

European Options Sensitivities via Monte Carlo Techniques

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Abstract

This paper proposes a unified approach to Monte Carlo estimation of sensitivity of European option premiums with respect to some arbitrary parameter. The classical framework assumes that the underlying parameter is some intrinsic parameter of the model, e.g., interest rate, volatility or time to maturity, in which case sensitivities are also known as "Greeks". Intrinsic parameters only induce variability in the dynamic of the stock-price(s) under consideration. The present approach allows the parameter under discussion to induce variability in the payoff function and also in the exercise rule of the option. Our leading examples come from the family of the so-called *digital options*, i.e., financial options which pay off some (apriori) fixed amount of money provided that the stock-price(s) at maturity lie in some certain region, such as (finite intersections of) polyhedra and/or spheres, in the n -dimensional space, where $n \geq 1$ denotes the number of underlying assets. For such options, the payoff and the exercise rule can be chosen independently. This approach essentially relies on differentiation of multiple integrals with parameter and appropriate formulas are established in a rather general setting. In our leading examples the underlying parameter dictates the exercise rule, i.e., integrals on moving domains have to be considered, and a direct appeal to surface integrals must be made. General Monte Carlo integration techniques for evaluating such integrals will be presented and illustrated by some examples. Finally, the connection between the main results of this paper and the concept of *weak differentiation* will be discussed.

Introduction

A financial option is a contract written by a seller that conveys to the buyer the right, but not the obligation, to buy or to sell a particular asset, shares of stock or some other underlying security at some maturity time T , or earlier. In return for granting this option, the seller collects a payment (the premium) from the buyer. Option pricing, which is determining the fair premium to be paid for such an option in an arbitrage-free market, along with hedging, is one of the key topics in mathematical finance. For risk-managers, however, equally important is to evaluate the sensitivity of the option premium w.r.t. various parameters such as volatility, interest rate, maturity time or strike price. Sensitivities of option premiums are known in the literature as Greeks (they are denoted by Greek letters) and, due to their importance, they have received very much attention in the mathematical finance literature in the last years.

By the most general definition, an option is characterized by a non-negative random variable Z , completely determined by the evolution of stock prices up to maturity T , which relates the profit of its holder to the (evolution of) stock prices. The random variable Z can be, for instance, a function of the stock price at time T (the simplest case) and, may involve several stock prices at some specified intermediate times or, in the most general case, it may depend on the whole evolution (sample path) of stock prices up to time T . Since an option modeled by Z brings a profit of Z units at maturity, its fair premium is given by the discounted expected value of Z , i.e.,

$$V := e^{-rT} \mathbb{E}[Z], \quad (1)$$

with the interest rate r assumed to be constant in the time interval $[0, T]$.

A wide range of financial options, having maturity T , are described via a *payoff function* $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ and a *feasibility function* $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$, as follows: the option entitles its holder to gain a profit of

$$Z = z(\mathbf{S}) = \varphi(\mathbf{S}) \mathbf{1}_{\{\psi(\mathbf{S}) > 0\}} \quad (2)$$

units by exercising the option at time T , where \mathbf{S} is a n -dimensional vector representing the stock prices of n underlying assets at time T . In words, the option holder will obtain a profit specified by the payoff function φ , conditioned on the fact that the vector of stock-prices \mathbf{S} lies within a pre-specified region in \mathbb{R}^n , called *feasibility domain*, at maturity time T . Therefore, the value of such an option is given by

$$V = e^{-rT} \mathbb{E}[z(\mathbf{S})] = e^{-rT} \mathbb{E}[\varphi(\mathbf{S}) \mathbf{1}_{\{\psi(\mathbf{S}) > 0\}}]. \quad (3)$$

Equation (2) suggests that we are considering financial options which bring a profit which can be expressed as a (measurable) function z of the terminal stock price \mathbf{S} . Since we are investigating differentiability properties of $\mathbb{E}[z(\mathbf{S})]$ w.r.t. a parameter which appears either in the expression of z or in the distribution of \mathbf{S} , generically denoted by θ , it is desirable to make some suitable assumptions over the function z . Assuming z to be a smooth function (in all variables) would be too restrictive since it does not cover any realistic model. As it will be shown later in this paper, it is more convenient to consider functions z which can be written as products between a smooth function and the indicator function of a

smooth domain. This is essentially the reason why we are considering functions z which can be expressed by means of payoff and feasibility functions such as in (2). For the time being we do not make any specific assumption over the parameter θ . Later in this paper we will distinguish between the case when θ is a parameter of the distribution of \mathbf{S} (intrinsic parameter) and the case when θ is a parameter of the payoff and/or feasibility functions (non-intrinsic).

Typically, the payoff function \wp and the feasibility function ψ agree since, for many usual financial options, the profit can be expressed as

$$\max\{\wp(\mathbf{S}), 0\} = \wp(\mathbf{S})\mathbf{1}_{\{\wp(\cdot) > 0\}}(\mathbf{S}).$$

However, this is not always the case since there are options for which \wp and ψ may be different. Instances of such options are the so-called *binary (digital)* options, e.g., the *asset/cash-or-nothing* options (AON/CON). The payoff of the AON option is a function of \mathbf{S} while for the CON option the payoff is constant. Both options, however, pay off only if the vector of stock-prices \mathbf{S} lies within a certain region in \mathbb{R}^n at the maturity time. Other (more complex) examples are *barrier* and *ladder* options which differ from the AON/CON options in that the payoff depends upon the whole evolution (is a path-functional) of the stock prices up to maturity time and not only on its terminal value \mathbf{S} .

Apart from a few well known types of options in a Black-Scholes market, option prices (and, consequently, Greeks) can be very rarely obtained in closed form expressions. Since the option price is, up to a multiplicative constant, given by an expectation of a certain random variable, Monte Carlo simulation seems to be the only reasonable way to estimate Greeks, since numerical methods involving numerical integration and/or algorithmic PDE solving become infeasible for large n . There are, however, several long established methods for estimating the sensitivity of V with respect to some specified parameter and in what follows we give a brief overview of these methods.

Most popular methods rely on approximating the derivatives via finite-differences (FD). That is, one uses the following approximation (for $\varepsilon \rightarrow 0$):

$$\frac{dV}{d\theta} \approx \frac{V(\theta + \varepsilon) - V(\theta - \varepsilon)}{2\varepsilon};$$

see, e.g., [15]. Note that this method requires re-simulation since both $V(\theta + \varepsilon)$ and $V(\theta - \varepsilon)$ are obtained by simulation. Another pitfall of the FD method is that, although the estimates converge, as $\varepsilon \rightarrow 0$, to the derivative $V'(\theta)$ when V is differentiable in θ , there is no indication on how small (close to 0) ε should be and this is directly influencing unbiasedness of the estimate.

Given the shortcomings of the FD method, two so-called direct methods, i.e., no re-simulation is needed, known as *infinitesimal perturbation analysis* (IPA) and *score-function method* (SF), respectively, were proposed; see, e.g., [2]. The IPA method essentially requires path-wise differentiation while the SF method requires differentiation of the density. Both methods lead to unbiased gradient estimates. However, IPA method is applicable only when the profit function Z is Lipschitz continuous w.r.t. θ , which is not the case in (3) since indicator functions induce discontinuities on the boundary of the corresponding sets. By differentiating the density, one can cope with this problem when the boundary $\{\mathbf{s} \in \mathbb{R}^n : \psi(\mathbf{s}) = 0\}$ does not depend on the parameter of interest and it is not charged by the distribution of \mathbf{S} ; see [10]. Therefore, the SF method can be

applied. Unfortunately, in many applications the feasibility domain depends also on the parameter of interest, as we will illustrate by means of several examples, and this is where the methods enumerated above show their limitation. For a detailed overview of these methods we refer to [5].

Estimation of Greeks can also be achieved by means of *Malliavin weighting function* (MF). This is a quite modern technique, based on Malliavin calculus, and determines a class of weighting functions which provide gradient estimates for the Greeks when multiplied with the payoff function. For some pioneering work on Malliavin Greek estimation we refer to [4]. The MF approach is essentially an extension of the SF method. More specifically, it has been shown in [1] that the score function appears as the Malliavin weighting function which induces the smallest total variance. Nevertheless, like SF method, the MF method does not cope with the case when the boundary of the feasibility set varies with respect to the parameter of interest.

In the following we propose a solution to estimate the sensitivity of V in (3) under some basic assumptions such as continuous differentiability of the mappings φ and ψ . To explain the reasoning behind this method we will address in the following the concept of *weak differentiation* of measures; see, e.g., [10] and we will embed the problem into a more general one.

Assume that μ_θ is a probability distribution on \mathbb{R} having density $f(\theta, x)$, for $x \in \mathbb{R}$, and X is a random variable distributed according to μ_θ . Then, if g is a bounded and continuous function, we have

$$\frac{d}{d\theta} \mathbb{E}_\theta[g(X)] = \frac{d}{d\theta} \int g(x) f(\theta, x) dx = \int g(x) \frac{\partial f}{\partial \theta}(\theta, x) dx, \quad (4)$$

provided that f is continuous and its derivative $\frac{\partial f}{\partial \theta}(\theta, x)$ exists for almost all x and is uniformly (w.r.t. θ) bounded by a Lebesgue integrable function $g(x)$. At this point one can still use the SF method; indeed, defining the score

$$L_\theta(x) := \frac{\partial}{\partial \theta} (\ln f(\theta, x)) = \frac{1}{f(\theta, x)} \cdot \frac{\partial f}{\partial \theta}(\theta, x),$$

we see that L_θ is defined almost everywhere and, according to (4), satisfies

$$\frac{d}{d\theta} \mathbb{E}_\theta[g(X)] = \mathbb{E}_\theta[L_\theta(X)g(X)].$$

On the other hand, by denoting $\mu'_\theta(dx) := \frac{\partial f}{\partial \theta}(\theta, x) dx$, i.e., the signed measure with density $\frac{\partial f}{\partial \theta}$, we obtain the following identity

$$\frac{d}{d\theta} \mathbb{E}_\theta[g(X)] = \frac{d}{d\theta} \int g(x) \mu_\theta(dx) = \int g(x) \mu'_\theta(dx). \quad (5)$$

The signed measure μ'_θ is called the *weak derivative* of μ_θ . It is unique and in general it can be written as the re-scaled difference of two probability measures, which makes (5) suitable for Monte Carlo simulation. This is known as the *measure-valued differentiation* (MVD) technique and it also leads to unbiased gradient estimation; see, e.g., [6, 7, 9, 10, 11]. Indeed, we have

$$\frac{d}{d\theta} \mathbb{E}_\theta[g(X)] = c_\theta \mathbb{E}_\theta [g(X^+) - g(X^-)],$$

provided that $\mu'_\theta = c_\theta(\mu_\theta^+ - \mu_\theta^-)$, where $c_\theta > 0$, μ_θ^\pm are probability measures and X^\pm are random variables distributed according to μ_θ^\pm , respectively.

The theoretical part of the present work is motivated by the following simple, but intriguing, observation. It is still possible to have a representation as in (5) even when f is not continuous, e.g., $f(\theta, x)$ is, as a function of x , the indicator function of some θ -dependent domain multiplied by a some normalization constant, i.e., some expression depending on θ , but not on x , such that μ_θ is a probability distribution. In this case, even though the partial derivative $\frac{\partial f}{\partial \theta}$ exists almost everywhere, (4) fails to hold true. A standard example is that of the uniform distribution on $[0, \theta]$, i.e.,

$$f(\theta, x) = \theta^{-1} \mathbf{1}_{[0, \theta]}(x), \quad \frac{\partial f}{\partial \theta}(\theta, x) = -\frac{1}{\theta^2} \mathbf{1}_{[0, \theta]}(x), \quad \text{a.e.}$$

Since the derivative $\frac{\partial f}{\partial \theta}$ exists a.e., a naive application of (4) would lead to

$$\frac{d}{d\theta} \left(\frac{1}{\theta} \int_0^\theta g(x) dx \right) = -\frac{1}{\theta^2} \int_0^\theta g(x) dx,$$

which, in general, is not true. In fact, by the Lebesgue integral rule, we obtain

$$\frac{d}{d\theta} \left(\frac{1}{\theta} \int_0^\theta g(x) dx \right) = \frac{1}{\theta} g(\theta) - \frac{1}{\theta^2} \int_0^\theta g(x) dx. \quad (6)$$

Therefore, the SF method is of no help in this situation but (6) shows that

$$\mu'_\theta = \frac{1}{\theta} (\delta_\theta - \mu_\theta), \quad (7)$$

where δ_θ denotes the Dirac distribution assigning mass 1 in θ , i.e., (5) holds true for μ'_θ defined by (6). Therefore, the MVD method still applies here. As a general rule, we conclude that weak derivatives of probability measures can be evaluated by simply differentiating the density $f(\theta, x)$ w.r.t. θ when the support of the distribution μ_θ does not depend on θ and by using the Lebesgue integral rule when the support of μ_θ depends on θ ; see [10]. In the first case the weak derivative μ'_θ is absolutely continuous w.r.t. the Lebesgue measure and both SF and MVD are applicable while in the latter case we obtain a singular component of μ'_θ and only MVD works.

For practical reasons one would be interested in investigating weak differentiability of multivariate, rather than univariate, distributions μ_θ . While the extension of the “density differentiation method” to multivariate distributions is straightforward, the generalization and interpretation of (6) in the multidimensional case is not quite clear and part of the present work is aimed to clarify this issue. In that sense, note that the uniform distribution presented above belongs to the class of so-called *truncated distributions*, i.e., probability measures which appear as conditional distributions induced by some measures on subsets of \mathbb{R} . In our case, the Lebesgue measure on \mathbb{R} is conditioned on the interval $[0, \theta]$. Moreover, θ appears as the only boundary point of $[0, \theta]$ which varies with θ . Now the intuition becomes somewhat clearer. Specifically, we can postulate that the weak derivative of a truncated distribution appears as a linear combination between the truncated distribution itself and some distribution supported on the “variable boundary” of the conditioning set.

Similar issues have been tackled in [12] and [14]. In [12] a measure-theoretic approach has been proposed for measures concentrated on n -dimensional polyhedra with variable (flat) boundaries while in [14] a pure analytical method is used to derive gradients of integrals on variable multidimensional domains. Here we propose a unified, general approach to this problem. A first step will be to establish a multidimensional counterpart for the Lebesgue integral rule. The formalism is as follows: we assume that \mathfrak{D}_θ is a piecewise smooth θ -variable domain, i.e., a finite intersection of smooth domains, in \mathbb{R}^n . To extend (6) to a multi-dimensional setting, let ℓ denote the Lebesgue measure on \mathbb{R}^n and define

$$\mu_\theta(d\mathbf{x}) = \frac{1}{\ell(\mathfrak{D}_\theta)} \mathbf{1}_{\mathfrak{D}_\theta}(\mathbf{x}) d\mathbf{x}, \quad f(\theta, \mathbf{x}) = \frac{g(\mathbf{x})}{\ell(\mathfrak{D}_\theta)}.$$

Then (6) generalizes to

$$\begin{aligned} \frac{d}{d\theta} \int g(\mathbf{x}) \mu_\theta(d\mathbf{x}) &= \frac{d}{d\theta} \int_{\mathfrak{D}_\theta} f(\theta, \mathbf{x}) d\mathbf{x} \\ &= \int_{\mathfrak{D}_\theta} \frac{\partial f}{\partial \theta}(\theta, \mathbf{x}) d\mathbf{x} + \partial_\theta \ell(\mathfrak{D}_\theta) \cdot \int_{\mathfrak{B}_\theta} f(\theta, \mathbf{x}) \varsigma_\theta(d\mathbf{x}), \end{aligned} \quad (8)$$

where, for simplicity, we denote by ∂_θ the differentiation w.r.t. θ , \mathfrak{B}_θ denotes the boundary of \mathfrak{D}_θ and ς_θ a distribution supported on the boundary \mathfrak{B}_θ . In general, \mathfrak{B}_θ is a k -dimensional smooth manifold (in general $k < n$; typically $k = n - 1$), i.e., there exist a one-to-one parametrization of \mathfrak{B}_θ with k -dimensional parameter¹, so that ς_θ is absolutely continuous with respect to the push-forward of the k -dimensional Lebesgue measure through the parametrization mapping. In accordance with the definition of the weak derivative, (8) reads

$$\mu'_\theta = \frac{\partial_\theta \ell(\mathfrak{D}_\theta)}{\ell(\mathfrak{D}_\theta)} \cdot (\varsigma_\theta - \mu_\theta) \quad (9)$$

and the analogy between (7) and (9) is clear now. However, it is not clear yet how to describe/determine the component ς_θ and this will be an important part of the technical work in this paper.

The formula in (8) can be regarded as a general differentiation formula for non-continuous, integrable functions and can be related to the concept of derivatives of generalized functions (distributions) in functional analysis. Then the product rule for generalized derivatives applies and yields

$$\partial_\theta \mathbf{1}_{\mathfrak{D}_\theta} = \partial_\theta (\ell(\mathfrak{D}_\theta) \cdot \mu_\theta) = \partial_\theta \ell(\mathfrak{D}_\theta) \cdot \mu_\theta + \ell(\mathfrak{D}_\theta) \cdot \mu'_\theta = \partial_\theta \ell(\mathfrak{D}_\theta) \cdot \varsigma_\theta,$$

i.e., $\partial_\theta \ell(\mathfrak{D}_\theta) \cdot \varsigma_\theta$ appears as the generalized derivative of the indicator $\mathbf{1}_{\mathfrak{D}_\theta}$.

Our next target will be to explain how the result can be adapted to simulation and we illustrate, by means of some examples, how to evaluate (simulate) the last integral appearing in (8). More specifically, we will show that one can evaluate surface integrals in \mathbb{R}^n by writing

$$\int g(\mathbf{x}) \varsigma_\theta(d\mathbf{x}) = \mathbb{E}_\theta[g(\vartheta(Y))],$$

¹Such k is unique. Manifolds of dimension 0 are simple points and we regard the Dirac distribution as the 0-dimensional Lebesgue measure.

where $\mathbf{x} = \vartheta(\mathbf{y})$ is a suitable parametrization of the boundary \mathfrak{B}_θ and Y has uniform distribution on some k -dimensional manifold.

Eventually, we apply the results to estimation of option price's sensitivities. For instance, if the vector of stock prices in some financial market model has the density $\rho(\theta, \mathbf{x})$ on \mathbb{R}^n then the premium of some option specified by payoff function $\varphi(\theta, \cdot)$ and feasibility function $\psi(\theta, \cdot)$ is given, according to (3), by

$$V = e^{-rT} \int \varphi(\theta, \mathbf{x}) \rho(\theta, \mathbf{x}) \mathbf{1}_{\mathfrak{D}_\theta}(\mathbf{x}) d\mathbf{x},$$

where $\mathfrak{D}_\theta := \{\mathbf{x} \in \mathbb{R}^n : \psi(\theta, \mathbf{x}) > 0\}$, i.e., $\mathfrak{B}_\theta = \{\mathbf{x} \in \mathbb{R}^n : \psi(\theta, \mathbf{x}) = 0\}$. Again, by the product rule, we obtain for the θ -sensitivity of V :

$$\partial_\theta V = e^{-rT} \cdot \int_{\mathfrak{D}_\theta} \partial_\theta \varphi(\theta, \mathbf{x}) \rho(\theta, \mathbf{x}) + \varphi(\theta, \mathbf{x}) \partial_\theta \rho(\theta, \mathbf{x}) d\mathbf{x} \quad (10)$$

$$+ e^{-rT} \cdot \partial_\theta \ell(\mathfrak{D}_\theta) \int_{\mathfrak{B}_\theta} \varphi(\theta, \mathbf{x}) \rho(\theta, \mathbf{x}) \varsigma_\theta(d\mathbf{x}). \quad (11)$$

Note that the integral in (10) arises from the classical product differentiation rule of the integrand in the expression of V while the integral in (11) appears due to variability of the domain \mathfrak{D}_θ w.r.t. the parameter θ .

The paper is organized as follows: in Section 1 we provide some basic concepts and notations which will be used throughout this paper. Section 2 provides a general theory for differentiation of integrals on variable domains with smooth boundaries. Some applications to financial options are presented in Section 3. An overview of relevant results regarding Gaussian vectors and their distributions is given in the Appendix.

1 Preliminaries and Notations

To present our analysis we will use the following notation and terminology. We consider the n -dimensional Euclidian space \mathbb{R}^n on which \mathbb{R}_+^n denotes the cone of *positive elements*, i.e.,

$$\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, \forall 1 \leq i \leq n\}.$$

On \mathbb{R}^n we denote by $\mathbf{0}$ the null vector, i.e., $\mathbf{0} := (0, \dots, 0)$ and we say that the vector \mathbf{v} is positive (in notation: $\mathbf{v} > \mathbf{0}$) if $\mathbf{v} \in \mathbb{R}_+^n$. This induces the following order relation on \mathbb{R}^n : $\mathbf{x} < \mathbf{y}$ if $\mathbf{y} - \mathbf{x} > \mathbf{0}$. For $\mathbf{x} := (x_1, \dots, x_n)$, $\mathbf{y} := (y_1, \dots, y_n) \in \mathbb{R}^n$ we denote their Euclidian product by $\langle \cdot | \cdot \rangle$. In formula,

$$\langle \mathbf{x} | \mathbf{y} \rangle := x_1 y_1 + \dots + x_n y_n.$$

Moreover, we denote by $|\mathbf{x}|$ the Euclidean norm of $\mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$, i.e.,

$$|\mathbf{x}| := \sqrt{\langle \mathbf{x} | \mathbf{x} \rangle} = \sqrt{x_1^2 + \dots + x_n^2}.$$

For $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$ we denote by $\vec{\mathbf{x}}$ the direction of \mathbf{x} , i.e., $\vec{\mathbf{x}} := |\mathbf{x}|^{-1} \mathbf{x}$.

For a function $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ we denote by $\partial f / \partial x_i$, for $1 \leq i \leq n$, the partial derivative with respect to the coordinate x_i ; on several occasions we will use alternative notations such as $\partial_{x_i} f$ or the short-hand notation $\partial_i f$, when the

order of the arguments is clear. If the partial derivatives $\partial f/\partial x_i$ exist for each i we will use the notation ∇f , or $\nabla_{\mathbf{x}}f$ when f depends on multiple arguments, for the gradient of f , i.e.,

$$\nabla f = \nabla_{\mathbf{x}}f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = (\partial_{x_1}f, \dots, \partial_{x_n}f) = (\partial_1f, \dots, \partial_nf).$$

A mapping $\Phi : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ will be called a vector field and it is usually defined via its coordinate mappings $\phi_i : \Omega \rightarrow \mathbb{R}$, for $1 \leq i \leq m$. The vector field Φ is differentiable if all its coordinates ϕ_i are Fréchet differentiable. If $\Phi := (\phi_1, \dots, \phi_m) : \Omega \rightarrow \mathbb{R}^m$ is a differentiable vector field we denote by Φ' its differential, which can be identified as the matrix $\Phi' := [\partial\phi_i/\partial x_j]_{1 \leq i \leq m, 1 \leq j \leq n}$ obtained from the m row-vectors $\{\nabla\phi_i\}_{1 \leq i \leq m}$. For $m = n$, the Jacobian of Φ in \mathbf{x} is defined as the determinant of $\Phi'(\mathbf{x})$ and we adopt the following notation:

$$\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(x_1, \dots, x_n)} := \det \Phi'.$$

If $W \subset \mathbb{R}^n$ we call the mapping $\Phi : \Omega \rightarrow W$ a diffeomorphism if it is bijective and both Φ and its inverse Φ^{-1} are differentiable on Ω ; in particular, the inverse Φ^{-1} is also a diffeomorphism. If $\Phi : \Omega \rightarrow \mathbb{R}^n$ is differentiable such that $\det \Phi' \neq 0$ on Ω then $\Phi : \Omega \rightarrow \Phi(\Omega)$ is a diffeomorphism; the converse implication, however, holds true only if Ω and W are simply connected. We say that Φ is a C^k -diffeomorphism, for $k \geq 1$, if all the (mixed) k^{th} -order derivatives of each ϕ_i , for $1 \leq i \leq n$, exist and are continuous.

On \mathbb{R}^n we denote by $\{\Pi_{\bar{i}} : 1 \leq i \leq n\}$ the family of the $(n-1)$ -dimensional projectors, i.e., $\Pi_{\bar{i}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ are defined by

$$\forall 1 \leq i \leq n : \Pi_{\bar{i}}(x_1, \dots, x_n) := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

and note that the mappings $\Pi_{\bar{i}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ are differentiable vector fields, for $1 \leq i \leq n$. Moreover, $\Pi_{\bar{i}}'$ can be obtained by removing the i^{th} row from the n -dimensional identity matrix.

A connected set $\mathcal{H} \subset \Omega$ will be called a k -surface in Ω , for $1 \leq k \leq n$, if for each $\mathbf{x} \in \mathcal{H}$ there exists an open set $W \subset \mathbb{R}^k$ and a differentiable vector field $\vartheta = \vartheta^{\mathbf{x}} := (\vartheta_1^{\mathbf{x}}, \dots, \vartheta_n^{\mathbf{x}}) : W \rightarrow \mathcal{H}$ such that the matrix $(\vartheta^{\mathbf{x}})'$ has full rank (that is, $\text{rank}(\vartheta^{\mathbf{x}})' = k$) on W and $\mathbf{x} \in \vartheta^{\mathbf{x}}(W)$. The mapping $\vartheta^{\mathbf{x}}$ is usually called a *local parametrization* of \mathcal{H} . Intuitively, this means that, locally on \mathcal{H} , exactly k coordinates are independent, i.e., the projection of \mathcal{H} on the corresponding k directions is the image of a k -dimensional open set via a diffeomorphism, while the remainder of $n-k$ coordinates are uniquely determined by the k independent ones. The number k is called the dimension of \mathcal{H} and $n-k$ is called the co-dimension of \mathcal{H} . It can be shown that the dimension k of a surface is uniquely determined. By convention, a 0-surface is just a point. A $(n-1)$ -surface is often called a hyper-surface and is typically defined by

$$\mathcal{H} = \{\mathbf{x} \in \Omega : \psi(\mathbf{x}) = 0\}, \quad (12)$$

where $\psi : \Omega \rightarrow \mathbb{R}$ is a differentiable function satisfying $\nabla_{\mathbf{x}}\psi \neq \mathbf{0}$ on Ω . The parametrization $\vartheta^{\mathbf{x}} : W \rightarrow \mathcal{H}$ is said to be (positively) *oriented* if the independent coordinates $x_{i_1} = \vartheta_{i_1}^{\mathbf{x}}(\mathbf{w}), \dots, x_{i_k} = \vartheta_{i_k}^{\mathbf{x}}(\mathbf{w})$ satisfy

$$\forall \mathbf{w} \in W : \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(w_1, \dots, w_k)}(\mathbf{w}) := \frac{\partial(\vartheta_{i_1}^{\mathbf{x}}, \dots, \vartheta_{i_k}^{\mathbf{x}})}{\partial(w_1, \dots, w_k)}(\mathbf{w}) > 0.$$

Let \mathcal{H} be a (smooth enough) k -surface in Ω . A canonical *surface measure* on \mathcal{H} , which will be denoted by $\sigma_{\mathcal{H}}$ and, intuitively, measures the k -dimensional volume on \mathcal{H} , can be defined by means of the Riemannian metric; the surface measure $\sigma_{\mathcal{H}}$ is absolutely continuous with respect to the push-forward through ϑ of the k -dimensional Lebesgue measure from W onto \mathcal{H} . In addition, its density can be locally expressed by means of the local parametrization $\vartheta^{\mathbf{x}}$. However, the surface measure does not depend on a particular parametrization in the sense that any other (local) parametrization generates precisely the same surface measure. If $k = 0$, i.e., if $H = \{\mathbf{x}\}$, then the surface measure coincides with the Dirac measure, denoted by $\delta_{\mathbf{x}}$. In Section 2.1 we will provide more details on $(n - 1)$ -dimensional surface measures and will present a general method to evaluate integrals with respect to $\sigma_{\mathcal{H}}$, for \mathcal{H} defined by (12).

For an arbitrary set $\Omega \subset \mathbb{R}^n$ we denote by Ω° the set of its interior points. By $\overline{\Omega}$ we denote the set of its adherent points and $\partial\Omega = \overline{\Omega} \setminus \Omega^\circ$ denotes its boundary, i.e., the set of points which are adherent to both Ω and its complementary $\mathbb{C}\Omega$. Finally, we denote by ℓ the Lebesgue measure on \mathbb{R}^n and let $\mathcal{L}^1(\Omega, \ell)$ denote the space of measurable functions which are Lebesgue integrable on Ω .

2 Differentiation Formulas for Integrals on Variable Domains

Let $\Theta \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ be open, connected and convex sets and $f : \Theta \times \Omega \rightarrow \mathbb{R}$ be such that $f(\theta, \cdot) \in \mathcal{L}^1(\Omega, \ell)$ for each $\theta \in \Theta$. If $\mathfrak{D} \subset \Omega$ then

$$\partial_\theta \int_{\mathfrak{D}} f(\theta, \mathbf{x}) d\mathbf{x} = \int_{\mathfrak{D}} \partial_\theta f(\theta, \mathbf{x}) d\mathbf{x}, \quad (13)$$

provided that the derivative $\partial_\theta f(\theta, \cdot)$ exists on Ω and satisfies some regularity assumptions. More specifically, we require that there exists some neighborhood V of θ such that

$$g(\mathbf{x}) := \sup_{\zeta \in V} |\partial_\theta f(\zeta, \mathbf{x})| \in \mathcal{L}^1(\Omega).$$

A successive application of the Mean Value and Dominated Convergence Theorems show that differentiation and integration can be interchanged in (13).

Nevertheless, (13) fails to hold true when the domain of integration \mathfrak{D} depends on θ and the natural question is how (13) generalizes to θ -variable domains. The aim of this section is to provide an answer to this question.

Basic Setup: We assume that $\Theta \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ are open, connected sets, Θ being convex and, for $\theta \in \Theta$, we denote by $\mathcal{L}_\theta^1(\Omega)$ the class of functions $f : \Theta \times \Omega \rightarrow \mathbb{R}$ such that $\sup_{\zeta \in V} |f(\zeta, \cdot)| \in \mathcal{L}^1(\Omega)$, for some neighborhood V of θ , and in general we will assume that $f : \Theta \times \Omega \rightarrow \mathbb{R}$ is a continuous function, continuously differentiable w.r.t. θ , such that $f \in \mathcal{L}_\theta^1(\Omega)$, $\partial_\theta f \in \mathcal{L}_\theta^1(\Omega)$.

Variability of the domain w.r.t. θ will be defined by means of a feasibility function $\psi : \Theta \times \Omega \rightarrow \mathbb{R}$, which will be assumed continuously differentiable such that its gradient $\nabla_{\mathbf{x}}\psi$ does not vanish on $\Theta \times \Omega$, i.e., $\nabla_{\mathbf{x}}\psi \neq \mathbf{0}$. In addition, we will require that there exist an open domain $\mathfrak{D} \subset \Omega$ with smooth boundary, such that $\overline{\mathfrak{D}} \subset \Omega$ and a continuously differentiable function $\Phi : \Theta \times \Omega \rightarrow \Omega$ such that for all $\theta \in \Theta$ the mapping $\Phi_\theta := \Phi(\theta, \cdot) : \mathfrak{D} \rightarrow \mathfrak{D}_\theta$ is a \mathcal{C}^2 -diffeomorphism, i.e.,

\mathfrak{D}_θ is a smooth transformation of some fixed domain² \mathfrak{D} ; see Figure 1. We will also require that $\ell(\mathfrak{D}_\theta) > 0$ and will denote by \mathfrak{B}_θ the boundary of \mathfrak{D}_θ in Ω . It is worth noting that Φ_θ maps the boundary $\partial\mathfrak{D}$ onto $\partial\mathfrak{D}_\theta$; more specifically, $\Phi_\theta : \partial\mathfrak{D} \rightarrow \partial\mathfrak{D}_\theta$ defines a bijection. Since most of our statements will deal with differentiability in some specific point $\theta \in \Theta$, we will assume w.l.o.g. that³ $\mathfrak{D} = \mathfrak{D}_\theta$ and Φ_θ (for this fixed θ) is just the identity function.

Finally, assuming that ψ is such that $\mathfrak{D}_\theta = \{\mathbf{x} \in \Omega : \psi(\theta, \mathbf{x}) > 0\}$, we have

$$\mathfrak{B}_\theta = \{\mathbf{x} \in \Omega : \psi(\theta, \mathbf{x}) = 0\}. \quad (14)$$

Indeed, the direct inclusion in (14) is true in general. On the other hand, the condition $\nabla_{\mathbf{x}}\psi \neq 0$ on $\Theta \times \Omega$ ensures that, for fixed θ , the function $\psi(\theta, \cdot)$ has no stationary points in Ω . Now if $\mathbf{x} \in \Omega$ is such that $\psi(\theta, \mathbf{x}) = 0$ then it is either a boundary point for \mathfrak{D}_θ or a local minimum for $\psi(\theta, \cdot)$ in Ω . But the second possibility is already ruled out by our assumption. Consequently, any $\mathbf{x} \in \Omega$ satisfying $\psi(\theta, \mathbf{x}) = 0$ belongs to the boundary of \mathfrak{D}_θ and this shows the converse inclusion, hence the equality, in (14).

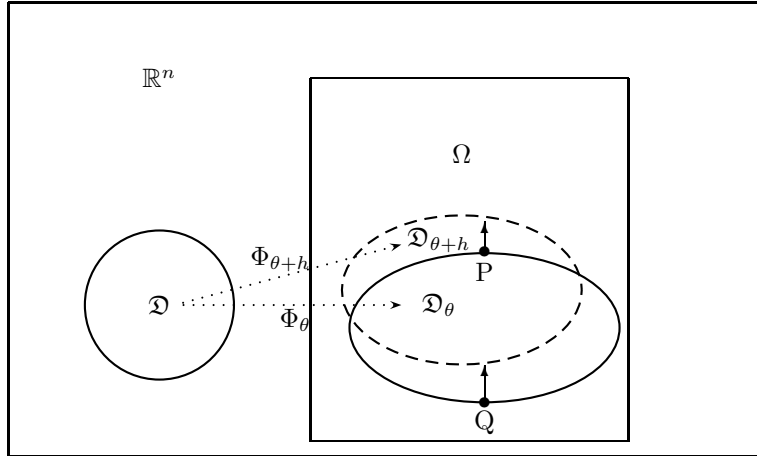


Figure 1: An example of θ -variable domain \mathfrak{D}_θ , diffeomorphic to a fixed one \mathfrak{D} . The arrows indicate the normal component of the velocity at the boundary in the corresponding points. When the direction pointing outwards the domain \mathfrak{D}_θ is considered the velocity is positive in P and negative in Q .

2.1 The Main Result

We consider first one-dimensional parameters, the extension to the multi-dimensional case is straightforward. More specifically, we assume that $\Theta \subset \mathbb{R}$ is an open

²In fluid dynamics such a phenomenon is called *moving flux* and models the time/space evolution of some deformable volume. Mathematically, it is described by the position $\Phi(\theta, \cdot)$ and velocity $\partial_\theta\Phi(\theta, \cdot)$ of individual particles, at time θ .

³One may always choose $\mathfrak{D} = \mathfrak{D}_\theta$ and define the family of C^2 -diffeomorphisms $\{\tilde{\Phi}_\vartheta\}_{\vartheta \in \Theta}$ as $\tilde{\Phi}_\vartheta := \Phi_\vartheta \circ \Phi_\theta^{-1} : \tilde{\mathfrak{D}} \rightarrow \mathfrak{D}_\vartheta$, which also satisfies the assumptions.

interval on the real line and we investigate whether the derivative

$$\partial_\theta \int f(\theta, \mathbf{x}) \mathbf{1}_{\{\psi(\theta, \cdot) > 0\}}(\mathbf{x}) d\mathbf{x} \quad (15)$$

exists, in which case, we are interested in estimating its value. In the multi-dimensional case one can obtain similar results concerning directional derivatives by using a proper parametrization of the model.

Starting point of our analysis is the following technical result which extends the well known *Leibniz integral rule* to multivariate functions. While in one dimension the result is known in real analysis as *the differentiation rule for integrals with variable endpoints*, in two and three dimensions, this result is better known in the field of fluid dynamics as the *Reynolds Transport Theorem*. For a general statement and proof, in the n -dimensional case, we refer to [3].

Lemma 1 *Let $\Theta \subset \mathbb{R}$ be an open interval and $\Omega \subset \mathbb{R}^n$ be an open connected subset. Let $\mathfrak{D} \subset \Omega$ be a connected domain with smooth boundary such that $\overline{\mathfrak{D}} \subset \Omega$ and suppose that there exists a continuously differentiable vector field $\Phi : \Theta \times \Omega \rightarrow \Omega$ such that for each $\theta \in \Theta$ the mapping*

$$\Phi_\theta := \Phi(\theta, \cdot) : \mathfrak{D} \rightarrow \mathfrak{D}_\theta := \Phi_\theta(\mathfrak{D}) \subset \Omega,$$

is a \mathcal{C}^2 -diffeomorphism. Then for each continuous function $f : \Theta \times \Omega \rightarrow \mathbb{R}$, continuously differentiable on Ω with respect to θ , such that $f \in \mathcal{L}_\theta^1(\Omega)$ and $\partial_\theta f \in \mathcal{L}_\theta^1(\Omega)$, it holds that

$$\partial_\theta \int_{\mathfrak{D}_\theta} f(\theta, \mathbf{x}) d\mathbf{x} = \int_{\mathfrak{D}_\theta} \partial_\theta f(\theta, \mathbf{x}) d\mathbf{x} + \int_{\mathfrak{B}_\theta} f(\theta, \mathbf{x}) \langle \dot{\mathbf{x}} | \vec{\mathbf{n}}_{\mathbf{x}} \rangle \sigma_\theta(d\mathbf{x}), \quad (16)$$

where⁴ $\dot{\mathbf{x}}$ is a short-hand notation for $\frac{\partial \mathbf{x}}{\partial \theta} = \frac{\partial \Phi}{\partial \theta}(\theta, \mathbf{x})$, $\vec{\mathbf{n}}_{\mathbf{x}}$ denotes the unit normal vector at the surface \mathfrak{B}_θ in \mathbf{x} , pointing outwards the domain \mathfrak{D}_θ and " $\sigma_\theta(d\mathbf{x})$ " denotes the infinitesimal area unit on \mathfrak{B}_θ ; that is, if $\vartheta^{\mathbf{x}} : W \rightarrow \partial \mathfrak{D}_\theta$, where $W \subset \mathbb{R}^{n-1}$, is an oriented (local) parametrization of \mathfrak{B}_θ in \mathbf{x} then⁵

$$\sigma_\theta(d\mathbf{x}) := \left| \sum_{i=1}^n (-1)^{i-1} \frac{\partial(\vartheta_1^{\mathbf{x}}, \dots, \vartheta_{i-1}^{\mathbf{x}}, \vartheta_{i+1}^{\mathbf{x}}, \dots, \vartheta_n^{\mathbf{x}})}{\partial(w_1, \dots, w_{n-1})} \nu_i \right| d\mathbf{w} \quad (17)$$

with $\mathbf{w} := (w_1, \dots, w_{n-1}) \in V$ and $\vec{\mathbf{n}}_{\mathbf{x}} := (\nu_1, \dots, \nu_n)$.

In [3] the result presented in Lemma 1 is stated in a much more general setting involving concepts such as differential form, exterior derivative and interior product and the proof essentially relies on Stokes' Theorem. In order to avoid a rather technical exposition of the above concepts, we have presented here a simpler version of this result, adjusted to our needs. The following result establishes the theoretical fundament of our framework.

Theorem 1 *Let $\Theta \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ be open and connected and $\psi : \Theta \times \Omega \rightarrow \mathbb{R}$ be continuously differentiable, such that $\nabla_{\mathbf{x}} \psi \neq \mathbf{0}$ on $\Theta \times \Omega$. If $f : \Theta \times \Omega \rightarrow \mathbb{R}$*

⁴In fluid dynamics the vector field $\dot{\mathbf{x}}$ is referred to as "Eulerian velocity at the boundary" or, simply, "Euler derivative". The Euler derivative

⁵Such a parametrization $\vartheta^{\mathbf{x}}$ exists and the representation in (17) is invariant on $\vartheta^{\mathbf{x}}$.

is continuous and continuously differentiable w.r.t. θ , such that $f \in \mathcal{L}_\theta^1(\Omega)$ and $\partial_\theta f \in \mathcal{L}_\theta^1(\Omega)$ then it holds that

$$\partial_\theta \int_{\mathfrak{D}_\theta} f(\theta, \mathbf{x}) d\mathbf{x} = \int_{\mathfrak{D}_\theta} \partial_\theta f(\theta, \mathbf{x}) d\mathbf{x} + \int_{\mathfrak{B}_\theta} f(\theta, \mathbf{x}) \partial_\theta^\psi(\mathbf{x}) \sigma_\theta(d\mathbf{x}). \quad (18)$$

where $\partial_\theta^\psi(\mathbf{x})$ denotes the projection of the velocity $\dot{\mathbf{x}}$ onto the normal direction to the surface \mathfrak{B}_θ in \mathbf{x} , i.e.,

$$\partial_\theta^\psi(\mathbf{x}) := \langle \dot{\mathbf{x}} | \bar{\mathbf{n}}_{\mathbf{x}} \rangle = \frac{\partial_\theta \psi}{|\nabla_{\mathbf{x}} \psi|}(\theta, \mathbf{x}). \quad (19)$$

Proof: For $\mathbf{x} \in \mathfrak{B}_\theta$ we have $\psi(\theta, \mathbf{x}) = 0$, for any $\theta \in \Theta$. Therefore, differentiating w.r.t. θ we obtain

$$\partial_\theta \psi + \langle \nabla_{\mathbf{x}} \psi | \dot{\mathbf{x}} \rangle = 0. \quad (20)$$

Since the normal direction to the surface $\{\mathbf{x} : \psi(\theta, \mathbf{x}) = 0\}$ is that of the gradient $\nabla_{\mathbf{x}} \psi$ which, in our case, points towards the interior of the domain \mathfrak{D}_θ (the sense of increment of ψ should be inside the domain $\{\mathbf{x} : \psi(\theta, \mathbf{x}) > 0\}$), it follows that $\bar{\mathbf{n}}_{\mathbf{x}} = -\nabla_{\mathbf{x}} \psi / |\nabla_{\mathbf{x}} \psi|$. Hence, we conclude from (20) that

$$\langle \dot{\mathbf{x}} | \bar{\mathbf{n}}_{\mathbf{x}} \rangle = -\frac{\langle \nabla_{\mathbf{x}} \psi | \dot{\mathbf{x}} \rangle}{|\nabla_{\mathbf{x}} \psi|} = \frac{\partial_\theta \psi}{|\nabla_{\mathbf{x}} \psi|} = \partial_\theta^\psi(\mathbf{x})$$

and the conclusion follows from Lemma 1. \square

If Ω is a bounded domain, i.e., $\ell(\Omega) < \infty$, by taking $f = 1$ in (18) yields

$$\partial_\theta \ell(\mathfrak{D}_\theta) = \int_{\mathfrak{B}_\theta} \partial_\theta^\psi(\mathbf{x}) \sigma_\theta(d\mathbf{x}). \quad (21)$$

Assume that ψ is strictly increasing w.r.t. θ , i.e., $\partial_\theta \psi > 0$ and $\partial_\theta \ell(\mathfrak{D}_\theta) \neq 0$. Then, for any bounded and continuous g it holds that

$$\partial_\theta \int_{\mathfrak{D}_\theta} \frac{g(\mathbf{x})}{\ell(\mathfrak{D}_\theta)} d\mathbf{x} = \frac{\partial_\theta \ell(\mathfrak{D}_\theta)}{\ell(\mathfrak{D}_\theta)} \left[\int_{\mathfrak{B}_\theta} g(\mathbf{x}) \frac{\partial_\theta^\psi(\mathbf{x})}{\partial_\theta \ell(\mathfrak{D}_\theta)} \sigma_\theta(d\mathbf{x}) - \int_{\mathfrak{D}_\theta} g(\mathbf{x}) \frac{1}{\ell(\mathfrak{D}_\theta)} d\mathbf{x} \right],$$

and we obtain the representation asserted in (9), with

$$\varsigma_\theta(d\mathbf{x}) := \mathbf{1}_{\mathfrak{B}_\theta}(\mathbf{x}) \frac{\partial_\theta^\psi(\mathbf{x})}{\partial_\theta \ell(\mathfrak{D}_\theta)} \cdot \sigma_\theta(d\mathbf{x}),$$

which, by (21), is a probability measure.

Extensions of the statement in Theorem 1 to multi-dimensional parameters, in terms of directional derivatives and gradient are straightforward. The result in Theorem 1 can be also extended to domains with piecewise smooth boundary. More specifically, if ψ_1, \dots, ψ_m are feasibility functions satisfying the assumed regularity conditions and one defines the domain

$$\mathfrak{D}_\theta := \{\mathbf{x} \in \Omega : \psi_j(\theta, \mathbf{x}) > 0, \forall 1 \leq j \leq m\},$$

then the boundary \mathfrak{B}_θ consists of all those \mathbf{x} 's in Ω for which $\psi_j(\theta, \mathbf{x}) \geq 0$, for all j 's and such that there exists (at least) one index j_0 such that $\psi_{j_0}(\theta, \mathbf{x}) = 0$. For some nonempty index set $J \subset \{1, 2, \dots, m\}$ we denote

$$\mathfrak{B}_\theta^J := \{\mathbf{x} \in \Omega : \psi_j(\theta, \mathbf{x}) = 0 (j \in J) \ \& \ \psi_j(\theta, \mathbf{x}) > 0 (j \notin J)\}.$$

It is easy to see that $\{\mathfrak{B}_\theta^J\}_{J \neq \emptyset}$ is a partition of \mathfrak{B}_θ and if $I \subset J$ then \mathfrak{B}_θ^I and \mathfrak{B}_θ^J are supported by the same hyper-surface, hence a consistent concept of surface measure on \mathfrak{B}_θ can be defined based on the canonical surface measures on each supporting surface $\{\mathbf{x} \in \Omega : \psi_j(\theta, \mathbf{x}) = 0\}$. A formula similar to (18) also holds in this case; however, the surface density on each \mathfrak{B}_θ^J , with $\#J \geq 2$ will be given by the minimal velocity $\partial_\theta^{\psi_j}$, for $j \in J$. Although the intuition behind this result is rather clear, the proof (in general) is rather technical and is beyond the purpose of this paper to present it. The particular case when all ψ_j 's are linear in \mathbf{x} , i.e., \mathfrak{D}_θ has flat boundaries, has been treated in [12].

2.2 Application and Examples

We have applied Lemma 1 to the domain $\mathfrak{D}_\theta = \{\mathbf{x} \in \Omega : \psi(\theta, \mathbf{x}) > 0\}$ and we have proved that (18) holds true for a suitable function f . While the first integral in the r.h.s. of (18) poses no problems, being a standard Lebesgue-Riemman integral, the natural question here is what is and how to deal with the second one⁶? The measure σ is often called *the surface measure* on the hyper-surface \mathfrak{B}_θ . Hence the question is how to calculate integrals with respect to the surface measure? In the following we aim to provide a satisfactory answer to this question.

Let us ignore for a while the parameter θ , which in (18) is fixed, and assume that $\psi : \Omega \rightarrow \mathbb{R}$ is a continuously differentiable function. We start by noting that, by assumption, $\nabla_{\mathbf{x}}\psi \neq \mathbf{0}$ on Ω . This means that, if we fix some $\mathbf{x} \in \Omega$, there exist $1 \leq i \leq n$ such that $\partial_i\psi(\mathbf{x}) \neq 0$ and we can assume w.l.o.g. that $i = n$. Consequently, the mapping ψ defines a hyper-surface of dimension $n - 1$ in \mathbb{R}^n by the equation $\psi(x_1, \dots, x_n) = 0$. Indeed, by the Implicit Function Theorem (see, e.g., [13]) it follows that there exist a local parametrization $x_n = \vartheta_n(x_1, \dots, x_{n-1})$ around⁷ \mathbf{x} , i.e.,

$$\mathbf{x} = \vartheta(x_1, \dots, x_{n-1}),$$

where $\vartheta := (\vartheta_1, \dots, \vartheta_n) : W \subset \Pi_{\bar{n}}(\Omega) \rightarrow \Omega$ with $\vartheta_i(x_1, \dots, x_{n-1}) = x_i$, for $1 \leq i \leq n$. Note that such a parametrization is always positively oriented since

$$\frac{\partial(\vartheta_1, \dots, \vartheta_{n-1})}{\partial(x_1, \dots, x_{n-1})} = \det \mathbf{I}_{n-1} = 1.$$

A surface of dimension 0, which is obtained for $n = 1$, is nothing but a point. Indeed, if $\psi : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with $\psi'(x) \neq 0$ then it defines a local bijection. Hence, the equation $\psi(x) = y$, for some y in the range of ψ , defines the 0-dimensional surface consisting of the point $\psi^{-1}(y)$. A measure on such a surface will be then given by a Dirac measure assigning mass to the corresponding point (possibly) multiplied by a weight (which may also depend on the point of reference).

⁶The second integral in (18) is a *surface integral*. For the particular cases $n = 2$ and $n = 3$ there are well known formulas (especially in some fields of physics and engineering) for calculating such integrals.

⁷It can be easily extended to the maximal connected component where $\partial_n\psi$ does not vanish. By the assumed continuity of the derivatives this maximal component includes a neighborhood of \mathbf{x} . For this reason we drop the superscript \mathbf{x} .

Going back to the general case, the Jacobians of the transformations in (17) are given by

$$\forall 1 \leq i \leq n : \frac{\partial(\vartheta_1, \dots, \vartheta_{i-1}, \vartheta_{i+1}, \dots, \vartheta_n)}{\partial(x_1, \dots, x_{n-1})} = \begin{cases} (-1)^{n+i-1} \frac{\partial \vartheta_n}{\partial x_i}, & 1 \leq i \leq n-1, \\ 1, & i = n. \end{cases}$$

Recalling that ϑ_n gives the n^{th} coordinate x_n as a function of (x_1, \dots, x_{n-1}) , determined from $\psi(x_1, \dots, x_{n-1}, x_n) = 0$, we obtain by implicit differentiation

$$\forall 1 \leq i \leq n-1 : \frac{\partial \vartheta_n}{\partial x_i} = - \left(\frac{\partial \psi}{\partial x_n} \right)^{-1} \frac{\partial \psi}{\partial x_i}. \quad (22)$$

On the other hand, since $\vec{\mathbf{n}} = -\nabla_{\mathbf{x}} \psi / |\nabla_{\mathbf{x}} \psi|$, it follows from (17) that

$$\begin{aligned} \sigma_{\theta}(d\mathbf{x}) &= \left| \frac{(-1)^{n-1}}{|\nabla_{\mathbf{x}} \psi|} \left[\left(\frac{\partial \psi}{\partial x_n} \right) + \sum_{i=1}^{n-1} (-1)^{2i-1} \frac{\partial \psi}{\partial x_i} \frac{\partial \vartheta_n}{\partial x_i} \right] \right| (\theta, \mathbf{x}) dx_1 \dots dx_{n-1} \\ &= \left| \frac{(-1)^{n-1}}{|\nabla_{\mathbf{x}} \psi|} \left[\left(\frac{\partial \psi}{\partial x_n} \right) + \left(\frac{\partial \psi}{\partial x_n} \right)^{-1} \sum_{i=1}^{n-1} \left(\frac{\partial \psi}{\partial x_i} \right)^2 \right] \right| (\theta, \mathbf{x}) dx_1 \dots dx_{n-1}, \end{aligned}$$

where the last equality follows from (22). Therefore, one concludes that

$$\sigma_{\theta}(d\mathbf{x}) = \left| \frac{\nabla_{\mathbf{x}} \psi}{\partial_n \psi} \right| (\theta, \mathbf{x}) dx_1 \dots dx_{n-1}. \quad (23)$$

Consequently, the last integral in the r.h.s. of (18) can be re-written as follows:

$$\int_{\mathfrak{B}_{\theta}} f(\theta, \mathbf{x}) \partial_{\theta}^{\psi}(\mathbf{x}) \sigma_{\theta}(d\mathbf{x}) = \int_{\Pi_{\bar{n}} \mathfrak{B}_{\theta}} \left[f \cdot \frac{\partial_{\theta} \psi}{|\partial_n \psi|} \right] (\theta, \mathbf{w}, \vartheta_n(\mathbf{w})) d\mathbf{w}, \quad (24)$$

where $\Pi_{\bar{n}}$ denotes the $(n-1)$ -dimensional projection and $\partial_n \psi = \partial \psi / \partial x_n$.

Example 1 *The half-space determined by a variable hyperplane:* Let $\Theta = \mathbb{R}$, $\Omega = \mathbb{R}^n$, for some $n \geq 1$, and let $b : \Theta \rightarrow \mathbb{R}$, $\mathbf{a} := (a_1, \dots, a_n) : \Theta \rightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$ be smooth functions. Take $\psi(\theta, \mathbf{x}) = b(\theta) - \langle \mathbf{a}(\theta) | \mathbf{x} \rangle$, i.e., \mathfrak{D}_{θ} appears as the half-space \mathcal{S}_{θ} with location parameter $b(\theta)$ and orientation parameter $\mathbf{a}(\theta)$, i.e.,

$$\mathcal{S}_{\theta} := \{\mathbf{x} \in \mathbb{R}^n : b(\theta) - \langle \mathbf{a}(\theta) | \mathbf{x} \rangle > 0\},$$

whose boundary is given by the hyperplane $\mathcal{H}_{\theta} := \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}(\theta) | \mathbf{x} \rangle = b(\theta)\}$. In addition, we have $\partial_{\theta} \psi(\theta, \mathbf{x}) = b'(\theta) - \langle \mathbf{a}'(\theta) | \mathbf{x} \rangle$, where $\mathbf{a}' = (a'_1, \dots, a'_n)$, and $\nabla_{\mathbf{x}} \psi(\theta, \mathbf{x}) = -\mathbf{a}(\theta)$ which yields

$$\partial_{\theta}^{\psi}(\mathbf{x}) = \frac{b'(\theta) - \langle \mathbf{a}'(\theta) | \mathbf{x} \rangle}{|\mathbf{a}(\theta)|}.$$

Fix θ and assume that $a_n(\theta) \neq 0$. Using the parametrization $\vartheta_n : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$,

$$\vartheta_n(x_1, \dots, x_{n-1}) = \frac{b(\theta) - \langle (a_1, \dots, a_{n-1})(\theta) | (x_1, \dots, x_{n-1}) \rangle}{a_n(\theta)},$$

it follows from Theorem 1 and (24) that

$$\begin{aligned} \partial_{\theta} \int_{\mathcal{S}_{\theta}} f(\theta, \mathbf{x}) d\mathbf{x} &= \int_{\mathcal{S}_{\theta}} \partial_{\theta} f(\theta, \mathbf{x}) d\mathbf{x} + \frac{1}{|a_n(\theta)|} \int_{\mathbb{R}^{n-1}} b'(\theta) \cdot f(\theta, \mathbf{w}, \vartheta_n(\mathbf{w})) d\mathbf{w} \\ &\quad - \frac{1}{|a_n(\theta)|} \int_{\mathbb{R}^{n-1}} \langle \mathbf{a}'(\theta) | (\mathbf{w}, \vartheta_n(\mathbf{w})) \rangle \cdot f(\theta, \mathbf{w}, \vartheta_n(\mathbf{w})) d\mathbf{w}, \end{aligned}$$

for suitable $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathbf{w} = \Pi_{\bar{n}}(\mathbf{x}) = (x_1, \dots, x_{n-1})$. In this case note that we have a global parametrization $\vartheta : W = \mathbb{R}^{n-1} \rightarrow \mathcal{H}_\theta$.

Example 2 *The disc with variable origin and radius:* Let us choose $\Theta = (0, \infty)$, $\Omega = \mathbb{R}^n \setminus \{\mathbf{0}\}$, for some $n \geq 1$, $r : \Theta \rightarrow (0, \infty)$ and $\mathbf{k} := (k_1, \dots, k_n) : \Theta \rightarrow \mathbb{R}^n$ be some smooth functions. Define

$$\psi(\theta, \mathbf{x}) = r(\theta)^2 - |\mathbf{x} - \mathbf{k}(\theta)|^2.$$

That is, we obtain \mathcal{D}_θ as the disc $\mathcal{D}_\theta^n := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{k}(\theta)| < r(\theta)\}$ centered in $\mathbf{k}(\theta)$ with radius $r(\theta)$. Therefore, $\partial\mathcal{D}_\theta^n = \mathcal{B}_\theta^n := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{k}(\theta)| = r(\theta)\}$ and by letting $\mathbf{k}' = (k'_1, \dots, k'_n)$ we have

$$\partial_\theta \psi(\theta, \mathbf{x}) = 2r'(\theta)r(\theta) + \left\langle \mathbf{k}'(\theta) \left| \overrightarrow{\mathbf{x} - \mathbf{k}(\theta)} \right. \right\rangle, \quad \nabla_{\mathbf{x}} \psi(\theta, \mathbf{x}) = - \left(\overrightarrow{\mathbf{x} - \mathbf{k}(\theta)} \right),$$

which yields $\partial_n \psi(\theta, \mathbf{x}) = -(x_n - k_n(\theta))/r(\theta)$, for $\mathbf{x} \in \mathcal{B}_\theta^n$. Therefore, we obtain

$$\forall \mathbf{x} \in \mathcal{B}_\theta^n : \frac{\partial_\theta \psi}{|\partial_n \psi|}(\theta, \mathbf{x}) = \frac{2r'(\theta)r(\theta)^2 + \langle \mathbf{k}'(\theta) | \mathbf{x} - \mathbf{k}(\theta) \rangle}{|x_n - k_n(\theta)|}. \quad (25)$$

Now consider the parametrizations $x_n = \vartheta_n^\pm(x_1, \dots, x_{n-1})$ defined as

$$\vartheta_n^\pm(\mathbf{w}) = k_n(\theta) \pm \sqrt{r(\theta)^2 - |\mathbf{w} - \Pi_{\bar{n}}\mathbf{k}(\theta)|^2},$$

for $\mathbf{w} := (x_1, \dots, x_{n-1}) \in \Pi_{\bar{n}}\mathcal{B}_\theta^n = \{\mathbf{w} \in \mathbb{R}^{n-1} : |\mathbf{w} - \Pi_{\bar{n}}\mathbf{k}(\theta)| < r(\theta)\}$, which are valid whenever $x_n \neq k_n(\theta)$; the latter condition fails on some set of dimension $(n-2)$, hence σ -negligible. Note, that none of the above parametrizations can be extended to the whole boundary \mathcal{B}_θ^n and one has to consider separately the two hemispheres determined from the conditions $x_n > k_n(\theta)$ (resp. $x_n < k_n(\theta)$) when calculating the integral in (24). More specifically, take $x_n = \vartheta_n^+(x_1, \dots, x_{n-1})$ for the hemisphere $\{\mathbf{x} \in \mathcal{B}_\theta^n : x_n > k_n(\theta)\}$ and $x_n = \vartheta_n^-(x_1, \dots, x_{n-1})$ for the hemisphere $\{\mathbf{x} \in \mathcal{B}_\theta^n : x_n < k_n(\theta)\}$. Then

$$\frac{\partial_\theta \psi}{|\partial_n \psi|}(\theta, \mathbf{w}, \vartheta^\pm(\mathbf{w})) = \frac{2r'(\theta)r(\theta)^2 + \sum_{i=1}^{n-1} [w_i - k_i(\theta)]k'_i(\theta)}{\sqrt{r(\theta)^2 - |\mathbf{w} - \Pi_{\bar{n}}\mathbf{k}(\theta)|^2}} \pm k'_n(\theta), \quad (26)$$

for any $\mathbf{w} \in \Pi_{\bar{n}}\mathcal{B}_\theta^n = \mathcal{D}_\theta^{n-1}$. Consequently, for suitable $f : (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, if $G_n^\pm(\theta, \mathbf{w}) := f(\theta, \mathbf{w}, \vartheta_n^+(\mathbf{w})) \pm f(\theta, \mathbf{w}, \vartheta_n^-(\mathbf{w}))$, we conclude from (24) and (26) that

$$\begin{aligned} \partial_\theta \int_{\mathcal{D}_\theta^n} f(\theta, \mathbf{x}) d\mathbf{x} &= \int_{\mathcal{D}_\theta^n} \partial_\theta f(\theta, \mathbf{x}) d\mathbf{x} \\ &+ \int_{\mathcal{D}_\theta^{n-1}} G_n^+(\theta, \mathbf{w}) \frac{2r'(\theta)r(\theta)^2 + \sum_{i=1}^{n-1} [w_i - k_i(\theta)]k'_i(\theta)}{\sqrt{r(\theta)^2 - |\mathbf{w} - \Pi_{\bar{n}}\mathbf{k}(\theta)|^2}} d\mathbf{w} \\ &+ \int_{\mathcal{D}_\theta^{n-1}} G_n^-(\theta, \mathbf{w}) k'_n(\theta) d\mathbf{w}. \end{aligned}$$

Of course, the above formula simplifies considerably when r or (some components of) \mathbf{k} do not depend on θ . In fact, since the role of x_n in this parametrization can be played by any other x_i , if there is a component k_i of \mathbf{k} not depending on θ , or, at least satisfying $k'_i(\theta) = 0$, one can simplify the calculations by taking x_i as a function of $\mathbf{w} = \Pi_{\bar{i}}\mathbf{x}$ on \mathcal{B}_θ^n and, in this way, $G_i^-(\theta, \cdot) = 0$, hence the last integral in the last display vanishes.

3 Application to Multi-Asset Digital Options

In this section we explain how the result derived in previous section can be applied to obtain gradient estimates for option premiums in the general Black-Scholes framework.

In the following, we consider a Black-Scholes model consisting of n bonds, in which the stock prices satisfy

$$\forall 1 \leq i \leq n : S_i(t) = s_i \exp \left(\left(r - \frac{\sigma_i^2}{2} \right) t + \sigma_i \sqrt{t} X_i \right), \quad (27)$$

at any time $t \geq 0$. In (27) r denotes the risk-free rate, $s_i = S_i(0) > 0$, $\sigma_i > 0$, for $1 \leq i \leq n$, denote the initial price and the volatility, respectively, of the i^{th} bond and $\mathbf{X} := (X_1, \dots, X_n)$ is a non-degenerate Gaussian n -dimensional vector with probability (Lebesgue) density given by

$$\forall \mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x}|\mathbf{0}, \mathbf{R}) := \frac{1}{\sqrt{(2\pi)^n \det \mathbf{R}}} \exp \left(-\frac{\mathbf{x}^* \mathbf{R}^{-1} \mathbf{x}}{2} \right); \quad (28)$$

here the superscript $*$ stands for the transpose operation and $\mathbf{R} := [\rho_{ij}]_{1 \leq i, j \leq n}$ denotes the “instantaneous” correlation matrix of the n Wiener processes governing the price dynamics of the n stocks. Note that, in this context, \mathbf{R} is a symmetric, positive definite $n \times n$ matrix and its diagonal elements satisfy $\rho_{ii} = 1$, for $1 \leq i \leq n$, while the non-diagonal elements are bounded by 1, i.e.⁸, $|\rho_{ij}| < 1$, for $1 \leq i < j \leq n$. When the n stocks vary independently then \mathbf{R} reduces to the identity matrix. In fact,

$$\forall i, j : \rho_{ij} := \mathbb{E}[X_i X_j] = \frac{1}{t} \text{Cov} [\sigma_i^{-1} \ln S_i(t), \sigma_j^{-1} \ln S_j(t)].$$

In the following we fix some arbitrary time horizon $t > 0$ and we denote by $\mathbf{S} := (S_1, \dots, S_n)$ the vector of stock prices at time t , i.e., the vector having as components $S_i(t)$ in (27). Note that the vector $\mathbf{S} = (S_1, \dots, S_n)$ has the property that $(\ln S_1, \dots, \ln S_n)$ is Gaussian with mean $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and covariance matrix $\mathbf{C} = [c_{ij}]_{1 \leq i, j \leq n}$ given by

$$\forall i, j : \mu_i := \ln s_i + \left(r - \frac{\sigma_i^2}{2} \right) t, \quad c_{ij} := t \sigma_i \sigma_j \rho_{ij}. \quad (29)$$

Therefore, \mathbf{S} follows a multivariate log-normal distribution on \mathbb{R}_+^n with density

$$\rho(\mathbf{x}|\boldsymbol{\mu}, \mathbf{C}) = \frac{1}{\sqrt{\det \mathbf{C}}} \left(\prod_{i=1}^n \frac{\mathbf{1}_{(0, \infty)}(x_i)}{x_i \sqrt{2\pi}} \right) \cdot \exp \left(-\frac{(\boldsymbol{\mu} - \ln \mathbf{x})^* \mathbf{C}^{-1} (\boldsymbol{\mu} - \ln \mathbf{x})}{2} \right), \quad (30)$$

where $\mathbf{x} := (x_1, \dots, x_n)$ and $\ln \mathbf{x} := (\ln x_1, \dots, \ln x_n)$. For ease of notation, we denote by $\boldsymbol{\Sigma}$ the diagonal matrix with elements σ_i , for $1 \leq i \leq n$, on the main diagonal and note that $\mathbf{C} = t \cdot \boldsymbol{\Sigma} \mathbf{R} \boldsymbol{\Sigma}$. Since, by assumption, all σ_i 's are strictly positive it follows that $\boldsymbol{\Sigma}$ is a nonsingular matrix; in fact, its inverse $\boldsymbol{\Sigma}^{-1}$ is a diagonal matrix with elements σ_i^{-1} on the main diagonal. In particular, it

⁸Actually we have $|\rho_{ij}| \leq 1$ but the additional non-degeneracy condition ensures that the inequality is strict for $i \neq j$.

follows that C is non-singular, hence (30) makes sense. Denoting now by $\bar{\varrho}_{ij}$ the elements of R^{-1} we have

$$C^{-1} = t^{-1} \cdot \Sigma^{-1} R^{-1} \Sigma^{-1} = \left[\frac{\bar{\varrho}_{ij}}{t\sigma_i\sigma_j} \right]_{1 \leq i, j \leq n}. \quad (31)$$

By a *European-style* option with maturity time t we mean a financial option which entitles its owner to a profit $g(\mathbf{S})$, where g is a measurable function, by exercising the option at time t . The present value (premium) of such an option will be given by

$$V = e^{-rt} \mathbb{E}[g(\mathbf{S})] = e^{-rt} \int_{\mathbb{R}^n} g(\mathbf{x}) \rho(\mathbf{x} | \mu, C) d\mathbf{x}. \quad (32)$$

The parameters appearing in the expressions of μ and C will be called *intrinsic* parameters of the model, since they appear in the distribution of the vector \mathbf{S} , and any other parameter (induced by g), such as strike price, will be called *non-intrinsic* parameters. Typically, the profit brought by a European option can be expressed as

$$g(\mathbf{S}) = \varphi(\mathbf{S}) \mathbf{1}_{\{\psi(\mathbf{S}) > 0\}}, \quad (33)$$

where φ is a smooth payoff function and ψ is some (piecewise) smooth feasibility function, none of them depending on intrinsic parameters. For instance, given a strike-price K , if we let

$$\varphi(S) = \psi(S) = S - K,$$

we recover the classical European call on a single asset. By changing $S - K$ into $K - S$ we obtain the European put. In multi-asset models the so-called *rainbow options*, which arise as a class of generalizations of the European call/put options on a single asset, are modeled in the same way. More specifically, their payoff can be expressed as in (33), or a sum of similar expressions. For instance, by letting in (33)

$$\varphi(\mathbf{S}) = \psi(\mathbf{S}) = \max\{S_1, \dots, S_n\} - K,$$

we obtain the *maximum option* and by changing the maximum to minimum in the above expression we obtain the *minimum option*. Furthermore, if we denote by $|\cdot|_p$ the p -norm on \mathbb{R}^n , i.e.,

$$\forall \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p},$$

by letting in (33) $\varphi(\mathbf{S}) = \psi(\mathbf{S}) = |\mathbf{S} - \mathbf{k}|_p - K$, for some positive vector $\mathbf{k} \in \mathbb{R}_+^n$, we obtain the *pyramid rainbow option*, for $p = 1$, and the *Madonna rainbow option*, for $p = 2$, etc.

Differentiating V in (32) w.r.t. some intrinsic parameter essentially reduces to differentiating the density ρ in (30) w.r.t. the corresponding parameter since the set of discontinuities of g , which is

$$\{\mathbf{x} \in \mathbb{R}^n : \psi(\mathbf{x}) = 0\},$$

has null Lebesgue measure and does not depend on intrinsic parameters. Consequently, the “density-differentiation” methods apply in this case yielding unbiased gradient estimates. To illustrate the utility of our approach, in our applications we focus on sensitivities w.r.t. non-intrinsic parameters of the model,

e.g., parameters induced by the exercise condition of a given option. That is, we assume that φ and ψ are functions of stock-price(s) and some parameter θ , in which case we have

$$V(\theta) = e^{-rt} \mathbb{E} [\varphi(\theta, \mathbf{S}) \mathbb{I}\{\psi(\theta, \mathbf{S}) > 0\}]. \quad (34)$$

To estimate $\partial_\theta V$ we apply Theorem 1 for $f(\theta, \mathbf{x}) := \varphi(\theta, \mathbf{x})\rho(\mathbf{x})$ and

$$\mathfrak{D}_\theta := \{\mathbf{x} : \psi(\theta, \mathbf{x}) > 0\}. \quad (35)$$

Moreover, bearing in mind the result in Theorem 1, we see that when the payoff and decision (exercise) functions agree, i.e., $\varphi = \psi$, the payoff function φ is null on the boundary \mathfrak{D}_θ , so that the surface integrand in (18) vanish and a gradient estimate for $\partial_\theta V$ can be obtained from

$$\partial_\theta V = e^{-rt} \int_{\mathfrak{D}_\theta} \partial_\theta \varphi(\theta, \mathbf{x}) \rho(\mathbf{x}) d\mathbf{x} = e^{-rt} \mathbb{E} [\partial_\theta \varphi(\theta, \mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\}],$$

since θ is assumed to be a non-intrinsic parameter, i.e., ρ does not depend on θ . A standard example is that of a European put with strike-price θ , i.e., $\varphi(\theta, S) = \psi(\theta, S) = \theta - S$, for which we have

$$\partial_\theta V = e^{-rt} \mathbb{E} [\mathbb{I}\{S < \theta\}] = e^{-rt} \mathbb{P}\{S < \theta\}.$$

Motivated by these remarks we will only consider options for which the payoff and exercise functions do not agree. This type of options are commonly known as *binary* or *digital* options. Apparently there is no clear distinction between the two concepts, both names being given to options described by discontinuous profit function which pay off only when the stock-price(s) at maturity lie in some feasibility region, although it is widely accepted that for a binary option the payoff is fixed once the option has been written whereas for a digital option the payoff is agreed upon at the maturity, provided that the stock-price(s) have reached the feasibility region. To comply with this definitions, we call *binary* an option with constant payoff function (in many cases $\varphi = 1$) and we call *digital* an option for which the payoff function φ depends on the stock-price(s). Hence, from a mathematical point of view, binary options are particular cases of digital ones and their premium is given by

$$V = e^{-rt} \mathbb{P}\{\psi(\theta, \mathbf{S}) > 0\}.$$

Most digital options are obtained from classical options, by changing the payoff function, and their name is typically given in accordance with the exercise rule (the type of the feasibility domain). Their binary counterparts are obtained simply by setting the payoff equal to 1 (or some other constant). Common examples of digital options are *digital spread-options*, *asset-or-nothing options*, *gap options* and *super-shares*. In this section we aim to illustrate the applicability of the method presented in this paper to sensitivity analysis (w.r.t. boundary parameters) of premiums of such options.

In order to make our approach fruitful for Monte-Carlo simulation one has to find a suitable interpretation, in terms of expected values, of the surface integrals appearing in differentiation formulas in Section 2. More specifically, consider an option with payoff function φ and smooth feasibility function ψ ,

its premium being given by (34). For the time being, assume that φ does not depend on θ , i.e.,

$$V(\theta) = e^{-rt} \mathbb{E} [\varphi(\theta, \mathbf{S}) \mathbb{I}\{\psi(\theta, \mathbf{S}) > 0\}] = e^{-rt} \mathbb{E} [\varphi(\mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\}], \quad (36)$$

with \mathfrak{D}_θ being defined in (35). By Theorem 1, we conclude that

$$\partial_\theta V = e^{-rt} \partial_\theta \int_{\mathfrak{D}_\theta} \varphi(\mathbf{x}) \rho(\mathbf{x}) d\mathbf{x} = e^{-rt} \int_{\mathfrak{B}_\theta} \varphi(\mathbf{x}) \rho(\mathbf{x}) \partial_\theta^\psi(\mathbf{x}) \sigma_\theta(d\mathbf{x}).$$

Provided that a global⁹ parametrization $\vartheta_n : \Pi_{\bar{n}} \mathfrak{B}_\theta \rightarrow \mathfrak{B}_\theta$ is available, giving the n^{th} component on the boundary \mathfrak{B}_θ , we obtain from (24)

$$\partial_\theta V = e^{-rt} \int_{\Pi_{\bar{n}} \mathfrak{B}_\theta} \varphi(\mathbf{w}, \vartheta_n(\mathbf{w})) \rho(\mathbf{w}, \vartheta(\mathbf{w})) \frac{\partial_\theta \psi}{|\partial_n \psi|}(\mathbf{w}, \vartheta_n(\mathbf{w})) d\mathbf{w},$$

with $\mathbf{w} := (x_1, \dots, x_{n-1}) \in \Pi_{\bar{n}} \mathfrak{B}_\theta$. Now note that ρ appears as the joint density of (\mathbf{w}, x_n) , hence, if $\rho_{\bar{n}}$ denotes the marginal density of (x_1, \dots, x_{n-1}) and $\rho_{n|\bar{n}}(\cdot|\mathbf{w})$ denotes the conditional density of x_n given $(x_1, \dots, x_{n-1}) = \mathbf{w}$, we conclude that $\rho(\mathbf{w}, \vartheta(\mathbf{w})) = \rho_{n|\bar{n}}(\vartheta(\mathbf{w})|\mathbf{w}) \cdot \rho_{\bar{n}}(\mathbf{w})$. In fact, we proved that

$$\Lambda_n(\theta, \mathbf{S}_{\bar{n}}) := \varphi(\mathbf{S}_{\bar{n}}, \vartheta_n(\mathbf{S}_{\bar{n}})) \rho_{n|\bar{n}}(\vartheta_n(\mathbf{S}_{\bar{n}})|\mathbf{S}_{\bar{n}}) \frac{\partial_\theta \psi}{|\partial_n \psi|}(\theta, \mathbf{S}_{\bar{n}}, \vartheta_n(\mathbf{S}_{\bar{n}})) \mathbf{1}_{\Pi_{\bar{n}} \mathfrak{B}_\theta}(\mathbf{S}_{\bar{n}}),$$

where $\mathbf{S}_{\bar{n}} = (S_1, \dots, S_{n-1})$, is an unbiased gradient estimator for $e^{rt}V$, i.e.,

$$\partial_\theta V = e^{-rt} \partial_\theta \mathbb{E} [\varphi(\mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\}] = e^{-rt} \mathbb{E} [\Lambda_n(\theta, \mathbf{S}_{\bar{n}})]. \quad (37)$$

Of course, the role of n in the above reasoning can be played by any other index i and the result remains valid after making the necessary changes. Moreover, the result in (37) extends to domains with piecewise smooth boundaries and payoff functions which depend on θ in accordance with the results presented in Section 2. Equation (37) relates surface integrals, appearing when differentiating option premiums w.r.t. boundary parameters, to expected values, so that sensitivities of option premiums can be estimated using Monte Carlo methods. In the following we derive closed-form expressions for such Λ for three types of digital options.

3.1 The Digital Spread-Option

A *digital spread-option* involves two underlying assets, i.e., $\mathbf{S} := (S_1, S_2)$, and a threshold $\theta > 0$. It entitles its holder to a profit of $\varphi(\mathbf{S}) > 0$, provided that $S_2 - S_1 > \theta$. In formula:

$$\psi(\theta, \mathbf{x}) := x_2 - x_1 - \theta.$$

The premium of such option is then given by

$$V = e^{-rt} \mathbb{E} [\varphi(\mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\}],$$

where $\mathfrak{D}_\theta := \{\mathbf{x} \in \mathbb{R}_+^2 : \psi(\theta, \mathbf{x}) = x_2 - x_1 - \theta > 0\}$ has the boundary given by $\mathfrak{B}_\theta := \{\mathbf{x} \in \mathbb{R}_+^2 : x_2 - x_1 = \theta\}$. In order to derive an expression for Λ which

⁹On can split the boundary into several “maximal parametrizations” in order to reduce the problem to global ones.

satisfies (37), we note that $\partial_\theta \psi = -1$, $\partial_{x_2} \psi = 1$ and on \mathfrak{B}_θ we have the global parametrization $x_2 = \vartheta(x_1) := \theta + x_1$ for any $x_1 \in \Pi_2 \mathfrak{B}_\theta$, i.e., $x_1 > 0$. Finally, to derive the conditional density of S_2 given $S_1 = x_1$, we note that, in this case, the matrix \mathbf{R} satisfies

$$\mathbf{R} := \begin{bmatrix} 1 & \varrho \\ \varrho & 1 \end{bmatrix} \Rightarrow \mathbf{R}^{-1} = \frac{1}{1 - \varrho^2} \begin{bmatrix} 1 & -\varrho \\ -\varrho & 1 \end{bmatrix},$$

where $\varrho = \varrho_{12} = \varrho_{21} \in (-1, 1)$. Consequently, according to (31),

$$\mathbf{C}^{-1} = \frac{1}{t(1 - \varrho^2)} \begin{bmatrix} \sigma_1^{-2} & \varrho(\sigma_1 \sigma_2)^{-1} \\ \varrho(\sigma_1 \sigma_2)^{-1} & \sigma_2^{-2} \end{bmatrix}.$$

Hence, by Corollary 1; see the Appendix, the conditional density $\rho_{2|1}(x_2|x_1)$ is given by

$$\rho_{2|1}(x_2|x_1) = \frac{\sqrt{1 - \varrho^2}}{\sigma_2 x_2 \sqrt{2\pi t}} \exp \left(-\frac{\left(\frac{\mu_2 - \ln x_2}{\sigma_2} - \varrho \frac{\mu_1 - \ln x_1}{\sigma_1} \right)^2}{2t(1 - \varrho^2)} \right) \mathbf{1}_{(0, \infty)}(x_2),$$

for any $x_2 > 0$, with μ_1, μ_2 given by (29). Finally, we conclude that

$$\Lambda_2(\theta, S_1) = \frac{-\sqrt{1 - \varrho^2}}{\sigma_2 \sqrt{2\pi t}} \frac{\wp(S_1, \theta + S_1)}{\theta + S_1} \exp \left(-\frac{\left(\frac{\mu_2 - \ln(\theta + S_1)}{\sigma_2} - \varrho \frac{\mu_1 - \ln S_1}{\sigma_1} \right)^2}{2t(1 - \varrho^2)} \right).$$

If, for instance, $\wp = 1$ and the stocks are varying independently, i.e., $\varrho = 0$, hence $V = e^{-rt} \mathbb{P}\{S_2 > \theta + S_1\}$, then the expression of Λ_2 reduces to

$$\Lambda_2(\theta, S_1) = \frac{-1}{\sigma_2 \sqrt{2\pi t} (\theta + S_1)} \exp \left(-\frac{\mu_2 - \ln(\theta + S_1)}{2\sigma_2^2 t} \right).$$

Exactly the same result is obtained by direct computation, using the fact that

$$\mathbb{P}\{S_2 > \theta + S_1\} = \mathbb{E} \left[\frac{1}{\sigma_2 \sqrt{2\pi t}} \int_{\ln(\theta + S_1)}^{\infty} \exp \left(-\frac{(\mu_1 - \ln x)^2}{2\sigma_2^2 t} \right) dx \right];$$

indeed, the θ -derivative of the above integrand coincides with $\Lambda_2(\theta, S_1)$.

3.2 The Digital Madonna Rainbow Option

Let us consider a digital option on n assets which pays off some amount $\wp(\mathbf{S})$ when $\|\mathbf{S}\|_2$ does not exceed a certain threshold $\theta > 0$. We formalize that by taking in (36)

$$\psi(\theta, \mathbf{x}) := \theta^2 - \|\mathbf{x}\|_2^2 = \theta^2 - x_1^2 + \dots + x_n^2.$$

Then \mathfrak{D}_θ is the interior of the ‘‘positive region’’ of the disc with radius θ , i.e.,

$$\mathfrak{D}_\theta := \{ \mathbf{x} \in \mathbb{R}_+^n : x_1^2 + \dots + x_n^2 < \theta^2 \}.$$

Hence, we are dealing with a put option. The boundary \mathfrak{B}_θ is then given by¹⁰

$$\mathfrak{B}_\theta := \{ \mathbf{x} \in \mathbb{R}_+^n : x_1^2 + \dots + x_n^2 = \theta^2 \}$$

¹⁰In fact, the boundary of \mathfrak{D}_θ contains also parts of the hyper-planes corresponding to $x_i = 0$. However, we are not interested in these parts of the boundary since they do not depend on θ . In fact, to comply with the theory put forward in Section 2, the velocity at the boundary in these points is 0, hence the corresponding surface integrals vanish.

and admits the global parametrization¹¹

$$x_n = \vartheta_n(\mathbf{w}) = \sqrt{\theta^2 - (x_1^2 + \dots + x_{n-1}^2)} = \sqrt{\theta^2 - \|\mathbf{w}\|_2^2},$$

which, cf. Example 2, is valid for

$$\mathbf{w} := (x_1, \dots, x_{n-1}) \in \Pi_{\bar{n}} \mathfrak{D}_\theta = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}_+^{n-1} : x_1^2 + \dots + x_{n-1}^2 < \theta^2\}.$$

Moreover, by Corollary 2 in the Appendix, the conditional density $\rho_{n|\bar{n}}(\cdot|\mathbf{w})$, for $\mathbf{w} \in \mathbb{R}_+^{n-1}$, is given by

$$\rho_{n|\bar{n}}(x_n|\mathbf{w}) = \frac{\sqrt{\bar{\varrho}_{nn}}}{\sigma_n x_n \sqrt{2\pi t}} \exp\left(-\frac{\left(\sum_{j=1}^n \bar{\varrho}_{jn} \frac{(\mu_j - \ln x_j)}{\sigma_j}\right)^2}{2t \bar{\varrho}_{nn}}\right) \mathbf{1}_{(0, \infty)}(x_n),$$

where $\mathbf{R}^{-1} = [\bar{\varrho}_{ij}]_{1 \leq i, j \leq n}$ and μ is given by (29). Taking now into account that $\partial_\theta \psi(\theta, \mathbf{x}) = 2\theta$ and $\partial_n \psi(\theta, \mathbf{x}) = -2x_n$, we conclude from (37) that

$$\Lambda_n(\theta, \mathbf{S}_{\bar{n}}) = \frac{\theta}{\sqrt{\theta^2 - \|\mathbf{S}_{\bar{n}}\|_2^2}} \wp\left(\mathbf{S}_{\bar{n}}, \sqrt{\theta^2 - \|\mathbf{S}_{\bar{n}}\|_2^2}\right) \rho_{n|\bar{n}}\left(\sqrt{\theta^2 - \|\mathbf{S}_{\bar{n}}\|_2^2} \middle| \mathbf{S}_{\bar{n}}\right).$$

The expression in the last display provides an estimator for the sensitivity w.r.t. θ of the digital Madonna rainbow (put) option. An estimator for the sensitivity of the corresponding call option, obtained for $\psi(\theta, \mathbf{x}) = \|\mathbf{x}\|_2^2 - \theta^2$, is the negative of the expression of $\Lambda_n(\theta, \mathbf{S}_{\bar{n}})$ in the last display. This can be either seen from the fact that the premiums of the call and put options sum up to a constant (w.r.t. θ) or directly deduced by repeating the above arguments for the new ψ . Finally, note that if one replaces n by any index $i = 1 \dots n$ the corresponding estimator $\Lambda_i(\theta, \mathbf{S}_{\bar{i}})$ has the same properties.

3.3 The Asset/Cash-or-Nothing Option

The AON/CON options derive from classical puts and calls and classify accordingly. Namely, if $\theta := (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n$ is a positive vector, they bring some profit only if the vector of stock prices at time t satisfies $\mathbf{S} \leq \theta$, i.e., if $S_i \leq \theta_i$, for any $1 \leq i \leq n$, for a AON put and $\mathbf{S} \geq \theta$ for a AON call. Therefore, the feasibility region for the AON call is given by¹²

$$(\mathbf{0}, \theta) := (0, \theta_1) \times \dots \times (0, \theta_n),$$

and the payoff function may depend on the vector \mathbf{S} , at most. The AON option is a digital option and one can define its binary counterpart, the *cash-or-nothing option*, by replacing \wp by a constant, e.g., $\wp = 1$.

Let $\wp : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function and consider the AON option which pays $\wp(\mathbf{S})$ units if the vector of stock prices is in the domain $(\mathbf{0}, \theta)$. Then its premium is given by

$$V(\theta) = e^{-rt} \mathbb{E}[\wp(\mathbf{S}) \mathbb{I}\{\mathbf{S} < \theta\}].$$

¹¹A global parametrization can be chosen because all components x_i , in particular x_n , are non-negative on \mathfrak{D}_θ .

¹²We can switch between $\mathbf{S} > \theta$ and $\mathbf{S} \geq \theta$ since the distribution of \mathbf{S} is continuous.

We want to estimate the sensitivity of V w.r.t. θ , i.e., the partial derivatives $\partial_{\theta_i} V(\theta)$, for $1 \leq i \leq n$. Let $i = n$, take $\Omega = \prod_{j=1}^{n-1} (0, \theta_j) \times (0, \infty)$ and let $\psi(\theta, \mathbf{x}) = \theta_n - x_n$. Note that the variable boundary of the domain is given by

$$\{\mathbf{x} \in \Omega : x_n = \theta_n\} \subset \mathcal{H}_n := \{\mathbf{x} \in \mathbb{R}^n : x_n = \theta_n\}.$$

Hence, the boundary of the domain is supported by the hyperplane \mathcal{H}_n and admits the global parametrization $\vartheta_n(x_1, \dots, x_{n-1}) = \theta_n$, on $\prod_{j=1}^{n-1} (0, \theta_j)$. In addition, for any $\mathbf{x} \in \mathcal{H}_n$ we have $\partial_{\theta_n} \psi(\theta, \mathbf{x}) = 1$ and $\partial_{x_n} \psi(\theta, \mathbf{x}) = -1$, hence

$$\Lambda_n(\theta, \mathbf{S}_{\bar{n}}) = \wp(\mathbf{S}_{\bar{n}}, \theta_n) \rho_{n|\bar{n}}(\theta_n | \mathbf{S}_{\bar{n}}) \prod_{j=1}^{n-1} \mathbb{I}\{S_j < \theta_j\},$$

where, in accordance with Corollary 2 (see the Appendix) we have

$$\rho_{n|\bar{n}}(\theta_n | \mathbf{S}_{\bar{n}}) = \frac{\sqrt{\bar{q}_{nn}}}{\sigma_n \theta_n \sqrt{2\pi t}} \exp \left(- \frac{\left(\bar{q}_{nn} \frac{\mu_n - \ln \theta_n}{\sigma_n} + \sum_{j=1}^{n-1} \bar{q}_{jn} \frac{\mu_j - \ln S_j}{\sigma_j} \right)^2}{2t \bar{q}_{nn}} \right).$$

A similar expression can be obtained for each partial derivative $\partial_{\theta_i} V(\theta)$, for any $1 \leq i \leq n$, by replacing in the above reasoning n by i . Finally, for the AON put, one should take the negative of the sensitivity of the AON call.

Conclusions and Future Research

In this paper we have developed a methodology for estimating sensitivity of option premiums, using Monte-Carlo simulation, in a rather general framework. The theoretical fundament for this methodology relies on establishing differentiation formulas for integrals with parameter, when the parameter may also induce variability of the integration domain. As pointed out in Section 3, our framework contains most common situations in financial pricing, e.g., sensitivity w.r.t. intrinsic parameters (Greeks). Apart from establishing a multi-dimensional version of the differentiation rule for integrals on moving domains (itself a result with a significant theoretical value which is scarcely available in the literature in such an elementary and intuitive form) which leads to a unified approach to gradient estimation of option premiums, the main contribution of this paper is that it extends the classical procedures to the study of the sensitivity of multi-asset digital/binary options w.r.t. non-intrinsic parameters. That is, if $\wp(\theta, \cdot)$ is a payoff function and \mathfrak{D}_θ is the exercise domain of the financial option then one can determine a measurable function F , depending on \wp and the "dynamics" of the domain \mathfrak{D}_θ only, satisfying

$$\partial_\theta \mathbb{E} [\wp(\theta, \mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\}] = \mathbb{E} [\partial_\theta \wp(\theta, \mathbf{S}) \mathbb{I}\{\mathbf{S} \in \mathfrak{D}_\theta\} + F(\theta, \mathbf{S})].$$

Recall that if the function $\wp(\theta, \cdot)$ is identically null on (the variable part of) the boundary of \mathfrak{D}_θ then $F(\theta, \mathbf{S}) = 0$; in principle, $F(\theta, \mathbf{S})$ can be derived whenever the joint density of the vector of stock prices is known. This technique of transforming surface integrals into (proper) expected values is a novelty, to the best of author's knowledge. As illustrated in Section 3, closed form expressions

for F can be found in most of the cases and they can be easily implemented whenever conditional distributions of stock prices can be computed.

An interesting topic for further research is to extend this methodology to sensitivity of barrier options, i.e., financial options for which the exercise rule depends upon the whole path of stock prices up to maturity and not only on the final value. For instance, consider a financial option on a single asset which pays at maturity t some amount of money, which may, or may not depend on $S(t)$, only if the stock-price, which is currently s , will not exceed a certain threshold $\theta > s$ up to maturity. In other words, the option becomes void if at some moment $\tau < t$ we have $S(\tau) \geq \theta$ even if at maturity we have $S(t) < \theta$.

From a measure-valued differentiation perspective, the results in this paper establish weak differentiability, as well as the expression of the weak derivatives, of a wide class of truncated distributions in the multi-dimensional Euclidian space. In practice, if \mathbf{X} is some random vector and \mathfrak{D}_θ is some variable domain satisfying some regularity assumptions, such that $\mathbb{P}(\mathbf{X} \in \mathfrak{D}_\theta) > 0$ locally, i.e., for all θ 's in some open set, then it turns out that the conditional distribution $\mathbb{P}(\mathbf{X} \in \cdot | \mathbf{X} \in \mathfrak{D}_\theta)$ is weakly differentiable and

$$\partial_\theta \mathbb{E}[f(\mathbf{X}) | \mathbf{X} \in \mathfrak{D}_\theta] = \partial_\theta \frac{\mathbb{E}[f(\mathbf{X}) \mathbb{I}\{\mathbf{X} \in \mathfrak{D}_\theta\}]}{\mathbb{P}(\mathbf{X} \in \mathfrak{D}_\theta)}$$

can be expressed as the re-scaled difference between two stochastic experiments, out of which one is the original one, i.e., $\mathbb{E}[f(\mathbf{X}) | \mathbf{X} \in \mathfrak{D}_\theta]$. Identifying and developing interesting applications of this result is also a topic for future research.

Appendix: Gaussian Vectors

The random vector¹³ $\mathbf{X} := (X_1, \dots, X_n)^*$ will be called a *Gaussian vector* if for any $\beta \in \mathbb{R}^n$ the random variable $\beta^* \mathbf{X}$ has normal distribution. Equivalently, \mathbf{X} is Gaussian if and only if there exist $\mu := (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ and a symmetric, positive semi-definite matrix $\Sigma = [\sigma_{ij}]_{1 \leq i, j \leq n}$ such that the characteristic function ϕ of \mathbf{X} satisfies

$$\forall \beta \in \mathbb{R}^n : \phi(\beta) := \mathbb{E} \left[e^{i \beta^* \mathbf{X}} \right] = \exp \left(i \beta^* \mu - \frac{1}{2} \beta^* \Sigma \beta \right). \quad (38)$$

When the vector \mathbf{X} is non-degenerate, i.e., Σ is non-singular, its probability density is given by

$$\forall \mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x} | \mu, \Sigma) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp \left(-\frac{1}{2} (\mathbf{x} - \mu)^* \Sigma^{-1} (\mathbf{x} - \mu) \right). \quad (39)$$

The vector μ is called the mean of \mathbf{X} and the matrix Σ is called the covariance matrix of \mathbf{X} since we have

$$\forall 1 \leq i, j \leq n : \mathbb{E}[X_i] = \mu_i, \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] = \sigma_{ij}.$$

The components X_1, \dots, X_n are mutually independent if and only if the matrix Σ is diagonal. The following result shows that the marginal distribution of every sub-vector of \mathbf{X} is Gaussian.

Lemma 2 *If $\mathbf{X} := (X_1, \dots, X_n)^*$ is a n -dimensional Gaussian vector then, for any $1 \leq i_1 < \dots < i_k \leq n$, for $k \leq n$, $\mathbf{Y} := (X_{i_1}, \dots, X_{i_k})^*$ is a k -dimensional Gaussian vector. If \mathbf{X} has mean μ and covariance matrix Σ then the mean and covariance matrix of \mathbf{Y} are obtained by removing from μ and Σ , respectively, the rows/columns corresponding to indices $j \notin \{i_1, \dots, i_k\}$.*

Proof: We can assume w.l.o.g. that $i_j = j$, for $1 \leq j \leq k$. By taking in (38) $\beta = (\xi, 0, \dots, 0)^*$, where $\xi \in \mathbb{R}^k$ is arbitrary, we obtain for the characteristic function of \mathbf{Y}

$$\mathbb{E} \left[e^{i \xi^* \mathbf{Y}} \right] = \mathbb{E} \left[e^{i \beta^* \mathbf{X}} \right] = \exp \left(i \beta^* \mu - \frac{1}{2} \beta^* \Sigma \beta \right).$$

To conclude the proof, note that the r.h.s. above equals

$$\exp \left(i \sum_{j=1}^n \beta_j \mu_j - \frac{1}{2} \sum_{i,j=1}^n \beta_i \sigma_{ij} \beta_j \right) = \exp \left(i \sum_{j=1}^k \beta_j \mu_j - \frac{1}{2} \sum_{i,j=1}^k \beta_i \sigma_{ij} \beta_j \right),$$

since, by assumption, $\beta_j = 0$ for $j > k$. □

Corollary 1 *If $\mathbf{X} := (X_1, \dots, X_n)^*$ is a n -dimensional Gaussian vector with mean μ and covariance matrix Σ then any component X_i , for $1 \leq i \leq n$, is normally distributed with mean μ_i and variance σ_{ii} .*

¹³In this section all vectors are assumed to be column-vectors and the superscript t denotes the transposition operator.

Proof: It is immediate from Lemma 2. \square

In the following we only consider non-degenerate Gaussian vectors. That is, we assume that the covariance matrix Σ is positive definite, hence non-singular, and the density is given by (39).

Lemma 3 *Let $\mathbf{X} := (X_1, \dots, X_n)^*$ be a n -dimensional Gaussian vector with mean μ and covariance matrix Σ and let $\mathbf{X}_{\bar{n}} := (X_1, \dots, X_{n-1})^*$. Let us consider the matrix $\Sigma_{\bar{n}} := [\sigma_{ij}]_{1 \leq i, j \leq n-1}$ and the vectors $\mu_{\bar{n}} := (\mu_1, \dots, \mu_{n-1})^*$ and $\mathbf{1} := (\sigma_{1n}, \dots, \sigma_{n-1n})^*$, i.e.,*

$$\mu = \begin{bmatrix} \mu_{\bar{n}} \\ \mu_n \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{\bar{n}} & \mathbf{1} \\ \mathbf{1}^* & \sigma_{nn} \end{bmatrix}.$$

Then the conditional distribution of X_n w.r.t. $\mathbf{X}_{\bar{n}} = \xi$ is normal with mean $m(\xi)$ and variance σ^2 where

$$\forall \xi \in \mathbb{R}^{n-1} : m(\xi) := \mu_n + (\xi - \mu_{\bar{n}})^* \Sigma_{\bar{n}}^{-1} \mathbf{1}, \quad \sigma^2 := \sigma_{nn} - \mathbf{1}^* \Sigma_{\bar{n}}^{-1} \mathbf{1}. \quad (40)$$

Proof: We start by noting that since Σ is a positive definite matrix so is $\Sigma_{\bar{n}}$ which shows that $\Sigma_{\bar{n}}$ is non-singular and it makes sense to consider its inverse in (40). By Lemma 2 we know that $\mathbf{X}_{\bar{n}}$ is a Gaussian vector having density

$$\gamma(\xi | \mu_{\bar{n}}, \Sigma_{\bar{n}}) = \frac{1}{\sqrt{(2\pi)^{n-1} \det \Sigma_{\bar{n}}}} \exp\left(-\frac{1}{2}(\xi - \mu_{\bar{n}})^* \Sigma_{\bar{n}}^{-1} (\xi - \mu_{\bar{n}})\right), \quad (41)$$

for $\xi \in \mathbb{R}^{n-1}$ and by assumption; see (39), the density of \mathbf{X} is given by

$$\forall \mathbf{x} \in \mathbb{R}^n : \gamma(\mathbf{x} | \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^* \Sigma^{-1} (\mathbf{x} - \mu)\right). \quad (42)$$

The conditional density of X_n w.r.t. $\mathbf{X}_{\bar{n}}$ can be calculated by dividing the expression in (42), for $\mathbf{x} = (\xi, x)$, by the expression in (41). That is, we obtain

$$f_{X_n}(x | \mathbf{X}_{\bar{n}} = \xi) = \frac{\gamma(\xi, x | \mu, \Sigma)}{\gamma(\xi | \mu_{\bar{n}}, \Sigma_{\bar{n}})}.$$

Hence, in accordance with (41) and (42), the conclusion is equivalent to

$$(\mathbf{x} - \mu)^* \Sigma^{-1} (\mathbf{x} - \mu) - (\xi - \mu_{\bar{n}})^* \Sigma_{\bar{n}}^{-1} (\xi - \mu_{\bar{n}}) = \sigma^{-2} (x - m(\xi))^2, \quad (43)$$

for $\mathbf{x} = (\xi, x) \in \mathbb{R}^n$ and $\det \Sigma = \sigma^2 \det \Sigma_{\bar{n}}$. Consider now the partitioning

$$\Sigma^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^* & c \end{bmatrix},$$

where \mathbf{A} is a $(n-1)$ -dimensional square matrix, \mathbf{b} is a $(n-1)$ -dimensional vector and c is a real number. Note that such a partitioning is possible since Σ^{-1} is a symmetric matrix. The l.h.s. in (43) then equals

$$c(x - \mu_n)^2 + 2(x - \mu_n)(\xi - \mu_{\bar{n}})^* \mathbf{b} + (\xi - \mu_{\bar{n}})^* (\mathbf{A} - \Sigma_{\bar{n}}^{-1}) (\xi - \mu_{\bar{n}}). \quad (44)$$

Now note that

$$\mathbf{I}_n = \Sigma \cdot \Sigma^{-1} = \begin{bmatrix} \Sigma_{\bar{n}} & \mathbf{1} \\ \mathbf{1}^* & \sigma_{nn} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^* & c \end{bmatrix},$$

where, for $n \geq 1$, we denote by \mathbf{I}_n the unit n -dimensional matrix. Writing explicitly the above equality yields the following linear system

$$\begin{cases} \Sigma_{\bar{n}}\mathbf{A} + \mathbf{I}\mathbf{b}^* = \mathbf{I}_{n-1}, & \text{(Eq. 1),} \\ \Sigma_{\bar{n}}\mathbf{b} + c\mathbf{1} = \mathbf{0}, & \text{(Eq. 2),} \\ \mathbf{A}\mathbf{1} + \sigma_{nn}\mathbf{b} = \mathbf{0}, & \text{(Eq. 3),} \\ \mathbf{b}^*\mathbf{1} + c\sigma_{nn} = 1, & \text{(Eq. 4).} \end{cases}$$

If $c = 0$ then, since $\Sigma_{\bar{n}}$ is non-singular, we conclude from (Eq. 2) that $\mathbf{b} = \mathbf{0}$, which violates (Eq. 4). Hence, we have $c > 0$. Dividing (Eq. 2) by c , we obtain

$$\Sigma_{\bar{n}}\mathbf{b} = -c\mathbf{1} \Rightarrow \mathbf{1} = -c^{-1}\Sigma_{\bar{n}}\mathbf{b}. \quad (45)$$

Substituting this value of $\mathbf{1}$ in (Eq. 1) we obtain

$$\Sigma_{\bar{n}}\mathbf{A} - c^{-1}\Sigma_{\bar{n}}\mathbf{b}\mathbf{b}^* = \mathbf{I}_{n-1} \Rightarrow \Sigma_{\bar{n}}^{-1} = \mathbf{A} - c^{-1}\mathbf{b}\mathbf{b}^*. \quad (46)$$

Substituting now the above value in (44) we obtain for the expression in (44)

$$c[(x - \mu_n) + c^{-1}(\xi - \mu_{\bar{n}})^*\mathbf{b}]^2 = c(x - m(\xi))^2, \quad (47)$$

since (45) implies $\mathbf{b} = -c\Sigma_{\bar{n}}^{-1}\mathbf{1}$. Now, substituting this value in (Eq. 4) yields

$$1 = c(\sigma_{nn} - \mathbf{1}^*\Sigma_{\bar{n}}^{-1}\mathbf{1}) = c\sigma^2 \Rightarrow c = \sigma^{-2}. \quad (48)$$

Finally, substituting this value in (47) proves (43). To conclude the proof, we note that $(\sigma_{nn} - \mathbf{1}^*\Sigma_{\bar{n}}^{-1}\mathbf{1})$ is the Schur complement of $\Sigma_{\bar{n}}$ in Σ ; see [?], hence

$$\det \Sigma = (\sigma_{nn} - \mathbf{1}^*\Sigma_{\bar{n}}^{-1}\mathbf{1}) \det \Sigma_{\bar{n}} = \sigma^2 \det \Sigma_{\bar{n}},$$

where, for establishing the last equality, we used again (Eq. 4). \square

Remark 1 Lemma 3 actually gives the conditional density of X_n given $\mathbf{X}_{\bar{n}}$. By a symmetry argument, a result similar to that in Lemma 3 holds true for any component X_i . More specifically, in (40) one replaces $\Sigma_{\bar{n}}$ by $\Sigma_{\bar{i}}$, obtained by removing the i^{th} row and the i^{th} column of Σ , $\mathbf{1}$ by the i^{th} column of Σ from which σ_{ii} is removed and $\mu_{\bar{n}}$ is replaced by $\mu_{\bar{i}}$, obtained by removing μ_i out of μ .

Remark 2 Analyzing the proof of Lemma 3; in fact, equations (45) and (48), we note that the conditional density of X_n given $\mathbf{X}_{\bar{n}} = \xi$ can be easier expressed in terms of the inverse of the covariance matrix. Namely, if \mathbf{X} is a non-degenerate gaussian vector with mean μ and covariance (non-singular) matrix Σ , such that

$$\Sigma^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^* & c \end{bmatrix},$$

then the conditional density of X_n given $\mathbf{X}_{\bar{n}} = \xi$ is Gaussian with parameters $(m(\xi), \sigma^2)$ given by

$$m(\xi) := \mu_n - c^{-1}(\xi - \mu_{\bar{n}})^*\mathbf{b}, \quad \sigma^2 = c^{-1}. \quad (49)$$

In equivalent formulation, if the Gaussian vector \mathbf{X} has mean μ and covariance matrix \mathbf{B}^{-1} , then the ‘‘conditional parameters’’ in (49) are given by

$$m(\xi) := \mu_n - b_{nn}^{-1}(\xi - \mu_{\bar{n}})^*\mathbf{b}_n, \quad \sigma^2 = b_{nn}^{-1},$$

where $\mathbf{B} = [b_{ij}]_{1 \leq i, j \leq n}$ and \mathbf{b}_n is the vector obtained by removing b_{nn} from the last column of \mathbf{B} , i.e.,

$$\mathbf{b}_n = (b_{1n}, \dots, b_{n-1n})^*.$$

By Remark 1, a similar fact holds true for any index i instead of n .

We say that a random vector $\mathbf{S} := (S_1, \dots, S_n) \in \mathbb{R}_+^n$ is (non-degenerate) log-normal with parameters $\mu \in \mathbb{R}^n$ and Σ , where Σ is a symmetric $n \times n$ matrix, if the random vector $\mathbf{X} := (X_1, \dots, X_n)$ with components $X_i := \ln S_i$ is a (non-degenerate) Gaussian vector with mean μ and covariance matrix Σ . In the particular case $n = 1$, the density of a log-normal variable with parameters μ and σ^2 is given by

$$f_{\mu, \sigma^2}(\xi) := \frac{1}{\sigma \xi \sqrt{2\pi}} \exp \left[-\frac{(\mu - \ln \xi)^2}{2\sigma^2} \right] \mathbf{1}_{(0, \infty)}(\xi).$$

The next statement follows immediately from Lemmas 2, 3 and Remark 2.

Corollary 2 *If \mathbf{S} is a non-degenerate log-normal n -dimensional vector with parameters μ and Σ then the vector $\mathbf{S}_{\bar{n}} := (S_1, \dots, S_{n-1})$ is also log-normal with parameters obtained from μ , resp. Σ , by removing the n^{th} element, resp. n^{th} row and column. Furthermore, if $\mathbf{B} = \Sigma^{-1}$, the conditional density of S_n , given that $\mathbf{S}_{\bar{n}} = (\xi_1, \dots, \xi_{n-1})$, is also log-normal with parameters*

$$m(\xi_1, \dots, \xi_{n-1}) := \mu_n + b_{nn}^{-1} \sum_{i=1}^{n-1} (\mu_i - \ln \xi_i) b_{in}, \quad \sigma^2 = b_{nn}^{-1},$$

where $\mathbf{B} = [b_{ij}]_{1 \leq i, j \leq n}$ and $\mathbf{b}_n = (b_{1n}, \dots, b_{n-1n})^*$. Furthermore, the conditional density of S_n , given that $\mathbf{S}_{\bar{n}} = (\xi_1, \dots, \xi_{n-1})$, can be expressed as

$$\rho_{n|\bar{n}}(\xi_n | \xi_1, \dots, \xi_{n-1}) = \frac{\sqrt{b_{nn}}}{\xi_n \sqrt{2\pi}} \exp \left[-\frac{(\sum_{i=1}^n (\mu_i - \ln \xi_i) b_{in})^2}{2b_{nn}} \right] \mathbf{1}_{(0, \infty)}(\xi_n).$$

By a symmetry argument, the above formula remains valid if n is replaced by any index $i = 1, 2, \dots, n-1$, after performing necessary modifications.

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