, \ldots, i_\kappa, i_{\kappa+1}, \ldots, i_n).$$

Then, we have

$$
\begin{aligned}
&m^{\theta^*}(v)(T^* \setminus (S \cup \{i_K\})) = \sum_{j=1}^{m} m^{\theta^*}_{r_j}(v) = \sum_{j=1}^{m} [v(P^{\theta^*}_{i_{r_j}} \cup \{i_{r_j}\}) - v(P^{\theta^*}_{i_{r_j}})] \\
&\leq \sum_{j=1}^{m-1} [v((P^{\theta^*}_{i_{r_m}} \setminus \{i_{r_j}, i_{r_{j+1}}, \ldots, i_{r_{m-1}}\}) \cup \{i_{r_j}\}) - v(P^{\theta^*}_{i_{r_m}} \setminus \{i_{r_j}, i_{r_{j+1}}, \ldots, i_{r_{m-1}}\})] \\
&\quad + v(P^{\theta^*}_{i_{r_m}} \cup \{i_{r_m}\}) - v(P^{\theta^*}_{i_{r_m}}) \\
&= v(P^{\theta^*}_{i_{r_m}} \cup \{i_{r_m}\}) - v(P^{\theta^*}_{i_{r_m}} \setminus \{i_{r_1}, i_{r_2}, \ldots, i_{r_{m-1}}\}) \\
&\leq v(P^{\theta^*}_{i_{r_m}} \cup \{i_{t^*}\} \cup \{i_{r_m}\}) - v((P^{\theta^*}_{i_{r_m}} \cup \{i_{t^*}\}) \setminus \{i_{r_1}, i_{r_2}, \ldots, i_{r_{m-1}}\}) \\
&\leq v(T^* \setminus \{i_K\}) - v(T^* \setminus \{i_{r_1}, i_{r_2}, \ldots, i_{r_m}, i_K\}) \\
&= v(T^* \setminus \{i_K\}) - v(S),
\end{aligned}
$$

where the first inequality follows from (2), the convexity of the subgame $(T^* \setminus \{i_K\}, v_{T^* \setminus \{i_K\}})$ and the fact that, for all $j = 1, \ldots, m-1$, we have

$$P^{\theta^*}_{i_{r_j}} \subseteq P^{\theta^*}_{i_{r_m}} \setminus \{i_{r_j}, i_{r_{j+1}}, \ldots, i_{r_{m-1}}\},$$

the second inequality follows from the convexity of the subgame $(T^* \setminus \{i_K\}, v_{T^* \setminus \{i_K\}})$, and the third one by taking in (3) $M = P^{\theta^*}_{i_{r_m}} \cup \{i_{t^*}\} \cup \{i_{r_m}\}$ and $M' = T^* \setminus \{i_{r_1}, i_{r_2}, \ldots, i_{r_m}, i_K\}$. Therefore, we obtain that $m^{\theta^*}(v)(T^* \setminus (S \cup \{i_K\})) \leq v(T^* \setminus \{i_K\}) - v(S)$. Using this inequality in (24) we obtain

$$
\begin{aligned}
v(T^* \setminus \{i_K\}) \leq \ell^{\theta^*}(v)(T^* \setminus \{i_K\}) &\leq m^{\theta^*}(v)(T^* \setminus (S \cup \{i_K\})) + \ell^{\theta^*}(v)(S) \\
&\leq v(T^* \setminus \{i_K\}) - v(S) + \ell^{\theta^*}(v)(S) = v(T^* \setminus \{i_K\}).
\end{aligned}
$$

Therefore, we conclude that $v(T^* \setminus \{i_K\}) = \ell^{\theta^*}(v)(T^* \setminus \{i_K\})$ which implies $i_K \in J^{\theta^*} \setminus S$, as we want to prove. □

Once we have proved the above subclaims, let us define the vector $\beta \in \mathbb{R}^N$ as

$$
\beta_i = \begin{cases} \ell^{\theta^*}_i(v) - \varepsilon_3 & \text{if } i \in J^{\theta^*} \\ \ell^{\theta^*}_i(v) & \text{if } i \in N \setminus J^{\theta^*}, \end{cases}
$$

where

$$0 < n \cdot \varepsilon_3 < \min_{\substack{M \subseteq N \\ \ell^{\theta^*}(v)(M) > v(M)}} \{\ell^{\theta^*}(v)(M) - v(M)\}.$$

By (23),

$$\beta_i \geq v(\{i\}), \text{ for all } i \in J^{\theta^*}. \tag{25}$$

Hence, define the game $(N \setminus T^*, \omega)$ as follows:

$$\omega(\varnothing) = 0$$
$$\omega(R) = \max_{Q \subseteq T^*} \{v(R \cup Q) - \beta(R \cup Q)\}, \text{ for all } \varnothing \neq R \subseteq N \setminus T^*.$$

Let us remark that $\omega(R) \leq |J^{\theta^*}| \cdot \varepsilon_3$, for any $\varnothing \neq R \subseteq N \setminus T^*$. To check it, simply notice that $\omega(R) = v(R \cup Q^*) - \beta(R \cup Q^*)$ for some $Q^* \subseteq T^*$, and thus $\omega(R) = v(R \cup Q^*) - \beta(R \cup Q^*) = v(R \cup Q^*) - \ell^{\theta^*}(R \cup Q^*) + |Q^* \cap J^{\theta^*}| \cdot \varepsilon_3 \leq |Q^* \cap J^{\theta^*}| \cdot \varepsilon_3 \leq |J^{\theta^*}| \cdot \varepsilon_3$. Moreover, for the case $R = N \setminus T^*$ we have $\omega(N \setminus T^*) = |J^{\theta^*}| \cdot \varepsilon_3$, just by taking $Q = T^*$ in its definition.

Next, define the subset Y of vectors in $\mathbb{R}^{N \setminus T^*}$ as follows:

$$Y = \{\alpha \in \mathbb{R}^{N \setminus T^*} \mid \alpha_i \geq 0, \text{ for all } i \in N \setminus T^* \text{ and } \alpha(N \setminus T^*) = \omega(N \setminus T^*) = |J^{\theta^*}| \cdot \varepsilon_3\}.$$

Notice that Y is a non-empty and compact subset of the preimputation set $I^*(N \setminus T^*, \omega)$, and thus, by Schmeidler (1969), the kernel[7] of the game $(N \setminus T^*, \omega)$ relative to Y is non-empty, i.e. $\mathcal{K}(N \setminus T^*, \omega, Y) \neq \varnothing$.

[7] Notice that the set Y is a non-empty box since it can be rewritten as

$$Y = \{\alpha \in \mathbb{R}^{N \setminus T^*} \mid 0 \leq \alpha_i \leq |J^{\theta^*}| \cdot \varepsilon_3, \text{ for all } i \in N \setminus T^*, \text{ and } \alpha(N \setminus T^*) = |J^{\theta^*}| \cdot \varepsilon_3\}.$$

.

Hence, select an element δ in the kernel of the game $(N \setminus T^*, \omega)$ relative to Y, i.e. $\delta \in \mathcal{K}(N \setminus T^*, \omega, Y)$, and define the vector $x \in \mathbb{R}^N$ as follows:

$$x_i = \begin{cases} \beta_i + \delta_i = \ell_i^{\theta^*}(v) + \delta_i & \text{if } i \in N \setminus T^* \\ \beta_i \qquad\;\; = \ell_i^{\theta^*}(v) & \text{if } i \in T^* \setminus J^{\theta^*} \\ \beta_i \qquad\;\; = \ell_i^{\theta^*}(v) - \varepsilon_3 & \text{if } i \in J^{\theta^*}. \end{cases}$$

The vector x is an imputation of the game (N, v): clearly, x is efficient, $x(N) = v(N)$; moreover, by definition of ε_3 and (23), we have $x_i = \ell_i^{\theta^*}(v) - \varepsilon_3 \geq v(\{i\})$, for all $i \in J^{\theta^*}$, $x_i = \ell_i^{\theta^*}(v) \geq v(\{i\})$, for all $i \in T^* \setminus J^{\theta^*}$ and, since $\delta_i \geq 0$, $x_i = \beta_i + \delta_i = \ell_i^{\theta^*}(v) + \delta_i \geq v(\{i\})$, for all $i \in N \setminus T^*$.

However, it is not in the core of the game (N, v) since $x(T^* \setminus \{i_{t^*}\}) = \ell^{\theta^*}(v)(T^* \setminus \{i_{t^*}\}) - (|J^{\theta^*}| - 1)\varepsilon_3 = v(T^* \setminus \{i_{t^*}\}) - (|J^{\theta^*}| - 1)\varepsilon_3 < v(T^* \setminus \{i_{t^*}\})$.

We finally check that x is in the steady bargaining set of the game (N, v). To this aim take $S \subseteq N$ such that $v(S) - x(S) > 0$. Notice that, since $x \in I(N, v)$, then $|S| \geq 2$. Furthermore, it holds that

$$S \cap J^{\theta^*} \neq \varnothing, \tag{26}$$

since otherwise $S \cap J^{\theta^*} = \varnothing$ and we would have

$$\begin{aligned} v(S) - x(S) &= v(S) - \beta(S) - \delta(S \cap (N \setminus T^*)) \leq v(S) - \beta(S) \\ &= v(S) - \ell^{\theta^*}(v)(S) \leq 0, \end{aligned}$$

reaching a contradiction with $v(S) - x(S) > 0$. Next, we shall prove there exists $M \subseteq N$ such that $M \setminus S \neq \varnothing$, $S \setminus M \neq \varnothing$, $S \cap M \neq \varnothing$ and $v(M) - x(M) \geq v(S) - x(S)$. We distinguish two cases.

A: $S \subseteq T^* = \{i_1, \ldots, i_{t^*}\}$. By the way we have defined ε_3, and being $S \subseteq T^*$, let us first see that $\ell^{\theta^*}(v)(S) = v(S)$. To check it, let us suppose that $\ell^{\theta^*}(v)(S) > v(S)$, then $v(S)-x(S)=v(S)-\ell^{\theta^*}(v)(S)+|S\cap J^{\theta^*}|\cdot\varepsilon_3 \leq v(S)-\ell^{\theta^*}(v)(S)+n\cdot\varepsilon_3<0$, which contradicts the hypothesis $v(S)-x(S)>0$. Moreover, by (26), $S\cap J^{\theta^*}\neq\varnothing$, and, by Subclaim 1.4, $J^{\theta^*}\setminus S\neq\varnothing$. Let $j\in J^{\theta^*}\setminus S$ and $i\in J^{\theta^*}\cap S$ and take $M=T^*\setminus\{i\}$. Notice that $j\in M\setminus S$, $i\in S\setminus M$ and, since $|S|\geq 2$, $M\cap S\neq\varnothing$. Furthermore, since $i\in J^{\theta^*}$ we have $\ell^{\theta^*}(v)(M)=v(M)$, and thus

$$v(M)-x(M)=v(M)-\ell^{\theta^*}(v)(M)+(|J^{\theta^*}|-1)\cdot\varepsilon_3$$

$$=v(S)-\ell^{\theta^*}(v)(S)+(|J^{\theta^*}|-1)\cdot\varepsilon_3\geq v(S)-x(S),$$

where the inequality follows since $j\in J^{\theta^*}\setminus S$.

B: $S\cap(N\setminus T^*)\neq\varnothing$. First let us remark that $S\cap(N\setminus T^*)\neq N\setminus T^*$, or equivalently $N\setminus(T^*\cup S)\neq\varnothing$; this holds since, otherwise, $S\cap(N\setminus T^*)=N\setminus T^*$ and

$$\begin{aligned}v(S)-x(S)&=v(S)-\beta(S)-\delta(N\setminus T^*)=v(S)-\beta(S)-\omega(N\setminus T^*)\\&=v(S)-\beta(S)-|J^{\theta^*}|\cdot\varepsilon_3\\&=v(S)-\ell^{\theta^*}(v)(S)+|S\cap J^{\theta^*}|\cdot\varepsilon_3-|J^{\theta^*}|\cdot\varepsilon_3\leq 0,\end{aligned}\tag{27}$$

reaching a contradiction.

Hence, let $i\in S\cap(N\setminus T^*)$ and select $j\in(N\setminus T^*)\setminus S=N\setminus(T^*\cup S)$ such that

$$s^{\omega}_{ji}(\delta)\geq s^{\omega}_{ij}(\delta).\tag{28}$$

Let us prove that such a player j exists. To check it, suppose that, given an arbitrary $k\in(N\setminus T^*)\setminus S$, we would have $s^{\omega}_{ki}(\delta)<s^{\omega}_{ik}(\delta)$. Since $\delta\in\mathcal{K}(N\setminus T^*,\omega,Y)$ then we would have that either $\delta_k=0$ or $\delta_i=|J^{\theta^*}|\cdot\varepsilon_3$. However, $\delta_i=|J^{\theta^*}|\cdot\varepsilon_3$ is not possible since, by a similar reasoning as in (27), we would reach a contradiction with $v(S)-x(S)>0$. Therefore, we obtain that $\delta_k=0$. Since k was chosen arbitrarily, we would conclude that $\delta_k=0$ for all $k\in(N\setminus T^*)\setminus S$

and thus

$$|J^{\theta^*}| \cdot \varepsilon_3 = \omega(N \setminus T^*) = \delta(N \setminus T^*) = \delta((N \setminus T^*) \cap S).$$

But then,

$$\begin{aligned} v(S) - x(S) &= v(S) - \beta(S) - \delta(S \cap (N \setminus T^*)) \\ &= v(S) - \ell^{\theta^*}(v)(S) + |S \cap J^{\theta^*}| \cdot \varepsilon_3 - |J^{\theta^*}| \cdot \varepsilon_3 \leq v(S) - \ell^{\theta^*}(v)(S) \leq 0, \end{aligned}$$

getting a contradiction with $v(S) - x(S) > 0$.

Now, by definition and taking agents i and j as in (28), we have

$$s^{\omega}_{ji}(\delta) = \omega(R^*) - \delta(R^*) = v(R^* \cup Q^*) - \beta(R^* \cup Q^*) - \delta(R^*) = v(R^* \cup Q^*) - x(R^* \cup Q^*),$$

for some $R^* \subseteq N \setminus T^*$, with $j \in R^*$ but $i \notin R^*$, and some $Q^* \subseteq T^*$. Hence, by (28), it follows that

$$\begin{aligned} v(R^* \cup Q^*) - x(R^* \cup Q^*) &= s^{\omega}_{ji}(\delta) \geq s^{\omega}_{ij}(\delta) \\ &\geq \omega(S \cap (N \setminus T^*)) - \delta(S \cap (N \setminus T^*)) \\ &\geq v(S) - x(S) > 0. \end{aligned} \tag{29}$$

Notice that $i \in S \setminus (R^* \cup Q^*)$ and $j \in (R^* \cup Q^*) \setminus S$. Furthermore, if $S \cap (R^* \cup Q^*) \neq \varnothing$, then take $M = R^* \cup Q^*$ and we are done. Otherwise, in case $S \cap (R^* \cup Q^*) = \varnothing$ we have, by (26), $(R^* \cup Q^*) \cap J^{\theta^*} \neq \varnothing$.

Hence, since we are supposing $S \cap (R^* \cup Q^*) = \varnothing$, $(R^* \cup Q^*) \cap J^{\theta^*} \neq \varnothing$ and $S \cap J^{\theta^*} \neq \varnothing$ (see (26)), we conclude that

$$S \cap J^{\theta^*} \subsetneq J^{\theta^*}. \tag{30}$$

Therefore,

$$\begin{aligned} v(S) - x(S) &= v(S) - \beta(S) - \delta(S \cap (N \setminus T^*)) \\ &= v(S) - \ell^{\theta^*}(v)(S) + |S \cap J^{\theta^*}| \cdot \varepsilon_3 - \delta(S \cap (N \setminus T^*)) \\ &\leq v(S) - \ell^{\theta^*}(v)(S) + |S \cap J^{\theta^*}| \cdot \varepsilon_3 \\ &\leq (|J^{\theta^*}| - 1) \cdot \varepsilon_3. \end{aligned}$$

Hence, it easily follows that

$$v(S) - x(S) \leq (|J^{\theta^*}| - 1) \cdot \varepsilon_3 = v(T^* \setminus \{k\}) - x(T^* \setminus \{k\}) \tag{31}$$

for all $k \in J^{\theta^*}$. Finally, by (26) and the fact that $J^{\theta^*} \subseteq T^*$, we have $S \cap T^* \neq \varnothing$. At this point we distinguish two cases:

- **B.1** If $|S \cap T^*| = 1$, i.e. $S \cap T^* = \{i'\}$, then take $M = T^* \setminus \{k\}$ where $k \in J^{\theta^*} \setminus S$ (such a player exists since $|J^{\theta^*}| \geq 2$, see (22)). In this subcase, $i' \in M \cap S$, $M \setminus S \neq \varnothing$, since by Subclaim 1.2, $t^* \geq 3$, and $S \setminus M \neq \varnothing$, by the hypothesis of case **B**.
- **B.2** If $|S \cap T^*| \geq 2$, then take $M = T^* \setminus \{k\}$ where $k \in J^{\theta^*} \cap S$ (such a player exists by (26)). In this subcase, $M \cap S \neq \varnothing$ since $|S \cap T^*| \geq 2$, $M \setminus S \neq \varnothing$ since, by (30), $S \cap J^{\theta^*} \subsetneq J^{\theta^*}$, and $S \setminus M \neq \varnothing$, by the hypothesis of case **B**.

In both cases **B.1** and **B.2**, $M = T^* \setminus \{k\}$, for some $k \in J^{\theta^*}$. Thus, by (31), we are done.

From both cases **A** and **B**, we have shown that $x \notin C(N, v)$, but $x \in \mathcal{SB}(N, v)$, getting a contradiction with the hypothesis $C(N, v) = \mathcal{SB}(N, v)$. □