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RESEARCH ARTICLE

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Component efficient solutions in line-graph games with applications

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Abstract Recently, applications of cooperative game theory to economic allocation problems have gained popularity. We investigate a class of cooperative games that generalizes some economic applications with a similar structure. These are the so-called line-graph games being cooperative TU-games in which the players are linearly ordered. Examples of situations that can be modeled like this are sequencing situations and water distribution problems. We define four properties with respect to deleting edges that each selects a unique component efficient solution on the class of line-graph games. We interpret these solutions and properties in terms of dividend distributions, and apply them to economic situations.

Keywords TU game · Line graph · Component efficiency · Shapley value · Harsanyi dividends

JEL Classfication Number C71

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1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N = \{1, \dots, n\}$ is a finite set of n players and $v: 2^N \to \mathbb{R}$ is a *characteristic function* on N such that $v(\emptyset) = 0$. In this paper we assume that N is a fixed set of players, allowing to denote a TU-game (N, v) shortly by its characteristic function v. For any coalition $S \subseteq N$, the real number $v(S)$ is the *worth* of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. A *payoff vector* of an n-person TU-game is an n-dimensional vector giving a payoff to any player $i \in N$. A *point-valued solution* (shortly refered to as *solution*) is a function f that assigns a single payoff vector $f(v) \in \mathbb{R}^n$ to any game v. A solution f is *efficient* if it for any game v precisely distributes the worth $v(N)$ of the grand coalition. An example of an efficient solution is the famous *Shapley value*, see Shapley (1953), being the average of the so-called *marginal value vectors*.

In standard cooperative game theory it is assumed that any coalition of players may form. On the other hand, in many situations the collection of possible coalitions is restricted by some social, hierarchical, economical, communicational or technical structure. Examples are *games in coalition structure* (see, e.g. Aumann and Drèze 1974; Owen 1977), *games with communication structure* (see, e.g. Myerson 1977; Owen 1986; Borm et al. 1994), *games with permission structure* (see Gilles et al. 1992; van den Brink and Gilles 1996; van den Brink 1997) and more general models of games restricted on combinatorial structures (see, e.g. Bilbao (2000)). In this paper we restrict ourselves to a special type of games with limited communication structure, called *line-graph games*, in which the communication (graph) structure is given by a linear ordering on the set of players. Following Myerson (1977) and Greenberg and Weber (1986), in such a line-graph game only consecutive players can communicate with each other.

For the class of line-graph games we define four different properties related to deleting edges in the graph. The first property is the fairness property introduced already by Myerson (1977) stating that deleting the edge between two players hurts (or benefits) both of them equally. The second property is upper equivalence stating that the payoff of a player does not depend on the presence of downward edges. The third property is lower equivalence meaning that the payoff of a player does not depend on the presence of upward edges. Finally, the equal loss property is an alternative to fairness, but instead of equalizing the change in individual payoffs for the players on the deleted edge, it states that the total payoff of the players at both sides of the deleted edge change by the same amount.

Together with component efficiency, each of these four properties uniquely selects a solution on the class of line-graph games. The first solution is the well-known Myerson value. The other three solutions are always in the core of the restricted game if the original game is superadditive. It also follows that each solution is a so-called Harsanyi solution, see Vasil'ev (1982, 2003) being solutions that distribute the *Harsanyi dividend* (Harsanyi 1959) of each coalition S among the players in S. Finally, these four solutions will be applied to sequencing games, see, e.g. Curiel (1988) and Curiel et al. (1993, 1994), and the water distribution problem, see Ambec and Sprumont (2002).

This paper is organised as follows: section 2 is a preliminary section containing concepts from cooperative game theory, including the concept of line-graph game. In section 3 we define the four edge properties for point-valued solutions on the class of line-graph games and we prove that each of these properties uniquely determines a component efficient solution on this class. In section 4 we discuss these four solutions in terms of the distribution of the Harsanyi dividends. In section 5 we discuss the applications mentioned above.

2 Preliminaries

We denote the collection of all TU-games on N (represented by their characteristic function) by $\mathcal{G}$. A TU-game v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of subsets $S, T \subseteq N$ such that $S \cap T = \emptyset$. Further, a TU-game v is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. A special class of convex games are unanimity games. For each nonempty $T \subseteq N$, the *unanimity* game u^{T} is given by $u^{\mathrm{T}}(S) = 1$ if $T \subseteq S$, and $u^{\mathrm{T}}(S) = 0$ otherwise. It is well-known that the unanimity games form a basis for $\mathcal{G}$. Denoting the collection of all nonempty subsets of N by Ω, it holds that $v = \sum_{T \in \Omega} \Delta^{\mathrm{T}}(v) u^{\mathrm{T}}$. Since

$$v(S) = \sum_{T \subseteq S} \Delta^{\mathrm{T}}(v) \qquad (1)$$

the Möbius inversion formula implies that

$$\Delta^{\mathrm{T}}(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S), \quad T \in \Omega.$$

Following Harsanyi (1959), we call $\Delta^S(v)$ the Harsanyi dividend of S in v. According to equation (1), the worth of a coalition S is equal to the dividend of S plus the sum of the dividends off all its proper subcoalitions. The dividend of a coalition S thus can be interpreted as the additional contribution of the cooperation among the players in S, that they did not already realize by cooperating in smaller coalitions.

For a permutation $\pi : N \to N$, assigning rank number $\pi(i) \in N$ to any player $i \in N$, we define $\pi^i = \{j \in N | \pi(j) \leq \pi(i)\}$, i.e., π^i is the set of all players with rank number at most equal to the rank number of i, including i itself. Then the *marginal value vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by $m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\})$, $i \in N$. The game v is convex if and only if the Core of v is equal to the convex hull of all marginal value vectors (see Shapley 1971; Ichiishi 1981), where the *Core* is the set-valued solution assigning to game v the (possibly empty) set

$$C(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n x_i = v(N), \quad \text{and} \quad \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N \right\}.$$

One of the most applied point-valued solutions for cooperative TU-games is the Shapley value, which is applied to economic allocation problems in Graham et al. (1990), Maniquet (2003) and Tauman and Watanabe (2006). The *Shapley value* (Shapley 1953) is the solution ψ that assigns to TU-game v the payoff vector $\psi(v)$, being the average of the marginal value vectors over the set Π of all permutations $\pi : N \rightarrow N$, and thus lies in the Core if the game is convex. Alternatively, the Shapley value can be defined as the solution that in every game equally distributes the Harsanyi dividends among the players in the corresponding coalitions, i.e., $\psi_i(v) = \sum_{\{T \subseteq N | i \in T\}} [\Delta^T(v)/|T|]$ for all $i \in N,\ v \in \mathcal{G}$.

Line-graph games are a special subclass of games with communication (graph) structure. We may assume without loss of generality that the players are ordered according to the natural ordering from 1 to n. The structure on the set of players then is given by a line-graph (N, L) with N the set of players and $L \subseteq L^c = \{\{i, i+1\} | i = 1, \ldots, n-1\}$ the set of *edges*. If $L = L^c$ then each pair of consecutive players is directly connected by an edge. For $i, j \in N$ with $i \leq j$ we denote by $[i, j]$ the set $\{i, i+1, \ldots, j-1, j\} \subseteq N$ of subsequent players. A coalition $S \subseteq N$ is *connected* in the graph (N, L) if and only if $S = [i, j]$ for some $i, j \in N$, $i \leq j$ and $\{k, k+1\} \in L$ for all $k \in [i, j-1]$. A line-graph $L \subset L^c$ consists of different components that each consists of consecutive linearly ordered players, where a set of players S is a *component* if and only if it is connected and $S \cup \{i\}$ is not connected for every $i \in N \setminus S$.

We shortly denote the game v with line-graph (N, L) as the *graph game* (v, L) and the collection of all line-graph games by $\mathcal{G} \times \mathcal{L}$, where $\mathcal{L} = \{L \mid L \subseteq L^c\}$ is the set of all line-graphs on N. Following Myerson (1977) and Greenberg and Weber (1986), in a graph game a coalition $S \subseteq N$ can only realise its worth $v(S)$ when S is connected. When S is not connected, the players in S can only realise the sum of the worths of the components of the subgraph $(S, L(S))$ with $L(S) = \{\{i, i+1\} \in L \mid \{i, i+1\} \subseteq S\}$. So, the resulting *restricted game* induced by the line-graph game (v, L) is given by

$$v^L(S) = \sum_{T \in C_L(S)} v(T), \tag{2}$$

where $C_L(S)$ is the collection of all components of S. The set S itself is the unique component of S if and only if S is connected. From Le Breton et al. (1992) it follows that the restricted game v^L of a superadditive cycle-free graph game has a nonempty Core and thus this holds for all superadditive line-graph games. The nonemptiness of the Core also follows from Granot and Huberman (1982), who showed that a permutational convex game has a nonempty core. More precisely, let u and ℓ be the two permutations on N defined by $u(i) = i, i = 1, \ldots, n$, respectively $\ell(i) = n + 1 - i, i = 1, \ldots, n$. Then it follows that when v is superadditive, the restricted game v^L satisfies the permutational convexity condition of Granot and Huberman for the two permutations u and ℓ, from which it follows that the two marginal value vectors $m^u(v^L)$ and $m^\ell(v^L)$ are in the Core of v^L.

3 Solutions for line-graph games

In this section we first give four properties of solutions on the class $\mathcal{G} \times \mathcal{L}$ of line-graph games. These properties say something about changes in payoffs as a result of deleting an edge from the line-graph. To state the properties, for $L \in \mathcal{L}$ and $i = 1, \ldots, n-1$, let $(N, L(i))$ be the graph on N with $L(i) = L \setminus \{\{i, i+1\}\}$

as the set of edges obtained by deleting the edge $\{i, i+1\}$ from L (if this edge is contained in L). Observe that $L \in \mathcal{L}$ implies that also $L(i) \in \mathcal{L}$.

Definition 3.1 1. *A solution f on $\mathcal{G} \times \mathcal{L}$ is called* fair *if for any $i = 1, \ldots, n-1$, $v \in \mathcal{G}$ and $L \in \mathcal{L}$ it holds that $f_i(v, L) - f_i(v, L(i)) = f_{i+1}(v, L) - f_{i+1}(v, L(i))$.*
2. *A solution f on $\mathcal{G} \times \mathcal{L}$ is called* upper equivalent *if for any $i = 1, \ldots, n-1$, $v \in \mathcal{G}$ and $L \in \mathcal{L}$ it holds that $f_j(v, L(i)) = f_j(v, L)$, $j = 1, \ldots, i$.*
3. *A solution f on $\mathcal{G} \times \mathcal{L}$ is called* lower equivalent *if for any $i = 1, \ldots, n-1$, $v \in \mathcal{G}$ and $L \in \mathcal{L}$ it holds that $f_j(v, L(i)) = f_j(v, L)$, $j = i+1, \ldots, n$.*
4. *A solution f on $\mathcal{G} \times \mathcal{L}$ is said to have the* equal loss property *if for any $i = 1, \ldots, n-1$, $v \in \mathcal{G}$ and $L \in \mathcal{L}$ it holds that $\sum_{j=1}^{i}(f_j(v, L) - f_j(v, L(i))) = \sum_{j=i+1}^{n}(f_j(v, L) - f_j(v, L(i)))$.*

The first property is the fairness property introduced already by Myerson (1977) and states that deleting the edge between i and $i+1$ hurts (or benefits) both players i and $i+1$ equally. Upper equivalence means that the payoff of a player does not depend on the presence of downward edges, while lower equivalence means that the payoff of a player does not depend on the presence of upward edges. The equal loss property is an alternative to fairness, but instead of equalizing the change in individual payoffs for the players on the edge that is deleted, it states that the total payoff of the players at both sides of the deleted edge change by the same amount. Which property is most appropriate depends on the application that is in mind and will be discussed in the sections 4 and 5.

The next theorem says that together with component efficiency, each of these four properties selects a unique solution on the class $\mathcal{G} \times \mathcal{L}$ of line-graph games. A solution is called *component efficient* if the sum of payoffs in any component S in (N, L) exactly equals the worth of that component, i.e.,

$$\sum_{i \in K} f_i(v, L) = v(K), \qquad \text{for all} \quad K \in C_L(N). \tag{3}$$

To state the theorem, let f^s, f^u, f^ℓ and f^e be the solutions on $\mathcal{G} \times \mathcal{L}$ defined by $f^s(v, L) = \psi(v^L)$, $f^u(v, L) = m^u(v^L)$, $f^\ell(v, L) = m^\ell(v^L)$ and $f^e(v) = \frac{1}{2}(m^u(v, L) + m^\ell(v, L))$, for all $(v, L) \in \mathcal{G} \times \mathcal{L}$, which respectively assign the Shapley value of v^L (this solution is also known as the *Myerson value*), the marginal value vector of v^L corresponding to the order $u(i) = i$, the marginal value vector corresponding to the order $l(i) = n + 1 - i$, and the average of these two vectors. Before stating the theorem, we first give the next lemma. Its proof is straightforward and therefore omitted.

Lemma 3.2 *For some permutation π, let $f : \mathcal{G} \times \mathcal{L} \to \mathbb{R}^n$ be the solution on the class $\mathcal{G} \times \mathcal{L}$ given by $f(v, L) = m^\pi(v^L)$. Then f satisfies component efficiency.*

Theorem 3.3 *Let $f : \mathcal{G} \times \mathcal{L} \to \mathbb{R}^n$ be a component efficient solution on the class $\mathcal{G} \times \mathcal{L}$. Then,*

1. *f is fair if and only if $f = f^s$.*
2. *f is upper equivalent if and only if $f = f^u$.*
3. *f is lower equivalent if and only if $f = f^\ell$.*
4. *f satisfies the equal loss property if and only if $f = f^e$.*

Proof Since each solution is a marginal value vector or a convex combination of marginal value vectors, from Lemma 3.2 it follows that each of the four solutions satisfies component efficiency. Below we show that each of the four solutions satisfies the corresponding edge property, and we also prove that each property together with component efficiency yields a unique solution. To do so, let $l = |L|$ be the number of edges and $c = |C_L(N)|$ the number of components of (N, L). Since $L \in \mathcal{L}$, we have that $l + c = n$.

1. That $f^s(v, L) = \psi(v^L)$ is fair follows from Myerson (1977) who characterized this solution by component efficiency and fairness on the class of all graph-games, see van den Brink (2001) for a related result on the class of TU-games. Since this does not imply uniqueness on the subclass of line-graph games, we show that there is a unique solution f on $\mathcal{G} \times \mathcal{L}$ satisfying component efficiency and fairness by induction on the number of edges. [This goes along similar lines as shown by Myerson (1977) on the class of all graph games.] First, let L be empty, i.e., all players are isolated. Then by component efficiency it follows that $f_i(v, L) = v(\{i\})$, $i = 1, \ldots, n$. Next, for some l, $1 \leq l \leq n-1$, suppose that $f(v, L')$ is uniquely determined for each L' with $|L'| \leq l-1$. Let $\{i, i+1\}$ be an edge in (N, L). Then the fairness property says that
$$f_i(v, L) - f_i(v, L(i)) = f_{i+1}(v, L) - f_{i+1}(v, L(i)), \tag{4}$$
where the values $f_h(v, L(i))$, $h = i, i+1$, are known by the induction hypothesis and $|L(i)| = l - 1$. Since there are c equations of type (3) and l equations of type (4) and all the $c + l = n$ equations are linearly independent, these equations uniquely determine $f(v, L)$.
2. To show that f^u is upper equivalent note that for any $v \in \mathcal{G}$ and for all $i = 1, \ldots, n-1$, $j = 1, \ldots, i$, we have by definition of the restricted game that $v^L[1, j] = v^{L(i)}[1, j]$.[1] Hence $f^u_j(v, L) = m^u_j(v^L) = v^L[1, j] - v^L[1, j-1] = v^{L(i)}[1, j] - v^{L(i)}[1, j-1] = m^u_j(v^{L(i)}) = f^u_j(v, L(i))$ for $j = 1, \ldots, i$. So, f^u is upper equivalent.
   Next, suppose that f satisfies component efficiency and the upper equivalent property. Let $\{i, i+1\}$ be an edge in a component K of (N, L) and let K^i be the component in $(N, L(i))$ containing i, i.e., $K^i = \{h \in K | h \leq i\}$. Then component efficiency implies that $\sum_{h \in K^i} f_h(v, L(i)) = v(K^i)$. Therefore, the upper equivalent property implies that
$$\sum_{h \in K^i} f_h(v, L) = \sum_{h \in K^i} f_h(v, L(i)) = v(K^i). \tag{5}$$
   Again, the c equations of type (3) and l equations of type (5) uniquely determine $f(v, L)$.[2]
3. The proof of this case is analogous to the case above. First, for any $v \in \mathcal{G}$ and for all $i = 1, \ldots, n-1$, $j = i+1, \ldots, n$, we have by definition of the restricted game that $v^L[j, n] = v^{L(i)}[j, n]$. Hence $f^\ell_j(v, L) = m^\ell_j(v^L) = v^L[j, n] -$

[1] Here and below we apply the shortening $v[i, j] = v([i, j])$.

[2] Because the upper equivalent property reduces to equations of the type (5), for this case it is not needed to use induction on the number of edges.

$v^L[j+1,n] = v^{L(i)}[j,n] - v^{L(i)}[j+1,n] = m_j^\ell(v^{L(i)}) = f_j^\ell(v, L(i))$ for $j = i+1, \ldots, n$. So, f^ℓ is lower equivalent.

Next, suppose that f satisfies component efficiency and the lower equivalent property. Let $\{i, i+1\}$ be an edge in a component K of (N, L) and let K^{i+1} be the component in $(N, L(i))$ containing $i+1$, i.e., $K^{i+1} = \{h \in K | h \geq i+1\}$. Then component efficiency implies that $\sum_{h \in K^{i+1}} f_h(v, L(i)) = v(K^{i+1})$. Therefore, the lower equivalent property implies that

$$\sum_{h \in K^{i+1}} f_h(v, L) = \sum_{h \in K^{i+1}} f_h(v, L(i)) = v(K^{i+1}). \tag{6}$$

Again the c equations of type (3) and l equations of type (6) uniquely determine $f(v, L)$.

4. First, observe that

$$\sum_{j=1}^{i} \left(m_j^u\left(v^L\right) - m_j^u\left(v^{L(i)}\right)\right) = 0$$
$$= \sum_{j=i+1}^{n} \left(m_j^\ell\left(v^L\right) - m_j^\ell\left(v^{L(i)}\right)\right)$$

and

$$\sum_{j=1}^{i} \left(m_j^\ell\left(v^L\right) - m_j^\ell\left(v^{L(i)}\right)\right) = v^L(N) - v^{L(i)}(N)$$
$$= \sum_{j=i+1}^{n} \left(m_j^u\left(v^L\right) - m_j^u\left(v^{L(i)}\right)\right).$$

These equations follow directly from the definition of m^ℓ, m^u, and the obvious equalities $v^L[1,i] = v^{L(i)}[1,i]$ and $v^L[i+1,n] = v^{L(i)}[i+1,n]$, $i = 1, \ldots, n-1$). Summing up these two equations and substituting $f^e(v, L) = \frac{1}{2}(m^u(v^L) + m^\ell(v^L))$, we obtain that

$$2\sum_{j=1}^{i}(f_j^e(v, L) - f_j^e(v, L(i))) = 2\sum_{j=i+1}^{n}(f_j^e(v, L) - f_j^e(v, L(i))),$$

showing that f^e satisfies the equal loss property.

Next, suppose that f satisfies component efficiency and the equal loss property. Let $\{i, i+1\}$ be an edge in a component K of (N, L) and let K^s be the component in $(N, L(i))$ containing s, $s = i, i+1$. Then component efficiency implies that $\sum_{h \in K^s} f_h(v, L(i)) = v(K^s)$, $s = i, i+1$, and, moreover, $\sum_{h \in K'} f_h(v, L(i)) = \sum_{h \in K'} f_h(v, L) = v(K')$ for all $K' \in C_L(N)$ such that $K' \neq K$. Therefore, the equal loss property implies that

$$\sum_{h \in K^i} f_h(v, L) - v(K^i) = \sum_{h \in K^{i+1}} f_h(v, L) - v(K^{i+1}). \tag{7}$$

Again the c equations of type (3) and l equations of type (6) uniquely determine $f(v, L)$.

□

The assertions of Theorem 3.3 show that any of the functions f^s, f^u, f^ℓ and f^e is characterized as a solution satisfying component efficiency and one other (edge) property. Observe that the induction on the number of edges is only needed in case of the first assertion.

To conclude this section, recall from the end of section 2 that for a superadditive v, the marginal value vectors $m^u(v^L)$ and $m^\ell(v^L)$ are in the Core of v^L and thus also the average of these two vectors. Therefore the upper equivalent property, the lower equivalent property and the equal loss property result in solutions that are stable in the sense that they belong to the Core of the restricted game. The fairness property may give a solution outside the Core. Core stability of the Shapley value is guaranteed when the (restricted) game is convex.

4 Distributing Harsanyi dividends

In this section we compare the four solutions in terms of the distribution of the Harsanyi dividends. A solution f on $\mathcal{G}$ is called a Harsanyi solution if it distributes the dividend of each coalition S in a TU-game among the players in S according to a given sharing system. To state this precisely, the collection of sharing systems is given by

$$P = \{p = \left(p^S\right)_{S\in\Omega} | p^S \in \mathbb{R}^n,\ p_i^S = 0 \text{ for } i \in N \setminus S,\ p_i^S \geq 0 \text{ for } i \in S,$$
$$\text{and} \sum_{j\in S} p_j^S = 1, \text{ for each } S \in \Omega\}$$

and a sharing system $p \in P$ assigns for each $S \in \Omega$ (the collection of nonempty coalitions) a nonnegative share p_i^S to every player $i \in S$ with the sum of these shares equal to one. For a given sharing system $p \in P$, the corresponding *Harsanyi solution*, see Vasil'ev (1982, 2003), is the solution ϕ^p on $\mathcal{G}$ given by

$$\phi_i^p(v) = \sum_{\{S\in\Omega|i\in S\}} p_i^S \Delta^S(v), \quad i \in N,$$

i.e., this solution assigns to player $i \in N$ in game $v \in \mathcal{G}$ the sum of i's share $p_i^S \Delta^S(v)$ in the Harsanyi dividends $\Delta^S(v)$ over all coalitions $S \in \Omega$ containing i. The payoff vector $\phi^p(v)$ is called a *Harsanyi payoff* vector. Since $v(N) = \sum_{S\in\Omega} \Delta^S(v)$, for each sharing system $p \in P$ it holds that $\sum_{i\in N} \phi_i^p(v) = v(N)$, and thus each Harsanyi solution is efficient. Distributing each dividend equally among the players of the corresponding coalition, i.e. taking the equal sharing system $p_i^S = (1/|S|)$, $i \in S$, $S \in \Omega$, yields the Shapley value as one of the Harsanyi solutions.

Also any function f that assigns to game v the marginal value vector $m^\pi(v)$ for some given permutation π is a Harsanyi solution. This can be seen as follows. For given π and coalition $S \subseteq N$, let $i(S)$ be the player in S with the highest rank number, i.e., $\pi(j) \leq \pi(i(S))$ for all $j \in S$. Let the sharing system $p(\pi) \in P$ be defined by $p_j^S(\pi) = 1$ if $j = i(S)$ and $p_j^S(\pi) = 0$ for all $j \neq i(S)$. Then it follows straightforward (see, e.g. Derks et al. 2006) that

$$m^\pi(v) = \phi^{p(\pi)}(v),$$

i.e., the marginal value vector $m^\pi(v)$ equals the Harsanyi payoff vector obtained by giving the dividend of any coalition S to the last player in S according to the permutation π. Hence, the solution f on $\mathcal{G}$ assigning $f(v) = m^\pi(v)$ to any $v \in \mathcal{G}$ is a Harsanyi solution with sharing system $p = p(\pi)$.

We are now ready to characterize the four solutions of section 3 in terms of the distribution of the dividends of the restricted game v^L. From Theorem 3.3 we know that component efficiency and the upper equivalent property yield the solution f^u assigning to each line-graph game (v, L) the marginal value vector $m^u(v^L)$ of the restricted game. From the discussion above it now follows that according to the upper equivalent property the dividend of a coalition $[h, k]$ in the restricted game v^L is fully given to the last player k. Similarly, the lower equivalent property implies that this dividend is fully given to the first player h. According to the equal loss property the dividend of coalition $[h, k]$ in v^L is equally divided among the two players h and k. Finally, the fairness property results in an equal division of the dividend of $[h, k]$ in v^L among all players in $[h, k]$.

Which of these rules is the most appropriate one, depends on the underlying situation. To give an example, suppose that $v[i, n] \geq v[i+1, n]$, $i = 1, \ldots, n-1$, and any edge $\{i, i+1\}$ is under control of player i, $i = 1, \ldots, n-1$, i.e., player i has the power to decide on whether to keep or to delete this edge from the graph. In case i deletes the edge, no cooperation between the players in front of the edge and the players after the edge is possible anymore. In this situation the lower equivalent property seems to be the most appropriate solution. The lower equivalent property says that, when i deletes the edge between i and $i+1$, the players after i are not hurt. So, only players j in front of i and i itself will suffer from deleting the edge $\{i, i+1\}$, giving player i an incentive not to delete the edge. (We will illustrate this with some specific examples in the next section.) Similarly, the upper equivalent property, and thus the solution f^u, seems to be more appropriate when $v[1, i+1] \geq v[1, i]$, $i = 1, \ldots, n-1$, and player $i+1$ is in control of the edge $\{i, i+1\}$, $i = 1, \ldots, n-1$. Consequently, the function f^e (which satisfies the equal loss property) may be appropriate when both i and $i+1$ have equal control on the edge $\{i, i+1\}$. Note that fairness equalizes the change in payoffs of only the two players on the deleted edge, while the resulting Shapley solution gives an equal distribution of the dividends among *all* players in the corresponding coalition. In contrast, the equal loss property equalizes the change in total payoffs of all players at both sides of the deleted edge, while the resulting solution equally shares the dividend between the two extreme players in the corresponding coalition.

We conclude this section with an explicit expression for the dividends of the restricted game v^L. It is well-known that in a restricted game any unconnected coalition has dividend zero. Applying a general formula given in Owen (1986) for cycle-free graphs to the case of line-graph games, it follows that the dividend of a connected coalition $[i, j]$, $j \geq i$, in the restricted game v^L is given by

$$\Delta^{[i,j]}\left(v^L\right) = v[i, j] - v[i+1, j] - v[i, j-1] + v[i+1, j-1], \tag{8}$$

(with the convention that $v[h, k] = v(\emptyset) = 0$ when $h > k$). We now say that a line-graph game (v, L) is *linear-convex* if for all i, j with $j > i$ it holds that $\Delta^{[i,j]}(v^L) \geq 0$. Clearly, a line-graph game (v, L) is linear-convex if game v is convex, since linear-convexity only requires the convexity conditions when $S = [i+1, j]$ and $T = [i, j-1]$ for some $j > i$. It is well-known that a game is convex when all dividends of coalitions containing at least two players are non-negative. With this it follows that a line-graph game (v, L) is linear-convex if and only if the restricted game v^L is convex. So, if (v, L) is linear-convex, fairness yields a payoff vector in the Core of the restricted game.

5 Applications

Many economic situations can be modeled as line-graph games. In this section we consider two applications, one-machine sequencing games and the water distribution problem. We will show that the modeling of economic situations as line-graph games can simplify the analysis of these situations considerably and helps to get more insight. This also shows that line-graph games are an interesting subset of the class of all graph games and motivates the study of this subclass of games.

5.1 One-machine sequencing games

A one-machine sequencing situation, see, e.g. Curiel et al. (1993, 1994) is described as a triple (N, p, q), where $N = \{1, \ldots, n\}$ is the set of jobs in a queue to be processed, $p \in \mathbb{R}^n_+$ is an n-vector with p_i the processing time of job i and $q = (q_i)_{i \in N}$ is a collection of cost functions $q_i : \mathbb{R}_+ \to \mathbb{R}_+$, specifying the costs $q_i(t)$ where t is the total time needed to complete job i. For a permutation[3] ρ on N describing the positions of the jobs in the queue, the completion time of job i is given by $T_i(\rho) = \sum_{\{j | \rho(j) \leq \rho(i)\}} p_j$, i.e. the sum of its waiting time and its own processing time. The cost of processing i is given by $C_i(\rho) = q_i(T_i(\rho))$. The total cost of completing the jobs in a coalition S of jobs given a permutation ρ is given by $C_S(\rho) = \sum_{i \in S} C_i(\rho)$. In the sequel we assume without loss of generality that the initial positions of the jobs are given by the permutation ρ^0 with $\rho^0(i) = i$ for all $i \in N$, with cost $C_S(\rho^0)$, $S \subseteq N$.

Each coalition S can obtain cost savings by rearranging the jobs amongst its members. The minimal cost of the grand coalition is given by $C_N = \min_\rho C_N(\rho)$. Members of any other coalition S can only rearrange their positions under the condition that the members of S are not allowed to 'jump' over jobs outside S. So, a permutation ρ is *admissible* for S if and only if for any $j \notin S$ the set of its predecessors does not change, i.e., for any $j \notin S$ it should hold that $\{k \in N | \rho(k) < \rho(j)\} = \{k \in N | k < j\}$. Let $\mathcal{A}(S)$ be the set of admissible permutations for S. Then the minimal cost of S is $C_S = \min_{\rho \in \mathcal{A}(S)} C_S(\rho)$. This gives the cost savings *sequencing* game v given by

$$v(S) = C_S(\rho^0) - C_S, \quad S \subseteq N. \tag{9}$$

[3] In this section we denote permutations representing an order in a queue by ρ to distinguish them from an order π in marginal value vectors.

Obviously, since only permutations in $\mathcal{A}(S)$ are admissible, only coalitions of consecutive players can realise cost reductions. So, taking the line-graph (N, L) with $L = L^c = \{\{i, i+1\} | i = 1, \ldots, n-1\}$, it follows immediately that the cost savings sequencing game v is equal to the restricted game v^L of the corresponding line-graph game (v, L) given by equation (2),

$$v(S) = v^L(S) = \sum_{T \in C_L(S)} v(T), \quad S \subseteq N.$$

In other words, a cost savings sequencing game can be seen as a line-graph game on the initial order in the queue[4]. By definition, v is superadditive and as mentioned at the end of section 2, this implies that each of the solutions f^u and f^ℓ, and thus also f^e, are Core stable.

Next, we compare the component efficient solutions discussed in this paper with other solutions proposed in the literature for the special case of linear cost, i.e. $q_i(t) = \alpha_i t$ for all $t \geq 0$, with $\alpha_i > 0$. In this case it is well-known that for each coalition $[i, j]$ holds

$$v[i, j] = \sum_{\{k,h \in [i,j] | k < h\}} g_{kh},$$

where $g_{kh} = \max(0, \alpha_h p_k - \alpha_k p_h)$ is the possible gain of a switch between players k and h in any permutation such that player k is directly in front of h, see, e.g. Curiel (1988). Applying equation (8) we obtain for any coalition $[i, j]$, $i < j$, that

$$\Delta^{[i,j]}(v) = \Delta^{[i,j]}(v^L) = v[i, j] - v[i+1, j] - v[i, j-1] \\ + v[i+1, j-1] = g_{ij} \geq 0.$$

So, all dividends of the game $v = v^L$ are nonnegative and thus in this case also the Shapley value is in the Core.

Curiel (1988) proposed the equal gain splitting (EGS) solution which has been characterized by Hamers (1995) as the unique solution satisfying efficiency, the equivalence property (when two initial orders only differ with respect to the mutual positions of the players in front of some player h, then this player gets the same payoff in both situations) and the switch property (if two consecutive players switch position, then both players get the same change of their payoffs in the new situation when compared with their payoffs in the original situation). It has been shown that this solution yields the payoff vector

$$f_h^{\mathrm{EGS}}(v) = \frac{1}{2}\left(\sum_{j>h} g_{hj} + \sum_{i<h} g_{ih}\right), \quad h \in N.$$

[4] The sequencing situation described here is different from the queueing situation considered by Maniquet (2003) and Chun (2005) in which there is no initial order of the players. They look for a fair allocation of utility (consisting of the waiting cost that can be compensated by monetary transfers) depending only on the waiting cost of the players. Maniquet's queueing game is not a line-graph game since any two-player coalition can have a positive dividend. On the other hand, all other coalitions have a zero dividend in his game, while in a line-graph game also coalitions of more than two players can have a positive dividend.

In the previous section we have seen that the component efficient solution f^e distributes the dividend g_{ij} of any coalition $[i, j]$ equally over the first and the last player of the coalition. Hence, it follows that $f^e(v) = f^{\mathrm{EGS}}(v)$, i.e., when applied to the linear cost one-machine sequencing game, the component efficient solution for line-graph games satisfying the equal loss property yields the Equal Gain Splitting rule.

Fernández et al. (2005) propose a cost assignment rule such that the net-cost of player h is given by the cost of the waiting time in the initial order minus the savings obtained from cooperation, i.e., the net-cost $c_h^u(v, L)$ of player h is given by

$$c_h^u(v, L) = C_h(\rho^0) - \sum_{j<h} g_{jh}, \quad h \in N. \tag{10}$$

They show that this cost-assignment rule is the unique solution that is Core stable and satisfies the so-called property of *drop out monotonicity* (DOM), stating that if one of the players leaves the queue, for each of the remaining players the costs are nonincreasing. DOM implies that when player k drops out, the players in front of k are not affected, while for the players after k the costs are decreasing, which seems to be a very appealing and reasonable property. However, recalling that the upper equivalent component efficient solution f^u assigns the dividend of any coalition $[i, j]$ to the last player j in the coalition, equation (10) shows that the DOM stable solution corresponds to the solution in which the cost savings are distributed among the players according to f^u.

As we have seen in the previous section, the upper equivalent solution f^u assigns the dividends of cooperation fully to the last player in the coalition, and therefore does not seem to be very reasonable in sequencing situations. Thus, the attractiveness of DOM of a stable solution must be reconsidered. It has the serious drawback that it does not give any incentive to a player i to cooperate with its successors in the queue. Suppose that player i is not willing to accept any permutation ρ that places i after a player $k > i$. This refusal of i to cooperate with the players after her implies that the dividends (cost savings) g_{hj}, $h \leq i$, $j > i$, can not be realised anymore. However, the DOM-stable solution assigns any cost savings g_{hj} to the last player j, so the refusal of i to cooperate with the players after her does not hurt the players $h \leq i$, in particular not player i herself.

The DOM-stable solution may also be criticized by a noncooperative argument. Consider the first two players in the queue and suppose $g_{12} > 0$, i.e., it is optimal to reverse the initial order and to place 2 in front of 1. The upper equivalent rule f^u fully assigns these cost savings to player 2. However, player 1 has the power to play the noncooperative ultimatum game and to offer the first place in the queue to player 2 if player 2 is willing to give all the gains of this change to player 1, i.e. player 1 can sell its place against a price equal to $\alpha_1 p_2$ (the additional cost of waiting for player 1) plus all gains g_{12} of this trade. Since player 2 is indifferent to accepting this offer or not, there is no reason to refuse, and certainly if player 1 offers its place against a slightly lower price it is beneficial for player 2 to accept the offer. In fact, player 2 needs the cooperation of player 1 to become the first player in the queue, or in words of the previous section, player 1 is in control of the edge $\{1, 2\}$. Extending this reasoning we could say that any player $i < n$ is in control of the edge $\{i, i+1\}$.

As argued in the previous section, in such a situation the lower equivalent solution f^ℓ assigning the full dividend g_{ij} of any coalition $[i, j]$ to its first player i looks more appropriate than f^u. Considering the structure of the sequencing situation and the dominance of player i over the edge $\{i, i+1\}$, any convex combination of the lower equivalent solution f^ℓ and the equal loss property solution f^e, giving player i at least half of any dividend g_{ij}, $j > i$ seems to be a reasonable solution.

5.2 The water distribution problem

In their paper 'Sharing a river', Ambec and Sprumont (2002) consider the problem of the optimal distribution of water to agents located along a river from upstream to downstream. Let $N = \{1, \ldots, n\}$ be the set of players representing the agents on the river, numbered successively from upstream to downstream, and let $e_i \geq 0$ be the flow of water entering the river between player $i-1$ and i, $i = 1, \ldots, n$, with e_1 the inflow before the most upstream player 1. Further it is assumed that each player has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$ where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i and $b^i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous nondecreasing function yielding the benefit $b^i(x_i)$ to player i of the consumption x_i of water. An allocation is a pair $(x, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ of water distribution and compensation scheme, satisfying

$$\sum_{i=1}^{n} t_i \leq 0 \quad \text{and} \quad \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i, \quad j = 1, \ldots, n.$$

The first condition is a budget condition and says that the total amount of compensations is nonpositive. The second condition reflects that any player can use the water that entered upstream, but that the water inflow downstream of some player can not be allocated to this player.

Because of the quasi-linearity and the possibility of making money transfers, an allocation is Pareto optimal (efficient) if and only if the distribution of the water streams maximizes the total benefits, i.e., the optimal water distribution $x^* \in \mathbb{R}^n_+$ solves the maximization problem

$$\max_{x_1,\ldots,x_n} \sum_{i=1}^{n} b^i(x_i) \text{ s.t. } \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i, \; j = 1, \ldots, n, \; \text{and } x_i \geq 0, i = 1, \ldots, n. \tag{11}$$

A welfare distribution distributes the total benefits $\sum_{i=1}^{n} b^i(x_i^*)$ of an optimal water distribution x^* among the players.

The problem to find a reasonable welfare distribution can be modeled as a line-graph game (v, L) with $L = L^c = \{\{i, i+1\} | i = 1, \ldots, n-1\}$. Obviously, for any pair of players i, j with $j > i$ it holds that water inflow entering the river before the upstream player i can only be allocated to the downstream player j if all players between i and j cooperate, otherwise any player between i and j can take the flow from i to j for its own use. Hence, only coalitions of consecutive players are admissible. To define the characteristic function v, put $v(N) = \sum_{i=1}^{n} b^i(x_i^*)$

with x^* a solution of (11). Further, for any connected coalition $[i, j]$ of consecutive players $v[i, j] = \sum_{h=i}^{j} b^h(x_h^{[ij]})$, where $x^{[ij]} = (x_h^{[ij]})_{h=i}^{j}$ solves

$$\max_{x_i,\ldots,x_j \geq 0} \sum_{h=i}^{j} b^h(x_h) \text{ s.t. } \sum_{k=i}^{h} x_k \leq \sum_{k=i}^{h} e_k, \quad h = i, \ldots, j. \tag{12}$$

For any unconnected coalition $S \subset N$ we have that

$$v(S) = \sum_{T \in C_L(S)} v(T),$$

so, similar as with one-machine sequencing cost-savings games, the restricted game v^L is equal to v. We denote without confusion the characteristic function v^L by v and refer to this game as the *river game*.

Clearly, a river game is superadditive and hence each of the functions f^u, f^ℓ and f^e is Core stable. In case all functions b^i are differentiable with derivative going to infinity as x_i tends to zero, strictly increasing and strictly concave, Ambec and Sprumont (2002) have shown that the game is convex and hence also f^s is Core stable. In fact, it should be noticed from Section 4 that it is sufficient to prove that v is linear-convex.

Under the conditions for convexity, Ambec and Sprumont (2002) propose as solution the marginal value vector $m^u(v)$ and show that it is the unique element in the Core satisfying the condition that any coalition gets at most its *aspiration level*, defined as the highest utility which it can obtain when it can use all the water of all the players $1, \ldots, \widehat{s}$, where $\widehat{s} = \max\{s | s \in S\}$. Clearly, this implies that any coalition $[1, j]$ can get at most $v[1, j]$, $j = 1, \ldots, n$, and it follows that indeed the marginal value vector $m^u(v)$ assigning $m_i^u(v) = v[1, i] - v[1, i-1]$, $i = 1, \ldots, n$, is the unique candidate in the Core satisfying the aspiration requirements.

As we have seen in section 3, the marginal value vector $m^u(v)$ is the payoff vector assigned by the component efficient upper equivalent solution f^u, and thus it assigns all dividends of cooperation fully to the downstream agents. Consequently, it has the property that when a player i does not want to cooperate, the players in front of i, including i itself, are not hurt. However, as in the sequencing game, this is a very counterintuitive outcome. Although any upstream coalition $[1, i]$ can prevent that coalition $[i+1, n]$ gets more than $v[i+1, n]$ by using all flows $e_1, \ldots, e_i$ by itself, all benefits from cooperating go to the downstream agents.

Again the upper equivalent solution has the serious drawback that it does not give any incentive to a player i to cooperate with its successors. Repeating the reasoning once more, again we consider a two player situation and suppose it is optimal to allocate a part of e_1 to the second player. The upper equivalent solution requires that player 1 is just compensated by player 2 for its loss of utility, i.e., player 1 receives a compensation t_1 such that $b^1(x_1^*) + t_1 = b^1(e_1)$. So, as in the sequencing game, there is no reason for player 1 to cooperate. However, again player 1 has the power to play the noncooperative ultimatum game and to pass the stream $e_1 - x_1^*$ to player 2 only if this latter player is willing to give up all the gains of cooperation, i.e., player 1 can sell this stream against a price (or compensation) equal to $t_1 = b^2(x_2^*) - b_2(e_2)$. The resulting payoff for player 2 is given by $b^2(x_2^*) - t_1 = b_2(e_2)$, making player 2 indifferent between accepting the offer or not. Also in this river game we may argue that player 1 is in control of the edge $\{1, 2\}$. In general, since along the river any player $i < n$ is in control of the edge $\{i, i+1\}$ with its downstream neighbour $i+1$, the lower equivalent solution is more appropriate than the upper equivalent solution. Also any convex combination of the lower equivalent solution and the equal loss property solution is appropriate, since each such combination gives upstream agents at least as much from the dividends of cooperation as downstream agents.

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