Braid invariants for non-linear differential equations

Braid invariants for non-linear differential equations ISBN: 978-94-6191-654-9

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

2010 Mathematics Subject Classification: 35A15, 35A16, 53D40

Printed by:
Ipskamp Drukkers, Enschede, The Netherlands

VRIJE UNIVERSITEIT

Braid invariants for non-linear differential equations

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. L.M. Bouter,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de Faculteit der Exacte Wetenschappen
op woensdag 10 april 2013 om 13.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

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To my family

"If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties" Francis Bacon

Acknowledgements

I would like to express my sincere gratitude to my advisors Rob Vandervorst and Jan Bouwe van den Berg for their warm guidance during this long journey. To me, they are an immense source of knowledge and their dedication is far beyond any requirement that a person should own.

I want to acknowledge Alberto Abbondandolo, Robert Ghrist, Federica Pasquotto and Mark Peletier for their useful suggestions to improve this thesis.

I cannot forget my colleagues, the other PhD students here in Amsterdam and the whole research group in Analysis.

I believe that behind the realization of someone's dream there is always the help and the sacrifices of his own family. I am grateful to Mamma Monica, Papà Pino and David, my brother and my friend. This entire work is dedicated to them. Thanks for sharing joyful and difficult moments and for believing always in me in spite of my complains and frustrations.

Simone Munaò
Amsterdam, February 2013

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1 Introduction

In this thesis we investigate topological properties and invariants of a special class of non-linear partial differential equations (PDEs) with applications to the theory of relative braid classes. The three main results can be summarized as follows:

- the Poincaré-Hopf Theorem for relative braid classes;
- the construction of an isomorphism between the braid Floer homology and the braid Morse homology;
- a generalization of the Poincaré-Bendixson Theorem for non-linear Cauchy-Riemann equations.

In the next sections we explain the interplay between braids and differential equations, and why we can exploit the topological properties therein contained. In order to put this into a more general context, we start off with some examples which involve the solution of analytical problems via topological tools.

1.1 Examples of topological invariants in analysis

In this thesis we use topology as a useful tool which can give information on the structure of dynamical systems. Perhaps the first user of topology in differential equations was Poincaré, who developed many of his topological methods while studying ordinary differential equations which arose from certain astronomy problems. His study of autonomous systems

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^2, \quad F \in C^1(\mathbb{R}^2;\mathbb{R}^2).$$

involved looking at the totality of all solutions rather than at particular trajectories as had been the case earlier. This is the context of the famous Poincaré-Bendixson Theorem.

The use of topological techniques in analysis is full of insightful examples. We provide three.

(i) The classical BROUWER DEGREE theory provides a tool that contains information about the zeroes of a continuous function. Its infinite-dimensional generalization, the LERAY-SCHAUDER DEGREE, applies to a special class of operators.

(ii) The POINCARÉ-HOPF FORMULA relates a purely topological concept, i.e. the Euler characteristic of a smooth manifold M, to the index of a vector field on M, which is a purely analytical concept.

(iii) MORSE THEORY also can be put in this framework: one of the consequences of the Morse inequalities is that the number of critical points of any Morse function on a smooth manifold is closely related to the homology of the underlying manifold.

1.1.1 The Brouwer degree and the Leray-Schauder degree

The analytical construction of the (localized) Brouwer degree $\deg(f, \Omega, p)$ of a smooth mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, with Ω open bounded and a regular value $p \notin f(\partial\Omega)$, is defined as

$$\deg(f, \Omega, p) := \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x).$$

Here J_f denotes the Jacobian of f. By approximation one can extend the definition to continuous functions and also to non-regular values.

For such maps the degree being non-vanishing implies the existence of an $x \in \Omega$ such that $f(x) = p$. More importantly the Brouwer degree is invariant under homotopies of functions and of domains. On the base of these properties one can show that degree theory has important implications, among which Brouwer's fixed point Theorem. In full generality the latter states that any Hausdorff topological space homeomorphic to the unit closed ball $B_1(0) \subset \mathbb{R}^n$ has the fixed point property [1].

A straightforward generalization of Brouwer's fixed point Theorem to infinite dimensions, i.e., using the unit ball of an arbitrary Banach space instead of Euclidean space, is not true. The main problem here is that the unit balls in infinite-dimensional Banach spaces are not compact. Nevertheless an infinite dimensional degree theory exists and has been developed by Leray and Schauder. They identified an important class of non-linear operators in a Banach space, the compact perturbations of the identity, for which the problem of contractibility of the sphere could be solved. This extension has been successfully applied to non-linear elliptic boundary value problems, see [41].

1.1.2 The Hairy Ball Theorem and the Poincaré-Hopf formula

The Hairy Ball Theorem states that there is no non-vanishing continuous tangent vector field on even dimensional n-spheres. For ordinary spheres, or 2-spheres, the latter can be rephrased as follows: whenever one attempts to comb a hairy ball flat, there will always be at least one tuft of hair at one point on the ball. The theorem was first stated by Poincaré in the late 19th century and proved in

[1] A Hausdorff topological space X has the fixed point property if every continuous mapping $g : X \to X$ has at least one fixed point.

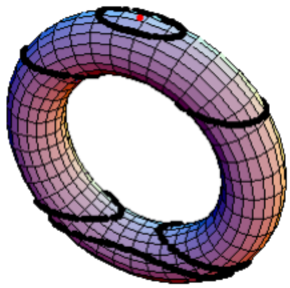


Figure 1.1: The standard two-torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$. In black the sublevel sets of the height function.

1912 by Brouwer. This is famously stated as *you can't comb a hairy ball flat without creating a cowlick,* or sometimes *you can't comb the hair on a coconut*.

From a more advanced point of view, every zero of a vector field has an *index* [2], and it can be shown that for an even dimensional sphere the sum of all of the indices at all of the zeros must be two. Therefore there must be at least one zero. This is a consequence of the Poincaré-Hopf formula. The latter has the form

$$\sum_i \mathrm{ind}_X(x_i) = \chi(M), \tag{1.1}$$

where M is a manifold, X a vector field on M, the sum of the indices is over all the isolated zeroes of X, and $\chi(M)$ is the Euler characteristic of M. One important consequence of (1.1) is that the index of a vector field does not depend on the choice of the vector field, but only on the topology of the manifold M. In the case of the torus, the Euler characteristic is 0; and it is possible to *comb a hairy doughnut flat*. In this regard, it follows that for any compact regular 2-dimensional manifold with non-zero Euler characteristic, any continuous tangent vector field has at least one zero.

1.1.3 Classical Morse Theory

The power of Morse theory is that it provides an analytical framework in which to study the topology of manifolds. One of the classical references is Milnor [37].

Consider the standard embedding of the two-torus $\mathbb{T}^2$ in $\mathbb{R}^3$, as shown in Figure 1.1 and the height function $h : \mathbb{T}^2 \to \mathbb{R}$ which returns the third coordinate of such embedding. This function has four critical points. By studying the sublevel

[2] defined, for an isolated zero, in terms of the mapping degree introduced in the previous section

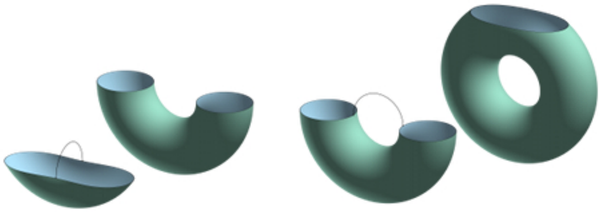


Figure 1.2: Handle decomposition of the sublevel sets of the embedded two-torus.

sets $M_c = h^{-1}((-\infty, c))$ we realize that the topology of M_c does not change as long as c does not pass a critical value of f. When c crosses a critical value, the topology changes. Morse theory is the study of this phenomenon. More generally, if a manifold has a non-trivial homotopy type, the sublevel set M_∞ has a non-trivial homotopy type and therefore f must have critical points. The Morse inequalities [9] are a concise formulation of this, relating the minimum number of critical points of a function to the homology of the underlying manifold. They imply furthermore (1.1). One can prove that the homotopy type of a sublevel set changes exactly by attaching an n-cell (or an n-handle as in Figure 1.2) where n is given by the nature of the critical point, i.e., depending whether the critical point is a minimum, maximum, or a saddle point. One builds up a CW-complex in this manner, which captures the homotopy type of the manifold. For this to work the function f needs to satisfy certain properties, which are contained in the concept of a Morse function.

1.2 Braids and braid diagrams

In this section we begin with informal definitions of braids, braid classes and relative braid classes in three different contexts. All three settings are closely related and we will point out their relations. We will mainly follow [52].

1.2.1 Braids on $\mathbb{D}^2$

Consider the standard 2-disc $\mathbb{D}^2$ (with coordinates $x = (p, q) \in \mathbb{D}^2$) in the plane and the cylinder $C = [0,1] \times \mathbb{D}^2$. An unordered collection of continuous functions $x = \{x^1(t), \dots, x^m(t)\}, x^k : [0,1] \to \mathbb{D}^2$ (called strands) is called a braid on the 2-disc $\mathbb{D}^2$ if:

(i) $x^k(t+1) = x^{\sigma(k)}(t)$ for some permutation $\sigma \in S_m$, and

(ii) $x^k(t) \neq x^h(t)$ for all $k \neq h$ and all $t \in [0,1]$.

The set of all braids on $\mathbb{D}^2$ homotopic to x is denoted by $[x]_{\mathbb{D}^2}$ and is called a braid class. We will often use, for such braids, the terminology *bounded* braids, since for all $x_0 \in [x]_{\mathbb{D}^2}$ we have $|x_0| \leq 1$. A way to visualize a braid is to consider a so-called braid diagram in the plane. The latter is obtained by projecting the cylinder C onto a plane of the form $[0,1] \times L$, where is $L \subset \mathbb{R}$ is a diameter of $\mathbb{D}^2$. If we denote the projection by $\pi : \mathbb{D}^2 \to L$, then two strands $x^k(t)$ and $x^h(t)$ have a positive crossing in the projection at $\pi x^k(t_0) = \pi x^h(t_0)$ if $x^k - x^h$ rotates counter clockwise about the origin, for small interval of times t around t_0. A negative crossing corresponds to a clockwise rotation. Now consider special collections of the form $\{x(t), y^1(t), \ldots, y^m(t)\}$, with $x = \{x(t)\}$ a periodic function on $[0,1]$, with values in $\mathbb{D}^2$ and $y = \{y^1(t), \ldots, y^m(t)\}$ as above. Denote such collections by $x \operatorname{rel} y$ and assume that they are braids with $1+m$ strands. Since we singled out two braid components we denote the braid class containing $x \operatorname{rel} y$ by $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The latter is called a relative braid class, abbreviated RBC. The component y is called the *skeleton* of the relative braid class. We refer to the x-component as the *free part*. If we take the skeleton y to be fixed, then the set of periodic functions x for which $x \operatorname{rel} y$ is a braid is denoted by $[x]_{\mathbb{D}^2} \operatorname{rel} y$ and is called a relative braid class fiber. On $[x]_{\mathbb{D}^2} \operatorname{rel} y$ we consider the C^0 topology. The space $[x \operatorname{rel} y]_{\mathbb{D}^2}$ is a fibered space over $[y]_{\mathbb{D}^2}$ and the relative braid class $[x]_{\mathbb{D}^2} \operatorname{rel} y$ is a fiber in $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The intertwining between x and y gives rise to different braid classes. A relative braid class is called PROPER if x can not be deformed, or 'collapsed', onto any y components, nor onto the boundary $\partial\mathbb{D}^2$. We abbreviate proper relative relative braid classes as PRBCes. In this thesis we consider only relative braids, whose free part is composed by only *one* strand, but we can easily generalize the notion of relative braid (classes) with x consisting of n strands.

1.2.2 Braid diagrams in dimension 1

In the special case that strands $x(t)$ are of the form $x_L(t) = (q_t(t), q(t))$ the projection onto the q-coordinate provides a representation of a braid in terms of graphs. Such strands satisfy the property that they lie in the kernel of the one-form

$$\alpha = dq - pdt,$$

which is known as the Legendrian property. An unordered collection of functions $Q = \{Q^1(t), \ldots, Q^m(t)\}, Q^j : [0,1] \to [-1,1], j = 1, \ldots, m$ is called a (bounded) braid diagram or, equivalently a (bounded) LEGENDRIAN braid if

(i) $Q^k(t+1) = Q^{\sigma(k)}(t)$ for some $\sigma \in S_m$, and

(ii) all graphs $Q^k(t)$ intersect transversally.

The set of all braid diagrams isotopic to Q is denoted by $[Q]_{[-1,1]}$. As before we also consider collections of the form $q \operatorname{rel} Q = \{q(t), Q^1(t), \ldots, Q^m(t)\}$ and the

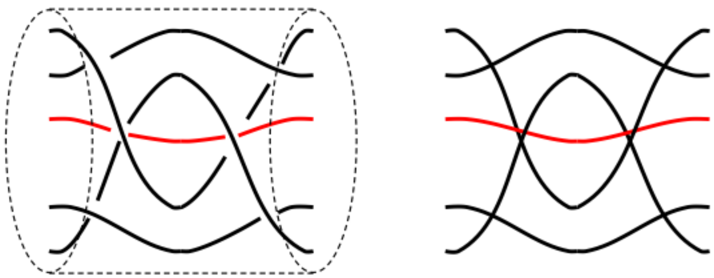


Figure 1.3: A positive relative braid and its Legendrian projection.

associated (bounded) relative braid classes $[q \operatorname{rel} Q]_{[-1,1]}$ and $[q]_{[-1,1]} \operatorname{rel} Q$ (fibers). In order to slim the notation for bounded Legendrian RBC we will write simply $[q \operatorname{rel} Q]$ and for fibers $[q] \operatorname{rel} Q$, instead of $[q \operatorname{rel} Q]_{[-1,1]}$ and $[q]_{[-1,1]} \operatorname{rel} Q$ respectively. It is immediate that these Legendrian braid classes are a subset of the braid classes on $\mathbb{D}^2$. The Legendrian constraints implies that all crossings of strands are positive.

As in the case of $\mathbb{D}^2$, the intertwining between q and Q yields different braid classes. In this case the notion of *proper* translates into the following condition. We say that a relative Legendrian braid class is PROPER if the strand q cannot be deformed onto any of the strands Q^k, for all $k = 1, \ldots, m$ nor onto the constant strands ± 1.

1.2.3 Discrete braid diagrams

Yet another simplification is obtained by considering piecewise linear functions connecting the points $q_i = q(i/d), i = 0, \ldots, d$. We represent such piecewise linear functions by sequences $q_D = \{q_i\}_{i=0,\ldots,d}$. Both the sequences and their linear piecewise extension will be denoted by the same symbol q_D. An unordered collection of sequences $Q_D = \{Q^1_D, \ldots, Q^m_D\} = \{\{Q^1_i\}, \ldots, \{Q^m_i\}\}_{i=0,\ldots,d}$ is called a discrete, or PIECEWISE LINEAR BRAID DIAGRAM if

(i) $Q^k_{i+1} = Q^{\sigma(k)}_i$, for some permutation $\sigma \in S_m$, and for all $i = 0, \ldots, d$

(ii) all the graphs $Q^k(t)$ intersect transversally [3].

The set of the equivalents classes, via isotopy, fixing the endpoints is denoted by $[Q_D]$. Crossing in this setting are also marked as positive. Collections of the form $q_D \operatorname{rel} Q_D = \{q_D, Q^1_D, \ldots Q^m_D\}$ and the associated relative braid class are denoted by $[q_D \operatorname{rel} Q_D]$ are the fibers by $[q_D] \operatorname{rel} Q_D$. As before, we say that a class

[3] in this setting we say that an intersection is transverse if $(Q^k_{i-1} - Q^{k'}_{i-1})(Q^k_{i+1} - Q^{k'}_{i+1}) > 0$ whenever $Q^k_i = Q^{k'}_i$.

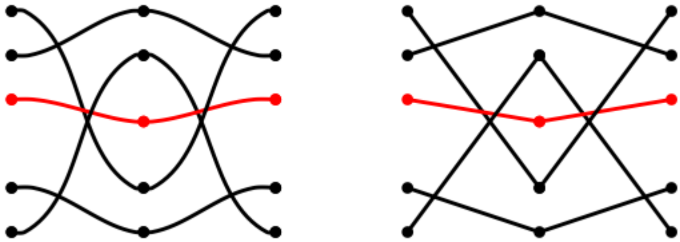


Figure 1.4: A Legendrian relative braid and its discretization.

of discrete braid diagrams is PROPER if the piecewise linear strand q_D cannot be deformed onto any of the strands Q_D^k, for all $k = 1, \dots, m$ and the strand q_D cannot de deformed onto the constant sequence ± 1.

1.2.1. Remark. Properness is a topological condition that descents from braids on $\mathbb{D}^2$ to discrete braids, i.e. properness of $[x \operatorname{rel} y]_{\mathbb{D}^2}$ implies

$$[x \operatorname{rel} y]_{\mathbb{D}^2} \implies [q \operatorname{rel} Q] \implies [q_D \operatorname{rel} Q_D].$$

The implications do not necessarily go in the opposite direction.

1.3 State of the art: braids and PDEs

The use of braids in dynamics is not without precedent (see e.g. [27], [28], [29], [38], [49]), in particular if applied to the theory of topological forcing in dimension two and three ([26], [49], [50]). How do braids evolve and under which equations this motion is ruled? We explain this in the next three paragraphs. An important motivation for using braid theory in dynamics comes from the comparison principle, which essentially states that if we evolve a braid in time, the complexity of the braid diminishes. The comparison principle motivates the choice of the Cauchy-Riemann equation for braids in $\mathbb{D}^2$, the choice of the heat flow for Legendrian braids in dimension 1, and the choice of discrete parabolic relations in the discrete case.

1.3.1 The Cauchy-Riemann equations

The non-linear equations

$$u_s - J(s,t)(u_t - X_H(t,u)), \quad u : \mathbb{R} \times S^1 \to \mathbb{D}^2 \tag{1.2}$$

are called the Cauchy-Riemann equations, or, abbreviated, non-linear CRE. The parameters J and H are called almost complex structure and Hamiltonian respectively. An almost complex structure is a smooth map $J : \mathbb{R} \times S^1 \to \mathrm{Sp}(2, \mathbb{R})$ such that $J(s,t)^2 = -\mathrm{Id}$, for all $(s,t) \in \mathbb{R} \times S^1$ (here $\mathrm{Sp}(2, \mathbb{R})$ denotes the symplectic group of degree 2 over $\mathbb{R}$). We consider the class of constant almost complex structures and we denote it by $\mathscr{J}$. Regarding the Hamiltonian function $H : S^1 \times \mathbb{D}^2 \to \mathbb{R}$, we assume that $H(t,x) = 0$ for all $x \in \partial\mathbb{D}^2$ and all $t \in \mathbb{R}$ and we call this class of Hamiltonians $\mathscr{H}$. The Hamiltonian function H gives rise to the *Hamiltonian* vector field X_H.

For a braid x the total crossing number $\mathrm{Cross}(x)$ is defined as the number of positive minus the number of negative crossings, i.e.

$$\mathrm{Cross}(x) := \#\{\text{positive crossings}\} - \#\{\text{negative crossings}\}.$$

For relative braids this number is denoted by $\mathrm{Cross}(x \,\mathrm{rel}\, y)$ and it is an invariant of the relative braid class $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$. Let $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ be a relative braid class fiber with skeleton y, then we can choose Hamiltonians H, such that the skeletal strands are solutions of the s-stationary equations $y_t = X_H(t,y)$. Let $u(s,\cdot) \,\mathrm{rel}\, y$ denote a local solution in s of the Cauchy-Riemann equations, then

$$\mathrm{Cross}(u(s_1,\cdot) \,\mathrm{rel}\, y)| \le \mathrm{Cross}(u(s_0,\cdot) \,\mathrm{rel}\, y) \quad \text{for all } s_1 \ge s_0.$$

This is also known in literature as the Monotonicity Lemma (see [49]): in essence along solutions $u(s,t)$ of the non-linear CRE (1.2), the number $\mathrm{Cross}(u(s,) \,\mathrm{rel}\, y)$ is non-increasing. In other words, along flow-lines of the non-linear CRE positive crossings can evolve into negative crossings, but not vice-versa. If we consider braid classes which are proper they yield isolating sets for the dynamics: a bounded solution $u(s,\cdot) \in [x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ with $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ a proper fiber stays away both from any of the y components and from $\partial\mathbb{D}^2$.

1.3.2 The heat flow

Consider the scalar parabolic equation, or the non-linear heat flow equation

$$v_s - v_{tt} + v - \partial_v W(t,v) = 0, \quad v : \mathbb{R} \times S^1 \to [-1,1]. \tag{1.3}$$

For the non-linearity W we assume the following hypotheses: $W \in C^\infty(S^1 \times [-1,1]; \mathbb{R})$ and $\partial_v W(t, \pm 1) = \pm 1$ for all $t \in S^1$. Equation (1.3), unlike (1.2), generates a local semi-flow ψ^s on the space of periodic function $C^0(S^1; [-1,1])$.

Let Q be a braid diagram of dimension 1 on m strands, we can define the analogue of the crossing number for x as the intersection number $\mathrm{I}(Q)$ as it follows:

$$\mathrm{I}(Q) := \#\{\text{total number of crossings}\}.$$

Since, by the Legendrian constraint, all intersections correspond to positive crossings, the total intersection number is equal to the crossing number defined above. This means that if we define $y = (Q_t, Q)$ then

$$\mathrm{Cross}(y) = \mathrm{I}(Q).$$

The classical lap-number property [6] of non-linear scalar heat equations states that the number of intersections between two graphs can only decrease in time s, as s increases.

We now apply this principle to Legendrian braid classes. Let $[q] \,\mathrm{rel}\, Q$ a Legendrian RBC fiber with skeleton Q and suppose that we can choose the non-linearity U such that the skeletal strands are solutions of the stationary equation $Q_{tt} - Q + \partial_Q W(t, Q) = 0$. Denote by $v(s, \cdot) \,\mathrm{rel}\, Q$ local solutions of the heat equation, then, as in the elliptic case, then

$$\mathrm{I}(v(s_1, \cdot) \,\mathrm{rel}\, Q) \leq \mathrm{I}(v(s_0, \cdot) \,\mathrm{rel}\, Q) \quad \text{for all } s_1 \geq s_0$$

If we consider Legendrian braid classes that are *proper* they yield isolating sets for the dynamics: also in this case a bounded solution $v(s, \cdot) \in [q] \,\mathrm{rel}\, Q$ with $[q] \,\mathrm{rel}\, Q$ a proper Legendrian fiber stays away both from each of the Q components and from the constant strands ± 1 (this property is also called isolation of proper braid classes).

1.3.3 Discrete parabolic relations

In the discrete setting the dynamics that respect the braids consists of the discrete parabolic equations. These are recurrence relations on the space of discretized braid diagram and consist of nearest neighbor interaction. They resemble spacial discretizations of parabolic equations. For a k-strand braid diagram on d points, the discrete parabolic relations are given by

$$\frac{d}{ds} v_i^\alpha = \mathcal{R}_i(v_{i-1}^\alpha, v_i^\alpha, v_{i+1}^\alpha), \quad \text{for all } i = 0, \ldots, d-1, \tag{1.4}$$

for every $\alpha = 1, \ldots, k$. On $\mathcal{R}_i$ we assume the following: $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$; $\mathcal{R}_{i+d} = \mathcal{R}_i$, for all i.

If we restrict the range of the sequences $v_D^\alpha, \alpha = 1, \dots k$ to the interval $[-1, 1]$, then Equation (1.4) generates a flow ϕ^s on the space $\mathcal{D}_d^k$ of k-tuples of d-periodic sequences. This flow will be referred as parabolic flow on $\mathcal{D}_d^k$. If we furthermore assume that $\mathcal{R}_i(-1, -1, -1) = \mathcal{R}_i(1, 1, 1) = 0$ for all i, the constant sequences ± 1 are stationary for the flow ϕ^s.

There is a discrete analogue of the crossing number and the intersection number. Recall that any discrete braid diagram (of k-strands) can be expressed in terms of the (positive) generators $\{\sigma_j\}_{j=1}^{k-1}$ of the braid group $\mathbf{B}_k$. While this word is not necessarily unique, the length of the word is, as one can easily see from the representation of $\mathbf{B}_k$. As in the previous cases, we consider piecewise linear braids that are composed by a free part and a skeletal part, and we denote them by $q_D \operatorname{rel} Q_D$. Note that the skeletal part may consist of multiple (say m) piecewise linear strands, i.e. $Q_D = \{Q_D^1, \dots, Q_D^m\}$, while we consider the free strands to be only of 1 strand. The length of a closed braid in the generators σ_j is thus precisely the word metric $\ell(Q_D)$ from geometric group theory. The geometric interpretation of $\ell(Q_D)$ for a piecewise linear braid Q_D is the number of pairwise strand crossings in the diagram Q_D. This means that if we discretize a positive braid diagram Q in dimension 1 and we call the discretization Q_D then

$$\ell(Q_D) = \mathrm{I}(Q).$$

A result in [28] shows that, as for the continuous case, the word length can only decrease as time s increases.

We now apply this principle to discrete braid diagram. Let $[q_D] \operatorname{rel} Q_D$ a discretized RBC fiber with skeleton Q_D and suppose that we can choose $\mathcal{R}_i$ such that the skeletal strands are solutions of the stationary equation for all $\alpha = 1, \dots, m$ $\mathcal{R}_i(Q_{i-1}^\alpha, Q_i^\alpha, Q_{i+1}^\alpha) = 0$, for $i = 0, \dots, d-1$ (and by periodicity $Q_0^\alpha = Q_d^\alpha$ for all $\alpha = 1, \dots, m$). Denote by $v_D(s) \operatorname{rel} Q$ local solutions of (1.4), then, as in the elliptic case, and in the continuous parabolic case we have

$$\ell(v_D(s_1) \operatorname{rel} Q) \leq \ell(v_D(s_0) \operatorname{rel} Q) \quad \text{for all } s_1 \geq s_0$$

This was shown in [28]. Also in this case, if we consider discrete braid classes which are proper they yield isolating sets for the dynamics.

1.4 Braid invariants

In the previous section we linked the three types of braid classes to natural dynamical systems associated with these braid classes. They all share the properties that proper braid classes yield isolating sets for the dynamics.

Floer's approach, used in the beginning to solve the Arnol'd conjecture, develops a Morse type theory for the Hamiltonian action

$$\mathscr{A}_H(x) = \int_0^1 \tfrac{1}{2}\langle Jx, x_t\rangle\, dt - \int_0^1 H(t, x(t))\, dt.$$

This applies to the non-linear CRE. The variational structure for the heat flow is given by the action

$$\mathscr{L}_U(q) = \int_0^1 \tfrac{1}{2}|q_t|^2 + \tfrac{1}{2}|q|^2\, dt - \int_0^1 U(t, q(t))\, dt$$

and takes the name of Lagrangian action functional. A discrete variational principle for discrete parabolic equations is given by the action

$$\mathscr{W}(\{q_i\}) = \sum_{i=0}^{d-1} S_i(q_i, q_{i+1}),$$

where S_i are smooth functions on $[-1,1]\times[-1,1]$ with the property that $\partial_1\partial_2 S_i > 0$. In this case $\mathcal{R}_i = \partial_2 S_{i-1} + \partial_1 S_i$. All the equations introduced above are now gradient flow equations and we carry out Floer's procedure.

1.4.1 Floer homology, Morse homology, Conley homology for proper relative braid classes

Let us explain the basic ingredients of Floer theory for the Cauchy-Riemann equations. The same applies to the other two cases. We should emphasize that the ingredients for obtaining respectively Floer homology, Morse homology and Conley homology are the same, but working out the details is very delicate and sometimes very tedious. Denote the set of bounded solutions of Equation (1.2) in a braid class fiber $[x]_{\mathbb{D}^2} \operatorname{rel} y$, that exists for all $s \in \mathbb{R}$, by $\mathscr{M}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)$. The image under the mapping $u \mapsto u(0,\cdot)$ is denoted by $\mathscr{S}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \subset C(S^1; \mathbb{R}^2)$.

- *Compactness.* Consider a PRBC and the set $\mathscr{M}$ of bounded solutions of the Cauchy-Riemann equations in the considered braid class. Elliptic regularity guarantees that the spaces $\mathscr{M}$ and $\mathscr{S}$ are compact with respect the appropriate topologies and properness insures that $\mathscr{S}$ is isolated. Compactness and isolation hold in all the three cases.

- *Genericity of critical points.* For a generic choice of Hamiltonians H in the class $\mathscr{H}$ (where $\mathscr{H}$ has been introduced in Section 1.3.1) for which the skeletal strands y are solutions of the associated Hamilton equations, the critical points of $\mathscr{A}_H$ in the proper relative braid class $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ are non-degenerate. Hence the set of critical points in $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$, which we denote by $\mathrm{Crit}_H([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$, consists only of finitely many isolated points. Notice that no non-degeneracy condition is imposed on the y strands. The fact that there are only finitely many isolated critical points in a class holds also for the other cases.

- *Genericity of connecting orbits.* The gradient structure of the Cauchy-Riemann equations implies that $\mathscr{M}$ is the union of the space of connecting orbit:

$$\mathscr{M}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H) = \bigcup_{x^{\pm}\in\mathrm{Crit}_H([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)} \mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H),$$

where $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H)$ is the subspace of bounded solutions of Equation (1.2) with limits x^- and x^+ for $s \to \pm\infty$. It can be proven that for generic choice of J and H, the space of connecting orbit are smooth finite dimensional manifolds without boundary.

- *Index function.* One can establish a grading $\mu(x)$ on the non-degenerate elements of $\mathrm{Crit}_H([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$ in such a way the the dimension of $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H)$ is given by the formula

$$\dim \mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H) = \mu(x^-) - \mu(x^+).$$

This theory is based on the theory of Fredholm operators and holds in all cases. For the Cauchy-Riemann equations we chose μ to be the Conley-Zender index, for the heat flow the classical Morse index and the same for the case of discrete parabolic equations.

- *Chain complex and its homology.* The construction of the chain complex and therefore the Floer homology has become a standard procedure ([23]). By the compactness and genericity $\mathrm{Crit}_H([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$ is finite and we define the chain groups $C_k([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$ as formal sum $\sum_j \alpha_j x_j$ with coefficients $\alpha_j \in \mathbb{Z}_2$. A boundary operator is defined by the formula

$$\partial_k x = \sum_{\mu(x')=k-1} n(x, x')x',$$

where $n(x, x')$ is the number of elements (modulo 2) in $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H)$ with $\mu(x^-) - \mu(x^+) = 1$. Genericity and compactness imply that this number is finite. Proving that ∂_k is a boundary operator is equivalent to showing that

$$\partial_{k-1} \circ \partial_k = 0.$$

The composition counts the number of broken trajectories, i.e. the number of elements in the set

$$\bigcup_{\mu(x')=k-1} \left(\mathscr{M}^{x^-,x'}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \times \mathscr{M}^{x',x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \right).$$

The space $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)/\mathbb{R}$, with $\mu(x^-) - \mu(x^+) = 2$, is a manifold without boundary of dimension 1 and the Floer's gluing construction reveals that if $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)/\mathbb{R}$ is not compact then the manifold can be compactified to manifold with boundary diffeomorphic to $[0, 1]$ by adding broken trajectories in

$$\bigcup_{\mu(x')=k-1} \left(\mathscr{M}^{x^-,x'}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \times \mathscr{M}^{x',x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \right).$$

The gluing construction also reveals that the procedure is surjective and thus the number of broken trajectories is even, thus $\partial_{k-1} \circ \partial_k = 0$. In the end this proves that (C_*, ∂_*) is a chain complex and its homology is well-defined and finite.

We define

$$\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) := H_k(C_*, \partial_*).$$

Different choices of $H \in \mathscr{H}$ and of $J \in \mathscr{J}$ ($\mathscr{H}$ and $\mathscr{J}$ have been defined in Section 1.3.1) yield isomorphic Floer homologies and

$$\mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y) = \varprojlim \mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y; H, J),$$

where the inverse limit is defined with respect to the canonical isomorphisms $a_k(H, H') : \mathrm{HF}_k([x] \operatorname{rel} y; H, J) \to \mathrm{HF}_k([x] \operatorname{rel} y; H', J)$ and $b_k(J, J') : \mathrm{HF}_k([x] \operatorname{rel} y; H, J) \to \mathrm{HF}_k([x] \operatorname{rel} y; H, J')$. Some properties are (see [49] for the proofs):

(i) the groups $\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y)$ are defined for all $k \in \mathbb{Z}$ and are finite, i.e. $\mathbb{Z}_2^d$ for some $d \geq 0$;

(ii) the groups $\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y)$ are invariants for the fibers in the same relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$, i.e. if $x \operatorname{rel} y \sim x' \operatorname{rel} y'$, then $\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y) \cong \mathrm{HF}_k([x']_{\mathbb{D}^2} \operatorname{rel} y')$. For this reason we will write $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})$;

(iii) if $(x \operatorname{rel} y) \cdot \Delta^{2\ell}$ denotes composition with ℓ full twists, then $\mathrm{HF}_k([(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{D}^2}) \cong \mathrm{HF}_{k-2\ell}([x \operatorname{rel} y]_{\mathbb{D}^2})$.

A similar construction can be carried out for the heat flow equation and the discrete parabolic equation leading to Morse and Conley homology, respectively

$$\mathrm{HM}_*([q \operatorname{rel} Q]) \quad \text{and} \quad \mathrm{HC}_*([q_D \operatorname{rel} Q_D]).$$

The latter is isomorphic to the homological Conley index. The former will be referred to as the Morse homology of $[q \operatorname{rel} Q]$ and the latter as the homological Conley index of $[q_D \operatorname{rel} Q_D]$. Note that properties (i) and (ii) continue to hold in the three different settings.

1.5 Discussion of the results

Since the construction of these three topological invariants is so similar in the three cases, the first question that arises is whether these three topological invariants are related. We give a (partial) answer to this question in this thesis. In the following subsections we analyze the main results contained in this manuscript. The first two results go towards the direction of linking topological invariants for discrete braids to those concerning continuous ones. We go even beyond this aim: in Chapter 2 we link the Euler-Floer characteristic to a non-variational problem, a novelty in the panorama of the Floer context. The last result, i.e. the Poincaré-Bendixson Theorem for non-linear Cauchy-Riemann equations, is more a topological property characterizing these equations.

1.5.1 The Euler-Floer characteristic and periodic point of two-dimensional diffeomorphisms

Endow the 2-disc $\mathbb{D}^2$ with the standard symplectic form $\omega = dp \wedge dq$ and choose a Hamiltonian function H in the class $\mathscr{H}$. Define the time-dependent Hamiltonian vector field X_H via the relation

$$dH = \omega(X_H, \cdot).$$

Solving the initial value problem associated to the vector field X_H, i.e.

$$\begin{cases} \dfrac{dx}{dt} = X_H(t,x) \\ x(0) = x_0 \end{cases} \tag{1.5}$$

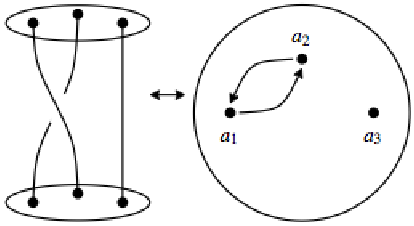


Figure 1.5: Here the braid on the left has two components and three strands. The diffeomorphism is on the right and has one point of period one and two points of period two. The two components of the braid are generated by two points of period two and one of period one. The fact that the braid has three strands does not necessarily implies that the diffeomorphism has one periodic point of period three.

gives rise to a smooth family of Hamiltonian symplectomorphisms (i.e. diffeomorphisms that preserve the area form ω and originated from X_H) denoted by $\psi_H : \mathbb{R} \times \mathbb{D}^2 \to \mathbb{D}^2$. The time-1 map $f = \psi_H(1, \cdot)$ is orientation preserving and exactly homotopic to the identity according to the nomenclature introduced in [13]. There is a one-to-one correspondence

$$k\text{-periodic points of } f \overset{1-1}{\longleftrightarrow} \text{period-}k \text{ orbits of } \psi_H.$$

The latter holds because $f^k(x) = x$ if and only if $\{\psi_H(t, x), t \in [0, 1]\}$ is a closed orbit of period k. The relation between braids and symplectomorphisms is explained as follows. Let $x \in \mathbb{D}^2$ be a k-periodic point, i.e. $f^k(x) = x$, $k \geq 1$, the minimal period. Then the set $A_k = \{x, f(x), \dots, f^{k-1}(x)\}$ satisfies $f(A_k) = \{f(x), f^2(x), \dots, f^k(x) = x\} = A_k$, and a periodic point is thus represented by an element $A_k \in \mathbf{C}_k(\mathbb{D}^2)$, the configuration space of k distinct points in $\mathbb{D}^2$. Any invariant set A_k of f of cardinality k is a point in $\mathbf{C}_k(\mathbb{D}^2)$ and gives rise to a k-strand braid via $t \mapsto \psi(t, A_k)$. Summarizing, a k-periodic point $x \in \mathbb{D}^2$ gives rise to an invariant set $A_k := \{f(x), \dots, f^{k-1}(x), f^k(x) = x\}$ for f, i.e. $f(A_k) = A_k$. On the other hand, if there exists a $k \in \mathbb{N}$ and distinct points $x_1, \dots, x_k \in \mathbb{D}^2$, such that the set $A_k := \{x_1, \dots, x_k\}$ is invariant for f, this does not imply necessarily that there exists one k-periodic point, but that there exists a collection of periodic points $x_1^1, \dots x_{k_1}^1, x_1^2, \dots x_{k_2}^2, \dots, x_1^\ell, \dots, x_{k_\ell}^\ell$ with $\sum_\ell k_\ell = k$.

In [49] the authors show that, under the hypotheses that f has an invariant set A_m representing the m-strand braid class $[y]_{\mathbb{D}^2}$, for any proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ for which the braid Floer homology $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \neq 0$, there exists an invariant set A'_n for f such that the union $A_m \cup A'_n$ represents the relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The latter is a forcing result: if the braid Floer homology of associated proper relative braid classes is non-trivial, then additional periodic points of the time-1 map of the Hamiltonian family of symplectomorphisms induced by the Hamilton equations are forced to exist. We stress that different braid classes yield different periodic points.

As explained so far, for any given proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ the Floer homology $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})$ is well-defined and applicable to Hamiltonian systems and area-preserving maps of the 2-disc. Two immediate questions that come to mind are: *Can the invariant be applied to more general systems and mappings of the 2-disc, and to what extend can the invariants be computed?*

We give a partial answer to this question in Chapter 2 and we summarize our result in this section.

The construction of HF_*, as it is presented in this thesis, fails when X is arbitrary. The main reason is simple: Equation (1.5) relies strongly on a variational principle, one-periodic solutions are critical points of an action functional. By replacing X_H with an arbitrary X the variational structure is lost, and, so far, Floer theory has never been applied in a non-variational setting. The project of building a non-variational Floer theory would certainly be challenging, and there is hope for this to work, also in light of our results presented in Chapter 4. Turning back to our problem, not everything is lost. By substituting in (1.5) a non-Hamiltonian vector field X we obtain

$$\begin{cases} \dfrac{dx}{dt} = X(t,x) \\ x(0) = x_0. \end{cases} \tag{1.6}$$

Under the hypotheses that X is one-periodic ($X(t,x) = X(t+1,x)$) and tangent to the boundary $\partial\mathbb{D}^2$ ($X(t,x)\cdot\nu = 0$ for all $x \in \partial\mathbb{D}^2$, where ν is the outward unit normal on $\partial\mathbb{D}^2$)[4], the system (1.6) gives rise to a smooth family of diffeomorphisms $\phi(t,\cdot) : \mathbb{D}^2 \to \mathbb{D}^2$, whose time 1-map $g = \phi(1,\cdot)$ is orientation preserving. The one-to-one correspondence between period-m points of g and m-periodic orbits of ϕ^t still holds. Note that g has less structure than f, namely it is *only* a diffeomorphism and *not* a Hamiltonian symplectomorphism in general. By assuming that g has an invariant set B_m that consists of m distinct points in $\mathbb{D}^2$, then B_m gives rise to a m-strand braid, exactly in the same manner as for symplectomorphims.

In Chapter 2 we show that Problems (1.5) and (1.6) can be rephrased into problems "à la Leray-Schauder" in the following way. Multiplying by J, adding $\mu x, \mu \neq 2\pi\mathbb{Z}$ on both sides of (1.5) and (1.6) and inverting $(J\frac{d}{dt} + \mu)$ we obtain respectively

$$\Phi_{\mu,H}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX_H(x,t) + \mu x) = 0$$

and

$$\Phi_{\mu}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX(x,t) + \mu x) = 0.$$

[4]In case $X = X_H$ this means that $H \in \mathscr{H}$

In this regard both maps $\Phi_{\mu,H}$ and Φ_μ are in the form "identity minus compact". Now, by assuming y to be the skeleton for X (i.e. $y^j_t = X(t, y^j), j = 1, \ldots, m$)and looking at periodic solutions in a *proper* relative braid class fiber $\Omega := [x]_{\mathbb{D}^2} \operatorname{rel} y$, we prove that isolation is preserved also for the non-variational case: in other words solutions that are contained in Ω stay away from the elements of the skeleton y and from the boundary $\partial\mathbb{D}^2$. The Leray-Schauder degree $\deg_{LS}(\Phi_\mu, \Omega, 0)$ is therefore well-defined. By assuming that y is also the skeleton for X_H (such a Hamiltonian function can always be constructed, see [49]) we perform a linear homotopy $X_\alpha = (1-\alpha)X + \alpha X_H,\ \alpha \in [0,1]$. For such homotopy X_α, y it is an admissible skeleton, since it is a skeleton for both X and X_H. Associated with the homotopy X_α we define the homotopy of maps

$$\Phi_{\mu,\alpha}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX_\alpha(x,t) + \mu x), \quad \alpha \in [0,1].$$

We observe that isolation is preserved *for all* $\alpha \in [0,1]$. By the homotopy invariance of the Leray-Schauder degree we have

$$\deg_{LS}(\Phi_\mu, \Omega, 0) = \deg_{LS}(\Phi_{\mu,\alpha}, \Omega, 0) = \deg_{LS}(\Phi_{\mu,H}, \Omega, 0).$$

By linearizing Φ_μ around a *non-degenerate* solution $x \in \Omega$ and gauging $\Phi'_{\mu,H}(x)$ with the operator $\operatorname{Id} - (J\frac{d}{dt} + \mu)^{-1}(\theta \operatorname{Id} + \mu), \theta \neq 2\pi\mathbb{Z}$ we prove that we can relate $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$ with the braid Floer homology: a delicate analysis of $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$ via spectral flow theory reveals that

$$\deg_{LS}(\Phi_{\mu,H}, \Omega, 0) = -\chi(\mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y)) = -\chi(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})), \tag{1.7}$$

where χ is the Euler characteristic of the braid Floer homology. In (1.7) the second equality follows from invariance of fibers of the braid Floer homology. The parallel with the finite dimensional case is clear. In case of finite dimensions, via the Morse inequalities one defines the Euler-Morse characteristic for *gradient* vector fields and extends it via the Brouwer degree to *arbitrary* vector fields. In our case we give meaning of the Euler-Floer characteristic, naturally associated to the *variational* problem (1.5) to *non-variational* systems such as (1.6), via infinite dimensional degree theory.

The above arguments lead to the definition of an *index* ι for non-degenerate and isolated one-periodic closed integral curves x of X. By using the theory of parity of index zero Fredholm operators we prove that for a non-degenerate and isolated one-periodic closed integral curves of X we have that $\deg_{LS}(\Phi_\mu, \Omega, 0)$ is independent of the choice of μ and of θ. More generally the index $\iota(x)$ is independent of the inversion of the operator $J\frac{d}{dt} + \mu$, and of the choice of any gauging matrix $\Theta \in \mathrm{M}_{2\times 2}(\mathbb{R})$, provided that $\sigma(\Theta) \cap 2\pi i\mathbb{Z} = \varnothing$. We then provide a derivation of a Poincaré-Hopf formula for relative braid classes. The latter has the form (recall (1.1))

$$\sum_{x_0} \iota(x_0) = \chi(\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2})).$$

The sum here is computed over all closed integral curves $x_0 \,\mathrm{rel}\, y$ in the proper relative braid class fiber $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$. The index formula can be used to obtain existence results for closed integral curves of arbitrary vector fields in proper relative braid classes and provides an extension of the already mentioned forcing result contained in [49]: if $\chi(\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2})) \neq 0$, this forces the existence of closed integral curves of arbitrary vector fields X in any proper relative braid class $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$. In the language of diffeomorphisms and periodic points the result can be reformulated as follows: under the hypotheses that a diffeomorphism g has an invariant set A_m representing the m-strand braid class $[y]_{\mathbb{D}^2}$, for any proper relative braid class $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$ for which the Euler characteristic of the braid Floer homology does not vanish, there exists a fixed point for g such that the union $A_m \cup \{x\}$ represents the relative braid class $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$. Note that we obtain results concerning fixed points of diffeomorphisms, but the same theory, with small but necessary adjustments, applies to periodic points of the diffeomorphisms. A further development would be to extend the result to any two-dimensional surfaces (with or without boundary).

The remaining part of Chapter 2 deals with computability of the Euler-Floer characteristic. The latter can indeed be determined via a discrete topological invariant. In this sense the challenge of constructing an isomorphism which links the Floer homology for proper relative braid classes to the Conley homology of proper discretized relative braid classes via Morse homology is not that far from being solved. On the level of the Euler characteristic of the three homology theories, the following holds

$$\chi\left(\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2})\right) = \chi\left(\mathrm{HM}_*([q \,\mathrm{rel}\, Q])\right) = \chi\left(\mathrm{HC}_*([q_D \,\mathrm{rel}\, Q_D])\right).$$

The idea behind the proof of this result is to first relate $\chi\left(\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2})\right)$ to mechanical Lagrangian systems and then use a discretization approach based on the method of broken geodesics. This result opens the door for computation of the Floer Homology (at least on the level of the Euler characteristic), since the problem of computing $\mathrm{HC}_*([q_D \,\mathrm{rel}\, Q_D])$ is combinatorial, and relates the Floer homology to finitely computable simplicial homology.

1.5.2 Braid Floer homology equals braid Morse homology

Chapter 3 consists of an isomorphism theorem between Floer homology for PRBC and Morse homology for Legendrian PRBC. Let $x = (p, q) \in \mathbb{D}^2$ and $y = (P, Q) \in \mathbb{D}^2$, such that $x \,\mathrm{rel}\, y$ is a proper relative braid. Compose $x \,\mathrm{rel}\, y$ with an integer ℓ

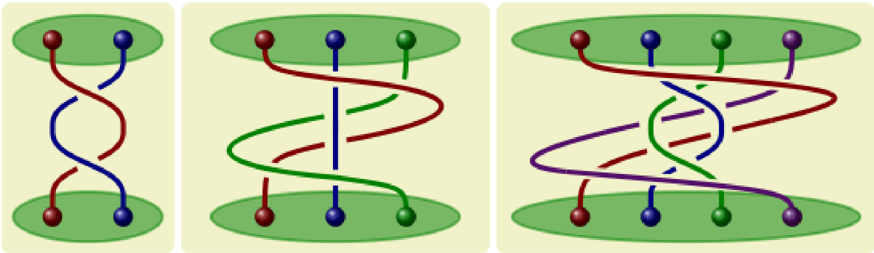


Figure 1.6: Representation of full twists of braids with 2, 3 and 4 strands.

of full twists Δ^2. A full twist can be explained informally in the following way: think of pieces of string attached to the tips of your fingers and rotate one hand by π; this is the half-twist, also also called the Garside element. Rotating the hand once more gives the full twist (see Figure 1.6).

If the number ℓ is chosen properly then $(x \operatorname{rel} y) \cdot \Delta^{2\ell}$ gives rise to a braid $x^+ \operatorname{rel} y^+$ with only positive crossings. The latter are called *positive* (relative) braids. This form for $(x \operatorname{rel} y) \cdot \Delta^{2\ell} = x^+ \operatorname{rel} y^+$ is called the Garside normal form, see [10] or [25].

Passing to braid classes, we obtain the following equality

$$[(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{D}^2} = [x^+ \operatorname{rel} y^+]_{\mathbb{D}^2} \tag{1.8}$$

By the shift property proved in [49] (Property (iii) of Section 1.4.1), on the level of the homology, this yields

$$\mathrm{HF}_{*-2\ell}([(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{D}^2}) \cong \mathrm{HF}_*([x^+ \operatorname{rel} y^+]_{\mathbb{D}^2}). \tag{1.9}$$

It follows from (1.9) that we can restrict ourselves to positive braids, hence from now on we will consider, without loss of generality, only positive relative braid classes. Positive braids enjoy, up to isotopy, the Legendrian property, in other words $x^+ \operatorname{rel} y^+$ is isotopic to a Legendrian relative braid $x^L \operatorname{rel} y^L$. The latter can be written as $x^L = (q_t, q)$ and $y^L = (Q_t, Q)$. Denoting by π_2 is the projection onto the second coordinate we can write (relative) Legendrian as $q \operatorname{rel} Q$. We denote by $[q \operatorname{rel} Q]$, all the (relative) braids which can be homotoped via a Legendrian isotopy to $q \operatorname{rel} Q$, and by $[q] \operatorname{rel} Q$ the associated fiber.

Having introduced the concepts used in the third chapter, in the following we summarize the content of Chapter 3, which consists of three different sections.

In the first section we define the braid Floer homology with respect to a new class of Hamiltonian functions. The construction is carried out by taking into account a broader class of braid classes. We consider relative classes that are homotopic to $x \,\mathrm{rel}\, y$ via homotopies in $\mathbb{R}^2$ instead of $\mathbb{D}^2$, and we denote them by $[x \,\mathrm{rel}\, y]_{\mathbb{R}^2}$. In this case the problem comes from the fact that fibers $[x]_{\mathbb{R}^2} \,\mathrm{rel}\, y$ are not a-priori bounded, since they are not a priori contained in compact subsets of $\mathbb{D}^2$ as it happens for $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$. To overcome the issue of non-compactness of $\mathbb{R}^2$, we consider a new class of Hamiltonian functions which we call hyperbolic. Following the construction summarized in Section 1.4.1 we obtain the definition of the *hyperbolic* Floer homology for *unbounded* proper relative braid class, which is denoted by

$$\mathrm{HHF}_*([x \,\mathrm{rel}\, y]_{\mathbb{R}^2}).$$

Even though we restrict our attention to positive braids, the hyperbolic braid Floer homology can be defined for all kind of braids, not only for positive ones. By following the arguments in [49] also in this case the shift theorem holds, i.e.

$$\mathrm{HHF}_{*-2\ell}([(x \,\mathrm{rel}\, y) \cdot \Delta^{2\ell}]_{\mathbb{R}^2}) \cong \mathrm{HHF}_*([x^+ \,\mathrm{rel}\, y^+]_{\mathbb{R}^2}). \tag{1.10}$$

The main result contained in the first section of Chapter 3 consists of proving that

$$\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2}) \cong \mathrm{HHF}_*([x \,\mathrm{rel}\, y]_{\mathbb{R}^2}). \tag{1.11}$$

The second part of Chapter 3 deals with Morse homology for braids. This is also a new result: so far, the formulation of a Morse theory for braids has been proven for piecewise linear braids in [28], not yet for continuous ones. By selecting a positive representative $x^+ \,\mathrm{rel}\, y^+$ in $[x \,\mathrm{rel}\, y]_{\mathbb{R}^2}$, and considering Legendrian isotopies, in the the second part of Chapter 3 we focus our attention on Legendrian relative braid classes $[q \,\mathrm{rel}\, Q]_{\mathbb{R}}$. For such braids we construct a Morse-type homology. The fact that, to build our theory, we can use the classical Morse index, instead of the Conley-Zehnder index, derives from the special properties of the Legendrian braid classes, where only positive crossings are admitted. As in the previous case we consider unbounded classes and a special class of Hamiltonians, which, in this case, are called mechanical. The latter allows to construct braid invariants with support on non-compact manifolds. At the end of the second part of Chapter 3, we define

$$\mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}),$$

i.e., the mechanical Morse homology for unbounded proper Legendrian braid classes. We observe, furthermore, that

$$\mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}) \cong \mathrm{HM}_*([q \,\mathrm{rel}\, Q]), \tag{1.12}$$

where the latter is the Morse analogue of $\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2})$.

In the last part of the chapter we prove that, for a (positive) proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$, the following holds:

$$\mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}) \cong \mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}). \tag{1.13}$$

The isomorphism (1.13) is proved using the machinery of [46], with some modifications that make it applicable to the theory of relative braid classes. In essence, to prove (1.13) we use a perturbation method, through which the solutions of the heat equation can be seen as limit as ε goes to zero of an ε dependent Cauchy-Riemann equation, see Section 3.4.1. As in [46] we prove that the bounded solutions of the Cauchy-Riemann equations are in one-to-one correspondence with the bounded solutions of the heat flow. The map which ensures the one-to-one correspondence takes the name of the Salamon-Weber map. We prove that this map respects the braid classes. As a consequence, the Morse complex defined for Legendrian braid classes agrees, up to isomorphisms, with the Floer complex defined for relative braid classes.

Putting together (1.11) (1.12) and (1.13) we obtain that for a proper positive braid class in $\mathbb{D}^2$ $[x \operatorname{rel} y]_{\mathbb{D}^2}$ it holds that

$$\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HM}_*([q \operatorname{rel} Q]).$$

By considering not-only positive proper relative braid classes, and considering the shift (1.9), this is a first step towards the conjecture that

$$\mathrm{HF}_{*-2\ell}([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HC}_*([q_D \operatorname{rel} Q_D]). \tag{1.14}$$

Equation (1.14) would link the Floer braid invariants to the discrete invariant for piecewise linear positive braid classes, and hence to finitely computable simplicial homology, opening finally the door for computation of the Floer homology.

1.5.3 Asymptotic behavior of the Cauchy-Riemann equations

Chapter 4 goes towards the direction of constructing a Floer homology theory in a non-variational setting. Our result is purely topological and exploits the structure of the Cauchy-Riemann equations. By looking at the construction of the Floer/Morse/Conley homology we see that the Cauchy-Riemann equations are obtained as formal L^2-gradient flow of the Hamiltonian action. In this case, bounded solutions will be, generically, connecting orbits between equilibria. As already mentioned, equilibria (i.e. the critical points of the action functional) are in this case periodic solutions of the equation

$$x_t = X_H(t, x), \quad x \in \mathbb{D}^2, t \in S^1, \tag{1.15}$$

for a chosen non-autonomous Hamiltonian vector field X_H on $\mathbb{D}^2$. For a *general* vector fields X we have shown furthermore that we can build a Poincaré-Hopf formula and give meaning to the Euler-Floer characteristic. By substituting a non-Hamiltonian vector field X in (1.15) we lose the variational structure, and, with it, the gradient-like behavior of the Cauchy-Riemann equations. In this case they become

$$u_s - J(u_t - X(t,u)) = 0, \quad u : \mathbb{R} \times S^1 \to \mathbb{D}^2, t \in S^1. \tag{1.16}$$

If $X = X_H$, then generically bounded solutions of the non-linear CRE are connecting orbits between one-periodic solutions of (1.15). If X is arbitrary, as in (1.16), then a priori bounded solutions do not have the connecting orbit structure, since (1.16) is not a gradient flow. Nevertheless, we have a result concerning the asymptotic of bounded solutions of (1.16). We prove that the asymptotics of (1.16) behaves surprisingly well as time s goes to infinity. More precisely, we prove that bounded solutions of Equation (1.16) admit Poincaré-Bendixson behavior.

The classical Poincaré-Bendixson Theorem describes the asymptotic behavior of flows in the plane. The topology of the plane puts severe restrictions on the behaviour of limit sets. Poincaré-Bendixson Theorem states for example that if the α- and the ω-limit set of a bounded trajectory of a smooth flow in $\mathbb{R}^2$ does not contain equilibria, then the limit set is a periodic orbit. In full generality the classical Poincaré-Bendixson Theorem can be formulated as follows.

1.5.1. Theorem (Poincaré-Bendixson (1906))**.** *Let R be a region of the plane which is closed and bounded. Consider a dynamical system $\dot{x} = X(x)$ in R where the vector field X is at least C^1. Assume that R contains no fixed points of X. Assume furthermore that there exists a trajectory γ of X (a solution of $\dot{x} = X(x)$) starting in R which stays in R for all future times. Then,*

(i) either γ is a closed orbit

(ii) or γ asymptotically approaches a closed orbit.

The classical proof of the Poincaré-Bendixson Theorem exploits the fact that, since the vector field X is autonomous, flow-lines can not intersect. As a consequence, the Jordan curve theorem is applicable and hence restricts the asymptotic behavior of flow-lines in two-dimensional domains. We stress that the above result is strictly linked to the dimensionality of the plane and essentially rules out chaos in the plane. However, it does not seem to hold for other configuration spaces or other types of dynamical systems.

Dynamical systems on two-dimensional manifolds other than the plane may well violate the Poincaré-Bendixson Theorem. Consider for instance the following vector field on the torus, which we identify with the unit square in the plane with opposite sides identified:

$$\dot{x} = 1 \text{ and } \dot{y} = \pi. \tag{1.17}$$

There is nothing special about the choice of π: any other irrational number would work just as well. Even though the torus is compact and the vector field (1.17) does not have any zeros, the orbits of (1.17) are not periodic: one can check that these orbits densely fill up the torus. This is referred to as quasi-periodic motion. Nevertheless, there is a generalization of the Poincaré-Bendixson for two-dimensional manifolds: either the classical dichotomy holds or the manifold is a torus.

In dimension three or higher, orbits may approach a very complicated limit set known as a *strange attractor*, which is characterized by a non-integer dimension and the fact that the dynamics on it are sensitive to initial conditions. In other words, chaos occurs. A celebrated example of a strange attractor is the *Lorentz attractor*.

However, the remarkable result by Fiedler and Mallet-Paret [18] establishes an extension of the Poincaré-Bendixson Theorem to infinite dimensional dynamical systems with a discrete positive Lyapunov function. They apply their result to scalar parabolic equations of the form

$$u_s = u_{tt} + f(x, u, u_t), \quad x \in S^1, f \in C^2. \tag{1.18}$$

For this equation the result of Matano ([35]) holds: it states that intersection between two solutions of (1.18) can only be destroyed (and not created) as times s increases. Here, the existence of a linear projection onto $\mathbb{R}^2$, of a discrete *positive* Lyapunov function combined with regularity of Equation (1.18) force solutions of (1.18) to have a Poincaré-Bendixson like behavior. As a matter of fact, the result contained in [18] does not only hold for the Equation (1.18), but for regular (semi) flows on Banach spaces endowed with a positive discrete Lyapunov function and a linear projection onto $\mathbb{R}^2$. The result is independent of the dimensionality of the system.

With our result, we establish a version of the Poincaré-Bendixson Theorem for bounded orbits of the non-linear Cauchy-Riemann equations in the plane. We prove that the asymptotic behavior, as s goes to infinity, of bounded solutions of Equation (1.16) is as simple as the limiting behavior of flows in $\mathbb{R}^2$. The non-linear Cauchy Riemann system is elliptic, and the Cauchy problem for elliptic equations is unstable with regard to small variations of data, i.e., it is ill-posed. As a consequence, there is no flow associated to (1.16). For this reason we consider the space of bounded flow-lines of (1.16). This space has nice properties, among which compactness and Hausdorffness. Since Equation (1.16) is autonomous in

s, we have that the space of bounded solution is invariant under s-translation. By translating flow-lines we can build a flow on such a space. The constructed flow is not regularizing, but at least it maintain the requirement of continuity. Furthermore, flow lines of equation (1.16) are endowed with a discrete Lyapunov function: as explained in the previous sections as times s increases, the winding number between two solutions decreases, possibly reaching negative values.

By embedding equation (1.16) in a more abstract setting, which include also equation (1.18), our result gives an abstract extension of the Poincaré-Bendixson Theorem to flows that allow a discrete Lyapunov function. We point out that the main differences between the results in [18] for parabolic equations and the results in Chapter 4, are that the Cauchy-Riemann equations do not define a well-posed initial value problem and, more importantly, the discrete Lyapunov functions that we consider are not bounded from below. Furthermore, our result does not assume differentiability of the flow, nor does the flow need to be defined on a compact Banach space. We only assume the space to be compact and Hausdorff. We also believe that most of the result contained in Chapter 4 can be extended to semi-flows.

Our result could be used to build a non-variational Floer theory. By proving that the asymptotic behavior of the non-linear Cauchy-Riemann equations is either a point or a periodic orbit we could build a Floer theory "à la Smale"[48] by incorporating in the chain complex periodic orbits and fixed points.

1.6 Conclusions and future work

The list of challenges we would like to solve is far from being complete. First of all, the question of transversality for a complete Morse theory for Legendrian PRBC has not been proved in the present work. We expect this to hold via modifying the proof for the Hamiltonian case and exploiting the Sard-Smale theory together with a version of the Implicit Function Theorem in infinite dimension. Second, we would like to fully prove the isomorphism (1.14). The first half is contained in this thesis, but the second half has a special meaning: it would open the door for computation of the Floer homology, via the construction of a finite cube complex. Developing computer algorithms would be a further step. Furthermore, extending the Floer theory to non-variational problems would be even more challenging. In fact, Floer theory has not been applied beyond the variational context, since it crucially uses the gradient structure of the Cauchy-Riemann equations. The above described Monotonicity Principle of the Cauchy-Riemann equations with respect to the crossing number $\mathrm{Cross}([x]_{\mathbb{D}^2} \operatorname{rel} y)$ remains valid for the non-variational Cauchy-Riemann equation (1.16). As in the variational case, bounded solution of the Cauchy-Riemann equation in a proper relative braid class are isolated. Notwithstanding, in order to link the Floer invariants $\mathrm{HF}_*([x] \operatorname{rel} y)$ to the non-variational Cauchy-Riemann equations we need to build a complex in a different way. In this sense the Poincaré-Bendixson Theorem for the non-linear Cauchy-Riemann equations suggests that limits as $s \to \infty$ are either periodic solutions of $x_t = X_H(t, x)$ or periodic solutions in s (and t). This first step establishes that the non-variational Cauchy-Riemann equations are generically a Morse-Smale system.

The next step would be, after putting the system in general position, to build an appropriate chain complex (C_*, ∂_*) incorporating periodic orbits and fixed points. If such an extension of the Floer homology can be developed then

$$H_*(C_*, \partial_*) \cong \mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}).$$

2 The Poincaré-Hopf Theorem for RBC

Braid Floer homology is an invariant of proper relative braid classes [49]. Closed integral curves of 1-periodic Hamiltonian vector fields on the 2-disc may be regarded as braids. If the Braid Floer homology of associated proper relative braid classes is non-trivial, then additional closed integral curves of the Hamiltonian equations are forced via a Morse type theory. In this article we show that certain information contained in the braid Floer homology — the Euler-Floer characteristic — also forces closed integral curves and periodic points of arbitrary vector fields and diffeomorphisms and leads to a Poincaré-Hopf type Theorem. The Euler-Floer characteristic for any proper relative braid class can be computed via a finite cube complex that serves as a model for the given braid class. The results in this paper are restricted to the 2-disc, but can be extended to two-dimensional surfaces (with or without boundary).

2.1 Introduction

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ denote the standard (closed) 2-disc in the plane with coordinates $x = (p, q)$ and let $X(x, t)$ be a smooth 1-periodic vector field on $\mathbb{D}^2$, i.e. $X(x, t + 1) = X(x, t)$ for all $x \in \mathbb{D}^2$ and $t \in \mathbb{R}$. The vector field X is tangent to the boundary $\partial\mathbb{D}^2$, i.e $X(x, t) \cdot \nu = 0$ for all $x \in \partial\mathbb{D}^2$, where ν the outward unit normal on $\partial\mathbb{D}^2$. The set of vector fields satisfying these hypotheses is denoted by $\mathcal{F}_{\|}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$. *Closed integral curves* $x(t)$ of X are integral curves[1] of X for which $x(t + \ell) = x(t)$ for some $\ell \in \mathbb{N}$. Every integral curve of X with minimal period ℓ defines a closed loop in the configuration space $\mathbf{C}_\ell(\mathbb{D}^2)$ of ℓ unordered distinct points. A collection of distinct closed integral curves with periods ℓ_j defines a closed loop in $\mathbf{C}_m(\mathbb{D}^2)$, with $m = \sum_j \ell_j$. As curves in the cylinder $\mathbb{D}^2 \times [0, 1]$ such a collection of integral curves represents a geometric braid which corresponds to a unique word $b_y \in \mathbf{B}_m$, modulo conjugacy and full twists:

$$b_y \sim b_y \Delta^{2k} \sim \Delta^{2k} b_y, \tag{2.1}$$

where Δ^2 is a full positive twist and $\mathbf{B}_m$ is the Artin Braid group on m strands.

Let y be a geometric braid consisting of closed integral curves of X, which will be referred to as a *skeleton*. The curves $y^i(t)$, $i = 1, \cdots, m$ satisfy the periodicity condition $y(0) = y(1)$ as point sets, i.e. $y^i(0) = y^{\sigma(i)}(1)$ for some permutation $\sigma \in S_m$. In the configuration space $\mathbf{C}_{n+m}(\mathbb{D}^2)$[2] we consider closed loops of the

[1] Integral curves of X are smooth functions $x : \mathbb{R} \to \mathbb{D}^2 \subset \mathbb{R}^2$ that satisfy the differential equation $x' = X(x, t)$.

[2] The space of continuous mapping $\mathbb{R}/\mathbb{Z} \to X$, with X a topological space, is called the free loop space of X and is denoted by $\mathcal{L}X$.

form $x \operatorname{rel} y := \{x^1(t), \cdots x^n(t), y^1(t), \cdots, y^m(t)\}$. The path component of $x \operatorname{rel} y$ of closed loops in $\mathcal{L}\mathbf{C}_{n+m}(\mathbb{D}^2)$ is denoted by $[x \operatorname{rel} y]$ and is called a *relative braid class*. The loops $x' \operatorname{rel} y' \in [x \operatorname{rel} y]$, keeping y' fixed, is denoted by $[x'] \operatorname{rel} y'$ and is called a *fiber*. Relative braid classes are path components of braids which have at least two components and the components are labeled into two groups: x and y. The intertwining of x and y defines various different braid classes. A relative braid class $[x \operatorname{rel} y]$ in $\mathbb{D}^2$ is *proper* if components $x_c \subset x$ cannot be deformed onto (i) the boundary $\partial\mathbb{D}^2$, (ii) itself,[3] or other components $x'_c \subset x$, or (iii) components in $y_c \subset y$, see [49] for details. In this paper we are mainly concerned with relative braids for which x has only one strand. To proper relative braid classes $[x \operatorname{rel} y]$ one can assign the invariants $\mathrm{HB}_*([x \operatorname{rel} y])$, with coefficients in $\mathbb{Z}_2$, called *Braid Floer homology*. In the following subsection we will briefly explain the construction of the invariants $\mathrm{HB}_*([x \operatorname{rel} y])$ in case that x consists of one single strand. See [49] for more details on Braid Floer homology.

2.1.1 A brief summary of Braid Floer homology

Fix a Hamiltonian vector field X_H in $\mathcal{F}_{||}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ of the form $X_H(x,t) = J\nabla H(x,t)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and H is a Hamiltonian function with the properties:

(i) $H \in C^\infty(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z}; \mathbb{R})$;

(ii) $H(x,t)|_{x \in \partial\mathbb{D}^2} = 0$, for all $t \in \mathbb{R}/\mathbb{Z}$.

For closed integral curves of X_H of period 1 we define the Hamilton action

$$\mathscr{A}_H(x) = \int_0^1 \tfrac{1}{2} Jx \cdot x_t - H(x,t)\, dt,$$

Critical points of the action functional $\mathscr{A}_H$ are in one-to-one correspondence with closed integral curves of period 1. Assume that $y = \{y^j(t)\}$ is a collection of closed integral curves of the Hamilton vector field X_H, i.e. periodic solutions of the $y^j_t = X_H(y^j, t)$. Consider a proper relative braid class $[x] \operatorname{rel} y$, with x 1-periodic and seek closed integral curves $x \operatorname{rel} y$ in $[x] \operatorname{rel} y$. The set of critical points of $\mathscr{A}_H$ in $[x] \operatorname{rel} y$ is denoted by $\mathrm{Crit}_{\mathscr{A}_H}([x] \operatorname{rel} y)$. In order to understand

[3] This condition is separated into two cases: (i) a component in x cannot be not deformed into a single strand, or (ii) if a component in x can be deformed into a single strand, then the latter necessarily intersects y or a different component in x.

the set $\mathrm{Crit}_{\mathscr{A}_H}([x] \,\mathrm{rel}\, y)$ we consider the negative L^2-gradient flow of $\mathscr{A}_H$. The L^2-gradient flow $u_s = -\nabla_{L^2}\mathscr{A}_H(u)$ yields the Cauchy-Riemann equations

$$u_s(s,t) - Ju_t(s,t) - \nabla H(u(s,t),t) = 0,$$

for which the stationary solutions $u(s,t) = x(t)$ are the critical points of $\mathscr{A}_H$.

To a braid y one can assign an integer $\mathrm{Cross}(y)$ which counts the number of crossings (with sign) of strands in the standard planar projection. In the case of a relative braid $x \,\mathrm{rel}\, y$ the number $\mathrm{Cross}(x \,\mathrm{rel}\, y)$ is an invariant of the relative braid class $[x \,\mathrm{rel}\, y]$. In [49] a monotonicity lemma is proved. The latter states that, along solutions $u(s,t)$ of the nonlinear Cauchy-Riemann equations, the number $\mathrm{Cross}(u(s,\cdot) \,\mathrm{rel}\, y)$ is non-increasing (the jumps correspond to 'singular braids', i.e. 'braids' for which intersections occur). As a consequence an isolation property for proper relative braid classes exists: the set bounded solutions of the Cauchy-Riemann equations in a proper braid class fiber $[x] \,\mathrm{rel}\, y$, denoted by $\mathscr{M}([x] \,\mathrm{rel}\, y; H)$, is compact and isolated with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^2$. These facts provide all the ingredients to follows Floer's approach towards Morse Theory for the Hamiltonian action [23]. For generic Hamiltonians which satisfy (i) and (ii) above and for which y is a skeleton, the critical points in $[x] \,\mathrm{rel}\, y$ of the action $\mathscr{A}_H$ are non-degenerate and the set of connecting orbits $\mathscr{M}_{x_-,x_+}([x] \,\mathrm{rel}\, y; H)$ are smooth finite dimensional manifolds. To critical in $\mathrm{Crit}_{\mathscr{A}_H}([x] \,\mathrm{rel}\, y)$ we assign a relative index $\mu^{CZ}(x)$ (the Conley-Zehnder index) and

$$\dim \mathscr{M}_{x_-,x_+}([x] \,\mathrm{rel}\, y; H) = \mu^{CZ}(x_-) - \mu^{CZ}(x_+).$$

Define the free abelian groups C_k over the critical points of index k, with coefficients in $\mathbb{Z}_2$, i.e.

$$C_k([x] \,\mathrm{rel}\, y; H) := \bigoplus_{\substack{x \in \mathrm{Crit}_{\mathscr{A}_H}([x] \,\mathrm{rel}\, y), \\ \mu(x)=k}} \mathbb{Z}_2\langle x\rangle,$$

and the boundary operator

$$\partial_k = \partial_k([x] \,\mathrm{rel}\, y; H) : C_k \to C_{k-1},$$

which counts the number of orbits (modulo 2) between critical points of index k and $k-1$ respectively. Analysis of the spaces $\mathscr{M}_{x_-,x_+}([x] \,\mathrm{rel}\, y; H)$ reveals that (C_*, ∂_*) is a chain complex, and its (Floer) homology is denoted by $\mathrm{HB}_*([x] \,\mathrm{rel}\, y; H)$. Different choices of H yields isomorphic Floer homologies and

$$\mathrm{HB}_*([x] \,\mathrm{rel}\, y) = \varprojlim \mathrm{HB}_*([x] \,\mathrm{rel}\, y; H),$$

where the inverse limit is defined with respect to the canonical isomorphisms $a_k(H, H') : \mathrm{HB}_k([x]\,\mathrm{rel}\,y, H) \to \mathrm{HB}_k([x]\,\mathrm{rel}\,y, H')$. Some properties are:

(i) the groups $\mathrm{HB}_k([x]\,\mathrm{rel}\,y)$ are defined for all $k \in \mathbb{Z}$ and are finite, i.e. $\mathbb{Z}_2^d$ for some $d \geq 0$;

(ii) the groups $\mathrm{HB}_k([x]\,\mathrm{rel}\,y)$ are invariants for the fibers in the same relative braid class $[x\,\mathrm{rel}\,y]$, i.e. if $x\,\mathrm{rel}\,y \sim x'\,\mathrm{rel}\,y'$, then $\mathrm{HB}_k([x]\,\mathrm{rel}\,y) \cong \mathrm{HB}_k([x']\,\mathrm{rel}\,y')$. For this reason we will write $\mathrm{HB}_*([x\,\mathrm{rel}\,y])$;

(iii) if $(x\,\mathrm{rel}\,y) \cdot \Delta^{2\ell}$ denotes composition with ℓ full twists, then $\mathrm{HB}_k([(x\,\mathrm{rel}\,y) \cdot \Delta^{2\ell}]) \cong \mathrm{HB}_{k-2\ell}([x\,\mathrm{rel}\,y])$.

2.1.2 The Euler-Floer characteristic and the Poincaré-Hopf Formula

Braid Floer homology is an invariant of conjugacy classes in $\mathbf{B}_{n+m}$ and can be computed from purely topological data. The *Euler-Floer characteristic* of $\mathrm{HB}_*\big([x\,\mathrm{rel}\,y]\big)$ is defined as follows:

$$\chi\big(x\,\mathrm{rel}\,y\big) = \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{HB}_k([x\,\mathrm{rel}\,y]). \tag{2.2}$$

In Section 2.7 we show that the Euler-Floer characteristic of $\mathrm{HB}_*\big([x\,\mathrm{rel}\,y]\big)$ can be computed from a finite cube complex which serves as a model for the braid class.

A 1-periodic function $x \in C^1(\mathbb{R}/\mathbb{Z})$ is an *isolated* closed integral curve of X if there exists an $\epsilon > 0$ such that x is the only solution of the differential equation

$$\mathscr{E}\big(x(t)\big) = \frac{dx}{dt}(t) - X\big(x(t), t\big), \tag{2.3}$$

in $B_\epsilon(x) \subset C^1(\mathbb{R}/\mathbb{Z})$. For isolated, and in particular non-degenerate closed integral curves we can define an *index* as follows. Let $\Theta \in \mathrm{M}_{2\times2}(\mathbb{R})$ be any matrix satisfying $\sigma(\Theta) \cap 2\pi k i\mathbb{R} = \varnothing$, for all $k \in \mathbb{Z}$ and let $\eta \mapsto R(t;\eta)$ be a curve in $C^\infty\big(\mathbb{R}/\mathbb{Z}; \mathrm{M}_{2\times2}(\mathbb{R})\big)$, with $R(t;0) = \Theta$ and $R(t;1) = D_xX(x(t),t)$ — the linearization of X at $x(t)$. Then $\eta \mapsto F(\eta) = \frac{d}{dt} - R(t;\eta)$ defines a curve in $\mathrm{Fred}_0(C^1, C^0)$. Denote by $\Sigma \subset \mathrm{Fred}_0(C^1, C^0)$ the set of non-invertible operators and by $\Sigma_1 \subset \Sigma$ the non-invertible operators with a 1-dimensional kernel. If the end points of F are invertible one can choose the path $\eta \mapsto R(t;\eta)$ such that $F(\eta)$ intersects Σ in Σ_1 and all intersections are transverse. If $\gamma = \#$ intersections of $F(\eta)$ with Σ_1, then

$$\iota(x) = -\,\mathrm{sgn}(\det(\Theta))(-1)^\gamma. \tag{2.4}$$

This definition is independent of the choice of Θ, see Section 2.6.

The above definition can be expressed in terms of the Leray-Schauder degree. Let $M \in \mathrm{GL}(C^0, C^1)$ be any isomorphism such that $\Phi_M(x) := M\mathscr{E}(x)$ is of the form 'identity + compact'. Then the index of an isolated closed integral curve is given by

$$\iota(x) = -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \deg_{LS}(\Phi_M, B_\epsilon(x), 0). \tag{2.5}$$

where $\beta_M(\Theta)$ is the number of negative eigenvalues of $M\frac{d}{dt} - M\Theta$ counted with multiplicity. The latter definition holds for both non-degenerate and isolated 1-periodic closed integral curves of X. In Section 2.6 we show that the two expressions for the index are the same and we show that they are independent of the choices of M and Θ.

2.1.1. Theorem (Poincaré-Hopf Formula)**.** *Let y be a skeleton of closed integral curves of a vector field $X \in \mathcal{F}_{||}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ and let $[x \operatorname{rel} y]$ be a proper relative braid class. Suppose that all 1-periodic closed integral curves of X are isolated, then for all closed integral curves $x_0 \operatorname{rel} y$ in $[x_0] \operatorname{rel} y$ it holds that*

$$\sum_{x_0} \iota(x_0) = \chi\big(x \operatorname{rel} y\big). \tag{2.6}$$

The index formula can be used to obtain existence resulst for closed integral curves in proper relative braid classes.

2.1.2. Theorem. *Let y be a skeleton of closed integral curves of a vector field $X \in \mathcal{F}_{||}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ and let $[x \operatorname{rel} y]$ be a proper relative braid class. If $\chi\big(x \operatorname{rel} y\big) \neq 0$, then there exist closed integral curves $x_0 \operatorname{rel} y$ in $[x] \operatorname{rel} y$.*

The analogue of Theorem 2.1.1 can also be proved for relative braid class $[x \operatorname{rel} y]$ in $\mathbf{C}_{n+m}(\mathbb{D}^2)$. Our theory also provides detailed information about the linking of solutions. In Section 2.8 we give various examples and compute the Euler-Floer characteristic. This does not provide a procedure for computing the braid Floer homology.

2.1.3. Remark. In this paper Theorem 2.1.1 is proved using the standard Leray-Schauder degree theory in combination with the theory of spectral flow and parity for operators on Hilbert spaces. The Leray-Schauder degree is related to the Euler characteristic of Braid Floer homology. Another approach is the use the degree theory developed by Fitzpatrick et al. [21].

2.1.3 Discretization and computability

The second part of the paper deals with the computability of the Euler-Floer characteristic. This is obtained through a finite dimensional model. A model is constructed in three steps:

(i) compose $x \,\mathrm{rel}\, y$ with $\ell \geq 0$ full twists Δ^2, such that $(x \,\mathrm{rel}\, y) \cdot \Delta^{2\ell}$ is isotopic to a positive braid $x^+ \,\mathrm{rel}\, y^+$;

(ii) relative braids $x^+ \,\mathrm{rel}\, y^+$ are isotopic to Legendrian braids $x_L \,\mathrm{rel}\, y_L$ on $\mathbb{R}^2$, i.e. braids which have the form $x_L = (q_t, q)$ and $y_L = (Q_t, Q)$, where $q = \pi_2 x$ and $Q = \pi_2 y$, and π_2 the projection onto the q–coordinate;

(iii) discretize q and $Q = \{Q^j\}$ to $q_d = \{q_i\}$, with $q_i = q(i/d), i = 0, \ldots, d$ and $Q_D = \{Q_D^j\}$, with $Q_D^j = \{Q_i^j\}$ and $Q_i^j = Q^j(i/d)$ respectively, and consider the piecewise linear interpolations connecting the *anchor* points q_i and Q_i^j for $i = 0, \ldots, d$. A discretization $q_D \,\mathrm{rel}\, Q_D$ is *admissible* if the linear interpolation is isotopic to $q \,\mathrm{rel}\, Q$. All such discretization form the discrete relative braid class $[q_D \,\mathrm{rel}\, Q_D]$, for which each fiber is a finite cube complex.

2.1.4. Remark. If the number of discretization points is not large enough, then the discretization may not be admissible and therefore not capture the topology of the braid. See [28] and Section 2.7.4 for more details.

For $d > 0$ large enough there exists an admissible discretization $q_D \,\mathrm{rel}\, Q_D$ for any Legendrian representative $x_L \,\mathrm{rel}\, y_L \in [x \,\mathrm{rel}\, y]$ and thus an associated discrete relative braid class $[q_D \,\mathrm{rel}\, Q_D]$. In [28] an invariant for discrete braid classes was introduced. Let $[q_D] \,\mathrm{rel}\, Q_D$ denote a fiber in $[q_D \,\mathrm{rel}\, Q_D]$, which is a cube complex with a finite number of connected components and their closures are denoted by N_j. The faces of the hypercubes N_j can be co-oriented in direction of decreasing the number of crossing in $q_D \,\mathrm{rel}\, Q_D$, and we define N_j^- as the closure of the set of faces with outward pointing co-orientation. The sets N_j^- are called *exit sets*. The invariant for a fiber is given by

$$\mathrm{HC}_*([q_D] \,\mathrm{rel}\, Q_D) = \bigoplus_j H_*(N_j, N_j^-).$$

This discrete braid invariant is well-defined for any $d > 0$ for which there exist admissible discretizations and is independent of both the particular fiber and the discretization size d. For the associated Euler characteristic we therefore write $\chi\big(q_D \,\mathrm{rel}\, Q_D\big)$. The latter is an Euler characteristic of a topological pair. The Euler characteristic of the Braid Floer homology $\chi(x \,\mathrm{rel}\, y)$ can be related to the Euler characteristic of the associated discrete braid class.

2.1.5. Theorem. *Let $[x \,\mathrm{rel}\, y]$ a proper relative braid class and $\ell \geq 0$ is an integer such that $(x \,\mathrm{rel}\, y) \cdot \Delta^{2\ell}$ is isotopic to a positive braid $x^+ \,\mathrm{rel}\, y^+$. Let $q_D \,\mathrm{rel}\, Q_D$ be an admissible discretization, for some $d > 0$, of a Legendrian representative $x_L \,\mathrm{rel}\, y_L \in [x^+ \,\mathrm{rel}\, y^+]$. Then*

$$\chi(x \,\mathrm{rel}\, y) = \chi(q_D \,\mathrm{rel}\, Q_D^*),$$

where Q_D^* is the augmentation of Q_D by adding the constant strands ± 1 to Q_D.

The idea behind the proof of Theorem 2.1.5 is to first relate $\chi(x \operatorname{rel} y)$ to mechanical Lagrangian systems and then use a discretization approach based on the method of broken geodesics. Theorem 2.1.5 is proved in Section 2.7. In Section 2.8 we use the latter to compute the Euler-Floer characteristic for various examples of proper relative braid classes.

2.1.4 Additional topological properties

In this paper we do not address the question whether the closed integral curves $x \operatorname{rel} y$ are non-constant, i.e. are not equilibrium points. By considering relative braid classes where x consists of more than one strand one can study non-constant closed integral curves. Braid Floer homology for relative braids with x consisting of n strands is defined in [49]. The ideas in this paper extend to relative braid classes with multi-strand braids x. In Section 2.8 we give an example of a multi-strand x in $x \operatorname{rel} y$ and explain how this yields the existence of non-trivial closed integral curves.

The invariant $\chi\big(q_D \operatorname{rel} Q_D\big)$ is a true Euler characteristic and

$$\chi\big(q_D \operatorname{rel} Q_D\big) = \chi\big([q_D] \operatorname{rel} Q_D, [q_D]^- \operatorname{rel} Q_D\big),$$

where $[q_D]^- \operatorname{rel} Q_D$ is the exit. A similar characterization does not a priori exist for $[x] \operatorname{rel} y$. This problem is circumvented by considering Hamiltonian systems and carrying out Floer's approach towards Morse theory (see [23]), by using the isolation property of $[x] \operatorname{rel} y$. The fact that the Euler characteristic of Floer homology is related to the Euler characteristic of a topological pair indicates that Floer homology is a good substitute for a suitable (co)-homology theory. For more details see Section 2.7 and Remark 2.7.6.

Braid Floer homology developed for the 2-disc $\mathbb{D}^2$ can be extended to more general 2-dimensional manifolds. This generalization of Braid Floer homology for 2-dimensional manifolds can then be used to extend the results in this paper to more general surfaces.

2.2 Closed integral curves

Let $X \in \mathcal{F}_{\|}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$, then closed integral curves of X of period 1 satisfy the differential equation

$$\begin{cases} \dfrac{dx}{dt} = X(x,t), \quad x \in \mathbb{D}^2,\ t \in \mathbb{R}/\mathbb{Z}, \\ x(0) = x(1). \end{cases} \tag{2.7}$$

Consider the unbounded operator $L_\mu : C^1(\mathbb{R}/\mathbb{Z}) \subset C^0(\mathbb{R}/\mathbb{Z}) \to C^0(\mathbb{R}/\mathbb{Z})$, defined by

$$L_\mu := -J\frac{d}{dt} + \mu, \quad \mu \in \mathbb{R}.$$

The operator is invertible for $\mu \neq 2\pi k, k \in \mathbb{Z}$ and the inverse $L_\mu^{-1} : C^0(\mathbb{R}/\mathbb{Z}) \to C^0(\mathbb{R}/\mathbb{Z})$ is compact. Transforming Equation (2.7), using L_μ^{-1}, yields the equation $\Phi_\mu(x) = 0$, where

$$\Phi_\mu(x) := x - L_\mu^{-1}\big(-JX(x,t) + \mu x\big).$$

If we set

$$K_\mu(x) := L_\mu^{-1}\big(-JX(x,t) + \mu x\big),$$

then Φ_μ is of the form $\Phi_\mu(x) = x - K_\mu(x)$, where K_μ is a (non-linear) compact operator on $C^0(\mathbb{R}/\mathbb{Z})$. Since X is a smooth vector field the mapping Φ_μ is a smooth mapping on $C^0(\mathbb{R}/\mathbb{Z})$.

2.2.1. Proposition. *A function $x \in C^0(\mathbb{R}/\mathbb{Z})$, with $|x(t)| \le 1$ for all t, is a solution of $\Phi_\mu(x) = 0$ if and only if $x \in C^1(\mathbb{R}/\mathbb{Z})$ and x satisfies Equation (2.7).*

Proof. If $x \in C^1(\mathbb{R}/\mathbb{Z}; \mathbb{D}^2)$ is a solution of Equation (2.7), then $\Phi_\mu(x) = 0$ is obviously satisfied. On the other hand, if $x \in C^0(\mathbb{R}/\mathbb{Z}; \mathbb{D}^2)$ is a zero of Φ_μ, then $x = K_\mu(x) \in C^1(\mathbb{R}/\mathbb{Z})$, since $R(L_\mu^{-1}) \subset C^1(\mathbb{R}/\mathbb{Z})$. Applying L_μ to both sides shows that x satisfies Equation (2.7). ■

Note that the zero set $\Phi_\mu^{-1}(0)$ does not depend on the parameter μ. In order to apply the Leray-Schauder degree theory we consider appropriate bounded, open subsets $\Omega \subset C^0(\mathbb{R}/\mathbb{Z})$, which have the property that $\Phi_\mu^{-1}(0) \cap \partial\Omega = \varnothing$. Let $\Omega = [x] \operatorname{rel} y$, where $[x] \operatorname{rel} y$ is a proper relative braid fiber, and $y = \{y^1, \cdots, y^m\}$ is a skeleton of closed integral curves for the vector field X.

2.2.2. Proposition. *Let $[x \operatorname{rel} y]$ be a proper relative braid class and let $\Omega = [x] \operatorname{rel} y$ be the fiber given by y. Then, there exists an $0 < r < 1$ such that*

$$|x(t)| < r, \ \ \textit{and} \quad |x(t) - y^j(t)| > 1 - r, \quad \forall\, j = 1, \cdots, m, \quad \forall\, t \in \mathbb{R},$$

and for all $x \in \Phi_\mu^{-1}(0) \cap \Omega = \{x \in \Omega \mid x = K_\mu(x)\}$.

Proof. Since $\Omega \subset C^0(\mathbb{R}/\mathbb{Z})$ is a bounded set and K_μ is compact, the solution set $\Phi_\mu^{-1}(0) \cap \Omega$ is compact. Indeed, let $x_n = K_\mu(x_n)$ be a sequence in $\Phi_\mu^{-1}(0) \cap \Omega$, then $K_\mu(x_{n_k}) \to x$, and thus $x_{n_k} \to x$, which, by continuity, implies that $K_\mu(x_{n_k}) \to K_\mu(x)$, and thus $x \in \Phi_\mu^{-1}(0) \cap \Omega$.

Let $x_n \in \Phi_\mu^{-1}(0) \cap \Omega$ and assume that such an $0 < r < 1$ does not exist. Then, by the compactness of $\Phi_\mu^{-1}(0) \cap \Omega$, there is a subsequence $x_{n_k} \to x$ such that one, or both of the following two possibilities hold: (i) $|x(t_0)| = 1$ for some t_0. By the uniqueness of solutions of Equation (2.7) and the invariance of the boundary $\partial\mathbb{D}^2$

$(X(x,t)$ is tangent to the boundary), $|x(t)| = 1$ for all t, which is impossible since $[x] \operatorname{rel} y$ is proper; (ii) $x(t_0) = y^j(t_0)$ for some t_0 and some j. As before, by the uniqueness of solutions of Equation (2.7), then $x(t) = y^j(t)$ for all t, which again contradicts the fact that $[x] \operatorname{rel} y$ is proper. ■

By Proposition 2.2.2 the Leray-Schauder degree $\deg_{LS}(\Phi_\mu, \Omega, 0)$ is well-defined. Consider the Hamiltonian vector field

$$X_H = J\nabla H, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

where $H(x,t)$ is a smooth Hamiltonian such that $X_H \in \mathcal{F}_{\|}(\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z})$ and y is a skeleton for X_H. Such a Hamiltonian can always be constructed, see [49], and the class of such Hamiltonians will be denote by $\mathcal{H}_{\|}(y)$. Since y is a skeleton for both X and X_H, it is a skeleton for the linear homotopy $X_\alpha = (1-\alpha)X + \alpha X_H$, $\alpha \in [0,1]$. Associated with the homotopy X_α of vector fields we define the homotopy

$$\Phi_{\mu,\alpha}(x) := x - L_\mu^{-1}\big(-JX_\alpha(x,t) + \mu x\big) = x - K_{\mu,\alpha}(x), \quad \alpha \in [0,1],$$

with $K_{\mu,\alpha}(x) = L_\mu^{-1}\big(-JX_\alpha(x,t) + \mu x\big)$. Proposition 2.2.2 applies for all $\alpha \in [0,1]$, i.e. by compactness there exists a uniform $0 < r < 1$ such that

$$|x(t)| < r, \ \text{ and } \quad |x(t) - y^j(t)| > 1 - r,$$

for all $t \in \mathbb{R}$, for all j and for all $x \in \Phi_{\mu,\alpha}^{-1}(0) \cap \Omega = \{x \in \Omega \mid x = K_{\mu,\alpha}(x)\}$ and all $\alpha \in [0,1]$. By the homotopy invariance of the Leray-Schauder degree we have

$$\deg_{LS}(\Phi_\mu, \Omega, 0) = \deg_{LS}(\Phi_{\mu,\alpha}, \Omega, 0) = \deg_{LS}(\Phi_{\mu,H}, \Omega, 0), \tag{2.9}$$

where $\Phi_{\mu,0} = \Phi_\mu$ and $\Phi_{\mu,1} = \Phi_{\mu,H}$. Note that the zeroes of $\Phi_{\mu,H}$ correspond to critical point of the functional

$$\mathscr{A}_H(x) = \int_0^1 \tfrac{1}{2} Jx \cdot x_t - H(x,t)dt, \tag{2.10}$$

and are denoted by $\operatorname{Crit}_{\mathscr{A}_H}([x] \operatorname{rel} y)$. In [49] invariants are defined which provide information about $\Phi_{\mu,H}^{-1}(0) \cap \Omega = \operatorname{Crit}_{\mathscr{A}_H}([x] \operatorname{rel} y)$ and thus $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$. These invariants are the Braid Floer homology groups $\mathrm{HB}_*\big([x] \operatorname{rel} y\big)$ as explained in the introduction. In the next section we examine spectral properties of the solutions of $\Phi_{\mu,\alpha}^{-1}(0) \cap \Omega$ in order to compute $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$ and thus $\deg_{LS}(\Phi_\mu, \Omega, 0)$.

2.2.3. Remark. There is obviously more room for choosing appropriate operators L_μ and therefore functions Φ_μ. In Section 2.6 this issue will be discussed in more detail.

2.3 Parity, Spectral flow and the Leray-Schauder degree

The Leray-Schauder degree of an isolated zero x of $\Phi_\mu(x) = 0$ is called the local degree. A zero $x \in \Phi_\mu^{-1}(0)$ is non-degenerate if $1 \notin \sigma(D_x K_\mu(x))$, where $D_x K_\mu(x) : C^0(\mathbb{R}/\mathbb{Z}) \to C^0(\mathbb{R}/\mathbb{Z})$ is the (compact) linearization at x and is given by $D_x K_\mu(x) = L_\mu^{-1}(-JD_x X(x,t) + \mu)$. If x is a non-degenerate zero, then it is an isolated zero and the degree can be determined from spectral information.

2.3.1. Proposition. *Let $x \in C^0(\mathbb{R}/\mathbb{Z})$ be a non-degenerate zero of Φ_μ and let $\epsilon > 0$ be sufficiently small such that $B_\epsilon(x) = \{\tilde{x} \in C^0(\mathbb{R}/\mathbb{Z}) \mid |\tilde{x}(t) - x(t)| < \epsilon, \forall t\}$ is a neighborhood in which x is the only zero. Then*

$$\deg_{LS}\big(\Phi_\mu, B_\epsilon(x), 0\big) = \deg_{LS}\big(\mathrm{Id} - D_x K_\mu(x), B_\epsilon(x), 0\big) = (-1)^{\beta_\mu(x)}$$

where

$$\beta_\mu(x) = \sum_{\sigma_j > 1,\ \sigma_j \in \sigma(D_x K_\mu(x))} \beta_j, \quad \beta_j = \dim\left(\bigcup_{i=1}^{\infty} \ker\big(\sigma_j \,\mathrm{Id} - D_x K_\mu(x)\big)^i\right),$$

which will be referred to as the Morse index of x, or alternatively the Morse index of linearized operator $D_x \Phi_\mu(x)$.

Proof. See [34]. ■

The functions $\Phi_{\mu,\alpha}(x) = x - K_{\mu,\alpha}(x)$ are of the form 'identity + compact' and Proposition 2.3.1 can be applied to non-degenerate zeroes of $\Phi_{\mu,\alpha}(x) = 0$. If we choose the Hamiltonian $H \in \mathcal{H}_{\|}^{\mathrm{reg}}(y)$ 'generically', then the zeroes of $\Phi_{\mu,H}$ are non-degenerate, i.e. $1 \notin \sigma(D_x K_{\mu,H}(x))$, where $D_x K_{\mu,H}(x) = D_x K_{\mu,1}(x)$. By compactness there are only finitely many zeroes in a fiber $\Omega = [x] \operatorname{rel} y$.

2.3.2. Lemma. *Let $x \in \Phi_{\mu,H}^{-1}(0) \cap \Omega$. Then following criteria for non-degeneracy are equivalent:*

- *(i)* $1 \notin \sigma(D_x K_{\mu,H}(x))$;
- *(ii) the operator $B = -J\frac{d}{dt} - D_x^2 H(x(t),t)$ is invertible;*
- *(iii) let $\Psi(t)$ be defined by $B\Psi(t) = 0$, $\Psi(0) = \mathrm{Id}$, then* $\det(\Psi(1) - \mathrm{Id}) \neq 0$.

Proof. A function ψ satisfies $D_x K_{\mu,H}(x)\psi = \psi$ if and only if $B\psi = 0$, which shows the equivalence between (i) and (ii). The equivalence between (ii) and (iii) is proved in [49]. ■

The generic choice of H follows from Proposition 7.1 in [49] based on criterion (iii). Hamiltonians for which the zeroes of $\Phi_{\mu,H}$ are non-degenerate are denoted by $\mathcal{H}_{\|}^{\mathrm{reg}}(y)$. Note that *no* genericity is needed for $\alpha \in [0,1)$! For the Leray-Schauder degree this yields

$$\deg_{LS}(\Phi_{\mu,\alpha}, \Omega, 0) = \deg_{LS}(\Phi_{\mu,H}, \Omega, 0) = \sum_{x \in \mathrm{Crit}_{\mathscr{A}_H}([x] \,\mathrm{rel}\, y)} (-1)^{\beta_{\mu,H}(x)}, \tag{2.11}$$

for all $\alpha \in [0,1]$ and where $\beta_{\mu,H}(x)$ is the Morse index of $\mathrm{Id} - D_x K_{\mu,H}(x)$.

The goal is to determine the Leray-Schauder degree $\deg_{LS}(\Phi_\mu, \Omega, 0)$ from information contained in the Braid Floer homology groups $\mathrm{HB}_*([x] \,\mathrm{rel}\, y)$. In order to do so we examen the Hamiltonian case. In the Hamiltonian case the linearized operator $D_x \Phi_{\mu,H}(x)$ is given by

$$A := D_x \Phi_{\mu,H}(x) = \mathrm{Id} - D_x K_{\mu,H}(x) = \mathrm{Id} - L_\mu^{-1}\big(D_x^2 H(x(t),t) + \mu\big),$$

which is a bounded operator on $C^0(\mathbb{R}/\mathbb{Z})$. The operator A extends to a bounded operator on $L^2(\mathbb{R}/\mathbb{Z})$. Consider the path $\eta \mapsto A(\eta)$, $\eta \in I = [0,1]$, given by

$$A(\eta) = \mathrm{Id} - L_\mu^{-1}(S(t;\eta) + \mu) = \mathrm{Id} - T_\mu(\eta), \tag{2.12}$$

where $S(t;\eta)$ a smooth family of symmetric matrices and $T_\mu(\eta) = L_\mu^{-1}(S(t;\eta) + \mu)$. The endpoints satisfy

$$S(t;0) = \theta \,\mathrm{Id}, \quad S(t;1) = D_x^2 H(x(t),t),$$

with $\theta \neq 2\pi k$, for some $k \in \mathbb{Z}$ and $D_x^2 H(x(t),t)$ is the Hessian of H at a critical point in $\mathrm{Crit}_{\mathscr{A}_H}([x] \,\mathrm{rel}\, y)$. The path of $\eta \mapsto A(\eta)$ is a path bounded linear Fredholm operators on $L^2(\mathbb{R}/\mathbb{Z})$ of Fredholm index 0, which are compact perturbations of the identity and whose endpoints are invertible.

2.3.3. Lemma. *The path $\eta \mapsto A(\eta)$ defined in (2.12) is a smooth path of bounded linear Fredholm operators in $H^s(\mathbb{R}/\mathbb{Z})$ of index 0, with invertible endpoints.*

Proof. By the smoothness of $S(t;\eta)$ we have that $\|S(t;\eta)x\|_{H^m} \leq C\|x\|_{H^m}$, for any $x \in H^m(\mathbb{R}/\mathbb{Z})$ and any $m \in \mathbb{N} \cup \{0\}$. By interpolation the same holds for all $x \in H^s(\mathbb{R}/\mathbb{Z})$ and the claim follows from the fact that $L_\mu^{-1} : H^s(\mathbb{R}/\mathbb{Z}) \to H^{s+1}(\mathbb{R}/\mathbb{Z}) \hookrightarrow H^s(\mathbb{R}/\mathbb{Z})$ is compact. ■

2.3.1 Parity of paths of linear Fredholm operators

Let $\eta \mapsto \Lambda(\eta)$ be a smooth path of bounded linear Fredholm operators of index 0 on a Hilbert space $\mathscr{H}$. A crossing $\eta_0 \in I$ is a number for which the operator $\Lambda(\eta_0)$ is not invertible. A crossing is simple if $\dim\ker\Lambda(\eta_0) = 1$. A path $\eta \mapsto \Lambda(\eta)$ between invertible ends can always be perturbed to have only simple crossings. Such paths are called generic. Following [19–22], we define the *parity* of a generic path $\eta \mapsto \Lambda(\eta)$ by

$$\text{parity}(\Lambda(\eta), I) := \prod_{\ker\Lambda(\eta_0)\neq 0} (-1) = (-1)^{\text{cross}(\Lambda(\eta), I)}, \tag{2.13}$$

where $\text{cross}(\Lambda(\eta), I) = \#\{\eta_0 \in I \ : \ \ker A(\eta_0) \neq 0\}$. The parity is a homotopy invariant with values in $\mathbb{Z}_2$. In [19–22] an alternative characterization of parity is given via the Leray-Schauder degree. For any Fredholm path $\eta \mapsto \Lambda(\eta)$ there exists a path $\eta \mapsto M(\eta)$, called a *parametrix*, such that $\eta \mapsto M(\eta)\Lambda(\eta)$ is of the form 'identity + compact'. For parity this gives:

$$\text{parity}(\Lambda(\eta), I) = \deg_{LS}\big(M(0)\Lambda(0)\big) \cdot \deg_{LS}\big(M(1)\Lambda(1)\big),$$

where $\deg_{LS}\big(M(\eta)\Lambda(\eta)\big) = \deg_{LS}\big(M(\eta)\Lambda(\eta), \mathscr{H}, 0\big)$, for $\eta = 0, 1$, and the expression is independent of the choice of parametrix. The latter extends the above definition to arbitrary paths with invertible endpoints. For a list of properties of parity see [19–22].

2.3.4. Proposition. *Let $\eta \mapsto A(\eta)$ be the path of bounded linear Fredholm operators on $H^s(\mathbb{R}/\mathbb{Z})$ defined by (2.12). Then*

$$\text{parity}(A(\eta), I) = (-1)^{\beta_{A(0)}} \cdot (-1)^{\beta_{A(1)}} = (-1)^{\beta_{A(0)} - \beta_{A(1)}}. \tag{2.14}$$

where $\beta_{A(0)}$ and $\beta_{A(1)}$ are the Morse indices of $A(0)$ and $A(1)$ respectively.

Proof. For $\eta \mapsto A(\eta)$ the parametrix is the constant path $\eta \mapsto M(\eta) = \text{Id}$. From Proposition 2.3.1 we derive that

$$\deg_{LS}\big(A(0)\big) = (-1)^{\beta_{A(0)}}, \quad \text{and} \quad \deg_{LS}\big(A(1)\big) = (-1)^{\beta_{A(1)}},$$

which proves the first part of the formula. Since $\beta(A(0)) - \beta(A(1)) = \big[\beta(A(0)) + \beta(A(1))\big] \mod 2$, the second identity follows. ■

2.3.5. Lemma. *For $\theta > 0$, the Morse index for $A(0)$ is given by $\beta_{A(0)} = 2\left\lceil \frac{\mu+\theta}{2\pi} \right\rceil$.*

Proof. The eigenvalues of the operator $A(0)$ are given by $\lambda = \frac{-\theta+2k\pi}{\mu+2k\pi}$ and all have multiplicity 2. Therefore number of integers k for which $\lambda < 0$ is equal to $\left\lceil \frac{\mu+\theta}{2\pi} \right\rceil$ and consequently $\beta_{A(0)} = 2\left\lceil \frac{\mu+\theta}{2\pi} \right\rceil$. ■

If $x \in \Phi^{-1}_{\mu,H}(0)$ is a non-degenerate zero, then its local degree can be expressed in terms of the parity of $A(\eta)$.

2.3.6. Proposition. *Let $x \in \Phi^{-1}_{\mu,H}(0)$ be a non-degenerate zero, then*

$$\deg_{LS}\big(\Phi_{\mu,H}, B_\epsilon(x), 0\big) = \text{parity}(A(\eta), I), \tag{2.15}$$

where $\eta \mapsto A(\eta)$ is given by (2.12).

Proof. From Proposition 2.3.1 we have that $\deg_{LS}\big(\Phi_{\mu,H}, B_\epsilon(x), 0\big) = (-1)^{\beta_{A(1)}}$ and by Equation (2.14), $\text{parity}(A(\eta), I) = (-1)^{\beta_{A(0)}} \cdot (-1)^{\beta_{A(1)}} = (-1)^{\beta_{A(1)}}$, which completes the proof. ■

2.3.2 Parity and spectral flow

The spectral flow is a more refined invariant for paths of selfadjoint operators. For $x \in H^s(\mathbb{R}/\mathbb{Z})$ we use the Fourier expansion $x = \sum_{k\in\mathbb{Z}} e^{2\pi Jkt}x_k$ and $\sum_{k\in\mathbb{Z}} |k|^{2s}|x_k|^2 < \infty$. From the functional calculus of the selfadjoint operator

$$-J\frac{d}{dt}x = \sum_{k\in\mathbb{Z}} \big(2\pi k\big)e^{2\pi Jkt}x_k,$$

we define the selfadjoint operators

$$N_\mu x = \sum_{k\in\mathbb{Z}} \big(2\pi|k| + \mu\big)e^{2\pi Jkt}x_k, \quad \text{and} \quad P_\mu x = \sum_{k\in\mathbb{Z}} \frac{2\pi k + \mu}{2\pi|k| + \mu}e^{2\pi Jkt}x_k. \tag{2.16}$$

For $\mu > 0$ and $\mu \neq 2\pi k$, $k \in \mathbb{Z}$, the operator P_μ is an isomorphism on $H^s(\mathbb{R}/\mathbb{Z})$, for all $s \geq 0$.[4] Consider the path

$$C(\eta) = P_\mu A(\eta) = P_\mu - N_\mu^{-1}(S(t;\eta) + \mu), \tag{2.17}$$

[4] As before $\|P_\mu x\|_{H^s} \leq \|x\|_{H^s}$ and $\|P_\mu^{-1}x\|_{H^{1/2}} \leq C(\mu)\|x\|_{H^{1/2}}$, $\mu > 0$ and $\mu \neq 2\pi k$.

which is a path of operators of Fredholm index 0. The constant path $\eta \mapsto M_\mu(\eta) = P_\mu^{-1}$ is a parametrix for $\eta \mapsto C(\eta)$ (see [21, 22]) and since $M_\mu C(\eta) = A(\eta)$, the parity of $C(\eta)$ is given by

$$\mathrm{parity}(C(\eta), I) = \mathrm{parity}(A(\eta), I). \tag{2.18}$$

Using N_μ, with $\mu > 0$ and $\mu \neq 2\pi k$, we define an equivalent norms on the Sobolev spaces $H^s(\mathbb{R}/\mathbb{Z})$:

$$(x, y)_{H^s} := \big(N_\mu^s x, N_\mu^s y\big)_{L^2}, \quad \forall x, y \in H^s(\mathbb{R}/\mathbb{Z}).$$

2.3.7. Lemma. *The operators $C(\eta)$ are selfadjoint on $\Big(H^{1/2}(\mathbb{R}/\mathbb{Z}), (\cdot,\cdot)_{H^{1/2}}\Big)$ for all $\eta \in I$, and $\eta \mapsto C(\eta)$ is a path of selfadjoint operators on $H^{1/2}(\mathbb{R}/\mathbb{Z})$.*

Proof. From the functional calculus we derive that

$$(P_\mu x, y)_{H^s} = \sum_{k \in \mathbb{Z}} p_\mu(k) n_\mu^{2s}(k) x_k y_k = (x, P_\mu y)_{H^s},$$

where $n_\mu(k) = 2\pi|k| + \mu$ and $p_\mu(k) = \frac{2\pi k + \mu}{2\pi|k| + \mu}$. For $s = 1/2$ we have that

$$\begin{aligned}\big(N_\mu^{-1}(S(t;\eta) + \mu)x, y\big)_{H^{1/2}} &= \big((S(t;\eta) + \mu)x, y\big)_{L^2} = \big(x, (S(t;\eta) + \mu)y\big)_{L^2} \\ &= \big(x, N_\mu^{-1}(S(t;\eta) + \mu)y\big)_{H^{1/2}},\end{aligned}$$

which completes the proof. ■

For a path $\eta \mapsto \Lambda(\eta)$ of *selfadjoint* operators on a Hilbert space $\mathscr{H}$, which is continuously differentiable in the (strong) operator topology we define the crossing operator $\mathbf{\Gamma}(\Lambda, \eta) = \pi \frac{d}{d\eta}\Lambda(\eta)\pi|_{\ker \Lambda(\eta)}$, where π is the orthogonal projection onto $\ker \Lambda(\eta)$. A crossing $\eta_0 \in I$ is a number for which the operator $\Lambda(\eta_0)$ is not invertible. A crossing is regular if $\mathbf{\Gamma}(\Lambda, \eta_0)$ is non-singular. A point η_0 for which $\dim\ker \Lambda(\eta_0) = 1$, is called a simple crossing. A path $\eta \mapsto \lambda(\eta)$ is called generic if all crossings are simple. A path $\eta \mapsto \Lambda(\eta)$ with invertible endpoints can always be chosen to be generic by a small perturbation. At a simple crossing η_0, there exists a C^1-curve $\lambda(\eta)$, for η near η_0, and $\lambda(\eta)$ is an eigenvalue of $\Lambda(\eta)$, with $\lambda(\eta_0) = 0$ and $\lambda'(\eta_0) \neq 0$, see [43, 44]. The spectral flow for a generic path is defined by

$$\mathrm{specflow}(\Lambda(\eta), I) = \sum_{\lambda(\eta_0)=0} \mathrm{sgn}(\lambda'(\eta_0)). \tag{2.19}$$

For a simple crossing η_0 the crossing operator is simply multiplication by $\lambda'(\eta_0)$ and

$$\mathbf{\Gamma}(\Lambda,\eta)\psi(\eta_0) = \Big(\frac{d}{d\eta}\Lambda(\eta_0)\psi(\eta_0),\psi(\eta_0)\Big)_{\mathscr{H}}\psi(\eta_0) = \lambda'(\eta_0)\psi(\eta_0), \tag{2.20}$$

where $\psi(\eta_0)$ is normalized in $\mathscr{H}$, and

$$\lambda'(\eta_0) = \Big(\frac{d}{d\eta}\Lambda(\eta_0)\psi(\eta_0)\psi(\eta_0)\Big)_{\mathscr{H}}. \tag{2.21}$$

The spectral flow is defined any for continuously differentiable path $\eta \mapsto \Lambda(\eta)$ with invertible endpoints. From the theory in [22] there is a connection between the spectral flow of $\Lambda(\eta)$ and its parity:

$$\mathrm{parity}(\Lambda(\eta), I) = (-1)^{\mathrm{specflow}(\Lambda(\eta), I)}, \tag{2.22}$$

which in view of Equation (2.13) follows from the fact that $\mathrm{cross}(\Lambda(\eta), I) = \mathrm{specflow}(\Lambda(\eta),\eta) \mod 2$ in the generic case.

The path $\eta \mapsto C(\eta)$ defined in (2.17) is a continuously differentiable path of operators on $H = H^{1/2}(\mathbb{R}/\mathbb{Z})$ with invertible endpoints, and therefore both parity and spectral flow are well-defined. If we combine Equations (2.15) and (2.18) with Equation (2.22) we obtain

$$\deg_{LS}(\Phi_{\mu,H}, B_\epsilon(x), 0) = \mathrm{parity}(A(\eta), I) = (-1)^{\mathrm{specflow}(C(\eta), I)}. \tag{2.23}$$

In the next section we link the spectral flow of $C(\eta)$ to the Conley-Zehnder indices of non-degenerate zeroes and therefore to the Euler-Floer characteristic.

2.4 The Conley-Zehnder index

We discuss the Conley-Zehnder index for Hamiltonian systems and mechanical systems, and explain the relation with the local degree and the Morse index for mechanical systems.

2.4.1 Hamiltonian systems

For a non-degenerate 1-periodic solution $x(t)$ of the Hamilton equations the Conley-Zehnder index can be defined as follows. The linearized flow Ψ is given by

$$\begin{cases} -J\dfrac{d\Psi}{dt} - D_x^2 H(x,t)\Psi = 0 \\ \Psi(0) = \mathrm{Id}, \end{cases}$$

By Lemma 2.3.2(iii), a 1-periodic solution is non-degenerate if $\Psi(1)$ has no eigenvalues equal to 1. The Conley-Zehnder index is defined using the symplectic path $\Psi(t)$. Following [44], consider the crossing form $\mathbf{\Gamma}(\Psi, t)$, defined for vectors $\xi \in \ker(\Psi(t) - \mathrm{Id})$,

$$\mathbf{\Gamma}(\Psi, t)\xi = \omega\big(\xi, \frac{d}{dt}\Psi(t)\xi\big) = (\xi, D_x^2 H(x(t), t)\xi). \tag{2.24}$$

A crossing $t_0 > 0$ is defined by $\det(\Psi(t_0) - \mathrm{Id}) = 0$. A crossing is regular if the crossing form is non-singular. A path $t \mapsto \Psi(t)$ is regular if all crossings are regular. Any path can be approximated by a regular path with the same endpoints and which is homotopic to the initial path, see [43] for details. For a regular path $t \mapsto \Psi(t)$ the Conley-Zehnder index is given by

$$\mu^{CZ}(\Psi) = \frac{1}{2}\operatorname{sgn} D_x^2 H(x(0), 0)) + \sum_{\substack{t_0 > 0, \\ \det(\Psi(t_0) - \mathrm{Id}) = 0}} \operatorname{sgn} \mathbf{\Gamma}(\Psi, t_0). \tag{2.25}$$

For a non-degenerate 1-periodic solution $x(t)$ we define the Conley-Zehnder index as $\mu^{CZ}(x) := \mu^{CZ}(\Psi)$, and the index is integer valued.

Let x be a 1-periodic solution and consider the path $\eta \mapsto B(\eta; x) = -J\frac{d}{dt} - S(t;\eta)$, where, as before, $S(t;\eta)$ is a smooth path of symmetric matrices with endpoints $S(t;0) = \theta\,\mathrm{Id}$ and $S(t;1) = D_x^2 H(x(t), t)$ with $\theta \neq 2\pi k, k \in \mathbb{Z}$. The operators $B(\eta) = B(\eta; x)$ are unbounded operators on $L^2(\mathbb{R}/\mathbb{Z})$, with domain $H^1(\mathbb{R}/\mathbb{Z})$. A path $\eta \mapsto B(\eta)$ is continuously differentiable in the (weak) operator topology of $\mathcal{B}(H^1, L^2)$ and Hypotheses (A1)-(A3) in [44] are satisfied. We now repeat the definition of spectral flow for a path of unbounded operators as developed in [44]. The crossing operator for a path $\eta \mapsto B(\eta)$ is given by $\mathbf{\Gamma}(B, \eta) = \pi\frac{d}{d\eta}B(\eta)\pi|_{\ker B(\eta)}$, where π is the orthogonal projection onto $\ker B(\eta)$. A crossing $\eta_0 \in I$ is a number for which the operator $B(\eta_0)$ is not invertible. A crossing is regular if $\mathbf{\Gamma}(B, \eta_0)$ is non-singular. A point η_0 for which $\dim\ker B(\eta_0) = 1$, is called a simple crossing. A path $\eta \mapsto B(\eta)$ is called generic if all crossing are simple. A path $\eta \mapsto B(\eta)$ can always be chosen to be generic. At a simple crossing η_0 there exists a C^1-curve $\ell(\eta)$, for η near η_0, and $\ell(\eta)$ is an eigenvalue of $B(\eta)$ with $\ell(\eta_0) = 0$ and $\ell'(\eta_0) \neq 0$. The spectral flow for a generic path is defined by

$$\text{specflow}(B(\eta), I) = \sum_{\ell(\eta_0)=0} \text{sgn}(\ell'(\eta_0)), \tag{2.26}$$

and at simple crossings η_0,

$$\mathbf{\Gamma}(B, \eta)\phi(\eta_0) = \Big(\frac{d}{d\eta}B(\eta_0)\phi(\eta_0), \phi(\eta_0)\Big)_{L^2}\phi(\eta_0) = \ell'(\eta_0)\phi(\eta_0), \tag{2.27}$$

after normalizing $\phi(\eta_0)$ in $L^2(\mathbb{R}/\mathbb{Z})$. As before the derivative of ℓ at η_0 is given by

$$\ell'(\eta_0) = -\big(\partial_\eta S(t;\eta_0)\phi(\eta_0), \phi(\eta_0)\big)_{L^2}. \tag{2.28}$$

2.4.1. Proposition. *Let $\eta \mapsto B(\eta), \eta \in I$, as defined above, be a generic path of unbounded self-adjoint operators with invertible endpoints, and let $\eta \mapsto \Psi(\eta; t)$ be the associated path of symplectic matrices defined by*

$$\begin{cases} -J\dfrac{d\Psi}{dt}(t;\eta) - S(t;\eta)\Psi(t;\eta) = 0 \\ \Psi(0;\eta) = \text{Id}, \end{cases}$$

Then

$$\text{specflow}(B(\eta), I) = \mu^{CZ}_{B(0)} - \mu^{CZ}_{B(1)} \tag{2.29}$$

where $\mu^{CZ}_{B(0)} = \mu^{CZ}(\Psi(t;0))$, $\mu^{CZ}_{B(1)} = \mu^{CZ}(\Psi(t;1))$.

Proof. The expression for the spectral flow follows from [44] and [49]. ■

In the case $\eta = 0$, the Conley-Zehnder index $\mu^{CZ}_{B(0)}$ can be computed explicitly. Recall that $B(0) = -J\frac{d}{dt} - S(0) = -J\frac{d}{dt} - \theta\,\text{Id}$.

2.4.2. Lemma. *Let $\theta > 0$ (fixed) and $\theta \neq 2\pi k$, then $\mu^{CZ}_{B(0)} = 1 + 2\lfloor\frac{\theta}{2\pi}\rfloor$.*

Proof. The solution to $B(0)\Psi(t) = 0$ is given by $\Psi(t) = e^{\theta J t}$ and $\det(\Psi(1) - \text{Id}) = 0$ exactly when $t = t_0 = \frac{2\pi k}{\theta}$. By (2.24) and (2.25) we have that $\mathbf{\Gamma}(\Psi, t)\xi = \theta|\xi|^2$ and therefore $\mu^{CZ}_{B(0)} = 1 + 2\lfloor\frac{\theta}{2\pi}\rfloor$, which proves the lemma. ■

The zeroes $x \in \Phi^{-1}_{\mu,H}(0)$ in $\Omega = [x]\,\text{rel}\,y$ can estimated by Braid Floer homology $\text{HB}_*([x]\,\text{rel}\,y)$ of $\Omega = [x]\,\text{rel}\,y$. The *Euler-Floer* characteristic of $\text{HB}_*([x]\,\text{rel}\,y)$ is defined as

$$\chi\big(\text{HB}_*([x]\,\text{rel}\,y)\big) := \sum_{k\in\mathbb{Z}} (-1)^k \dim \text{HB}_k([x]\,\text{rel}\,y). \tag{2.30}$$

In [49] the following analogue of the Poincaré-Hopf formula is proved.

2.4.3. Proposition. *For a proper braid class* $[x]\,\mathrm{rel}\,y$ *and a generic Hamiltonian* $H \in \mathcal{H}^{\mathrm{reg}}_{\|}(y)$*, it holds that*

$$\chi\big(\mathrm{HB}_*([x]\,\mathrm{rel}\,y)\big) = \sum_{x\in\Phi^{-1}_{\mu,H}(0)} (-1)^{\mu^{CZ}(x)}.$$

It remains to show that $\chi\big(\mathrm{HB}_*([x]\,\mathrm{rel}\,y)\big)$ and $\deg_{LS}(\Phi_{\mu,H},\Omega,0)$ are related.

2.4.4. Proposition. *For a proper braid class* $[x]\,\mathrm{rel}\,y$ *and a generic Hamiltonian* $H \in \mathcal{H}^{\mathrm{reg}}_{\|}(y)$*, we have that*

$$\chi\big(\mathrm{HB}_*([x]\,\mathrm{rel}\,y)\big) = -\sum_{x_i\in\Phi^{-1}_{\mu,H}(0)} (-1)^{-\,\mathrm{specflow}(B(\eta;x),I)}, \tag{2.31}$$

where $\eta \mapsto B(\eta;x)$ *is given above for* $x \in \Phi^{-1}_{\mu,H}(0)$.

Proof. By Proposition 2.4.1 and Lemma 2.4.2 the spectral flow satisfies,

$$\begin{aligned}\mu^{CZ}(x) = \mu^{CZ}_{B(1;x)} &= \mu^{CZ}_{B(0)} - \mathrm{specflow}(B(\eta;x),I)\\ &= 1+2\left\lfloor\tfrac{\theta}{2\pi}\right\rfloor - \mathrm{specflow}(B(\eta;x),I).\end{aligned}$$

This implies

$$(-1)^{\mu^{CZ}(x)} = -(-1)^{-\,\mathrm{specflow}(B(\eta;x),I)},$$

which completes the proof. ■

2.4.2 Mechanical systems

A mechanical system is defined as the Euler-Lagrange equations of the Lagrangian density $L(q,t) = \frac{1}{2}q_t^2 - V(q,t)$. The linearization at a critical points $q(t)$ of the Lagrangian action is given by the unbounded opeartor

$$-\frac{d^2}{dt^2} - D_q^2 V(q(t),t) : H^2(\mathbb{R}/\mathbb{Z}) \subset L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z}).$$

Consider a path of unbounded self-adjoint operators on $L^2(\mathbb{R}/\mathbb{Z})$ given by $\eta \mapsto D(\eta) = -\frac{d^2}{dt^2} - Q(t;\eta)$, with $Q(t;\eta)$ smooth. If $D(0)$ and $D(1)$ are invertible, then the spectral flow is well-defined.

2.4.5. Proposition. *Assume that the endpoints of $\eta \mapsto D(\eta)$ are invertible. Then*

$$\text{specflow}(D(\eta), I) = \beta_{D(0)} - \beta_{D(1)}, \tag{2.32}$$

where $\beta_{D(0)}$ and $\beta_{D(1)}$ are the Morse indices of $D(0)$ and $D(1)$ respectively.

Proof. In [44] the concatenation property of the spectral flow is proved. We use concatenation as follows. Let $c > 0$ be a sufficiently large constant such that $D(0) + c\,\text{Id}$ and $D(1) + c\,\text{Id}$ are positive definite self-adjoint operators on $L^2(\mathbb{R}/\mathbb{Z})$. Consider the paths $\eta \mapsto D_1(\eta) = D(0) + \eta c\,\text{Id}$ and $\eta \mapsto D_2(\eta) = D(1) + (1-\eta)c\,\text{Id}$. Their concatenation $D_1 \# D_2$ is a path from $D(0)$ to $D(1)$ and $\eta \mapsto D_1\#D_2$ is homotopic to $\eta \mapsto D(\eta)$. Using the homotopy invariance and the concatenation property of the spectral flow we obtain

$$\text{specflow}(D(\eta), I) = \text{specflow}(D_1\#D_2, I) = \text{specflow}(D_1, I) + \text{specflow}(D_2, I).$$

Since $D(0)$ is invertible, the regular crossings of $D_1(\eta)$ are given by $\eta_i^1 = -\frac{\lambda_i}{c}$, where λ_i are negative eigenvalues of $D(0)$. By the positive definiteness of $D(0) + c\,\text{Id}$, the negative eigenvalues of $D(0)$ satisfy $0 > \lambda_i > -c$. For the crossing η_i this implies

$$0 < \eta_i = -\frac{\lambda_i}{c} < 1,$$

and therefore the number of crossings equals the number of negative eigenvalues of $D(0)$ counted with multiplicity. By the choice of c, we also have that $\frac{d}{d\eta} D_1(\eta) = c\,\text{Id}$ is positive definite and therefore the signature of the crossing operator of $D_1(\eta)$ is exactly the number of negative eigenvalues of $D(0)$, i.e. $\text{specflow}(D_1, I) = \beta_{D(0)}$. For $D_2(\eta)$ we obtain, $\text{specflow}(D_2, I) = -\beta_{D(1)}$. This proves that $\text{specflow}(D(\eta), I) = \beta_{D(0)} - \beta_{D(1)}$. ■

For a mechanical system we have the Hamiltonian $H(x,t) = \frac{1}{2}p^2 + V(q,t)$. As such the Conley-Zenhder index of a critical point q can be defined as the Conley-Zehnder index of $x = (q_t, q)$ using the mechanical Hamiltonian, see also [1] and [16].

2.4.6. Lemma. *Let q be a critical point of the mechanical Lagrangian action, then the associated Conley-Zehnder index $\mu^{CZ}(x)$ is well-defined, and $\mu^{CZ}(x) = \beta(q)$, where $\beta(q)$ is the Morse index of q.*

Proof. As before, consider the curves $\eta \mapsto B(\eta)$ and $\eta \mapsto D(\eta)$, $\eta \in I = [0,1]$ given by

$$B(\eta) = -J\frac{d}{dt} - \begin{pmatrix} 1 & 0 \\ 0 & Q(t;\eta) \end{pmatrix}, \quad D(\eta) = -\frac{d^2}{dt^2} - Q(t;\eta).$$

The crossing forms of the curves are the same — $\mathbf{\Gamma}(B,\eta) = \mathbf{\Gamma}(D,\eta)$ — and therefore also the crossings η_0 are identical. Indeed, $B(\eta_0)$ is non-invertible if and only if $D(\eta_0)$ is non-invertible. Consequently, $\mathrm{specflow}\big(B(\eta), I\big) = \mathrm{specflow}\big(D(\eta), I\big)$ and the Propositions 2.4.1 and 2.4.5 then imply that

$$\beta_{D(0)} - \beta_{D(1)} = \mu^{CZ}_{B(0)} - \mu^{CZ}_{B(1)}. \tag{2.33}$$

Now choose $Q(t;\eta)$ such that $Q(t;0) = d^2V(q(t),t) + c$ and $Q(t;1) = D_q^2V(q(t),t)$ and such that $\eta \mapsto B(\eta)$ and $\eta \mapsto D(\eta)$ are regular curves. If $c \ll 0$, then $\beta_{D(0)} = 0$. In order to compute $\mu^{CZ}_{B(0)}$ we invoke the crossing from $\mathbf{\Gamma}(\Psi, t)$ for the associated symplectic path $\Psi(t)$ as explained in Section 2.4. Crossings at $t_0 \in (0,1]$ correspond to non-trivial solutions of the equation $D(0)\psi = 0$ on $[0, t_0]$, with periodic boundary conditions. To be more precise, let $\Psi = (\phi, \psi)$, then $B(0)\Psi = 0$ is equivalent to $\psi_t = \phi$ and $-\phi_t - \big(D_q^2V(q(t),t) + c\big)\psi = 0$, which yields the equation $D(0)\psi = 0$. For the latter the kernel is trivial for any $t_0 \in (0,1]$. Indeed, if ψ is a solution, then $\int_0^{t_0} |\psi_t|^2 = \int_0^{t_0} (D_q^2V(q,t) + c)\psi^2 < 0$, which is a contradiction. Therefore, there are no crossing $t_0 \in (0,1]$. As for $t_0 = 0$ we have that $\big(D_q^2V(q(0),0) + c\big) < 0$, which implies that $\mathrm{sgn}\, S(0;0) = 0$ and therefore $\mu^{CZ}_{B(0)} = 0$, which proves the lemma. ■

2.5 The spectral flows are the same

In order to show that the spectral flows are the same we use the fact that the paths $\eta \mapsto C(\eta)$ and $\eta \mapsto B(\eta)$ for a non-degenerate zero $x_i \in \Phi^{-1}_{\mu,H}(0)$ are chosen to have only simple crossings for their crossing operators, i.e. zero eigenvalues are simple. In this case the spectral flows are determined by the signs of the derivatives of the eigenvalues at the crossings. For $\eta \mapsto B(\eta)$ the expression given by Equation (2.28) and from Equation (2.21) a similar expression for $\eta \mapsto C(\eta)$ can be derived and is given by

$$\lambda'(\eta_0) = -\big(N_\mu^{-1}\partial_\eta S(t;\eta_0)\psi(\eta_0), \psi(\eta_0)\big)_{H^{1/2}} = -\big(\partial_\eta S(t;\eta_0)\psi(\eta_0), \psi(\eta_0)\big)_{L^2} \tag{2.34}$$

2.5.1. Lemma. *The sets* $\{\eta \in [0,1] : C(\eta)\psi(\eta) = 0\}$ *and* $\{\eta \in [0,1] : B(\eta)\phi(\eta) = 0\}$ *are the same, and the operators* $C(\eta)$ *and* $B(\eta)$ *have the same eigenfunctions at crossings* η_0*. In particular,* $\eta \mapsto B(\eta)$ *is generic if and only if* $\eta \mapsto C(\eta)$ *is generic.*

Proof. Given $\eta_0 \in [0,1]$ such that $C(\eta_0)\psi(\eta_0) = 0$, then

$$P_\mu\psi(\eta_0) - N_\mu^{-1}(S(\eta_0;t) + \mu)\psi(\eta_0) = 0,$$

and thus $\psi(\eta_0) - L_\mu^{-1}(S(\eta_0;t) + \mu)\psi(\eta_0) = 0$, which is equivalent to the equation $\left(-J\frac{d}{dt} - S(t;\eta_0)\right)\psi(\eta_0) = 0$, i.e. $B(\eta_0)\psi(\eta_0) = 0$. ■

2.5.2. Lemma. *For all* $\mu > 0$*, with* $\mu \neq 2\pi k$*,* $k \in \mathbb{Z}$*,* $\operatorname{sgn}\lambda'(\eta_0) = \operatorname{sgn}\ell'(\eta_0)$ *for all crossings at* η_0*.*

Proof. The eigenfunctions $\psi(\eta_0)$ in Equation (2.34) for $\lambda'(\eta_0)$ are normalized in $H^{1/2}(\mathbb{R}/\mathbb{Z})$. Therefore they relate to the eigenfunctions $\phi(\eta_0)$ in Equation (2.28) for $\ell'(\eta_0)$ as follows:

$$\psi(\eta_0) = \frac{\phi(\eta_0)}{\|\phi(\eta_0)\|_{H^{1/2}}}, \quad \|\phi(\eta_0)\|_{L^2} = 1.$$

Combining Equations (2.28) and (2.34) then gives

$$\begin{aligned}\lambda'(\eta_0) &= -\left(\partial_\eta S(t;\eta_0)\psi(\eta_0), \psi(\eta_0)\right)_{L^2} \\ &= -\frac{1}{\|\phi(\eta_0)\|^2_{H^{1/2}}}\left(\partial_\eta S(t;\eta_0)\phi(\eta_0), \phi(\eta_0)\right)_{L^2} = \frac{\ell'(\eta_0)}{\|\phi(\eta_0)\|^2_{H^{1/2}}},\end{aligned}$$

which proves the lemma. ■

Lemma 2.5.2 implies that for any non-degenerate $x \in \Phi^{-1}_{\mu,H}(0) \cap \Omega$

$$\operatorname{specflow}(C(\eta;x), I) = \operatorname{specflow}(B(\eta;x), I), \tag{2.35}$$

where $B(\eta;x)$ and $C(\eta;x)$ are the above described path associated with x. Therefore

$$\operatorname{parity}(A(\eta;x), I) = (-1)^{\operatorname{specflow}(C(\eta;x),I)} = (-1)^{\operatorname{specflow}(B(\eta;x),I)}, \tag{2.36}$$

which yields the following proposition.

2.5.3. Proposition. *The Leray-Schauder degree satisfies*

$$\deg_{LS}(\Phi_{\mu,H}, \Omega, 0) = -\chi\big(\mathrm{HB}_*([x] \operatorname{rel} y)\big).$$

Proof. For any Hamiltonian $H \in \mathcal{H}_{\|}(y)$ there exists a generic Hamiltonian $\tilde{H} \in \mathcal{H}^{\mathrm{reg}}_{\|}(y)$ such all zeroes $x_i \in \Phi^{-1}_{\mu,\tilde{H}}(0) \cap \Omega$ are non-degenerate. Since $\Omega = [x] \operatorname{rel} y$ is isolating for all Hamiltonians in $\mathcal{H}_{\|}(y)$, the invariance if the Leray-Schauder degree yields $\deg_{LS}\big(\Phi_{\mu,H},\Omega,0\big)=\deg_{LS}\big(\Phi_{\mu,\tilde H},\Omega,0\big)$. From the Propositions 2.3.6 and 2.4.4 and Equation (2.36) we conclude that

$$\begin{aligned}\deg_{LS}\big(\Phi_{\mu,H},\Omega,0\big)&=\deg_{LS}\big(\Phi_{\mu,\tilde H},\Omega,0\big)\\&=\sum_{x\in\Phi^{-1}_{\mu,\tilde H}(0)}\deg_{LS}\big(\Phi_{\mu,\tilde H},B_\epsilon(x),0\big)=\sum_{x\in\Phi^{-1}_{\mu,\tilde H}(0)}\mathrm{parity}(A(\eta;x),I)\\&=\sum_{x\in\Phi^{-1}_{\mu,\tilde H}(0)}(-1)^{\mathrm{specflow}(B(\eta;x),I)}=\sum_{x\in\Phi^{-1}_{\mu,\tilde H}(0)}(-1)^{-\,\mathrm{specflow}(B(\eta;x),I)}\\&=-\chi\big(\mathrm{HB}_*([x]\,\mathrm{rel}\,y)\big),\end{aligned}$$

which completes the proof. ■

2.5.4. Remark. As $\mu\gg 1$ it holds that $\ell'(\eta_0)\sim\mu\lambda'(\eta_0)$. Indeed, $\|\phi(\eta_0)\|^2_{H^{1/2}}=\sum_k(2\pi|k|+\mu)a_k^2$, where a_k are the Fourier coefficients of $\phi(\eta_0)$ and $\sum_k a_k^2=1$. Since $\phi(\eta_0)$ are smooth functions the Fourier coefficients satisfy the following properties. For any $\delta>0$ and any $s>0$, there exists $N_{\delta,s}>0$ such that $\sum_{|k|\geq N}|k|^{2s}|a_k|^2\leq\delta$, for all $N\geq N_{\delta,s}$. From the latter it follows that $\sum_k 2\pi|k|a_k^2\leq C$, with $C>0$ independent of η_0 and μ. We derive that $\mu\leq\|\phi(\eta_0)\|^2_{H^{1/2}}\leq C+\mu$ and

$$1\leftarrow\frac{\mu}{C+\mu}\leq\frac{\mu\lambda'(\eta_0)}{\ell'(\eta_0)}=\frac{\mu}{\|\phi(\eta_0)\|^2_{H^{1/2}}}\leq\frac{\mu}{\mu}=1,$$

as $\mu\to\infty$, which proves our statement.

2.6 The proof of Theorems 2.1.1 and 2.1.2

We start with the proof of Theorem 2.1.2. Since $\mathrm{HB}_*([x]\,\mathrm{rel}\,y)$ is an invariant of the proper braid class $[x\,\mathrm{rel}\,y]$ it does not depend on a particular fiber $[x]\,\mathrm{rel}\,y$. Therefore we denote the Euler-Floer characteristic by $\chi(x\,\mathrm{rel}\,y):=\chi\big(\mathrm{HB}_*([x]\,\mathrm{rel}\,y)\big)$. Recall the homotopy invariance of the Leray-Schauder degree as expressed in Equation (2.9)

$$\deg_{LS}(\Phi_\mu,\Omega,0)=\deg_{LS}(\Phi_{\mu,\alpha},\Omega,0)=\deg_{LS}(\Phi_{\mu,H},\Omega,0).$$

By Proposition 2.5.3 we have that

$$\deg_{LS}(\Phi_\mu,\Omega,0)=\deg_{LS}(\Phi_{\mu,H},\Omega,0)=-\chi(x\,\mathrm{rel}\,y),$$

and $\chi(x \operatorname{rel} y) \neq 0$, then implies that $\Phi_\mu^{-1}(0) \cap \Omega \neq \varnothing$. Therefore there exist closed integral curves in any relative braid class fiber of $[x \operatorname{rel} y]$, whenever $\chi(x \operatorname{rel} y) \neq 0$, and this completes the proof of Theorem 2.1.2.

The remainder of this section is to prove the Poincaré-Hopf Formula in Theorem 2.1.1 for closed integral curves in proper braid fibers. The mapping

$$\mathscr{E} : C^1(\mathbb{R}/\mathbb{Z}) \to C^0(\mathbb{R}/\mathbb{Z}), \quad \mathscr{E}(x) = \frac{dx}{dt} - X(x,t),$$

is smooth (nonlinear) Fredholm mapping of index 0. Let $M \in \mathrm{GL}(C^0, C^1)$ be an isomorphism such that $M\mathscr{E}(x)$ is of the form $M\mathscr{E}(x) = \Phi_M(x) = x - K_M(x)$, with $K_M : C^1(\mathbb{R}/\mathbb{Z}) \to C^1(\mathbb{R}/\mathbb{Z})$ compact. Such isomorphisms M (constant parametrices) obviously exist. For example $M = \left(\frac{d}{dt} + 1\right)^{-1}$, or $M = -JL_\mu^{-1}$. The mappings $\Phi_M : C^1(\mathbb{R}/\mathbb{Z}) \to C^1(\mathbb{R}/\mathbb{Z})$ are Fredholm mappings of index 0.

Let $x \in C^1(\mathbb{R}/\mathbb{Z})$ be a non-degenerate zero of $\mathscr{E}$ and recall the index $\iota(x)$:

$$\iota(x) = -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \deg_{LS}\big(\Phi_M, B_\epsilon(x), 0\big),$$

where $\Theta \in \mathrm{M}_{2\times 2}(\mathbb{R})$, with $\sigma(\Theta) \cap 2\pi k i \mathbb{R} = \varnothing$, $k \in \mathbb{Z}$ and $\beta_M(\Theta)$ is the Morse index of $\mathrm{Id} - K_M(0)$.

2.6.1. Lemma. *The index $\iota(x)$ for a non-degenerate zero of $\mathscr{E}$ is well-defined, i.e. independent of the choices of $M \in \mathrm{GL}(C^0, C^1)$ and $\Theta \in \mathrm{M}_{2\times 2}(\mathbb{R})$.*

Proof. Consider smooth paths $\eta \mapsto F_\Theta(\eta)$, defined by $F_\Theta(\eta) = \frac{d}{dt} - R(t;\eta)$, where $R(t;0) = \Theta$ and $R(t;1) = D_x X(x(t), t)$. The path

$$F_\Theta : [0,1] \to \mathrm{Fred}_0(C^1, C^0)$$

has invertible end points, and by the theory in [19, 20] we have that the parity of $\eta \mapsto F_\Theta(\eta)$ is well-defined and independent of M, i.e.

$$\begin{aligned} \operatorname{parity}(F_\Theta(\eta), I) &= \operatorname{parity}(D_{M,\Theta}(\eta), I) = (-1)^{\beta_M(\Theta)}(-1)^{\beta_M(x)} \\ &= (-1)^{\beta_M(\Theta)} \deg_{LS}\big(\Phi_M, B_\epsilon(x), 0\big), \end{aligned}$$

where $D_{M,\Theta}(\eta) = MF_\Theta(\eta)$ and $\beta_M(x)$ is the Morse index of $D_{M,\Theta}(1) = \mathrm{Id} - K_M(1)$. It remains to show that the index $\iota(x)$ is independent with respect to Θ. Let Θ and Θ' be admissible matrices and let $\eta \mapsto G(\eta)$ be a path connecting $G(0) = \frac{d}{dt} - \Theta$ and $G(1) = \frac{d}{dt} - \Theta'$. For the parities it holds that

$$\operatorname{parity}(F_\Theta(\eta), I) = \operatorname{parity}(G(\eta), I) \cdot \operatorname{parity}(F_{\Theta'}(\eta), I).$$

To compute $\text{parity}(G(\eta), I)$ we consider a special parametrix M_μ, given by $M_\mu = \left(\frac{d}{dt} + \mu\right)^{-1}$, $\mu > 0$. From the definition of parity we have that

$$\text{parity}(G(\eta), I) = \text{parity}(M_\mu G(\eta), I) = \deg_{LS}\big(M_\mu G(0)\big) \cdot \deg_{LS}\big(M_\mu G(1)\big).$$

We now compute the Leray-Schauder degrees of $M_\mu G(0)$ and $M_\mu G(1)$. We start with Θ and in order to compute the degree we determine the Morse index. Consider the eigenvalue problem

$$M_\mu G(0)\psi = \lambda\psi, \quad \lambda \in \mathbb{R},$$

which is equivalent to $(1-\lambda)\frac{d\psi}{dt} = \big(\Theta + \lambda\mu\big)\psi$. Non-trivial solutions are given by $\psi(t) = \exp\left(\frac{\Theta+\lambda\mu}{1-\lambda}t\right)\psi_0$, which yields the condition $\frac{\theta+\lambda\mu}{1-\lambda} = 2\pi k i$, $k \in \mathbb{Z}$, where θ is an eigenvalues of Θ. We now consider three cases:

(i) $\theta_\pm = a \pm ib$. In case of a negative eigenvalue λ we have $\frac{a+\lambda\mu}{1-\lambda} = 0$ and $\frac{b}{1-\lambda} = 2\pi k$. The same $\lambda < 0$ also suffices for the conjugate eigenvalue via $\frac{-b}{1-\lambda} = -2\pi k$. This implies that any eigenvalue $\lambda < 0$ has multiplicity 2, and thus $\deg_{LS}\big(M_\mu G(0)\big) = 1$.

(ii) $\theta_\pm \in \mathbb{R}$, $\theta_- \cdot \theta_+ > 0$. In case of a negative eigenvalue λ we have $\frac{\theta_\pm+\lambda\mu}{1-\lambda} = 0$ and thus $\lambda_\pm = -\frac{\theta_\pm}{\mu}$, which yields two negative or two positive eigenvalues. As before $\deg_{LS}\big(M_\mu G(0)\big) = 1$.

(iii) $\theta_\pm \in \mathbb{R}$, $\theta_- \cdot \theta_+ < 0$. From case (ii) we easily derive that there exist two eigenvalues $\lambda_\pm$, one positive and one negative, and therefore $\deg_{LS}\big(M_\mu G(0)\big) = -1$.

These cases combined impliy that $\deg_{LS}\big(M_\mu G(0)\big) = \text{sgn}(\det(\Theta))$ and

$$\text{parity}(G(\eta), I) = \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta')).$$

From the latter we derive:

$$\begin{aligned}\text{sgn}(\det(\Theta)) \cdot \text{parity}(F_\Theta(\eta), I) & \\ &= \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta)) \cdot \text{sgn}(\det(\Theta')) \cdot \text{parity}(F_{\Theta'}(\eta), I) \\ &= \text{sgn}(\det(\Theta')) \cdot \text{parity}(F_{\Theta'}(\eta), I),\end{aligned}$$

which proves the independence of Θ. ■

Lemmas 2.6.1 shows that the index of a non-degenerate zero of $\mathscr{E}$ is well-defined. We now show that the same holds for isolated zeroes.

2.6.2. Lemma. *The index $\iota(x)$ for an isolated zero of $\mathscr{E}$ is well-defined and for a fixed choice of M and Θ the index is given by*

$$\iota(x) = -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \deg_{LS}\big(\Phi_M, B_\epsilon(x), 0\big),$$

where $\epsilon > 0$ is small enough such that x is the only zero of $\mathscr{E}$ in $B_\epsilon(x)$.

Proof. By the Sard-Smale Theorem one can choose an arbitrarily small $h \in C^0(\mathbb{R}/\mathbb{Z})$, $\|h\|_{C^0} < \epsilon'$, such that h is a regular value of $\mathscr{E}$ and $\mathscr{E}^{-1}(h) \cap B_\epsilon(x)$ consists of finitely many non-degenerate zeroes in x_h. Set $\widetilde{\mathscr{E}}(x) = \mathscr{E}(x) - h$ and define

$$\iota(x) = \sum_{x_h \in \widetilde{\mathscr{E}}^{-1}(0) \cap B_\epsilon(x)} \iota(x_h). \tag{2.37}$$

We now show that $\iota(x)$ is well-defined. Choose a fixed parametrix M (for $\mathscr{E}$) and fixed $\Theta \in \mathrm{M}_{2\times 2}(\mathbb{R})$, and let $\widetilde{\Phi}_M = M\widetilde{\mathscr{E}}$, then

$$\sum_{x_h} \iota(x_h) = -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \sum_{x_h} \deg_{LS}(\widetilde{\Phi}_M, B_{\epsilon_h}(x_h), 0),$$

where $B_{\epsilon_h}(x_h)$ are sufficiently small neighborhoods containing only one zero. From Leray-Schauder degree theory we derive that

$$\sum_{x_h} \deg_{LS}(\widetilde{\Phi}_M, B_{\epsilon_h}(x_h), 0) = \deg_{LS}(\widetilde{\Phi}_M, B_\epsilon(x), 0) = \deg_{LS}(\Phi_M, B_\epsilon(x), 0),$$

which proves the lemma. ∎

Theorem 2.1.1 now follows from the Leray-Schauder degree. Suppose all zeroes of $\mathscr{E}$ in $\Omega = [x] \operatorname{rel} y$ are isolated, then Lemma 2.6.2 implies that

$$\begin{aligned}\sum_{x \in \mathscr{E}^{-1}(0) \cap \Omega} \iota(x) &= -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \sum_{x} \deg_{LS}\big(\Phi_M, B_\epsilon(x), 0\big) \\ &= -\operatorname{sgn}(\det(\Theta))(-1)^{\beta_M(\Theta)} \deg_{LS}\big(\Phi_M, \Omega, 0\big)\end{aligned}$$

Since the latter expression is independent of M and Θ we choose $M = L_\mu^{-1}$ and $\Theta = \theta J$. Then, $\Phi_M = \Phi_\mu$, and for the indices we have $\operatorname{sgn}(\det(\theta J)) = 1$ and by Lemma 2.3.5, $(-1)^{\beta_{L_\mu^{-1}}(\theta J)} = 1$. By Proposition 2.5.3, $\deg_{LS}(\Phi_\mu, \Omega, 0) = -\chi\big(x \operatorname{rel} y\big)$, which, by substitution of these choices into the index formula, yields

$$\sum_{x \in \mathscr{E}^{-1}(0) \cap \Omega} \iota(x) = \chi\big(x \operatorname{rel} y\big),$$

completing the proof Theorem 2.1.1.

2.7 Computing the Euler-Floer characteristic

In section we prove Theorem 2.1.5 and show that the Euler-Floer characteristic can be determined via a discrete topological invariant.

2.7.1 Hyperbolic Hamiltonians on $\mathbb{R}^2$

Consider Hamiltonians of the form

$$H(x,t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + h(x,t), \tag{2.38}$$

where h satisfies the following hypotheses:

(h1) $h \in C^\infty(\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z})$;

(h2) $\text{supp}(h) \subset \mathbb{R} \times [-R, R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 0$;

(h3) $\|h\|_{C_b^2(\mathbb{R}^2\times\mathbb{R}/\mathbb{Z})} \le c$.

2.7.1. Lemma. *Let H be given by (2.38), with h satisfying (h1)-(h3). Then, there exists a constant $R' \ge R > 0$, such any 1-periodic solution of x of $x' = X_H(x,t)$ satisfies the estimate*

$$|x(t)| \le R', \quad \textit{for all } \ t \in \mathbb{R}/\mathbb{Z}.$$

Proof. The Hamilton equation in local coordinates are given by

$$p_t = q - h_q(p,q,t), \quad q_t = p + h_p(p,q,t).$$

Since h is smooth we can rewrite the equations as

$$q_{tt} = h_{pq}(p,q,t)q_t + \big(1 + h_{pp}(p,q,t)\big)\big(q - h_q(p,q,t)\big) + h_{pt}(p,q,t). \tag{2.39}$$

If $x(t)$ is a 1-periodic solution to the Hamilton equations, and suppose there exists an interval $I = [t_0, t_1] \subset [0,1]$ such that $|q(t)| > R$ on $\text{int}(I)$ and $|q(t)|\big|_{\partial I} = R$. The function $q|_I$ satisfies the equation $q_{tt} - q = 0$, and obviously such solutions do not exist. Indeed, if $q|_I \ge R$, then $q_t(t_0) \ge 0$ and $q_t(t_1) \le 0$ and thus $0 \ge q_t|_{\partial I} = \int_I q \ge R|I| > 0$, a contradiction. The same holds for $q|_I \le -R$. We conclude that

$$|q(t)| < R, \quad \text{for all } \ t \in \mathbb{R}/\mathbb{Z}.$$

We now use the a priori q-estimate in combination with Equation (2.39) and Hypothesis (h3). Multiplying Equation (2.39) by q and integrating over $[0,1]$ gives:

$$\int_0^1 q_t^2 = -\int_0^1 h_{pq} q_t q - \int_0^1 (1+h_{pp})(q-h_q)q - \int_0^1 h_{pt} q$$
$$\leq C\int_0^1 |q_t| + C \leq \epsilon \int_0^1 q_t^2 + C_\epsilon,$$

which implies that $\int_0^1 q_t^2 \leq C(R)$. The L^2-norm of the right hand side in (2.39) can be estimated using the L^∞ estimate on q and the L^2-estimate on q_t, which yields $\int_0^1 q_{tt}^2 \leq C(R)$. Combining these estimates we have that $\|q\|_{H^2(\mathbb{R}/\mathbb{Z})} \leq C(R)$ and thus $|q_t(t)| \leq C(R)$, for all $t \in \mathbb{R}/\mathbb{Z}$. From the Hamilton equations it follows that $|p(t)| \leq |q_t(t)| + C$, which proves the lemma. ■

2.7.2. Lemma. *If $H(x,t;\alpha)$, $\alpha \in [0,1]$ is a (smooth) homotopy of Hamiltonians satisfying (h1)-(h3) with uniform constants $R > 0$ and $c > 0$, then $|x_\alpha(t)| \leq R'$, for all 1-periodic solutions and for all $\alpha \in [0,1]$.*

Proof. The a priori H^2-estimates in Lemma 2.7.1 hold with uniform constants with respect to $\alpha \in [0,1]$. This then proves the lemma. ■

2.7.2 Braids on $\mathbb{R}^2$ and Legendrian braids

In Section 2.1 we defined braid classes as path components of closed loops in $\mathcal{L}\mathbf{C}_n(\mathbb{D}^2)$, denoted by $[x]$. If we consider closed loops in $\mathbf{C}_n(\mathbb{R}^2)$, then the braid classes will be denoted by $[x]_{\mathbb{R}^2}$. The same notation applies to relative braid classes $[x \operatorname{rel} y]_{\mathbb{R}^2}$. A relative braid class is proper if components $x_c \subset x$ cannot be deformed onto (i) itself, or other components $x'_c \subset x$, or (ii) components $y_c \subset y$. A fiber $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is *not* bounded!

In order to compute the Euler-Floer characteristic of $[x \operatorname{rel} y]$ we assume without loss of generality that $x \operatorname{rel} y$ is a positive representative. If not we compose $x \operatorname{rel} y$ with a sufficient number of positive full twists such that the resulting braid is positive, i.e. only positive crossings, see [49] for more details. The Euler-Floer characteristic remains unchanged. We denote a positive representative $x^+ \operatorname{rel} y^+$ again by $x \operatorname{rel} y$.

Define an augmented skeleton y^* by adding the constant strands $y_-(t) = (0,-1)$ and $y_+(t) = (0,1)$. For proper braid classes it holds that $[x \operatorname{rel} y] = [x \operatorname{rel} y^*]$. For notational simplicity we denote the augmented skeleton again by y. We also choose the representative $x \operatorname{rel} y$ with the additional the property that $\pi_2 x \operatorname{rel} \pi_2 y$ is a relative braid diagram, i.e. there are no tangencies between the strands, where π_2 the projection onto the q-coordinate. We denote the projection by $q \operatorname{rel} Q$, where

$q = \pi_2 x$ and $Q = \pi_2 y$. Special braids on $\mathbb{R}^2$ can be constructed from (smooth) positive braids. Define $x_L = (q_t, q)$ and $y_L = (Q_t, Q)$, where the subscript t denotes differentiating with respect to t. These are called *Legendrian braids* with respect to $\theta = pdt - dq$.

2.7.3. Lemma. *For positive braid $x \operatorname{rel} y$ with only transverse, positive crossings, the braids $x_L \operatorname{rel} y_L$ and $x \operatorname{rel} y$ are isotopic as braids on $\mathbb{R}^2$. Moreover, if $x_L \operatorname{rel} y_L$ and $x'_L \operatorname{rel} y'_L$ are isotopic Legrendrian braids, then they are isotopic via a Legendrian isotopy.*

Proof. By assumption $x \operatorname{rel} y$ is a representative for which the braid diagram $q \operatorname{rel} Q$ has only positive transverse crossings. Due to the transversality of intersections the associated Legendrian braid $x_L \operatorname{rel} y_L$ is a braid $[x \operatorname{rel} y]_{\mathbb{R}^2}$. Consider the homotopy

$$\zeta^j(t,\tau) = \tau p^j(t) + (1-\tau) q_t^j,$$

for every strand q^j. At q-intersections, i.e. times t_0 such that $q^j(t_0) = q^{j'}(t_0)$ for some $j \neq j'$, it holds that $p^j(t_0) - p^{j'}(t_0)$ and $q_t^j(t_0) - q_t^{j'}(t_0)$ are non-zero and have the same sign since all crossings in $x \operatorname{rel} y$ are positive! Therefore, $\zeta^j(t_0,\tau) \neq \zeta^{j'}(t_0,\tau)$ for any intersection t_0 and any $\tau \in [0,1]$, which shows that $x \operatorname{rel} y$ and $x_L \operatorname{rel} y_L$ are isotopic. Since $x_L \operatorname{rel} y_L$ and $x'_L \operatorname{rel} y'_L$ have only positive crossings, a smooth Legendrian isotopy exists. ∎

The associated equivalence class of Legendrian braid diagrams is denoted by $[q \operatorname{rel} Q]$ and its fibers by $[q] \operatorname{rel} Q$.

2.7.3 Lagrangian systems

Legendrian braids can be described with Lagrangian systems and Hamiltonians of the form $H_L(x,t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q,t)$. On the potential functions g we impose the following hypotheses:

(g1) $g \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$;

(g2) $\operatorname{supp}(g) \subset [-R, R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 1$.

In order to have a straightforward construction of a mechanical Lagrangian we may consider a special representation of y. The Euler-Floer characteristic $\chi(x \operatorname{rel} y)$ does not depend on the choice of the fiber $[x] \operatorname{rel} y$ and therefore also not on the skeleton y. We assume that y has linear crossings in y_L. Let $t = t_0$ be a crossing and let $I(t_0)$ be the set of labels defined by: $i, j \in I(t_0)$, if $i \neq j$ and $Q^i(t_0) = Q^j(t_0)$. A crossing at $t = t_0$ is *linear* if

$$Q_t^i(t) = \text{constant}, \quad \forall i \in I(t_0), \text{ and } \quad \forall t \in (-\epsilon + t_0, \epsilon + t_0),$$

for some $\epsilon = \epsilon(t_0) > 0$. Every skeleton Q with transverse crossings is isotopic to a skeleton with linear crossings via a small local deformation at crossings. For

Legendrian braids $x_L \operatorname{rel} y_L \in [x \operatorname{rel} y]_{\mathbb{R}^2}$ with linear crossings the following result holds:

2.7.4. Lemma. *Let y_L be a Legendrian skeleton with linear crossings. Then, there exists a Hamiltonian of the form $H_L(x,t) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q,t)$, with g satisfying Hypotheses (g1)-(g2), and $R > 0$ sufficiently large, such that y_L is a skeleton for $X_{H_L}(x,t)$.*

Proof. Due to the linear crossings in y_L we can follow the construction in [49]. For each strand Q^i we define the potentials $g^i(t,x) = -Q^i_{tt}(t)q$. By construction Q^i is a solution of the equation $Q^i_{tt} = -g^i_q(t,Q^i)$. Now choose small tubular neighborhoods of the strands Q^i and cut-off functions ω^i that are equal to 1 near Q^i and are supported in the tubular neighborhoods. If the tubular neighborhoods are narrow enough, then $\operatorname{supp}(\omega^i g^i) \cap \operatorname{supp}(\omega^j g^j) = \varnothing$, for all $i \neq j$, due to the fact that at crossings the functions g^i in question are zero. This implies that all strands Q^i satisfy the differential equation $Q^i_{tt} = -\sum_i \omega^j(t) g^j_q(Q^i,t)$ and on $[-1,1] \times \mathbb{R}/\mathbb{Z}$, the function is $\sum_i \omega^i(t) g^i(q,t)$ is compactly supported. The latter follows from the fact that for the constant strands $Q^i = \pm 1$, the potentials g^i vanish. Let $R > 1$ and define

$$\tilde{g}^i(t,q) = \begin{cases} g^i(t,q) & \text{for } \ |q| \le 1,\ t \in \mathbb{R}/\mathbb{Z}, \\ -\frac{1}{2m}q^2 & \text{for } |q| \ge R,\ t \in \mathbb{R}/\mathbb{Z}. \end{cases}$$

where $m = \#Q$, which yields smooth functions $\tilde{g}^i$ on $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Now define

$$g(q,t) = \frac{1}{2}q^2 + \sum_{i=1}^{m} \tilde{g}^i(q,t).$$

By construction $\operatorname{supp}(g) \subset [-R,R] \times \mathbb{R}/\mathbb{Z}$, for some $R > 1$ and the strands Q^i all satisfy the Euler-Lagrange equations $Q^i_{tt} = Q^i - g_{qq}(Q^i,t)$, which completes the proof. ■

The Hamiltonian H_L given by Lemma 2.7.4 gives rise to a Lagrangian system with the Lagrangian action given by

$$\mathscr{L}_g = \int_0^1 \frac{1}{2}q_t^2 + \frac{1}{2}q^2 - g(q,t)dt. \tag{2.40}$$

The braid class $[q] \operatorname{rel} Q$ is bounded due to the special strands ± 1 and all free strands q satisfy $-1 \le q(t) \le 1$. Therefore, the set of critical points of $\mathscr{L}_g$ in $[q] \operatorname{rel} Q$ is a compact set. The critical points of $\mathscr{L}_g$ in $[q] \operatorname{rel} Q$ are in one-to-one correspondence with the zeroes of the equation

$$\Phi_{\mu,H_L}(x) = x - L_\mu^{-1}\big(\nabla H_L(x,t) + \mu x\big) = 0,$$

in the set $\Omega_{\mathbb{R}^2} = [x_L]_{\mathbb{R}^2} \operatorname{rel} y_L$, which implies that Φ_{μ,H_L} is a proper mapping on $\Omega_{\mathbb{R}^2}$. From Lemma 2.7.1 we derive that the zeroes of Φ_{μ,H_L} are contained in ball in $\mathbb{R}^2$ with radius $R' > 1$, and thus $\Phi^{-1}_{\mu,H_L}(0) \cap \Omega_{\mathbb{R}^2} \subset B_{R'}(0) \subset C^1(\mathbb{R}/\mathbb{Z})$. Therefore the Leray-Schauder degree is well-defined and in the generic case Lemma 2.4.6 and Equations (2.23), (2.29) and (2.35) yield

$$\deg_{LS}(\Phi_{\mu,H_L}, \Omega_{\mathbb{R}^2}, 0) = -\sum_{x \in \Phi^{-1}_{\mu,H_L}(0) \cap \Omega_{\mathbb{R}^2}} (-1)^{\mu^{CZ}(x)} = -\sum_{q \in \operatorname{Crit}(\mathscr{L}_g) \cap ([q] \operatorname{rel} Q)} (-1)^{\beta(q)}. \tag{2.41}$$

We are now in a position to use a homotopy argument. We can scale y to a braid ρy such that the rescaled Legendrian braid ρy_L is supported in $\mathbb{D}^2$. By Lemma 2.7.3, y is isotopic to y_L and scaling defines an isotopy between y_L and ρy_L. Denote the isotopy from y to ρy_L by y_α. By Proposition 2.5.3 we obtain that for both skeletons y and ρy_L it holds that

$$\deg_{LS}(\Phi_{\mu,H}, \Omega, 0) = -\chi\big(x \operatorname{rel} y\big) = \deg_{LS}(\Phi_{\mu,H_\rho}, \Omega_\rho, 0),$$

where $\Omega_\rho = [\rho x_L] \operatorname{rel} \rho y_L \subset [x \operatorname{rel} y]$ and $H_\rho \in \mathcal{H}_{\|}(\rho y_L)$. Now extend H_ρ to $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$, such that Hypotheses (h1)-(h3) are satisfied for some $R > 1$. We denote the Hamiltonian again by H_ρ. By construction all zeroes of Φ_{μ,H_ρ} in $[\rho x_L] \operatorname{rel} \rho y_L$ are supported in $\mathbb{D}^2$ and therefore the zeroes of Φ_{μ,H_ρ} in $[\rho x_L]_{\mathbb{R}^2} \operatorname{rel} \rho y_L$ are also supported in $\mathbb{D}^2$. Indeed, any zero intersects $\mathbb{D}^2$, since the braid class is proper and since $\partial\mathbb{D}^2$ is invariant for the Hamiltonian vector field, a zero is either inside or outside $\mathbb{D}^2$. Combining these facts implies that a zero lies inside $\mathbb{D}^2$. This yields

$$\deg_{LS}(\Phi_{\mu,H_\rho}, \Omega_{\rho,\mathbb{R}^2}, 0) = \deg_{LS}(\Phi_{\mu,H_\rho}, \Omega_\rho, 0) = -\chi\big(x \operatorname{rel} y\big),$$

where $\Omega_{\rho,\mathbb{R}^2} = [\rho x_L]_{\mathbb{R}^2} \operatorname{rel} \rho y_L$. For the next homotopy we keep the skeleton ρy_L fixed as well as the domain $\Omega_{\rho,\mathbb{R}^2}$. Consider the linear homotopy of Hamiltonians

$$H_1(x,t;\alpha) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + (1-\alpha)h_\rho(x,t) + \alpha g_\rho(q,t),$$

where $H_{\rho,L}(t,x) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g_\rho(q,t)$ given by Lemma 2.7.4. This defines an admissible homotopy since ρy_L is a skeleton for all $\alpha \in [0,1]$. The uniform estimates are obtained, as before, by Lemma 2.7.2, which allows application of the Leray-Schauder degree:

$$\deg_{LS}(\Phi_{\mu,H_{\rho,L}}, \Omega_{\rho,\mathbb{R}^2}, 0) = \deg_{LS}(\Phi_{\mu,H_\rho}, \Omega_{\rho,\mathbb{R}^2}, 0) = -\chi\big(x \operatorname{rel} y\big).$$

Finally, we scale ρy_L to y_L via $y_{\alpha,L} = (1-\alpha)\rho y_L + \alpha y_L$ and we consider the homotopy

$$H_2(x,t;\alpha) = \frac{1}{2}p^2 - \frac{1}{2}q^2 + g(q,t;\alpha),$$

between H_L and $H_{\rho,L}$, where $g(q,t;\alpha)$ is found by applying Lemma 2.7.4 to $y_{\alpha,L}$. The uniform estimates from Lemma 2.7.2 allows us to apply the Leray-Schauder degree:

$$\deg_{LS}(\Phi_{\mu,H_L},\Omega_{\mathbb{R}^2},0) = \deg_{LS}(\Phi_{\mu,H_{\rho,L}},\Omega_{\rho,\mathbb{R}^2},0) = -\chi\big(x \operatorname{rel} y\big).$$

Combining the equalities for the various Leray-Schauder degrees with (2.41) yields:

$$-\deg_{LS}(\Phi_{H_L},\Omega_{\mathbb{R}^2},0) = \chi\big(x \operatorname{rel} y\big) = \sum_{q\in\mathrm{Crit}(\mathscr{L}_g)\cap([q]\operatorname{rel} Q)} (-1)^{\beta(q)}. \tag{2.42}$$

2.7.4 Discretized braid classes

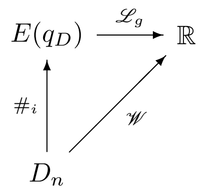
The Lagrangian problem (2.40) can be treated by using a variation on the method of broken geodesics. If we choose $1/d > 0$ sufficiently small, the integral

$$S_i(q_i,q_{i+1}) = \min_{\substack{q(t)\in E_i(q_i,q_{i+1})\\ |q(t)|\le 1}} \int_{\tau_i}^{\tau_{i+1}} \frac{1}{2}q_t^2 + \frac{1}{2}q^2 - g(q,t)dt, \tag{2.43}$$

has a unique minimizer q^i, where $E_i(q_i,q_{i+1}) = \big\{q \in H^1(\tau_i,\tau_{i+1}) \mid q(\tau_i) = q_i,\ q(\tau_{i+1}) = q_{i+1}\big\}$, and $\tau_i = i/d$. Moreover, if $1/d$ is small, then the minimizers are non-degenerate and S_i is a smooth function of q_i and q_{i+1}. Critical points q of $\mathscr{L}_g$ with $|q(t)| \le 1$ correspond to sequences $q_D = (q_0,\cdots,q_d)$, with $q_0 = q_n$, which are critical points of the discrete action

$$\mathscr{W}(q_D) = \sum_{i=0}^{n-1} S_i(q_i,q_{i+1}). \tag{2.44}$$

A concatenation $\#_i q^i$ of minimizers q^i is continuous and is an element in the function space $H^1(\mathbb{R}/\mathbb{Z})$, and is referred to as a *broken geodesic*. The set of broken geodesics $\#_i q^i$ is denoted by $E(q_D)$ and standard arguments using the non-degeneracy of minimizers q^i show that $E(q_D) \hookrightarrow H^1(\mathbb{R}/\mathbb{Z})$ is a smooth, d-dimensional submanifold in $H^1(\mathbb{R}/\mathbb{Z})$. The submanifold $E(q_D)$ is parametrized by sequences $D_d = \{q_D \in \mathbb{R}^d \mid |q_i| \le 1\}$ and yields the following commuting diagram:



In the above diagram $\#_i$ is regarded as a mapping $q_D \mapsto \#_i q^i$, where the minimizers q_i are determined by q_D. The tangent space to $E(q_D)$ at a broken geodesic $\#_i q^i$ is identified by

$$T_{\#_i q^i} E(q_D) = \big\{\psi \in H^1(\mathbb{R}/\mathbb{Z}) \mid -\psi_{tt} + \psi - g_{qq}(q^i(t), t)\psi = 0,$$
$$\psi(\tau_i) = \delta q_i,\ \ \psi(\tau_{i+1}) = \delta q_{i+1},\ \ \delta q_i \in \mathbb{R}, \forall i\big\},$$

and $\#_i q^i + T_{\#_i q^i} E(q_D)$ is the tangent hyperplane at $\#_i q^i$. For $H^1(\mathbb{R}/\mathbb{Z})$ we have the following decomposition for any broken geodesic $\#_i q^i \in E(q_D)$:

$$H^1(\mathbb{R}/\mathbb{Z}) = E' \oplus T_{\#_i q^i} E(q_D), \tag{2.45}$$

where $E' = \{\eta \in H^1(\mathbb{R}/\mathbb{Z}) \mid \eta(\tau_i) = 0,\ \forall i\}$. To be more specific the decomposition is orthogonal with respect to the quadratic form

$$D^2\mathscr{L}_g(q)\phi\widetilde{\phi} = \int_0^1 \phi_t\widetilde{\phi}_t + \phi\widetilde{\phi} - g_{qq}(q(t), t)\phi\widetilde{\phi}dt, \quad \phi, \widetilde{\phi} \in H^1(\mathbb{R}/\mathbb{Z}).$$

Indeed, let $\eta \in E'$ and $\psi \in T_{\#_i q^i} E(q_D)$, then

$$\begin{aligned} D^2\mathscr{L}_g(\#_i q^i)\eta\psi &= \sum_i \int_{\tau_i}^{\tau_{i+1}} \eta_t\psi_t + \eta\psi - g_{qq}(q^i(t), t)\xi\eta dt \\ &= \sum_i \psi_t\eta\Big|_{\tau_i}^{\tau_{i+1}} - \sum_i \int_{\tau_i}^{\tau_{i+1}} \big[-\psi_{tt} + \psi + g_{qq}(q^i(t), t)\psi\big]\eta dt = 0. \end{aligned}$$

Let $\phi = \eta + \psi$, then

$$D^2\mathscr{L}_g(\#_i q^i)\phi\widetilde{\phi} = D^2\mathscr{L}_g(\#_i q^i)\eta\widetilde{\eta} + D^2\mathscr{L}_g(\#_i q^i)\psi\widetilde{\psi},$$

by the above orthogonality. By construction the minimizers q^i are non-degenerate and therefore $D^2\mathscr{L}_g|_{E'}$ is positive definite. This implies that the Morse index of a (stationary) broken geodesic is determined by $D^2\mathscr{L}_g|_{T_{\#_i q^i} E(q_D)}$. By the commuting diagram for $\mathscr{W}$ this implies that the Morse index is given by quadratic form $D^2\mathscr{W}(q_D)$. We have now proved the following lemma that relates the Morse index of critical points of the discrete action $\mathscr{W}$ to Morse index of the 'full' action $\mathscr{L}_g$.

2.7.5. Lemma. *Let q be a critical point of $\mathscr{L}_g$ and q_D the corresponding critical point of $\mathscr{W}$, then the Morse indices are the same i.e. $\beta(q) = \beta(q_D)$.*

For a 1-periodic function $q(t)$ we define the mapping

$$q \xrightarrow{D_d} q_D = (q_0, \cdots, q_d), \quad q_i = q(i/d), \ \ i = 0, \cdots, d,$$

and q_D is called the discretization of q. The linear interpolation

$$q_D \mapsto \ell_{q_D}(t) = \#_i\Big[q_i + \frac{q_{i+1} - q_i}{d} t\Big],$$

reconstructs a piecewise linear 1-periodic function. For a relative braid diagram $q \operatorname{rel} Q$, let $q_D \operatorname{rel} Q_D$ be its discretization, where Q_D is obtained by applying D_d to every strand in Q. A discretization $q_D \operatorname{rel} Q_D$ is *admissible* if $\ell_{q_D} \operatorname{rel} \ell_{Q_D}$ is homotopic to $q \operatorname{rel} Q$, i.e. $\ell_{q_D} \operatorname{rel} \ell_{Q_D} \in [q \operatorname{rel} Q]$. Define the *discrete* relative braid class $[q_D \operatorname{rel} Q_D]$ as the set of 'discrete relative braids' $q'_D \operatorname{rel} Q'_D$, such that $\ell_{q'_D} \operatorname{rel} \ell_{Q'_D} \in [q \operatorname{rel} Q]$. The associated fibers are denoted by $[q_D] \operatorname{rel} Q_D$. It follows from [28], Proposition 27, that $[q_D \operatorname{rel} Q_D]$ is guaranteed to be connected when

$$d > \#\{\text{ crossings in } q \operatorname{rel} Q\},$$

i.e. for any two discrete relative braids $q_D \operatorname{rel} Q_D$ and $q'_D \operatorname{rel} Q'_D$, there exists a homotopy $q_D^\alpha \operatorname{rel} Q_D^\alpha$ (discrete homotopy) such that $\ell_{q_D^\alpha} \operatorname{rel} \ell_{Q_D^\alpha}$ is a path in $[q \operatorname{rel} Q]$. Note that fibers are not necessarily connected! For a braid classes $[q \operatorname{rel} Q]$ the associated discrete braid class $[q_D \operatorname{rel} Q_D]$ may be connected for a smaller choice of d.

We showed above that if $1/d > 0$ is sufficiently small, then the critical points of $\mathscr{L}_g$, with $|q| \le 1$, are in one-to-one correspondence with the critical points of $\mathscr{W}$, and their Morse indices coincide by Lemma 2.7.5. Moreover, if $1/d > 0$ is small enough, then for all critical points of $\mathscr{L}_g$ in $[q] \operatorname{rel} Q$, the associated discretizations are admissible and $[q_D \operatorname{rel} Q_D]$ is a connected set. The discretizations of the critical points of $\mathscr{L}_g$ in $[q] \operatorname{rel} Q$ are critical points of $\mathscr{W}$ in the discrete braid class fiber $[q_D] \operatorname{rel} Q_D$.

Now combine the index identity with (2.42), which yields

$$\chi(x \operatorname{rel} y) = \sum_{q \in \mathrm{Crit}(\mathscr{L}_g) \cap ([q] \operatorname{rel} Q)} (-1)^{\beta(q)} = \sum_{q_D \in \mathrm{Crit}(\mathscr{W}) \cap ([q_D] \operatorname{rel} Q_D)} (-1)^{\beta(q_D)}. \quad (2.46)$$

2.7.5 The Conley index for discrete braids

In [28] an invariant for discrete braid classes $[q_D \,\mathrm{rel}\, Q_D]$ is defined based on the Conley index. The invariant $\mathrm{HC}_*([q_D]\,\mathrm{rel}\, Q_D)$ is independent of the fiber and can be described as follows. A fiber $[q_D]\,\mathrm{rel}\, Q_D$ is a finite dimensional cube complex with a finite number of connected components. Denote the closures of the connected components by N_j. The faces of the hypercubes N_j can be co-oriented in direction of decreasing the number of crossing in $q_D \,\mathrm{rel}\, Q_D$, and define N_j^- as the closure of the set of faces with outward pointing co-orientation. The sets N_j^- are called *exit sets*. The invariant is given by

$$\mathrm{HC}_*([q_D]\,\mathrm{rel}\, Q_D) = \bigoplus_j H_*(N_j, N_j^-).$$

The invariant is well-defined for any $d > 0$ for which there exist admissible discretizations and is independent of both the fiber and the discretization size. From [28] we have for any Morse function $\mathscr{W}$ on a proper braid class fiber $[q_D]\,\mathrm{rel}\, Q_D$,

$$\sum_{q_D \in \mathrm{Crit}(\mathscr{W}) \cap ([q_D]\,\mathrm{rel}\, Q_D)} (-1)^{\beta(q_D)} = \chi\big(\mathrm{HC}_*([q_D]\,\mathrm{rel}\, Q_D)\big) =: \chi\big(q_D \,\mathrm{rel}\, Q_D\big). \quad (2.47)$$

The latter can be computed for any admissible discretization and is an invariant for $[q \,\mathrm{rel}\, Q]$. Combining 2.46 and 2.47 gives

$$\chi\big(x \,\mathrm{rel}\, y\big) = \chi\big(q_D \,\mathrm{rel}\, Q_D\big). \quad (2.48)$$

In this section we assumed without loss of generality that $x \,\mathrm{rel}\, y$ is augmented and since the Euler-Floer characteristic is a braid class invariant, an admissible discretization is construction for an appropriate augmented, Legendrian representative $x_L \,\mathrm{rel}\, y_L$. Summarizing

$$\chi\big(x \,\mathrm{rel}\, y\big) = \chi\big(x_L \,\mathrm{rel}\, y_L^*\big) = \chi\big(q_D \,\mathrm{rel}\, Q_D^*\big).$$

Since $\chi\big(q_D \,\mathrm{rel}\, Q_D^*\big)$ is the same for any admissible discretization, the Euler-Floer characteristic can be computed using any admissible discretization, which proves Theorem 2.1.5.

2.7.6. Remark. The invariant $\chi\big(q_D \,\mathrm{rel}\, Q_D\big)$ is a true Euler characteristic of a topological pair. To be more precise

$$\chi\big(q_D \,\mathrm{rel}\, Q_D\big) = \chi\big([q_D]\,\mathrm{rel}\, Q_D, [q_D]^- \,\mathrm{rel}\, Q_D\big),$$

where $[q_D]^- \operatorname{rel} Q_D$ is the exit set a described above. A similar characterization does not a priori exist for $[x] \operatorname{rel} y$. Firstly, it is more complicated to designate the equivalent of an exit set $[x]^- \operatorname{rel} y$ for $[x] \operatorname{rel} y$, and secondly it is not straightforward to develop a (co)-homology theory that is able to provide meaningful information about the topological pair $\big([x] \operatorname{rel} y, [x]^- \operatorname{rel} y\big)$. This problem is circumvented by considering Hamiltonian systems and carrying out Floer's approach towards Morse theory (see [23]), by using the isolation property of $[x] \operatorname{rel} y$. The fact that the Euler characteristic of Floer homology is related to the Euler characteristic of topological pair indicates that Floer homology is a good substitute for a suitable (co)-homology theory.

2.8 Examples

We will illustrate by means of two examples that the Euler-Floer characteristic is computable and can be used to find closed integral curves of vector fields on the 2-disc.

2.8.1 Example

Figure 2.1[left] shows the braid diagram $q \operatorname{rel} Q$ of a positive relative braid $x \operatorname{rel} y$. The discretization with $q_D \operatorname{rel} Q_D$, with $d = 2$, is shown in Figure 2.1[right]. The chosen discretization is admissible and defines the relative braid class $[q_D \operatorname{rel} Q_D]$. There are five strands, one is free and four are fixed. We denote the points on the free strand by $q_D = (q_0, q_1)$ and on the skeleton by $Q_D = \{Q^1, \cdots, Q^4\}$, with $Q^i = (Q^i_0, Q^i_1)$, $i = 1, \cdots, 4$.

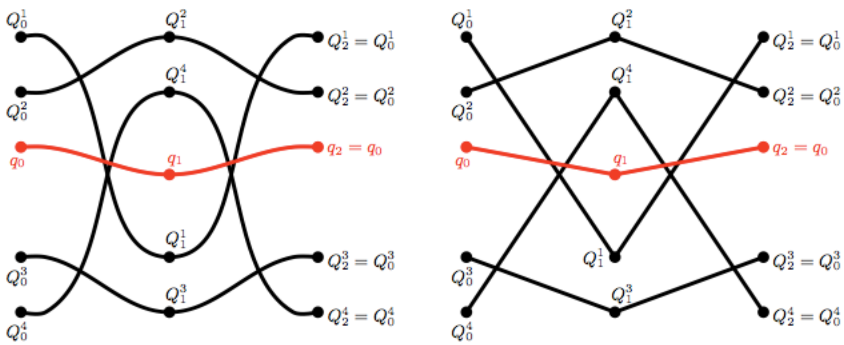


Figure 2.1: A positive braid diagram [left] and an admissible discretization [right].

In Figure 2.2[left] the braid class fiber $[q_D] \operatorname{rel} Q_D$ is depicted. The coordinate q_0 is allowed to move between Q_0^3 and Q_0^2 and q_1 remains in the same braid class if it varies between Q_1^1 and Q_1^4. For the values $q_0 = Q_0^3$ and $q_0 = Q_0^2$ the relative braid becomes singular and if q_0 crosses these values two intersections are created. If q_1 crosses the values Q_1^1 or Q_1^4 two intersections are destroyed. This provides the desired co-orientation, see Figure 2.2[middle]. The braid class fiber $[q_D] \operatorname{rel} Q_D$ consists of 1 component and we have that

$$N = \operatorname{cl}([q_D \operatorname{rel} Q_D]) = \{(q_0, q_1) : Q_0^3 \leq q_0 \leq Q_0^2, Q_1^1 \leq q_1 \leq Q_1^4\},$$

and the exit set is

$$N^- = \{(q_0, q_1) : q_1 = Q_1^1, \text{ or } q_1 = Q_1^4\}.$$

For the Conley index this gives:

$$\operatorname{HC}_k([q_D] \operatorname{rel} Q_D) = H_k(N, N^-; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

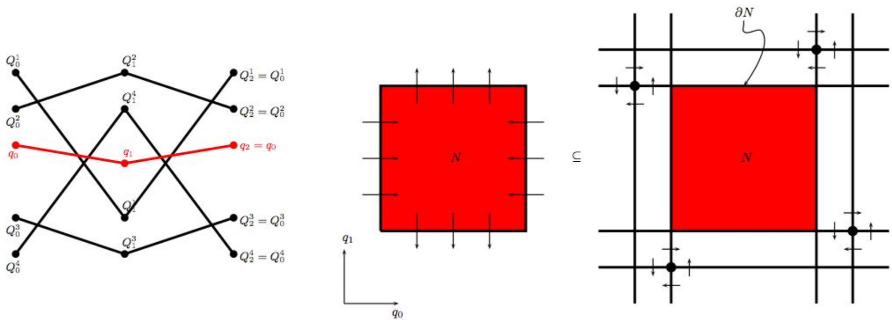


Figure 2.2: The relative braid fiber $[q_D] \operatorname{rel} Q_D$ and $N = \operatorname{cl}([q_D] \operatorname{rel} Q_D)$.

The Euler characteristic of $\big([q_D] \operatorname{rel} Q_D, [q_D]^- \operatorname{rel} Q_D\big)$ can be computed now and the Euler-Floer characteristic $\big(x \operatorname{rel} y\big)$ is given by

$$\chi(x \operatorname{rel} y) = \chi\big([q_D] \operatorname{rel} Q_D, [q_D]^- \operatorname{rel} Q_D\big) = -1 \neq 0$$

From Theorem 2.1.2 we derive that any vector field for which y is a skeleton has at least 1 closed integral curve $x_0 \operatorname{rel} y \in [x] \operatorname{rel} y$. Theorem 2.1.2 also implies that any orientation preserving diffeomorphism f on the 2-disc which fixes the set of four points A_4, whose mapping class $[f; A_4]$ is represented by the braid y has an additional fixed point.

2.8.2 Example

The theory can also be used to find additional closed integral curves by concatenating the skeleton y. As in the previous example y is given by Figure 2.1. Glue ℓ copies of the skeleton y to its ℓ-fold concatenation and a reparametrize time by $t \mapsto \ell \cdot t$. Denote the rescaled ℓ-fold concatenation of y by $\#_\ell y$. Choose $d = 2\ell$ and discretize $\#_\ell y$ as in the previous example. For a given braid class $[x \,\mathrm{rel}\, \#_\ell y]$,

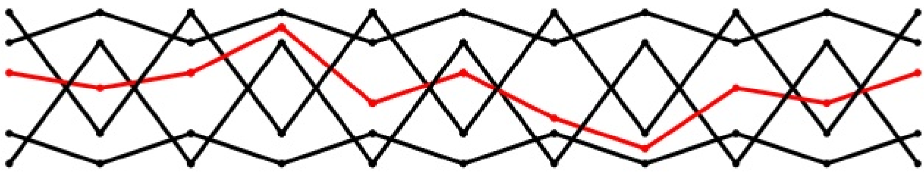


Figure 2.3: A discretization of a braid class with a 5-fold concatenation of the skeleton y. The number of odd anchor points in middle position is $\mu = 3$.

Figure 2.3 below shows a discretized representative $q_D \,\mathrm{rel}\, \#_\ell Q_D$, which is admissible. For the skeleton $\#_\ell Q_D$ we can construct $3^\ell - 2$ proper relative braid classes in the following way: the even anchor points of the free strand q_D are always in the middle and for the odd anchor points we have 3 possible choices: bottom, middle, top (2 braids are not proper). We now compute the Conley index of the $3^\ell - 2$ different proper discrete relative braid classes and show that the Euler-Floer characteristic is non-trivial for these relative braid classes.

The configuration space $N = \mathrm{cl}\big([q_D] \,\mathrm{rel}\, \#_\ell Q_D\big)$ in this case is given by a cartesian product of 2ℓ closed intervals, and therefore a 2ℓ-dimensional hypercube. We now proceed by determining the exit set N^-. As in the previous example the co-orientation is found by a union of faces with an outward pointing co-orientation. Due to the simple product structure of N, the set N^- is determined by the odd anchor points in the middle position. Denote the number of middle positions at odd anchor points by μ. In this way N^- consists of opposite faces at at odd anchor points in middle position, see Figure 2.3. Therefore

$$\mathrm{HC}_k([q_D] \,\mathrm{rel}\, \#_\ell Q_D) = H_k(N, N^-) = \begin{cases} \mathbb{Z}_2 & k = \mu \\ 0 & k \neq \mu, \end{cases}$$

and the Euler-Floer characterisc is given by

$$\chi\big(x \,\mathrm{rel}\, \#_\ell y\big) = (-1)^\mu \neq 0.$$

Let $X(x,t)$ be a vector field for which y is a skeleton of closed integral curves, then $\#_\ell y$ is a skeleton for the vector field $X^\ell(x,t) := \ell X_\ell(x, \ell t)$. From Theorem 2.1.2 we derive that there exists a closed integral curve in each of the $3^\ell - 2$ proper relative classes $[x] \,\mathrm{rel}\, y$ described above. For the original vector field X this yields $3^\ell - 2$ distinct closed integral curves. Using the arguments in [51] one can find a compact invariant set for X with positive topological entropy, which proves that the associated flow is 'chaotic' whenever y is a skeleton of given integral curves.

2.8.1. Remark. The computations of the Conley homology in the above examples can be found in [28].

2.8.3 Example

So far we have not addressed the question whether the closed integral curves $x \,\mathrm{rel}\, y$ are non-trivial, i.e. not equilibrium points of X. The theory can also be extended in order to find non-trivial closed integral curves. This paper restricts to relative braids where x consists of just one strand. Braid Floer homology for relative braids with x consisting of n strands is defined in [49]. To illustrate the importance of multi-strand braids we consider the discrete braid class in Figure 2.4.

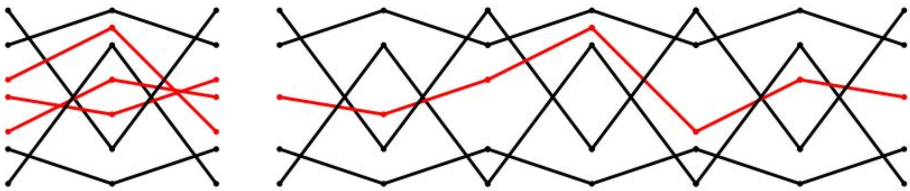


Figure 2.4: A discretization of a braid class with a 3-fold concatenation of the skeleton y. The number of odd anchor points in middle position is $\mu = 2$ [right]. If we represent all translates of x we obtain a proper relative braid class where x is a 3-strand braid [left]. The latter provides additional linking information.

The braid class depicted in Figure 2.4[right] is discussed in the previous example and the Euler-Floer characteristic is equal to 1. By considering all translates of x on the circle $\mathbb{R}/\mathbb{Z}$, we obtain the braids in Figure 2.4[left]. The latter braid class is proper and encodes extra information about q_D relative to Q_D. The braid class fiber is a 6-dimensional cube with the same Conley index as the braid class in Figure 2.4[right]. Therefore,

$$\chi(q_D \operatorname{rel} Q_D) = (-1)^2 = 1.$$

As in the 1-strand case, the discrete Euler characteristic can used to compute the associated Euler-Floer characteristic of $x \operatorname{rel} y$ and $\chi(x \operatorname{rel} y) = 1$. The skeleton y thus forces solutions $x \operatorname{rel} y$ of the above described type. The additional information we obtain this way is that for braid classes $[x \operatorname{rel} y]$, the associated closed integral curves for X cannot be constant and therefore represent non-trivial closed integral curves.

3 Floer and Morse homology for RBC

This Chapter consists of three parts. Recall that in [49] a Floer type theory applied to proper relative braid classes with support in $\mathbb{D}^2$ was introduced. In the first part of this Chapter, for a particular class of Hamiltonian systems, called hyperbolic, we extend the same construction as in [49] to unbounded proper relative braid classes. These are braid classes whose representatives are not necessarily supported in a compact set of $\mathbb{R}^2$, like $\mathbb{D}^2$. We prove that this construction leads to the definition of a braid Floer homology which is isomorphic to the classical braid Floer homology defined in [49]. In the second part, we define a Morse type theory for a special class of unbounded braids, called Legendrian. The latter is obtained by applying, in a parabolic setting, the construction we have used in the first part of the Chapter. In the last part, we apply the techniques of [46] to establish an isomorphism between the two homology theories for braids we have introduced.

3.1 Introduction

Floer homology theory plays nowadays an important role in Geometry and Analysis. It has been used in many fields with different aims and different results, from the solution of the well-known Arnol'd Conjecture [23], to many applications in symplectic field theory [17], symplectic homology [24], [42], [53], heat flows [46], elliptic systems [7] and strongly indefinite functionals on Hilbert spaces [2]. Roughly speaking the Floer theory is an extension of the Morse theory to a fully infinite-dimensional setting with a strongly indefinite functional. Floer homology considers a formal gradient flow and studies its set of bounded flowlines. Floer's initial work studied the elliptic non-linear Cauchy-Riemann equations, which occur as a formal L^2-gradient flow of a (strongly indefinite) Hamiltonian action. As in the construction of Morse homology one builds a complex by grading the critical points via the Fredholm index and constructs a boundary operator by counting heteroclinic flowlines between points with difference one in index. The homology of this complex, called Floer homology, satisfies a continuation principle and remains unchanged under suitable (large) perturbations.

In [49] the above mentioned Floer theory was applied for the first time to the algebraic theory of braids, with many different interesting results, such as the Monotonicity Lemma and the definition of the braid Floer homology for bounded proper relative braid classes.

As mentioned already in Chapter 1 and Chapter 2 braids on $\mathbb{D}^2$ may be represented as closed loops in the configuration space $\mathbf{C}_\ell(\mathbb{D}^2)$. This may be extended to any two-dimensional manifold. In particular in the following notes we will use braids on $\mathbb{R}^2$. When we will consider explicitly $\mathbb{R}^2$ instead of $\mathbb{D}^2$, in order not to create ambiguities, we will use the adjective UNBOUNDED.

Recall briefly from Chapter 2 that relative braids, which we denote by $x \operatorname{rel} y$ are those braids which consist of two components: x and y. In this thesis we are concerned with relative braids such that the x component consists only on 1 strands, while y consists of m strands. The path component of $x \operatorname{rel} y$ of closed loops in $\mathcal{L}\mathbf{C}_{1+m}(\mathbb{R}^2)$[1] is denoted by $[x \operatorname{rel} y]_{\mathbb{R}^2}$ and is called a relative braid class. The intertwining of x and y defines various different braid classes. A braid class $[x \operatorname{rel} y]_{\mathbb{R}^2}$ is PROPER if x cannot be deformed onto components in $y_c \subset y$. We abbreviate relative braid classes by RBC and proper relative braid classes by PRBC. Recall that a bounded braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ is PROPER if x cannot be deformed (via homotopies in $\mathbb{D}^2$) onto components in either $y_c \subset y$ or the boundary $\partial\mathbb{D}^2$.

3.1.1. Remark. Properness is a topological condition that descents from braids on $\mathbb{D}^2$ to braids on $\mathbb{R}^2$, i.e. properness of $[x \operatorname{rel} y]_{\mathbb{D}^2}$ implies properness of $[x \operatorname{rel} y]_{\mathbb{R}^2}$. The implication does not necessarily go in the opposite direction.

The aim of this chapter is threefold:

(1) extend, for a special class of Hamiltonians, the construction of the braid Floer homology of [49] from bounded PRBC to unbounded PRBC and establish an isomorphism between them;

(2) define a Morse thoery for a special type of (unbounded) braids, called (unbounded) Legendrian;

(3) establish an isomorphism between Floer homology for bounded PRBC and Morse homology for unbounded Legendrian PRBC, using the construction of [46] adapted to our setting.

In the next three paragraphs we give a short description of the three main parts of the Chapter.

3.1.1 Uniform estimates for unbounded proper relative braid classes

Let $\mathbb{R}^2$ be the plane with coordinates $x = (p, q) \in \mathbb{R}^2$. The construction of braid Floer homology proposed in [49] cannot be directly extended from $\mathbb{D}^2$ to $\mathbb{R}^2$, with the same class of Hamiltonians as in [49]. This is due to the fact that $\mathbb{R}^2$ is not a compact manifold, and hence loops in $\mathbf{C}_m(\mathbb{R}^2)$ are not a priori contained in a compact set of $\mathbb{R}^2$, as it happens for those in $\mathbf{C}_m(\mathbb{D}^2)$. In order to construct a well

[1]The space of continuous mapping $S^1 \to X$, with X a topological space, is called the free loop space of X and is denoted by $\mathcal{L}X$.

defined Floer homology theory for unbounded PRBC, we assume the Hamiltonian function is HYPERBOLIC. Let $H_V : S^1 \times \mathbb{R}^2 \to \mathbb{R}$ defined as follows

$$H_V(t,p,q) = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 + V(t,p,q). \tag{3.1}$$

The function $V : S^1 \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions:

(V1) $V \in C^\infty(S^1 \times \mathbb{R}^2; \mathbb{R})$;

(V2) for $j = 0,1,2$ there exists a $G^j \in C_c^\infty(\mathbb{R})$ such that

$$|\partial^j V(t,p,q)| \le G^j(q), \quad \text{for all } (t,p,q) \in S^1 \times \mathbb{R}^2.$$

We denote by $\mathscr{V}$ the space of functions $V : S^1 \times \mathbb{R}^2$ such that (V1) and (V2) holds, and by $\mathscr{H}_{\text{hyp}}$ the space of Hamiltonians with $V \in \mathscr{V}$. For $V \in \mathscr{V}$ and $H_V \in \mathscr{H}_{\text{hyp}}$ define the action functional as follows

$$\mathscr{A}_{H_V}(x) = \int_0^1 \tfrac{1}{2}\langle Jx, x_t\rangle - H_V(t,x)\, dt. \tag{3.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$, and J is the standard symplectic matrix, i.e.

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

The requested uniform bounds are obtained by exploiting the properties of generic hyperbolic Hamiltonians. With respect to this class of special Hamiltonians, the construction of A. Floer [23] applies and yields

$$\mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y; H_V, J).$$

Here the almost complex structure J plays the role of a parameter. We prove, by construction (Theorem 3.2.18), that for a specific $H \in C_0^\infty(\mathbb{D}^2) \subset \mathscr{H}$ there exists $\widehat{V} \in \mathscr{V}$ and $H_{\widehat{V}} \in \mathscr{H}_{\text{hyp}}$ such that

$$\mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y; H, J) = \mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y; H_{\widehat{V}}, J).$$

Different choices of $H_V \in \mathscr{H}_{\text{hyp}}$ and of constants $J \in \mathscr{J}$ (defined in Section 1.3.1) yields isomorphic Floer homologies and

$$\mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y) = \varprojlim \mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y; H_V, J),$$

where the inverse limit is defined with respect to the canonical isomorphisms $a_k(H_V, H_{V'}) : \mathrm{HHF}_k([x]\,\mathrm{rel}\,y; H_V, J) \to \mathrm{HHF}_k([x]\,\mathrm{rel}\,y; H_{V'}, J)$ and $b_k(J, J') : \mathrm{HHF}_k([x]\,\mathrm{rel}\,y; H_V, J) \to \mathrm{HHF}_k([x]\,\mathrm{rel}\,y; H_V, J')$. This implies, in particular, that

$$\mathrm{HHF}_*([x\,\mathrm{rel}\,y]_{\mathbb{R}^2}) \cong \mathrm{HF}_*([x\,\mathrm{rel}\,y]_{\mathbb{D}^2}). \tag{3.4}$$

3.1.2 Mechanical Morse homology for unbounded Legendrian PRBC

Property (iii) of Section 1.4.1 can be shown to hold for $\mathrm{HF}_*([x]_{\mathbb{R}^2}\,\mathrm{rel}\,y, H_V)$. It shows that if we compose $x\,\mathrm{rel}\,y$ with $\ell \geq 0$ full twists Δ^2, the homology groups of $[x\,\mathrm{rel}\,y]_{\mathbb{R}^2}$ are isomorphic to the ones of $[(x\,\mathrm{rel}\,y)\cdot\Delta^{2\ell}]_{\mathbb{R}^2}$ (up to shifting).
Let $x\,\mathrm{rel}\,y$ be a RBC supported in $\mathbb{R}^2$ and compose $x\,\mathrm{rel}\,y$ with $\ell \geq 0$ fulltwists Δ^2, such that $(x\,\mathrm{rel}\,y)\cdot\Delta^{2\ell}$ is isotopic to a positive braid $x^+\,\mathrm{rel}\,y^+$. By Property (iii) of Section 1.4.1 we have

$$\mathrm{HHF}_*([x\,\mathrm{rel}\,y]_{\mathbb{R}^2}) \cong \mathrm{HHF}_{*-2\ell}([(x^+\,\mathrm{rel}\,y^+)\cdot\Delta^{2\ell}]_{\mathbb{R}^2}). \tag{3.5}$$

By (3.5) we will assume from now on, without loss of generality, that our relative braids have a positive representative. Positive relative braids $x^+\,\mathrm{rel}\,y^+$ are isotopic to Legendrian braids $x^L\,\mathrm{rel}\,y^L$ on $\mathbb{R}^2$, i.e. braids of the form $x^L = (q_t, q)$ and $y^L = (Q_t, Q)$, where $q = \pi_2 x$ and $Q = \pi_2 y$, and π_2 the projection onto the q-coordinate, see Sections 2.7.2 and 2.7.3. The associated equivalence class of unbounded Legendrian braids is denoted by $[q\,\mathrm{rel}\,Q]_{\mathbb{R}}$ and its fibers by $[q]_{\mathbb{R}}\,\mathrm{rel}\,Q$. Legendrian braids can be described via mechanical systems and Hamiltonians of the form

$$H_U(t,p,q) = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 + U(t,q). \tag{3.6}$$

On the potential $U : S^1 \times \mathbb{R} \to \mathbb{R}$ we assume the following hyphotheses:

($U1$) $U \in C^\infty(\mathbb{R}\times S^1;\mathbb{R})$

($U2$) for $j = 0,1,2$ there exists a $G^j \in C_c^\infty(\mathbb{R})$ such that

$$|\partial^j U(t,q)| \leq G^j(q), \quad \text{for all } (t,q) \in S^1 \times \mathbb{R}.$$

We will denote by $\mathscr{U}$ the space of functions $U : \mathbb{R}\times S^1 \to \mathbb{R}$ such that (U1) and (U2) holds, and by $\mathscr{H}_{\mathrm{mech}}$ the class of Hamiltonians H_U such that $U \in \mathscr{U}$. The action functional in this setting becomes

$$\mathscr{L}_U(q) = \int_0^1 \tfrac{1}{2}q_t^2 + \tfrac{1}{2}q^2 - U(t,q)\,dt. \tag{3.7}$$

We note that $\mathscr{L}_U$ is not strongly indefinite. Then, as in classical Morse theory, to any critical point q of $\mathscr{L}_U$ one can assign a finite Morse index, which we denote by $\beta(q)$. Since also in this case we are working with unbounded Legendrian PRBC, also in this case we need to find uniform estimates. To obtain uniform bounds, we use Fourier estimates and parabolic bootstrapping. We emphasize that in this pages we do not develop the proofs of the genericity properties for critical points and connecting orbits. The construction of the Morse homology follows the steps summarized in Section 1.4.1. It follows that

$$\mathrm{HHM}_*([q]_{\mathbb{R}} \,\mathrm{rel}\, Q; U)$$

is well-defined and independent of the chosen potential $U \in \mathscr{U}$ and of the fiber $[q]_{\mathbb{R}} \,\mathrm{rel}\, Q$. For these reasons we will write $\mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}})$. At the end of Section 3.2, in Section 3.3.3, we introduce the class of potentials $\mathscr{W}$. These are functions such that

(W1) $W \in C^\infty(S^1 \times \mathbb{R}; \mathbb{R})$

(W2) $\partial_q W(t, \pm 1) = \pm 1$.

For these choice of potentials we can define $\mathrm{HM}_*([q \,\mathrm{rel}\, Q])$, where $[q \,\mathrm{rel}\, Q]$ is a bounded Legendrian proper relative braid class. We also show that

$$\mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}) \cong \mathrm{HM}_*([q \,\mathrm{rel}\, Q]).$$

3.1.3 Isomorphism between braid Floer and Morse homology

In the third part of the chapter we establish the isomorphism

$$\mathrm{HHF}_*([x \,\mathrm{rel}\, y]_{\mathbb{R}^2}) \cong \mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}), \tag{3.8}$$

where $[x \,\mathrm{rel}\, y]_{\mathbb{R}^2}$ is an unbounded proper relative braid class and $[q \,\mathrm{rel}\, Q]_{\mathbb{R}}$ its (unbounded) proper Legendrian projected class. We notice that the solutions of the Cauchy-Riemann equations which have been perturbed by changing the symplectic matrix J into

$$J^\varepsilon := \begin{pmatrix} 0 & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}, \quad \text{for some } \varepsilon > 0,$$

formally converge when $\varepsilon \to 0$ to the solutions of the heat equations, see Section 3.4.1. A deeper analysis of the structure of the equations (see Proposition 3.4.8) shows that the convergence is indeed in $C^1_{\mathrm{loc}}(\mathbb{R} \times S^1)$. This allows to build a one-to-one map between connecting orbits of the heat equation and connecting orbits of the Cauchy-Riemann equations. We call this map the Salamon-Weber map, see [46]. We prove finally that the map respects the braid classes, provided the RBC are proper. From this, the required isomorphism follows.

Summarizing the main result of this Chapter is the proof of the chain of isomorphisms

$$\mathrm{HF}_*([x \,\mathrm{rel}\, y]_{\mathbb{D}^2}) \cong \mathrm{HHF}_*([x \,\mathrm{rel}\, y]_{\mathbb{R}^2}) \cong \mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}) \cong \mathrm{HM}_*([q \,\mathrm{rel}\, Q]) \quad (3.9)$$

3.2 Hyperbolic braid Floer homology

In this Section we extend the construction of the classical braid Floer homology of [49] from bounded PRBC to unbounded PRBC with respect to hyperbolic Hamiltonians. At the end, Theorem 3.2.18 establishes an isomorphism between them, and hence the first part of the chain isomorphism (3.9).

3.2.1 Set-up: hyperbolic Hamiltonians in $\mathbb{R}^2$.

Let $V \in \mathscr{V}$ and $H_V \in \mathscr{H}_{\mathrm{hyp}}$ a hyperbolic Hamiltonian. Let $\mathscr{A}_{H_V}$ the action functional defined in (3.2). Denote by Crit_{H_V} the space of critical points of $\mathscr{A}_{H_V}$ and endow Crit_{H_V} with the compact-open topology on S^1. Since S^1 is compact this is equivalent to the strong $C^0(S^1)$ topology. Hyperbolic Hamiltonian systems have special properties, among which the existence of uniform a priori estimates for critical points of $\mathscr{A}_{H_V}$. This is the content of the following lemma.

3.2.1. Lemma. *Let $H_V \in \mathscr{H}_{\mathrm{hyp}}, V \in \mathscr{V}$ then there exists a uniform constant $c > 0$ (dependent only on the support of G^j (in (V2))) such that for every critical point $x \in \mathrm{Crit}_{H_V}$, it holds*

$$||x||_{C^0(S^1)} \leq c. \quad (3.10)$$

Hence there exists $C > 0$ such that for all $x \in \mathrm{Crit}_{\mathscr{A}_H}$

$$|\mathscr{A}_{H_V}(x)| \leq C. \quad (3.11)$$

Proof. Lemma 2.7.1, or [39, lemma 7.1], proves (3.10). Then (3.11) immediately follows. ■

For critical points of hyperbolic Hamiltonians the following compactness result holds.

3.2.2. Proposition. *The space Crit_{H_V} is compact with respect to the compact-open topology.*

Proof. Let x be in Crit_{H_V} then, writing the Euler Lagrange equation for x and passing to the C^0 norm we obtain

$$||x_t||_{C^0(S^1)} \leq ||x||_{C^0(S^1)} + ||\nabla V||_{C^0(S^1)}.$$

The right hand side is uniformly bounded because of Lemma 3.2.1 and because $V \in \mathscr{V}$. Hence x is uniformly bounded in $C^1(S^1)$. By the Theorem of Arzelà-Ascoli, $C^1(S^1) \hookrightarrow C^0(S^1)$ is compact. Hence Crit_{H_V} is compact with respect to the strong C^0 topology on S^1. Since S^1 is a compact set the compact-open topology is equivalent to the strong C^0 topology on S^1, and the result follows. ■

3.2.2 Uniform estimates for bounded solutions

The negative L^2 gradient flow of the action functional $\mathscr{A}_{H_V}$ yields the non-linear Cauchy-Riemann equations i.e.

$$u_s - J(u_t - X_{H_V}(t,u)) = 0. \tag{3.12}$$

Since $H_V \in \mathscr{H}_{\mathrm{hyp}}, V \in \mathscr{V}$, Equation (3.12) can be written as

$$\partial_{J,R} u - \nabla V(t,u) = 0, \tag{3.13}$$

where $\partial_{J,R}$ is the linear operator represented by

$$\partial_{J,R} = \partial_s - J\partial_t + R.$$

Here

$$R := Q - P \tag{3.14}$$

and $P, Q : \mathbb{R}^2 \to \mathbb{R}^2$ are the projections defined as follows

$$P(p,q) = (p,0), \quad Q(p,q) = (0,q). \tag{3.15}$$

Let $V \in \mathscr{V}$ and $x^{\pm}$ be in Crit_{H_V}, we define the space of connecting orbits between x^- and x^+ as

$$\mathscr{M}_{H_V}^{x^-,x^+} := \left\{ u : \mathbb{R} \times S^1 \to \mathbb{R}^2, u \text{ satisfies (3.12) } \lim_{s \to \pm\infty} u(s,\cdot) = x^{\pm} \right\}.$$

Let $0 < a < \infty$, we define the space of bounded solution as follows

$$\mathscr{M}^a_{H_V} := \left\{ u : \mathbb{R} \times S^1 \to \mathbb{R}^2, u \text{ satisfies (3.12)} : \sup_{s \in \mathbb{R}} |\mathscr{A}_{H_V}(u(s,\cdot))| \leq a \right\}.$$

We endow $\mathscr{M}^{x^-,x^+}_{H_V}$ and $\mathscr{M}^a_{H_V}$ with the compact-open topology on $\mathbb{R} \times S^1$. In this Section we show that if $V \in \mathscr{V}$, there exists $a > 0$ large enough such that

$$\mathscr{M}^{x^-,x^+}_{H_V} \subseteq \mathscr{M}^a_{H_V},$$

for every $x^\pm \in \mathrm{Crit}_{H_V}$. We prove furthermore that elements in $\mathscr{M}^{x^-,x^+}_{H_V}$ are uniformly bounded in $C^r(\mathbb{R} \times S^1)$, for every $r \in \mathbb{N}$. In order to prove this we need some preliminary lemmas.

3.2.3. Lemma. *There exists a constant $C > 0$ such that for every $u \in C^1_c(\mathbb{R}^2)$ it holds*

$$||u||_{H^1(\mathbb{R}^2)} \leq C||\partial_{J,R} u||_{L^2(\mathbb{R}^2)}. \tag{3.16}$$

Proof. Invoking the Fourier transform on $\mathbb{R}^2$ we obtain that

$$\widehat{\partial_R} = A = \begin{pmatrix} i\xi - 1 & -i\eta \\ i\eta & i\xi + 1 \end{pmatrix},$$

where (ξ, η) are the variables in the Fourier space and the Fourier transform is defined by

$$\widehat{u}(\xi,\eta) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i(s\xi + t\eta)} u(s,t)\, dsdt.$$

The inverse of A is given by

$$A^{-1} = \frac{-1}{\xi^2 + \eta^2 + 1} \begin{pmatrix} i\xi + 1 & i\eta \\ -i\eta & i\xi - 1 \end{pmatrix}.$$

Since $|\det A| \geq 1$ we get

$$\frac{\xi^2 + 1}{(\det A)^2} \leq 1, \quad \frac{\eta^2}{(\det A)^2} \leq 1, \tag{3.17}$$

which implies that $||A^{-1}||_{L(\mathbb{R}^2;\mathbb{R}^2)} \leq 1$, where $||\cdot||_{L(\mathbb{R}^2;\mathbb{R}^2)}$ is the norm of the bounded operators in from $\mathbb{R}^2$ into itself. By the Plancherel isometry we have

$$\begin{aligned} ||u||_{L^2(\mathbb{R}^2)} &= ||\widehat{u}||_{L^2(\mathbb{R}^2)} = ||A^{-1}\widehat{\partial_{J,R} u}||_{L^2(\mathbb{R}^2)} \\ &\leq ||A^{-1}||_{L(\mathbb{R}^2;\mathbb{R}^2)}||\widehat{\partial_{J,R} u}||_{L^2(\mathbb{R}^2)} \leq ||\partial_{J,R} u||_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.18}$$

We still need to prove the L^2 estimates on the derivatives. These are given in the Fourier space by $i\xi A^{-1}$ and $i\eta A^{-1}$. We have that

$$i\xi A^{-1} = \frac{-1}{\xi^2+\eta^2+1}\begin{pmatrix} i\xi-\xi^2 & -\xi\eta \\ \xi\eta & -i\xi-\xi^2 \end{pmatrix}.$$

As in (3.17) the norms of the matrix entries are bounded by 1. The case for $i\eta A^{-1}$ is analogous. As in (3.18) we obtain estimates for $||u_s||_{L^2(\mathbb{R}^2)}$ and $||u_t||_{L^2(\mathbb{R}^2)}$. ■

3.2.4. Lemma. *Let be $G \subset \mathbb{R}^2$ be compact and $K \subset\subset G$. For any function $u \in C^1(\mathbb{R}\times S^1;\mathbb{R}^2)$ there exists a constant $C_{K,G} > 0$ such that*

$$||u||_{H^1(K)} \le C_{K,G}\left(||\partial_{J,R}u||_{L^2(G)} + ||u||_{L^2(G)}\right), \tag{3.19}$$

Proof. First, extend u via periodic extension to a function on $\mathbb{R}^2$ in the t direction. Let ε be positive such that $\varepsilon < \mathrm{dist}(K,\partial G)$. By compactness, K can be covered by finitely many open balls of radius $\varepsilon/2$:

$$K \subset \bigcup_{i=1}^{N_\varepsilon} B_{\varepsilon/2}(x_i).$$

We consider a partition of unity $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots,N_\varepsilon}$ on K subordinate to $\{B_\varepsilon(x_i)\}_{i=1,\dots,N_\varepsilon}$. In particular the support of ρ_{ε,x_i} is contained in $B_\varepsilon(x_i)$, for any $\varepsilon > 0$ and any $i = 1\dots N_\varepsilon$. Then, for any u, any $\varepsilon > 0$ and any $i = 1\dots N_\varepsilon$, the function $v_{\varepsilon,i} := \rho_{\varepsilon,x_i}u$ belongs to $H_0^k(\mathbb{R}^2)$, for any $k \in \mathbb{N}$. Recall that ρ_{ε,x_i} is a scalar function (and set $\partial_J = \partial_s - J\partial_t$). Using Lemma 3.2.3 we get

$$\begin{aligned} ||v_{\varepsilon,i}||_{H^1(\mathbb{R}^2)} &= ||v_{\varepsilon,i}||_{H^1(B_\varepsilon(x_i))} \le C||\partial_{J,R}v_{\varepsilon,i}||_{L^2(B_\varepsilon(x_i))} \\ &= C||\rho_{\varepsilon,x_i}\partial_{J,R}u + \rho_{\varepsilon,x_i}Ru + u\partial_J\rho_{\varepsilon,x_i}||_{L^2(B_\varepsilon(x_i))} \\ &\le C||\partial_{J,R}u||_{L^2(G)} + C||u||_{L^2(G)}, \end{aligned} \tag{3.20}$$

where C is the constant that appears in (3.16). As $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots,N_\varepsilon}$ is a partition of unity it follows that

$$||u||_{H^1(K)} = \left|\left|\sum_{i=1}^{N_\varepsilon} v_{\varepsilon,i}\right|\right|_{H^1(K)} \le \sum_{i=1}^{N_\varepsilon}||v_{\varepsilon,i}||_{H^1(B_\varepsilon(x_i))}. \tag{3.21}$$

Putting together (3.20) and (3.21) we obtain (3.19). ■

3.2.5. Lemma. *Let $V \in \mathscr{V}$ and $a < \infty$. There exists $C_a > 0$ such that*

$$||u_s||_{L^2(\mathbb{R}\times S^1)} \le C_a, \tag{3.22}$$

for every $u \in \mathscr{M}^a_{H_V}$.

Proof. Let $u \in \mathscr{M}^a_{H_V}$, for some $0 < a < \infty$. By the gradient flow structure of the Cauchy-Riemann equations we obtain

$$\frac{d}{ds}\mathscr{A}_{H_V}(u(s,\cdot)) = -\int_0^1 |u_s(s,t)|^2\, dt.$$

Let $S > 0$ be arbitrary, by integrating over $[-S, S]$ we have

$$\int_{-S}^{S}\int_0^1 |u_s(s,t)|^2\, dtds = \mathscr{A}_{H_V}(u(-S,\cdot)) - \mathscr{A}_{H_V}(u(S,\cdot)).$$

The right hand side is bounded by $2a$. Since this bound is independent of the chosen $u \in \mathscr{M}^a_{H_U}$, passing to the limit $S \to \infty$, we obtain (3.22). ∎

3.2.6. Corollary. *Let* $V \in \mathscr{V}$. *There exists* $c > 0$ *such that*

$$||u_s||_{L^2(\mathbb{R}\times S^1)} \le c, \tag{3.23}$$

for every $x^\pm \in \mathrm{Crit}_{H_V}$ *and every* $u \in \mathscr{M}^{x^-,x^+}_{H_V}$.

Proof. Follow the proof of Lemma 3.2.5. We obtain

$$||u_s||^2_{L^2(\mathbb{R}\times S^1)} \le |\mathscr{A}_{H_V}(x^-)| + |\mathscr{A}_{H_V}(x^+)|.$$

The right hand side is uniformly bounded by (3.11) of Lemma 3.2.1, hence the same estimate holds for every $u \in \mathscr{M}^{x^-,x^+}_{H_V}$ and every $x^\pm \in \mathrm{Crit}_{H_V}$. ∎

3.2.7. Proposition. *Let* $V \in \mathscr{V}$ *and* $a < \infty$. *There exists a positive constant* $C_a > 0$ *such that*

$$||u||_{C^r(\mathbb{R}\times S^1)} \le C_a, \quad \textit{for all } r \in \mathbb{N}, \tag{3.24}$$

for every $u \in \mathscr{M}^a_{H_V}$. *Furthermore, the space* $\mathscr{M}^a_{H_V}$ *is compact in the compact-open topology on* $\mathbb{R} \times S^1$.

Proof. We start multiplying Equation (3.13) by Ru. We obtain

$$\langle \partial_{J,R}u, Ru\rangle = -p_s p + q_s q - (pq)_t + p^2 + q^2 = \partial_q V(t,p,q)q - \partial_p V(t,p,q)p. \tag{3.25}$$

Let $S > 3$ be arbitrary. We integrate Eq. (3.13) over the rectangle $R^1_S := [-S, S] \times [-2, 3]$. Because of (3.25), $V \in \mathscr{V}$ and u is one-periodic in t we obtain

$$\begin{aligned} ||u||^2_{L^2(R^1_S)} &= \iint_{R^1_S} |u|^2\, ds\, dt = \iint_{R^1_S} |p^2 + q^2|\, dsdt \\ &\leq \iint_{R^1_S} |p_s p|\, dsdt + \iint_{R^1_S} |q_s q| dsdt \\ &+ \iint_{R^1_S} |V_q(t, p, q)q|\, dsdt + \iint_{R^1_S} |V_p(t, p, q)p|\, dsdt \\ &\leq C_S ||u||_{L^2(R^1_S)}. \end{aligned} \tag{3.26}$$

Here we have used the Cauchy-Schwartz inequality and the uniform bounds on $||u_s||_{L^2(\mathbb{R}\times S^1)}$ given by Lemma 3.2.5. This implies

$$||u||_{L^2(R^1_S)} \leq C_S. \tag{3.27}$$

We point out that the constant C_S in this proof changes from line to line, it is uniform in u but it may depend on S. We can now start with elliptic bootstrapping. Since u satisfies (3.13) we have that

$$\partial_{J,R} u = -\nabla V(t, u) = f(s, t).$$

Because $V \in \mathscr{V}$

$$||\partial_{J,R} u||_{L^2(R^1_S)} = ||f||_{L^2(R^1_S)} \leq C_S \tag{3.28}$$

By Lemma 3.2.4, (3.27) and (3.28) if we choose $R^2_S; = [-(S-1), S-1] \times [-1, 2]$, then $R^2_S \Subset R^1_S$ we obtain

$$||u||_{H^1(R^2_S)} \leq C_S. \tag{3.29}$$

Differentiating the equation (3.13) with respect to t and s we obtain respectively

$$\partial_{J,R} u_t = g^1(s, t) \text{ and } \partial_{J,R} u_s = g^2(s, t),$$

fome some $g^1(s, t) := \partial_t(\nabla V(t, u(s, t)))$ and $g^2(s, t) := \partial_s(\nabla V(t, u(s, t)))$. Because $V \in \mathscr{V}$, also $g^i \in L^2(Q_S)$, for $i = 1, 2$. Let $R^3_S = [-(S-2), S-2] \times [0, 1]$ then $R^3_S \Subset R^2_S \Subset R^1_S$ By (3.19) we obtain that there exists $C_S > 0$ such that

$$||u_t||_{H^1(R^3_S)} \leq C_S \left(||g^1||_{L^2(R^2_S)} + ||u_t||_{L^2(R^2_S)} \right)$$

and

$$||u_s||_{H^1(R^3_S)} \leq C_S \left(||g^2||_{L^2(R^2_S)} + ||u_s||_{L^2(R^2_S)} \right).$$

By (3.29) we obtain

$$||u||_{H^2(R^3_S)} \leq C_S.$$

Using [5, Theorem 4.12, part 1, Case A] the continuous Sobolev embedding ($n = p = 2, q = \infty$)

$$H^2(R_S^3) \hookrightarrow C^0(R_S^3).$$

we obtain

$$||u||_{C^0(R_S^3)} \le C_S,$$

for a uniform constant C_S. Let σ be in $\mathbb{R}$, define $R_{S+\sigma} := [-(S-2)+\sigma,(S-2)+\sigma] \times [0,1]$. Since solutions of the Cauchy-Riemann equations are s-translation invariant, we have

$$||u||_{C^0(R_S^3)} = ||u||_{C^0(R_{S+\sigma})}$$

for every $\sigma \in \mathbb{R}$. From this it follows

$$||u||_{C^0(\mathbb{R}\times S^1)} \le C.$$

Let $K \Subset R_S^3$, by differentiating again Equation (3.13) with respect to t and s we get uniform $H^3(K)$ estimates. By using again [5, Theorem 4.12, part 1, Case A], this yields uniform local C^1 estimate. Since all these estimates are s-translation invariant, they can be extended to global C^1 estimate. By further differentiations of Eq. (3.13) we obtain uniform bounds for every derivative of u. From this (3.30) follows. For the remainder of the proof, it is enough to take $K_0 \Subset K$ compact, and use the compact Sobolev embedding (see [5, Theorem 6.3 Part III])

$$H^3(K) \hookrightarrow C^0(K_0).$$

This proves the final assertion. ■

3.2.8. Corollary. *Let $V \in \mathscr{V}$. There exists a positive constant $C > 0$ such that*

$$||u||_{C^r(\mathbb{R}\times S^1)} \le C, \quad \textit{for all } r \in \mathbb{N}, \tag{3.30}$$

for every $u \in \mathscr{M}_{H_V}^{x^-,x^+}$ and every $x^\pm \in \mathrm{Crit}_{H_V}$.

Proof. The same as Proposition 3.2.7. Start the bootstrapping with the bound (3.23) of Corollary 3.2.6. The latter depends only on the constant that appear in (3.10), and it is independent of the critical points $x^-, x^+ \in \mathrm{Crit}_{H_V}$. ■

3.2.9. Remark. The proof of Proposition 3.2.7 uses the same bootstrapping argument as in the proof of Proposition 4.2.1 of Chapter 4. We use here the Hilbert Sobolev spaces H^k instead of the Banach Sobolev spaces $W^{k,p}$, $k \in \mathbb{N}, p \ge 1$. This is a consequence of the Fourier estimates of Lemma 3.2.3.

3.2.3 Compactness and isolation for critical points in a PRBC

Recall that in Chapter 1, if x and y do not have tangencies[2], we have defined the integer $\mathrm{Cross}(x \operatorname{rel} y)$, in terms of the winding number of $x \operatorname{rel} y$ (with the convention that the crossing is positive if the $x - y^i$ rotates counter clockwise rotation about the origin, and negative if $x - y^i$ rotates clockwise). For un unbounded braid the same definition can be given and the following result hold.

3.2.10. Lemma (Monotonicity lemma). *Let $a > 0, V \in \mathscr{V}$, $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ and $u \in \mathscr{M}^a_{H_V}$. The function $s \mapsto \mathrm{Cross}(u(s,\cdot) \operatorname{rel} y)$ is (when well-defined) a non-increasing function of s with values in $\mathbb{Z}$. To be more precise, if there exists $(s_0, t_0) \in \mathbb{R} \times S^1$ such that $u(s_0, t_0) = y(t_0)$ then there exists an $\varepsilon_0 > 0$ such that*

$$\mathrm{Cross}(u(s_0 - \varepsilon, \cdot) \operatorname{rel} y) > \mathrm{Cross}(u(s_0 + \varepsilon, \cdot) \operatorname{rel} y),$$

for all $0 < \varepsilon \le \varepsilon_0$.

Proof. Follow line-by-line [49, Lemma 5.4]. The same argument holds since it does not require elements $x \operatorname{rel} y$ to be uniformly bounded. ■

Let y be in $\mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$. Consider an unbounded relative braid class (RBC) $[x]_{\mathbb{R}^2} \operatorname{rel} y$. Recall that y are called skeleton and $x \in [x]_{\mathbb{R}^2}$ are called free strands. If the number of free strands of an element in $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is one, the space $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is a subspace of $C^0(S^1; \mathbb{R}^2)$. Define the space of critical points of $\mathscr{A}_{H_V}$ inside $[x]_{\mathbb{R}} \operatorname{rel} y$ as

$$\mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) := \mathrm{Crit}_{H_V} \cap ([x]_{\mathbb{R}^2} \operatorname{rel} y).$$

For $V \in \mathscr{V}$, and $x^{\pm} \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ define the space of connecting orbits in $[x]_{\mathbb{R}^2} \operatorname{rel} y$ as

$$\mathscr{M}^{x^-,x^+}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) = \left\{ u \in \mathscr{M}^{x^+,x^-}_{H_V} : u(s,\cdot) \in [x]_{\mathbb{R}^2} \operatorname{rel} y, \text{ for all } s \in \mathbb{R} \right\}.$$

Let $a \in \mathbb{R}$ be positive. The space of bounded solutions of Equation (3.13) is defined as follows

$$\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) = \left\{ u \in \mathscr{M}^a_{H_V} : u(s,\cdot) \in [x]_{\mathbb{R}^2} \operatorname{rel} y, \text{ for all } s \in \mathbb{R} \right\}.$$

It follows from Lemma 3.2.10 that if $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is an unbounded RBC, and $a > 0$, for $u \in \mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ we have that $\mathrm{Cross}(u(s,\cdot) \operatorname{rel} y)$ is an invariant of the unbounded relative braid class. This means that $\mathrm{Cross}(u(s,\cdot) \operatorname{rel} y)$ is constant on connected components of $[x]_{\mathbb{R}^2} \operatorname{rel} y$.

[2] more precisely if x and y^i for all $i = 1, \dots, m$ do not have tangencies

3.2.11. Remark. Let $0 < a < \infty, V \in \mathscr{V}$ and $[x]_{\mathbb{R}^2} \operatorname{rel} y$ an unbounded RBC fiber, with $y \in \mathcal{L}(\mathbf{C}_m(\mathbb{D}^2))$. By Proposition 3.2.7 we have uniform bounds in C^r for every r for the space $\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$.

Recall that for a relative braid class fiber with skeleton y, we say that $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is proper, if the free strand x cannot be deformed onto itself, or components in $y_c \subset y$. For an unbounded PROPER relative braid class $[x]_{\mathbb{R}^2} \operatorname{rel} y$, the space

$$\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$$

has special properties. The most important is that unbounded PROPER RBCes isolate the space of critical points.

3.2.12. Proposition (properness implies isolation). *Let $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ and $V \in \mathscr{V}$. If $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is an unbounded* PROPER *RBC, then the space $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is compact (with respect to the compact-open topology) and isolated in $[x]_{\mathbb{R}^2} \operatorname{rel} y$.*

Proof. We first prove that $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is compact. Let $x_n \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$, then in particular $x_n \in \operatorname{Crit}_{H_V}$. Since $\operatorname{Crit}_{H_V}$ is compact (by Proposition 3.2.2), up to a subsequence

$$x_n \to x_0 \in \operatorname{Crit}_{H_V}.$$

Since $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is proper, by Lemma 3.2.10, $\operatorname{Cross}(x_n \operatorname{rel} y)$ is well defined for all $n \in \mathbb{N}$ and, by continuity, $\operatorname{Cross}(x_n \operatorname{rel} y) = \operatorname{Cross}(x_0 \operatorname{rel} y)$. This implies that $x_0 \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$. It follows that $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is closed, and hence compact. We now prove that $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is isolated. By Remark 3.2.11, we have that there exists $c > 0$ such that $||x||_{C^0} \le c$, for every $x \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$. We will show that there exists $0 < r < c$ such that

$$|x(t)| < r \quad \text{and} \quad |x(t) - y^j(t)| > c - r, \quad \text{for all } j = 1, \dots, m \tag{3.31}$$

for all $x \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$. By contradiction, suppose that such r does not exist. Let $x_n \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$, by compactness of $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ (up to a subsequence) there exists $x_0 \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$, such that $x_n \to x_0$. If such r in (3.31) does not exist, then there exists $t_0 \in S^1$ such that

$$x_0(t_0) = y^j(t_0) \quad \text{for some } j \in \{1, \dots, m\}.$$

By uniqueness of the solutions of the Euler-Lagrange equations we have $x(t) = y^j(t)$, for all $t \in S^1$ which contradicts the fact that $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is proper. ■

3.2.4 Genericity for critical points of hyperbolic Hamiltonians

Let $V \in \mathscr{V}, y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ and $[x]_{\mathbb{R}^2} \operatorname{rel} y$ an unbounded RBC. We say that $x \in \operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is a non-degenerate critical point if the operator

$$J\frac{d}{dt} + D^2 H_V(t,x) : H^1(S^1) \to L^2(S^1)$$

is invertible. We prove now that if V is chosen "generically" in $\mathscr{V}$ then all critical points are non-degenerate. This is the content of the following proposition. We first give some nomenclature. For $H_V \in \mathscr{H}_{\text{hyp}}, V \in \mathscr{V}$, we say that $y = \{y^1(t), \ldots, y^m(t)\} \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ is a solution curve of X_{H_V}[3] if for every $i = 1, \ldots, m$ y^i a solution of

$$y^i_t = X_{H_V}(t, y^i), \quad y^i(0) = y^{\sigma(i)}(1), \text{ for some } \sigma \in S_m. \tag{3.32}$$

Recall that $||\cdot||_{C^\infty}$ is defined as

$$||h||_{C^\infty} := \sum_{k \in \mathbb{N}} \varepsilon_k ||h||_{C^k},$$

for a sufficiently fast decaying positive sequence ε_k.

3.2.13. Proposition. *Let y be in $\mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ and $[x]_{\mathbb{R}^2} \operatorname{rel} y$ be an unbounded PRBC. For every hyperbolic Hamiltonian $H_V \in \mathscr{H}_{\text{hyp}}, V \in \mathscr{V}$ with y a solution curve of X_{H_V} there exists a $\delta_* > 0$ with the following significance. For every $\delta < \delta_*$ there exists a $V' \in \mathscr{V}$ (and $H_{V'} \in \mathscr{H}_{\text{hyp}}$) with the property that*

(i) $||H_V - H_{V'}||_{C^\infty} < \delta$;

(ii) *y is a solution curve of $X_{H_{V'}}$.*

and such that $\operatorname{Crit}_{H_{V'}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ consists only of non degenerate critical points. This implies, by compactness, that the space $\operatorname{Crit}_{H_{V'}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ consists only of finitely many isolated points.

Proof. By Lemma 3.2.1 the space $\operatorname{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is uniformly bounded, by a constant $C > 0$. Let $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$, then y has m strands. Let $B_\varepsilon(y^i)$ a tubular neighborhood of y^i of radius ε. Define

$$N_\varepsilon(y) = \bigcup_{k=1,\ldots,m} B_\varepsilon(y^i).$$

[3]the Hamiltonian vector field is defines as $X_{H_V} = J\nabla H_V$

If $\varepsilon > 0$ is chosen small enough, N_ε consists of m disjoints cylinders.

$$D_C := \{x \in \mathbb{R}^2 : |x| \leq C\},$$

Let

$$\mathbb{T}_C^2 := S^1 \times D_C$$

be the two-torus. By the uniform bounds on Crit_{H_V}, if C is chosen large enough and $\varepsilon > 0$ small enough, $N_\varepsilon(y) \subset \mathbb{T}_C^2$. Define

$$A_{\varepsilon,C} := \mathbb{T}_C^2 \setminus N_\varepsilon(y).$$

Since $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is proper there exists an $\varepsilon_* > 0$ such that for every $\varepsilon \leq \varepsilon_*$ it holds that

$$\mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) \subset \operatorname{int}(A_{2\varepsilon,C}).$$

Fix now $\varepsilon \in (0, \varepsilon_*]$ and, for $\delta > 0$ small define the space

$$\mathcal{V}_{\delta,\varepsilon} := \{v \in C^\infty(S^1 \times \mathbb{R}^2; \mathbb{R}), \operatorname{supp} v \subset A_{\varepsilon,C}, ||v||_{C^\infty} \leq \delta\}$$

of perturbations of V. Let v_δ be in $\mathcal{V}_{\varepsilon,\delta}$ and define Let $V' := V + v_\delta$. Construct a Hamiltonian

$$H_{V'} = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 + V + v_\delta.$$

By construction $H_{V'} \in \mathscr{H}_{\mathrm{hyp}}$, $V' \in \mathscr{V}$, y is a solution curve of $X_{H_{V'}}$, and

$$||H_V - H_{V'}||_{C^\infty} < \delta.$$

Using the same arguments as in Proposition 3.2.12 we have that the compact set

$$\mathrm{Crit}_{H_{V'}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$$

is isolated for all perturbations $v_\delta \in \mathcal{V}_{\delta,\varepsilon}$. This shows that

$$\mathrm{Crit}_{H_{V+v_\delta}}([x]_{\mathbb{R}^2} \operatorname{rel} y) \to \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$$

in the Hausdorff metric[4] as $\delta \to 0$. Therefore there exists a $\delta_* > 0$ such that

$$\mathrm{Crit}_{H_{V+v_\delta}}([x]_{\mathbb{R}^2} \operatorname{rel} y) \subset \operatorname{int}(A_{2\varepsilon,C}) \tag{3.33}$$

for all $0 \leq \delta \leq \delta_*$. Now fix $\delta \in (0, \delta_*]$. The Hamilton equations for $H_{V'}$ are $-Jx_t + \nabla H_V(t,x) + \nabla v_\delta(t,x) = 0$, with periodic boundary conditions. Now, in order to

[4]If X, Y are non-empty subsets of a metric space (M, d) we define their Hausdorff distance as

$$d_H(X,Y) := \max\{\sup_{x\in X} \inf_{y\in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y)\}$$

conclude the proof, we need to show that we can choose $v_\delta \in \mathcal{V}_{\delta,\varepsilon}$ generically such that for every $x \in \mathrm{Crit}_{H_{V'}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ the operator $Jx_t + D^2 H_{V'}(t,x)$ is invertible. Without going into details (see [49, Proposition 7.1] for a complete proof), we only say that using the fact that $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is proper, and, in particular (3.33), for a large (dense) class of $v_\delta \in \mathcal{V}_{\delta,\varepsilon}$, the Sard-Smale Theorem [47] implies that the operator $Jx_t + D^2 H_{V'}(t,x)$ is invertible. ■

3.2.14. Remark. The previous Proposition 3.2.13 shows that if we choose V generically, then critical points are non-degenerate, which implies that they are isolated points. Compactness of critical points, (Proposition 3.2.12) shows furthermore that they are finitely many.

We now prove that the constant C_a which appears in (3.30) is independent of a, provided a is chosen large enough and $V \in \mathscr{V}$ is generic. This is a consequence of the fact that generically the space of bounded solutions equals the space of connecting orbits (inside a proper relative braid class).

3.2.15. Proposition (uniform bounds). *Let $H_V \in \mathscr{H}_{\mathrm{hyp}}$ with $V \in \mathscr{V}$ generic. Let y be in $\mathcal{L}\mathbf{C}_m(\mathbb{D}^2)$ and $[x]_{\mathbb{R}^2} \operatorname{rel} y$ be an unbounded PRBC. If $a > 0$ is sufficiently large it holds*

$$\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) = \bigcup_{x^\pm \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)} \mathscr{M}^{x^-,x^+}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y). \tag{3.34}$$

This implies that there exists $C > 0$ such that

$$||u||_{C^r(\mathbb{R}\times S^1)} \le C,$$

for all $a > 0$ sufficiently large and all $u \in \mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$.

Proof. By Corollary 3.2.8 we obtain that there exists (a uniform) $a > 0$

$$\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) \supseteq \mathscr{M}^{x^-,x^+}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$$

for all $x^\pm \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$. Hence taking the union over all $x^\pm \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ we obtain

$$\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y) \supseteq \bigcup_{x^\pm \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)} \mathscr{M}^{x^-,x^+}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y).$$

It remain to establish the opposite inclusion. Let $u \in \mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$, then by Proposition 3.2.7 we obtain that there exists $C_a > 0$ such that $||u||_{C^r(\mathbb{R}\times S^1)} \le C_a$. Using this bound and and genericity of $V \in \mathscr{V}$ as in [45, Proposition 4.2] we establish that solutions in $\mathscr{M}^a_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ have limits, i.e.

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t), \quad \text{for all } t \in S^1.$$

for some $x^\pm \in C^1(S^1)$. If $a > 0$ is large enough then $x^\pm \in \mathrm{Crit}_{H_V}$. Hence $u \in \mathscr{M}_{H_V}^{x^-,x^+}$, for some $x^\pm \in \mathrm{Crit}_{H_V}$. Isolation of proper relative braid classes implies that $x^\pm \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$. by the Monotonicity Lemma (Lemma 3.2.10) we have $u \in \mathscr{M}_{H_V}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ and the other inclusion follows.

To prove the final assertion, we note that elements on the right hand side of (3.34) are uniformly bounded by a constant C that is independent of a. ■

3.2.16. Remark. Let $a > 0$ be large enough $V \in \mathscr{V}$ generic and $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is an unbounded PRBC. It follows from Proposition 3.2.15 that the space $\mathscr{M}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is compact in the following sense. Let $\{u_\nu\}_{\nu\in\mathbb{N}} \subseteq \mathscr{M}_{H_V}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ be a sequence, then there exists a subsequence (still denoted by u_ν) and sequences of times $s_\nu^i \in \mathbb{R}, i = 0, \dots, k$ such that $u_\nu(\cdot + s_\nu^i, \cdot)$ converges with its derivatives uniformly on compact sets to $u^i \in \mathscr{M}_{H_V}^{x^i,x^{i-1}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$, where $x^i \in \mathrm{Crit}_{H_V}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ for $i = 0, \dots, k$ and $x^0 = x^-$ and $x^k = x^+$. This type of convergence is called in literature geometric convergence (see [45]).

3.2.5 Hyperbolic braid Floer homology and its isomorphism with the classical braid Floer homology

We are now ready to construct the Floer homology for hyperbolic Hamiltonian systems in the setting of unbounded PRBC. We follow the steps of Section 1.4.1. Recall that the image of $\mathscr{M}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$ under the mapping $u \mapsto u(0,\cdot)$ is called $\mathscr{S}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$. We have proved compactness of the space $\mathscr{M}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$. The proof of compactness of $\mathscr{S}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$ follows from the same arguments as in [49], and from the compactness of $\mathscr{M}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$. The same arguments as in [49] show that $\mathscr{S}_{H_V}^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$ is isolated. We have shown genericity properties of critical points of hyperbolic Hamiltonian systems. For connecting orbits, the arguments are the same as in [49]. This yields that the space of connecting orbits consists of smooth dimensional manifolds without boundary. For non-degenerate critical points of hyperbolic Hamiltonians we can define the Conley-Zenhder index as in Section 2.4. By grading the critical points of a generic hyperbolic Hamiltonian with the Conley-Zenhder index, we define a chain complex $C_k([x]_{\mathbb{R}^2} \operatorname{rel} y)$ with coefficients in $\mathbb{Z}_2$ and a boundary ∂_k operator. Its homology is denoted by

$$\mathrm{HHF}_k([x]_{\mathbb{R}^2} \operatorname{rel} y; H_V) := H_k(C_*, \partial_*).$$

The latter are called hyperbolic braid Floer homology groups. By the same arguments as in [49] we can show that $\mathrm{HHF}_k([x]_{\mathbb{R}^2} \operatorname{rel} y; H_V)$ is independent of the chosen hyperbolic Hamiltonian and of the fiber. Hence we can write

$$\mathrm{HHF}_k([x \operatorname{rel} y]_{\mathbb{R}^2}).$$

We now prove the isomorphism (3.4). But before we perform an extension from bounded proper relative braid classes to unbounded ones in the following way. Let $[x\,\mathrm{rel}\,y]_{\mathbb{D}^2}$ be a bounded proper relative braid class and $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ its fiber. Extend now $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ to $[x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$ by considering homotopies in $\mathbb{R}^2$ instead of $\mathbb{D}^2$. Since $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ is proper, by Remark 3.1.1, $[x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$ is proper. The extended unbounded proper relative braid classes $[x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$ inherit furthermore from $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ the topological property that can not be deformed onto $\partial\mathbb{D}^2$. Extensions of proper relative braids class fibers enjoy the following property.

3.2.17. Lemma. *Let $y \in \mathcal{L}\mathbf{C}_m(\mathbb{D}^2)$ and $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ be a bounded proper relative braid class fiber. Let $[x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$ its proper unbounded extension. If there exists a loop $\tilde{x}$ in $\mathbb{R}^2$ with support entirely contained in $(\mathbb{D}^2)^c := \{x \in \mathbb{R}^2 : |x| > 1\}$ then*

$$\tilde{x} \notin [x]_{\mathbb{R}^2}\,\mathrm{rel}\,y.$$

Proof. We have $y \in \mathcal{L}(\mathbf{C}_m(\mathbb{D}^2))$. Then all the skeleton strands have support in $\mathbb{D}^2$. Fix a representative $x_0 \in [x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$. Since, as sets, $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y \subset [x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$, then we can assume that $x_0 \in [x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$ has support entirely contained in $\mathbb{D}^2$. Suppose, by contradiction, that there exists $\tilde{x} \in [x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$, with support entirely contained in $(\mathbb{D}^2)^c$. Then there exists a homotopy $g : [0,1] \times \mathcal{L}(\mathbb{R}^2) \to \mathcal{L}(\mathbb{R}^2)$ such that $g(0,x) = x_0$ and $g(1,x) = \tilde{x}$ and $g(\lambda,x) \in [x]_{\mathbb{R}^2}\,\mathrm{rel}\,y$, for all $\lambda \in [0,1]$. Then there are only two topological configurations for $\tilde{x}$.

(i) $\tilde{x}$ has non-trivial homotopy type. Without loss of generality, since $\mathrm{supp}(\tilde{x})$ is entirely contained in $(\mathbb{D}^2)^c$ and $\tilde{x}$ has non-trivial homotopy type, we can assume that there exists $R > 1$ such that $\tilde{x} = \partial\mathbb{D}^2_R = \{x \in \mathbb{R}^2 : |x| = R\}$. If we now perform the following homotopy

$$h(\lambda,x) = \begin{cases} g(\lambda,x) & \text{for all } \lambda : |g(\lambda,x)| \le 1, \text{ for all } x \in \mathcal{L}(\mathbb{R}^2) \\ g(\lambda,\bar{x})\#\pi_{\partial\mathbb{D}^2}g(\lambda,\bar{x}) & \text{for all } \lambda : \text{ there exists } \bar{x} : |g(\lambda,\bar{x})| \ge 1 \end{cases}$$

where $\pi_{\partial\mathbb{D}^2} : C([0,\tau];\mathbb{R}^2) \to \partial\mathbb{D}^2$ is the continuous function that makes every curve γ in $\mathbb{R}^2$ with intersection with $\partial\mathbb{D}^2$ collapse to the segment (inside γ) in $\partial\mathbb{D}^2$ with endpoints the intersection between the curve and $\partial\mathbb{D}^2$, and $\#$ denote the concatenation between two curves. By construction h is continuous, $|h(\lambda,x)| \le 1$, for all $\lambda \in (0,1)$, and all $x \in \mathcal{L}(\mathbb{R}^2)$, and $h(0,x) = x_0$ and $\tilde{x} = h(1,x) = \pi_{\partial\mathbb{D}^2}g(1,x) = \partial\mathbb{D}^2$, for all $|x| > 1$. Hence we have found a homotopy $h(\lambda,x) \in [x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ for all $\lambda \in [0,1]$ with endpoints x_0 and $\partial\mathbb{D}^2$. This contradicts the fact that $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ is proper.

(ii) $\tilde{x}$ has trivial homotopy type. Without loss of generality, since $\mathrm{supp}(\tilde{x})$ is entirely contained in $(\mathbb{D}^2)^c$ and $\tilde{x}$ has trivial homotopy type, we can assume that $\tilde{x} \in \partial\mathbb{D}^2$. Hence $g(\lambda,x) \in [x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ for all $\lambda \in [0,1]$ with endpoints x_0 and $\tilde{x} \in \partial\mathbb{D}^2$. This contradicts again the fact that $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$ is proper. ■

3.2.18. Theorem. *Let $[x \operatorname{rel} y]_{\mathbb{D}^2}$ a proper relative braid class and $[x \operatorname{rel} y]_{\mathbb{R}^2}$ its unbounded extension. Then*

$$\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}). \tag{3.35}$$

Proof. The proof is constructive. Let $[x \operatorname{rel} y]_{\mathbb{D}^2}$ be a bounded proper relative braid class with fiber $[x]_{\mathbb{D}^2} \operatorname{rel} y$ and $y \in \mathcal{L}(\mathbf{C}_m(\mathbb{D}^2))$. Let $H \in \mathscr{H}$ (defined in Section 1.3.1) be generic, then

$$\mathrm{HF}_k([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y; H).$$

Since the left hand side is independent (up to isomorphims) of the chosen Hamiltonian $H \in \mathscr{H}$, without loss of generality we can choose $H(t, \cdot) \in C_0^\infty(\mathbb{D}^2)$. Extend $H(t, \cdot)$ smoothly to $\mathbb{R}^2$ in the following way

$$\widehat{H}(t,x) := \begin{cases} H(t,x) & |x| \le 1 \\ 0 & |x| \ge 1. \end{cases}$$

Note that $\widehat{H}(t, \cdot) \in C_0^\infty(\mathbb{R}^2)$. Let g be in $C^\infty(\mathbb{R}^+)$ such that

$$g(r) = \begin{cases} 1 & r \le 2 \\ 0 & r \ge 3, \end{cases}$$

and define

$$\widehat{V}(t,p,q) = g(|x|)\left(-\tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \widehat{H}(t,x)\right).$$

By construction $\widehat{V} \in \mathscr{V}$ and hence

$$H_{\widehat{V}}(t,x) = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 + \widehat{V}(t,p,q)$$

is in $\mathscr{H}_{\mathrm{hyp}}$. We have furthermore, by construction of $H_{\widehat{V}}$, that every constant x_0 with $1 \le |x_0| \le 2$ is a solution of the Hamilton equations of $H_{\widehat{V}}$, hence an element of $\mathrm{Crit}_{H_{\widehat{V}}}$. Since the Hamilton equations of $H_{\widehat{V}}$ have no closed orbits for $|x| \ge 3$, there are no critical points of $\mathscr{A}_{H_{\widehat{V}}}$ for $|x| \ge 3$.

The proof now proceeds in two steps.

(i) We prove that

$$\mathrm{Crit}_H([x]_{\mathbb{D}^2} \operatorname{rel} y) = \mathrm{Crit}_{H_{\widehat{V}}}([x]_{\mathbb{R}^2} \operatorname{rel} y). \tag{3.36}$$

It suffices to show that every $x \in \mathrm{Crit}_{H_{\widehat{V}}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ has support in $\mathrm{int}(\mathbb{D}^2)$. Suppose, by contradiction, that there exists $x \in \mathrm{Crit}_{H_{\widehat{V}}}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ and t_0 such that $x(t_0) = x_0$ for some $x_0 \in \partial\mathbb{D}^2$. Uniqueness of the initial value problem of the Hamilton equations yields $|x(t)| = x_0$ for all $t \in S^1$. This is a contradiction, since $[x]_{\mathbb{R}^2} \operatorname{rel} y$ is an extension of the proper relative braid class $[x]_{\mathbb{D}^2}\operatorname{rel}y$ (that can not be deformed onto $\partial\mathbb{D}^2$). This proves that, for every $x \in \operatorname{Crit}_{H_{\widehat{V}}}([x]_{\mathbb{R}^2}\operatorname{rel}y)$, x has support either in the interior or in the exterior of $\mathbb{D}^2$. Lemma 3.2.17 proves that the support of x can not be contained entirely in $(\mathbb{D}^2)^c$, and hence (3.36).

(ii) And now we show that

$$\mathscr{M}_H^{x^-,x^+}([x]_{\mathbb{D}^2}\operatorname{rel}y) = \mathscr{M}_{H_{\widehat{V}}}^{x^-,x^+}([x]_{\mathbb{R}^2}\operatorname{rel}y). \tag{3.37}$$

By (i) if $u \in \mathscr{M}_{H_{\widehat{V}}}^{x^-,x^+}([x]_{\mathbb{R}^2}\operatorname{rel}y)$ then there exists $S > 0$ such that $u(s,\cdot) \in [x]_{\mathbb{D}^2}\operatorname{rel}y$ for all $|s| > S$. The Monotonicity Lemma (Lemma 3.2.10) implies that $u(s,\cdot) \in [x]_{\mathbb{D}^2}\operatorname{rel}y$ for every $s \in \mathbb{R}$. Since $H_{\widehat{V}}|_{\mathbb{D}^2} = H_{\mathbb{D}^2}$ we have that $u \in \mathscr{M}_H^{x^-,x^+}([x]_{\mathbb{D}^2}\operatorname{rel}y)$. This implies (3.37).

To conclude the proof we have, by (3.37) homotopy invariance of the Hamiltonians $H \in \mathscr{H}$ and $H_V \in \mathscr{H}_{\text{hyp}}$, homotopy invariance of the fiber $[x]_{\mathbb{D}^2}\operatorname{rel}y$ in $[x\operatorname{rel}y]_{\mathbb{D}^2}$ and of $[x]_{\mathbb{R}}\operatorname{rel}y$ in $[x\operatorname{rel}y]_{\mathbb{R}^2}$, and homotopy invariance of the chosen constant almost complex structure $J \in \mathscr{J}$, that

$$\begin{aligned}\mathrm{HF}_*([x\operatorname{rel}y]_{\mathbb{D}^2}) &\cong \mathrm{HF}_*([x]_{\mathbb{D}^2}\operatorname{rel}y; H, J)\\ &= \mathrm{HHF}_*([x]_{\mathbb{R}^2}\operatorname{rel}y; H_{\widehat{V}}, J) \cong \mathrm{HHF}_*([x\operatorname{rel}y]_{\mathbb{R}^2}).\end{aligned}$$

This shows (3.35) and concludes the proof. ■

3.3 Mechanical braid Morse homology

In this Section we define, for a special type of braids, called Legendrian, a Morse type thoery. The construction will follow the step summarized in Section 1.4.1 and it is carried out in full details except for the genericity properties for critical points and connecting orbits. Since we consider Legendrian unbounded braid class, we need to obtain also in this case we need uniform estimates. These estimates will be obtained in the same manner as in Section 3.2, but in a parabolic setting.

Consider now a simplification of the hyperbolic Hamiltonian system (3.1). Let $U \in \mathscr{U}$ (defined in Section 3.1.2) and $H_U \in \mathscr{H}_{\text{mech}}$. Recall that H_U has the form

$$H_U(t,p,q) = \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2 - U(t,q)$$

where U satisfies the hypotheses (U1) and (U2) of Section 3.1.2. The action functional has the form (3.7). Note that $\mathscr{L}_U(q) = \mathscr{A}_{H_U}(q_t, q)$. Critical points of $\mathscr{L}_U$ are denoted by Crit_U.

3.3.1. Remark. Since mechanical systems are special hyperbolic hamiltonian systems, for mechanical systems, Lemma 3.2.1 and Proposition 3.2.2 continue to hold. Hence there exists constants $c > 0$ and $C > 0$ such that

$$||q||_{C^0(S^1)} \leq c \tag{3.38}$$

and

$$|\mathscr{L}_U(q)| \leq C, \tag{3.39}$$

for all $q \in \mathrm{Crit}_U$. Furthermore the set Crit_U is compact with respect the compact-open topology.

The choice of H_U of the form (3.6), implies that critical points of $\mathscr{L}_U$ are restricted to be $x(t) = (p(t), q(t)) = (q_t(t), q(t))$. In the setting of braids, it can be easily seen that such strands lay in the kernel of the one-form $\alpha = dq - pdt$. This property is known as the Legendrian property, and we will refer to these braids as LEGENDRIAN BRAIDS. It it important to highlight that for a Legendrian braid (with multiple strands) the Legendrian constraint implies that all intersections correspond to positive crossings.

3.3.1 Uniform bounds (and compactness) for bounded solutions

The negative L^2 gradient flow of the action functional $\mathscr{L}_U$ gives rise to the nonlinear heat equation

$$v_s - v_{tt} + v - \partial_v U(t, v) = 0, \tag{3.40}$$

Solutions of Equation (3.40) lay in the space

$$C^{1,2}(\mathbb{R} \times S^1) := \{v \in C^0(\mathbb{R} \times S^1) : v_s, v_t, v_{tt} \in C^0(\mathbb{R} \times S^1)\}.$$

As for the Cauchy-Riemann equations, for $U \in \mathscr{U}$ and $q^\pm \in \mathrm{Crit}_U$ define the space of connecting orbits between q^- and q^+ as

$$\mathscr{N}_U^{q^+,q^-} := \left\{ v \in C^{1,2}(\mathbb{R} \times S^1) : v \text{ satisfies (3.40) and } \lim_{s \to \pm\infty} v(s, \cdot) = q^\pm \right\}.$$

Let $a > 0$ define the space of bounded solutions as

$$\mathscr{N}_U^a := \left\{ v \in C^{1,2}(\mathbb{R} \times S^1) : v \text{ satisfies (3.40) and } \sup_{s \in \mathbb{R}} |\mathscr{L}_U(v(s, \cdot))| \leq a \right\}.$$

In order to prove the requested uniform estimates it is useful to deal with the so-called anisotropic Sobolev spaces. We define

$$H_2^{1,2}(\mathbb{R}^2) := \{v \in L^2(\mathbb{R}^2) : v_t, v_s, v_{tt} \in L^2(\mathbb{R}^2)\},$$

The norm associated to the space $H_2^{1,2}(\mathbb{R}^2)$ is the following

$$||v||_{H_2^{1,2}(\mathbb{R}^2)} = ||v||_{L^2} + ||v_s||_{L^2} + ||v_t||_{L^2} + ||v_{tt}||_{L^2}.$$

Define then recursively

$$H_2^{k,2k}(\mathbb{R}^2) := \{v \in L^2(\mathbb{R}^2) : v_t, v_s, v_{tt} \in H_2^{k-1}(\mathbb{R}^2)\}.$$

We define also the linear heat operator $\partial_{\text{heat}} : H_2^{1,2}(\mathbb{R} \times S^1) \to L^2(\mathbb{R} \times S^1)$ as follows

$$\partial_{\text{heat}} v := v_s - v_{tt} + v.$$

In order to obtain uniform estimates for connecting orbits we need some preliminary lemmas. We now prove a parabolic version of Lemma 3.2.3.

3.3.2. Lemma. *It holds*

$$||v||_{H_2^{1,2}(\mathbb{R}^2)} \le ||\partial_{\text{heat}} v||_{L^2(\mathbb{R}^2)}, \tag{3.41}$$

for every $v \in C_c^{1,2}(\mathbb{R}^2)$.

Proof. As in Lemma 3.2.3 we denote by $\hat{v}(\xi, \eta) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i(s\xi + t\eta)} v(s,t)\, dsdt$ the Fourier transform of v. By using the Fourier transform we have

$$||\hat{v}||^2_{L^2(\mathbb{R}^2)} = ||\tfrac{1}{i\eta + \xi^2 + 1} \widehat{\partial_{\text{heat}} v}||^2_{L^2(\mathbb{R}^2)}.$$

Hence, using the Plancherel identity we obtain

$$||v||_{L^2(\mathbb{R}^2)} \le ||\partial_{\text{heat}} v||_{L^2(\mathbb{R}^2)}.$$

The functions v_s, v_t, v_{tt} in the Fourier space become respectively $i\eta\hat{v}, i\xi\hat{v}$ and $-\xi^2\hat{v}$. We have

$$\begin{array}{rcccl} |i\eta\hat{v}|^2 & = & \frac{\eta^2}{\eta^2 + (\xi^2+1)^2} |\widehat{\partial_{\text{heat}} v}|^2 & \le & |\widehat{\partial_{\text{heat}} v}|^2 \\ |i\xi\hat{v}|^2 & = & \frac{\xi^2}{\eta^2 + (\xi^2+1)^2} |\widehat{\partial_{\text{heat}} v}|^2 & \le & \frac{1}{2} |\widehat{\partial_{\text{heat}} v}|^2 \\ |\xi^2\hat{v}|^2 & = & \frac{\xi^4}{\eta^2 + (\xi^2+1)^2} |\widehat{\partial_{\text{heat}} v}|^2 & \le & |\widehat{\partial_{\text{heat}} v}|.^2 \end{array}$$

By using again the Plancherel identity (3.41) follows. ∎

And now we prove a parabolic version of Lemma (3.2.4).

3.3.3. Lemma. *Let $G \subset \mathbb{R}^2$ be compact and $K \Subset G$. There exists a constant $C_{K,G} > 0$ such that*

$$||v||_{H_2^{1,2}(K)} \leq C_{K,G} \left(||\partial_{\text{heat}} v||_{L^2(G)} + ||v||_{L^2(G)} + ||v_t||_{L^2(G)} \right), \tag{3.42}$$

for every $u \in C^{1,2}(\mathbb{R} \times S^1)$.

Proof. Extend v via periodic extension in t to a function $v \in C^{1,2}(\mathbb{R} \times S^1)$. The proof of estimate (3.42) is equal to the proof of Lemma 3.2.4. Interchange in that proof the role of the Cauchy-Riemann operator $\partial_{J,R}$ with the heat operator ∂_{heat} and use Lemma 3.3.2, when Lemma 3.2.3 is used. The extra term ($||v_t||_{L^2(G)}$) arises because the heat operator has two derivatives in t. ■

3.3.4. Remark. Let $U \in \mathscr{U}$, $a < \infty$ and $q^\pm \in \text{Crit}_U$. The estimates (3.22) and (3.23) continue to hold respectively for elements in $\mathscr{N}_U^a$ and $\mathscr{N}_U^{q^-,q^+}$. Their proofs use only the fact that the Cauchy-Riemann equations are a gradient flow system and the bounds on the functional. Since these two conditions are satisfied by the heat flow we obtain that for every a there exists $c_a > 0$ such that

$$||v_s||_{L^2(\mathbb{R}\times S^1)} \leq c_a, \tag{3.43}$$

for every $v \in \mathscr{N}_U^a$. We get also that there exists $c > 0$ such that

$$||v_s||_{L^2(\mathbb{R}\times S^1)} \leq c. \tag{3.44}$$

for every $v \in \mathscr{N}_U^{q^-,q^+}$

3.3.5. Proposition. *Let $U \in \mathscr{U}$ and $a < \infty$. There exists a constant $C_a > 0$ such that*

$$||v||_{C^r(\mathbb{R}\times S^1)} \leq C_a \text{ for all } r \in \mathbb{N}, \tag{3.45}$$

for every $v \in \mathscr{N}_{H_U}^a$. Furthermore, the space $\mathscr{N}_U^a$ is compact with respect to the compact-open topology on $\mathbb{R} \times S^1$.

Proof. The proof resembles the proof of Proposition 3.2.7. Nevertheless we provide all the details. We will obtain (3.45) via bootstrapping argument as Proposition 3.2.7.
Let $S > 3$ and $R_S^1 := [-S, S] \times [-2, 3]$. We start multiplying Equation (3.40) by v and integrating over R_S and using the periodic boundary conditions. We obtain

$$\iint_{R_S^1} v_s v \, dsdt = \iint_{R_S^1} v_{tt} v - v^2 + \partial_v U(t,v) v \, dsdt = \iint_{R_S^1} -v_t^2 - v^2 + \partial_v U(t,v) \, dsdt.$$

Rearranging the terms, using the Cauchy-Schwartz inequality, (3.43) and the fact that $U \in \mathscr{U}$ we obtain

$$\begin{aligned} ||v||^2_{L^2(R^1_S)} + ||v_t||^2_{L^2(R^1_S)} &= \iint_{R^1_S} v^2 + v_t^2 \, dsdt \\ &\leq \iint_{R^1_S} |v_s v| \, dsdt + \iint_{R^1_S} \partial_v U(t,v) v \, dsdt \\ &\leq C_S(||v||^2_{L^2(R^1_S)} + ||v_t||^2_{L^2(R^1_S)})^{1/2}. \end{aligned}$$

From this it follows that

$$||v||_{L^2(R^1_S)} \leq C_S, \tag{3.46}$$

and

$$||v_t||_{L^2(R^1_S)} \leq C_S. \tag{3.47}$$

For $v \in \mathscr{N}^a_U$ we have

$$v_s - v_{tt} + v = \partial_v U(v,t) = f(s,t). \tag{3.48}$$

Because $U \in \mathscr{U}$ then f satisfies

$$||\partial_{\text{heat}} v|| = ||f||_{L^2(R^1_S)} \leq C_S. \tag{3.49}$$

Let $R^2_S := [-S+1, S-1] \times [-1, 2]$, then $R^2_S \Subset R^1_S$. Using (3.42), (3.46), (3.47), and (3.49) we obtain

$$||v||_{H^{1,2}_2(R^2_S)} \leq C_S \left(||f||_{L^2(R^1_S)} + ||v||_{L^2(R^1_S)} + ||v_t||_{L^2(R^1_S)} \right) < C_S, \tag{3.50}$$

We continue with parabolic bootstrapping. By differentiating the equation (3.48) with respect to t

$$\partial_{\text{heat}} v_t = g^1(s,t) \quad \text{and} \quad \partial_{\text{heat}} v_s = g^2(s,t),$$

where $g^1 = \partial_t(\partial_v U)$ and $g^2 = \partial_s(\partial_v U)$. Because $U \in \mathscr{U}$ then $g^i \in L^2(\mathbb{R}^2), i = 1, 2$. Let now $R^3_S := [-S+2, S-2] \times [0, 1]$, then $R^3_S \Subset R^2_S$. By (3.42) we obtain, using (3.50)

$$||v_t||_{H^{1,2}_2(R^3_S)} \leq C(||g^1||_{L^2(R^2_S)} + ||v_t||_{L^2(R^3_S)} + ||v_{tt}||_{L^2(R^3_S)}) \leq C.$$

This implies that

$$||v_{ts}||_{L^2(R^3_S)} \leq C \quad \text{and} \quad ||v_{ttt}||_{L^2(R^3_S)} \leq C, \tag{3.51}$$

for a uniform constant $C > 0$. By (3.42) and (3.51) we obtain

$$||v_s||_{H_2^{1,2}(R_S^3)} \leq C(||g^2||_{L^2(R_S^2)} + ||v_s||_{L^2(R_S^3)} + ||v_{st}||_{L^2(R_S^3)}) \leq C,$$

for a uniform $C > 0$. Hence $v \in H^{2,4}(R_S^3)$. By composing the embedding $H^{2,4}(R_S^3) \hookrightarrow H^2(R_S^3)$ with the continuous Sobolev embedding $H^2(R_S^3) \hookrightarrow C^0(R_S^3)$, we obtain,

$$||v||_{C^0(R_S^3)} \leq C_S.$$

Since the Equation (3.40) is autonomous then all the estimates can be extended globally on $\mathbb{R} \times S^1$, hence

$$||v||_{C^0(\mathbb{R}\times S^1)} \leq C.$$

By continuing bootstrapping we can get uniform (local) bounds in $H^{s,2s}$, for every $s \in \mathbb{N}$. This implies a bound for derivative of v of every order (i.e. (3.45)) and, using the Rellich-Kondrachov Theorem (see [5, Theorem 6.3, Part II],) the final assertion is proven. ■

3.3.6. Corollary. *Let $U \in \mathscr{U}$. There exists a positive constant $C > 0$ such that*

$$||v||_{C^r(\mathbb{R}\times S^1)} \leq C, \quad \text{for all } r \in \mathbb{N}, \tag{3.52}$$

for every $v \in \mathscr{N}_U^{q^-,q^+}$ and every $q^\pm \in \mathrm{Crit}_U$.

Proof. Use the estimate (3.44) and start with parabolic bootstrapping. ■

3.3.2 Mechanical braid Morse homology

As for braids with support in $\mathbb{R}^2$ we now focus our attention to relative braids, i.e. those braids which have at least two strands, labeled into two group: q and Q. The elements denoted by $Q = \{Q^1, \ldots, Q^m\}$ are called skeleton and correspond to the (skeleton) elements $y = \{y^1, \ldots, y^m\}$ in the case of braids supported in the plane. If $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$, it holds that $Q = \pi_2(y)$, where π_2 is the standard projection on the second coordinate. For the free strands we will always assume that they consist of a single strand. Therefore $q = \{q\}$. We recall that such braids that have only positive crossings. The associated equivalence class of unbounded Legendrian braids and their fibers are denoted by

$$[q \,\mathrm{rel}\, Q]_{\mathbb{R}} \quad \text{and} \quad [q]_{\mathbb{R}} \,\mathrm{rel}\, Q.$$

These are elements which are invariant under Legendrian isotopies. Let now $[q]_{\mathbb{R}} \operatorname{rel} Q$, an unbounded Legendrian relative class fiber. The space of critical points of $\mathscr{L}_U$ in $[q]_{\mathbb{R}} \operatorname{rel} Q$ is denoted by

$$\operatorname{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q) := \operatorname{Crit}_U \cap [q]_{\mathbb{R}} \operatorname{rel} Q.$$

For $U \in \mathscr{U}$ and $q^{\pm} \in \operatorname{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)$, define the space of connecting orbits on $[q]_{\mathbb{R}} \operatorname{rel} Q$ by

$$\mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q) := \left\{ v \in \mathscr{N}_U^{q^-,q^+} : v(s,\cdot) \in [q]_{\mathbb{R}} \operatorname{rel} Q, \text{ for all } s \in \mathbb{R} \right\}.$$

Let $a > 0$. The space of bounded solutions in $[q]_{\mathbb{R}} \operatorname{rel} Q$ is defined as

$$\mathscr{N}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q) := \{ v \in \mathscr{N}_U^a : v(s,\cdot) \in [q]_{\mathbb{R}} \operatorname{rel} Q, \text{ for all } s \in \mathbb{R} \}.$$

3.3.7. Remark. Let $a > 0$ and $U \in \mathscr{U}$ and $[q]_{\mathbb{R}^2} \operatorname{rel} y$ a RBC fiber. By Proposition 3.3.5 we have uniform bounds in C^r for every r for the space $\mathscr{N}_U^a([x]_{\mathbb{R}^2} \operatorname{rel} y)$

As for unbounded RBCes, for Legendrian unbounded RBCes, an invariant can be defined. If $x \operatorname{rel} y$ is a relative braid then we have defined the integer $\operatorname{Cross}(x \operatorname{rel} y)$ in function of the winding number of $x \operatorname{rel} y$. For Legendrian braids the same definition can be given. For the Legendrian property all the crossings are constraint to be positive. If we define $\mathrm{I}(q \operatorname{rel} Q)$ the number of crossings between q and Q and $x = (q_t, q)$ and $y = (Q_t, Q)$ then we have

$$\operatorname{Cross}(x \operatorname{rel} y) = \mathrm{I}(q \operatorname{rel} Q),$$

For parabolic equations like (3.40) a similar principle of Lemma 3.2.10 holds.

3.3.8. Lemma (parabolic Monotonicity Lemma)**.** *Let $a > 0, U \in \mathscr{U}, y \in \mathcal{L}\mathbf{C}_m(\mathbb{D}^2), Q = \pi_2(y)$ and $v \in \mathscr{N}_U^a$. The function $s \mapsto I(v(s,\cdot) \operatorname{rel} Q)$ is (when well defined) a non-increasing function of s with values in $\mathbb{N}$. In particular, if there exists $(s_0, t_0) \in \mathbb{R} \times S^1$ such that $v(s_0, t_0) = Q(t_0)$ then there exists an $\varepsilon_0 > 0$ such that*

$$\mathrm{I}(v(s_0 - \varepsilon, \cdot) \operatorname{rel} Q) > \mathrm{I}(v(s_0 + \varepsilon, \cdot) \operatorname{rel} Q),$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. See, for instance [6] [12], [18] and [35]. ■

3.3.9. Remark. The implications of the parabolic Monotonicity Lemma (Lemma 3.3.8) (and Remark 3.3.7) are the same as the implications of the Monotonicity

Lemma (i.e. Lemma 3.2.10) (and Remark 3.2.11). In particular, by the same argument as in the proof of Proposition 3.2.12, we can prove that if $[q]_{\mathbb{R}^2} \operatorname{rel} Q$ is an unbounded PRBC, and $U \in \mathscr{U}$ then the space $\mathrm{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)$ is compact and isolated in $[q]_{\mathbb{R}} \operatorname{rel} Q$.

Let $U \in \mathscr{U}$ and $[q]_{\mathbb{R}} \operatorname{rel} Q$ an unbounded relative braid class we say that $q \in \mathrm{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)$ is non-degenerate if the operator

$$A_q := \partial_{tt} - \mathrm{Id} + \partial_q^2 U(t,q) : H^2(S^1) \to L^2(S^1)$$

is invertible. This operator is self-adjoint with respect the standard L^2 inner product. The number of negative eigenvalues is finite, and it is denoted by $\beta(A_q)$ and called the Morse index of A_q. If $q \in \mathrm{Crit}_U$ we define its Morse index by

$$\beta(q) := \beta(A_q).$$

By analogy with the hyperbolic Hamiltonian case, for $H_U \in \mathscr{H}_{\mathrm{mech}}, U \in \mathscr{U}$ and $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ of the form $y = (Q_t, Q)$ we say that $Q = \{Q^1(t), \ldots, Q^m(t)\} = \pi_2(y)$ is a solution curve of X_{H_U} (this is abuse of notation) if $y = (Q_t, Q)$ is a solution curve of the vector field X_{H_U} (see (3.32)). Because $Q = \pi_2(y)$ Equation (3.32) means that for every $i = 1, \ldots, m$ Q^i a solution of

$$Q^i_{tt} - Q^i + \partial_{Q^i} U(t, Q^i) = 0 \quad Q^i(0) = Q^{\sigma(i)}(1), \quad \text{for some } \sigma \in S_m.$$

In these notes the proofs for genericity properties of critical points of the Lagrangian action is not given. We will state this property as a conjecture.

3.3.10. Conjecture. *Let $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ of the form $y = (Q_t, Q)$ with Q a solution curve of X_{H_U} and $[q]_{\mathbb{R}} \operatorname{rel} Q$ be an unbounded Legendrian PRBC fiber. Then for every mechanical Hamiltonian $H_U \in \mathscr{H}_{\mathrm{mech}}, U \in \mathscr{U}$, there exists δ_* with the following significance. For every $\delta < \delta_*$ there exists a $U' \in \mathscr{U}$ with the property that*

(i) $||H_U - H_{U'}||_{C^\infty} < \delta$

(ii) *Q is a solution curve of X_{H_U}*

and such that $\mathrm{Crit}_{U'}([q]_{\mathbb{R}} \operatorname{rel} Q)$ consists only of non degenerate critical points. This implies, by compactness, that the space $\mathrm{Crit}_{U'}([q]_{\mathbb{R}} \operatorname{rel} Q)$ consists only of finitely many isolated points.

3.3.11. Remark. The idea for the proof of Conjecture 3.3.10 is to use a Sard-Smale argument to show that the space of certain variations of the potential function U (which would give non-degeneracy of critical points) is dense in the set of all variations. The argument is standard in transversality theory and can be found for instance in [3, Section 2.11]. It turns out that we need only to prove that the linearization (around (q, V)) of the map $\mathcal{F}(q^\varepsilon, v^\delta) := q^\varepsilon_{tt} - q^\varepsilon + \partial_{q^\varepsilon}(V(t, q^\varepsilon) + v^\delta(t, q^\varepsilon))$ such that q^ε is an ε perturbation of q (that stays away from Q) with $\varepsilon > 0$ small, and v^δ is a δ perturbation of V, with $\delta > 0$ small, is surjective. Since we want to keep Q as a solution curve of X_{H_U} we do not want to perturb our potential close to Q^i, for all $i = 1, \ldots, m$. By the Hahn-Banach Theorem one can prove that $d\mathcal{F}(q, V)$ is surjective, by proving that the image has trivial orthogonal complement. In order to prove this, one can use the same argument as in [49].

We say that $\mathscr{L}_U$ is Morse if all the critical points are non-degenerate. By the above conjecture $\mathscr{L}_U$ is a Morse function for a *generic* choice of a potential function $U \in \mathscr{U}$. By compactness, this implies that $\mathrm{Crit}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$ is a finite. If we assume Conjecture 3.3.10 to hold, then we can obtain uniform bounds for unbounded Legendrian PRBCes, in the same as the elliptic case.

3.3.12. Proposition (uniform bounds for the parabolic case)**.** *Let $U \in \mathscr{U}$ with $\mathscr{L}_U$ Morse and let Q is a solution curve of X_{H_U}. Let $[q]_\mathbb{R} \,\mathrm{rel}\, Q$ be an unbounded Legendrian relative braid class fiber. Then there exists $C > 0$ such that*

$$||v||_{C^r(\mathbb{R}\times S^1)} \leq C$$

for all $a > 0$ sufficiently large and all $v \in \mathscr{N}^a_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$.

Proof. As in the elliptic case we have

$$\mathscr{N}^a_U([q]_\mathbb{R} \,\mathrm{rel}\, Q) = \bigcup_{q^\pm \in \mathrm{Crit}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)} \mathscr{N}^{q^-,q^+}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q),$$

if $U \in \mathscr{U}$ and $\mathscr{L}_U$ is Morse and $a > 0$ sufficiently large. By Corollary 3.3.6 we have that elements on the right hand side are uniformly bounded in C^r by a constant C that is independent of a. Hence, the same holds for elements in the right hand side. ■

3.3.13. Remark. Let $a > 0$ be large enough, $U \in \mathscr{U}$ such that $\mathscr{L}_U$ is Morse, let $[q]_\mathbb{R} \,\mathrm{rel}\, Q$ be an unbounded Legendrian PRBC and $q^\pm \in \mathrm{Crit}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$. It follows from Proposition 3.3.12, that as in Remark 3.2.16 for the elliptic case, also in the parabolic setting compactness of $\mathscr{N}^a_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$ is to be meant as compactness to broken trajectories. More precisely Let $\{v_\nu\}_{\nu\in\mathbb{N}} \subseteq \mathscr{N}^{q^-,q^+}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$ be a sequence, then there exists a subsequence (still denoted by v_ν) and sequences of times $s^i_\nu \in \mathbb{R}, i = 0, \ldots, k$ such that $v_\nu(\cdot + s^i_\nu, \cdot)$ converges with its derivatives uniformly on compact sets to $v^i \in \mathscr{N}^{q^i,q^{i-1}}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$, where $q^i \in \mathrm{Crit}_U([q]_\mathbb{R} \,\mathrm{rel}\, Q)$ for $i = 0, \ldots, k$ and $q^0 = q^-$ and $q^k = q^+$.

As for critical points, non-degeneracy can also be defined for connecting orbits. Let $U \in \mathscr{U}$ and $q^{\pm} \in \mathrm{Crit}_U$. A connecting orbit $v \in \mathscr{N}_U^{q^-,q^+}$ is said to be non-degenerate, if the linearized heat operator

$$\partial_s - \partial_{tt} + \mathrm{Id} - \partial_q U(t,q) : H_2^{1,2}(\mathbb{R} \times S^1) \to L^2(\mathbb{R} \times S^1) \qquad (3.53)$$

is surjective. This is an analytical condition that states that the stable manifold of q^- and the unstable of q^+ intersect transversely. If the transversality condition holds we say that $\mathscr{L}_U$ is Morse-Smale.

3.3.14. Remark. Note that $\xi \in H_2^{1,2}(\mathbb{R} \times S^1)$ admit limits $\lim_{s\to\pm\infty} \xi(s,t) = 0$ uniformly in t. This holds because, by the (anisotropic) Sobolev embedding, $H_2^{1,2}(\mathbb{R} \times S^1) \hookrightarrow L^\infty(\mathbb{R} \times S^1)$ (see [32, Lemma 2.3])

3.3.15. Proposition. *Let $y \in \mathcal{L}\mathbf{C}_m(\mathbb{R}^2)$ of the form $y = (Q_t, Q)$ with Q a solution curve of X_{H_U} and $[q]_{\mathbb{R}} \operatorname{rel} Q$ be an unbounded Legendrian PRBC fiber. Under the assumption that all critical points of $\mathscr{L}_U$ are non-degenerate, i.e. $\mathscr{L}_U$ is Morse, it holds that*

(i) $\mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q)$ *consists only of non-degenerate connecting orbits*

(ii) $\mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q)$ *are smooth manifolds without boundary with*

$$\dim(\mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q)) = \beta(q^-) - \beta(q^+).$$

Proof. In [33] the authors prove that the heat flow with periodic boundary conditions is automatically Morse-Smale, if one assumes that $\mathscr{L}_U$ is Morse. From this (i) and (ii) follow. ■

We are now ready to construct the braid Morse homology for mechanical Hamiltonian systems in the setting of unbounded Legendrian PRBCes. We follow the steps of Section 1.4.1. Let $a > 0$ sufficiently large $U \in \mathscr{U}$, and $[q]_{\mathbb{R}} \operatorname{rel} Q$ be an unbounded Legendrian PRBC. Recall that the image of $\mathscr{N}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$ under the mapping $v \mapsto v(0,\cdot)$ is called $\mathscr{S}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$. We have already proved compactness for the space $\mathscr{N}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$ Compactness of $\mathscr{S}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$ follows from compactness of $\mathscr{N}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$ Properness shows furthermore that $\mathscr{S}_U^a([q]_{\mathbb{R}} \operatorname{rel} Q)$ is isolated. By grading the critical points in a generic hyperbolic mechanical Hamiltonian system with the Morse index, it follows that the Morse chain complex $C_k = C_k([q]_{\mathbb{R}} \operatorname{rel} Q)$ with $\mathbb{Z}_2$ coefficients and the associated boundary operator $\partial_k : C_k \to C_{k-1}$ are well defined. Its homology is denoted, in analogy with the hyperbolic braid Floer homology by $\mathrm{HHM}_*([q]_{\mathbb{R}} \operatorname{rel} Q; H_U)$. The latter can be denoted by

$$\mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}),$$

after proving that $\mathrm{HHM}_*([q]_{\mathbb{R}} \operatorname{rel} Q; H_U)$. is independent of the skeleton.

3.3.3 Braid Morse homology and its isomorphism with the mechanical braid Morse homology

A similar construction can be carried out if we assume that the class of potential satisfies the properties (W1) and (W2) stated at the end of Section 3.1.2. Recall that the class of function that satisfies (W1) and (W2) is denoted by $\mathscr{W}$. For this class of potentials we can restrict ourselves to bounded Legendrian proper relative braid classes. Condition (W2) implies that $q \equiv \pm 1$ are stationary for the Lagrangian action $\mathscr{L}_W$. In this case, we restrict to Legendrian relative braids with support in $[-1, 1]$. This means that we consider only solutions of the heat flow which are uniformly bounded by 1. We then do not need all the extra-work we did for the unbounded case to obtain uniform bounds. Following the steps in Section 1.4.1, by considering proper (bounded) Legendrian relative braid class fiber $[q] \operatorname{rel} Q$, and a $W \in \mathscr{W}$ we define

$$\mathrm{HM}_*([q] \operatorname{rel} Q; W).$$

The latter is invariant, up to isomorphism, of the choice of a generic $W \in \mathscr{W}$ and of the fiber $[q] \operatorname{rel} Q$. This implies that

$$\mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HM}_*([q] \operatorname{rel} Q; W).$$

We now show that this invariant is isomorphic to $\mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}})$, in the spirit of Theorem 3.2.18. We first give meaning to extension of bounded Legendrian relative braid classes in the following way. If $[q \operatorname{rel} Q]$ is a (bounded) Legendrian relative braid class, then we extend $[q \operatorname{rel} Q]$ to $[q \operatorname{rel} Q]_{\mathbb{R}}$ by considering Legendrian isotopies in $\mathbb{R}$ instead of $[-1, 1]$.

3.3.16. Theorem. *Let $[q \operatorname{rel} Q]$ be a bounded Legendrian proper relative braid class and let $[q \operatorname{rel} Q]_{\mathbb{R}}$ its unbounded extension. Then*

$$\mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}) \tag{3.54}$$

Proof. This is only a simpler case of Theorem 3.2.18, we will be brief. Extend $[q] \operatorname{rel} Q$ to $[q]_{\mathbb{R}} \operatorname{rel} Q$ by considering isotopies in $\mathbb{R}$ (and not only in $[-1, 1]$). The unbounded relative braid class fiber $[q]_{\mathbb{R}} \operatorname{rel} Q$ inherit from $[q] \operatorname{rel} Q$ the proper condition: in particular elements in $[q]_{\mathbb{R}} \operatorname{rel} Q$ can not be deformed onto the constants ± 1. Let $W \in \mathscr{W}$ be generic. We have

$$\mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HM}_*([q] \operatorname{rel} Q; W)$$

Let $\varepsilon > 0$ small and extend W to $\widehat{W}$ such that $\widehat{W}(t,\cdot) \in C_0^\infty([-1-\varepsilon, 1+\varepsilon])$. Extend again $\widehat{W}$ to $\widehat{U}$ in the following way

$$\widehat{U} = \begin{cases} \widehat{W}(t,q) & |q| \leq 1+\varepsilon \\ 0 & |q| \geq 1+\varepsilon \end{cases}$$

its extension. Note that $\widehat{U} \in C_0^\infty(\mathbb{R})$ and hence $\in \mathscr{U}$. By similar arguments as in the proof of Theorem 3.2.18 we obtain

$$\mathrm{Crit}_W([q] \,\mathrm{rel}\, Q) = \mathrm{Crit}_{\widehat{U}}([q]_{\mathbb{R}} \,\mathrm{rel}\, Q)$$

and

$$\mathscr{N}_W^{q^-,q^+}([q] \,\mathrm{rel}\, Q) = \mathscr{N}_{\widehat{U}}^{q^-,q^+}([q]_{\mathbb{R}} \,\mathrm{rel}\, Q).$$

Hence

$$\begin{aligned} \mathrm{HM}_*([q \,\mathrm{rel}\, Q]) &\cong \mathrm{HM}_*([q] \,\mathrm{rel}\, Q; W) \\ &= \mathrm{HHM}_*([q] \,\mathrm{rel}\, Q; \hat{U}) \cong \mathrm{HHM}_*([q \,\mathrm{rel}\, Q]_{\mathbb{R}}), \end{aligned}$$

which proves (3.54) ■

3.4 Hyperbolic braid Floer homology equals mechanical braid Morse homology

The aim of this Section is to establish an isomorphism between the hyperbolic braid Floer homology introduced in Section 3.2 and the mechanical braid Morse homology developed in Section 3.3. We use the techniques of [46] and we show that they can be applied to the setting of braids. In order to make a self contained thesis, we will incorporate some of the proofs contained in [46] in Appendix 3.A.

3.4.1 The adiabatic limit

Let $[x \,\mathrm{rel}\, y]_{\mathbb{R}^2}$ be an unbounded proper relative braid class, then by the arguments shown in Section 3.2 the hyperbolic braid Floer homology groups $\mathrm{HHM}_*([x \,\mathrm{rel}\, y]_{\mathbb{R}^2})$ are well-defined, and independent, up to isomorphisms, of the fiber $[x]_{\mathbb{R}^2} \,\mathrm{rel}\, y$, of the hyperbolic Hamiltonian $H_V \in \mathscr{H}_{\mathrm{hyp}}$ and of the constant almost complex structure $J \in \mathscr{J}$. Let now $U \in \mathscr{U}$, then $H_U \in \mathscr{H}_{\mathrm{mech}} \subset \mathscr{H}_{\mathrm{hyp}}$. Let $\varepsilon > 0$, define J^ε as follows

$$J^\varepsilon = \begin{pmatrix} 0 & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}.$$

We have that $J^\varepsilon \in \mathscr{J}$. It follows that

$$\mathrm{HHF}_*([x\,\mathrm{rel}\,y]_{\mathbb{R}^2}) \cong \mathrm{HHF}_*([x]_{\mathbb{R}^2}\,\mathrm{rel}\,y; H_U, J^\varepsilon).$$

Fix now $H_U \in \mathscr{H}_{\mathrm{hyp}}$ generically and J^ε as above.

In this section we show that the heat equation can be seen as a formal limit of the Cauchy-Riemann equations. With such choice of $J^\varepsilon \in \mathscr{J}$ and H_U as above, the Cauchy-Riemann equations become

$$u_s - J^\varepsilon(u_t - X_{H_U}(t,u)) = 0. \tag{3.55}$$

If we write them in (p,q)-coordinates, Equation (3.55)

$$\begin{cases} p_s + \varepsilon^{-1} q_t - \varepsilon^{-1} p = 0 \\ q_s - \varepsilon p_t + \varepsilon q - \varepsilon \partial_q U = 0. \end{cases}$$

Under the scaling

$$\sigma = \varepsilon s, \tag{3.56}$$

we obtain $\varepsilon p_\sigma = p_s$ and $\varepsilon q_\sigma = q_s$. Hence we get

$$\begin{cases} \varepsilon^2 p_\sigma + q_t - p = 0 \\ q_\sigma - p_t + q - \partial_q U = 0. \end{cases} \tag{3.57}$$

Differentiating the first equation with respect to t, for $\varepsilon = 0$ Equation (3.57) becomes

$$q_\sigma = q_{tt} - q + \partial_q U(t,q).$$

Labeling the variable σ by s we obtain Equation (3.40). With a terminology due to [46], we say then the heat equation is the adiabatic limit for $\varepsilon \to 0$ of the Cauchy-Riemann equations.

3.4.2 More estimates of Cauchy-Riemann equations

In this section we show the existence of uniform bounds for the ε-dependent Cauchy-Riemann equations introduced in (3.55). The estimates carried out in Section 3.2 are independent of u, but not necessarily independent on $\varepsilon > 0$. In this Section we show that those estimates are also independent of $\varepsilon > 0$. After performing the scaling (3.56) Equation (3.55) can be written in the form

$$D_\varepsilon u^\varepsilon - \iota(\partial_{q^\varepsilon} U(t, q^\varepsilon)) = 0. \tag{3.58}$$

Here $D_\varepsilon := B_\varepsilon \partial_s - J\partial_t + R$, where J is the symplectic matrix (3.3), R the operator defined in (3.14), B_ε is the following

$$B_\varepsilon = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\iota : \mathbb{R} \to \mathbb{R}^2$ is the embedding $\iota(q) = (0, q)$. Because Equation (3.58) is ε-dependent it is useful to work with ε-dependent Sobolev norms. Define for $u \in H^1(\mathbb{R}^2; \mathbb{R}^2)$ with coordinate $u = (p, q)$ the equivalent Sobolev norm

$$||u||_{H^1_\varepsilon} := ||u||_{L^2} + \varepsilon||p_s||_{L^2} + ||q_s||_{L^2} + ||u_t||_{L^2}. \tag{3.59}$$

To prove uniform estimates we introduce some preliminary lemmas.

3.4.1. Lemma. *There exists a constant $C > 0$ such that, for every $u \in C^1_c(\mathbb{R}^2)$ and every $0 < \varepsilon \leq 1$, it holds*

$$||u||_{H^1_\varepsilon(\mathbb{R}^2)} \leq C||D_\varepsilon u||_{L^2(\mathbb{R}^2)}. \tag{3.60}$$

Proof. This is an ε-version of Lemma 3.2.3. We repeat the arguments. Using the Fourier transform we obtain

$$\widehat{D_\varepsilon u} = A_\varepsilon \widehat{u} = \begin{pmatrix} i\xi\varepsilon^2 + 1 & i\eta \\ -i\eta & i\xi - 1 \end{pmatrix} \begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix}.$$

The inverse of A_ε is given by

$$A_\varepsilon^{-1} = \frac{1}{\det A_\varepsilon} \begin{pmatrix} i\xi - 1 & i\eta \\ -i\eta & i\varepsilon^2\xi + 1 \end{pmatrix}$$

where $\det A_\varepsilon = -(\varepsilon^2\xi^2 + 1 + \eta^2 + i(\varepsilon^2\xi - \xi))$. For the square norms of the matrix entries we obtain

$$\frac{\xi^2 + 1}{|\det A_\varepsilon|^2} \leq 1, \quad \frac{\eta^2}{|\det A_\varepsilon|^2} \leq \frac{1}{2}, \quad \frac{\varepsilon^4\xi^2 + 1}{|\det A_\varepsilon|^2} \leq 1,$$

which implies $||A_\varepsilon^{-1}|| \leq 1$, and therefore, using the Plancherel isometry we obtain

$$||u||_{L^2(\mathbb{R}^2)} \leq ||D_\varepsilon u||_{L^2(\mathbb{R}^2)}. \tag{3.61}$$

To prove the remainder of the lemma we need to estimate the matrix norms of $i\xi A_\varepsilon^{-1}$ and $i\eta A_\varepsilon^{-1}$. By the Plancherel isometry we have

$$\varepsilon||p_s||_{L^2} = \varepsilon||\widehat{p_s}||_{L^2} = \varepsilon||i\xi\widehat{p}||_{L^2} = \varepsilon||i\xi P A_\varepsilon^{-1}\widehat{D_\varepsilon u}||_{L^2}$$

and, similarly,

$$||q_s||_{L^2} = ||\widehat{q_s}||_{L^2} = ||i\xi\widehat{q}||_{L^2} = ||i\xi Q A_\varepsilon^{-1}\widehat{D_\varepsilon u}||_{L^2}.$$

Here P and Q are the operators defined in (3.15). In order to estimate the matrix norm of PA_ε^{-1} and QA_ε^{-1} we need to bound the following terms

$$\frac{\varepsilon^2(\xi^4+\xi^2)}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le 1, \quad \frac{\varepsilon^2\xi^2\eta^2}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le \frac{1}{2}$$

and

$$\frac{\xi^2\eta^2}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le \frac{1}{2}, \quad , \frac{\xi^2+\varepsilon^4\xi^4}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le 1.$$

And for the norm u_t we need to estimate

$$\frac{\eta^2+(\eta\xi)^2}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le 2, \quad \frac{\eta^4}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le 1$$

and

$$\frac{\varepsilon^4\xi^2\eta^2+\eta^2}{(\varepsilon^2\xi^2+1+\eta^2)^2+\xi^2(\varepsilon^2-1)^2} \le 2.$$

These bounds show (3.60). ■

3.4.2. Lemma. *Let $G \subset \mathbb{R}^2$ be compact and $K \Subset G$. There exists s constant $C_{K,G} > 0$ such that*

$$||u||_{H^1_\varepsilon(K)} \le C_{K,G}\left(||D_\varepsilon u||_{L^2(G)} + ||u||_{L^2(G)}\right) \tag{3.62}$$

for every $u \in C^1(\mathbb{R} \times S^1; \mathbb{R}^2)$ and every $0 < \varepsilon \le 1$

Proof. The proof is the same as the proof of Lemma 3.2.4. Replace the norm H^1 with the norm H^1_ε and use Lemma 3.4.1 when in the proof of Lemma 3.2.4 Lemma 3.2.3 is used. ■

For $U \in \mathscr{U}$ and $q^\pm \in \mathrm{Crit}_U$ we denote, as usual, the space of connecting orbits between q^- and q^- of (3.58) by $\mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$, and for $a > 0$ the space of bounded solutions of (3.58) by $\mathscr{M}^a_{H_U,\varepsilon}$.

3.4.3. Remark. Let $U \in \mathscr{U}$. By denoting $q \in \mathrm{Crit}_U$ we mean that $x = (q_t, q) \in \mathrm{Crit}_{H_U}$. With abuse of notation we will also denote by $\mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$ the set of connecting orbits between $x^- = (q_t^-, q^-)$ and $x^+ = (q_t^+, q^+)$. Sometimes we will also write $\mathscr{M}^{x^-,x^+}_{H_U,\varepsilon}$ with the same meaning as $\mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$. Note furthermore that for Legendrian braid class fibers we have that $[x]_{\mathbb{R}^2}\,\mathrm{rel}\,y = [q]_{\mathbb{R}}\,\mathrm{rel}\,Q$. Hence we can write also $\mathscr{M}^{x^-,x^+}_{H_U,\varepsilon}([x]_{\mathbb{R}^2}\,\mathrm{rel}\,y) = \mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}([q]_{\mathbb{R}}\,\mathrm{rel}\,Q)$.

3.4.4. Remark. For $U \in \mathscr{U}$, we have that, in particular, $U \in \mathscr{V}$. Let $q^{\pm} \in \mathrm{Crit}_U$, then Lemma 3.2.1 holds. We point out that an ε-version of Lemma 3.2.5 and Corollary 3.2.6 holds if we interchange the role of $\mathscr{M}^a_{H_V}$ with $\mathscr{M}^a_{H_U,\varepsilon}$ and of $\mathscr{M}^{x^-,x^+}_{H_V}$ with $\mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$. Their proofs, indeed, rely only on Lemma 3.2.1 and on the gradient-flow structure of the Cauchy-Riemann equations. Hence, the same argument as in the proof of Lemma 3.2.5 gives that for every $a > 0$ we obtain a constant c_a such that

$$\varepsilon||p^\varepsilon_s||_{L^2(\mathbb{R}\times S^1)} + ||q^\varepsilon_s||_{L^2(\mathbb{R}\times S^1)} \leq c_a, \tag{3.63}$$

for all $u \in \mathscr{M}^a_{H_U}$, for every $0 < \varepsilon \leq 1$. The same argument as in the proof of Corollary 3.2.6 gives that

$$\varepsilon||p^\varepsilon_s||_{L^2(\mathbb{R}\times S^1)} + ||q^\varepsilon_s||_{L^2(\mathbb{R}\times S^1)} \leq c \tag{3.64}$$

for every $0 < \varepsilon \leq 1$, every $q^{\pm} \in \mathrm{Crit}_U$ and every $u^\varepsilon \in \mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$.

3.4.5. Proposition. *Let $U \in \mathscr{U}$ and $0 < a < \infty$. Then there exists a uniform positive constant $C_a > 0$, such that*

$$||u^\varepsilon||_{C^r(\mathbb{R}\times S^1)} \leq C_a, \ \textit{for every } r \in \mathbb{N} \tag{3.65}$$

for every $0 < \varepsilon \leq 1$ and every $u^\varepsilon \in \mathscr{M}^a_{H_U,\varepsilon}$. Furthermore, the space $\mathscr{M}^a_{H_U,\varepsilon}$ is compact in the $C^\ell_{loc}(\mathbb{R}\times S^1)$ topology, for every $\ell \in \mathbb{N}$.

Proof. This is a variation of the proof of Proposition 3.2.7. We give a condensed version. By the same token as in (3.25), by the fact that $U \in \mathscr{U}$ and using (3.63) instead of (3.22), we obtain a local uniform (in both u and ε) L^2-estimate for u^ε. Using now the bound (3.62) and the fact that $U \in \mathscr{U}$ we obtain a uniform local H^1_ε for u^ε. This implies that we have local uniform L^2-estimates (in both ε and u^ε) for $p^\varepsilon_t, q^\varepsilon_t$ and q^ε_s, but not yet for p^ε_s, as in the proof of Proposition 3.2.7. This is due to the fact that we are using the ε dependent Sobolev norm H^1_ε (3.59). To obtain local uniform estimates for p^ε_s and further derivatives of q^ε and p^ε we differentiate the Equation (3.58) with respect to s, and we start bootstrapping. Using the fact that $U \in \mathscr{U}$ and the bound (3.62) we obtain local uniform L^2-estimates (in both ε and u^ε) for p^ε_s and for $p^\varepsilon_{st}, q^\varepsilon_{st}$ and q^ε_{ss}. And by differentiating Equation (3.58) with respect to t, we get bounds for for $p^\varepsilon_{tt}, q^\varepsilon_{tt}$. Differentiating even more we can obtain uniform bounds in H^s, for every $s \in \mathbb{N}$. This implies (3.65), by using the Sobolev continuous embedding $H^s \hookrightarrow C^{s-2}$, and the s-translation invariance of the Cauchy-Riemann equations. As in the proof of Proposition 3.2.7, by using the bounds obtained so far, we can apply the Rellich-Kondrachov Theorem and the last assertion follows. ■

3.4.6. Remark. If $U \in \mathscr{U}$ is generic (it is suffcient that $\mathscr{A}_{H_U}$ is Morse, this would imply non-degeneracy of $\mathscr{N}_U^{q^-,q^+}$ and hence of $\mathscr{M}_{H_U,\varepsilon}^{q^-,q^+}$) and $0 < a < \infty$ is sufficiently large, the constant which appears in (3.65) is also independent of a, provided we consider elements $u \in \mathscr{M}_{H_U,\varepsilon}^a([q]_{\mathbb{R}} \operatorname{rel} Q)$, where $[q]_{\mathbb{R}} \operatorname{rel} Q$ is a proper (unbounded) Legendrian relative braid class fiber. This follows from the fact that under these hypotheses Conjecture 3.3.10 holds. It follow then, from the same arguments as in the proof of Proposition 3.2.15 that

$$\mathscr{M}_{H_U,\varepsilon}^a([q]_{\mathbb{R}} \operatorname{rel} Q) = \bigcup_{q \in \operatorname{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)} \mathscr{M}_{H_U,\varepsilon}^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q). \tag{3.66}$$

3.4.7. Remark. Compactness of $\mathscr{M}_{H_U,\varepsilon}^a$ implies furthermore that if $u^{\varepsilon_i} \in \mathscr{M}_{H_U,\varepsilon_i}^a$, where ε_i is a sequence of positive numbers converging to zero, then $u^{\varepsilon_i} \to (w, v)$ in C^1_{loc} (up to a subsequence). The adiabatic limit argument implies that v satisfies (3.40). Regularity of the heat equation implies that $w = v_t$.

3.4.8. Proposition. *Let $U \in \mathscr{U}$ and $\mathscr{L}_U$ be Morse, let Q be a $[q]_{\mathbb{R}} \operatorname{rel} Q$ be an unbounded Legendrian PRBC, and $q^\pm \in \operatorname{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)$, with $\mu^{CZ}(x^-) - \mu^{CZ}(x^+) = 1$. If $u^{\varepsilon_i} \in \mathscr{M}_{H_U,\varepsilon_i}^{q^-,q^-}([q]_{\mathbb{R}} \operatorname{rel} y)$, where ε_i is a sequence of positive numbers converging to zero, then $u^{\varepsilon_i} \to v$ in C^1_{loc} (up to a subsequence) with $v \in \mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q)$.*

Proof. Since $\mathscr{M}_{H_U}^{q^-,q^+} \subseteq \mathscr{M}_{H_U}^a$, for some $a > 0$, by compactness of the space $\mathscr{M}_{H_U}^a$ (see Remark 3.4.7), we have that there exists v such that $u^{\varepsilon_i} \to v$ and v satisfies (3.40). For every $s \in \mathbb{R}$ we have

$$\begin{aligned}
\mathscr{L}_U(v(s,\cdot)) &= \int_0^1 \tfrac{1}{2}|v_t(s,t)|^2 + \tfrac{1}{2}|v(s,t)|^2 - U(t, v(s,t))\, ds \\
&= \lim_{i\to\infty} \int_0^1 \tfrac{1}{2}|p^{\varepsilon_i}(s,t)|^2 + \tfrac{1}{2}|q^{\varepsilon_i}(s,t)|^2 - U(t, q^{\varepsilon_i}(s,t))\, dsdt \\
&= \lim_{i\to\infty} \mathscr{A}_{H_U}(u^{\varepsilon_i}(s,\cdot))
\end{aligned}$$

From this it follows, by the fact that $u^\varepsilon \in \mathscr{M}_{H_U}^a$, that $v \in \mathscr{N}_U^a$. Since $U \in \mathscr{U}$ is generic (because $\mathscr{L}_U$ is Morse, and hence Morse-Smale), by the uniform estimates of Proposition 3.3.12 we establish that v has limits. Hence there exist $q^0, q^1 \in \operatorname{Crit}_U$ such that $v(s,\cdot) \to q^0$ when $s \to \infty$ and $v(s,\cdot) \to q^1$ when $s \to -\infty$. Isolation of proper relative braid classes shows that $q^0, q^1 \in \operatorname{Crit}_U([q]_{\mathbb{R}} \operatorname{rel} Q)$. The parabolic Monotonicity Lemma (Lemma 3.3.8) implies that $v \in \mathscr{N}_{H_U}^{q^0,q^1}([q]_{\mathbb{R}} \operatorname{rel} Q)$. To finish the proof we show that $q^0 = q^-$ and $q^1 = q^+$, if $\mu^{CZ}(x^-) - \mu^{CZ}(x^+) = 1$. Suppose by contradiction that either of the $q^i, i = 0, 1$ is different from $q^\pm$. Without loss of generality we can assume that $q^0 = q^-$ and $q^1 \neq q^+$. Then, by [39] (Lemma 2.4.6), we have

$$\mu^{CZ}(x^-) - \mu^{CZ}(x^+) = \beta(q^-) - \beta(q^+) = \beta(q^0) - \beta(q^1) + \beta(q^1) - \beta(q^+) \geq 2.$$

This is a contradiction and concludes the proof. ■

3.4.3 The Salamon-Weber map and the isomorphism

For $U \in \mathscr{U}$ and $q^\pm \in \mathrm{Crit}_U$ define $x^\pm = (q_t^\pm, q^\pm)$. By construction $x^\pm \in \mathrm{Crit}_U$. In this section we prove the isomorphism (3.31). The key idea is to use, when $\varepsilon > 0$ small, the so-called Salamon-Weber map [46]. This is a bijective map between $\mathscr{N}_U^{q^-,q^+}$ and $\mathscr{M}_{H_U,\varepsilon}^{x^-,x^+}$, where $U \in \mathscr{U}$ is chosen generic. This shows that for every connecting orbit of the heat equation we can detect a nearby connecting orbit of the ε-dependent Cauchy-Riemann equation. We will show furthermore that the Salamon-Weber map respect the braid classes, hence the isomorphism (3.31) follows.

Before introducing the Salamon-Weber map we introduce some nomenclature for the linearized operators. For $q \in \mathbb{R}$, recall the embedding $\iota : \mathbb{R} \to \mathbb{R}^2$ given by $\iota(q) = (0, q)$. By linearizing Equation (3.58) around a solution $u^\varepsilon = (p^\varepsilon, q^\varepsilon)$ we obtain the first order linear differential operator $C^\varepsilon_{(p^\varepsilon,q^\varepsilon)} : H^1(\mathbb{R} \times S^1; \mathbb{R}^2) \to L^2(\mathbb{R} \times S^1; \mathbb{R}^2)$ by

$$C^\varepsilon_{(p^\varepsilon,q^\varepsilon)} = B^\varepsilon \partial_s - J\partial_t + R + \iota(\partial^2_{q^\varepsilon} U(t, q^\varepsilon)) = D_\varepsilon + \iota(\partial^2_{q^\varepsilon} U(t, q^\varepsilon)).$$

Since $C^\varepsilon_{(p^\varepsilon,q^\varepsilon)}$ does not depend on p^ε we will write C^ε_q and drop the ε superscript over q. We furthermore denote by $(C^\varepsilon_q)^* = -B_\varepsilon \partial_s - J\partial_t + R + \iota(\partial^2_q U(t, q^\varepsilon))$ the adjoint in the L^2 norm of C^ε_q.

We proceed now with the definition of the Salamon-Weber map. In Appendix 3.A we will repeat the estimates of [46] in our case to show that the map is well defined, injective and surjective.

3.4.9. Theorem (Salamon-Weber, [46])**.** *Assume that $U \in \mathscr{U}$ is generic. There exists an $\varepsilon_0 > 0$ with the following significance. There is a one-to-one correspondance between $\mathscr{N}_U^{q^-,q^+}$ and $\mathscr{M}_{H_U,\varepsilon}^{q^-,q^+}$, for every $\varepsilon \in (0, \varepsilon_0)$ and every $q^\pm \in \mathrm{Crit}_U$ with $\beta(q^-) - \beta(q^+) = 1$.*

Proof. See [46, Theorems 4.1, Theorem 4.3 and Theorem 10.1], and Appendix 3.A. ■

The proof of Theorem 3.4.9 is constructive. It shows that for every $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is defined in Theorem 3.4.9, there exists a unique element $\zeta^\varepsilon = (\eta^\varepsilon, \xi^\varepsilon) \in \mathrm{im}(C_q^\varepsilon)^*$ such that

$$u^\varepsilon = (p^\varepsilon, q^\varepsilon) \in \mathscr{M}_{H_U}^{q^-,q^+}, \quad p^\varepsilon = v_t + \eta^\varepsilon, \;\; q^\varepsilon = v + \xi^\varepsilon.$$

for each $v \in \mathscr{N}_U^{q^-,q^+}$ such that $\beta(q^-) - \beta(q^+) = 1$. This gives rise to the Salamon-Weber map defined by

$$\begin{array}{rcccl} \mathcal{T}^\varepsilon & : & \mathscr{N}_U^{q^-,q^+} & \to & \mathscr{M}_{H_U,\varepsilon}^{q^-,q^+} \\ & & v & \mapsto & u^\varepsilon = (v_t + \eta^\varepsilon, v + \xi^\varepsilon), \quad (\eta^\varepsilon, \xi^\varepsilon) \in \mathrm{im}(C_v^\varepsilon)^*, \end{array} \tag{3.67}$$

Theorem 3.4.9 demonstrates that the map $\mathcal{T}^\varepsilon$ is well-defined, injective and surjective.

We apply now the Salamon-Weber map to the setting of the braids.

3.4.10. Proposition. *Assume that $U \in \mathscr{U}$ is generic. Let $[q]_\mathbb{R} \operatorname{rel} Q$ be an unbounded Legendrian PRBC fiber and let ε_0 defined in Theorem 3.4.9. For every $\varepsilon \in (0, \varepsilon_0)$ the Salamon-Weber map*

$$\mathcal{T}^\varepsilon : \mathscr{N}_U^{q^-,q^+}([q]_\mathbb{R} \operatorname{rel} Q) \to \mathscr{M}_{H_U,\varepsilon}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y),$$

given by $v \mapsto u^\varepsilon$ is bijective.

Proof. By injectivity of the map $\mathcal{T}^\varepsilon$ we have that, when $\varepsilon \in (0, \varepsilon_0)$, for every $v \in \mathscr{N}_U^{q^-,q^+}([q]_\mathbb{R} \operatorname{rel} Q)$ there exists a unique $u^\varepsilon \in \mathscr{M}_{H_U,\varepsilon}^{q^-,q^+}$. By the Monotonicity Lemma (3.2.10), since $q^\pm \in \mathrm{Crit}_U([q]_\mathbb{R} \operatorname{rel} Q)$, we obtain that $u^\varepsilon \in \mathscr{M}_{H_U,\varepsilon}^{q^-,q^+}([q]_\mathbb{R} \operatorname{rel} Q)$. Since $[q]_\mathbb{R} \operatorname{rel} Q$ is Legendrian we have that $[q]_\mathbb{R} \operatorname{rel} Q = [x]_{\mathbb{R}^2} \operatorname{rel} Q$ hence $u^\varepsilon \in \mathscr{M}_{H_U,\varepsilon}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ and then the injectivity is proved. By surjectivity of the map $\mathcal{T}^\varepsilon$ we have that, when $\varepsilon \in (0, \varepsilon_0)$, for every $u^\varepsilon \in \mathscr{M}_{H_U,\varepsilon}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y)$ there exists a unique $v \in \mathscr{N}_U^{q^-,q^+}$ such that $u^\varepsilon = \mathcal{T}^\varepsilon(v)$. Take a sequence ε_i of positive numbers converging to zero, then by Proposition 3.4.8 $u^{\varepsilon_i} \to v$ and $v \in \mathscr{N}_U^{q^-,q^+}([q]_\mathbb{R} \operatorname{rel} Q)$. Hence surjectivity follows. ■

3.4.11. Theorem. *Let $[q \operatorname{rel} Q]_\mathbb{R}$ an unbounded Legendrian PRBC. Denote $[x \operatorname{rel} y]_{\mathbb{R}^2}$ the unbounded proper relative braid class associated to $[q \operatorname{rel} Q]_\mathbb{R}$. Then*

$$\mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}) \cong \mathrm{HHM}_*([q \operatorname{rel} Q]_\mathbb{R}). \tag{3.68}$$

Proof. Denote by $[q]_{\mathbb{R}} \operatorname{rel} Q$ a proper (unbounded) fiber associated to the class $[q \operatorname{rel} Q]^L_{\mathbb{R}}$. Then, we have

$$\mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}) \cong \mathrm{HHM}([q]_{\mathbb{R}} \operatorname{rel} Q; U),$$

for a generic choice of $U \in \mathscr{U}$. Fix now $U \in \mathscr{U}$. Fix $0 < \varepsilon < \varepsilon_0$, where ε_0 appears in Proposition 3.4.10, and consider $J^\varepsilon \in \mathscr{J}$. By Proposition 3.4.10 we obtaint

$$\#\mathscr{N}_U^{q^-,q^+}([q]_{\mathbb{R}} \operatorname{rel} Q)/\mathbb{R} = \#\mathscr{M}_{H_U,\varepsilon}^{x^-,x^+}([x]_{\mathbb{R}^2} \operatorname{rel} y)/\mathbb{R}.$$

It follows that the chain complex defined by counting (modulo 2) the solutions of (3.57) agrees with the Morse boundary operator defined by counting solutions of (3.40). This proves the isomorphism

$$\mathrm{HHM}_*([q]_{\mathbb{R}} \operatorname{rel} Q; U) \cong \mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y, J_\varepsilon, H_U).$$

Since the right hand side is independent of $J^\varepsilon \in \mathscr{J}$, $H_U \in \mathscr{H}_{\mathrm{hyp}}$ and of the fiber we obtain

$$\mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y, J_\varepsilon, H_U) \cong \mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}).$$

Summarizing, we have proved the following chain of isomorphisms

$$\begin{aligned} \mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}) &\cong \mathrm{HHM}_*([q]_{\mathbb{R}} \operatorname{rel} Q; U) \\ &\cong \mathrm{HHF}_*([x]_{\mathbb{R}^2} \operatorname{rel} y; H_U, J^\varepsilon) \quad \cong \quad \mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}), \end{aligned}$$

which shows (3.68) and concludes the proof. ■

3.4.12. Corollary. *Let $[q \operatorname{rel} Q]_{\mathbb{R}}$ a Legendrian PRBC and $[q \operatorname{rel} Q]_{\mathbb{R}}$ its unbounded extension. Denote by $[x \operatorname{rel} y]_{\mathbb{R}^2}$ the unbounded proper relative braid class associated to $[q \operatorname{rel} Q]_{\mathbb{R}}$, and by $[x \operatorname{rel} y]_{\mathbb{D}^2}$ the bounded one. Then we have the following chain of isomorphisms:*

$$\mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}) \cong \mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}) \cong \mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}).$$

Proof. The first isomorphism follows from Theorem 3.3.16, the second from Theorem 3.4.11 and the third from Theorem 3.4.11 . ■

3.A The Salamon-Weber map

In this appendix we construct the Salamon-Weber map of [46] in our setting. We follow the lines of [46]. See also [54], for an even more detailed reference. The construction of the map relies on a variation of the Newton's method in infinite dimensions. In order to construct the Salamon-Weber map we proceed as follows. Let $U \in \mathscr{U}$ be generic (it is enough to assume that $\mathscr{L}_U$—and hence $\mathscr{A}_{H_U}$— is Morse), $q \in \mathrm{Crit}_U$ and $v \in \mathscr{N}_U^{q^-,q^+}$. Define $p = v_t$ and $q = v$ and consider $u^0 = (p, q)$. Then u^0 satisfies (3.57) for $\varepsilon = 0$, but it does not satisfy (3.57) for $\varepsilon > 0$. The idea is to perturb u^0 to $u^0 + \zeta^\varepsilon$, for a $\zeta^\varepsilon \in H^1(\mathbb{R} \times S^1; \mathbb{R}^2)$ such that $u^0 + \zeta^\varepsilon$ is a solution of (3.57) and then find ζ^ε.

Let $\varepsilon \geq 0$, define $(p^\varepsilon, q^\varepsilon)$ via

$$p^\varepsilon = v_t + \eta^\varepsilon, \quad q^\varepsilon = v + \xi^\varepsilon, \quad (\eta^\varepsilon, \xi^\varepsilon) = \zeta^\varepsilon \in H^1(\mathbb{R} \times S^1; \mathbb{R}^2). \tag{3.69}$$

Here ζ^ε is a zero of the non-linear map $\mathcal{F}^\varepsilon_{u^0} : H^1(\mathbb{R} \times S^1; \mathbb{R}^2) \to L^2(\mathbb{R} \times S^1; \mathbb{R}^2)$, defined as follows

$$\mathcal{F}^\varepsilon_{u^0}(\zeta^\varepsilon) = \mathcal{F}^\varepsilon_{u^0}(\eta^\varepsilon, \xi^\varepsilon) = \begin{pmatrix} \varepsilon^2 v_{ts} + \varepsilon^2 \eta^\varepsilon_s + \xi^\varepsilon_t - \eta^\varepsilon \\ \xi^\varepsilon_s - \eta^\varepsilon_t + \xi^\varepsilon + \partial_v U(t, v) - \partial_v U(t, v + \xi^\varepsilon) \end{pmatrix}. \tag{3.70}$$

The map $\mathcal{F}^\varepsilon_{u^0}$ is obtained by substituting $(p^\varepsilon, q^\varepsilon)$ in (3.57) and using the fact that v satisfies (3.40). Note that, for $\varepsilon = 0$ and $\zeta^\varepsilon = 0$, we have $\mathcal{F}^\varepsilon_{u^0}(0) = 0$. Now, suppose that there exists $\varepsilon > 0$ such that $\mathcal{F}^\varepsilon_{u^0}(\zeta^\varepsilon) = 0$. Then, by defining $u^\varepsilon := (p^\varepsilon, q^\varepsilon)$ as in (3.69) we have u^ε is a solution of (3.55) in the sense of L^2. Regularity of the heat equation and of the Cauchy-Riemann equations will show then that $u^\varepsilon \in \mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$, for every $\varepsilon \in (0, \varepsilon_0)$, which we will demonstrate later on.

Finding a unique zero of the map (3.70) is therefore essential to define the map (3.67). Hence, Equation (3.69) gives the key to define the Salamon-Weber map (3.67). We will use the classical Newton's method to find the zero of the map $\mathcal{F}^\varepsilon_{u^0}$.

In the following we denote by X, Y Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Y$ and by $||\cdot||_{\mathcal{L}(X,Y)}$ the operator norm.

3.A.1. Theorem (Newton's method). *Let X and Y be Banach spaces and let $f : X \to Y$ a continuously differentiable map. Let $x_0 \in X$ and suppose that the linearization at x_0 of f, i.e. $df(x_0)$, is onto with right inverse T. Assume furthermore that there exist constants $\delta, c > 0$ such that*

$$||f(x_0)||_Y \leq \tfrac{\delta}{2c}, \quad ||T||_{\mathcal{L}(X,Y)} \leq c, \quad ||df(x) - df(x_0)||_{\mathcal{L}(X,Y)} \leq \tfrac{1}{2c}, \tag{3.71}$$

whenever $||x - x_0||_X \leq \delta$. Then there exists a unique $\tilde{x} \in X$ with $f(\tilde{x}) = 0$, $||x_0 - \tilde{x}|| \leq \delta$ and $\tilde{x} - x_0 \in \mathrm{im}(T)$.

Proof. See [54, Theorem C.2.9]. ■

As explained in Theorem 3.A.1 there are three ingredients that have to be controlled: a small initial value of $\mathcal{F}^\varepsilon_{u^0}(0, 0)$, a uniformly bounded right inverse $\mathcal{Q}^\varepsilon_{u^0}$

of $d\mathcal{F}^{\varepsilon}_{u^0}(0)$, and the variation of derivatives $d\mathcal{F}^{\varepsilon}_{u^0}(\zeta^{\varepsilon}) - d\mathcal{F}^{\varepsilon}_{u^0}(0)$. We proceed now with verifying the three conditions.

(i) We have that there exists $c > 0$

$$||\mathcal{F}^{\varepsilon}_{u^0}(0,0)||_{L^2(\mathbb{R}\times S^1)} = \left|\left| \begin{pmatrix} \varepsilon^2 v_{ts} \\ 0 \end{pmatrix} \right|\right|_{L^2(\mathbb{R}\times S^1)} \leq \varepsilon^2 ||v_{ts}||_{L^2(\mathbb{R}\times S^1)} \leq c\varepsilon^2.$$

This follows from regularity estimates of the heat equation. If v satisfies the heat equation then v_t satisfies the linearized heat equation. Because of the gradient flow structure of the heat equation, by similar arguments as in Remark 3.3.4, we obtain that there exists $c > 0$ such that $||v_{st}||_{L^2(\mathbb{R}\times S^1)} \leq c$, for every $v \in \mathcal{N}_U^{q^-,q^+}$ and every $q^{\pm} \in \mathrm{Crit}_U$.

(ii) We have that the operator

$$d\mathcal{F}^{\varepsilon}_{u^0}(0,0) \begin{pmatrix} \eta^{\varepsilon} \\ \xi^{\varepsilon} \end{pmatrix} = \begin{pmatrix} \varepsilon^2 \eta^{\varepsilon}_s + \xi^{\varepsilon}_t + \eta^{\varepsilon} \\ \xi^{\varepsilon}_s - \eta^{\varepsilon}_t + \xi^{\varepsilon} - \partial^2_q U(t,v)\xi^{\varepsilon} \end{pmatrix}$$

correspond to the usual linearized Cauchy-Riemann operator (with almost complex structure J^{ε}). Hence for every $\varepsilon > 0$ the operator $d\mathcal{F}^{\varepsilon}_{u^0}(0,0)$ is bounded (if $\zeta^{\varepsilon} \in H^1$). Since $U \in \mathscr{U}$ is generic then $\partial_s - \partial_{tt} + \mathrm{Id} + \partial^2_v U(t,v)$ is onto (if we assume only that $U \in \mathscr{U}$ is such that $\mathscr{L}_U$ is Morse, then by using the result of [33] we have that $\partial_s - \partial_{tt} + \mathrm{Id} + \partial^2_v U(t,v)$ is onto). It follows that, by [46, Thoerem 3.3], the linearized Cauchy-Riemann operator $d\mathcal{F}^{\varepsilon}_{u^0}(0,0)$ is also onto. Therefore it admits a right inverse $\mathcal{Q}^{\varepsilon}_{u^0} : L^2 \to H^1$. Since $d\mathcal{F}^{\varepsilon}_{u^0}(0,0)$ is onto then $(d\mathcal{F}^{\varepsilon}_{u^0}(0)d\mathcal{F}^{\varepsilon}_{u^0}(0)^*)$ is onto and one-to-one (see [11]), hence it has an inverse (where $(d\mathcal{F}^{\varepsilon}_{u^0}(0)^*$ is the adjoint of $d\mathcal{F}^{\varepsilon}_{u^0}(0)$ with respect to the L^2 norm). Then the right inverse of $d\mathcal{F}^{\varepsilon}_{u^0}(0,0)$ can be expressed in the form

$$\mathcal{Q}^{\varepsilon}_{u^0}(\zeta) = d\mathcal{F}^{\varepsilon}_{u^0}(0)^*(d\mathcal{F}^{\varepsilon}_{u^0}(0)d\mathcal{F}^{\varepsilon}_{u^0}(0)^*)^{-1}\zeta.$$

Being a composition of two bounded operators, $\mathcal{Q}^{\varepsilon}_{u^0}$ is itself a bounded operator. To see the boundedness of $(d\mathcal{F}^{\varepsilon}_{u^0}(0)d\mathcal{F}^{\varepsilon}_{u^0}(0)^*)$ consider

$$H^2 \xrightarrow{d\mathcal{F}^{\varepsilon}_{u^0}(0)^*} H^1 \xrightarrow{d\mathcal{F}^{\varepsilon}_{u^0}(0)} L^2.$$

We have, by [54, Lemma 4.4.3], that $\ker(d\mathcal{F}^{\varepsilon}_{u^0}(0)) = \mathrm{coker}(d\mathcal{F}^{\varepsilon}_{u^0}(0)^*)$. Hence, the operator $d\mathcal{F}^{\varepsilon}_{u^0}(0)d\mathcal{F}^{\varepsilon}_{u^0}(0)^*$ is a bounded bijection from $H^2 \to L^2$, hence it has a bounded inverse by the Open Mapping Theorem. We need to show that the constants are ε-independent. By Lemma 3.4.1 we have that there exists $C > 0$ for every $0 < \varepsilon \leq 1$ and every $\zeta^\varepsilon \in C_c^\infty(\mathbb{R} \times S^1; \mathbb{R}^2)$

$$||\zeta^\varepsilon||_{H^1_\varepsilon} \leq C||d\mathcal{F}^\varepsilon_{u^0}(0)\zeta^\varepsilon||_{L^2},$$

and by a similar argument we can prove that

$$||\zeta^\varepsilon||_{H^1_\varepsilon} \leq C||d\mathcal{F}^\varepsilon_{u^0}(0)^*\zeta^\varepsilon||_{L^2}.$$

Hence, by surjectivity of $d\mathcal{F}^\varepsilon_{u^0}(0,0)$ we have

$$||(d\mathcal{F}^\varepsilon_{u^0}(0)d\mathcal{F}^\varepsilon_{u^0}(0)^*)^{-1}\zeta^\varepsilon||_{H^1_\varepsilon} \leq 4||\zeta^\varepsilon||_{L^2}.$$

(iii) We have that for $\zeta^\varepsilon = (\eta^\varepsilon, \xi^\varepsilon)$

$$(d\mathcal{F}^\varepsilon_{u^0}(\zeta^\varepsilon) - d\mathcal{F}^\varepsilon_{u^0}(0)) = \begin{pmatrix} 0 \\ -\partial_v^2 U(t, v+\xi^\varepsilon) + \partial_v^2 U(t,v) \end{pmatrix},$$

is independent of ε. The operator is bounded, because $U \in \mathscr{U}$.

By (i)-(ii)-(iii) we have that if $U \in \mathscr{U}$ for every $0 < \varepsilon \leq 1$, for every $q^\pm \in \mathrm{Crit}_U$ and every $v \in \mathscr{N}_U^{q^- q^+}$ there exists a unique ζ^ε such that $\mathcal{F}^\varepsilon_{u^0}(\zeta^\varepsilon) = 0$. Therefore, by defining $u^\varepsilon = (p^\varepsilon, q^\varepsilon)$ as in (3.69), we obtain that u^ε is a solution of the Cauchy-Riemann equations. Now, the fact that $\zeta^\varepsilon \in H^1(\mathbb{R}\times S^1; \mathbb{R}^2)$ does not guarantee that $\zeta^\varepsilon(s,\cdot)$ admits limits for $s \to \pm\infty$, and the limit is 0. Hence we can not conclude immediately that $u^\varepsilon \in \mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$. However, since $\zeta^\varepsilon \in H^1$ and $v \in \mathscr{N}_U^{q^-,q^+}$ then

$$\int_{-\infty}^{\infty}\int_0^1 |u^\varepsilon_s|^2 dtds \leq \int_{-\infty}^{\infty}\int_0^1 |v_{st}|^2 + |v_s|^2 + |\eta^\varepsilon_s|^2 + |\xi^\varepsilon_s|^2 \, dsdt \leq C.$$

Using the same arguments as in the proof of Proposition 3.2.7 we obtain regularity for u^ε, in particular $u^\varepsilon \in C^r(\mathbb{R} \times S^1; \mathbb{R}^2)$, for every $r \in \mathbb{N}$. It follows that $\mathscr{A}_{H_U}(u^\varepsilon(s,\cdot))$ is well-defined for every $s \in \mathbb{R}$ and uniformly bounded by C. We deduce that

$$\sup_{s\in\mathbb{R}} |\mathscr{A}_{H_U}(u^\varepsilon(s,\cdot))| \leq C.$$

It follows that $u^\varepsilon \in \mathscr{M}^C_{H_U}$. If $U \in \mathscr{U}$ and hence $H_U \in \mathscr{H}_{\mathrm{mech}}$ is generic, by (3.66) we establish that u^ε has limits for $s \to \infty$ uniformly in t. It follows that

$$\lim_{s\to\pm\infty} \zeta^\varepsilon(s,\cdot) = \lim_{s\to\pm\infty} u^\varepsilon(s,\cdot) - u^0(s,\cdot)$$

exists. Now suppose that the limit $\lim_{s\to\infty}|\zeta^\varepsilon(s,\cdot)| = |f(\cdot)| > C \geq 0$. Then by choosing S sufficiently large we obtain

$$\iint_{\{|s|>S\}\times S^1} |\zeta^\varepsilon(s,t)|^2\, dtds \geq \iint_{\{|s|>S\}\times S^1} (C-\delta)^2 > \infty,$$

where $0 < \delta < C$. This implies that $\lim_{s\to\pm\infty}\zeta^\varepsilon(s,\cdot) = 0$, uniformly in t. Hence u^ε has the right limits and $u^\varepsilon \in \mathscr{M}^{q^-,q^+}_{H_U,\varepsilon}$. From the above arguments it follows that the map $\mathcal{T}^\varepsilon : \mathscr{N}^{q^-,q^+}_U \to \mathscr{M}^{x^-,x^+}_{H_U,\varepsilon}$ defined by (3.67) is well defined and injective. Theorem 3.A.1 also says that $(\eta^\varepsilon,\xi^\varepsilon) \in \mathrm{im}\,\mathcal{Q}^\varepsilon_{u^0} \subseteq \mathrm{im}(d\mathcal{F}_{u^0}(0))^*$.

In order to prove that the map is surjective we will show that there exists ε_0 such that for every $\varepsilon \in (0,\varepsilon_0)$ the map $\mathcal{T}^\varepsilon$ admits an inverse. Let $\varepsilon > 0, U \in \mathscr{U}$ generic, $x^\pm \in \mathrm{Crit}_{H_U}$ and $q^\pm = \pi_2(x^\pm)$ and $u^\varepsilon = (p^\varepsilon, q^\varepsilon) \in \mathscr{M}^{x^-,x^+}_{H_U}$. Define the non-linear map $\mathcal{G}^\varepsilon_{u^\varepsilon} : H^{1,2}_2(\mathbb{R}\times S^1;\mathbb{R}) \to L^2(\mathbb{R}\times S^1;\mathbb{R})$ by

$$\mathcal{G}^\varepsilon_{u^\varepsilon}(\xi^\varepsilon) = \xi^\varepsilon_s - \xi^\varepsilon_{tt} + \xi^\varepsilon_t + \partial_q U(t, q^\varepsilon - \xi^\varepsilon) - \partial_q U(t,q^\varepsilon) + \varepsilon^2 p^\varepsilon_{st}. \tag{3.72}$$

By similar arguments as in (i)-(ii)-(iii) (where we interchange the role of the heat equation to the one of the Cauchy-Riemann equations) we can apply the Newton's method to the map (3.72) which yields the existence of an ε_0 and a unique ξ^ε, for every $\varepsilon \in (0,\varepsilon_0)$ such that $\mathcal{G}^\varepsilon_{u^\varepsilon}(\xi^\varepsilon) = 0$. Define now the map

$$\begin{array}{cccc} \mathcal{S}^\varepsilon & : & \mathscr{M}^{x^-,x^+}_{H_U} & \to & \mathscr{N}^{q^-,q^+}_{H_U,\varepsilon} \\ & & u^\varepsilon = (p^\varepsilon,q^\varepsilon) & \mapsto & q^\varepsilon - \xi^\varepsilon, \end{array}$$

The Newton's method and regularity of the heat equation imply that the map $\mathcal{S}^\varepsilon$ is well defined and injective. By construction $\mathcal{S}^\varepsilon$ is the inverse of $\mathcal{T}^\varepsilon$.

4 A Poincaré-Bendixson result for CRE

In [18] Fiedler and Mallet-Paret prove a version of the classical Poincaré-Bendixson Theorem for scalar parabolic equations. We prove that a similar result holds for bounded solutions of the non-linear Cauchy-Riemann equations. The latter is an application of an abstract theorem for flows with an (unbounded) discrete Lyapunov function.

4.1 Introduction

The classical Poincaré-Bendixson Theorem describes the asymptotic behavior of flows in the plane. The topology of the plane puts severe restrictions on the behaviour of limit sets. The Poincaré-Bendixson Theorem states for example that if the α- and the ω-limit set of a bounded trajectory of a smooth flow in $\mathbb{R}^2$ does not contain equilibria, then the limit set is a periodic orbit. Several generalizations of this theorem have appeared in the literature. For instance in [8], the Poincaré-Bendixson Theorem is generalized to two-dimensional manifolds. In [30] an extension to continuous (two-dimensional) flows is obtained, and [14] provides a generalization to semi-flows. The remarkable result by Fiedler and Mallet-Paret [18] establishes an extension of the Poincaré-Bendixson Theorem to infinite dimensional dynamical systems with a positive Lyapunov function. They apply their result to *scalar* parabolic equations of the form

$$u_s = u_{xx} + f(x, u, u_x), \quad x \in S^1, f \in C^2. \tag{4.1}$$

In this paper we establish a version of the Poincaré-Bendixson Theorem for bounded orbits of the nonlinear Cauchy-Riemann equations in the plane. A bounded orbit of the nonlinear Cauchy-Riemann equation in the plane is a (smooth) bounded function $u\colon \mathbb{R} \times S^1 \to \mathbb{R}^2$, which satisfies the equation

$$u_s - J(u_t - F(t, u)) = 0, \tag{4.2}$$

with $u(s,t) = \big(p(s,t), q(s,t)\big)$, $s \in \mathbb{R}$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Here $F(t,u)$ is a smooth non-autonomous vector field on $\mathbb{R}^2$ and J is the symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We prove that the asymptotic behavior, as s goes to infinity, of bounded solutions of Equation (4.2) is as simple as the limiting behavior of flows in $\mathbb{R}^2$. Equation (4.2) arises in many different contexts, in particular in Floer Homology literature (see, for instance [36]), where the vector field has the form $F(t,u) = F_H(t,u)$, i.e. F_H is *Hamiltonian*. The latter implies that there exists a time-dependent Hamiltonian function $H(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, such that $F_H(t,u) = J\nabla H(t,u)$. In the Hamiltonian case the Cauchy-Riemann equations are the L^2-gradient flow of the Hamilton action and as such bounded solutions of (4.2) will, generically, be connections orbits between equilibria. The Hamilton action is an $\mathbb{R}$-valued Lyapunov function for the Cauchy-Riemann equations. In this paper we obtain a result about the asymptotic behavior of orbits for *general* vector fields F in the Cauchy-Riemann Equations.

The main result for the Cauchy-Riemann equations in this paper concerns the asymptotic behavior of bounded solutions. A bounded solution of the Cauchy-Riemann equations is a smooth function u with $|u(s,t)| \le C$. Let X be the set of solutions bounded by a fixed (but arbitrary) constant (in the present work we will always choose $C = 1$). Endowed with the compact-open topology X is a compact Hausdorff space. The translation invariance of the Cauchy-Riemann equations defines a flow ϕ^σ on X by translating solutions in the s-variable. A bounded solution u can be identified with its orbit $\gamma(u)$, and $\alpha(u)$ and $\omega(u)$ are well-defined elements of X. In Section 4.2 we given a detailed account of the space X and the flow ϕ^σ in the context of the Cauchy-Riemann equations.

4.1.1. Theorem. *Let u be a bounded solution of the Cauchy-Riemann Equations (4.2). Then, for the ω-limit set $\omega(u)$ the following dichotomy holds:*

(i) either $\omega(u)$ consists of exactly one periodic orbit, or

(ii) $\alpha(v) \subseteq E$ and $\omega(v) \subseteq E$, for every $v \in \omega(u)$,

where E denotes the set of 1-periodic solutions of the vector field $F(x,t)$. The same dichotomy holds for the α-limit set $\alpha(u)$.

As in the classical Poincaré-Bendixson Theorem, alternative (ii) allows for $\omega(u)$ (or $\alpha(u)$) to consist of homoclinic and/or heteroclinic solutions joining equilibria. An important reason why a generalization of the Poincaré-Bendixson holds for the Cauchy-Riemann equations is that there exists a continuous projection onto $\mathbb{R}^2$, which is defined as follows. Let $t_0 \in S^1$ be arbitrary, then define

$$\begin{array}{cccc} \pi_{t_0} : & X & \to & \mathbb{R}^2 \\ & u = (p,q) & \mapsto & \pi_{t_0}(u) = \big(p(0,t_0), q(0,t_0)\big). \end{array} \tag{4.3}$$

4.1.2. Theorem. *Under the assumptions of Theorem 4.1.1 the projection*

$$\pi_{t_0} : \omega(u) \to \pi_{t_0}\omega(u)$$

is a homeomorphism onto its image.

In general, if a flow allows a continuous Lyapunov function, then limit sets of orbits consist only of equibria. Such flows are referred to as gradient-like flows. Theorem 4.3.1 in this paper gives an abstract extension of the Poincaré-Bendixson Theorem to flows that allow a *discrete* Lyapunov function. In particular Theorem 4.3.1 implies Theorem 4.1.1. Note that Theorem 4.1.2 together with the classical Poincaré-Bendixson Theorem implies Theorem 4.1.1.

The main differences between the results in [18] for parabolic equations and the results in this paper, are that the Cauchy-Riemann equations do not define a well-posed initial value problem and, more importantly, the discrete Lyapunov functions that are considered in this paper are *not* bounded from below. Furthermore, the results obtained in this paper do not assume differentiability of the flow, nor does the flow need to be defined on a Banach space. We believe that most of the results in this paper can be extended to semi-flows, e.g. [14].

This paper is structured as follows. In Section 4.2 we analyze the main properties of the Cauchy-Riemann equations (4.2), with additional details given in Section 4.6. In Section 4.3, we set up an abstract setting which generalizes the properties of the Cauchy-Riemann equations. In Sections 4.4 and 4.5 a full proof of the Poincaré-Bendixson Theorem is given, adapted to the abstract setting introduced in Section 4.3.

4.2 The Cauchy-Riemann Equations

Since the initial value problem of Equation (4.2) is ill-posed, we restrict our attention to bounded solutions, i.e. functions $u \in C^1(\mathbb{R} \times S^1; \mathbb{R}^2)$, that satisfy Equation (4.2), and for which

$$|u(s,t)| < \infty, \quad \text{for all } (s,t) \in \mathbb{R} \times S^1. \tag{4.4}$$

Since we can consider each bounded solution separately, it suffices to consider the space X of functions $u \in C^1(\mathbb{R} \times S^1; \mathbb{R}^2)$ satisfying Equation (4.2), and for which

$$|u(s,t)| \le C, \quad \text{for all } (s,t) \in \mathbb{R} \times S^1,$$

for some fixed arbitrary constant $C > 0$. Note that, without loss of generality, we can choose $C = 1$. On X we consider the compact-open topology, i.e.

$$u^n \xrightarrow{X} u \quad \Longleftrightarrow \quad u^n \xrightarrow{C^0_{\text{loc}}} u, \tag{4.5}$$

where the latter indicates uniform convergence on compact subsets of $S^1 \times \mathbb{R}$. Since $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, endowed with the compact-open topology, is Hausdorff (see [40, §47]), and $X \subset C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, also X is a Hausdorff space.

4.2.1. Proposition. *The solution space X is a compact Hausdorff space.*

Proof. See Section 4.6. ■

Identify the translation mapping $(s,t) \mapsto (s+\sigma, t)$ by $\sigma \in \mathbb{R}$ and consider the evaluation mapping

$$\mathbb{R} \times C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(\mathbb{R} \times S^1; \mathbb{R}^2), \quad (\sigma, u) \mapsto \phi^\sigma(u) = u \circ \sigma. \tag{4.6}$$

4.2.2. Lemma. *The evaluation mapping $(\sigma, u) \mapsto \phi^\sigma(u)$ is continuous with respect to the compact-open topology on $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$.*

Proof. Since $\mathbb{R} \times S^1$ is a locally compact Hausdorff space, the composition of mappings

$$C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1) \times C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(\mathbb{R} \times S^1; \mathbb{R}^2),$$

is continuous with respect to the compact-open topologies on $C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1)$ and $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, see [40, §46]. The translation σ as defined above is a continuous mapping in $C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1)$, which proves the lemma. ■

Since the Cauchy-Riemann Equations are s-translation invariant we have that $u \in X$ implies that $\phi^\sigma(u) \in X$. We therefore obtain a continuous mapping $\mathbb{R} \times X \to X$, again denote by $\phi^\sigma(u)$. Also,

$$\phi^\sigma\big(\phi^{\sigma'}(u)\big) = (u \circ \sigma') \circ \sigma = u \circ (\sigma + \sigma') = \phi^{\sigma+\sigma'}(u),$$

which shows that ϕ^σ defines a continuous flow on X. A continuous flow on X is a continuous mapping $(\sigma, u) \mapsto \phi^\sigma(u) \in X$, such that $\phi^0(u) = u$ and $\phi^{\sigma+\sigma'}(u) = \phi^\sigma(\phi^{\sigma'}(u))$, for all $\sigma, \sigma' \in \mathbb{R}$ and for all $u \in X$.

Consider the evaluation mapping $\iota : C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(S^1; \mathbb{R}^2)$, defined by

$$u(\cdot, \cdot) \mapsto u(0, \cdot).$$

By a similar argument as in Lemma 4.2.2 it follows that the mapping ι is a continuous mapping with respect to the compact-open topology on $C^0(S^1; \mathbb{R}^2)$.

4.2.3. Proposition. *The mapping $\iota : X \to \mathscr{X}$, with $\mathscr{X} = \iota(X)$, is a homeomorphism.*

Proof. See Section 4.6. ■

For ϕ^σ we have the following commuting diagram:

$$\begin{array}{ccc} \mathbb{R} \times X & \xrightarrow{\phi^\sigma} & X \\ \downarrow{\scriptstyle \mathrm{id}\times\iota} & & \downarrow{\scriptstyle \iota} \\ \mathbb{R} \times \mathscr{X} & \xrightarrow{T^\sigma} & \mathscr{X}, \end{array}$$

with $u(0,\cdot) \mapsto T^\sigma(u(0,\cdot)) = u(\sigma,\cdot)$, and T^σ defines a flow on $\mathscr{X}$.

The principal tool in the proof Theorem 4.1.1 is the existence of an unbounded, discrete Lyapunov function, which decreases along orbits of the flow ϕ^σ. Let $u^1, u^2 \in X$ be two solutions, with $u^1 \neq u^2$, such that the function $t \mapsto u^1(s,t) - u^2(s,t)$ is nowhere zero. Then define $w := u^1 - u^2 \in C^0(\mathbb{R}\times S^1;\mathbb{R}^2)$. The s-dependent winding number $\mathscr{W}$ of the pair (u^1,u^2) is defined as the winding number of w about the origin, i.e.

$$\mathscr{W}(u^1(s,\cdot), u^2(s,\cdot)) := \mathscr{W}(w(s,\cdot),0) = \frac{1}{2\pi}\int_{S^1} w^*\theta, \tag{4.7}$$

where $\theta = \frac{-qdp+pdq}{p^2+q^2}$ is a closed one-form on $\mathbb{R}^2 \setminus \{0\}$ (see [49] for more details). A pair of solutions $(u^1,u^2) \in X \times X$ is said to be *singular*, if they belong to the "crossing" set defined by

$$\Sigma_X := \{(u^1,u^2) \in X \times X : \exists\, s \in \mathbb{R} \;:\; u^1(s,t) = u^2(s,t) \text{ for some } t \in S^1\},$$

and $W\colon (X \times X) \setminus \Sigma_X \to \mathbb{Z}$, is defined by

$$W(u^1,u^2) := \mathscr{W}\big(\iota(u^1), \iota(u^2)\big). \tag{4.8}$$

The Lyapunov function W is continuous on $(X \times X) \setminus \Sigma_X$ and constant on connected components. The set Σ_X is a closed in $X \times X$, since uniform convergence on compact sets implies point-wise convergence. The function W is a symmetric:

$$W(u^1,u^2) = W(u^2,u^1), \quad \text{for all } (u^1,u^2) \notin \Sigma_X.$$

The *diagonal* in $X \times X$ is defined by

$$\Delta := \{(u^1,u^2) \in X \times X : u^1 = u^2\},$$

and $\Delta \subset \Sigma_X$. The flow ϕ^σ induces a product flow on $X \times X$, via $(u^1, u^2) \mapsto \big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big)$, and the diagonal Δ is invariant for the product flow. For the action of the flow on W we have

$$\begin{aligned} W\big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big) &= \mathscr{W}\big(\iota \circ \phi^\sigma(u^1), \iota \circ \phi^\sigma(u^2)\big) \\ &= \mathscr{W}\big(T^\sigma(\iota(u^1)), T^\sigma(\iota(u^2))\big) = \mathscr{W}(u^1(\sigma, \cdot), u^2(\sigma, \cdot)). \end{aligned}$$

In [49] it is proved that the set $\Sigma_X \setminus \Delta$ is "thin" in $X \times X$, which is the content of the following proposition.

4.2.4. Proposition ([49]). *For every singular solution pair $(u^1, u^2) \in \Sigma_X \setminus \Delta$, there exists an $\varepsilon_0 = \varepsilon(u^1, u^2) > 0$, such that $(\phi^\sigma(u^1), \phi^\sigma(u^2)) \notin \Sigma_X$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$.*

Orbits which intersect Σ_X "transversely" (and thus are not in the diagonal) instantly escape from Σ_X and the diagonal Δ is the maximal invariant set contained in Σ_X. The following proposition indicates W is a discrete Lyapunov function.

4.2.5. Proposition ([49]). *For every pair of singular solutions $(u^1, u^2) \in \Sigma_X \setminus \Delta$, there exists an $\varepsilon_0 = \varepsilon(u^1, u^2) > 0$, such that $W(\phi^\sigma(u^1), \phi^\sigma(u^2)) > W(\phi^{\sigma'}(u^1), \phi^{\sigma'}(u^2))$, for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$.*

For a given $u \in X$ define the α- and ω-limit sets as:

$$\begin{aligned} \omega(u) &:= \{w \in X : \phi^{\sigma_n}(u) \xrightarrow{X} w, \text{ for some } \sigma_n \to \infty\}, \\ \alpha(u) &:= \{w \in X : \phi^{\sigma_n}(u) \xrightarrow{X} w, \text{ for some } \sigma_n \to -\infty\}. \end{aligned}$$

The sets $\alpha(u)$ and $\omega(u)$ are closed invariant sets for the flow ϕ^σ, see [30, Lemma 4.6 Chapter IV]. Since X is compact, also $\alpha(u)$ and $\omega(u)$ are compact. Compactness of X implies furthermore that $\alpha(u)$ and $\omega(u)$ are non-empty, see [30, Theorem 4.7 Chapter IV]. The Hausdorff property of X and the continuity of the flow ϕ^σ imply that $\alpha(u)$ and $\omega(u)$ are connected sets, see [30, Theorem 4.7 Chapter IV]. Define the equilibria of ϕ^σ by

$$E := \{u \in X : \phi^\sigma(u) = u \text{ for all } \sigma \in \mathbb{R}\}.$$

Equilibria are functions $u = u(t)$ which satisfy the stationary equation $u_t = F(t, u)$.

4.3 The abstract Poincaré-Bendixson Theorem

The concepts introduced so far can be embedded in a more abstract setting, which generalizes the work by Fiedler and Mallet-Paret in [18]. Let ϕ^σ be a continuous flow on a *compact* Hausdorff space X. In the case of the Cauchy-Riemann equations the flow ϕ^σ was defined in (4.6), where the space X can be either the full solution space, or else, the space which consists of the closure of a single entire (bounded) orbit.

The notions of α- and ω-limit sets, defined in Section 4.2 remain unchanged, and $\alpha(u)$ and $\omega(u)$ are non-empty, compact, connected, invariant sets.

Let $\Delta = \{(u^1, u^2) \in X \times X : u^1 = u^2\}$ be invariant for the product flow induced by ϕ^σ. We assume that there exist a closed "thin" singular set Σ, with $\Delta \subset \Sigma \subset X \times X$, and functions $W : (X \times X) \setminus \Sigma \to \mathbb{Z}$ and $\pi : X \to \pi(X) \subset \mathbb{R}^2$, which satisfy the following axioms:

(A1) the map $W : X \times X \setminus \Sigma \to \mathbb{Z}$, is continuous and symmetric;

(A2) the map $\pi : X \to \pi(X) \subset \mathbb{R}^2$, is a continuous projection onto its (compact) image;

(A3) the set $\{(u^1, u^2) \in X \times X : \pi(u^1) = \pi(u^2)\}$ is a subset of Σ;

(A4) for every $(u^1, u^2) \in \Sigma \setminus \Delta$, there exists an $\varepsilon_0 > 0$, depending on (u^1, u^2), such that $(\phi^\sigma(u^1), \phi^\sigma(u^2)) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$;

(A5) for every $(u^1, u^2) \in \Sigma \setminus \Delta$, there exists an $\varepsilon_0 > 0$, depending on (u^1, u^2), such that
$$W(\phi^\sigma(u^1), \phi^\sigma(u^2)) > W(\phi^{\sigma'}(u^1), \phi^{\sigma'}(u^2)),$$
for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$.

Axioms (A1)-(A5) are modeled on the properties of the non-linear Cauchy-Riemann Equations discussed in Section 4.2, with $\pi = \pi_{t_0}$ defined in (4.3). The above axioms also generalize the conditions in the work of Fiedler and Mallet-Paret in [18]. Note that the function W is a priori unbounded in the present case and the flow ϕ^σ does not necessarily regularize. Under these assumptions we prove the following Theorem.

4.3.1. Theorem (Poincaré-Bendixson). *Let ϕ^σ be a continuous flow on a compact Hausdorff space X. Let Σ be a closed subset of $X \times X$, and let $W : (X \times X) \setminus \Sigma \to \mathbb{Z}$ and $\pi : X \to \pi(X) \subset \mathbb{R}^2$ be mappings as defined above, and which satisfy Axioms (A1)-(A5). Then for $\omega(u)$ we have the following dichotomy*

(i) either $\omega(u)$ consists of precisely one periodic orbit, or else

(ii) $\alpha(w) \subseteq E$ and $\omega(w) \subseteq E$, for every $w \in \omega(u)$.

The same dichotomy holds for $\alpha(u)$.

As in [18], the proof of Theorem 4.3.1 will be divided into three Propositions, namely Proposition 4.3.2, Proposition 4.3.3 and Proposition 4.3.4. For the following three Proposition and throughout the paper we assume the hypotheses of Theorem 4.3.1.

4.3.2. Proposition (Soft version). *Let u be in X and let $w \in \omega(u)$, then $\omega(w)$ contains a periodic solution or an equilibrium. The same holds for $\alpha(w)$.*

Proposition 4.3.2 implies that, since $\omega(w)$ and $\alpha(w)$ are both subsets of $\omega(u)$, also $\omega(u)$ contains a periodic solution or an equilibrium.

4.3.3. Proposition. *Let u be in X and let $w \in \omega(u)$. Then either,*

(i) $\alpha(w)$ and $\omega(w)$ consist only of equilibria, or else

(ii) $\gamma(w)$ is a periodic orbit.

4.3.4. Proposition. *Let u be X. If $\omega(u)$ contains a periodic orbit, then $\omega(u)$ is a single periodic orbit.*

The proof of Proposition 4.3.2 is given in Section 4.4 and the proofs of Propositions 4.3.3 and 4.3.4 are carried out in Section 4.5.2. Section 4.5.2 also provides the proof of Theorem 4.1.2, with a formulation adapted to the abstract setting. Propositions 4.3.3 and 4.3.4 together imply Theorem 4.3.1, while Proposition 4.3.2 will be used to prove Proposition 4.3.3. Theorem 4.3.1 can be applied directly to the Cauchy-Riemann equations and therefore implies Theorem 4.1.1. Subsection 4.5.1 of Section 4.5 contains a number of technical lemmas. Finally, Section 4.6 provides the proofs of Propositions 4.2.1 and 4.2.3.

4.4 The soft version

This section deals with the soft version of the Poincaré-Bendixson theorem given by Proposition 4.3.2. In the remainder of this text we adopt the hypotheses of Section 4.3.

4.4.1. Lemma. *For every pair $(u^1, u^2) \in (X \times X) \setminus \Delta$, the set*

$$A_{(u^1,u^2)} := \{\sigma \in \mathbb{R} \colon \big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big) \in \Sigma\}$$

consists of isolated points only. Moreover, the mapping

$$\sigma \mapsto W(\phi^\sigma(u^1), \phi^\sigma(u^2)),$$

defined for $\sigma \in \mathbb{R} \setminus A_{(u^1,u^2)}$, is a non-increasing function of σ and constant on the connected components of $\mathbb{R} \setminus A_{(u^1,u^2)}$.

Proof. Suppose there exists an accumulation point $\sigma_n \to \sigma_*$, for $\sigma_n \in A_{(u^1,u^2)}$. By definition $\big(\phi^{\sigma_n}(u^1), \phi^{\sigma_n}(u^2)\big) \in \Sigma \setminus \Delta$, since Δ is invariant and $(u^1, u^2) \notin \Delta$. By the continuity of ϕ^σ we have that

$$\big(\phi^{\sigma_n}(u^1), \phi^{\sigma_n}(u^2)\big) \xrightarrow{n\to\infty} \big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big) \in \Sigma,$$

since Σ is closed. This proves that $\sigma_* \in A_{(u^1,u^2)}$. The invariance of Δ implies that $\big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big) \in \Sigma \setminus \Delta$. By Axiom (A4) there exists an $\varepsilon_0 > 0$, depending on $\big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big)$, such that $(\phi^{\sigma_*+\varepsilon}(u^1), \phi^{\sigma_*+\varepsilon}(u^2)) \notin \Sigma$, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$. This contradicts the fact that σ_* is an accumulation point.

The set $A_{(u^1,u^2)}$ is a discrete and ordered set. Let $\sigma' < \sigma''$ be two consecutive points in $A_{(u^1,u^2)}$. By Axiom (A1), W is continuous and $\mathbb{Z}$-valued, and therefore $W(\phi^\sigma(u^1), \phi^\sigma(u^2))$ is constant on $\sigma \in (\sigma', \sigma'')$. The fact that W is non-increasing then follows from (A5), since $W(\phi^\sigma(u^1), \phi^\sigma(u^2))$ drops at points in $A_{(u^1,u^2)}$. ■

4.4.2. Lemma. *Let $u \in X$ and $w \in \omega(u)$. For every $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$ with $w^1 \neq w^2$, it holds that $(w^1, w^2) \notin \Sigma$.*

Proof. For contradiction, suppose $(w^1, w^2) \in \Sigma \setminus \Delta$, then, by the Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$, such that $\big(\phi^\sigma(w^1), \phi^\sigma(w^2)\big) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^\sigma(w^1), \phi^\sigma(w^2)) > W(\phi^{\sigma'}(w^1), \phi^{\sigma'}(w^2)),$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Since $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, there exist $s_1, s_2 \in \mathbb{R}$, such that $\big(\phi^{s_1\pm\varepsilon}(w), \phi^{s_2\pm\varepsilon}(w)\big) \notin \Sigma$ and $\big(\phi^{s_1\pm\varepsilon}(w), \phi^{s_2\pm\varepsilon}(w)\big)$ is close to $\big(\phi^{\pm\varepsilon}(w^1), \phi^{\pm\varepsilon}(w^2)\big)$. The continuity of W (Axiom (A1)) then implies

$$\begin{aligned} W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) &= W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)) \\ &< W(\phi^{-\varepsilon}(w^1), \phi^{-\varepsilon}(w^2)) \\ &= W(\phi^{s_1-\varepsilon}(w), \phi^{s_2-\varepsilon}(w)). \end{aligned} \tag{4.9}$$

Since $\gamma(w) \subset \omega(u)$ is an invariant subset of $\omega(u)$, the definition of ω-limit set and the continuity of ϕ^σ imply that there exists a sequence $\sigma_n \to \infty$, as $n \to \infty$, such that

$$\phi^{\sigma_n+s_1-s_2\pm\varepsilon}(u) \to \phi^{s_1\pm\varepsilon}(w), \quad \text{and} \quad \phi^{\sigma_n\pm\varepsilon}(u) \to \phi^{s_2\pm\varepsilon}(w). \tag{4.10}$$

Since σ_n is divergent, we may assume

$$\sigma_{n+1} > \sigma_n + 2\varepsilon, \qquad \text{for all } n. \tag{4.11}$$

Inequality (4.9), the convergence in (4.10), Axiom (A1) (continuity) and the fact that W is locally constant (see Lemma 4.4.1), imply, for $\sigma_n \to \infty$, that

$$\begin{aligned} W(\phi^{\sigma_n+s_1-s_2+\varepsilon}(u), \phi^{\sigma_n+\varepsilon}(u)) &= W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) \\ &< W(\phi^{s_1-\varepsilon}(w), \phi^{s_2-\varepsilon}(w)) \\ &= W(\phi^{\sigma_n+s_1-s_2-\varepsilon}(u), \phi^{\sigma_n-\varepsilon}(u)). \end{aligned}$$

By combining the latter with (4.11) and the fact that W is non-increasing, we obtain

$$W(\phi^{\sigma_{n+1}+s_1-s_2-\varepsilon}(u), \phi^{\sigma_{n+1}-\varepsilon}(u)) < W(\phi^{\sigma_n+s_1-s_2-\varepsilon}(u), \phi^{\sigma_n-\varepsilon}(u)),$$

for all n. From this inequality we deduce that $\sigma \mapsto W(\phi^{\sigma+s_1-s_2}(u), \phi^{\sigma}(u))$ has infinitely many jumps and therefore

$$W(\phi^{\sigma+s_1-s_2}(u), \phi^{\sigma}(u)) \to -\infty, \quad \text{as} \quad \sigma \to \infty.$$

On the other hand, by continuity of W and (4.10) we have, for $\sigma_n \to \infty$, that

$$W(\phi^{\sigma_n+s_1-s_2+\varepsilon}(u), \phi^{\sigma_n+\varepsilon}(u)) = W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) > -\infty,$$

which is a contradiction. ■

4.4.3. Lemma. *Let $u \in X$ and $w \in \omega(u)$, then*

$$\pi\colon\ \mathrm{cl}\,(\gamma(w)) \to \pi\,\mathrm{cl}(\gamma(w)) \subset \mathbb{R}^2$$

is a homeomorphism onto its image. Hence, $\pi \circ \phi^{\sigma} \circ \pi^{-1}$ is a continuous flow on $\pi\,\mathrm{cl}(\gamma(w))$.

Proof. By Axiom (A2), the projection $\pi\colon\ \mathrm{cl}\,(\gamma(w)) \to \pi\,\mathrm{cl}(\gamma(w))$ is continuous. Since $\mathrm{cl}\,(\gamma(w))$ is compact and $\pi\,\mathrm{cl}(\gamma(w))$ is Hausdorff, it is sufficient to show that π is bijective, see [40, §26, Thm. 26.6]. The projection π is surjective and it remains to show that π is injective on $\mathrm{cl}(\gamma(w))$. Suppose π is not injective, then there exist $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, such that $w^1 \neq w^2$ and $\pi(w^1) = \pi(w^2)$. Axiom (A3) then implies that $(w^1, w^2) \in \Sigma \setminus \Delta$. On the other hand, Lemma 4.4.2 implies that $(w^1, w^2) \notin \Sigma$, which is a contradiction. This establishes the injectivity of π. ■

For the projected flow on $\pi\,\mathrm{cl}(\gamma(w))$ we have the following commuting diagram:

$$\begin{array}{ccc} \mathbb{R} \times \mathrm{cl}(\gamma(w)) & \xrightarrow{\ \phi^{\sigma}\ } & \mathrm{cl}(\gamma(w)) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{R} \times \pi\,\mathrm{cl}(\gamma(w)) & \xrightarrow{\ \psi^{\sigma}\ } & \pi\,\mathrm{cl}(\gamma(w)), \end{array} \tag{4.12}$$

where $\psi^{\sigma} = \pi \circ \phi^{\sigma} \circ (\mathrm{id} \times \pi)^{-1}$.

4.4.4. Corollary. *The equilibria of the planar flow $\psi^{\sigma} := \pi \circ \phi^{\sigma} \circ (\mathrm{id} \times \pi)^{-1}$ on $\pi\,\mathrm{cl}(\gamma(w))$ are in one-to-one correspondence with the equilibria of the flow ϕ^{σ} in $\mathrm{cl}(\gamma(w))$.*

Let $(\sigma, x) \mapsto \psi^\sigma(x)$, $x \in \mathbb{R}^2$, be a continuous flow on $\mathbb{R}^2$. A subset $\mathscr{Q} \subset \mathbb{R}^2$ is a *section* for ψ^σ, if there is a $\delta > 0$, such that

$$\psi^{\sigma_1}(\mathscr{Q}) \cap \psi^{\sigma_2}(\mathscr{Q}) = \varnothing, \quad \text{for all } 0 \le \sigma_1 < \sigma_2 \le \delta,$$

where δ is called a *σ-extent* of $\mathscr{Q}$. The definition of section does *not* require differentiability of the flow ψ^σ. If there exists a section $\mathscr{Q}$ that is a continuum (i.e. a compact, connected set containing at least two points), then $\mathscr{Q}$ is a curve in $\mathbb{R}^2$ see [30, Theorem 1.6 ch. VII]. A curve in $\mathbb{R}^2$ that is a section is called a *transversal*.

4.4.5. Lemma (Flow-box Theorem for planar flows). *Let ψ^σ be a planar flow and let $x \in \mathbb{R}^2$ not be an equilibrium of ψ^σ. Then,*

(i) there exists a transversal $\mathscr{C}$ containing x, of extent δ, for some $\delta > 0$. The transversal $\mathscr{C}$ is given as the image of the embedding $r : [-\varepsilon, \varepsilon] \to \mathbb{R}^2$ with $r(0) = x$, for $\varepsilon > 0$ sufficiently small. The set

$$\mathscr{U} := \left\{ \psi^\sigma(r(\tau)) \; : \; \tau \in [-\varepsilon, \varepsilon], \sigma \in \left[-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\right] \right\},$$

is a neighborhood of x;

(ii) there exists a homeomorphism $h : \mathscr{U} \to U := [-\varepsilon, \varepsilon] \times \left[-\frac{1}{2}\delta, \frac{1}{2}\delta\right]$, such that for every $\tau \in [-\varepsilon, \varepsilon]$,

$$h \circ \psi^\sigma(r(\tau)) = (\tau, \sigma) \in U, \quad \textit{for all } \; \sigma \in \left[-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\right].$$

Proof. See [30, Proposition 2.5 chapter VII]. ■

4.4.6. Remark. The homeomorphism h is also referred to as the *canonical homeomorphism* for ψ^σ. Under its image the flow trivialises to the parallel flow as indicated in Figure 4.1. The set U is referred to as a *canonical domain*, for which h is a change of coordinates, that transforms the flow ψ^σ to the parallel flow $h \circ \psi^\sigma \circ h^{-1}$.

4.4.7. Remark. In our case the projection π does not induce a planar flow on the full $\mathbb{R}^2$ (or an open subset of it), but only on the closed invariant subset $\pi\,\mathrm{cl}(\gamma(w)) \subset \mathbb{R}^2$. In fact, as we will see later, we need to apply a variant of Lemma 4.4.5 to forward invariant closed subsets of the form

$$\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \ge \sigma_*\}),$$

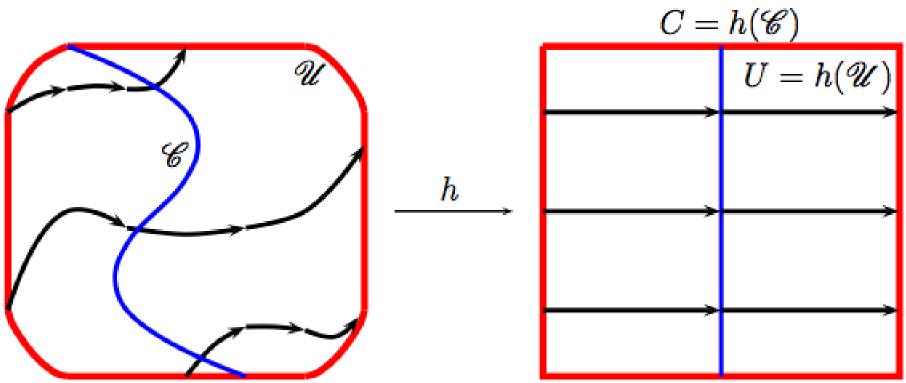


Figure 4.1: The transversal $\mathscr{C}$ and the canonical change of coordinates h that maps to the parallel flow.

where $u \in X, w \in \omega(u), \sigma_* \in \mathbb{R}$. On $\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_*\})$ we have a commuting diagram similar to (4.12). In order to have a bi-directional local flow, we define the slightly smaller set

$$\mathscr{V} := \pi\,\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_* + \delta\}), \tag{4.13}$$

for $\delta > 0$ small. Then, if $x \in \mathscr{V}$ is not an equilibrium for ψ^σ, Lemma 4.4.5 continues to hold, provided we replace $\mathscr{U}$ with

$$\tilde{\mathscr{U}} := \left\{\psi^\sigma(r(\tau)) : \tau \in r^{-1}(\mathscr{C} \cup \mathscr{V}), \sigma \in \left[-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\right]\right\}.$$

Note that $\tilde{\mathscr{U}} := \mathscr{U}|_{\tau \in r^{-1}(\mathscr{C} \cup \mathscr{V})}$.

4.4.8. Proposition (Soft version). *Let u be in X and $w \in \omega(u)$, then $\omega(w)$ contains a periodic orbit or an equilibrium. The same holds for $\alpha(w)$.*

Proof. Suppose $\omega(w)$ does not contain any equilibria. Choose $\zeta \in \omega(w)$ and $\zeta^* \in \omega(\zeta)$, then

$$\omega(\zeta) \subseteq \omega(\omega(w)) = \omega(w) \subseteq \omega(\gamma(w)) = \mathrm{cl}(\gamma(w)). \tag{4.14}$$

Since ζ^* is not an equilibrium, then $\pi(\zeta^*)$ is not an equilibrium for $\psi^\sigma = \pi \circ \phi^\sigma \circ (\mathrm{id} \times \pi)^{-1}$ by Corollary 4.4.4. According to Lemma 4.4.5 there exists a transversal $\mathscr{C}$ for ψ^σ, through $x = \pi(\zeta^*)$. Let U be the canonical domain, which is the image of $\mathscr{U}$ (neighbourhood of $\pi(\zeta^*)$) under h, and

$$h \circ \psi^\sigma \circ h^{-1}\big(h(\pi(\zeta))\big) = h \circ \psi^\sigma(\zeta) = h \circ \pi \circ \phi^\sigma(\zeta), \quad \text{for all } \pi(\zeta) \in \mathscr{U}.$$

Since $\zeta^* \in \omega(\zeta)$, there exist times $\sigma_n \to \infty$, such that $\phi^{\sigma_n}(\zeta) \to \zeta^*$. By the Flow-box Theorem (Lemma 4.4.5) these times can be chosen such that $\pi \circ \phi^{\sigma_n}(\zeta) \in \mathscr{C}$, for σ_n sufficiently large, and moreover $\pi \circ \phi^{\sigma}(\zeta) \notin \mathscr{C}$ for $\sigma \in (\sigma_n, \sigma_{n+1})$. We consider two cases.

Case 1. For some $n \neq n'$, we have $\pi \circ \phi^{\sigma_n}(\zeta) = \pi \circ \phi^{\sigma_{n'}}(\zeta)$. Then, since π is a homeomorphism on $\mathrm{cl}(\gamma(w))$ (see Lemma 4.4.3) and since $\omega(\zeta) \subset \mathrm{cl}(\gamma(w))$ (see Equation (4.14)), it follows that $\phi^{\sigma_n}(\zeta) = \phi^{\sigma_{n'}}(\zeta)$, and thus $\phi^{\sigma}(\zeta)$ is a periodic orbit.

Case 2. All $\pi \circ \phi^{\sigma_n}(\zeta)$ are mutually distinct. Choose n^* sufficiently large, such that $\pi \circ \phi^{\sigma_n}(\zeta)$ and $\pi \circ \phi^{\sigma_{n+1}}(\zeta)$ lie in $\mathscr{U}$, for all $n \geq n^*$. Consider the closed Jordan curve $\mathscr{J}$ in $\mathbb{R}^2$ that is the union of the sets $\pi \circ \phi^{[\sigma_n, \sigma_{n+1}]}(\zeta)$ and $\mathscr{S}$, where $\mathscr{S} = h^{-1}\big([\tau_n, \tau_{n+1}] \times \{0\}\big)$, with τ_n and τ_{n+1} the first coordinates of the points $h \circ \pi \circ \phi^{\sigma_n}(\zeta)$ and $h \circ \pi \circ \phi^{\sigma_{n+1}}(\zeta)$, respectively.

Consider the orbit $\pi \circ \phi^{\sigma}(w)$. Since $\zeta^* \in \omega(w)$ and π is continuous, we have that $\pi(\zeta^*)$ is an ω-limit point of $\pi(w)$ under ψ^{σ}. This implies, for σ sufficiently large, that every time $\pi \circ \phi^{\sigma}(w)$ enters $\mathscr{U}$, it crosses $\mathscr{C}$ exactly once. By Lemma 4.4.3, ψ^{σ} is a planar flow on $\pi(\gamma(w))$ and therefore $\pi \circ \phi^{\sigma}(w)$ cannot intersect $\pi \circ \phi^{[\sigma_n, \sigma_{n+1}]}(\zeta)$. Thus, for σ sufficiently large, $\pi \circ \phi^{\sigma}(w)$ is either in the interior or in the exterior of the Jordan curve $\mathscr{J}$ i.e. the interior of $\mathscr{J}$ is either forward or backward invariant with respect to ψ^{σ}. Therefore there are only four possibilities:

(i) the interior of $\mathscr{J}$ is forward invariant. For all σ sufficiently large, $\pi \circ \phi^{\sigma}(w)$ is inside $\mathscr{J}$;

(ii) the interior of $\mathscr{J}$ is forward invariant. For all σ sufficiently large, $\pi \circ \phi^{\sigma}(w)$ is outside $\mathscr{J}$;

(iii) the interior of $\mathscr{J}$ is backward invariant. For all σ sufficiently large, $\pi \circ \phi^{\sigma}(w)$ is inside $\mathscr{J}$;

(iv) the interior of $\mathscr{J}$ is backward invariant. For all σ sufficiently large, $\pi \circ \phi^{\sigma}(w)$ is outside $\mathscr{J}$.

By (4.14) and the invariance of $\omega(w)$, we have that $\phi^{\sigma_n}(\zeta) \in \omega(w)$, for all $n \in \mathbb{N}$, and hence $\pi \circ \phi^{\sigma_n}(\zeta) \in \pi\omega(w)$. Consequently, for all $n \in \mathbb{N}$, $\pi \circ \phi^{\sigma_n}(\zeta)$ are ω-limit points of $\pi \circ \phi^{\sigma}(w)$. Hence there exist $\sigma_k^1 \to \infty$ and $\sigma_k^2 \to \infty$ as $k \to \infty$ with $\phi^{\sigma_k^1}(w) \to \phi^{\sigma_n}(\zeta)$ and $\phi^{\sigma_k^2}(w) \to \phi^{\sigma_{n+1}}(\zeta)$, as $k \to \infty$. By the Flow-Box Theorem (Lemma 4.4.5) we may choose $\sigma_k^j, j = 1, 2$ such that $\pi \circ \phi^{\sigma_k^j}(w) \in \mathscr{C}, j = 1, 2$. Let $\eta > 0$ be small. Now, either $\pi \circ \phi^{\sigma_k^1 - \eta}(w)$ is outside $\mathscr{J}$ and $\pi \circ \phi^{\sigma_k^2 + \eta}(w)$ is inside $\mathscr{J}$ (see Figure 4.2 [left]), or $\pi \circ \phi^{\sigma_k^1 - \eta}(w)$ is inside $\mathscr{J}$ and $\pi \circ \phi^{\sigma_k^2 + \eta}(w)$ is outside $\mathscr{J}$ (see Figure 4.2 [right]). Since $\sigma_k^j \to \infty, j = 1, 2$, this contradicts all four cases above. Consequently, Case 2 cannot occur, which implies Case 1 and thus a periodic orbit. ■

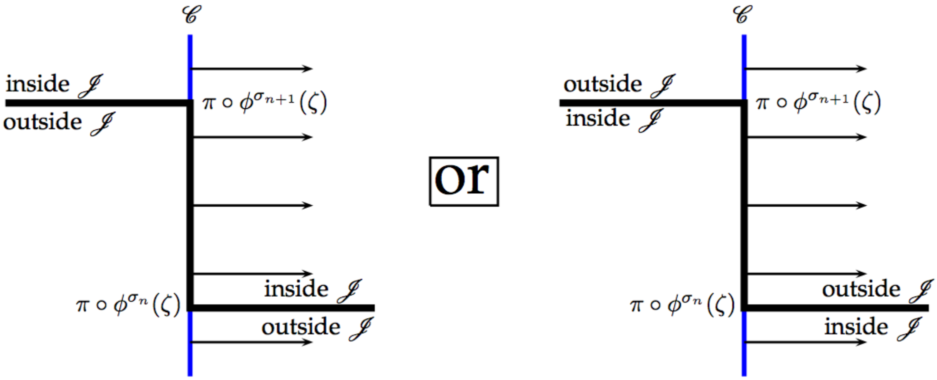


Figure 4.2: The two possible local geometries near the section $\mathscr{C}$ (in canonical coordinates), for the Jordan curve $\mathscr{J}$. On the left the interior of $\mathscr{J}$ is forward invariant, while on the right it is backward invariant.

4.4.9. Remark. In [18, Proposition 2] the "soft version" was proved using both smoothness of the flow and fact that there exists a non-negative discrete Lyapunov function. The extension given by Proposition 4.4.8 makes it applicable to the Cauchy-Riemann equations, for which a $\mathbb{Z}$-valued Lyapunov function exists.

4.5 The strong version

This section is subdivided into two subsections. In the first subsection we show some preliminary lemmas that will be used to prove the strong version of the Poincaré-Bendixson Theorem. The proof of Proposition 4.3.3 occupy the second subsection. Proofs are as in [18], but worked out in more details, and eventually adjusted to our setting.

4.5.1 Technical lemmas

4.5.1. Lemma. *Let $u \in X$, then for every $w \in \omega(u)$ there exists an integer $k(w)$, such that*

$$W(w^1, w^2) = k(w),$$

for all $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$, with $w^1 \neq w^2$.

Proof. See also [18, Lemma 3.1]. Since we consider two distinct $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$, we may exclude the case that w is an equilibrium. We therefore distinguish two cases: (i) $\gamma(w)$ is a periodic orbit, or (ii) $\sigma \mapsto \phi^\sigma(w)$ is injective. Lemma 4.4.2 implies that $(w^1, w^2) \notin \Sigma$, and therefore $(w^1, w^2) \mapsto W(w^1, w^2)$ is a continuous $\mathbb{Z}$-valued function on $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$.

(i) If $\gamma(w)$ is a periodic orbit, then, $\mathrm{cl}(\gamma(w)) = \gamma(w)$, which is homeomorphic to S^1, and $\gamma(w) \times \gamma(w)$ is therefore homeomorphic to the 2-torus $\mathbb{T}^2$. Therefore $(w^1, w^2) \mapsto W(w^1, w^2)$ induces a continuous $\mathbb{Z}$-valued function on $\mathbb{T}^2 \setminus S^1$. Since the latter is connected, it follows that W is constant on $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$.

(ii) If $\sigma \to \phi^\sigma(w)$ is injective, then $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$ has two connected components given by $(\phi^{\sigma_1}(w), \phi^{\sigma_2}(w))$, with $\sigma_1 > \sigma_2$, and $\sigma_1 < \sigma_2$, respectively. Since W is symmetric (Axiom (A1)) we conclude that W is constant on $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$. Note that $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$ is the closure of $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$ in $(X \times X) \setminus \Delta$. Since W is continuous on $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$, it is also constant, which proves the lemma. ■

4.5.2. Lemma. *Assume that $u \in X$ and $w \in \omega(u)$. Let $k(w)$ be defined as in Lemma 4.5.1. If $\alpha(w) \cap \omega(w) = \varnothing$, then there exists a $\sigma_* \geq 0$, such that*

$$W(u^1, w^1) = k(w) \tag{4.15}$$

for every $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$ and every $w^1 \in \mathrm{cl}(\gamma(w))$, such that $u^1 \neq w^1$. In particular, if $\pi(u^1) = \pi(w^1)$ for some $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$ and $w^1 \in \mathrm{cl}(\gamma(w))$, then $u^1 = w^1$. Hence

$$\pi \circ \phi^\sigma(u) \notin \pi\, \mathrm{cl}(\gamma(w)) \ \textit{ for all } \sigma \geq \sigma_*. \tag{4.16}$$

Proof. See [18, Lemma 3.2]. We start by observing that it is enough to prove that (4.15) holds for $u^1 \in \phi^\sigma(u), \sigma \geq \sigma_*$. Then by continuity of W, the statement follows for all $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$.

Suppose there exist sequences $\sigma_n \to \infty$, $w_n \in \mathrm{cl}(\gamma(w))$, with

$$\phi^{\sigma_n}(u) \neq w_n, \quad k_n := W(\phi^{\sigma_n}(u), w_n) \neq k(w).$$

We may assume, passing to a subsequence if necessary, that for all n we have that either $k_n > k(w)$ or $k_n < k(w)$. We will split the proof in two cases.

Case 1: $k_n < k(w)$. Again passing to a subsequence if necessary, we may assume that either $w_n \in \alpha(w)$ for all n or else $w_n \in \mathrm{cl}(\gamma(w)) \setminus \alpha(w)$ for all n. Since $\alpha(w)$ and $\omega(w)$ are disjoint by assumption, it follows that $\mathrm{cl}(\gamma(w)) \setminus \alpha(w) = \gamma(w) \cup \omega(w)$. Choose now $w^1 \in \omega(w)$ in case $w_n \in \alpha(w)$, and $w^1 \in \alpha(w)$ in case $w_n \in \gamma(w) \cup \omega(w)$. In both cases we have $w^1 \in \omega(u)$, hence we can choose a sequence $\tilde{\sigma}_n$ with $\tilde{\sigma}_n > \sigma_n$, for every n such that

$$w^1 := \lim_{n \to \infty} \phi^{\tilde{\sigma}_n}(u).$$

In case $w_n \in \gamma(w) \cup \omega(w)$ we may assume that $\tilde{\sigma}_n - \sigma_n$ is so large that $\phi^{\tilde{\sigma}_n - \sigma_n}(w_n) \in \mathrm{cl}\{\phi^\sigma(w), \sigma > 0\}$. For a further subsequence, we have convergence of $\phi^{\tilde{\sigma}_n - \sigma_n}(w_n)$. Call

$$w^2 := \lim_{n\to\infty} \phi^{\tilde{\sigma}_n - \sigma_n}(w_n).$$

Note that $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, and $w^1 \neq w^2$ since $\alpha(w) \cap \omega(w) = \varnothing$. In fact, by construction it follows that either $w^1 \in \omega(w)$ and $w^2 \in \alpha(w)$, or else $w^1 \in \alpha(w)$ and $w^2 \in \mathrm{cl}\{\gamma(w), \sigma \geq 0\} = \{\phi^\sigma(w), \sigma \geq 0\} \cup \omega(w)$. By Lemma 4.5.1 there exists $k(w) \in \mathbb{Z}$ such that

$$W(w^1, w^2) = k(w).$$

Now, for n big enough, by continuity of W we obtain

$$\begin{aligned} k_n < k(w) &= W(w^1, w^2) = W(\phi^{\tilde{\sigma}_n}(u), \phi^{\tilde{\sigma}_n - \sigma_n}(w_n)) \\ &= W(\phi^{\sigma_n + (\tilde{\sigma}_n - \sigma_n)}(u), \phi^{\tilde{\sigma}_n - \sigma_n}(w^n)) \\ &\leq W(\phi^{\sigma_n}(u), w_n) = k_n, \end{aligned}$$

which is a contradiction.

The final assertion (4.16) follows from the following observation. Suppose, for contradiction that there exist a $u^1 = \phi^{\sigma_1}(u)$, for some $\sigma_1 \geq \sigma_*$ and $w^1 \in \mathrm{cl}(\gamma(w))$, such that $\pi(u^1) = \pi(w^1)$. By what we have just proved, we then have $u^1 = w^1$. Since $w^1 \in \mathrm{cl}(\gamma(w))$ and, by assumption, the sets $\alpha(w)$, $\gamma(w)$ and $\omega(w)$ are disjoint, there are only three different possibilities.

(1) $w^1 \in \omega(w)$. Then $\phi^{\sigma_1}(u) \in \omega(w)$. By invariance $\omega(u) \subseteq \omega(\omega(w)) = \omega(w)$. Since $\alpha(w) \subseteq \omega(u) \subseteq \omega(w)$, this contradicts $\alpha(w) \cap \omega(w) = \varnothing$.

(2) $w^1 \in \alpha(w)$. Then $\phi^{\sigma_1}(u) \in \alpha(w)$. By invariance $\omega(u) \subseteq \omega(\alpha(w)) = \alpha(w)$. Since $\omega(w) \subseteq \omega(u) \subseteq \alpha(w)$, this contradicts $\alpha(w) \cap \omega(w) = \varnothing$.

(3) $w^1 \in \gamma(w)$. Then $\phi^{\sigma_1}(u) \in \gamma(w)$. By invariance $\omega(u) = \omega(w)$. But $\alpha(w) \subseteq \omega(u) = \omega(w)$, again contradicting $\alpha(w) \cap \omega(w) = \varnothing$.

Case 2: $k_n > k(w)$. This case is analogous to the previous one. It is enough to exchange the roles of $\alpha(w)$ and $\omega(w)$. See [18, Lemma 3.2] for further details. ■

4.5.3. Remark. Lemma 4.5.2 implies that the commutative diagram (4.12) extends from $\mathrm{cl}(\gamma(w))$ to $\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_*\})$, if $\alpha(w) \cap \omega(w) = \varnothing$. Additionally, by Lemma 4.4.5 and by Remark 4.4.7 the Flow-box Theorem holds for every $x \in \mathscr{V}$ (defined in (4.13)) that is not an equilibrium.

4.5.4. Lemma. *Let $u \in X$ and let γ_1 and γ_2 be (not necessarily distinct) stationary or periodic orbits in $\omega(u)$. Then, there exists a $k = k(\gamma_1, \gamma_2)$, $k \in \mathbb{Z}$, such that*

$$W(p^1, p^2) = k, \tag{4.17}$$

for every $p^j \in \gamma_j, p^1 \neq p^2$. In particular, the projections of disjoint periodic orbits are disjoint.

Proof. See [18, Lemma 3.3]. We consider the case where γ_1 and γ_2 are both periodic, the others are analogous or even simpler. We first claim that $W(p^1, p^2)$ is defined for every $p^1 \in \gamma^1$ and every $p^2 \in \gamma^2$ with $p^1 \neq p^2$. Suppose, for contradiction, that there exist $p^1 \in \gamma^1$ and $p^2 \in \gamma^2$ with $p^1 \neq p^2$ such that $(p^1, p^2) \in \Sigma \setminus \Delta$. Then, by Axiom (A4) and (A5) there exists an $\varepsilon_0 > 0$, such that $(\phi^\sigma(p^1), \phi^\sigma(p^2)) \notin \Sigma$ for every $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^{\sigma'}(p^1), \phi^{\sigma'}(p^2)) < W(\phi^\sigma(p^1), \phi^\sigma(p^2)), \tag{4.18}$$

for $\sigma' \in (0, \varepsilon_0)$ and $\sigma \in (-\varepsilon_0, 0)$. Set $\sigma' = \frac{\varepsilon_0}{2}$ and $\sigma = -\frac{\varepsilon_0}{2}$. By continuity of W there exists an $\eta \in (0, \frac{\varepsilon_0}{2})$ such that W is constant on the set

$$\mathcal{U} = \left\{(\phi^{\sigma_1}(p^1), \phi^{\sigma_2}(p^2)) \mid -\tfrac{\varepsilon_0}{2} - \eta < \sigma_1, \sigma_2 < \tfrac{\varepsilon_0}{2} + \eta\right\}.$$

By periodicity of γ^1 and γ^2 there is a $\sigma_3 > \varepsilon_0$ such that $(\phi^{\sigma_3}(p^1), \phi^{\sigma_3}(p^2)) \in \mathcal{U}$ (both in the periodic and the quasi-periodic case). Now, by (4.18)

$$W(\phi^{\varepsilon_0/2}(p^1), \phi^{\varepsilon_0/2}(p^2)) < W(\phi^{-\varepsilon_0/2}(p^1), \phi^{-\varepsilon_0/2}(p^2)) = W(\phi^{\sigma_3}(p^1), \phi^{\sigma_3}(p^2)).$$

Since $\sigma_3 > \frac{\varepsilon_0}{2}$, this contradicts Lemma 4.4.1. Hence $(p^1, p^2) \notin \Sigma$ and $W(p^1, p^2)$ is well defined for every $p^1 \in \gamma^1$ and every $p^2 \in \gamma^2$, with $p^1 \neq p^2$.

This implies, by continuity of W, that the map

$$(p^1, p^2) \to W(p^1, p^2)$$

is locally constant on

$$\{(p^1, p^2) \in \gamma_1 \times \gamma_2 \mid p^1 \neq p^2\}.$$

This set is connected, which proves (4.17). ■

4.5.5. Lemma. *Let $u \in X$ and $e \in E$. For every $w \in \omega(u)$ with $w \neq e$ it holds $(w, e) \notin \Sigma$. If, furthermore, $e \neq \omega(u)$ then there exists a $\bar{\sigma} \in \mathbb{R}$ such that the map $\sigma \mapsto W(\phi^\sigma(u), e)$ is constant for $\sigma > \bar{\sigma}$.*

Proof. The proof resembles the one of Lemma 4.4.2. We repeat the argument. Let $w \in \omega(u)$. Since $w \neq e$, we can assume that $(w, e) \notin \Delta$. Suppose, for contradiction, that $(w, e) \in \Sigma \setminus \Delta$, then by Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$ such that $(\phi^\sigma(w), e) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^\sigma(w), e) > W(\phi^{\sigma'}(w), e),$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Then we have

$$W(\phi^{-\varepsilon}(w), e) > W(\phi^{\varepsilon}(w), e). \tag{4.19}$$

By definition of the ω-limit set and the fact that ω is invariant, we have that there exists a sequence $\sigma_n \to \infty$, as $n \to \infty$ such that

$$\phi^{\sigma_n \pm \varepsilon}(u) \to \phi^{\pm\varepsilon}(w). \tag{4.20}$$

Since σ_n is divergent we assume that

$$\sigma_{n+1} > \sigma_n + 2\varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{4.21}$$

Inequality (4.19), convergence in (4.20) and Lemma 4.4.1 imply, for $\sigma_n \to \infty$, that

$$\begin{aligned} W(\phi^{\sigma_n+\varepsilon}(u), e) &= W(\phi^{+\varepsilon}(w), e) \\ &< W(\phi^{-\varepsilon}(w), e) \\ &= W(\phi^{\sigma_n-\varepsilon}(u), e). \end{aligned}$$

Combining the latter with (4.21) and the fact that W is non-increasing, we obtain

$$W(\phi^{\sigma_{n+1}-\varepsilon}(u), e) < W(\phi^{\sigma_n-\varepsilon}(u), e),$$

for all n. From this, we deduce that $\sigma \mapsto W(\phi^{\sigma}(u), e)$ has infinitely many jumps and therefore

$$W(\phi^{\sigma}(u), e) \to -\infty \quad \text{as} \quad \sigma \to \infty.$$

On the other hand, continuity of W and (4.20) imply, for $\sigma_n \to \infty$, that

$$W(\phi^{\sigma_n+\varepsilon}(u), e) = W(\phi^{\varepsilon}(w), e) > -\infty,$$

which is a contradiction.

To prove the final assertion, suppose, by contradiction, that such a $\bar{\sigma}$ does not exist. Then there exists a sequence $\sigma_n \to \infty$ such that $(\phi^{\sigma_n}(u), e) \in \Sigma$. Now choose a $w \in \omega(u) \setminus \{e\} \neq \varnothing$. There exists a sequence $\tilde{\sigma}_n \to \infty$ such that $\phi^{\tilde{\sigma}_n}(u) \to w$. By the first part of the lemma, $W(w, e) \in \mathbb{Z}$. We may choose $\tilde{\sigma}_n > \sigma_n$ without loss of generality. By continuity of W and axiom (A5) it follows that

$$W(w, e) = \lim_{n\to\infty} W(\phi^{\tilde{\sigma}_n}(u), e) = -\infty,$$

a contradiction. This concludes the proof. ∎

4.5.6. Lemma. *Let u be in X. There exists an integer $k_0 \in \mathbb{Z}$ such that*

$$W(w, e) = k_0 \tag{4.22}$$

for every $w \in \omega(u)$, and for every equilibrium $e \in \omega(u)$ such that $w \neq e$.

Proof. Fix $e \in E \cap \omega(u)$. Let $w \in \omega(u) \setminus \{e\}$. According to Lemma 4.5.5, $W(w, e)$ is well-defined. Since $\phi^{\sigma_n}(u) \to w$ for some $\sigma_n \to \infty$,

$$\begin{aligned} W(w, e) &= \lim_{n \to \infty} W(\phi^{\sigma_n}(u), e) \\ &= \lim_{\sigma \to \infty} W(\phi^{\sigma}(u), e) = k_e, \end{aligned}$$

where the second limit exists by Lemma 4.5.5. Since the above statement holds for any $w \in \omega(u) \setminus \{e\}$, this implies that $W(w, e)$ is independent of $w \in \omega(u) \setminus \{e\}$.

We still need to show that $W(w, e)$ is independent of $e \in E \cap \omega(u)$. Therefore let $e, \tilde{e} \in E \cap \omega(u), e \neq \tilde{e}$. Then, by Axiom (A1), by the fact that $e, \tilde{e} \in \omega(u)$, and by Lemma 4.5.5 it holds that

$$k_e = W(w, e) = W(\tilde{e}, e) = W(e, \tilde{e}) = W(w, \tilde{e}) = k_{\tilde{e}}.$$

This shows (4.22) and concludes the proof. ■

4.5.2 Proof of the strong version

In this section we prove Propositions 4.3.3 and 4.3.4. This completes the proof of Theorem 4.3.1. We finish by proving Theorem 4.1.2 (in the abstract setting of Section 4.3).

Proof of Proposition 4.3.3. Suppose that $w^* \in \omega(w)$ is not an equilibrium and suppose furthermore that $\gamma(w)$ is not periodic. Lemma 4.4.3 implies that $\pi \circ \phi^\sigma$ is a planar flow on the set $\omega(w) \subseteq \text{cl}(\gamma(w))$. By Corollary 4.4.4, the point $\pi(w^*)$ is not an equilibrium for $\pi \circ \phi^\sigma$. According to Lemma 4.4.5 there exist a transversal $\mathscr{C}$ through $\pi(w^*)$ and a canonical domain U centered in $\pi(w^*)$, in which the flow has the form shown in Figure 4.1 (as in Lemma 4.4.8 we will identify $\pi(w^*)$ with $h \circ \pi(w^*)$ and the neighborhood $\mathscr{U}$ of $\pi(w^*)$ with its image under h, i.e. $U = h(\mathscr{U})$). Consider first $\pi \circ \phi^\sigma(w)$ and recall that by Lemma 4.4.3 the map $\sigma \to \pi \circ \phi^\sigma(w)$ is one-to-one since $\gamma(w)$ is not periodic. Let $\sigma_n \to \infty$ denote those positive times for which $\pi \circ \phi^{\sigma_n}(w) \in \mathscr{C}$, and note that $\{\pi \circ \phi^{\sigma_n}(w)\}_{n=1}^{\infty}$ are all distinct. By construction, for all $n \in \mathbb{N}$ we have $\pi \circ \phi^{\sigma_n}(w), \pi \circ \phi^{\sigma_{n+1}}(w) \in U$. Consider the Jordan curve $\mathscr{J}$ consisting of $\pi \circ \phi^{[\sigma_n, \sigma_{n+1}]}(w)$ together with the subinterval of $\mathscr{C}$ with endpoints $\pi \circ \phi^{\sigma_n}(w)$ and $\pi \circ \phi^{\sigma_{n+1}}(w)$. As in the proof of Proposition 4.4.8, the region inside $\mathscr{J}$, is either forward or backward invariant for the flow $\pi \circ \phi^\sigma(w)$, and thus $\mathscr{J}$ separates $\pi\omega(w)$ from $\pi\alpha(w)$. By Lemma 4.4.3 we obtain $\alpha(w) \cap \omega(w) = \varnothing$. The assumptions of Lemma 4.5.2 are satisfied and hence there exists a time σ_*, such that the curve $\pi \circ \phi^\sigma(u)$ ($\sigma \geq \sigma_*$) cannot cross the curve $\pi \circ \phi^\sigma(w)$. In particular it cannot cross $\pi \circ \phi^{[\sigma_n, \sigma_{n+1}]}(w)$, for large n. When $\pi \circ \phi^\sigma(u)$ enters U it crosses $\mathscr{C}$ in the same direction as $\pi(w^*)$, hence $\pi \circ \phi^\sigma(u)$ is either inside or outside $\mathscr{J}$ for all large $\sigma > \sigma_{**} \geq \sigma_*$. Since both $\omega(w)$ and $\alpha(w)$ are contained in $\omega(u)$ the curve $\pi \circ \phi^\sigma(u)$ will have ω-limit points when $\sigma \to \infty$ in both $\pi\alpha(w)$ and $\pi\omega(w)$. These two sets are separated by $\mathscr{J}$ and hence the forward orbit $\pi \circ \phi^\sigma(u), \sigma > \sigma_{**}$ has to cross $\mathscr{J}$. This is a contradiction. ■

Proof of Proposition 4.3.4. See [18, Proposition 2]. Suppose that $\omega(u)$ strictly contains a periodic orbit $\gamma(p)$. Let $V \subseteq X$ be a closed tubular neighborhood of $\gamma(p)$. Choose V small enough such that it does not contain equilibria and such that $\omega(u)$ still has elements outside V. Since there are accumulation points (for $\phi^\sigma(u)$ when σ goes to infinity) both inside and outside V, then $\phi^\sigma(u)$ must enter and leave V infinitely often. Let $\sigma_n \to \infty$ be a sequence such that

$$p = \lim_{n\to\infty} \phi^{\sigma_n}(u)$$

and such that $\phi^\sigma(u)$ leaves V between any two consecutive times σ_n. Let $I_n := [\sigma_n - \alpha_n, \sigma_n + \beta_n]$ be the maximal time interval containing σ_n such that

$$\phi^\sigma(u) \in V \quad \text{for all } \sigma \in I_n.$$

Since ∂V is closed, we may assume convergence (passing to a subsequence, if necessary) of $\phi^{\sigma_n-\alpha_n}(u)$. Note that $\sigma_{n-1} < \sigma_n - \alpha_n$ thus $\sigma_n - \alpha_n \to \infty$. Let

$$q := \lim_{n\to\infty} \phi^{\sigma_n-\alpha_n}(u) \in \omega(u).$$

We have that $q \in \partial V$. Moreover we may assume that $\alpha_n + \beta_n \to \infty$ (at least for a subsequence) since $\omega(u)$ contains a periodic orbit in the interior of V. We have thus

$$\omega(q) \subseteq \mathrm{cl}(\phi^\sigma(q)) \subseteq V, \ \ \sigma > 0.$$

By Proposition 4.3.3 we have that $\gamma(q)$ is periodic. By construction $\gamma(q)$ and $\gamma(p)$ are distinct and $\gamma(q)$ is contained in V. By continuity of the flow and the projection π, and compactness of $V, \pi\gamma(p)$ and $\pi\gamma(q)$ are close to each other with the standard topology of $\mathbb{R}^2$, provided that we take the tubular neighborhood V sufficiently small. From this it follows that $\pi\gamma(q)$ and $\pi\gamma(p)$ are nested closed curves. Reducing V to separate $\gamma(p)$ from $\gamma(q)$, a periodic solution $\gamma(r)$ can be constructed in the same way. Note once more that $\pi\gamma(q), \pi\gamma(p)$ and $\pi\gamma(r)$ are nested closed curves. Applying Lemma 4.5.4 to the trajectories $\gamma(p)$ and $\gamma(q)$ we conclude that there exists a $k \in \mathbb{Z}$ such that

$$W(p^1, q^1) = k,$$

for all $p^1 \in \gamma(p)$ and $q^1 \in \gamma(q)$. By continuity of W (Axiom (A1)) this implies that

$$W(p^1, \phi^{\sigma_n-\alpha_n}(u)) = k,$$

for all $p^1 \in \gamma(p)$ when n is big enough, since $\phi^{\sigma_n-\alpha_n}(u) \to q \in \gamma(q)$. By Assumption (A5) we get $\pi \circ \phi^\sigma(u) \notin \pi\gamma(p)$ for every σ in the open interval with endpoints $\sigma_n - \alpha_n, \sigma_m - \alpha_m$, provided n, m are chosen large enough. Since $\sigma_m - \alpha_m \to \infty$, as $m \to \infty$, it follows that $\pi \circ \phi^\sigma(u) \notin \pi\gamma(p)$, for any σ large enough. In an analogous manner we can prove that, for σ large enough, the curve $\pi \circ \phi^\sigma(u)$ can never intersect $\pi\gamma(q)$ and $\pi\gamma(r)$, but this is a contradiction since $\pi \circ \phi^\sigma(u)$ has ω-limit points as $\sigma \to \infty$ in the three nested curves $\pi\gamma(p), \pi\gamma(q), \pi\gamma(r)$. ■

Finally we are able to prove the following Theorem 4.5.7. Since, by Section 4.2, the Cauchy-Riemann Equations satisfy the Axioms in Section 4.3, Theorem 4.1.2 follows from Propostion 4.5.7.

4.5.7. Proposition. *Let $u \in X$ then*

$$\pi : \omega(u) \to \pi(\omega(u))$$

is a homeomorphism onto its image. Hence $\pi \circ \phi^\sigma$ is a flow on $\pi(\omega(u))$.

Proof. See also [18, Theorem 2]. By Axiom (A5) it is enough to show that there exists a $k_0 \in \mathbb{Z}$ such that

$$W(w^1, w^2) = k_0, \tag{4.23}$$

for all $w^1, w^2 \in \omega(u), w^1 \neq w^2$. We now apply Theorem 4.3.1 (Poincaré-Bendixson). If $\omega(u)$ consists of a single periodic orbit, then (4.23) holds by Lemma 4.5.4. We may therefore assume for the remainder of the proof that for every $w \in \omega(u)$ we have $\alpha(w), \omega(w) \subseteq E$. If either w^1 or w^2 is an equilibrium then (4.23) holds with k_0 defined in Lemma 4.5.6. We may therefore assume that $w^1 \notin E$. Suppose now, for contradiction, that there exist $(w^1, w^2) \in \Sigma \setminus \Delta$. By Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$, such that $\left(\phi^\sigma(w^1), \phi^\sigma(w^2)\right) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^{\sigma'}(w^1), \phi^{\sigma'}(w^2)) < W(\phi^\sigma(w^1), \phi^\sigma(w^2))$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Since $w^1 \in \omega(u)$, there exists $\sigma_n \to \infty$ such that

$$w^1 = \lim_{n\to\infty} \phi^{\sigma_n}(u),$$

and

$$0 < \sigma_{n+1} - \sigma_n \to \infty, \text{ as } n \to \infty.$$

Define $\hat{\sigma}_n := (\sigma_{n+1} - \sigma_n) \to \infty$ then, passing to a subsequence if necessary, the limits

$$e := \lim_{n\to\infty} \phi^{-\hat{\sigma}_n}(\phi^{-\varepsilon}(w^2)) \quad \text{and} \quad \tilde{e} := \lim_{n\to\infty} \phi^{\hat{\sigma}_n}(\phi^{\varepsilon}(w^2))$$

exist, and $e, \tilde{e} \in E$, since $\alpha(w^2) \subseteq E$ and $\omega(w^2) \subseteq E$. By Axiom (A1), Lemma 4.4.1, Lemma 4.5.6 and the fact that $w^1 \notin E$ we infer that, for n sufficiently large (slightly shifting ε if necessary to make W well-defined for all relevant pairs)

$$\begin{aligned}
W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)) &< W(\phi^{-\varepsilon}(w^1), \phi^{-\varepsilon}(w^2)) \\
&= W(\phi^{\sigma_{n+1}-\varepsilon}(u), \phi^{-\varepsilon}(w^2)) \\
&\leq W(\phi^{\sigma_{n+1}-\hat{\sigma}_n-\varepsilon}(u), \phi^{-\hat{\sigma}_n-\varepsilon}(w^2)) \\
&= W(\phi^{\sigma_n-\varepsilon}(u), e) \\
&= W(\phi^{-\varepsilon}(w^1), e) \\
&= W(\phi^{-\varepsilon}(w^1), \tilde{e}) \\
&= W(\phi^{\varepsilon}(w^1), \tilde{e}) \\
&= W(\phi^{\sigma_{n+1}+\varepsilon}(u), \tilde{e}) \\
&= W(\phi^{\hat{\sigma}_n+\sigma_n+\varepsilon}(u), \phi^{\hat{\sigma}_n+\varepsilon}(w^2)) \\
&\leq W(\phi^{\sigma_n+\varepsilon}(u), \phi^{\varepsilon}(w^2)) \\
&= W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)),
\end{aligned}$$

which is a contradiction. In the sixth and in the seventh (in)equality we have used Lemma 4.5.6. ■

4.6 Proofs of Propositions 4.2.1 and 4.2.3

Consider the operators

$$\partial = \partial_s - J\partial_t \text{ and } \overline{\partial} = \partial_s + J\partial_t,$$

and recall the following regularity estimates:

4.6.1. Lemma. *Let g be a function in $\in C_c^\infty(\mathbb{R} \times S^1; \mathbb{R}^2)$. For every $1 < p < \infty$, there exists a constant $C_p > 0$, such that*

$$||\nabla g||_{L^p(\mathbb{R}\times S^1)} \leq C_p ||\overline{\partial} g||_{L^p(\mathbb{R}\times S^1)}. \tag{4.24}$$

The same estimate holds for ∂ via $t \mapsto -t$.

Proof. See [4], [15] [31], [36, appendix B]. ■

Proof of Proposition 4.2.1. For a solution $u \in X$, we can write

$$\overline{\partial} u = -JF(t, u) = f(s, t), \tag{4.25}$$

where F, and therefore f, are uniformly bounded since for every $u \in X$ we have $|u(s,t)| \leq 1$ for all $(s,t) \in \mathbb{R} \times S^1$, i.e. u satisfies the a priori estimate

$$||u||_{L^\infty(\mathbb{R}\times S^1)} \leq 1. \tag{4.26}$$

Extend f and u via periodic extension to a function on $\mathbb{R}^2$ in the t-direction. By (4.26) we obtain the existence of a constant $M > 0$, such that

$$||f||_{L^\infty(\mathbb{R}^2)} \leq M. \tag{4.27}$$

We use (4.24) to obtain the interior regularity estimates for the Cauchy-Riemann operators.

Let K, L, G be compact sets contained in $\mathbb{R}^2$ such that $K \Subset L \Subset G \subset \mathbb{R}^2$, and let ε be positive such that $\varepsilon < \operatorname{dist}(L, \partial G)$. By compactness, L can be covered by finitely many open balls of radius $\varepsilon/2$:

$$L \subset \bigcup_{i=1}^{N_\varepsilon} B_{\varepsilon/2}(x_i).$$

Consider a partition of unity $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots,N_\varepsilon}$ on L subordinate to $\{B_\varepsilon(x_i)\}_{i=1,\dots,N_\varepsilon}$. In particular the supports of ρ_{ε,x_i} are contained in $B_\varepsilon(x_i)$, for every $i = 1\dots N_\varepsilon$. Then, for every u, every small $\varepsilon > 0$ and every $i = 1\dots N_\varepsilon$, the function $v_{\varepsilon,i} := \rho_{\varepsilon,x_i} u$ belongs to $W_0^{k,p}(\mathbb{R}^2)$, for every $p \geq 1$, and every $k \in \mathbb{N}$. Using the Poincaré inequality and Lemma 4.6.1 we get (with C changing from line to line)

$$\begin{aligned} ||v_{\varepsilon,i}||_{W^{1,p}(\mathbb{R}^2)} &= ||v_{\varepsilon,i}||_{W^{1,p}(B_\varepsilon(x_i))} \leq C||v_{\varepsilon,i}||_{W_0^{1,p}(B_\varepsilon(x_i))} \\ &\leq C||\overline{\partial} v_{\varepsilon,i}||_{L^p(B_\varepsilon(x_i))} \\ &\leq C||\rho_{\varepsilon,x_i}\overline{\partial} u||_{L^p(B_\varepsilon(x_i))} + C||u\overline{\partial}\rho_{\varepsilon,x_i}||_{L^p(B_\varepsilon(x_i))} \\ &\leq C||\overline{\partial} u||_{L^p(G)} + C||u||_{L^p(G)}. \end{aligned} \tag{4.28}$$

As $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots,N_\varepsilon}$ is a partition of unity it follows that

$$||u||_{W^{1,p}(L)} = \left|\left|\sum_{i=1}^{N_\varepsilon} v_{\varepsilon,i}\right|\right|_{W^{1,p}(L)} \leq \sum_{i=1}^{N_\varepsilon} ||v_{\varepsilon,i}||_{W^{1,p}(B_\varepsilon(x_i))}. \tag{4.29}$$

Putting together (4.28) and (4.29) we obtain

$$||u||_{W^{1,p}(L)} \leq C_{p,L,G}\left(||\overline{\partial} u||_{L^p(G)} + ||u||_{L^p(G)}\right). \tag{4.30}$$

Using (4.30) and (4.25), (4.26) and (4.27) we obtain

$$||u||_{W^{1,p}(L)} \leq C_{p,L,G}\left(||f||_{L^p(G)} + ||u||_{L^p(G)}\right) \leq C^1_{p,L,G} < \infty, \tag{4.31}$$

where the constant $C^1_{p,L,G}$ depends on p, L, G, but not on u. Therefore, we have that if we take a sequence $\{u_n\} \subset X$ then, from (4.31), u_n is uniformly bounded in $W^{1,p}(L)$. It follows from the Sobolev compact embedding

$$W^{1,p}(L) \hookrightarrow C^0(\overline{L_0}), \tag{4.32}$$

for every $L_0 \Subset L$ (see e.g. [5, theorem 6.3 part II]) that u_n has a converging subsequence in $C^0_{\mathrm{loc}}(L_0)$. Since this holds for every $L \subset \mathbb{R}^2$, we obtain that u_n converges (up to a subsequence) in X to a continuous u. We still need to prove that the limit u solves Equation (4.2). To prove that $u \in X$ we will show that the convergence of u_n to u is stronger, more precisely C^1_{loc}. In order to obtain further regularity, we consider a partition of unity of $K \Subset L$, which we call again $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots N_\varepsilon}$, where now $0 < \varepsilon < \mathrm{dist}(K, \partial L)$. On balls $B_\varepsilon(x_i)$ we obtain

$$\begin{aligned} ||\rho_{\varepsilon,x_i} u||_{W^{2,p}(B_\varepsilon)} &\leq C||\rho_{\varepsilon,x_i} u||_{W_0^{2,p}(B_\varepsilon(x_i))} \leq C||\overline{\partial}(\rho_{\varepsilon,x_i} u)||_{W^{1,p}(B_\varepsilon(x_i))} \\ &\leq C\left(||\rho_{\varepsilon,x_i}\overline{\partial}u||_{W^{1,p}(B_\varepsilon(x_i))} + ||u\overline{\partial}\rho_{\varepsilon,x_i}||_{W^{1,p}(B_\varepsilon(x_i))}\right) \\ &\leq C\left(||\overline{\partial}u||_{L^\infty(L)} + ||\overline{\partial}u||_{W^{1,p}(L)} + ||u||_{L^\infty(L)} + ||u||_{W^{1,p}(L)}\right). \end{aligned}$$

As in (4.29), using (4.25) we obtain

$$||u||_{W^{2,p}(K)} \leq \tilde{C}_{p,K,L,G}\left(||f||_{L^\infty(L)} + ||f||_{W^{1,p}(L)} + ||u||_{L^\infty(L)} + ||u||_{W^{1,p}(L)}\right). \tag{4.33}$$

To estimate the three terms $||f||_{L^\infty(L)}$, $||u||_{L^\infty(L)}$ and $||u||_{W^{1,p}(L)}$ we use respectively (4.27), (4.26), and (4.31). In order to estimate $||f||_{W^{1,p}(L)}$ we differentiate the smooth vector field F and we obtain

$$\begin{aligned} f_s(s,t) &= (F(t,u))_s = D_{t,u}X(t,u)(0,u_s) \\ f_t(s,t) &= (F(t,u))_t = D_{t,u}X(t,u)(1,u_t). \end{aligned}$$

Both right hand sides lie in $L^p(L)$, and hence also $Df = (f_s, f_t)$ is in $L^p(L)$. From this it follows, by using (4.33), that there exists a constant $C^2_{p,K,L,G}$, dependent on p, K, L, G and uniform in u, such that

$$||u||_{W^{2,p}(K)} \leq C^2_{p,K,L,G} < \infty,$$

for compact domains $K \Subset L \Subset G$. By the compact Sobolev immersion

$$W^{2,p}(K) \hookrightarrow C^1(\overline{K_0}), \tag{4.34}$$

for every $K_0 \Subset K$ [5, theorem 6.3 part II] we get, by choosing $p > 2$, and passing to a subsequence that the limit $u \in X$. ■

Proof of Proposition 4.2.3. As in the proof of Lemma 4.4.3 if suffices to show that ι is injective. Suppose there exist $u_1, u_2 \in X$ such that $\iota(u_1) = \iota(u_2)$. By definition of ι we have

$$u_1(0, \cdot) = u_2(0, \cdot). \tag{4.35}$$

Define $v(s,t) := u_1(s,t) - u_2(s,t)$, for all $(s,t) \in \mathbb{R} \times S^1$. By (4.35) we have $v(0,t) = 0$ for all $t \in S^1$. By smoothness of the vector field F we can write

$$F(t, u_1) = F(t, u_2) + R(t, u_1, u_2 - u_1)(u_2 - u_1),$$

where R_1 is a smooth function of its arguments. Upon substitution this gives

$$v_s - Jv_t + A(s,t)v = 0, \quad v(0,t) = 0 \quad \text{for all } t \in S^1, \tag{4.36}$$

and $A(s,t) = R(t, u_1(s,t), v(s,t))$ is (at least) continuous on $\mathbb{R} \times S^1$. Evaluating (4.36) in $t = 0$ we obtain, in particular,

$$v_s - Jv_t + A(s,t)v = 0, \quad v(0,0) = 0 \tag{4.37}$$

Introducing complex coordinates $z := s + it$, (4.37) becomes

$$\partial_{\overline{z}} v + A(z)v = 0, \quad v(0) = 0, \tag{4.38}$$

where the operator $\partial_{\overline{z}} := \partial_s - i\partial_t$ is the standard anti-holomorphic derivative and we have used the identification between the complex structure J in $\mathbb{R}^2$ and i in $\mathbb{C}$. Multiplying (4.38) by $e^{\int_0^z A(\zeta)d\zeta}$ and defining

$$w(z) := e^{\int_0^z A(\zeta)d\zeta} v(z),$$

we obtain

$$\partial_{\overline{z}} w = 0, \quad w(0) = 0.$$

which means that w is analytic. This implies that either 0 is an isolated zero for w, or there exists a $\delta > 0$, such that $w(z) = 0$, on $U_\delta := \{z \in \mathbb{C} : |z| \leq \delta\}$. Because of (4.36) we conclude that 0 cannot be an isolated zero for w, hence $w \equiv 0$ in $U_\delta := \{z \in \mathbb{C} : |z| \leq \delta\}$. Repeating these arguments we obtain that $w(s,t) = 0$ for all $(s,t) \in \mathbb{R} \times S^1$ and hence $v \equiv 0$. This implies $u_1 = u_2$, which concludes the proof. ∎

4.6.2. Remark. The same proof can be carried out in case J is a smooth map $\mathbb{R} \times S^1 \to \mathrm{Sp}(2, \mathbb{R})$ such that $J^2 = -\mathrm{Id}$ (i.e. J is an almost complex structure and $\mathrm{Sp}(2, \mathbb{R})$ denotes the symplectic group of degree 2 over $\mathbb{R}$). In this case one can prove that the equation $u_s - J(s,t)(u_t - F(t,u)) = 0$ can be tranformed into (4.2) using [31, Theorem 12, Appendix A.6].

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Samenvatting (Dutch Summary)

Braid invarianten voor niet-lineaire differentiaalvergelijkingen

In dit proefschrift worden topologische eigenschappen en invarianten van speciale typen niet-lineaire partiële differentiaalvergelijkingen onderzocht welke gebruikt worden voor het vinden van oplossingen in bepaalde 'braid' klassen van krommen — krommen met de structuur van een vlechtwerk. De belangrijkste resultaten kunnen in drie punten worden samengevat:

- een uitbreiding van de Poincaré-Hopf stelling voor braid klassen;
- de constructie van een isomorfisme tussen Floer homologie en Morse homologie voor braid klassen;
- een generalisatie van de Poincaré-Bendixson stelling voor niet-lineaire Cauchy-Riemann vergelijkingen.

Periodieke banen van 1-periodieke Hamilton vectorvelden op de eenheidsbol $\mathbb{D}^2$ kunnen als braids (vlechtwerken) worden beschouwd. Voor bepaalde braid klassen kan er een algebraïsch topologische invariant worden gedefinieerd, de Floer homologie van de braid klasse, zie [49]. Als de Floer homologie van een braid klasse niet triviaal is, worden via Morse theorie aanvullende periodieke banen van de Hamilton vergelijkingen geforceerd.

Het eerste resultaat in Hoofdstuk 2 toont aan dat de Euler karakteristiek van de Floer homologie het bestaan van periodieke banen en periodieke punten voor willekeurige vectorvelden en diffeomorfismen geeft. Dit resultaat kan worden geformuleerd in termen van een Poincaré-Hopf stelling. Floer homologie en dus ook de bijbehorende Euler-Floer karakteristiek zijn abstract gedefinieerd en moeilijk te berekenen. We beschrijven een methode hoe de Euler-Floer karakteristiek voor een willekeurige relatieve braid klassen berekend kan worden via een (eindig) simplicial complex.

In Hoofdstuk 3 geven we de definitie van de Morse homologie voor braids en het isomorfisme naar de bijbehorende Floer homologie. De constructie wordt uitgevoerd met behulp van verschillende technieken en met de keuze van speciale Hamilton functies. Dit is een eerste stap richting een isomorfisme tussen de Floer homologie en de Conley index voor relatieve braid klassen. Op het niveau van Euler karakteristieken is dit reeds uitgevoerd in Hoofdstuk 2. Een isomorfisme naar de Conley index geeft een soortgelijke berekenbaarheid van de Floer homologie via simplicial complexes.

Hoofdstuk 4 betreft de karakterisering van het asymptotisch gedrag van de Cauchy-Riemann vergelijkingen. Fiedler en Mallet-Paret (cf. [18]) bewijzen een versie van de klassieke Poincaré-Bendixson stelling voor scalaire parabolische vergelijkingen. Wij tonen aan dat een vergelijkbaar resultaat geldt voor begrensde oplossingen van de niet-lineaire Cauchy-Riemann vergelijkingen. Om dit laatste te bereiken, bewijzen we een abstracte Poincaré-Bendixson stelling voor flows op compacte Hausdorff ruimten die een discrete Lyapunov functie hebben. Deze stelling is een generalisatie van het resultaat van Fiedler en Mallet-Paret. De stelling van Fiedler en Mallet-Paret kan niet toegepast worden op de Cauchy-Riemann vergelijkingen.